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Holographic RG flows on curved manifolds and quantum phase transitions

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ABSTRACT: Holographic RG flows dual to QFTs on maximally symmetric curved manifolds (dS_d , AdS_d , and S^d) are considered in the framework of Einstein-dilaton gravity in $d + 1$ dimensions. A general dilaton potential is used and the flows are driven by a scalar relevant operator. The general properties of such flows are analyzed and the UV and IR asymptotics computed. New RG flows can appear at finite curvature which do not have a zero curvature counterpart. The so-called ‘bouncing’ flows, where the β -function has a branch cut at which it changes sign, are found to persist at finite curvature. Novel quantum first-order phase transitions are found, triggered by a variation in the d -dimensional curvature in theories allowing multiple ground states.

KEYWORDS: AdS-CFT Correspondence, Gauge-gravity correspondence, Renormalization Group

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1 Introduction, summary of results and outlook

Renormalization group (RG) flows in Quantum Field Theory (QFT) are usually studied in flat space. There are, however, many reasons to consider QFTs on curved manifolds and study the associated RG flows.

One reason is that curved manifolds are considered in order to render QFTs well defined or well controlled in the IR, by taming IR divergences. There are many facets of this idea, going back to [1], and to [2] for a similar approach of regulating IR divergences in string theory. On the holographic side of QFTs, this is the role played by global AdS space. There, the QFT lives on $R \times S^d$, where the spatial part is a sphere.

Curved manifolds also provide IR modifications to supersymmetric QFTs in a way that supersymmetric indices or other supersymmetric observables are well-defined, [3, 4]. Moreover, they provide control parameters on which supersymmetric observables can depend upon. Non-trivial backgrounds in general can be useful in deriving non-trivial results like exact β -functions, [5]. Curved manifolds (spheres) were used to regulate otherwise singular holographic solutions in [6]. Holographic flows with de Sitter slices were discussed in [7].

Further objects of interest are partition functions on curved manifolds. In particular, the partition function on spheres was argued to serve as an analogue of the c-functions in odd dimensions. The case of three dimensions is well known, [8, 9] but from holographic arguments the case can be made also for other dimensions, [10, 11]. The dynamics of QFTs on curved manifolds may have interesting and different structure from that on flat manifolds, especially in the case of QFTs on AdS manifolds, [12].

Cosmology is another important context where quantum field theories on curved spacetimes may be considered. There is a vast body of work in this context. In connection to the renormalization group flow aspect, for example, the non-perturbative RG group on de Sitter backgrounds was studied for large- N scalar field theories in [13–19].

There is a folk theorem which says that RG flows of QFTs on curved manifolds are very similar to those on flat manifolds. The argument for this is that β -functions are determined by UV/short-distance divergences and the short distance structure of a given QFT is independent of its curvature. Although the leading intuition of such statements is basically correct, the folk theorem fails on several grounds. Indeed, the leading UV divergences are independent of curvature. However, subleading ones do depend on curvature. We will see this clearly and in a controlled fashion in this paper, although it is well known to experts, [20–22]. A further observation is that already for CFTs, curvature is a source of breaking of scale invariance via the conformal anomaly, [23]. For general QFTs driven by

relevant couplings, the β functions do in general depend on curvature, [24]. The same is true for the vacuum expectation values of operators. More generally, one may expect that curvature becomes very important in the IR and this expectation is, in general, correct.

In this work we will study the RG behavior of QFTs on maximally symmetric curved space-times using the framework of holography. The holographic correspondence, [25–27], provides a map between QFT and gravity/string-theories in higher dimensions, at least in the limit of large N . In this context, the holographic dimension serves as an effective RG scale in the dual QFT, thus geometrizing the notion of RG flow. In essence, RG flows can be understood as bulk evolution in the holographic dimensions [10, 28–45].

In this framework, there is additional motivation for studying RG flows in the presence of curvature.

- Vacuum expectation values depend crucially on curvature. In perturbation theory, vevs are trivial and therefore they do not affect the RG running. In holographic QFTs however, they are non-trivial generically, they do depend on curvature, and they do affect the RG flows. In this work we will show many non-trivial modifications, driven by curvature, of (holographic) RG flows. These modifications may be so significant that they can trigger quantum phase transitions between different ground states of the theory. For example, in [46] quantum phase transitions were observed for a supersymmetric field theory on $\mathbb{R} \times S^3$. In this work we show the presence of such a quantum phase transition in an explicit example for a field theory on dS_4 .¹
- Holography provides a means of calculating the induced action for the background metric in QFT. Integrating out a QFT coupled to a background metric provides a Schwinger functional for the metric that can be turned into an effective action for the expectation value of the stress tensor. This is the starting point for many cosmology setups, like that of Starobinsky, [48], and its generalizations, [49]. In the case of a CFT, this effective action is rather simple as the only scale is the UV cutoff, and there are only a finite number (depending on the space-time dimension) of curvature-related terms that can appear. For example in four dimensions one obtains a cosmological constant at order $\mathcal{O}(\Lambda^4)$, an Einstein term at order $\mathcal{O}(\Lambda^2)$ and the conformal anomaly at order $\mathcal{O}(\log \Lambda^2)$. The curvature squared term is scheme dependent due to the form of the conformal anomaly. All other terms are suppressed by the cutoff.

In more general QFTs, the situation is more complex as now the QFT has more dimensionful parameters, that can enter the effective action for the metric/curvature. Such effective actions are largely unexplored, as computing them in perturbation theory is a difficult task. Here, in the context of holographic theories we will provide a concrete framework to compute them. More detailed results will appear in a forthcoming publication [50].

In particular, we will be working at constant curvature $\mathcal{R}$, and the effective action we will compute is a function of $\mathcal{R}$. In this computation, derivatives of the curvature

¹Phase transitions triggered by a change in space-time curvature in CFTs on de Sitter space *at finite temperature* were addressed in [47].

are neglected and therefore the effective action is valid for manifolds whose curvature scale is sufficiently slowly varying. In that sense it is similar to the DBI action for gauge fields in string theory. This approach is complementary to the one followed in e.g. [44], where RG flows were studied in theories where the curvature tensor is small but generic (i.e. the space-time is not maximally symmetric).

- In the Hartle-Hawking proposal for the wave-function of the universe, [51], such a wave-function is determined by a gravitational path integral. It has been proposed in [52, 53] that holography can be used to define such a path integral and generate the relevant wave-functions. Our results provide the spatial curvature dependence of such wave-functions.
- In [54] a new approach was proposed using holography in order to implement the self-tuning idea for the cosmological constant. In order to further address solutions for the universe brane with constant (positive or negative) curvature one must use for the bulk the type of solutions we are studying in this paper. Their application to the self-tuning problem will be discussed in a forthcoming publication.

In the following we summarize our setup and results.

1.1 Summary of results

We use a two-derivative $(d+1)$ -dimensional model described by the action:

$$S[g, \varphi] = \int du d^d x \sqrt{-g} \left(R^{(g)} - \frac{1}{2} \partial_a \varphi \partial^a \varphi - V(\varphi) \right). \quad (1.1)$$

We consider solutions with a scalar field profile and a metric ansatz given by:

$$\varphi = \varphi(u), \quad ds^2 = du^2 + e^{2A(u)} \zeta_{\mu\nu} dx^\mu dx^\nu. \quad (1.2)$$

We will study solutions where $\zeta_{\mu\nu}$ is a fixed (i.e. u -independent) metric on a maximally symmetric d -dimensional space-time with positive or negative curvature. With our work, we extend to these space-times the systematic analysis of flat RG flows presented in [45]. Via gauge/gravity duality, the ansatz (1.2) describes RG flows of field theories defined on manifolds with constant positive curvature (S^d or dS_d , depending on the signature²) or constant negative curvature (H^d , AdS_d).

The scalar field profile $\varphi(u)$ is interpreted as the running coupling associated to a scalar operator $\mathcal{O}$ of the field theory. We will focus our attention on scalar potentials that display extrema with negative cosmological constant, which can serve as a UV/IR fixed points for the dual field theory. More specifically, we will consider geometries which contain an asymptotic boundary region, where the scalar field reaches the maximum of the potential and the full metric asymptotes to (the boundary region of) $d+1$ -dimensional Anti-de Sitter space-time.

From the dual field theory standpoint, these flows describe a UV CFT which is defined on a curved space-time and deformed by a relevant operator. In this language, the two

²The form of the solution as well as most of our final results do not depend on the space-time signature.

Figure 1. Family of solutions $W_{C,R}(\varphi)$ in the vicinity of a UV fixed point located at a maximum of the potential. For a field theory with a given UV value of the source and fixed R^{uv} only a finite number of these solutions (here only one) can be completed into a flow reaching an IR end point at φ_0 . The shaded region below the blue curve is not accessible for $R^{\text{uv}} \geq 0$.

parameters defining the UV theory are R^{uv} (the scalar curvature of the space-time where and the UV CFT is defined) and the leading coefficient in the scalar field near-boundary expansion, φ_- (the coefficient of the relevant deformation). They appear in the near-boundary asymptotic expansion as:

$$ds^2 \simeq du^2 + e^{-2u/\ell} \gamma_{\mu\nu}^{\text{uv}} dx^\mu dx^\nu, \quad \varphi \simeq \varphi_- \ell^{(d-\Delta)} e^{(d-\Delta)u/\ell}, \quad u \rightarrow -\infty. \quad (1.3)$$

where γ^{uv} is a metric whose constant curvature is R^{uv} . Both parameters R^{uv} and φ_- break conformal invariance explicitly. One of our main goals will be to characterize the interplay between these two different kinds of deformation. The asymptotic analysis of the solution (1.2) will also define precisely what exactly is the curvature of the space-time seen by the UV CFT.

In the rest of this subsection we give a short summary and discussion of the results which are derived and presented in the main body of the paper.

First order formalism (section 2.2). The connection between the bulk geometry and RG flow in the dual field theory is made by using a first order formalism adapted to the case where the induced metric on constant u slices has constant curvature.³ One defines two independent scalar functions (superpotentials) $W(\varphi)$ and $S(\varphi)$ which satisfy:

$$W(\varphi(u)) \equiv -2(d-1) \frac{dA(u)}{du}, \quad S(\varphi(u)) \equiv \frac{d\varphi(u)}{du}, \quad (1.4)$$

and which are solutions of a set of non-linear first order differential equations (2.21) and (2.22). Assuming the scale factor plays the same role of energy scale as in the flat

³A similar first order formalism was developed in [55] by defining complex superpotentials. The latter version of the first order formalism is equivalent to ours, as we show in appendix B.

Figure 2. The solid lines show the scalar field (left) and scale factor (right) profiles of a positive curvature RG flow geometry, from the UV ($u \rightarrow -\infty$, $\varphi \rightarrow 0$) to the IR endpoint ($u = u_0$, $\varphi = \varphi_0$). The dashed lines represent the solutions with zero curvature, extending all the way to $u \rightarrow +\infty$ and to the IR fixed point at $\varphi = \varphi_{\text{IR}}$.

case, the holographic β -function can be expressed in terms of $W(\varphi)$ and $S(\varphi)$:

$$\beta(\varphi) \equiv \frac{d\varphi}{dA} = -2(d-1) \frac{S(\varphi)}{W(\varphi)}. \quad (1.5)$$

The space of regular solutions for $W(\varphi)$ and $S(\varphi)$ coincides with the space of possible RG flows up to a choice of two initial conditions: (1) The value of the deformation parameter in the UV, φ_- , which is encoded in the leading scalar field asymptotics in equation (1.3). It is called the source and corresponds to the UV coupling constant of the relevant operator dual to the scalar. (2) The UV value of the scalar curvature R^{uv} of the manifold on which the field theory is defined. Therefore, by classifying all solutions $W(\varphi)$, $S(\varphi)$ to the equations (2.21) and (2.22) for a given potential $V(\varphi)$, we can characterize all possible RG flows corresponding to a given bulk theory.

What we find is as follows:

UV fixed points (section 3). It is a well-known result from holography that, when the boundary theory is in flat space, UV fixed points are associated with extrema of the potential V . This persists at finite boundary curvature ($R^{\text{uv}} \neq 0$).

- *Maxima of V :* for maxima of V , we find solutions describing a relevant deformation away from a UV fixed point. Such solutions come as *two-parameter families* $W_{C,\mathcal{R}}(\varphi)$, $S_{C,\mathcal{R}}(\varphi)$, where C and $\mathcal{R}$ are dimensionless parameters related to the vev of the deforming operator and to the UV value of the scalar curvature R^{uv} , respectively. At most a finite subset of these solutions can be extended to globally regular solutions corresponding to RG flows. This is shown schematically in figure 1.
- *Minima of V :* flows away from a minimum of V also exist and are driven by the vev of an irrelevant operator [45]. There is only one free parameter $\mathcal{R}$ related to R^{uv} . These solutions are all *badly singular* in generic theories and can only be extended to globally regular solutions in special cases.

Figure 3. The solid lines show the scalar field (left) and scale factor (right) profiles of a negative curvature RG flow geometry, from the left boundary ($u \rightarrow -\infty$, $\varphi \rightarrow 0$) to the turning point ($u = u_0$, $\varphi = \varphi_0$), to the right boundary ($u \rightarrow +\infty$, $\varphi \rightarrow 0$). The solution is symmetric around u_0 . The dashed lines represent the zero-curvature solution interpolating from the UV boundary to the IR fixed point at $\varphi = \varphi_{\text{IR}}$, and featuring a monotonic scale factor.

IR limit (section 4). The behavior in the innermost region of the geometry (far from the UV boundary) depends on the sign of the curvature R^{uv} of the boundary metric. Recall that, in the flat case, the IR endpoint of a holographic RG flow corresponds to the scalar field asymptoting to a minimum of the bulk potential. In that case, in the far IR region, the bulk geometry asymptotes to AdS_{d+1} in the interior.

- *Field theory on manifolds with $R^{uv} > 0$ (section 4.1):* here, non-singular flows stop at an IR endpoint, where $e^A \rightarrow 0$, φ takes on a finite value φ_0 , and $S(\varphi) = \dot{\varphi} = 0$.

At the endpoint, the function W diverges as $W \sim |\varphi - \varphi_0|^{-1/2}$. This behavior is shown schematically in figures 1 (superpotential W) and 2 (scalar field and scale factor). The bulk geometry is regular and becomes approximately AdS_{d+1} when approaching the IR endpoint. An important remark is that, when the field theory curvature is non-vanishing, the IR endpoint *cannot* be located at a minimum of the scalar potential.

- *Field theory on manifold with $R^{uv} < 0$ (4.2):* in this case, the flow eventually reaches a turning point φ_0 where both $A(u)$ and $\varphi(u)$ invert their direction, but the value of e^A remains finite. These points are characterized by $S(\varphi_0) = W(\varphi_0) = 0$. As u increases past the turning point, the geometry connects back to the boundary of AdS_{d+1} . Therefore, in this case, there is no IR endpoint, but rather an IR “throat” connecting two boundary regions. This behavior is shown schematically in figure 3. This situation was already discussed in [56] in the case of pure gravity. As it was pointed out there, the two UV regions are part of the same AdS_{d+1} boundary, and are connected via the AdS_d boundary of the codimension-one slices.

Complete non-singular RG flows (section 5). In general, any point φ_0 can be an IR end point ($R^{uv} > 0$) or turning point ($R^{uv} < 0$), as long as φ_0 is not an extremum of

Figure 4. Schematic structure of a field theory with an RG flow exhibiting a “bounce”.

V . However, for a given curvature R^{uv} , and for every value φ_0 , there is a unique solution corresponding to a flow ending at φ_0 and connecting to the UV boundary. Such points, for $R^{\text{uv}} > 0$, are the fixed points of the flow, where the associated β -function, defined in (1.5), vanishes as discussed in section 5.2

We construct several examples of curved holographic RG flow solutions that start from a UV fixed point at a maximum of the potential, and have a regular interior where the geometry displays the general features described in the previous paragraphs.

The value φ_0 (be it an endpoint or a turning point) for a given regular flow is determined by the values of R^{uv} and φ_- governing the leading UV asymptotics in equation (1.3). More precisely, the value φ_0 is completely determined by the dimensionless combination $\mathcal{R} \equiv R^{\text{uv}}|\varphi_-|^{2/(d-\Delta)}$, where Δ is the dimension of the dual operator corresponding to φ .

In particular, the following observations hold for generic potentials and either sign of the curvature: *Increasing* $|R^{\text{uv}}|$ while keeping the UV source φ_- fixed causes the IR endpoint (or turning point) φ_0 to move closer to the starting point of the flow at the UV maximum. Conversely, *decreasing* $|R^{\text{uv}}|$ while keeping the UV source φ_- fixed causes φ_0 to move away from the starting point of the flows at the UV maximum. For $R^{\text{uv}} \rightarrow 0$ the IR endpoint (or turning point) approaches a minimum of V .

In the simplest RG flow solutions, the scalar field is monotonic between the UV fixed point and the IR critical point φ_0 , which typically is reached before the (would-be flat) IR fixed point (as in figure 2a). This is, however, not always the case. For example, already in the flat case it was shown in [45] that holographic RG flows may exhibit phenomena which seem exotic from the point of view of QFT perturbation theory. For one, holographic RG flows may *bounce*, i.e. the flow may reverse direction in coupling constant space (see figure 4) without stopping.

In this work we find that flows exhibiting bounces persist for field theories on curved manifolds. Interestingly, for field theories on AdS_4 we find that bounces are generic features of RG flows if the UV operator dimension is $3 < \Delta < 4$. This is unlike the situation in the flat or positively curved case, where this does not occur generically.

Quantum Phase Transitions (section 6). A field theory on a curved manifold, defined by a value of the UV source φ_- and of the UV curvature R^{uv} may display several saddle points, which in the holographic dual framework correspond to different RG flow geometries. The question then remains which one of the ground states is the true vacuum of the theory. This can be determined by calculating the free energies of the flows corresponding to the various saddle points: the solution with the lowest free energy is the true vacuum.

Figure 5. Schematic structure of a field theory which presents multiple RG-flows: in particular, there are two flows starting at the fixed point UV_1 , one going to the closer IR fixed point IR_1 , the second skipping IR_1 and ending at IR_2 .

If we vary the value of R^{uv} , the RG flows deform and the vevs and free energies of the various vacua change. Interestingly, under a variation of R^{uv} the identification of the true vacuum may change, i.e. the system may exhibit phase transitions. This corresponds to a *quantum* phase transition as our system is at zero temperature.⁴ The control parameter in this case is the curvature R^{uv} .

We find further evidence that curvature-driven first order quantum phase transitions *do* arise in a holographic framework. Previously, this was observed holographically for supersymmetric field theories on $\mathbb{R} \times S^3$ in [46]. In section 6 we present an explicit example that exhibits this phenomenon.⁵ The example we discuss here is based on a potential which was already studied in [45] in the flat case. Its main feature is to allow two regular flows in the zero-curvature case, one connecting two neighboring (UV and IR) fixed points, the second *skipping* the closest available IR fixed point and ending at a fixed point at larger field value. This is represented schematically in figure 5.

In [45], it was found that, for the theory in flat space, the preferred ground state is the skipping solution. Turning on a small positive curvature, the skipping and non-skipping solutions persist, their free-energy difference decreases up to a critical value $\mathcal{R}_c$ where it changes sign: for $\mathcal{R} > \mathcal{R}_c$ the dominant saddle point is now the shorter (non-skipping) flow. Finally, skipping flows disappear beyond a finite value larger than $\mathcal{R}_c$ (see figure 32).

This is the typical structure of a first order phase transition, where a stable phase becomes unstable then disappears, as a function of a control parameter. Usually these kinds of phase transitions are driven by temperature (e.g. Hawking-Page transition for global AdS-Schwarzschild black holes). Here it takes place at zero temperature and is driven by one of the “couplings” (the curvature is a non-normalizable deformation in the AdS/CFT language) of the UV theory.

1.2 Open questions and outlook

From the point of view of field theory, this work offers a new (holographic) insight on RG flows on curved manifolds. In particular, we have seen that positive curvature flows end at points of maximal symmetry. From the gravity side, this symmetry is just the full set of AdS isometries. However, it is unclear how this symmetry is realized on the field

⁴A richer structure may be obtained at finite temperature, but we will not pursue this here.

⁵A somewhat similar phenomenon, although one cannot talk about a proper quantum phase transition, was already observed since the very early days of holography in [57] for $N = 4$ SYM: the flat-space theory is in the deconfined phase at arbitrarily low temperature, whereas the theory on $\mathbb{R} \times S^3$ is in the confined phase up to a finite temperature set by the curvature scale.

theory side, since (unlike in the case of flat IR fixed points) it does not obviously reduce to conformal transformations (or rather conformal isometries) on the field theory coordinates. It would be interesting to investigate this further.

One important question is whether the solutions discussed in this work are perturbatively stable. In the case of flat sections, it was shown in [45] that, regardless of the details of the bulk geometries, solutions which reach an IR fixed point are stable under small perturbations. This includes bouncing solutions, as the bounce does not introduce any particular features in the fluctuation equations which may trigger instabilities. In the curved case the situation is less clear. To reach the same conclusion one would have to develop a complete fluctuation analysis around the ansatz (1.2). This is beyond the scope of this work. Nevertheless, in section 4.5 we present a general discussion which allows to draw some partial conclusions and suggests that perturbative stability is preserved when we introduce a non-vanishing curvature.

Another important question concerns the boundary conditions and stability for solutions in which the field theory is on a negatively curved space. When the field theory space-time is AdS_d , it turns out that the solution is not single valued on the boundary, but it must contain a defect along the portion of the AdS_{d+1} boundary which corresponds to the AdS_d boundary (see the discussion in section 4.4). This can be avoided by quotienting the AdS_d slices by a finite group, but as argued in [56], this may introduce perturbative instabilities. It would be interesting to understand these issues in more detail.

Another interesting application of our analysis is in the context of 3-dimensional field theories, and the associated F -theorem [9]. The original definition of the F -theorem is in terms of the free energy $F = -\ln Z_{S^3}$ of a field theory on S^3 . However, it is an entanglement entropy [10, 11] (in particular the renormalized entanglement entropy of [58]) rather than the free energy which gives rise to a suitable F -function, which decreases monotonically from UV to IR [59]. An interesting open question is whether a suitable F -function can be defined using RG flows in curved space-times. In holography, this question was addressed in [60] in the large curvature limit and it was shown that the free energy on S^3 does in general not give rise to a suitable F -function. We will extend this analysis in an upcoming publication [50] and describe how a monotonic F -function can be recovered from holographic RG flows on S^3 .

As we have mentioned earlier, one of the original motivations for this work was to find curved space-time solutions (eventually time-dependent) in the self-tuning framework developed in [54]. In that case, the interesting solutions would be two distinct RG flow solutions (one containing the UV, the other containing an IR region) glued by a domain wall in the transverse direction. The study of the junction conditions in [54] will then be extended to the curved case, and will give an indication of the dynamics of the domain wall away from the flat ground state solution.

Finally, it would be interesting to study examples of Einstein-dilaton gravity where the flat space-time field theory is confining, as those described in [37], and investigate the phase structure (and possible phase transitions) in the presence of curvature (see also [47] for a related discussion of holographic confining theories on space-times with non-zero curvature).

2 Holographic space-times with curved slicing and RG flows

Our object of study is an Einstein-scalar theory in $(d + 1)$ -dimensions with Euclidean or Lorentzian metric (in the latter case we use the convention $(-, +, +, \dots, +)$). In the Lorentzian case, it is described by the following two-derivative action:

$$S[g, \varphi] = \int du d^d x \sqrt{|g|} \left(R^{(g)} - \frac{1}{2} \partial_a \varphi \partial^a \varphi - V(\varphi) \right) + S_{\text{GHY}}, \quad (2.1)$$

where we also included the Gibbons-Hawking-York term S_{GHY} . The Euclidean action is defined by setting $S_E = -S$ and changing the metric to positive signature.

In this work we will be interested in boundary field theories defined on curved maximally symmetric d -dimensional space-times. Then, without loss of generality, we can employ domain wall coordinates and choose the following ansatz for φ and the $(d + 1)$ -dimensional metric (for both Euclidean and Lorentzian signatures):

$$\varphi = \varphi(u), \quad ds^2 = du^2 + e^{2A(u)} \zeta_{\mu\nu} dx^\mu dx^\nu. \quad (2.2)$$

Here, $A(u)$ is a scale factor that depends on the coordinate u only, while $\zeta_{\mu\nu}$ is a metric describing a d -dimensional maximally symmetric space-time. As a consequence of maximal symmetry we have⁶

$$R_{\mu\nu}^{(\zeta)} = \kappa \zeta_{\mu\nu}, \quad R^{(\zeta)} = d\kappa, \quad \text{with} \quad \kappa = \begin{cases} \frac{(d-1)}{\alpha^2} & \text{dS}_d \text{ or } S^d \\ 0 & \mathcal{M}_d \\ -\frac{(d-1)}{\alpha^2} & \text{AdS}_d \end{cases}, \quad (2.3)$$

where α is the curvature length scale and $\mathcal{M}_d$ denotes d -dimensional Minkowski space. We also define the induced metric $\gamma_{\mu\nu}$ on a d -dimensional slice at constant u which is given by

$$\gamma_{\mu\nu} \equiv e^{2A(u)} \zeta_{\mu\nu}. \quad (2.4)$$

In the following, we will also adhere to the following shorthand notation. Derivatives with respect to u will be denoted by a dot while derivatives with respect to φ will be denoted by a prime, i.e.:

$$\dot{f}(u) \equiv \frac{df(u)}{du}, \quad g'(\varphi) \equiv \frac{dg(\varphi)}{d\varphi}. \quad (2.5)$$

Varying the action (2.1) with respect to the metric and the scalar φ gives rise to the equations of motion:

$$2(d-1)\ddot{A} + \dot{\varphi}^2 + \frac{2}{d} e^{-2A} R^{(\zeta)} = 0, \quad (2.6)$$

$$d(d-1)\dot{A}^2 - \frac{1}{2}\dot{\varphi}^2 + V - e^{-2A} R^{(\zeta)} = 0, \quad (2.7)$$

$$\ddot{\varphi} + d\dot{A}\dot{\varphi} - V' = 0. \quad (2.8)$$

⁶In the case of a foliation with positive curvature slices, these can be given by d -dimensional de Sitter space or a d -sphere. In the rest of the paper we will mainly refer to the case of the sphere, keeping in mind that the results will also hold for de Sitter.

These equations are the same for both Lorentzian and Euclidean signatures, so all our results will hold for both cases (unless stated explicitly).

Holographic RG flows are in one-to-one correspondence with regular solutions to the equations of motion (2.6)–(2.8). Hence, in the following we will be interested in solutions to these equations for various choices of the potential $V(\varphi)$.

To be specific, we will assume that $V(\varphi)$ has at least one maximum, where V takes a negative value. This ensures that there exists a conformal fixed point, and a family of asymptotically AdS solutions which correspond to deforming the theory away from the fixed point by a relevant operator.

In addition, $V(\varphi)$ may have other maxima and/or minima representing distinct UV or IR fixed points for the dual CFT. The aim of the next subsection is to study the solutions *at* the fixed points in the presence of curvature, but with the relevant deformation set to zero.

2.1 Conformal fixed points

A conformal fixed point corresponds to a solution of (2.6)–(2.8) with $\varphi = \text{const.}$, i.e. $\dot{\varphi} = 0 = \ddot{\varphi}$. Such solutions are associated with extrema φ_{ext} of the potential, at which $V'(\varphi_{\text{ext}}) = 0$.

In this case, the solution (2.2) is always the space-time AdS_{d+1} , written in different coordinate systems, regardless of the curvature of the d -dimensional slices with metric $\zeta_{\mu\nu}$. Indeed, AdS_{d+1} admits d -dimensional de Sitter (or sphere, in the Euclidean version), Minkowski or AdS slicings of the form (2.2), with (see e.g. [61]):

$$e^{A(u)} = \begin{cases} \frac{\ell}{\alpha} \sinh\left(-\frac{u+c}{\ell}\right), & -\infty < u \leq -c, & \text{dS}_d \text{ or } S^d \\ \exp\left(-\frac{u+c}{\ell}\right), & -\infty < u < +\infty, & \mathcal{M}_d \\ \frac{\ell}{\alpha} \cosh\left(\frac{u+c}{\ell}\right), & -\infty < u < +\infty, & \text{AdS}_d \end{cases} . \quad (2.9)$$

Here we introduced the AdS_{d+1} length ℓ which is defined via $V(\varphi_{\text{ext}}) = -\frac{d(d-1)}{\ell^2}$, while the length scale α was introduced in (2.3). We also chose the boundary of AdS_{d+1} to be located at $u \rightarrow -\infty$. The parameter c is an integration constant.

Although the bulk space-time is the same, the asymptotic boundary is different in the three cases. This leads to inequivalent boundary theories. As the d -dimensional boundaries of the space-times (2.9) are all conformally equivalent, at the fixed point the effect of curvature is completely encoded in the conformal anomaly. This will change when we consider RG-flows and introduce an explicit breaking of conformal invariance.

In the case of Minkowski slices, c can be removed by a conformal rescaling of the boundary metric and we can set $c = 0$. However, for dS and AdS slices the constant c contains physical information and cannot be removed. In particular, c determines the curvature of the d -dimensional slices. To see this, first note that the length scale α is unphysical and can be chosen freely. It is an artifact of splitting the induced metric $\gamma_{\mu\nu}$

on the d -dimensional slices into a scale factor $e^{A(u)}$ and $\zeta_{\mu\nu}$. This can be seen e.g. by evaluating $R^{(\gamma)}$, i.e. the scalar curvature associated with the induced metric $\gamma_{\mu\nu}$ on the d -dimensional slices, which is given by

$$R^{(\gamma)} = e^{-2A(u)} R^{(\zeta)}. \quad (2.10)$$

For a conformal fixed point we hence find

$$R^{(\gamma)} = \begin{cases} \frac{d(d-1)}{\ell^2} \sinh^{-2} \frac{u+c}{\ell} & \text{dS}_d \text{ or } S^d \\ -\frac{d(d-1)}{\ell^2} \cosh^{-2} \frac{u+c}{\ell} & \text{AdS}_d \end{cases}, \quad (2.11)$$

i.e. the parameter α has dropped out from the expressions. Instead, it is the parameter c which contains information about the curvature.

An important quantity will be the scalar curvature R^{uv} , which we define as the scalar curvature associated with the UV limit of the (rescaled) induced metric $\gamma_{\mu\nu}^{\text{uv}} \equiv \lim_{u \rightarrow -\infty} e^{2u/\ell} \gamma_{\mu\nu}$. Indeed, the metric $\gamma_{\mu\nu}^{\text{uv}}$ is the leading boundary data appearing in the Fefferman-Graham expansion near the boundary, and in holography it is interpreted as the metric of the dual field theory. The associated curvature R^{uv} is given by:

$$R^{\text{uv}} = \lim_{u \rightarrow -\infty} e^{-2u/\ell} R^{(\gamma)}, \quad (2.12)$$

For the case of a conformal fixed point we find

$$R^{\text{uv}} = \pm \frac{4d(d-1)}{\ell^2} e^{2c/\ell}, \quad (2.13)$$

where the positive (negative) sign is appropriate for dS (AdS) curvature.

For later use, it will be convenient to expand the scale factor (2.9) in the vicinity of the boundary $u \rightarrow -\infty$. Restricting attention to the cases of S^d/AdS_d slicings we obtain

$$\begin{aligned} A(u) &= \ln \left(\frac{\ell}{2\alpha} \right) - \frac{u+c}{\ell} \mp e^{2(u+c)/\ell} + \mathcal{O}(e^{4u/\ell}) \\ &= \frac{1}{2} \ln \left(\frac{R^{(\zeta)}}{R^{\text{uv}}} \right) - \frac{u}{\ell} - \frac{\ell^2 R^{\text{uv}}}{4d(d-1)} e^{2u/\ell} + \mathcal{O}(e^{4u/\ell}) \end{aligned} \quad (2.14)$$

where the last line is valid for both S^d/AdS_d slicings.

For most practical purposes it will be convenient to set $R^{(\zeta)} = R^{\text{uv}}$, which we can always achieve by choosing $\alpha = \frac{\ell}{2} e^{-c/\ell}$ for the arbitrary parameter α . This is the convention that we will adopt throughout this paper. Note that in this case the constant term in (2.14) vanishes. This also holds for space-times which only asymptote to AdS_{d+1} near the boundary: we can always ensure that $R^{(\zeta)} = R^{\text{uv}}$ for an asymptotically AdS space-time by setting the constant term in the near-boundary expansion of $A(u)$ to zero.

Summary. A solution of the form (2.2) in which φ is fixed to an extremum of the scalar potential is dual to a CFT living on a manifold whose curvature can be arbitrary, and depends on the choice of the leading term in the boundary asymptotics (2.14). As a final remark, the definition (2.12) of R^{uv} is also valid for asymptotically AdS_{d+1} space-times, i.e. space-times where $A(u)$ approaches the form (2.9) at the boundary $u \rightarrow -\infty$, but may depart from that form in the interior.

2.2 The first order formalism

We now turn our attention to solutions where the scalar field has a non-trivial dependence on the coordinate u . Our goal is to interpret solutions to the equations (2.6)–(2.8) in terms of RG flows.

To this end it will be convenient to rewrite the second-order Einstein equations as a set of first-order equations, which will allow an interpretation as gradient RG flows. This is locally always possible, except at special points where $\dot{\varphi} = 0$, which we will later refer to as bounces, as previously observed in [45]. Given a solution, as long as $\dot{\varphi}(u) \neq 0$, we can invert the relation between u and $\varphi(u)$ and define the following *scalar* functions of φ : we define,

$$W(\varphi) \equiv -2(d-1)\dot{A}, \quad (2.15)$$

$$S(\varphi) \equiv \dot{\varphi}, \quad (2.16)$$

$$T(\varphi) \equiv e^{-2A}R^{(\zeta)} = R^{(\gamma)}. \quad (2.17)$$

where the expressions on the right hand side are evaluated at $u = u(\varphi)$.

In terms of the functions defined above, the equations of motion (2.6)–(2.8) become

$$S^2 - SW' + \frac{2}{d}T = 0, \quad (2.18)$$

$$\frac{d}{2(d-1)}W^2 - S^2 - 2T + 2V = 0, \quad (2.19)$$

$$SS' - \frac{d}{2(d-1)}SW - V' = 0, \quad (2.20)$$

which are coordinate independent, first-order non-linear differential equations in field space. In the flat case $T = 0$ and we recover the usual superpotential equation for W by setting $S = W'$. Thus, the difference between S and W' is a measure of the curvature.

Note that equations (2.18)–(2.20) are algebraic in T . We can hence partially solve this system by eliminating T so that we are left with the following two equations

$$\frac{d}{2(d-1)}W^2 + (d-1)S^2 - dSW' + 2V = 0, \quad (2.21)$$

$$SS' - \frac{d}{2(d-1)}SW - V' = 0. \quad (2.22)$$

Next, we can solve the second equation (2.22) for W algebraically. Substituting into the first equation (2.21) we obtain

$$dS^3S'' - \frac{d}{2}S^4 - S^2(S')^2 - \frac{d}{d-1}S^2V + (d+2)SS'V' - dS^2V'' - (V')^2 = 0 \quad (2.23)$$

This is a second order equation in S and its integration requires two integration constants. As we will see in the next section, one integration constant will be related to the vev of the perturbing operator, while the other integration constant will be related to the UV curvature.

In the following we will work both with the full set of equations (2.18)–(2.20), with the two equations (2.21)–(2.22) or with (2.23) choosing whichever is more convenient.

A few important properties of the functions W , S and T are discussed below (see appendix C for more details):

1. At a generic point in field space, there exist two branches of solutions, with opposite signs of S and W' . In each branch, S and W' have the same sign.
2. We define the *critical curve* $B(\varphi)$ as:

$$B(\varphi) = \sqrt{-\frac{4(d-1)}{d}V(\varphi)} \quad (2.24)$$

In the flat case, $W(\varphi)$ has to satisfy $|W(\varphi)| \geq B(\varphi)$, and equality can be reached only where $W' = 0$. For positive non-zero curvature, the bound is stricter, and the critical curve cannot be reached, except in the UV where $T \rightarrow 0$. On the other hand, for negative curvature W can cross the critical curve.

3. Curvature invariants are finite as long as $S(\varphi)$ and $V(\varphi)$ are both finite. On the other hand a divergent $W(\varphi)$ does not necessarily imply that the solution is singular.
4. Points where $S \rightarrow 0$ (i.e. $\varphi(u)$ has an extremum) correspond to points of enhanced symmetry: in fact, around these points the metric is approximately maximally symmetric, i.e.

$$S(\varphi_*) = 0 \quad \Rightarrow \quad R_{ab} = \frac{V(\varphi_*)}{d-1}g_{ab}. \quad (2.25)$$

This has to be confronted with generic points, where $R_{\mu\nu}^{(g)} = kg_{\mu\nu}$, $R_{uu}^{(g)} = k'g_{uu}$, with $k' - k = S^2/2$.

3 Perturbative analysis near extrema of the potential

We will now examine solutions of our system in the vicinity of extremal points of V . We then proceed to determining the near-boundary geometry. Without loss of generality we take the extremum to be at $\varphi = 0$. It will then be sufficient to consider the potential

$$V = -\frac{d(d-1)}{\ell^2} - \frac{m^2}{2}\varphi^2 + \mathcal{O}(\varphi^3), \quad (3.1)$$

where we will choose $m^2 > 0$ for maxima and $m^2 < 0$ for minima. In the following we will solve equations (2.18)–(2.20) for $W(\varphi)$, $S(\varphi)$ and $T(\varphi)$ near $\varphi = 0$. The relevant calculations are performed in appendix D. Here we present and discuss the results.

3.1 Expansion near maxima of the potential

We work in an expansion in φ about the maximum at $\varphi = 0$. Like in the case of zero boundary curvature discussed in e.g. [45], there are two branches of solutions to equations (2.18)–(2.20), and we will distinguish them by the subscripts (+) and (−). The (−)

solutions are:

$$W_-(\varphi) = \frac{1}{\ell} \left[2(d-1) + \frac{\Delta_-}{2} \varphi^2 + \mathcal{O}(\varphi^3) \right] + \frac{\mathcal{R}}{d\ell} |\varphi|^{\frac{2}{\Delta_-}} [1 + \mathcal{O}(\varphi) + \mathcal{O}(\mathcal{R})] \\ + \frac{C}{\ell} |\varphi|^{\frac{d}{\Delta_-}} [1 + \mathcal{O}(\varphi) + \mathcal{O}(C) + \mathcal{O}(\mathcal{R})], \quad (3.2)$$

$$S_-(\varphi) = \frac{\Delta_-}{\ell} \varphi [1 + \mathcal{O}(\varphi)] + \frac{Cd}{\Delta_- \ell} |\varphi|^{\frac{d}{\Delta_-}-1} [1 + \mathcal{O}(\varphi) + \mathcal{O}(C)], \quad (3.3) \\ + \frac{1}{\ell} \mathcal{O} \left(\mathcal{R} |\varphi|^{\frac{2}{\Delta_-}+1} \right) + \frac{1}{\ell} \mathcal{O} \left(\mathcal{R} C |\varphi|^{\frac{2+d}{\Delta_-}-1} \right)$$

$$T_-(\varphi) = \ell^{-2} \mathcal{R} |\varphi|^{\frac{2}{\Delta_-}} [1 + \mathcal{O}(\varphi) + \mathcal{O}(C) + \mathcal{O}(\mathcal{R})], \quad (3.4)$$

where C and $\mathcal{R}$ are integration constants, and we have defined:

$$\Delta_{\pm} = \frac{1}{2} \left(d \pm \sqrt{d^2 - 4m^2 \ell^2} \right) \quad \text{with} \quad 0 < m^2 < \frac{d^2}{4\ell^2} \quad (3.5)$$

The (+) solution is given by:

$$W_+(\varphi) = \frac{1}{\ell} \left[2(d-1) + \frac{\Delta_+}{2} \varphi^2 + \mathcal{O}(\varphi^3) \right] + \frac{\mathcal{R}}{d\ell} |\varphi|^{\frac{2}{\Delta_+}} [1 + \mathcal{O}(\varphi) + \mathcal{O}(\mathcal{R})], \quad (3.6)$$

$$S_+(\varphi) = \frac{\Delta_+}{\ell} \varphi [1 + \mathcal{O}(\varphi)] + \mathcal{O} \left(\mathcal{R} |\varphi|^{\frac{2}{\Delta_+}+1} \right), \quad (3.7)$$

$$T_+(\varphi) = \ell^{-2} \mathcal{R} |\varphi|^{\frac{2}{\Delta_+}} [1 + \mathcal{O}(\varphi) + \mathcal{O}(\mathcal{R})]. \quad (3.8)$$

The above expressions describe two continuous families of solutions, whose structure is a *universal* analytic expansion in integer powers of φ , plus a series of non-analytic, subleading terms which, in principle, depend on two (dimensionless) integration constants C and $\mathcal{R}$. Note that the (−)-branch of solutions depends on two integration constants C and $\mathcal{R}$, while only the integration constant $\mathcal{R}$ appears in the solutions of the (+)-branch. The notation $\mathcal{O}(C)$ and $\mathcal{O}(\mathcal{R})$ does not imply that C or $\mathcal{R}$ have to be small. Rather, this is shorthand to indicate that these terms will be accompanied by higher powers of φ thus justifying their omission.⁷

The solutions (3.2)–(3.4) generalize the near-extremum superpotential solutions that arise in the flat case, where only the integration constant C was present. Indeed, setting $\mathcal{R} = 0$ we recover the flat result for $W(\varphi)$ and we also find $S = W'$ and $T = 0$. This indicates that $\mathcal{R} = 0$ is strictly related to the curvature of the d -dimensional metric, a fact that will be confirmed explicitly below.

Given our results for W , S and T , we are now in a position to solve for $\varphi(u)$ and $A(u)$. For the (−)-branch we solve (2.16) and (2.15) subject to (3.3) and (3.2) to obtain

$$\varphi(u) = \varphi_- \ell^{\Delta_-} e^{\Delta_- u/\ell} \left[1 + \mathcal{O} \left(\mathcal{R} |\varphi_-|^{2/\Delta_-} e^{2u/\ell} \right) + \dots \right] \\ + \frac{Cd |\varphi_-|^{\Delta_+/\Delta_-}}{\Delta_- (d - 2\Delta_-)} \ell^{\Delta_+} e^{\Delta_+ u/\ell} \left[1 + \mathcal{O} \left(\mathcal{R} |\varphi_-|^{2/\Delta_-} e^{2u/\ell} \right) + \dots \right] + \dots, \quad (3.10)$$

⁷For example, the solution for W on the (−)-branch can be written schematically as the following triple expansion:

$$W_-(\varphi) = \frac{1}{\ell} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} A_{l,m,n} \left(C |\varphi|^{d/\Delta_-} \right)^l \left(\mathcal{R} |\varphi|^{2/\Delta_-} \right)^m \varphi^n \quad (3.9)$$

$$\begin{aligned}
 A(u) = & \bar{A}_- - \frac{u}{\ell} - \frac{\varphi_-^2 \ell^{2\Delta_-}}{8(d-1)} e^{2\Delta_- u/\ell} - \frac{\mathcal{R}|\varphi_-|^{2/\Delta_-} \ell^2}{4d(d-1)} e^{2u/\ell} \\
 & - \frac{\Delta_+ C |\varphi_-|^{d/\Delta_-} \ell^d}{d(d-1)(d-2\Delta_-)} e^{du/\ell} + \dots
 \end{aligned} \tag{3.11}$$

Here, we introduced the integration constants φ_- and $\bar{A}_-$. We can repeat the analysis for the (+)-branch of solutions by solving (2.16) and (2.15) subject to (3.7) and (3.6). We find:

$$\varphi(u) = \varphi_+ \ell^{\Delta_+} e^{\Delta_+ u/\ell} \left[1 + \mathcal{O} \left(\mathcal{R} |\varphi_+|^{2/\Delta_+} e^{2u/\ell} \right) + \dots \right] + \dots, \tag{3.12}$$

$$A(u) = \bar{A}_+ - \frac{u}{\ell} - \frac{\varphi_+^2 \ell^{2\Delta_+}}{8(d-1)} e^{2\Delta_+ u/\ell} - \frac{\mathcal{R} |\varphi_+|^{2/\Delta_+} \ell^2}{4d(d-1)} e^{2u/\ell} + \dots, \tag{3.13}$$

where we introduced the integration constants φ_+ and $\bar{A}_+$. A few comments are in order.

- As our solutions for W , S and T are only valid for small φ , the above results are the leading terms in $\varphi(u)$ and $A(u)$ for $u \rightarrow -\infty$.
- For both the (+) and (−)-branch, the result for $A(u)$ exhibits the behavior expected for a scale factor in the near-boundary region of an AdS_{d+1} space-time with length scale ℓ , as can be seen by comparing with (2.14). As explained in section 2.1, for asymptotically AdS space-times we can always choose a metric ansatz such that $R^{(\zeta)} = R^{\text{uv}}$. This amounts to setting $\bar{A}_{\pm} = 0$.
- For the (−)-branch of solutions, we identify φ_- as the source for the scalar operator $\mathcal{O}$ in the boundary field theory associated with φ . The vacuum expectation value of $\mathcal{O}$ depends on C and is given by

$$\langle \mathcal{O} \rangle_- = \frac{Cd}{\Delta_-} |\varphi_-|^{\Delta_+/\Delta_-}. \tag{3.14}$$

- For the (+)-branch of solutions, the bulk field φ is also associated with a scalar operator $\mathcal{O}$ in the boundary field theory. However, in this case the source is identically zero, yet there is a non-zero vev given by

$$\langle \mathcal{O} \rangle_+ = (2\Delta_+ - d) \varphi_+. \tag{3.15}$$

There is also an associated moduli space of vevs, as φ_+ is arbitrary, being an integration constant of the first order flow equation.

- We can learn even more by comparing the two expressions for $A(u)$ in (3.11) and (2.14). Matching the coefficients of $e^{2u/\ell}$ implies

$$\mathcal{R} \equiv \begin{cases} R^{\text{uv}} |\varphi_-|^{-2/\Delta_-} & (-)\text{-branch} \\ R^{\text{uv}} |\varphi_+|^{-2/\Delta_+} & (+)\text{-branch} \end{cases}. \tag{3.16}$$

Thus, the integration constant $\mathcal{R}$ is related to the curvature R^{uv} of the manifold on which the UV QFT is defined.

- Interestingly, the $(-)$ and $(+)$ -branches of solutions are not completely unrelated. One can check that given a solution of the $(-)$ -branch, we can arrive at a solution on the $(+)$ -branch by performing the following rescaling:

$$\varphi_- \rightarrow 0, \quad C \rightarrow +\infty, \quad C|\varphi_-|^{\frac{\Delta_+}{\Delta_-}} = \text{const.}, \quad \mathcal{R}|\varphi_-|^{\frac{2}{\Delta_-}} = \text{const.} \quad (3.17)$$

To be specific, under this rescaling the solutions in (3.10) and (3.11) can be brought into the form of (3.12) and (3.13). This gives rise to another interpretation of the $(+)$ -branch of solutions. In particular, for fixed R^{uv} a W_+ solution is the upper envelope of the family of W_- solutions parameterized by C . This is similar to the flat case [35, 45].

Overall, the above findings imply that, as in the flat case, *maxima of the potential are associated with UV fixed points*. The bulk space-time asymptotes to AdS_{d+1} and reaching the maximum of the potential is equivalent to reaching the boundary. Moving away from the boundary corresponds to a flow leaving the UV. Flows corresponding to solutions on the $(-)$ -branch are driven by the existence of a non-zero source φ_- for the perturbing operator $\mathcal{O}$. Flows corresponding to solutions on the $(+)$ -branch are driven purely by a non-zero vev for $\mathcal{O}$, as the source vanishes identically.

Finally, notice that as expected near a UV fixed point, the curvature terms proportional to $\mathcal{R}$ only enter as subleading corrections in W , S and the bulk solution (3.10)–(3.13).

3.2 Expansion near minima of the potential

In the following, we will display solutions for $W(\varphi)$, $S(\varphi)$ and $T(\varphi)$ corresponding to flows that either leave or arrive at minima of the potential. We will again consider the potential (3.1), but now we have $m^2 < 0$. In the following, we present the results, while detailed calculations can be found in appendix D. Again, we find that there exist $(+)$ and $(-)$ -branches in the space of solutions, which we will discuss in turn.

$(+)$ -branch. For the $(+)$ -branch the expansions around a minimum of V take the same form as in the vicinity of a maximum. Therefore, we obtain:

$$W_+(\varphi) = \frac{1}{\ell} \left[2(d-1) + \frac{\Delta_+}{2} \varphi^2 + \mathcal{O}(\varphi^3) \right] + \frac{\mathcal{R}}{d\ell} |\varphi|^{\frac{2}{\Delta_+}} [1 + \mathcal{O}(\varphi) + \mathcal{O}(\mathcal{R})], \quad (3.18)$$

$$S_+(\varphi) = \frac{\Delta_+}{\ell} \varphi [1 + \mathcal{O}(\varphi)] + \frac{1}{\ell} \mathcal{O} \left(\mathcal{R} |\varphi|^{\frac{2}{\Delta_+}+1} \right), \quad (3.19)$$

$$T_+(\varphi) = \ell^{-2} \mathcal{R} |\varphi|^{\frac{2}{\Delta_+}} [1 + \mathcal{O}(\varphi) + \mathcal{O}(\mathcal{R})], \quad (3.20)$$

with Δ_+ defined as before in (3.5), but now we have that $m^2 < 0$. The integration constant $\mathcal{R}$ is continuous and gives rise to a family of solutions. As before, we can integrate to obtain:

$$\varphi(u) = \varphi_+ \ell^{\Delta_+} e^{\Delta_+ u/\ell} \left[1 + \mathcal{O} \left(\mathcal{R} |\varphi_+|^{2/\Delta_+} e^{2u/\ell} \right) + \dots \right] + \dots, \quad (3.21)$$

$$A(u) = \bar{A}_+ - \frac{u}{\ell} - \frac{\varphi_+^2 \ell^{2\Delta_+}}{8(d-1)} e^{2\Delta_+ u/\ell} - \frac{\mathcal{R} |\varphi_+|^{2/\Delta_+} \ell^2}{4d(d-1)} e^{2u/\ell} + \dots, \quad (3.22)$$

This solution has the following interpretation:

- Recall that the expansions of W , S and T are valid only for small φ . As $\Delta_+ > d$ at a minimum, it follows from (3.21) that small φ requires $u \rightarrow -\infty$. Using (3.22), this in turn implies that $e^{A(u)} \rightarrow \infty$ when approaching the minimum of the potential. This is the behavior expected when approaching a UV fixed point.
- In the boundary QFT, the bulk field φ will be associated with an operator $\mathcal{O}$. However, the absence of a term $\sim e^{\Delta_- u/\ell}$ in (3.21) implies that the source of this operator vanishes. On the other hand, there is a non-zero vev given by

$$\langle \mathcal{O} \rangle_+ = (2\Delta_+ - d)\varphi_+ . \quad (3.23)$$

The solution (3.21) is to be interpreted as a flow purely driven by a vev.

- Matching the expression (3.22) with the near-boundary expansion (2.14) we again find that $\mathcal{R}$ is related to the curvature R^{uv} of the background manifold of the UV QFT:

$$\mathcal{R} = R^{\text{uv}}|\varphi_+|^{-2/\Delta_+} . \quad (3.24)$$

To summarize, as in the flat case, for solutions of the (+)-branch *minima of the potential correspond to UV fixed points*. Flows from such UV fixed points are not driven by a source for the perturbing operator $\mathcal{O}$, but rather by its vev $\langle \mathcal{O} \rangle_+$. This corresponds to a spontaneous breaking of conformal invariance.

As in the flat case, these solutions are generically singular in the IR, because unlike the (−)-branch solutions departing from a UV *maximum*, they have no continuous adjustable parameter (beside the UV data R^{uv}) which one may vary to select a solution with regular interior geometry. In the (−)-branch on the other hand, this role is played by the extra integration constant C . Nevertheless, one can construct specific examples where regular solutions of the (+) type exist [45].

(−)-branch. Interestingly, we will need to distinguish between the two cases where the boundary field theory is defined on a curved manifold ($R^{\text{uv}} \neq 0$) and a flat manifold ($R^{\text{uv}} = 0$). A key result is that for $R^{\text{uv}} \neq 0$ the (−)-branch of solutions does not exist. A proof (at the level of the functions $\varphi(u)$ and $A(u)$) can be found in appendix E. More specifically, there are no solutions in the (−)-branch corresponding to flows that either leave or arrive at a minimum of V for $R^{\text{uv}} \neq 0$. Such a solution only exists if $R^{\text{uv}} = 0$ and we recover the result from [45], where RG flows for QFTs on flat manifolds were studied.

For $R^{\text{uv}} = 0$ we have that $T_-(\varphi) = 0$ identically. In addition, (2.6) implies that $S(\varphi) = W'(\varphi)$ and the solution is completely specified by $W(\varphi)$. Our findings can be summarized as follows:

$$R^{\text{uv}} \neq 0 : \quad \text{No solution.} \quad (3.25)$$

$$R^{\text{uv}} = 0 : \quad W_-(\varphi) = \frac{1}{\ell} \left[2(d-1) + \frac{\Delta_-}{2} \varphi^2 + \mathcal{O}(\varphi^3) \right] . \quad (3.26)$$

For the case $R^{\text{uv}} = 0$ we can determine $\varphi(u)$ and $A(u)$ by integrating (2.15), (2.16) subject to (3.26) and $S = W'$. One obtains

$$\varphi(u) = \varphi_- \ell^{\Delta_-} e^{\Delta_- u/\ell} + \dots, \quad (3.27)$$

$$A(u) = \bar{A} - \frac{u}{\ell} - \frac{\varphi_-^2}{8(d-1)} \ell^{2\Delta_-} e^{2\Delta_- u/\ell} + \dots, \quad (3.28)$$

with $\bar{A}$ and φ_- integration constants. We recover the known result that for $R^{\text{uv}} = 0$ *minima of the potential correspond to IR fixed points*, as φ small now requires $u \rightarrow +\infty$ (as $\Delta_- < 0$ at a minimum) which in turn implies $e^{A(u)} \rightarrow 0$. On the other hand, no such IR limit exist in the presence of curvature, unlike around a UV fixed point, where curvature only added subleading corrections. One is then led to wonder what is the fate of curved RG flows in the interior where curvature becomes the dominant driving parameter. This is the subject of the next section.

4 The geometry in the interior

After having analyzed the behavior close to the UV boundary, in this section we analyze the geometry in the interior. In particular, we will be interested in the way the space-time can “end” in a regular way, i.e. where the scale factor shrinks to zero but the bulk curvature invariants are finite. These are the curved analogs of flat IR fixed points.

Unlike what happens close to the AdS boundary, where curvature leads only to subleading corrections to the near-boundary asymptotics, in the interior curvature can drastically change the geometry with respect to the flat case. As we will see, for positive curvature, the RG flow reaches an end before the (would-be) IR fixed point of the flat theory; for negative curvature, on the other hand, both the scalar field and the scale factor turn around and the flow reaches the boundary on both sides. Non-zero curvature can also deform the geometry near *bounces* (points where the scalar field inverts its flow direction [45] but the scale factor is still monotonically decreasing).

In this analysis, a key role is played by the critical points in the 1st order equations. These are points where $\dot{\phi} = 0$. Before discussing the non-zero curvature case, we briefly review the analysis of critical points for zero curvature.

In the flat case critical points correspond to points along the flow with vanishing W' . Extrema of W in the interior of the geometry belong to two classes: regular IR fixed points, and bounces [45].

- *Flat IR fixed points.* A regular IR endpoint of the geometry is attained, in the flat case, when the flow asymptotes to a minimum (which we denote by φ_{IR}) of the scalar potential. In this case, $\dot{\phi} \rightarrow 0$ and the geometry in the interior is asymptotically AdS_{d+1} , with

$$e^A(u) \simeq e^{-u/\ell_{\text{IR}}}, \quad \varphi(u) \rightarrow \varphi_{\text{IR}}, \quad u \rightarrow +\infty \quad (4.1)$$

In the language of the superpotential, as $\varphi \rightarrow \varphi_{\text{IR}}$, $W \rightarrow 2(d-1)/\ell_{\text{IR}}$ and $W' \rightarrow 0$, and the (flat) holographic β -function vanishes:

$$\beta(\varphi) = -2(d-1) \frac{W'}{W} \rightarrow 0, \quad \varphi \rightarrow \varphi_{\text{IR}}. \quad (4.2)$$

The condition $W'(\varphi_{\text{IR}}) = 0$ by itself does not imply that the flow has reached a fixed point: for that, we also need $W''(\varphi)$ to be finite, and this only happens when $V'(\varphi_{\text{IR}}) = 0$, i.e. φ_{IR} is also a minimum of the bulk potential as we have assumed above. If the latter condition is not met, then the geometry features a bounce, as discussed below.

- *Flat bounces.* A generic extremum of W , i.e. a point φ_B in field space such that $W'(\varphi_B) = 0$ but $V'(\varphi_B) \neq 0$, corresponds to a *bounce*, i.e. a point where the flow inverts its direction [45]. The superpotential becomes singular because φ is not a good coordinate around such points. The flow however can be continued by gluing to another branch, where $\dot{\varphi}$ has the opposite sign. Close to a bounce, the superpotential behaves as:

$$W(\varphi) \simeq W_b \pm c(\varphi_B - \varphi)^{3/2} + O((\varphi_B - \varphi)^2), \quad (4.3)$$

where c is a constant and we have supposed that the flow reaches the bounce from below ($\varphi < \varphi_B$). The two signs correspond to the two branches of the superpotential along the RG flow before and after the bounce, which can be glued at $\varphi = \varphi_B$ giving rise to a regular solution $\varphi(u), A(u)$, in which all curvature invariants are finite. The holographic β function still vanishes at φ_B , but it becomes multivalued:

$$\beta(\varphi) \simeq \mp \frac{3(d-1)}{W_b} \sqrt{\varphi_B - \varphi} + O((\varphi_B - \varphi)), \quad (4.4)$$

It is natural to ask what happens to IR fixed points and bounces when the boundary theory lives on a curved space-time. In the maximally symmetric case analyzed in this paper, the condition $\dot{\varphi} = 0$ is now the vanishing of the function $S(\varphi)$, which in the flat case coincides with W' . The analysis of the possible ways in which S can vanish is presented in appendix F. As it is shown there, the asymptotic behavior of the functions W , S and T always takes the general form of an expansion in half-integer powers of $x \equiv (\varphi_0 - \varphi)$ (which we assume to be positive, for simplicity):

$$S(x) = \sqrt{x} (S_0 + S_1\sqrt{x} + S_2x + \dots), \quad (4.5)$$

$$W(x) = \frac{1}{\sqrt{x}} (W_0 + W_1\sqrt{x} + W_2x + \dots), \quad (4.6)$$

$$T(x) = \frac{1}{x} (T_0 + T_1\sqrt{x} + T_2x + \dots), \quad (4.7)$$

On the other hand, we have assumed that the potential has a regular expansion around φ_0 :

$$V(x) = V_0 + V_1x + V_2x^2 + \dots. \quad (4.8)$$

Depending on the sign of the curvature and on the values of the coefficients in (4.5)–(4.7), the solution around $\varphi = \varphi_0$ can be of three possible types:

1. Fixed points (positive curvature only)
2. Reflection points (negative curvature only)
3. Bounces (both signs of the curvature)

We proceed to discuss each one in turn.

4.1 Positive curvature flows: IR endpoints

Let us suppose that φ_0 is such that $S(\varphi_0) = 0$, and the leading coefficients S_0 , W_0 and T_0 are all non-vanishing (this corresponds to case (b) in appendix F). Then, $T = R^{(\zeta)} e^{-2A}$ diverges at φ_0 , implying that the scale factor e^A shrinks to zero size. However, this does not imply a singularity but as we will see below, it represents a coordinate horizon (or a regular end of space in the Euclidean signature).

To leading order in $\varphi_0 - \varphi$, we have:

$$S(\varphi) \simeq S_0(\varphi_0 - \varphi)^{1/2}, \quad W(\varphi) \simeq \frac{W_0}{(\varphi_0 - \varphi)^{1/2}}, \quad (4.9)$$

$$T \simeq \frac{T_0}{(\varphi_0 - \varphi)} \quad (4.10)$$

where the coefficients are fixed by the superpotential equations to be:

$$S_0^2 = \frac{2V_1}{d+1}, \quad W_0 = (d-1)S_0, \quad T_0 = \frac{d(d-1)}{4(d+1)} S_0^2 \quad (4.11)$$

We see that this solution, obtained assuming $\varphi < \varphi_0$, makes sense only for $V_1 > 0$, i.e. $V'(\varphi_0) < 0$ (cfr. equation (4.8)). Similarly, it is easy to show that, for $V'(\varphi_0) > 0$ we have to reach φ_0 from above. In both cases, equation (4.10) implies $T > 0$. From the definition (2.17), this in turn implies that such behavior can occur only for *positive curvature*.

With expressions (4.9)–(4.10) for W , S and T , we can integrate equations (2.15)–(2.16) order by order in $(\varphi_0 - \varphi)$ to find the expressions for the scale factor $A(u)$ and the scalar field $\varphi(u)$. We define the “end of space” point u_0 where

$$\varphi(u_0) = \varphi_0, \quad e^{A(u_0)} = 0, \quad (4.12)$$

To lowest order, one finds:

$$\varphi = \varphi_0 - \frac{S_0^2}{4}(u - u_0)^2 + O((u - u_0)^3), \quad A(u) = \ln(u - u_0) + A_0 + O(u - u_0). \quad (4.13)$$

The parameter A_0 is an integration constant for equation (2.15), but it is determined algebraically by the asymptotic form of the superpotential $T(\varphi)$ close to φ_0 :

$$T \equiv R^{(\zeta)} e^{-2A(u)} \simeq \frac{T_0}{\varphi_0 - \varphi} \quad (4.14)$$

Inserting the expansions (4.13), and using the relations (4.11) and the fact that $R^{(\zeta)} = d(d-1)/\alpha^2$, one easily finds:

$$A_0 = -\log \alpha \quad (4.15)$$

Interestingly, it is precisely (and only) for this value of A_0 that there is no singularity at u_0 . Recall that the metric is given by:

$$ds^2 = du^2 + e^{2A(u)} ds_{\text{dS}_\alpha}^2 = du^2 + \alpha^2 e^{2A(u)} ds_{\text{dS}_{\alpha=1}}^2, \quad (4.16)$$

where $ds_{\text{dS}_\alpha}^2$ is the de Sitter metric with radius α . Near $u \rightarrow u_0$, this becomes,

$$ds^2 \simeq du^2 + \alpha^2 e^{2A_0} (u - u_0)^2 ds_{\text{dS}_{\alpha=1}}^2. \quad (4.17)$$

For $A_0 = -\log \alpha$, the four-dimensional part of the metric vanishes exactly like it does at the “end of space” in the dS slicing of AdS_{d+1} , as can be seen from equation (2.9) by expanding the scale factor around the point $u = -c$. Equivalently, if we go to Euclidean signature, unit-curvature dS_d becomes the unit sphere S_d , and for $A_0 = -\log \alpha$ the metric close to u_0 is approximately:

$$ds_E^2 = d\rho^2 + \rho^2 dS_d^2, \quad \rho \equiv |u - u_0| \quad (4.18)$$

i.e. the metric around the origin of Euclidean $d + 1$ -space in spherical coordinates. In the Minkowski version, $u = u_0$ is a coordinate singularity (it is a horizon).

Notice that the qualitative behavior of the solution is different, and the IR endpoint discussed here does not exist, if $V'(\varphi_0) = 0$: indeed, in this case $T_0 = 0$. Thus, solutions with positive slice curvature cannot reach a would-be flat IR fixed point φ_{IR} , which corresponds to a minimum of the scalar potential. This is in agreement with the result found in section 3, that curved RG flows cannot end at minima of the potential.

From the above analysis we can infer the general structure of a positive curvature RG flow in a model with a bulk potential having several maxima and minima. Consider a *flat* RG flow connecting a UV maximum at $\varphi = \varphi_{\text{UV}}$ to an IR minimum at $\varphi = \varphi_{\text{IR}}$. In the region $\varphi_{\text{UV}} < \varphi < \varphi_{\text{IR}}$, we have $V'(\varphi) < 0$ by construction. Thus, adding positive curvature forces the flow to stop at an intermediate point $\varphi_{\text{UV}} < \varphi_0 < \varphi_{\text{IR}}$. This behavior is illustrated in figure 2.

The position of the endpoint in field space is determined by equations (4.11) and by the value of the UV data, i.e. the source term φ_- in the scalar field UV asymptotics, and the boundary curvature R^{uv} . This can be seen as follows: from the IR equations, choosing the endpoint φ_0 determines all the expansion coefficients in the superpotentials, (4.11), through the values of $V(\varphi)$ and its derivatives at φ_0 . On the other hand, the integration constant A_0 in the scale factor is fixed by the curvature parameter by equation (4.15). Since all the integration constants are fixed, if we integrate the equations towards the UV we will find a given value of the UV source and vev parameters, φ_- and φ_+ . Turning it around, for a given set of UV data φ_- and R^{uv} , there exist a single regular solution with a specific vev $\varphi_+(\varphi_-, R^{\text{uv}})$ and a specific endpoint $\varphi_0(\varphi_-, R^{\text{uv}})$.

As we will see in the next section, and as expected on general grounds, the position of the endpoint actually depends only on the dimensionless combination $\mathcal{R} \equiv R^{\text{uv}}(\varphi_-)^{-2/\Delta_-}$. Indeed, from the UV expansions (3.2)–(3.4) it is clear that the superpotentials know only about the integration constant $\mathcal{R}$, and do not depend separately on the source φ_- . Thus it is the combination $\mathcal{R}$ which determines the endpoint. As $\mathcal{R}$ ranges from zero to infinity, the endpoint moves from φ_{IR} to φ_{UV} .

Finally, the IR endpoint discussed in this section, constitutes a special case of the general result we present as property 5 of superpotentials in appendix C, namely that around a point where $S = 0$, the bulk metric is approximately maximally symmetric. In

other words, the IR fixed point is a point of enhanced symmetry. This is similar to the flat case, where at an IR fixed point, Poincaré symmetry is upgraded to conformal invariance. Here, from the boundary point of view, the invariance is conformal symmetry written in “unusual” coordinates.

4.2 Negative curvature flows: AdS throat

The endpoint behavior presented in the previous section requires positive curvature slices. If the curvature is negative, then we must consider (a) in appendix F. This generically corresponds to a bounce (which will be discussed in the next section) *except* in the special case $W_0 = W_1 = 0$ (referred to as case (c) in appendix F):

$$S_0^2 = 2V_1, \quad W_0 = W_1 = 0, \quad T_0 = T_1 = 0 \quad (4.19)$$

$$W_2 = \frac{4V_0}{dS_0}, \quad T_2 = V_0. \quad (4.20)$$

At leading order the superpotentials are (assuming $\varphi < \varphi_0$):

$$S \simeq S_0(\varphi_0 - \varphi)^{1/2} + \dots, \quad W \simeq W_2(\varphi_0 - \varphi)^{1/2} + \dots, \quad T \simeq T_2 + \dots \quad (4.21)$$

This solution is possible only for negative slice curvature, since $T = e^{-2A}R^{(\zeta)}$ has the same sign as the potential $V(\varphi_0)$, which we assumed to be negative definite. Notice that now W vanishes at φ_0 , and $T = e^{-2A}R^{(\zeta)}$ remains finite. This implies that $\dot{A} = 0$ and A is finite at φ_0 , i.e. the scale factor has a turning point.

By integrating the first order flow equations, one can show that this solution corresponds to the following geometry for $u \rightarrow u_0$:

$$\varphi(u) \simeq \varphi_0 - \frac{S_0^2}{4}(u - u_0)^2, \quad (4.22)$$

$$ds^2 \simeq du^2 + e^{2A_0} \left(1 + \frac{(u - u_0)^2}{\ell_{\text{IR}}^2} \right) ds_{\text{AdS}_d, \alpha}^2, \quad (4.23)$$

where A_0 is an integration constant, $ds_{\text{AdS}_d, \alpha}^2$ is a metric on AdS_d with length scale α and $\ell_{\text{IR}}^2 = d(d-1)/|V(\varphi_0)|$. The behavior around u_0 matches that of space AdS_{d+1} , with length scale ℓ_{IR} , sliced with AdS_d hypersurfaces, as seen in (2.9).

The conditions $\dot{A}(u_0) = \dot{\varphi}(u_0) = 0$ imply that we can continue the geometry for $u > u_0$ by gluing its reflected image around u_0 . The resulting geometry is regular since the metric and scalar field, as well as their derivatives, are all continuous across u_0 . If the solution starts at a UV fixed point reached as $u \rightarrow -\infty$, the resulting doubled geometry will reach a “second” UV boundary again as $u \rightarrow +\infty$, connected by a throat where the scale factor reaches a minimum, as shown in figure 3.

This geometry does not really have two disconnected boundaries: rather, the two boundaries at $u = \pm\infty$ are connected through the boundary of the lower-dimensional AdS_d slices. This was discussed in [56] in the case of the AdS_{d+1} geometry written in AdS_d slices, and we will return to this point in subsection 4.4.

As was the case for the endpoint in the positive curvature case, the turning point φ_0 is completely determined by the UV data φ_- and R^{uv} .

Figure 6. φ and A vs. u for an example exhibiting a bounce at $u = u_B$.

4.3 Bounces

The generic situation of such a point φ_B with vanishing S and finite W corresponds to $W_0 = 0$ but $W_1 \neq 0$ (case (a) in appendix F). To leading order, the superpotentials behave as (assuming $(\varphi < \varphi_B)$):

$$S \simeq S_0(\varphi_B - \varphi)^{1/2} + \dots, \quad W \simeq W_1 + W_2(\varphi_B - \varphi)^{1/2} + \dots, \quad T \simeq T_2 + \dots \quad (4.24)$$

where

$$S_0^2 = 2V_1, \quad T_2 = V_0 + \frac{dW_1^2}{4(d-1)}, \quad (4.25)$$

and W_1 is arbitrary ($W_1 = 0$ corresponds to the special case of the AdS throat analyzed in the previous subsection). In this case, the scalar field has a turning point, but the scale factor has a non-vanishing derivative. Since $V_0 < 0$, T_2 does not have a definite sign and therefore this solution is allowed for both positive and negative curvature.

The solution (4.25) actually describes two branches, with $S_0 = \pm\sqrt{V_1}$, corresponding to the two branches of the superpotential equations discussed in section 2.2: at a bounce, these two branches can be glued, giving rise to a regular geometry, exactly as in the flat case [45]. Integrating the superpotential equations we find the bulk solution close to φ_B in terms of the u coordinate (where $u = u_B$ is the bounce point):

$$\varphi(u) \simeq \varphi_B - \frac{S_0^2}{4}(u - u_B)^2 + \dots \quad A(u) = A_B - \frac{W_1}{2(d-1)}(u - u_B) + \dots \quad (4.26)$$

This solution is regular both for $u < u_B$ ($S(\varphi) > 0$) and $u > u_B$ ($S(\varphi) > 0$). The superpotentials are singular because at φ_B because φ ceases to be a good coordinate at φ_B , but the geometry (4.26) is smooth. The typical scalar field profile is illustrated in figure 6a. Like in the flat case, we refer to this as a bouncing flow.

The fact that the geometry is regular at the bounce can be also be seen from the fact that all the curvature invariants can be written in terms of the potential $V(\varphi)$ and the function $S(\varphi)$ only. As the behavior of $S(\varphi)$ is regular near the bounce, all the curvature invariants are finite, as one can see from the expressions in appendix A.

We can relate the value of the scale factor at the bounce by evaluating the relation $T = R^{(\zeta)} e^{-2A}$ at $u = u_B$, using equations (4.24) and (4.26):

$$T_2 = R^{(\zeta)} e^{-2A_B}. \quad (4.27)$$

Using the expression of T_2 from eq. (4.25), we find a relation between A_B , W_1 and $V(\varphi_B)$:

$$A_B = -\ln(\alpha) - \frac{1}{2} \ln \left| \left(\frac{W_1}{2(d-1)} \right)^2 - \frac{1}{\ell_B^2} \right| \quad (4.28)$$

where we have defined

$$V(\varphi_B) = -\frac{d(d-1)}{\ell_B^2}. \quad (4.29)$$

As $S = 0$ at the bounce, the bulk geometry is approximately maximally symmetric, as shown in appendix C.

4.4 The geometry of curved RG flows

In this section we clarify some aspects about the global geometry of the RG flow solutions discussed above. In particular, we are interested in the geometry close to the boundary and in the deep interior (the endpoint of the flow).

For simplicity, we will consider explicitly the situation where the scalar field does not run: such solutions are just AdS_{d+1} space-time written in an unusual coordinate system, and the analysis can be performed analytically. However, all the qualitative conclusions carry on to non-trivial RG flow solutions both close to the boundary and close to the flow endpoint, where the geometry is approximately AdS_{d+1} .

We start by recalling some facts about various coordinate systems in AdS_{d+1} . The “flat” slicing coordinates $(z, t, \vec{x})$ cover the Poincaré patch of AdS_{d+1} , and in these coordinates the metric is given by:

$$ds^2 = \frac{\ell^2}{z^2} \left(dz^2 - dt^2 + \sum_{i=1}^{d-1} dx^i dx^i \right), \quad z > 0. \quad (4.30)$$

Global coordinates (ρ, ψ, Ω_i) , on AdS_{d+1} are related to the Poincaré coordinates by:

$$\cosh \rho = \frac{[\Delta^2 + (2\ell t)^2]^{1/2}}{2\ell z}, \quad \sin \psi = \frac{2\ell t}{[\Delta^2 + (2\ell t)^2]^{1/2}} \quad (4.31)$$

$$\Omega_i = \frac{2\ell x_i}{[\Delta^2 + (2\ell t)^2 - (2\ell z)^2]^{1/2}}, \quad i = 1 \dots d-1, \quad (4.32)$$

$$\Delta \equiv z^2 + \ell^2 + |\vec{x}|^2 - t^2.$$

In global coordinates the metric reads:

$$ds^2 = \ell^2 [d\rho^2 - \cosh^2 \rho d\psi^2 + \sinh^2 \rho d\Omega^2]. \quad (4.33)$$

The AdS boundary is located at $\rho \rightarrow +\infty$ in global coordinates, and it is reached as $z \rightarrow 0$ (with the (t, x_i) coordinates fixed) in the Poincaré patch. In addition, a single point on

the boundary (corresponding to $\psi = 0, \Omega_i = 0$ in global coordinates) can also be reached as $z \rightarrow +\infty$. This point also corresponds to spatial infinity ($|\vec{x}| \rightarrow +\infty$) on any fixed- z slice. The Poincaré horizon is reached as $z \rightarrow \infty, t \rightarrow \pm\infty, z = \pm t + \text{subleading terms}$, and corresponds to ρ finite (e.g. $\vec{x} = 0$ on the Poincaré horizon is located at $\rho = 0, \psi = \pi/2, \Omega_i = 0$). For a detailed discussion of how the AdS boundary is described in Poincaré coordinates, the reader is referred to [62]. Finally, recall that one can go to conformal global coordinates by changing variables to $\tan \Theta = \sinh \rho$, which leads to:

$$ds^2 = \frac{\ell^2}{\cos^2 \Theta} [d\Theta^2 - d\psi^2 + \sin^2 \Theta d\Omega^2]. \quad (4.34)$$

In these coordinates the AdS boundary is at $\Theta = \pi/2$ and it is manifestly conformal to $R_\psi \times S^{d-1}$.

Positive curvature. We now consider the positive curvature slicing of AdS_{d+1} ,

$$ds^2 = du^2 + \ell^2 \sinh^2 \frac{u}{\ell} \left[\frac{-d\tau^2 + \delta_{ij} dx^i dx^j}{\tau^2} \right], \quad \tau \in (-\infty, 0), \quad u \in (-\infty, 0). \quad (4.35)$$

This metric corresponds to the scale factor given in (2.9) for the dS_d case, where for simplicity we have chosen $c = 0$. For the positive curvature submanifolds at fixed u , we have chosen to write the metric as the cosmological patch of de Sitter space dS_d in conformal time. We emphasize that for pure AdS this is a choice of coordinates, whereas in the case of RG flows the metric ansatz below corresponds to a different solution from flat-section RG flows.

One can check that the coordinate transformation between the forms (4.35) and (4.30) of the metric is given by:

$$t = -\tau \coth \frac{u}{\ell}, \quad z = \frac{\tau}{\sinh \frac{u}{\ell}}, \quad (4.36)$$

$$u = -\ell \cosh^{-1} \left(-\frac{t}{z} \right), \quad \tau = -(t^2 - z^2)^{1/2}. \quad (4.37)$$

As is clear from these equations, the dS coordinates only cover a wedge of the Poincaré patch, corresponding to $t \in (-\infty, 0)$ and $0 < z \leq |t|$. It is also instructive to go to hyperbolic coordinates:

$$t = \tau \cosh \theta, \quad z = \tau \sinh \theta, \quad \cosh \frac{u}{\ell} = -\coth \theta. \quad (4.38)$$

The metric in these coordinates reads:

$$ds^2 = \frac{\ell^2}{\sinh^2 \theta} \left[d\theta^2 + \frac{-d\tau^2 + \delta_{ij} dx^i dx^j}{\tau^2} \right], \quad \tau \in (-\infty, 0), \quad \theta \in (-\infty, 0). \quad (4.39)$$

with the boundary located at $\theta = 0$. The curves $\tau = \text{const}$ are branches of hyperboles, whereas the curves $u = \text{const}$ are straight lines through the origin which can be parametrized by θ (see figure 7).

We can identify the following limits:

Figure 7. Relation between Poincaré coordinates (t, z) and dS-slicing coordinates (τ, u) . Constant u curves are half straight lines all ending at the origin ($\tau \rightarrow 0^-$); constant τ curves are branches of hyperbolas ending at $u = 0$ (null infinity on the $z = -t$ line). The boundary $z = 0$ corresponds to $u \rightarrow -\infty$.

- As $u \rightarrow -\infty$, ($\theta = 0$) the two time coordinates become equal, $t = \tau$, and $z \rightarrow 0$. This corresponds to going to the boundary.
- The limit $\tau \rightarrow 0^-$, i.e. the far future in dS, also corresponds to the boundary $z = 0$, as well as the future end of the t half-line, $t = 0$.
- The “endpoint” $u \rightarrow 0$, ($\theta = -\infty$) corresponds to the line $z = |t|$, with both $t \rightarrow \infty$ and $t \rightarrow -\infty$. This is the past horizon of Poincaré AdS.

The same remarks apply to the same regions in the RG flow geometry.

To have a better understanding of the “endpoint” $u \rightarrow 0$, it is convenient to see what it corresponds to in global coordinates. Using equations (4.31) and (4.38) we find the transformation between global and dS-slicing coordinates, which at $\vec{x} = 0$ reads:

$$\cosh \rho = \frac{\left[(\ell^2 - \tau^2)^2 + 4\tau^2 \ell^2 \cosh^2 \theta \right]^{1/2}}{2\ell\tau \sinh \theta}, \quad (4.40)$$

$$\sin \psi = -\frac{2\ell\tau \cosh \theta}{\left[(\ell^2 - \tau^2)^2 + 4\tau^2 \ell^2 \cosh^2 \theta \right]^{1/2}}. \quad (4.41)$$

In the IR limit $\theta \rightarrow -\infty$, and for any τ fixed, we reach the point $\rho = 0, \psi = -\pi/2$, which is a point in the interior of AdS space-time. On the other hand, the far past and future are on the boundary at $\psi = -\pi$ and $\psi = 0$ respectively. This embedding is illustrated in figure 8. Figure 9 represents the conformal diagram of the same geometry, where the AdS boundary is now at $\Theta = \pi/2$.

Figure 8. Embedding of the dS patch in global coordinates. The flow endpoint $u = 0$ corresponds to the point $\rho = 0, \psi = -\pi/2$ in global coordinates. the AdS boundary is at $\rho = +\infty$ and it is reached along u as $u \rightarrow -\infty$, and along τ both as $\tau \rightarrow -\infty$ and as $\tau \rightarrow 0$.

Figure 9. Embedding of the dS patch in global conformal coordinates, $\tan \Theta = \sinh \rho$, where each point is a $d - 1$ sphere “filled” by Θ . The boundary is at $\Theta = \pi/2$. The dashed lines correspond to the Poincaré patch embedded in global conformal coordinates. The flow endpoint $u = 0$ is situated on the Poincaré horizon.

Negative curvature. We now repeat the analysis for the case negative curvature. We write the metric of AdS_{d+1} using the scale factor from equation (2.9) in the negative curvature case,

$$ds^2 = du^2 + \ell^2 \cosh^2 \frac{u}{\ell} \left[\frac{d\xi^2 - dt^2 + \eta_{ij} dx^i dx^j}{\xi^2} \right], \quad (4.42)$$

$$\xi \in (0, +\infty), \quad u \in (-\infty, +\infty), \quad i = 1, \dots, d-2.$$

We have set the constant $c = 0$ for simplicity, and written the AdS_d slice metric in Poincaré coordinates.

As we discussed in the previous subsections, the metric (4.42) seems to have two distinct boundaries, at $u = \pm\infty$. Below we will clarify the meaning of the boundary structure, and argue that the boundary conditions at the two endpoints must be identified.

First, let us write the coordinate transformation that brings the Poincaré AdS metric (4.30) to the form (4.42). To this end, we single out one spatial coordinate $x \equiv x^1$ which will be traded for the AdS_d radial coordinate ξ (the other coordinates $x^2 \dots x^{d-1}$ being spectators). Then, the appropriate transformation is:

$$x = \xi \tanh \frac{u}{\ell}, \quad z = \frac{\xi}{\cosh \frac{u}{\ell}}, \quad (4.43)$$

$$u = \ell \sinh^{-1} \left(\frac{x}{z} \right), \quad \xi = (x^2 + z^2)^{1/2}. \quad (4.44)$$

It is instructive to use polar coordinates:

$$x = \xi \sin \theta, \quad z = \xi \cos \theta, \quad \sinh \frac{u}{\ell} = \tan \theta. \quad (4.45)$$

We notice the following features:

- The lines of constant θ are straight lines in the half-plane $(x, z > 0)$, reaching the origin as $\xi \rightarrow 0$ (Poincaré boundary of the lower-dimensional AdS_d slice) (see figure 10).
- The Poincaré boundary $z = 0, x \in (-\infty, +\infty)$ is split in two parts: $x > 0$ corresponds to $u \rightarrow +\infty$ ($\theta = \pi/2$), whereas $x < 0$ is reached as $u \rightarrow -\infty$ ($\theta = -\pi/2$).
- The lines of constant ξ are circles joining the two half-boundaries from $\theta = -\pi/2$ to $\theta = \pi/2$.
- The two halves of the boundary are connected through the surface $\xi = 0$ (which is on the boundary for any value of u (or θ)). This surface corresponds to the boundary of the *lower dimensional* AdS space.

Therefore, the two boundaries at $u = \pm\infty$ are not really disconnected, but they are connected through the lower-dimensional AdS boundary. Because of this, the geometry really has only one boundary, as was already noted in [56] in the case of the pure AdS geometry. This fact carries over to the our RG flow solutions, however there is a subtlety due to the non-trivial scalar field profile. In our case, the scalar field is constant on each

Figure 10. Relation between Poincaré coordinates (x, z) and AdS-slicing coordinates (ξ, u) . Constant u curves are half straight lines all ending at the origin ($\xi \rightarrow 0^-$); constant ξ curves are semicircle joining the two halves of the boundary at $u = \pm\infty$.

radial slice. In the coordinates (4.45), this is a function $\varphi(\theta)$, which does not vanish at the origin $\xi = 0$. Thus, the solution can only be defined up to a cut-off $\xi = \epsilon$, but it cannot be extended to a regular function all the way to the *slice* AdS boundary $\xi = 0$. The full boundary is thus the two half lines $\theta = \pm\pi/2$ starting at $x = \pm\epsilon$, connected by the half circle $\xi = \epsilon$ with $-\pi/2 < \theta < \pi/2$. The fact that one cannot take the limit $\epsilon \rightarrow 0$ in a regular way suggests that, on the field theory side, the theory lives on an AdS space with a defect (or a source) on the boundary. The precise nature of such a defect is a matter of speculation, and we will not investigate it further here.

There is another setup that is described by the AdS-sliced metrics we consider. As already suggested in [56], we may orbifold the AdS slices by discrete Fuchsian groups in order to make them have finite volume. For AdS_2 for example one obtains genus $g \geq 2$ Riemann surfaces with negative curvature. In such a case, the full bulk space-time has now two disconnected boundaries. There are possible instabilities in that case studied for the full AdS geometry in [56]. Our results are applicable to cases where such instabilities do not arise.

Notice that regularity of the RG flows implies that the solution must be symmetric around $\theta = 0$, since at that point both the derivative of the scale factor and the scalar field vanish. Thus, the UV boundary conditions on $x < 0$ must be the same as for $x > 0$.

Finally, notice that knowledge of the bulk solution automatically imposes boundary conditions for the fields in the lower dimensional theory on AdS_d .

4.5 Comments on stability

To conclude this section we briefly comment on the issue whether the solutions we are discussing are stable under small perturbations. Here we will not perform a complete stability analysis, which is technically involved and is beyond the scope of this paper. Nevertheless, we can draw some hints about stability from general considerations and previous experience with the same issue in flat-slicing solutions. This will not be conclusive, but it will lay the ground for a more thorough future analysis. We start with some preliminary considerations, which we will then illustrate explicitly in the case of tensor fluctuations.

In $(d + 1)$ -dimensional Einstein-scalar theories where the metric ansatz respects d -dimensional maximal symmetry, physical fluctuations can be decomposed into tensor, vector and the scalar components with respect to the d -dimensional constant curvature metric $\zeta_{\mu\nu}$. Linearized diffeomorphisms play the role of gauge transformations, and one can show that the gauge-invariant propagating fluctuations reduce to a single propagating, gauge-invariant scalar mode, as well as a single tensor mode. The latter is obtained by perturbing the background slice metric by a symmetric transverse traceless 2-tensor,

$$\zeta_{\mu\nu} \rightarrow \zeta_{\mu\nu} + h_{\mu\nu}, \quad \nabla^{(\zeta)\cdot\mu} h_{\mu\nu} = 0 = \zeta^{\mu\nu} h_{\mu\nu}. \quad (4.46)$$

The definition of the single gauge-invariant scalar fluctuation $\xi(u, x^\mu)$ is much more complex, as it involves the background functions $A(u)$ and $\varphi(u)$.

The question of stability can be phrased in terms of the positivity of eigenvalues $\lambda = m^2$ of suitable Hamiltonians for a set of Schrödinger-like problems (one for each independent bulk perturbation). The Hamiltonian is the operator governing the radial evolution. The corresponding eigenvalues m^2 represent the mass spectrum in a Kaluza-Klein decomposition in terms of d -dimensional modes, and stability requires that they are above a certain minimal value (which is zero in flat space). Absence of ghosts on the other hand is automatically guaranteed by the fact that all sources satisfy the null-energy condition.

It is known (see e.g. [45] for a self-contained discussion) that, in scalar-tensor models of type (2.1), asymptotically AdS solutions with *flat* constant- u induced metric, are stable under small perturbations, at least when the Schrödinger problem is well defined without imposing extra boundary conditions in the IR. This condition is satisfied by a large class of bulk potentials, as long as they do not grow too fast at large φ , and it certainly applies when there is an AdS fixed point in the IR. In the latter case, one can show that the spectrum of m^2 for both tensor and scalar fluctuations is continuous and starting at zero (excluded). This holds regardless of the details of the bulk geometry, and it applies in particular to geometries which include a bounce [45]. In other words, in the flat case the presence of a bounce does neither destabilize the tensor nor the scalar sector.

Let us now move on to the curved case. Intuitively, when we put the theory on a positively curved manifold, we may expect to gap a spectrum which was gapless in the flat theory. We will see that this expectation is reflected in a modification of the IR asymptotic behavior of the effective Schrödinger potential for the tensor perturbations. Furthermore, experience with the flat-slicing case suggests that the tensor and scalar Schrödinger potentials obey the same UV and IR asymptotics, which is universally governed by the asymptotically AdS form of the metric. Therefore, we expect that a similar IR modification will

also gap the scalar fluctuations. The presence of a bounce is not expected to introduce specific stability problems, since none were there in the flat case to start with.

Let us now discuss the negatively curved case. One might expect that in this case the spectra become discrete, since the bulk is enclosed by two UV AdS boundaries which behave like infinite potential walls. However the situation is complicated by the fact that, as discussed in section 4.4, one needs to regulate the boundary on both sides, in order to obtain a consistent asymptotic behavior on the slice-AdS boundary. This will again make the spectrum discrete, but the eigenvalues will crucially depend on the boundary conditions one imposes at the regulated boundary.

To illustrate some of the previous points we now present a more quantitative analysis for the tensor perturbations, which turn out to be quite tractable. The linearized second order equation governing the tensor fluctuations is universal, i.e. independent of the fields which source the background. It only involves the scale factor, and in our ansatz it reads (see e.g. [61]):

$$\left[-\partial_u^2 - d(\partial_u A)\partial_u - \left(\nabla^{(\zeta),\mu} \nabla_\mu^{(\zeta)} - 2K \right) \right] h_{\mu\nu}(u, x^\mu) = 0. \quad (4.47)$$

where $K = \alpha^{-2}$ for positive curvature, and $K = -\alpha^{-2}$ for negative curvature. One then decomposes this equation in terms of d -dimensional modes satisfying the appropriate linear equation for a tensor of mass m^2 , plus a radial equation for the profile,

$$h_{\mu\nu} = h(u)\epsilon_{\mu\nu}(x), \quad \left(\nabla^{(\zeta),\mu} \nabla_\mu^{(\zeta)} \pm \frac{2}{\alpha^2} \right) \epsilon_{\mu\nu} = m^2 \epsilon_{\mu\nu}, \quad (4.48)$$

$$\left[-\frac{d^2}{du^2} - d\frac{dA}{du}\frac{d}{du} \right] h(u) = m^2 h(u). \quad (4.49)$$

Equation (4.48) reduces to the linearized Einstein equation for $m^2 = 0$. Thus, in this language, m^2 corresponds to the Pauli-Fierz mass parameter.

One then performs the following transformation on both the radial coordinate and the radial profile $h(u)$,

$$du = e^{A(r)} dr, \quad \psi(r) = e^{\frac{(d-1)}{2}A(r)} h(r), \quad (4.50)$$

to obtain the Schrödinger-like equation

$$\left[-\frac{d^2}{dr^2} + V(r) \right] \psi(r) = m^2 \psi(r), \quad V(r) = \frac{(d-1)^2}{4} \left(\frac{dA}{dr} \right)^2 + \frac{(d-1)}{2} \frac{d^2 A}{dr^2}. \quad (4.51)$$

We now discuss the positive and negative curvature cases separately.

- *Positive curvature:* in this case $r \rightarrow 0$ as $u \rightarrow -\infty$ (UV boundary) and $r \rightarrow +\infty$ as $u \rightarrow u_c$ (IR endpoint). One then finds

$$A(r) \rightarrow \begin{cases} -\log \frac{r}{\ell} & r \rightarrow 0 \\ -\frac{r}{\alpha} & r \rightarrow +\infty \end{cases}, \quad V(r) \rightarrow \begin{cases} \frac{d^2 - 1}{4r^2} & r \rightarrow 0 \\ \frac{(d-1)^2}{4\alpha^2} & r \rightarrow +\infty \end{cases} \quad (4.52)$$

In the UV, as in the zero-curvature case, the potential diverges as $1/r^2$. In the IR, it now asymptotes a positive constant instead of tending to zero as in the flat case. The asymptotics (4.52) imply that the continuous spectrum is gapped by $(d-1)^2/4\alpha^2$. With this asymptotic behavior one can show following [45] that the Hamiltonian operator (4.51) is strictly positive, which implies $m^2 > 0$. This however is not enough, because a massive, transverse traceless symmetric tensor must in addition satisfy the Higuchi bound [63, 64],

$$m^2 \geq \frac{(d-2)}{\alpha^2} \quad (4.53)$$

Since $(d-1)^2/4 \geq (d-2)$ for any d , the modes in the continuous spectrum *do* satisfy the Higuchi bound (4.53). It is suggestive that the gap in the continuous spectrum is sufficient to satisfy the bound, although we cannot exclude the existence of discrete eigenmodes with masses in the range $0 < m^2 < (d-2)/\alpha^2$. This issue can only be settled with a detailed study of the full solution, which most likely has to be performed numerically.

- *Negative curvature:* in this case the Schrödinger problem is in a finite size box $0 < r < L$ whose endpoints are the images of the two boundaries at $u \rightarrow \pm\infty$. The potential has the following asymptotic behavior

$$V(r) \rightarrow \begin{cases} \frac{d^2-1}{4r^2} & r \rightarrow 0 \\ \frac{d^2-1}{4(L-r)^2} & r \rightarrow L \end{cases}. \quad (4.54)$$

Taken at face value, the Hamiltonian operator is positive and all modes respect the unitarity bound for symmetric tensors in AdS_d , which is just $m^2 > 0$ (there is no equivalent of the Higuchi bound in AdS). However, as we have mentioned above, we must consider cutting off the two boundaries at finite u . The spectrum then depends on the boundary conditions. Therefore, without a more complete theory of how these solutions arise and what imposes the boundary conditions, it is premature to draw definite conclusion about the positivity of the spectrum.⁸

The analysis above strongly hints at the fact that no unstable tensor modes are present. However, obtaining the equation for the scalar perturbation is much more complicated: one expects to obtain an equation of a similar form as (4.51), in which the coefficients will involve not only the scale factor but also the scalar-field profile. Nevertheless we may expect (as experience with the flat case suggests) that the behavior of the Schrödinger potential close to the UV boundary and, in the positive curvature case, to the IR endpoint to be similar to the one observed for tensor modes. This in turn would imply that the scalar

⁸One can expect that if the boundary condition is imposed by some physical well-behaved source (e.g. a lower-dimensional brane with positive energy) no instability should arise from the boundary conditions. For example, if we impose vanishing of the wave-function at the endpoint (which is the extension of the normalizability condition in the full range of r , and corresponds to an infinitely massive extended source at the boundary) then positivity of m^2 still holds.

spectrum is positive (with a gapped continuous component). We leave these questions for future investigation.

We conclude this section with a comment about “bouncing” solutions, where the scalar field is non-monotonic. Notice that nothing special happens to the Schrödinger potential (4.51) at points corresponding to bounces: the scale factor there is finite, as well as its first and second derivatives, and no special features arise. One can argue that the same holds for scalar perturbations, even without knowing the precise form of the scalar perturbation equation. This is known to be the case for the flat-slicing geometries, in which indeed the bounce does not destabilize the fluctuations [45]. Adding curvature will most likely only modify the perturbation equation by further mass-like terms, as it is the case for the tensor equation (4.47). This will not dramatically change the fact that the presence of the bounce does not significantly affect stability.

5 Examples of complete RG flows

In this section, we will display solutions corresponding to full RG flows for different choices of potentials. The examples are chosen to illustrate various properties of RG flows of theories on curved manifolds described in the previous sections. The flows will originate from a UV fixed point at a maximum of the potential, which we choose to locate at $\varphi = 0$ for convenience, and will end at an IR point φ_0 , which does not need to be an extremum of the potential. We will distinguish between flows where φ changes monotonically between the UV and IR points and situations where the flow in φ changes direction, an effect referred to as a bounce. In addition, we will describe how the flows are deformed by changes to UV data such as the dimensionless curvature $\mathcal{R}$.

5.1 Generic flows

We will refer to flows as generic if they exhibit the following two properties: they originate at a maximum of the potential and end in the region between this maximum and the nearest minimum. In addition, φ changes monotonically along the flow from UV to IR. We call such solutions ‘generic’ as they arise for generic potentials as long as they possess at least one maximum.

To illustrate generic flows, a potential exhibiting a maximum and at least one minimum will be sufficient.⁹ For this purpose we consider the following quadratic-quartic potential:

$$V(\varphi) = -\frac{d(d-1)}{\ell^2} - \frac{m^2}{2}\varphi^2 + \lambda\varphi^4. \quad (5.1)$$

This potential has one maximum at $\varphi_{\max} = 0$. We will also find it convenient to choose $\lambda = m^2/4$ such that the minima occur at $\varphi_{\min} = \pm 1$. We then proceed to studying RG flows by solving (2.21)–(2.22) numerically for $W(\varphi)$ and $S(\varphi)$. In practice, it is easiest to specify boundary conditions for W and S at or close to the IR end point φ_0 . The relevant

⁹In fact, the existence of a minimum at a finite value of φ is not strictly necessary, but will be useful for illustrating further properties of these flows. Similar conclusions apply also to potentials that extend to infinity in field space, which in the flat case were also discussed in [45].

Figure 11. Solutions for $W(\varphi)$ with $\mathcal{R} \geq 0$ and for the potential (5.1) with $\Delta_- = 1.2$. The five solutions $W_i(\varphi)$ with $i = 1, \dots, 5$ differ in the value of their IR endpoint φ_0 . The critical curve is defined as $B(\varphi) = \sqrt{-3V(\varphi)}$. In the case of zero and positive curvature, the superpotential $W(\varphi)$ cannot enter the area below the critical curve, which is depicted as the shaded region.

boundary conditions for RG flows are described in sections 4.1 and 4.2 for the two cases of a QFT on a sphere or AdS. Given the symmetry of the setup, we restrict our attention to flows that end in the region $\varphi_0 \in [0, 1]$.

QFT on S^d : $\mathcal{R} > 0$. In figure 11 we exhibit solutions for the superpotential $W(\varphi)$ corresponding to generic RG flows for a bulk potential given by (5.1) and for $\mathcal{R} > 0$. To be specific, we have set $\Delta_- = 1.2$, but our observations will hold more generally.

- The main result is that for every value of φ_0 between $\varphi_{\max} = 0$ and $\varphi_{\min} = 1$ there exists a unique solution to the superpotential equations (2.18)–(2.20) corresponding to an RG flow originating from the UV fixed point at $\varphi_{\max} = 0$ and ending at φ_0 . For illustration, in figure 11 we have chosen to display five such solutions labelled by $W_i(\varphi)$ with $i = 1, \dots, 5$, that differ in the value of the IR endpoint φ_0 . However, there is no solution for a flow with $\mathcal{R} > 0$ with an endpoint at $\varphi_0 = \varphi_{\min}$ exactly. In contrast, such a flow arises for $\mathcal{R} = 0$ and we have included the corresponding superpotential W_{flat} in the figure.
- Note that the solutions $W_i(\varphi)$ diverge when approaching their corresponding IR endpoints. The divergence is of the form $\sim (\varphi_0 - \varphi)^{-1/2}$ as expected for a RG flow with $\mathcal{R} > 0$ (see section 4.1). Recall that this divergence does not imply a singularity in the bulk geometry.
- Returning to figure 11, in the vicinity of the UV fixed point the solutions are described by the family of solutions collectively denoted by $W_-(\varphi)$ in section 3. These solutions

Figure 12. QFT on S^d : dimensionless curvature $\mathcal{R}$ and dimensionless vev C as a function of IR endpoint φ_0 for the potential (5.1) with $\Delta_- = 1.2$. Both these quantities diverge as φ_0 approaches the UV fixed point at $\varphi = 0$.

depend on the two continuous parameters $\mathcal{R}$ and C . This is consistent with the existence of a unique solution corresponding to a RG flow for every φ_0 . Picking a solution with the correct IR behavior for a RG flow fixes one combination of the two parameters. The remaining freedom is then equivalent to the choice of IR endpoint φ_0 .

- Given a numerical solution, we can extract the corresponding values of $\mathcal{R}$ and C explicitly by fitting the UV region with the asymptotics (3.2)–(3.4). An interesting observation is that there exists an inverse relation between $\mathcal{R}$ and φ_0 , i.e. flows with endpoints closer to the UV fixed points exhibit larger values of $\mathcal{R}$. Further, $\mathcal{R}$ diverges when φ_0 approaches the UV fixed point. On the other hand, when the IR endpoint approaches $\varphi_{\min}$ the value of $\mathcal{R}$ asymptotes to zero. In fact, for the potential considered here the evolution of $\mathcal{R}$ with φ_0 is monotonic and displayed in figure 12a.
- For completeness, we also exhibit C as a function of φ_0 in figure 12b. We find that C is negative, but its behavior is otherwise similar to that of $\mathcal{R}$. In particular, its absolute value increases with decreasing φ_0 and diverges when $\varphi_0 \rightarrow 0$. However, as $\varphi_0 \rightarrow \varphi_{\min}$ we find numerically that $C \neq 0$.
- Given a solution $W_i(\varphi)$, we can also determine the corresponding scale factor $e^{A(u)}$ and scalar field profile $\varphi(u)$. This introduces the remaining integration constant φ_- into the solution. In figure 13 we plot the scale factors $e^{A(u)}$ and the scalar field profile $\varphi(u)$ corresponding to the solutions $W_1(\varphi)$, $W_2(\varphi)$ and $W_3(\varphi)$ in figure 11. The scale factor $e^{A(u)}$ shrinks to zero at a finite value of u , in agreement with our identification of φ_0 with an IR endpoint. The scalar field arrives at the IR value φ_0 at that point.

The above findings can be summarized as follows. Consider a field theory with fixed source φ_- on a background with fixed curvature R^{uv} , such that $\mathcal{R} = R^{\text{uv}}|\varphi_-|^{-2/\Delta_-}$ is fixed.

Figure 13. (a): scale factor e^A as a function of u for the flows denoted by 1,2,3 in figure 11. For the flat case, the scale factor is depicted as the dark red curve and only shrinks to zero at $u \rightarrow \infty$. For the positive curvature cases, the scale factor e^A shrinks to zero at finite values of u . (b): scalar field $\varphi(u)$ as a function of u . For the flat case, the scalar field arrives at the minimum of the potential at $\varphi = 1$ at the end of the flow. Here it is denoted by the darker red line. For the curved cases, the scalar field does not reach the minimum. The figure shows the scalar field profile for flows denoted by 1,2,3 in figure 11.

The flow will end at a value φ_0 , which is determined by $\mathcal{R}$. The larger the value of $\mathcal{R}$, the closer this end point φ_0 will be to $\varphi_{\max}$. This is in agreement with QFT intuition. The remaining parameter C is then determined by the requirement that the solution is has the correct (regular) behavior at the endpoint.

QFT on AdS_d : $\mathcal{R} < 0$. We repeat the above analysis for the case with $\mathcal{R} < 0$. We will find that many properties observed for positive $\mathcal{R}$ also hold for negative $\mathcal{R}$. In this case however the IR endpoint at φ_0 is replaced by an IR *turning point* for both φ and A (see section 4.2). There are also other important differences which we will highlight. To be specific, we will again work with the potential given by (5.1). In the following, we collect our observations. It will be convenient to distinguish the cases $\Delta_- < 1$ and $\Delta_- > 1$.

- For $\Delta_- < 1$ there exists a unique solution to equations (2.18)–(2.20) corresponding to a *monotonic* RG flow for every value of IR turning point $\varphi_0 \in [0, \varphi_{\min}]$. This is identical to what we have found for flows with $\mathcal{R} > 0$, and for any Δ_- .
- For $\Delta_- > 1$ an RG flow solution exists for every value of IR turning point $\varphi_0 \in [0, \varphi_{\min}]$, but these are not necessarily monotonic. In particular, we find that generic flows can only have $\varphi_0 \in [\varphi_c, \varphi_{\min}]$, where $\varphi_c > \varphi_{\max}$ is determined by the specific choice of potential. On the other hand, flows with $\varphi_0 \in [0, \varphi_c]$ exist, but are non-monotonic: the scalar field $\varphi(u)$ starts from 0 towards *negative* values, before turning around, crossing $\varphi = 0$ again and reaching a value $\varphi_0 \in [0, \varphi_c]$. We will refer to such solutions as *bouncing* flows and they will be discussed in more detail in section 5.3. Note that these bouncing flows are dual to a different class of theories than the monotonic flows, since the UV source must have a different sign in the two cases.

Figure 14. Solutions for $W(\varphi)$ with $\mathcal{R} < 0$ and for the potential (5.1) with $\Delta_- = 1.2$. The three solutions $W_i(\varphi)$ with $i = 1, 2, 3$ have different IR turning points φ_0 . For negative curvature case, the superpotential $W(\varphi)$ can go into the area below the critical curve $B(\varphi) = \sqrt{-3V(\varphi)}$ which is the shaded region.

- In figure 14 we display three solutions for $W(\varphi)$ with $\mathcal{R} < 0$. The results are obtained for the potential (5.1) with $\Delta_- = 1.2$. The three solutions labelled by W_1 , W_2 and W_3 differ in the value of their corresponding IR turning point φ_0 . Note that $W(\varphi)$ goes to zero at φ_0 , with infinite slope as expected from the analytical results in section 4.2. The inverse scale factor $e^{A(u)}$ reaches a minimum at the loci where W vanishes (see figure 15). This is due to the fact that A reaches a minimum at the turning point.
- Returning to figure 14, in the vicinity of the UV fixed point the solutions are again described by the family of solutions collectively denoted by $W_-(\varphi)$ in section 3. We can extract the constants $\mathcal{R}$ and C for generic flows. As in the positive curvature case, we find that the absolute value $|\mathcal{R}|$ increases monotonically with a decrease in the value of the IR (turning) point φ_0 . For $\Delta_- < 1$ we find that the increase in $|\mathcal{R}|$ continues until $\varphi_0 \rightarrow 0$ at which point $|\mathcal{R}|$ diverges. This changes for $\Delta_- > 1$. In this case $|\mathcal{R}|$ already diverges when $\varphi_0 \rightarrow \varphi_c$ for some positive value φ_c . This is shown in figure 16a, where we display $\mathcal{R}$ as a function of φ_0 for the potential (5.1) with $\Delta_- = 1.2$. In this example we find that $\varphi_c \approx 0.49$. The reason for this behavior is that the flow ending at φ_c corresponds to a theory with vanishing source φ_- , which in turn implies that $\mathcal{R} = R^{uv}|\varphi_-|^{-2/\Delta_-}$ diverges. The rose-colored area in figure 16a corresponds to $\varphi < \varphi_c$, which is the region where direct flows cannot arrive. Flows with turning points in this region leave the UV fixed point at $\varphi = 0$ to the left before reversing direction. We will discuss such bouncing flows in more detail in section 5.3.
- For completeness, we display C as a function of φ_0 in figure 16b. The results are again obtained for the potential (5.1) with $\Delta_- = 1.2$. For $\varphi_0 \rightarrow \varphi_{\min}$ the parameter C

Figure 15. (a): $e^{A(u)}$ vs. u and (b) $\varphi(u)$ vs. u corresponding to the flows in figure 14. The coloring is the same as in figure 14. The dark red curve corresponds to the solution for the flat case. The vertical lines represent the reflection points.

Figure 16. QFT on AdS_d : dimensionless curvature $\mathcal{R} = R^{uv}|\varphi_-|^{-2/\Delta_-}$ and dimensionless vev $C = \frac{\Delta_-}{d}\langle\mathcal{O}\rangle|\varphi_-|^{-\Delta_+/\Delta_-}$ vs. φ_0 for the potential (5.1) with $\Delta_- = 1.2$. Flows with turning points in the rose-colored region leave the UV fixed point at $\varphi = 0$ to the left before bouncing and finally ending at positive φ_0 . Flows with turning points in the white region are direct: they leave the UV fixed point at $\varphi = 0$ to the right and do not exhibit a reversal of direction. The flow with turning point φ_c on the border between the bouncing/non-bouncing regime corresponds to a theory with vanishing source φ_- . As a result, both $\mathcal{R}$ and C diverge at this point.

takes some finite value. The value of C then increases monotonically with decreasing φ_0 until C diverges for $\varphi_0 \rightarrow \varphi_c$. This can be understood as the divergence of $C = \frac{\Delta_-}{d}\langle\mathcal{O}\rangle|\varphi_-|^{-\Delta_+/\Delta_-}$ for $\varphi_- \rightarrow 0$. The rose-colored region again corresponds to the area $\varphi < \varphi_c$ where direct flows cannot end.

To summarize, generic flows for $\mathcal{R} < 0$ exhibit many phenomena already observed for $\mathcal{R} > 0$. In particular, the inverse relation between $|\mathcal{R}|$ and φ_0 persists. A new phenomenon is the appearance of bouncing solutions for $\Delta_- > 1$. We will discuss these solutions in detail in the following section.

5.2 The holographic β -function

Having discussed generic flow solutions for $W(\varphi)$ and $S(\varphi)$, let us now turn to the holographic β -function given by

$$\beta(\varphi) \equiv \frac{d\varphi}{dA} = -2(d-1) \frac{S(\varphi)}{W(\varphi)}. \quad (5.2)$$

In figure 17 we plot examples of the holographic β -function for generic flows in the potential (5.1) with $\Delta_- = 1.2$. The dark red curve is $\beta_{\text{flat}}(\varphi) = -2(d-1)W'(\varphi)/W(\varphi)$ for a flat flow. The remaining curves $\beta_i(\varphi)$ with $i = 1, 2, 3$ correspond to RG flow solutions with $\mathcal{R} = 0.09, 1.29, 7.14$ respectively. Note that the β -functions show the same asymptotic behavior in the UV, i.e.

$$\beta(\varphi) = -\Delta_- \varphi + \dots + \mathcal{O}\left(\mathcal{R} |\varphi|^{1+\frac{2}{\Delta_-}}\right), \quad (5.3)$$

where we have indicated the first non-trivial curvature correction at small curvature. Moving away from the UV fixed point at $\varphi = 0$, the β -functions begin to change due to the curvature. The departure from β_{flat} is the more pronounced, the higher the value of $\mathcal{R}$. Most notably, depending on the value of $\mathcal{R}$, the various β -functions vanish at different values of φ_0 . Using results from sections 3.2 and 4.1, we find that the β -functions exhibit the following asymptotic form near their respective IR end-points:

$$\mathcal{R} = 0 : \quad \beta_{\text{flat}} = \Delta_-^{\min}(\varphi_{\min} - \varphi) + \mathcal{O}((\varphi_{\min} - \varphi)^2), \quad (5.4)$$

$$\mathcal{R} > 0 : \quad \beta_i = -2(\varphi_{0,i} - \varphi) + \mathcal{O}((\varphi_{0,i} - \varphi)^{3/2}), \quad \text{for } i = 1, 2, 3, \quad (5.5)$$

where $\varphi_{0,i}$ are the respective end-points of the flows. The difference in IR behavior between flat flows and flows with $\mathcal{R} > 0$ can be understood as follows. For flat flows, IR fixed-points are determined by the potential, as they correspond to minima of V . Correspondingly, the vanishing of β_{flat} at the fixed point is set by $\Delta_-^{\min}$, which depends on the potential at the minimum. In contrast, for flows with $\mathcal{R} > 0$, it is the existence of non-zero curvature which cuts off the flows in the IR, rather than a property of the potential. As a result, the behavior of β_i in the IR does not depend on the behavior of the potential at the IR end-point. In fact, the behavior $\beta_i = -2(\varphi_{0,i} - \varphi)$ for $\mathcal{R} > 0$ is universal regardless of V and $\mathcal{R}$. The effect of non-zero curvature $\mathcal{R}$ is to fix the value of the IR end point $\varphi_{0,i}$.

Last, we comment on flows with $\mathcal{R} < 0$. The behavior of $\beta(\varphi)$ in the UV is again given by (5.3). However, at the turning point φ_0 the β -function does not vanish, but remains finite. Using our expressions from section 4.2 one finds

$$\beta(\varphi_0) = d(d-1) \frac{|V'(\varphi_0)|}{V(\varphi_0)}. \quad (5.6)$$

This indicates that the turning points φ_0 in the negative curvature case are not fixed points of an RG flow.

5.3 Bouncing solutions

Bouncing solutions are made possible by the fact that at a generic point $W(\varphi)$ and $S(\varphi)$ are typically multivalued. We have already observed in the previous section that bounces are generic for negative curvature. Here we present a few more examples.

Figure 17. Solutions for $\beta(\varphi) = -2(d-1)\frac{S(\varphi)}{W(\varphi)}$ with $\mathcal{R} \geq 0$ and for the potential (5.1) with $\Delta_- = 1.2$. The five solutions $\beta_i(\varphi)$ with $i = 1, \dots, 5$ differ in the value of their IR endpoint φ_0 .

QFT on S^d : $\mathcal{R} > 0$. We now consider the potential:

$$V = -\frac{\Delta_+(d-\Delta_+)}{2\ell^2}\varphi^2 + \frac{d(d-1)}{\ell^2}\left(\frac{\Gamma}{2}\varphi^2 - \cosh(\Gamma\varphi)\right) - \frac{|c|}{\ell^2}\left(e^{-\frac{(\varphi-\varphi_*)^2}{2\sigma}} + e^{-\frac{(\varphi+\varphi_*)^2}{2\sigma}} + e^{-\frac{\varphi_*^2}{2\sigma}}\left(\frac{1}{\sigma} - \frac{\varphi_*^2}{\sigma^2}\right)\varphi^2 - 2e^{-\frac{\varphi_*^2}{2\sigma}}\right). \quad (5.7)$$

which depends on the parameters Δ_+ , Γ , φ_* and c . The potential is an inverted parabola superimposed on an inverted cosh. In addition, there are two inverted Gaussians with peaks at $\pm\varphi_*$ and whose height and width can be adjusted using c and σ . The potential thus has a maximum at $\varphi = 0$. In addition, we choose the parameters Γ , c and σ such that there are also two minima close to $\pm\varphi_*$. The parameter Δ_+ is consistent with the definition of $\Delta_{\pm}$ in (3.5), i.e. it is given by (3.5) with $m^2 = V''|_{\varphi=0}$. A plot of this potential is shown in figure 18 for parameters $\Delta_+ = 2.91$, $\sigma = 0.01$, $c = 2000$, $\Gamma = 2/\sqrt{3}$, $\varphi_* = 4$. This potential is constructed with the intention to give rise to bouncing solutions. The two Gaussians introduce two steep features in the potential, which increase the likelihood of bounces.

Figure 19 shows two bouncing solutions $W_1(\varphi)$ and $W_2(\varphi)$ with $\mathcal{R} > 0$. For comparison, we also plot W_{flat} , which is the bouncing solution in the same potential for $\mathcal{R} = 0$. The flows in figure 19 originate from a UV fixed point at a maximum of V at $\varphi_{\text{max}} = 0$, departing towards negative values of φ . The flows then exhibit a bounce, reverse direction and eventually flow to the regime with $\varphi > 0$. Depending on the precise form of the potential for $\varphi > 0$, the flows may bounce again (even repeatedly), diverge, or approach an IR end point.

For $\mathcal{R} = 0$ the bounce locus is the critical curve, i.e. $W_{\text{flat}}(\varphi_B) = B(\varphi_B)$. In addition $W'_{\text{flat}} \sim \pm(\varphi - \varphi_B)^{1/2}$ in the vicinity of a bounce. As shown in [45], these two properties are shared by all bouncing solutions for $\mathcal{R} = 0$.

Figure 18. Plot of the potential (5.7) with parameters $\Delta_+ = 2.91$, $\sigma = 0.01$, $c = 2000$, $\Gamma = 2/\sqrt{3}$, $\varphi_* = 4$.

Figure 19. Solutions for $W(\varphi)$ for flows exhibiting a single bounce for the potential (5.7) with $\Delta_+ = 2.91$, $\sigma = 0.01$, $c = 2000$, $\Gamma = 2/\sqrt{3}$, $\varphi_* = 4$. $W_{1,2}$ are solutions with $\mathcal{R}_2 > \mathcal{R}_1 > 0$, while W_{flat} is the solution in the same potential with $\mathcal{R} = 0$. The shaded area is the region below the critical curve $B(\varphi) = \sqrt{-3V(\varphi)}$.

For $\mathcal{R} > 0$ we can make the following observations:

- For $\mathcal{R} > 0$ the bounce locus does not lie on the critical curve (2.24). Instead we find that $W(\varphi_B) > B(\varphi_B)$ for $\mathcal{R} > 0$ (see figure 19). Another interesting observation is that $W' \sim \pm(\varphi - \varphi_B)^{-1/2}$ in the vicinity of a bounce for $\mathcal{R} \neq 0$. Thus, while W' changes sign at the bounce, it also diverges at this point.
- We can also determine the values $\mathcal{R}_1$ and $\mathcal{R}_2$ corresponding to the two flows W_1 and W_2 . Here we have $\mathcal{R}_2 > \mathcal{R}_1$. Studying further bouncing flows with $\mathcal{R} \neq 0$ the following behavior emerges: the larger the value of $|\mathcal{R}|$, the closer bounces occur to the UV fixed point.
- For completeness, we also plot the functions S and the inverse scale factor e^{-2A} corresponding to the two solutions W_1 and W_2 . This is shown in figures 20a and 20b respectively. Note that e^{-2A} takes finite values throughout, including at the bouncing

Figure 20. Plots of $S_{1,2}(\varphi)$ and $e^{-2A_{1,2}(\varphi)}$ corresponding to the two solutions $W_{1,2}(\varphi)$ in figure 19. The solutions are obtained for the potential (5.7) with $\Delta_+ = 2.91$, $\sigma = 0.01$, $c = 2000$, $\Gamma = 2/\sqrt{3}$, $\varphi_* = 4$.

locus φ_B . The bulk geometry is thus perfectly regular along the flow including the bounces.

We can summarize our findings as follows. Consider a field theory with fixed UV source φ_- and defined on a manifold with UV curvature R^{uv} . One observation of this section is that for a suitable potential V , solutions corresponding to RG flows with bounces exist. Decreasing R^{uv} (and thus $\mathcal{R} = R^{\text{uv}}|\varphi_-|^{-2/\Delta_-}$), the bounce moves further away from the UV fixed point and approaches the critical curve $B(\varphi)$. For $R^{\text{uv}} \rightarrow 0$ the bounce eventually reaches the critical curve. On the contrary, increasing R^{uv} moves the bounce towards the UV fixed point.

QFT on AdS_d : $\mathcal{R} < 0$. Bouncing solutions also exist for $\mathcal{R} < 0$, i.e. for field theories defined on negatively curved manifolds. Our observations from above also hold here: bounce loci φ_B do not occur on the critical curve, but now we have $W(\varphi_B) < B(\varphi_B)$ for $\mathcal{R} < 0$ (see figure 21). Similarly, increasing $|\mathcal{R}|$ moves φ_B towards the UV fixed point. For example, for the two flows W_1 and W_2 in figure 21 we have $\mathcal{R}_2 > \mathcal{R}_1$.

An important difference to the case $\mathcal{R} > 0$ is that for $\mathcal{R} < 0$ bouncing solutions are generic. Let us explain what we mean by this. For $\mathcal{R} > 0$ bouncing solutions may exist in suitable potentials, but given a generic potential, one does not expect the existence of bouncing solutions. This is to be contrasted with the situation for $\mathcal{R} < 0$. There any potential with at least one maximum with $\Delta_- > 1$ will allow for bouncing solutions.

This can be understood analytically from the results of appendix G. There we study solutions for flows in purely quadratic potentials, where we restricted our focus to flows that end or turn close to the UV fixed point where they originated from. The main observation of relevance here is as follows: we find that for $\mathcal{R} < 0$ such flows will always bounce if $\Delta_- > 1$. Now note that sufficiently close to a maximum the potential can always be approximated by a quadratic function, and the results from appendix G will hold. Thus we expect that for $\Delta_- > 1$ flows with turning points sufficiently close to this maximum will necessarily bounce. This is indeed what we observe numerically.

Thus, for $\mathcal{R} < 0$ bounces can occur in very simple potentials, like the quadratic-quartic potential in (5.1) and which we will discuss in the following. However, now we will consider

Figure 21. Plot of solutions for $W(\varphi)$ for flows exhibiting a single bounce for the potential (5.7) with $\Delta_+ = 2.91$, $\sigma = 0.01$, $c = 2000$, $\Gamma = 2/\sqrt{3}$, $\varphi_* = 4$. $W_{1,2}$ are solutions with $\mathcal{R} < 0$, and $|\mathcal{R}_2| > |\mathcal{R}_1|$.

this potential for the range $\varphi \in [-\varphi_{\min}, \varphi_{\min}] = [-1, 1]$. As we have seen in section 5.1, for negative curvature and $\Delta_- > 1$ there exists a critical value φ_c for the IR turning point below which the solution goes through a bounce before reaching the turning point. The critical value φ_c depends on the details of the potential: for example, a numerical analysis shows that, for $\Delta_- = 1.2$, $\varphi_c = 0.49$, while for $\Delta_- = 1.4$ $\varphi_c = 0.63$.

We will now examine how the quantities $\mathcal{R}$ and C vary as the endpoint of the flow φ_0 is varied from $\varphi_{\max} = 0$ to $\varphi_{\min} = 1$. For $\Delta_- = 1.2$ this has already been discussed for the non-bouncing flows with $\varphi_0 \gtrsim 0.49$. We will now complement these plots with the results for bouncing flows, i.e. for $|\varphi_0| \lesssim 0.49$. This is shown in figure 16.

Interestingly, in addition to the usual divergence in $\mathcal{R}$ and C for $\varphi_0 \rightarrow \varphi_{\max}$ we find that both $\mathcal{R}$ and C also diverge when $\varphi_0 \rightarrow \varphi_c$, i.e. when φ_0 approaches the boundary between bouncing and non-bouncing flows. This can be understood as follows. Non-bouncing flows with turning points in $\varphi_0 > \varphi_c$ correspond to flows that leave the UV fixed point to the right, and as a consequence have $\varphi_- > 0$. Bouncing flows with $\varphi_0 < \varphi_c$ leave the UV fixed point to the left before turning around, and thus have $\varphi_- < 0$. Thus, when φ_0 passes through φ_c the source has to change sign. Note that in terms of the physical UV curvature R^{uv} and the vev $\langle \mathcal{O} \rangle$ the two quantities $\mathcal{R}$ and C can be written as:

$$\mathcal{R} = \frac{R^{\text{uv}}}{|\varphi_-|^{2/\Delta_-}}, \quad C = \frac{\Delta_-}{d} \frac{\langle \mathcal{O} \rangle}{|\varphi_-|^{\Delta_+/\Delta_-}}, \quad (5.8)$$

The divergence for $\varphi_0 \rightarrow \varphi_c$ can thus be interpreted as follows. To change sign the source has to pass through zero. As long as R^{uv} and $\langle \mathcal{O} \rangle$ remain finite, letting $\varphi_- \rightarrow 0$ makes both $\mathcal{R}$ and C diverge.

Figure 22. QFT on AdS_d : $C/\mathcal{R}^{\Delta_+/2}$ vs. φ_0 for the potential (5.1) with $\Delta_- = 1.2$. Flows with turning points in the rose-colored region exhibit a bounce.

This interpretation can be checked by plotting the quantity

$$\chi \equiv \frac{\langle \mathcal{O} \rangle}{(R^{\text{uv}})^{\Delta_+/2}} = \frac{d}{\Delta_-} \frac{C}{\mathcal{R}^{\Delta_+/2}}, \quad (5.9)$$

which is manifestly independent of φ_- and should thus be finite for $\varphi_0 \rightarrow \varphi_c$. This is indeed what we observe. In figure 22 we plot $\frac{\Delta_- \chi}{d}$ vs. φ_0 for the potential (5.1) with $\Delta_- = 1.2$. We find that χ is finite and continuous when φ_0 passes through φ_c .

6 A holographic quantum phase transition driven by curvature

In the framework of holography there may be multiple RG flows that originate from a single UV fixed point but flow to different IR end points. From the field theory point of view this corresponds to a theory that exhibits multiple vacua, with a one-to-one correspondence between the number of flows and the number of vacua.

This behavior was recently described in [45] for holographic RG flows of field theories on flat manifolds. There it was observed that in addition to a flow from a maximum to the nearest minimum, there may also exist flows that *skip* the nearest minimum to end at the next minimum. We find that this situation persists if we add non-zero curvature: at a fixed curvature and UV source there may be multiple RG flow emanating from the same UV fixed point, but terminating at different IR end points. The different flows are distinguished by the subleading term which determines the vev of the dual operator.

The various flows correspond to different saddle points of the action and are hence in one-to-one correspondence with vacua of the field theory. In the case of multiple flows the question then arises which of these ground states is the true vacuum. In the dual gravitational picture this is equivalent to the questions which saddle point is dominant in the gravitational path integral. This can be answered by comparing the free energies associated with the various vacua and identifying the ground state with the lowest free energy as the true vacuum.

If we vary the value of R^{uv} the RG flows deform and the vevs and free energies of the various vacua change. Interestingly, under a variation of R^{uv} the identification of the true vacuum may change, i.e. the system may exhibit a phase transition. This corresponds to

a *quantum* phase transition as our system is at zero temperature. The control parameter in this case is the curvature R^{uv} .

In the following, we review the calculation of the free energy for a given RG flow. We then turn to a specific example based on a suitably chosen potential exhibiting multiple RG flows originating from the same UV fixed point. For this example we study the evolution of the free energies for the various flows under a change of curvature. Ultimately, we will observe a change of sign of the free energy difference between the different saddle points, indicating the existence of a phase transition driven by curvature.

6.1 The free energy

The free energy is given by the Euclidean on-shell action on the gravitational side. This can be determined by starting from the action (2.1) and evaluating it on a given solution. One arrives at the expression:¹⁰

$$S_{\text{on-shell}} = M_P^{d-1} \int d^d x \sqrt{-\zeta} \left(\left[e^{dA} W \right]_{\text{UV}} + \frac{2R^{\text{uv}}}{d} \int_{\text{UV}}^{\text{IR}} du e^{(d-2)A(u)} \right), \quad (6.1)$$

where the subscript UV denotes a quantity to be evaluated at the UV boundary. We also reinstated the Planck mass M_P . Whenever we write $S_{\text{on-shell}}$ this corresponds to the on-shell action for a theory on a manifold with Lorentzian signature. We can obtain the corresponding Euclidean action by simply multiplying with (-1) .

In the following, let us briefly sketch how (6.1) can be derived. The starting point is the action (2.1). First, rewrite the bulk curvature scalar $R^{(g)}$ in terms of $\dot{\varphi}^2$ and the potential V as shown in appendix A. In the next step insert for the potential V using the equation of motion (2.7). An important contribution comes from the Gibbons-Hawking-York boundary term S_{GHY} which we have left implicit so far. This is given by

$$S_{\text{GHY}} = 2M_P^{d-1} \left[\int d^d x \sqrt{-\gamma} K \right]_{\text{UV}} = -2dM_P^{d-1} \int d^d x \sqrt{-\zeta} \left[e^{dA} \dot{A} \right]_{\text{UV}}, \quad (6.2)$$

where $K = -d\dot{A}_{\text{UV}}$ is the extrinsic curvature of the boundary. Here $\gamma_{\mu\nu}$ is the induced metric defined in (2.4). Putting everything together we arrive at (6.1). The second term in (6.1) is only present for field theories on curved manifolds, i.e. when $R^{\text{uv}} \neq 0$. For field theories on flat space-time only the first term in (6.1) contributes.

The on-shell action is divergent and the divergences are originating from the UV boundary at $u \rightarrow 0$, which we choose to coincide with $\varphi = 0$. To regulate we define a UV cut-off as $\varphi = \varphi_{\text{UV}}$. The divergences can then be removed by the addition of suitable counter-terms. The number of required counter-terms depends on the number of space-time dimensions. For example for $d = 4$ the required counter-terms are [35]:

$$S_{\text{ct}}^{(0)} = -M_P^{d-1} \int_{\text{UV}} d^d x \sqrt{-\gamma} W_{0,\text{ct}}(\varphi), \quad (6.3)$$

$$S_{\text{ct}}^{(1)} = M_P^{d-1} \int_{\text{UV}} d^d x \sqrt{-\gamma} R^{(\gamma)} U_{\text{ct}}(\varphi), \quad (6.4)$$

¹⁰As explained in section 2.1 we choose to set $R^{(\zeta)} = R^{\text{uv}}$. This can be done by a suitable choice for the integration constant entering $A(u)$.

Figure 23. Plot of the degree-12 potential which allows skipping solutions. This potential has several extrema which are denoted as UV_1 , UV_2 , IR_1 and IR_2 .

$$S_{ct}^{(2)} = -M_P^{d-1} \int_{UV} d^d x \sqrt{-\gamma} (R^{(\gamma)})^2 \frac{\ell^3}{48\Delta_-} \log(\varphi_{UV}) . \quad (6.5)$$

$S_{ct}^{(n)}$ is the counter-term required at the order $(R^{(\gamma)})^n$. Also note that the counter-terms are manifestly covariant. The functions $W_{0,ct}(\varphi)$ and $U_{ct}(\varphi)$ satisfy the following equations [35, 44]:

$$\frac{d}{4(d-1)} W_{0,ct}^2(\varphi) - \frac{1}{2} (W'_{0,ct}(\varphi))^2 = -V, \quad (6.6)$$

$$W'_{0,ct}(\varphi) U'_{ct}(\varphi) - \frac{d-2}{2(d-1)} W_{0,ct}(\varphi) U_{ct}(\varphi) = 1. \quad (6.7)$$

Near $\varphi = 0$, we can find these functions as a series in φ , [65–67]:

$$W_{0,ct}(\varphi) \simeq \frac{2(d-1)}{\ell} + \frac{\Delta_-}{2\ell} \varphi^2 + \frac{C_{ct}}{\ell} |\varphi|^{d/\Delta_-} + \dots, \quad (6.8)$$

$$U_{ct}(\varphi) \simeq \frac{\ell}{2-d} + \frac{d\ell^3}{2-d} B_{ct} |\varphi_-|^{2/\Delta_-} |\varphi|^{(d-2)/\Delta_-} + \dots. \quad (6.9)$$

Here C_{ct} and B_{ct} are the integration constants for the counter-terms. A particular choice for C_{ct} and B_{ct} corresponds to a renormalization scheme. Here and in the following we will choose $C_{ct} = B_{ct} = 0$ which we will refer to as “minimal subtraction”.

We can now write down an expression for the free energy F . It is given by the renormalized Euclidean on-shell action S_E^{ren} , i.e. the Euclidean on-shell action including the counter-terms. In case we are working in a space-time with a Lorentzian signature, a Wick rotation is needed to pass to Euclidean signature. In this circumstances the on-shell action picks up a minus sign. We thus have

$$F = S_E^{\text{ren}}, \quad \text{with} \quad S_E^{\text{ren}} = -S_{\text{on-shell}}^{\text{ren}}. \quad (6.10)$$

Figure 24. The plot represents the superpotentials corresponding to the three regular holographic RG flow solutions arising from the potential in figure 23. Two of them are standard interpolating from UV_1 to IR_1 and from UV_2 to IR_1 . The third one skips IR_1 and ends up at IR_2 . The shaded region is the forbidden region below the critical curve $B(\varphi)$. The arrows represent the direction of the flow from the UV to the IR.

6.2 Skipping flows and a quantum phase transition driven by curvature

To illustrate the concept of skipping flows, we set $d = 4$ and consider a potential which is a polynomial of degree 12, whose explicit form is not essential and can be found in [45]. A plot of (part of) this potential is shown in figure 23. It has several extrema, denoted by UV_1 , UV_2 , IR_1 and IR_2 . In this section, we will determine the possible RG flows as we increase the dimensionless UV curvature parameter, $\mathcal{R}$.

Flat case: $\mathcal{R} = 0$. We first consider the RG flow solutions for the QFT in flat space-time. The corresponding solutions for $W(\varphi)$ are shown in figure 24. In this case, there are three RG flows in total, two originating from UV_1 and one from UV_2 . We will focus on the flows emanating from UV_1 . Among the two RG flows emanating from the UV_1 , the one denoted by $W_{11}(\varphi)$ interpolates between UV_1 and the nearest minimum to the right IR_1 . The second flow denoted $W_{12}(\varphi)$ skips the fixed point IR_1 and ends at IR_2 .

To decide which solution is the dominant saddle point in the path integral, we need to compare their corresponding free energies at fixed boundary condition φ_- . As described in [45] the flow with the higher value of vev parameter C is the dominant saddle point: this is the skipping flow, as the solution with larger C has a larger $W(\varphi)$ in the UV. Consequently, the skipping flow represents the true ground-state of the dual QFT in flat space-time.

Figure 25. The solid lines represent the superpotential $W(\varphi)$ corresponding to the three different solutions starting from UV_1 which exist at small positive curvature. Two of them (red and green curves) are skipping flows and the third one (orange curve) is non-skipping. For comparison, we also show the flat RG flows (dashed curves)

Although our focus will be on flows starting at UV_1 , for completeness we briefly discuss the flow from UV_2 , also depicted in figure 24. There is a single flow which starts from UV_2 , denoted as $W_{21}(\varphi)$, and it interpolates between the UV_2 and IR_1 . There is no solution starting from UV_2 and ending at IR_2 as there can be at most one flow that can end at a given IR point from a given direction. This is because regular solutions in the IR do not admit small deformations [45]. Here the fixed point IR_2 is already “taken” by the skipping flow $W_{12}(\varphi)$.

We now turn on positive curvature on the d -dimensional slice and see how this affects the various RG flows solutions presented above.

Finite curvature $\mathcal{R} > 0$; flows from UV_1 . Turning on a small positive curvature, there are now three flows (instead of the two at $\mathcal{R} = 0$) emanating from UV_1 and shown in the figure 25. Two of them are skipping the nearest possible IR region (between $\varphi = 0$ and IR_1) and end up near IR_2 . These two are denoted as $W_{s,1}(\varphi)$ and $W_{s,2}(\varphi)$ and represented as red and green curves respectively in the figure 25. The third flow is the non-skipping flow and denoted as $W_{ns}(\varphi)$. It starts from UV_1 and ends up in the region between $\varphi = 0$ and IR_1 . The two flows $W_{ns}(\varphi)$ and $W_{s,1}(\varphi)$ follow the general pattern seen in section 6. They are deformations of the corresponding flat flows (represented as the dotted black and red curves in figure 25) and end slightly before the respective IR fixed points. The flow $W_{s,2}(\varphi)$ on the other hand is a new branch which only exists for non-zero curvature.

As $\mathcal{R}$ is increased, the flow are deformed as shown in figure 26. The IR end points of the skipping flows $W_{s,2}(\varphi)$ and $W_{s,1}(\varphi)$, move towards each other, up to a critical value of $\mathcal{R}$ where the two solutions merge. Above this value, the skipping solution disappears and

Figure 26. As the curvature is increased, IR end points of the skipping flows move toward each other. The non-skipping IR end point moves closer to UV_1 .

only the non-skipping flow is left. The IR end point of the non-skipping flow $W_{ns}(\varphi)$ moves to the left from the IR_1 and it approaches the UV fixed point UV_1 as $\mathcal{R}$ is increased, as expected in the generic case.

Finite curvature: $\mathcal{R} > 0$; flows from UV_2 . For completeness, we will also describe the flows starting from UV_2 as we increase the curvature. In the flat case, there was only one such flow, with a negative source (see $W_{21}(\varphi)$ in figure 24). If the curvature is slightly increased, this flow splits into two. One starts from UV_2 and ends in the region between IR_1 and UV_2 . This is shown as the red curve $W_{2nb,n}(\varphi)$ in figure 27. The other flow, represented as the red curve $W_{2b,n}(\varphi)$ in figure 28, starts from UV_2 , *bounces* in the region between IR_1 and UV_2 and ends in the region $\varphi > UV_2$. These two flows, namely the non-bouncing flow and the bouncing flow with negative source, exist for all values of curvature.

Beyond a certain value of the curvature, two new solutions appear that emanate from UV_2 in addition to the bouncing and the non-bouncing flows. These have a positive source and have no $\mathcal{R} = 0$ counterpart. They end in region $\varphi > UV_2$. They are shown as the red curves $W_{2nb,p}(\varphi)$ in figure 29.

Summary and observations. The full space of solutions is represented in figure 30, which shows the dimensionless UV curvature parameter $\mathcal{R}$ as a function of the IR end point φ_0 of the flow. Non-skipping flows start from UV_1 and end in the leftmost blue region. This type of solution exists for all values of curvature. On the other hand, skipping flows from UV_1 end up in the rightmost purple region, and they come in two branches (i.e. there are two different endpoints in the right purple region for a given value

Figure 27. Superpotential $W_{2nb,n}(\varphi)$ for a non-bouncing flows starting from UV_2 . In this case the source is negative. This type of flow comes from a curvature deformation of the flat case flow W_{21} and exists for all values of the curvature.

Figure 28. Superpotential $W_{2b,n}(\varphi)$ for a bouncing flows starting from UV_2 . For bouncing flows, the source is negative. This type of flow also comes from a curvature deformation of the flat case flow W_{21} (dotted line) and exist for all values of the curvature.

of $\mathcal{R}$). These skipping flows exist up to a maximum value of curvature, where the two branches merge.

We also display the results for flows starting from UV_2 in figure 30. Flows with a negative source (i.e. going towards the left of UV_2) either end in the light brown region, or bounce and then end in the red region. Both these bouncing and non-bouncing solutions

Figure 29. Two non-bouncing flows $W_{2nb,p}(\varphi)$ from UV_2 with identical value of $\mathcal{R}$. Interestingly, flows of this type only exist for $\mathcal{R} > \mathcal{R}_{\min}$, i.e. when the dimensionless curvature is larger than some minimal value. As a result, such flows do not come from a curvature deformation of a $\mathcal{R} = 0$ flow.

Figure 30. Dimensionless curvature $\mathcal{R}$ vs. flow end point φ_0 . The leftmost blue region is the IR region where non-skipping flows from UV_1 can end. The rightmost purple area is the IR region for skipping flows from UV_1 . The light brown region is the IR region for flows starting from UV_2 with a negative source and no bounces along the flow. The green area is the IR region for flows starting from UV_2 with a positive source. The red region is the IR region for flows from UV_2 with a negative source but which exhibit a bounce. The blue line is the critical curve $B(\varphi) = \sqrt{-3V(\varphi)}$.

originate as deformations of the “flat” flow $W_{21}(\varphi)$ (see figures 27 and 28), and exist for any positive value of $\mathcal{R}$.

A particularly interesting set of flows from UV_2 are those with end points in the green region in figure 30. Note that all flows in this region have $\mathcal{R} \geq \mathcal{R}_{\min}$ for some minimal value of the dimensionless curvature. This implies that these flows do not arise from a continuous deformation of a flat flow. In fact, there is no flat flow from UV_2 with positive source. This is explained in [45] as arising from the fact that there is already a regular flow terminating at IR_2 from the left, and therefore no other regular flows can also arrive from the same side. The solutions with end points in the green region in figure 30 thus correspond to new vacua which do not exist for $\mathcal{R} = 0$, but appear when $\mathcal{R}$ becomes large enough.

The fact that there is a regular flow starting at UV_2 for negative coupling (towards the left of UV_2) but not positive coupling (towards the right of UV_2) has analogues in QFT. Both for YM and $\lambda\phi^4$ theory the coupling must have a fixed sign. The theories do not exist for negative g_{YM}^2 or negative λ . Something similar also happens here. The unusual occurrence however is that the theory with the ‘wrong’ sign exists for sufficiently large space-time curvature. In a sense positive curvature cures the sickness of the flat theory. It would be interesting to understand this phenomenon better and find examples of a similar behavior in QFT.

Notice that there are several values of the endpoint where the curvature parameter diverges. Two of them correspond to the UV fixed points UV_1 and UV_2 , in line with the generic behavior found in section 5. The third one (namely the point φ_* in figure 30 separating the green and red regions) is more interesting. Across this point the source changes sign. For the endpoint exactly at φ_* the solution must have zero source, implying that for finite R^{uv} , the quantity $\mathcal{R}$ diverges (recall the definition in equation (3.16)) as we approach φ_* . This fact has a remarkable consequence: it implies that, above some positive threshold value for the curvature, there exist RG flows with no source (i.e. of the W_+ branch, the ones driven by a vev) starting at UV_2 and ending at φ_* . Compare this to the flat case, where regular vev flows are highly non-generic, and require a fine-tuned potential.

Note that there seems to exist an endpoint for flows with $\mathcal{R} = 0$ that does not coincide with an extremum of the potential. This is the point $\varphi_!$ in figure 30 separating the purple and red regions. However, one finds that this endpoint cannot be reached by a flow from UV_1 or UV_2 . To illustrate this, let us consider what happens for flows ending in both the red and purple regions of figure 30 when we let $\varphi_0 \rightarrow \varphi_!$. Flows ending in the red region of figure 30 leave UV_2 to the left before bouncing and reversing direction (see figure 28). Taking $\varphi_0 \rightarrow \varphi_!$ the bounce point occurs closer and closer to IR_1 , coinciding with IR_1 for $\varphi_0 = \varphi_!$. Once this happens the two branches of the previously bouncing solution become two independent RG flows. For one, there is the flow starting from UV_2 and ending at IR_1 (denoted by W_{21} in figure 31). This is a zero-curvature flow from extremum to extremum. The 2nd branch of the bouncing flow now becomes a flow starting at the minimum associated with IR_1 and ending at $\varphi_!$. This is shown as W_{3+} in figure 31. The minimum thus plays the role of a UV fixed point, which we label UV_3 in figure 31. This corresponds to a W_+ solution in the language of section 3 with fixed value $\mathcal{R} =$

Figure 31. RG flows with IR endpoint $\varphi_0 \rightarrow \varphi_!$. When the endpoint φ_0 approaches $\varphi_!$ flows from both UV_1 and UV_2 pass by closely to IR_1 , passing through IR_1 exactly for $\varphi_0 = \varphi_!$. This is shown by the purple and red curves. Beyond IR_1 both these solutions coincide, which is denoted by the colored dashed curve. These have the following interpretation. The flows from UV_1 and UV_2 should not be continued beyond IR_1 , which becomes the IR endpoint for the zero curvature flows W_{11} and W_{21} . The remaining branch (the colored dashed curve) is now an independent flow denoted by W_{3+} . This is a flow from a UV fixed point at a minimum of the potential (denoted by UV_3 above) to $\varphi_!$ and corresponds to a W_+ solution in the notation of section 3 with fixed value $\mathcal{R} = R^{uv}|\varphi_+|^{-2/\Delta_+} \neq 0$. While flows from UV_1 and UV_2 can end arbitrarily close to $\varphi_!$, the endpoint $\varphi_0 = \varphi_!$ cannot be reached from UV_1 or UV_2 .

$R^{uv}|\varphi_+|^{-2/\Delta_+} \neq 0$. We can make an analogous observation for flows ending in the purple region, which originate from UV_1 and skip IR_1 . Taking $\varphi_0 \rightarrow \varphi_!$ the flows miss IR_1 with ever decreasing distance, and pass through IR_1 for $\varphi_0 = \varphi_!$. Again, in this case we should not continue the flow beyond IR_1 , which becomes the endpoint. This corresponds to the solution W_{11} in figure 31.

A quantum phase transition. We have observed that there are more than one flows with the same dimensionless UV curvature parameter, $\mathcal{R}$, and now we will focus on flows which start from UV_1 and have the same source. These correspond to three branches of solutions contained in the blue and purple regions in figure 30. Beyond a certain value of the curvature, only the non-skipping solution in the blue region exists, so this solution it is necessarily the ground state. On the other hand, as we mentioned at the beginning of this section, at zero curvature one of the *skipping* solutions is the true ground state. This suggests that varying $\mathcal{R}$ we should encounter a phase transition.

A numerical analysis shows that this expectation is correct, and the phase transition is first order. To compare the free energies, we evaluate expression (6.1) numerically,

Figure 32. Free energy difference between the skipping and the non-skipping solution. For a fixed curvature, there are two skipping solutions for which the superpotential $W(\varphi)$ was denoted as $W_{s,1}(\varphi)$ and $W_{s,2}(\varphi)$ (they were denoted as red and green curves respectively in figures 25 and 26). In this figure, the red curve corresponds to the on-shell action difference between the $W_{s,1}(\varphi)$ solution and the non-skipping solution. The green curve corresponds to the on-shell action difference between the $W_{s,2}(\varphi)$ solution and the non-skipping solution $W_{ns}(\varphi)$.

supplemented by the appropriate counter-terms (6.3)–(6.5), for each of the solutions.¹¹ The numerical evaluation of the free energy difference between the two skipping solutions and the non-skipping solution is shown in figure 32, which clearly shows that we are in presence of a first order quantum phase transition driven by space-time curvature.

At zero curvature, one of the skipping solutions dominates, and the second one becomes degenerate with the non-skipping solution. As the curvature is increased, the difference between the free energies decreases. At a certain value of the dimensionless curvature, $\mathcal{R} = \mathcal{R}_c$, the free energy difference changes sign. (In our example we find $\mathcal{R}_c \approx 80$.) Now, the non-skipping solution is the dominant saddle point in the path integral. Finally, the two skipping solutions merge at a maximal value $\mathcal{R}$ and disappear for larger values of the curvature.

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A Curvature Invariants

In this appendix we record expressions for various curvature invariants. In particular, for the metric (2.2), we will compute R , $R_{AB}R^{AB}$ and $R_{ABCD}R^{ABCD}$ and we will express them in terms of the functions W , S and T defined in (2.15)–(2.17).

¹¹In this case, there is no analytic result for all values of curvature because of the bulk integral in expression (6.1) which is non-trivial to evaluate.

Ricci scalar. The expression for the Ricci scalar is found as,

$$R = - \left[2d\ddot{A}(u) + d(d+1)\dot{A}^2(u) \right] + e^{-2A(u)} R^{(\zeta)} \quad (\text{A.1})$$

which in terms of $W(\varphi)$, $S(\varphi)$ and $T(\varphi)$ can be written as,

$$R = \frac{d}{d-1} W' S - \frac{d(d+1)}{4(d-1)^2} W^2 + T = \frac{S^2}{2} + \frac{d+1}{d-1} V. \quad (\text{A.2})$$

In the last step we have used equations (2.18)–(2.20) to simplify. This shows that the scalar curvature diverges when the potential $V(\varphi)$ diverges or when $S(\varphi)$ diverges.

Ricci squared. The square of the Ricci tensor is given as,

$$\begin{aligned} R_{AB} R^{AB} = & d^2 \left(\ddot{A}(u) + \dot{A}^2(u) \right)^2 + \frac{(R^{(\zeta)})^2}{d} e^{-4A(u)} - 2R^{(\zeta)} e^{-2A(u)} (\ddot{A}(u) \\ & + d\dot{A}^2(u)) + d \left(\ddot{A}(u) + d\dot{A}^2(u) \right)^2. \end{aligned} \quad (\text{A.3})$$

On shell, this can be written in terms of V and S as

$$R_{AB} R^{AB} = d^2 \left(\frac{S^2}{2d} + \frac{V}{d(d-1)} \right)^2 + \frac{dV^2}{(d-1)^2}. \quad (\text{A.4})$$

This also diverges when $V(\varphi) \rightarrow \infty$ or when $S(\varphi)$ diverges.

Riemann squared. The so-called Kretschmann scalar is found as,

$$R_{ABCD} R^{ABCD} = 4d \left(\ddot{A}(u) + \dot{A}^2(u) \right)^2 + 2d(d-1) \left(\frac{e^{-2A(u)}}{\alpha^2} - \dot{A}^2(u) \right)^2. \quad (\text{A.5})$$

Using the equations of motion this can be written as,

$$R_{ABCD} R^{ABCD} = \frac{4}{d} \left(\frac{S^2}{2} + \frac{V}{d-1} \right)^2 + \frac{1}{2d(d-1)} (S^2 - 2V)^2. \quad (\text{A.6})$$

Again, this diverges whenever the potential $V(\varphi)$ or the function $S(\varphi)$ diverge.

B Relation to the complex first order formalism

In [55] a first order formalism using a complex (‘superpotential’) function Z was introduced. In this appendix, we will relate quantities calculated using the complex first order formalism of [55] to expressions calculated in the first order formalism employing the real functions W , S and T used in this work.

The bulk theory considered in [55] is Einstein-dilaton gravity with a dilaton potential, just like in this work. In the conventions of [55] the Lagrangian is given by

$$\mathcal{L} = \sqrt{-g} \left[R - \frac{1}{2} (\partial\sigma)^2 - V(\sigma) \right]. \quad (\text{B.1})$$

and the following ansatz is used:

$$ds^2 = \eta(f(z)e^{\alpha\varphi(z)})^2 dz^2 + e^{2\beta\varphi(z)} ds_d^2, \quad \sigma = \sigma(z). \quad (\text{B.2})$$

To make contact with the approach in this paper, the following map is required:

$$\eta = 1 \quad (\text{B.3})$$

$$f(z) = e^{-\alpha\varphi(z)} \quad (\text{B.4})$$

$$z \rightarrow u \quad (\text{B.5})$$

$$\beta\varphi(z) \rightarrow A(u) \quad (\text{B.6})$$

$$\sigma(z) \rightarrow \varphi(u) \quad (\text{B.7})$$

where $\alpha = \sqrt{\frac{d}{2(d-1)}}$ and $\beta = \sqrt{\frac{1}{2d(d-1)}}$.

In the Hamiltonian formulation, the following Hamiltonian is used in [55]:

$$\mathcal{H} = \frac{1}{2} (\pi^2 - p^2) + e^{2\alpha\varphi} \left(\eta V - \frac{\eta k}{2\beta^2} e^{-2\beta\varphi} \right), \quad (\text{B.8})$$

with $k = -1, 0, 1$ and where the conjugate momenta are defined as

$$f^{-1}\dot{\varphi} = \pi, \quad f^{-1}\dot{\sigma} = -p. \quad (\text{B.9})$$

Using the relations (B.3)–(B.7) the Hamiltonian constraint $\mathcal{H} = 0$ reduces to

$$d((d-1)\dot{A}^2 - \frac{1}{2}\dot{\varphi}^2 + V - R^{(\zeta)} e^{-2A} = 0, \quad (\text{B.10})$$

with $R^{(\zeta)} = d(d-1)k$. This means that the bulk is sliced by dS_d/S^d ($k = 1$) or AdS_d ($k = -1$) space-times of unit radius or Minkowski space $\mathcal{M}_d$ ($k = 0$). eq. (B.10) is identical to the equation of motion (2.7) in our conventions. The other two equations of motion from Hamilton's equations can be shown to reproduce the remaining two equations of motion (2.6) and (2.8) in our notation.

In the following, we want to make contact between expressions using W and S in our conventions and expressions using Z . To this end note that the complex function Z is related to W and S as follows:

$$W = \pm 2 \left[\frac{\text{Re}(\bar{Z}Z')}{|Z'|} \right], \quad (\text{B.11})$$

$$S = \pm 2|Z'|. \quad (\text{B.12})$$

Writing

$$Z = X + iY, \quad \bar{Z} = X - iY, \quad Z' = X' + iY', \quad \bar{Z}' = X' - iY' \quad (\text{B.13})$$

the expressions (B.11)–(B.12) can be written as

$$W = \pm 2 \frac{XX' + YY'}{\sqrt{(X')^2 + (Y')^2}}, \quad (\text{B.14})$$

$$S = \pm 2\sqrt{(X')^2 + (Y')^2}. \quad (\text{B.15})$$

Now consider our equation of motion (2.20) in the first order formalism, which can be written as

$$\frac{1}{2} \frac{d}{d\varphi} S^2 - \frac{d}{2(d-1)} SW = V'. \quad (\text{B.16})$$

Using (B.11) and (B.12) we can write eq. (B.16) as

$$\begin{aligned} \frac{1}{2} \frac{d}{d\varphi} S^2 - 4\alpha^2 \text{Re}(\bar{Z}Z') &= V', \\ \Rightarrow \frac{1}{2} \frac{d}{d\varphi} S^2 - 2\alpha^2 \frac{d}{d\varphi} |Z|^2 &= V'. \end{aligned} \quad (\text{B.17})$$

Integrating this equation we find

$$V = \frac{1}{2} S^2 - 2\alpha^2 |Z|^2 + 2\alpha^2 v, \quad (\text{B.18})$$

where $2\alpha^2 v$ is an integration constant. Using $S(\varphi) = 2|Z'|$, we can write eq. (B.18) as:

$$V = 2|Z'|^2 - 2\alpha^2(|Z|^2 - v). \quad (\text{B.19})$$

This is eq. (2.7) in [55]. Overall, we conclude that the first order formalism using a complex function Z and the approach presented in this paper are two equivalent descriptions. The results of this section can be used to translate between the two approaches.

C Properties of the functions W , S and T

C.1 Positive curved case (S^d or dS_d)

1. There are two branches of solutions W and S at a generic point. On a single branch the signs of S and W' coincide. This can be seen from eq. (2.18) which we can write as:

$$SW' = S^2 + \frac{2}{d} T. \quad (\text{C.1})$$

The r.h.s. is always positive. Therefore the signs of S and W' must be same. The two possible signs for S and W' give rise to the two branches.

2. The absolute value of $W(\varphi)$ is bounded by the critical curve $B(\varphi) = \sqrt{-\frac{4(d-1)}{d} V(\varphi)}$ and on this critical curve the two functions $S(\varphi)$ and $T(\varphi)$ go to zero. This can be shown from eq. (2.19) which we can write as

$$\frac{d}{2(d-1)} W^2 = S^2 + 2T - 2V \geq -2V. \quad (\text{C.2})$$

Therefore, we can write,

$$W^2(\varphi) \geq -\frac{4(d-1)}{d} V(\varphi) = B^2(\varphi). \quad (\text{C.3})$$

On the critical curve $\frac{d}{2(d-1)}W^2(\varphi) = -2V(\varphi)$. So eq. (2.19) can be written as

$$S^2 + 2T = 0 . \quad (\text{C.4})$$

This is a sum of two positive numbers which can be zero iff the individual contributions are zero. Hence, on the critical curve, the functions $S(\varphi)$ and $T(\varphi)$ vanish.

3. Bounces cannot happen on the critical curve. Let us assume that along a flow originating from a UV fixed point a bounce occurs at $\varphi = \varphi_*$, which happens to lie on the critical curve. From property 2, we see that at $\varphi = \varphi_*$, $T(\varphi_*) = 0$. But we started from the UV, which is also on the critical curve and consequently had $T(\varphi_{\text{UV}}) = 0$. So $T(\varphi)$ is starting from a value of 0 and ending with a value of 0. This cannot happen because $\frac{dT}{du} = \frac{TW}{d-1} \geq 0$ (assuming $W \geq 0$). Instead, a bounce can happen when $S = 0$ and

$$W^2 = \frac{2(d-1)}{d} (2T - 2V) \quad \text{where } T > 0 . \quad (\text{C.5})$$

C.2 Positive and negative curved case

4. The functions $W(\varphi)$, $S(\varphi)$ and $T(\varphi)$ satisfy the following relation:

$$T'S = \frac{TW}{d-1} . \quad (\text{C.6})$$

This comes from the definition of the functions $W(\varphi)$, $S(\varphi)$ and $T(\varphi)$. We can take a u -derivative of $T = e^{-2A(u)}R^{(\zeta)}$ and then use the definitions (2.17)–(2.15) .

5. At the zeros of the function $S(\varphi)$, the geometry is approximately maximally symmetric. We can express the Ricci tensor in terms of the function $S(\varphi)$. The only non-zero components of the Ricci tensor are:

$$R_{uu} = \left(\frac{S^2}{2} + \frac{V}{d-1} \right) g_{uu}, \quad (\text{C.7})$$

$$R_{\mu\nu} = \frac{V}{d-1} g_{\mu\nu}. \quad (\text{C.8})$$

Let S be zero at a value $\varphi = \varphi_*$. Then at this point the Ricci tensor can be written as

$$R_{AB} = \frac{V(\varphi_*)}{d-1} g_{AB}. \quad (\text{C.9})$$

Therefore the space is maximally symmetric near the zeros of S . As the potential is negative, this space is approximately AdS.

6. Symmetry properties. Let $W(\varphi)$, $S(\varphi)$ and $T(\varphi)$ satisfy the equations (2.18)–(2.20) for a generic potential $V(\varphi)$. Then

$$\bar{S}(\varphi) = -S(\varphi), \quad (\text{C.10})$$

$$\bar{W}(\varphi) = -W(\varphi), \quad (\text{C.11})$$

$$\bar{T}(\varphi) = T(\varphi) \quad (\text{C.12})$$

are also solutions of the equations (2.18)–(2.20). The scale factor and the curvature invariants behave as

$$\bar{A}(u) = A(u), \quad (\text{C.13})$$

$$\bar{R}(\varphi) = R(\varphi), \quad (\text{C.14})$$

$$\bar{R}_{AB}\bar{R}^{AB}(\varphi) = R_{AB}R^{AB}(\varphi), \quad (\text{C.15})$$

$$\bar{R}_{ABCD}\bar{R}^{ABCD}(\varphi) = R_{ABCD}R^{ABCD}(\varphi). \quad (\text{C.16})$$

This means that we have two equivalent copies of geometries below and above the φ axis.

If $V(\varphi)$ is an even function of φ , then

$$\tilde{S}(\varphi) = -S(-\varphi), \quad (\text{C.17})$$

$$\tilde{W}(\varphi) = W(-\varphi), \quad (\text{C.18})$$

$$\tilde{T}(\varphi) = T(-\varphi), \quad (\text{C.19})$$

satisfy the equations (2.18)–(2.20). This is because the above relations imply

$$\tilde{S}'(\varphi) = S'(-\varphi), \quad (\text{C.20})$$

$$\tilde{W}'(\varphi) = -W'(-\varphi), \quad (\text{C.21})$$

and we also have that $V'(\varphi) = -V'(-\varphi)$. The scale factor and the curvature invariants behave as

$$\tilde{A}(u) = A(u), \quad (\text{C.22})$$

$$\tilde{R}(\varphi) = R(-\varphi), \quad (\text{C.23})$$

$$\tilde{R}_{AB}\tilde{R}^{AB}(\varphi) = R_{AB}R^{AB}(-\varphi), \quad (\text{C.24})$$

$$\tilde{R}_{ABCD}\tilde{R}^{ABCD}(\varphi) = R_{ABCD}R^{ABCD}(-\varphi). \quad (\text{C.25})$$

This shows we have two equivalent copies of geometries on the positive and negative side of φ axis.

D Near boundary solution: small curvature expansion

In this appendix we will collect analytical expressions for $W(\varphi)$, $S(\varphi)$ and $T(\varphi)$ describing the geometry in the vicinity of the boundary of the asymptotically AdS_{d+1} space-time.

D.1 The small curvature expansion

It will be advantageous to organize the solution as a perturbative expansion in R^{uv} , expanding about the result for $R^{\text{uv}} = 0$. The reason is as follows. We will find that for asymptotically AdS_{d+1} space-times *a near-boundary expansion of the quantities φ , A , W , S and T will automatically give rise to an expansion in R^{uv}* . We have already encountered this in (2.14) where we expanded the scale factor $A(u)$ for AdS_{d+1} in the vicinity

of the boundary. We found that R^{uv} enters the expansion exclusively in the combination $R^{\text{uv}} e^{2u/\ell}$. Expanding $A(u)$ in powers of $e^{2u/\ell}$ in the vicinity of the boundary at $u \rightarrow -\infty$ is thus equivalent to expanding in powers of R^{uv} . This observation will also hold more generally for asymptotically AdS_{d+1} space-times.

Consequently, we will expand $W(\varphi)$, $S(\varphi)$ and $T(\varphi)$ as follows:

$$W(\varphi) = \sum_{n=0}^{\infty} W_n (\ell^2 R^{\text{uv}})^n, \quad (\text{D.1})$$

$$S(\varphi) = \sum_{n=0}^{\infty} S_n (\ell^2 R^{\text{uv}})^n, \quad (\text{D.2})$$

$$T(\varphi) = \sum_{n=0}^{\infty} T_n (\ell^2 R^{\text{uv}})^{n+1}, \quad (\text{D.3})$$

where ℓ will be identified with the AdS length scale of the asymptotically AdS_{d+1} bulk space-time. Note that the expansion of $T(\varphi) \equiv R^{\text{uv}} e^{-2A(\varphi)}$ starts at linear order in R^{uv} . We proceed by inserting the above expansions into the equations of motion (2.18)–(2.20) and solve these order by order in R^{uv} .

Calculating S_n . Inserting the above expansions into (2.18) the resulting equations at order $(R^{\text{uv}})^0$ and $(R^{\text{uv}})^1$ can be rewritten as

$$S_0 = W'_0, \quad (\text{D.4})$$

$$S_1 = -\frac{2}{d} \frac{T_0}{W'_0} + W'_1, \quad (\text{D.5})$$

where we assumed $S_0 \neq 0$ as we are interested in non-zero flows. In virtue of (D.4) we will freely replace S_0 by W'_0 whenever convenient. Eq. (D.5) will allow us to calculate S_1 once T_0 , W_0 and W_1 have been determined.

Calculating T_n . Note that we can write

$$\begin{aligned} -2A &= -2A_0 + \frac{1}{d-1} \int_{\varphi_0}^{\varphi} \frac{W}{S} d\varphi \\ &= -2A_0 + \frac{1}{d-1} \int_{\varphi_0}^{\varphi} \frac{W_0}{S_0} d\varphi - \frac{\ell^2 R^{\text{uv}}}{d-1} \int_{\varphi_0}^{\varphi} \frac{S_1 W_0 - S_0 W_1}{S_0^2} d\varphi + \dots \end{aligned} \quad (\text{D.6})$$

$$= -2A_0 + \frac{1}{d-1} \int_{\varphi_0}^{\varphi} \frac{W_0}{W'_0} d\varphi - \frac{\ell^2 R^{\text{uv}}}{d-1} \int_{\varphi_0}^{\varphi} \frac{S_1 W_0 - W'_0 W_1}{(W'_0)^2} d\varphi + \dots, \quad (\text{D.7})$$

where we defined $A_0 \equiv A(\varphi_0)$ for some φ_0 . Despite the explicit appearance φ_0 above, the result is independent of φ_0 . Then, from the defining expression $T \equiv R^{\text{uv}} e^{-2A}$ we obtain

$$T_0 = \ell^{-2} e^{-2A_0} e^{\frac{1}{d-1} \int_{\varphi_0}^{\varphi} \frac{W_0}{W'_0} d\varphi}, \quad (\text{D.8})$$

$$T_1 = -\ell^{-2} e^{-2A_0} e^{\frac{1}{d-1} \int_{\varphi_0}^{\varphi} \frac{W_0}{W'_0} d\varphi} \frac{1}{d-1} \int_{\varphi_0}^{\varphi} \frac{S_1 W_0 - W'_0 W_1}{(W'_0)^2} d\varphi. \quad (\text{D.9})$$

We will calculate T_0 explicitly in section D.2.

Calculating W_n . Inserting the expansions into (2.19) we obtain the following two equations at the orders $(R^{\text{uv}})^0$ and $(R^{\text{uv}})^1$:

$$\frac{d}{4(d-1)}W_0^2 - \frac{1}{2}(W_0')^2 = -V, \quad (\text{D.10})$$

$$W_1' - \frac{d}{2(d-1)}\frac{W_0}{W_0'}W_1 + \frac{d-2}{d}\frac{T_0}{W_0'} = 0. \quad (\text{D.11})$$

Inserting our expression (D.8) for T_0 into the last equation, this can be integrated to give

$$W_1 = e^{\frac{d}{2(d-1)}\int_{\varphi_0}^{\varphi}\frac{W_0}{W_0'}} \left(\tilde{C}_1 + \frac{2-d}{d}e^{-2A_0}\int_{\varphi_0}^{\varphi}\frac{e^{\frac{2-d}{2(d-1)}\int_{\varphi_0}^{\varphi}\frac{W_0}{W_0'}}}{W_0'} \right), \quad (\text{D.12})$$

where $\tilde{C}_1$ is an integration constant.

Overall, given a solution for W_0 and W_0' we can now determine W_1 and T_0 , which in turn will allow us to calculate S_1 and finally T_1 . While these results will be sufficient for the scope of this paper, the analysis in this appendix can in principle be extended to determine the coefficients W_n , S_n and T_n to an arbitrarily high order.

D.2 Extrema of V

Here we will derive solutions for W , S and T in the vicinity of extrema of the potential, using the small curvature expansion introduced in section D.1.¹² To this end, it will be sufficient to consider the potential

$$V = -\frac{d(d-1)}{\ell^2} - \frac{m^2}{2}\varphi^2 + \mathcal{O}(\varphi^3), \quad (\text{D.13})$$

where we will choose $m^2 > 0$ for maxima and $m^2 < 0$ for minima.

Calculating W_0 . At order $(R^{\text{uv}})^0$ the analysis of our system reduces to a study of holographic RG flows for field theories on flat manifolds. This has been studied extensively in the past and we can hence be brief. For details we will refer readers to e.g. [45]. To be specific, at order $(R^{\text{uv}})^0$ we have $W = W_0$, $S = S_0 = W_0'$ and $T = 0$. Thus at order $(R^{\text{uv}})^0$ the solution is completely determined by W_0 .

We can determine W_0 by solving (D.10). In the vicinity of the extremum of V at $\varphi = 0$ this can be done by writing W_0 as a regular expansion in powers of φ . There exist two independent solutions, which we will label by the two subscripts $(\pm)$:

$$W_{0,\pm}^{\text{reg}} = \frac{2(d-1)}{\ell} + \frac{\Delta_{\pm}}{2\ell}\varphi^2 + \mathcal{O}(\varphi^3), \quad (\text{D.14})$$

with $\Delta_{\pm} = \frac{1}{2}\left(d \pm \sqrt{d^2 - 4m^2\ell^2}\right).$

Terms in $W_{0,\pm}^{\text{reg}}$ of order φ^3 and higher will depend on cubic and higher terms in V , which we ignore in the vicinity of extrema.

¹²This is consistent as long as extrema of the potential coincide with the boundary of the bulk space-time. We check this explicitly when discussing the resulting geometries in section 3.

The solution permits a continuous deformation δW_{01} as long as the deformation is subleading compared to $W_{0,\pm}^{\text{reg}}$. We will make this condition precise at the end of this section. The deformation can be determined by inserting $W_0 = W_{0,\pm}^{\text{reg}} + \delta W_{01}$ into (D.10) and solving for δW_{01} :

$$\delta W_{01} = \frac{C}{\ell} \exp \left(\frac{d}{2(d-1)} \int^\varphi d\varphi \frac{W_{0,\pm}^{\text{reg}}}{(W_{0,\pm}^{\text{reg}})' } \right) = \frac{C}{\ell} |\varphi|^{\frac{d}{\Delta_\pm}} [1 + \mathcal{O}(\varphi)] , \quad (\text{D.15})$$

where we introduced the dimensionless integration constant C . Putting everything together, we thus obtain:

$$W_0 = \frac{2(d-1)}{\ell} + \frac{\Delta_\pm}{2\ell} \varphi^2 + \mathcal{O}(\varphi^3) + \frac{C}{\ell} |\varphi|^{\frac{d}{\Delta_\pm}} [1 + \mathcal{O}(\varphi) + \mathcal{O}(C)] . \quad (\text{D.16})$$

From this we also have that

$$S_0 = W_0' = \frac{\Delta_\pm}{\ell} \varphi + \mathcal{O}(\varphi^2) + \frac{Cd}{\Delta_\pm \ell} |\varphi|^{\frac{d}{\Delta_\pm}-1} [1 + \mathcal{O}(\varphi) + \mathcal{O}(C)] . \quad (\text{D.17})$$

We can now return to the question under which circumstances the deformation $\propto C$ is permitted. To this end consider eq. (2.23) which involves S only and which any solution for S must satisfy. It is easy to check that the corresponding equation at order $(\mathcal{R}^{\text{uv}})^0$ is only satisfied by S_0 if the linear term in (D.17) dominates over the term involving C . This implies that solutions for S and W permit a deformation only if

$$\frac{d}{\Delta_\pm} > 2 . \quad (\text{D.18})$$

If this is not the case, no deformation is permitted, which is equivalent to setting $C = 0$.

Calculating T_0 : given the expression for W_0 in (D.16), we can now proceed to determining T_0 from (D.8). In particular, the exponent in (D.8) is given by

$$\begin{aligned} & -2A(\varphi_0) + \frac{1}{(d-1)} \int_{\varphi_0}^\varphi d\varphi \frac{W_0}{W_0'} \\ &= -2A(\varphi_0) + \frac{2}{\Delta_\pm} \int_{\varphi_0}^\varphi \frac{d\varphi}{\varphi} \left[1 + \mathcal{O}(\varphi) + \mathcal{O}(C|\varphi|^{d/\Delta_\pm-2}) \right] , \end{aligned} \quad (\text{D.19})$$

where we have used (D.16) to get to the second line. One unattractive feature is the explicit appearance of the arbitrary parameter φ_0 . In particular, there is physical information in $A(\varphi_0)$ which is obscured by this notation. Thus, in the following, we will explain how we can remove φ_0 from the expression for T_0 . The idea is to trade the arbitrary parameter φ_0 for φ_- or φ_+ which are physical parameters of the boundary field theory.

To this end, given our expression (D.16) for W_0 , let us calculate the corresponding solutions for $A(u)$ and $\varphi(u)$ using $W_0 = -2(d-1)\dot{A}$ and $W_0' = \dot{\varphi}$:

$$\varphi(u) = \begin{cases} \varphi_+ \ell^{\Delta_+} e^{\Delta_+ u/\ell} + \mathcal{O}(e^{2\Delta_- u/\ell}) & (+)\text{-branch,} \\ \varphi_- \ell^{\Delta_-} e^{\Delta_- u/\ell} + \mathcal{O}(e^{2\Delta_- u/\ell}, e^{\Delta_+ u/\ell}) & (-)\text{-branch,} \end{cases} \quad (\text{D.20})$$

$$A(u) = \bar{A} - \frac{u}{\ell} + \mathcal{O}(e^{2\Delta_\pm u/\ell}, e^{du/\ell}) , \quad (\text{D.21})$$

where we introduced φ_+ , φ_- and $\bar{A}$ as integration constants.¹³ We can always set $\bar{A} = 0$ as argued in section 2.1 and we will do so in the following. As explained in section 3, φ_- is interpreted as the UV value of source for the operator $\mathcal{O}$ in the boundary field theory. The parameter φ_+ is related to the vev of $\mathcal{O}$ as follows: $\langle \mathcal{O} \rangle = (2\Delta_+ - d)\varphi_+$. Using these expressions we can now invert $\varphi(u)$ and insert the result into $A(u)$ to arrive at an expression for $A(\varphi_0)$:

$$\begin{aligned} A(\varphi_0) &= -\frac{1}{\Delta_{\pm}} \ln \left(\frac{\varphi_0}{\varphi_{\pm} \ell^{\Delta_{\pm}}} \right) + \mathcal{O}(\varphi_0^a) \\ &= -\frac{1}{\Delta_{\pm}} \int_{\varphi_{\pm} \ell^{\Delta_{\pm}}}^{\varphi_0} \frac{d\varphi}{\varphi} + \mathcal{O}(\varphi_0^a). \end{aligned} \quad (\text{D.22})$$

Here a is an exponent that will depend on the precise values of $\Delta_{\pm}$. The most important fact for this analysis is that $a > 0$, as can be verified explicitly. Inserting this expression for $A(\varphi_0)$ into (D.19) we find

$$\begin{aligned} &-2A(\varphi_0) + \frac{2}{\Delta_{\pm}} \int_{\varphi_0}^{\varphi} \frac{d\varphi}{\varphi} \left[1 + \mathcal{O}(\varphi) + \mathcal{O}(C|\varphi|^{d/\Delta_{\pm}-2}) \right] \\ &= \frac{2}{\Delta_{\pm}} \int_{\varphi_{\pm} \ell^{\Delta_{\pm}}}^{\varphi_0} \frac{d\varphi}{\varphi} + \frac{2}{\Delta_{\pm}} \int_{\varphi_0}^{\varphi} \frac{d\varphi}{\varphi} + \int_{\varphi_0}^{\varphi} d\varphi \left[\mathcal{O}(\varphi^0) + \mathcal{O}(C\varphi^{d/\Delta_{\pm}-3}) \right] + \mathcal{O}(\varphi_0^a) \\ &= \frac{2}{\Delta_{\pm}} \int_{\varphi_{\pm} \ell^{\Delta_{\pm}}}^{\varphi} \frac{d\varphi}{\varphi} + \int_{\varphi_0}^{\varphi} d\varphi \left[\mathcal{O}(\varphi^0) + \mathcal{O}(C\varphi^{d/\Delta_{\pm}-3}) \right] + \mathcal{O}(\varphi_0^a) \\ &= \frac{2}{\Delta_{\pm}} \ln \left(\frac{\varphi}{\varphi_{\pm} \ell^{\Delta_{\pm}}} \right) + \int_{\varphi_0}^{\varphi} d\varphi \left[\mathcal{O}(\varphi^0) + \mathcal{O}(C\varphi^{d/\Delta_{\pm}-3}) \right] + \mathcal{O}(\varphi_0^a). \end{aligned} \quad (\text{D.23})$$

The main observation is that we can now remove φ_0 by setting $\varphi_0 \rightarrow 0$.¹⁴ There is no danger of picking up a divergence from the integral. (Recall that the term involving C is only present if $d/\Delta_{\pm} > 2$.) The lower limit of the integral simply becomes 0 while the term $\mathcal{O}(\varphi_0^a)$ simply vanishes as $a > 0$. We are thus finally in a position to state the result for T_0 . Using (D.23) in (D.8) we obtain:

$$T_0 = \ell^{-4} \left(\frac{\varphi}{\varphi_{\pm}} \right)^{\frac{2}{\Delta_{\pm}}} \left[1 + \mathcal{O}(\varphi) + \mathcal{O}(C|\varphi|^{d/\Delta_{\pm}-2}) \right]. \quad (\text{D.24})$$

Thus, in the vicinity of an extremum we find that T has an expansion of the form

$$T_{\pm}(\varphi) = \ell^{-2} \left(R^{\text{uv}} |\varphi_{\pm}|^{-2/\Delta_{\pm}} \right) |\varphi|^{2/\Delta_{\pm}} \left[1 + \mathcal{O}(\varphi) + \mathcal{O}(C|\varphi|^{d/\Delta_{\pm}-2}) + \mathcal{O}(\ell^2 R^{\text{uv}}) \right]. \quad (\text{D.25})$$

Here we wish to highlight the appearance of the dimensionless combination of quantities $R^{\text{uv}} |\varphi_{\pm}|^{-2/\Delta_{\pm}}$. This is not a coincidence. In fact, we will find that in W , S and T the

¹³The leading terms in $\varphi(u)$ and $A(u)$ will turn out to be universal, i.e. higher order corrections in R^{uv} to W and S will not affect the leading terms.

¹⁴However, we could in principle also choose some finite value for φ_0 . It is easy to check that this would have the same effect as a shift in the integration constant $\bar{A}$.

quantity R^{uv} will exclusively appear in the combination $R^{\text{uv}}|\varphi_{\pm}|^{-2/\Delta_{\pm}}$. Thus, it will be useful in assigning a label to this particular combination and we hence define

$$\mathcal{R} = \begin{cases} R^{\text{uv}}|\varphi_+|^{-2/\Delta_+} & \text{on the (+)-branch} \\ R^{\text{uv}}|\varphi_-|^{-2/\Delta_-} & \text{on the (-)-branch} \end{cases}. \quad (\text{D.26})$$

Strictly speaking, we should understand the small curvature expansions in (D.1)–(D.3) as expansions in $\mathcal{R}$.

Calculating W_1 and S_1 : given our expressions (D.16) and (D.24) for W_0 and T_0 we are now in a position to calculate W_1 and S_1 from (D.12) and (D.5), respectively. Starting with W_1 , after some work one finds

$$W_1 = \frac{1}{d\ell^3} \left(\frac{\varphi}{\varphi_{\pm}} \right)^{\frac{2}{\Delta_{\pm}}} \left[1 + \mathcal{O}(\varphi) + \mathcal{O}(C|\varphi|^{d/\Delta_{\pm}-2}) \right] + \frac{C_1}{\ell} |\varphi|^{d/\Delta_{\pm}} \left[1 + \mathcal{O}(\varphi) + \mathcal{O}(C|\varphi|^{d/\Delta_{\pm}-2}) \right] \quad (\text{D.27})$$

Most importantly, any dependence on the arbitrary parameter φ_0 appearing in (D.12) can be absorbed into the integration constant C_1 . Another observation is that the term $C_1|\varphi|^{d/\Delta_{\pm}}$ in W_1 combines with the term $C|\varphi|^{d/\Delta_{\pm}}$ in W_0 and thus C_1 does not represent an independent integration constant.

We are now in a position to calculate S_1 . We obtain:

$$S_1 = \frac{C_1 d}{\Delta_{\pm} \ell} |\varphi|^{\frac{d}{\Delta_{\pm}}-1} \left[1 + \mathcal{O}(\varphi) + \mathcal{O}\left(C|\varphi|^{\frac{d}{\Delta_{\pm}}-2}\right) \right] + \frac{1}{\ell} \mathcal{O}\left(|\varphi|^{\frac{2}{\Delta_{\pm}}+1}\right) + \frac{1}{\ell} \mathcal{O}\left(C|\varphi|^{\frac{2+d}{\Delta_{\pm}}-1}\right). \quad (\text{D.28})$$

Here, the term involving C_1 will combine with the corresponding term in S_0 . We also determined the leading order in φ of the terms not containing C_1 . One can check that the exact numerical coefficients will depend on the cubic and quartic terms in the potential, and which we hence leave implicit.

D.3 Summary

Putting everything together, we are now in a position to write down expressions for W , S and T in the vicinity of an extremum of V and up to order R^{uv} (or $\mathcal{R}$):

$$W_{\pm}(\varphi) = \frac{1}{\ell} \left[2(d-1) + \frac{\Delta_{\pm}}{2} \varphi^2 + \mathcal{O}(\varphi^3) \right] \quad (\text{D.29})$$

$$\begin{aligned} & + \frac{\mathcal{R}}{d\ell} |\varphi|^{\frac{2}{\Delta_{\pm}}} [1 + \mathcal{O}(\varphi) + \mathcal{O}(C) + \mathcal{O}(\mathcal{R})] \\ & + \frac{C}{\ell} |\varphi|^{\frac{d}{\Delta_{\pm}}} [1 + \mathcal{O}(\varphi) + \mathcal{O}(C) + \mathcal{O}(\mathcal{R})], \\ S_{\pm}(\varphi) &= \frac{\Delta_{\pm}}{\ell} \varphi [1 + \mathcal{O}(\varphi)] + \frac{Cd}{\Delta_{\pm} \ell} |\varphi|^{\frac{d}{\Delta_{\pm}}-1} [1 + \mathcal{O}(\varphi) + \mathcal{O}(C)], \\ & + \frac{1}{\ell} \mathcal{O}\left(\mathcal{R}|\varphi|^{\frac{2}{\Delta_{\pm}}+1}\right) + \frac{1}{\ell} \mathcal{O}\left(\mathcal{R}C|\varphi|^{\frac{2+d}{\Delta_{\pm}}-1}\right) \end{aligned} \quad (\text{D.30})$$

$$T_{\pm}(\varphi) = \ell^{-2} \mathcal{R} |\varphi|^{\frac{2}{\Delta_{\pm}}} [1 + \mathcal{O}(\varphi) + \mathcal{O}(C) + \mathcal{O}(\mathcal{R})], \quad (\text{D.31})$$

where $\mathcal{R}$ has been defined in (D.26). While we write $\mathcal{O}(C)$ and $\mathcal{O}(\mathcal{R})$ to remove clutter when possible, it is to be understood that C and $\mathcal{R}$ are always accompanied by an appropriate power in φ . Also, we absorbed the integration constant C_1 into C . Further, if $d/\Delta_{\pm} < 2$ we have to set $C = 0$, otherwise S is not a solution of (2.23). While this is the most general result for an extremum of V , we now look at maxima and minima of V in turn.

Maxima of V . At a maximum of V we have:

$$\frac{d}{2} < \Delta_+ < d, \quad 0 < \Delta_- < \frac{d}{2}, \quad (\text{D.32})$$

$$1 < \frac{d}{\Delta_+} < 2, \quad \frac{d}{\Delta_-} > 2. \quad (\text{D.33})$$

As $d/\Delta_+ < 2$ we have to set $C = 0$ in the (+)-branch of solutions. This is not required for the (−)-branch as we have $d/\Delta_- > 2$. The resulting solutions are shown in discussed in the main body of the paper in section 3.1.

Minima of V . At a minimum of V we have:

$$\Delta_+ < d, \quad \Delta_- < 0, \quad (\text{D.34})$$

$$\frac{d}{\Delta_+} < 1, \quad \frac{d}{\Delta_-} < 0. \quad (\text{D.35})$$

As $d/\Delta_+ < 2$ we have to set $C = 0$ in the (+)-branch of solutions. However, on the (−)-branch we have $\Delta_- < 0$ and as a result, any term involving C or $\mathcal{R}$ in (D.29)–(D.31) diverges! To arrive at an acceptable solution, we have to set both C and R^{uv} (and thus $\mathcal{R}$) to zero. We then recover the solution for RG flows for field theories on flat manifolds. A more comprehensive discussion of the solutions on both the (+) and (−)-branches can be found in the main text in section 3.2.

E Solution in the vicinity of minima of V

Here we will examine solutions for $A(u)$ and $\varphi(u)$ in the vicinity of a minimum. In particular, we will be interested in flows starting or ending at the minimum. The potential is given by (3.1) with $m^2 < 0$. As we work in the vicinity of the minimum, we will express solutions as a perturbation about the solution for an AdS fixed point:

$$A(u) = A_0(u) + \epsilon^2 A_2(u) + \mathcal{O}(\epsilon^4), \quad (\text{E.1})$$

$$\varphi(u) = 0 + \epsilon \varphi_1(u) + \mathcal{O}(\epsilon^3), \quad (\text{E.2})$$

with

$$A_0(u) = \begin{cases} \ln \left(-\frac{\ell}{\alpha} \sinh \frac{u+c}{\ell} \right) & S^d \text{ or } \text{dS}_d \\ -\frac{u+c}{\ell} & \mathcal{M}_d \\ \ln \left(\frac{\ell}{\alpha} \cosh \frac{u+c}{\ell} \right) & \text{AdS}_d \end{cases}. \quad (\text{E.3})$$

With this ansatz we proceed to solving the equations of motion (2.6)–(2.8) order by order in ϵ . To be specific, at order $\mathcal{O}(\epsilon)$ we obtain:

$$\ddot{\varphi}_1 + d\dot{A}_0\dot{\varphi}_1 + m^2\varphi_1 = 0. \quad (\text{E.4})$$

Minima as UV fixed points. First, we wish to show that a minimum can indeed give rise to a UV fixed point. At a UV fixed point the scale factor diverges as $A(u) \rightarrow \infty$ which is the case for $u \rightarrow -\infty$. A flow leaving a UV fixed point will thus correspond to a solution to (E.4) subject to the boundary conditions $\varphi_1(u \rightarrow -\infty) \rightarrow 0$ and $\dot{\varphi}_1(u \rightarrow -\infty) \rightarrow 0$.

We can discuss the case of dS, AdS and Minkowski slices in a unified way, due to $A_0(u \rightarrow -\infty) \rightarrow -\frac{u}{\ell}$ for all three cases. Thus, for $u \rightarrow -\infty$ (E.4) becomes

$$\ddot{\varphi}_1 - \frac{d}{\ell}\dot{\varphi}_1 + m^2\varphi_1 = 0, \quad (\text{E.5})$$

which is solved by

$$\varphi_1(u) = c_1 e^{\Delta_- u/\ell} + c_2 e^{\Delta_+ u/\ell} + \dots, \quad (\text{E.6})$$

where $c_{1,2}$ are integration constants and $\Delta_{\pm}$ have been defined before in (3.5). This is just the usual asymptotic form of the scalar field in the vicinity of a UV fixed point, with c_1 related to the source of the perturbing operator and c_2 related to its vev. However, note that for $m^2 < 0$ we have $\Delta_- < 0$ while $\Delta_+ > d$. The boundary conditions hence require that we set $c_1 = 0$. As a result, flows away from a UV fixed point at a minimum of the potential are purely driven by a vev. As the leading behavior of φ is determined by Δ_+ , solutions with minima as UV fixed points are associated with the (+)-branch of solutions for W , S and T .

Minima as IR fixed points. For the case of Minkowski slicings it is well-known that minima of the potential can be identified as IR fixed points, a fact we will confirm presently. We define an IR endpoint as the locus where the scale factor diverges as $A(u) \rightarrow -\infty$. For the case of flat slicings this occurs for $u \rightarrow \infty$. The relevant equation is again (E.5), but the boundary conditions now read $\varphi_1(u \rightarrow \infty) \rightarrow 0$ and $\dot{\varphi}_1(u \rightarrow \infty) \rightarrow 0$. The general solution is, as before, given by (E.6), but the boundary conditions now require that $c_2 = 0$ instead. Thus, for $R^{\text{uv}} = 0$ minima of the potential can play the role of IR endpoints of RG flows. When rewritten in terms of W , S and T , this is a solution on the (−)-branch.

In the following we will show that in space-times with a curved foliation RG flows cannot end at a minimum of the potential. For the case of dS and AdS slicings the IR is identified with $u \rightarrow -c_{\text{IR}}$ where the bulk geometry asymptotes to AdS_{d+1} :

$$A_0(u) \xrightarrow{u \rightarrow -c_{\text{IR}}} \begin{cases} \ln \left(-\frac{\ell}{\alpha} \sinh \frac{u + c_{\text{IR}}}{\ell_{\text{IR}}} \right) & S^d \text{ or } \text{dS}_d \\ \ln \left(\frac{\ell}{\alpha} \cosh \frac{u + c_{\text{IR}}}{\ell_{\text{IR}}} \right) & \text{AdS}_d \end{cases}. \quad (\text{E.7})$$

For the following analysis it will be useful to introduce the coordinate $w \equiv \frac{u + c_{\text{IR}}}{\ell_{\text{IR}}}$. In the remainder of this subsection we will also define $\cdot \equiv \frac{d}{dw}$. RG flows ending at an IR endpoint

at the minimum of the potential will then correspond to solutions to (E.4) subject to the boundary conditions $\dot{\varphi}_1(w=0) = 0$ and $\varphi(w=0) = 0$.

In the vicinity of the IR locus equation (E.4) becomes

$$S^d \text{ or } dS_d: \quad \ddot{\varphi}_1 + d \left(\frac{1}{w} + \frac{w}{3} + \mathcal{O}(w^3) \right) \dot{\varphi}_1 + m^2 \ell_{\text{IR}}^2 \varphi_1 = 0, \quad (\text{E.8})$$

$$\text{AdS}_d: \quad \ddot{\varphi}_1 + d \left(w - \frac{w^3}{3} + \mathcal{O}(w^5) \right) \dot{\varphi}_1 + m^2 \ell_{\text{IR}}^2 \varphi_1 = 0. \quad (\text{E.9})$$

Solving and implementing the boundary condition $\dot{\varphi}_1(w=0) = 0$ we find

$$S^d \text{ or } dS_d: \quad \varphi_1(w) = c_1 \left(1 - \frac{m^2 \ell_{\text{IR}}^2}{2(d+1)} w^2 + \mathcal{O}(w^4) \right), \quad (\text{E.10})$$

$$\text{AdS}_d: \quad \varphi_1(w) = c_1 \left(1 - \frac{m^2 \ell_{\text{IR}}^2}{2} w^2 + \mathcal{O}(w^4) \right), \quad (\text{E.11})$$

with c_1 the remaining constant of integration. Note that the only way of satisfying the second boundary condition $\varphi_1(w=0) = 0$ is to set $c_1 = 0$, which causes $\varphi(w)$ to vanish identically. *Hence there are no solutions that smoothly arrive at a minimum of the potential for S^d/dS_d and AdS_d slicings.* The only solutions that exist for $R^{\text{uv}} \neq 0$ are solutions with $\varphi = 0$, i.e. conformal fixed points. Most importantly, there are no RG flows for $R^{\text{uv}} \neq 0$ that end at a fixed point at a minimum of the potential. This is equivalent to the absence of the $(-)$ -branch of solutions for W , S and T in the vicinity of a minimum of V for $R^{\text{uv}} \neq 0$.

F Extremal points of the first order flow equations

We will start by examining the critical power behavior for the first order functions that is allowed by the equations of motion (2.18)–(2.20). Then we will proceed to examine the solutions in the regime where the flow stops. To find where the flow of φ stops, we need to find (finite) points φ_0 where S vanishes.

We parametrize S as

$$S \simeq C_0 (\varphi_0 - \varphi)^a, \quad \varphi \rightarrow \varphi_0, \quad a > 0 \quad (\text{F.1})$$

We assume without loss of generality that φ is approaching to φ_0 from the left, i.e. $\varphi < \varphi_0$. As the flow does not end up at the minimum at the presence of curvature, φ_0 is a generic point and therefore

$$V(\varphi) = V_0 + V_1(\varphi_0 - \varphi) + V_2(\varphi_0 - \varphi)^2 + \mathcal{O}((\varphi_0 - \varphi)^3). \quad (\text{F.2})$$

Near φ_0 the various terms of (2.23) behave as follows

$$\begin{aligned} S^4 &\sim (\varphi_0 - \varphi)^{4a}, & S^3 S'' &\sim (\varphi_0 - \varphi)^{4a-2}, & S^2 V &\sim (\varphi_0 - \varphi)^{2a}, \\ S S' V' &\sim (\varphi_0 - \varphi)^{2a-1}, & S^2 V'' &\sim (\varphi_0 - \varphi)^{2a}, & V'^2 &\sim (\varphi_0 - \varphi)^0. \end{aligned} \quad (\text{F.3})$$

It is apparent that near φ_0 , if $a = 1/2$, then to leading order eq. (2.23) is satisfied. After getting the leading order power law behavior, it is clear that the superpotentials have

a square root series expansion near the IR end-point. For brevity, let us write $x = \varphi_0 - \varphi$. Let us also write the following expansions near $x = 0$:

$$S(x) = \sqrt{x} (S_0 + S_1\sqrt{x} + S_2x + \cdots) , \quad (\text{F.4})$$

$$W(x) = \frac{1}{\sqrt{x}} (W_0 + W_1\sqrt{x} + W_2x + \cdots) , \quad (\text{F.5})$$

$$T(x) = \frac{1}{x} (T_0 + T_1\sqrt{x} + T_2x + \cdots) , \quad (\text{F.6})$$

$$V(x) = V_0 + V_1x + V_2x^2 + \cdots . \quad (\text{F.7})$$

Now we can use eqs. (2.21) and (2.22) to find the various coefficients. There are two cases which are generic.

Case (a):

$$S_0^2 = 2V_1 , \quad S_1 = \frac{-dW_1}{3(d-1)} , \quad S_2 = \frac{36(d-1)((d-1)V_2 + V_0) + d(d+9)W_1^2}{36(d-1)^2S_0} \quad (\text{F.8})$$

$$W_0 = 0 , \quad W_1 = \text{arbitrary} , \quad W_2 = -\frac{4(d-1)V_0 + dW_1^2}{d(d-1)S_0} \quad (\text{F.9})$$

$$T_0 = 0 , \quad T_1 = 0 , \quad T_2 = \frac{dW_1^2}{4(d-1)} + V_0 , \quad T_3 = -\frac{W_1(4(d-1)V_0 + dW_1^2)}{2(d-1)^2S_0} \quad (\text{F.10})$$

In this case $\ddot{\varphi} = \frac{1}{2} \frac{d}{dx} S^2 = S_0^2 = 2V_1 \neq 0$. So the flow does not stop here. We will find that this corresponds to a bouncing solution. Note that $S_0 = \pm\sqrt{2V_1}$, i.e. there are two branches. These will be the two branches those meet at the bouncing point. As $V_0 < 0$ we find that T can be both positive or negative, depending on the value of W_1 . Hence bounces are expected to exist for both positively and negatively curved slices.

Case (b):

$$S_0^2 = \frac{2V_1}{d+1} , \quad S_1 = 0 , \quad S_2 = \frac{V_0 + 3(d-1)V_2}{(d-1)(d+3)S_0} \quad (\text{F.11})$$

$$W_0 = (d-1)S_0 , \quad W_1 = 0 , \quad W_2 = -\frac{(d+4)V_0 - (d-1)dV_2}{d(d+3)S_0} \quad (\text{F.12})$$

$$T_0 = \frac{d(d-1)}{4} S_0^2 , \quad T_1 = 0 , \quad T_2 = \frac{dS_0W_2}{2} + V_0 . \quad (\text{F.13})$$

In this case, $\ddot{\varphi} = \frac{1}{2} \frac{d}{dx} S^2 = S_0^2 = \frac{2V_1}{d+1} \neq 0$. Although the flow does not stop here, the space-time ends. Hence this is an IR solution. Again, it seems that there are two possible solutions with $S_0 = \pm\sqrt{\frac{2V_1}{d+1}}$. However, note that $W_0 = (d-1)S_0$. As we have chosen $W > 0$, only the solution with $S_0 = +\sqrt{\frac{2V_1}{d+1}}$ is acceptable. Furthermore, here we have tacitly assumed that $V_1 > 0$. As $T_0 > 0$, this solution only occurs for positive curvature (e.g. a dS slicing).¹⁵

¹⁵If $V_1 < 0$ we can replace V_1 by $|V_1|$ in all expressions. The solution persists and corresponds to a space-time with positively curved d -dimensional slices.

Case (c): It is worthwhile to highlight another solution, which corresponds to a special case of situation (a) described above. In particular, this is the case with $W = 0$ at $x = 0$, i.e. we have $W_0 = 0 = W_1$. The solution is given by

$$S_0^2 = 2V_1, \quad (F.14)$$

$$W_0 = 0, \quad W_1 = 0, \quad W_2 = \frac{4V_0}{dS_0}, \quad (F.15)$$

$$T_0 = 0 \quad T_1 = 0, \quad T_2 = V_0. \quad (F.16)$$

As $V_0 < 0$ we find $T < 0$ in this case. Hence this corresponds to the IR solution with negatively curved slices.

G The large curvature solution

In sections 3 and 4 we examined solutions of the Einstein-dilaton system in the vicinity of individual points, studying extrema of the potential and generic points in turn. In this section, we will present (approximate) solutions for complete flows from an UV fixed point to an IR endpoint. The flows we will derive will originate from a UV fixed point at a maximum of the potential, which we locate at $\varphi = 0$ and end at a generic point, which we will denote by φ_0 . Most importantly, in this section we restrict our attention to flows, which terminate close to the maximum at $\varphi = 0$, i.e. in a regime where the potential is well described by a quadratic approximation:

$$V \simeq -V_0 - \frac{1}{2}m^2\varphi^2. \quad (G.1)$$

Furthermore, we will require that

$$\varphi_0^2 \ll \frac{|V_0|}{m^2}, \quad (G.2)$$

i.e. the potential is approximately constant along the whole flow from $\varphi = 0$ to $\varphi = \varphi_0$. This will have important consequences. For one, the existence of a small parameter φ_0 will simplify the task of solving this system, as we will be able to organize the solution perturbatively in φ_0 . In addition, we will later find that for small φ_0 , the value of the endpoint φ_0 is in one-to-one correspondence with the dimensionless curvature parameter $\mathcal{R}$. To be specific, we will find that $\mathcal{R} \sim \varphi_0^{-2/\Delta-}$. Hence a flow terminating at a small value of φ_0 will describe an RG flow for *large* values of the curvature $\mathcal{R}$. In turn, our perturbative solution will thus correspond to a large curvature expansion, with $\mathcal{R}^{-\Delta-/2}$ the expansion parameter.

The approximation and the method in this section here are the same which were presented in [60] in the case $d = 3$, where curved RG flows on spheres with endpoints near the UV boundary were studied. In this section we obtain similar results for $d = 4$.

G.1 Perturbative solution

For the task at hand, it will be most convenient to solve the second order differential equations for $A(u)$ and $\varphi(u)$ directly, rather than working with $W(\varphi)$, $S(\varphi)$ and $T(\varphi)$. Hence, the relevant equations of motion in this section are (2.6)–(2.8).

Here we will outline the solution strategy and present the results. To solve the Einstein-dilaton system, we exploit the fact that in the regime of interest, flows remain in the vicinity of the maximum of the potential. This implies that we can organize the solution as an expansion about the solution at a maximum of the potential, which we derived before in section 2.1. The solution at a maximum is characterized by $\varphi = 0$, while $A(u)$ takes the form of the AdS scale factor (2.9), which we reproduce below for convenience:

$$S^d/dS_d: \quad A(u) = \log \left(-\frac{\ell}{\alpha} \sinh \frac{u+c}{\ell} \right), \quad (\text{G.3})$$

$$\text{AdS}_d: \quad A(u) = \log \left(\frac{\ell}{\alpha} \cosh \frac{u+c}{\ell} \right), \quad (\text{G.4})$$

where we distinguish between space-times with S^d/dS_d slicings and AdS_d slicings. Inserting this ansatz for $A(u)$ into the Klein-Gordon equation (2.8) then results in a differential equation for $\varphi(u)$, which will allow us to determine the leading order contribution to flows $\varphi(u)$. The resulting equations take the form:

$$S^d/dS_d: \quad \ddot{\varphi} + \frac{d}{\ell} \coth \left(\frac{u+c}{\ell} \right) \dot{\varphi} + m^2 \varphi = 0, \quad (\text{G.5})$$

$$\text{AdS}_d: \quad \ddot{\varphi} + \frac{d}{\ell} \tanh \left(\frac{u+c}{\ell} \right) \dot{\varphi} + m^2 \varphi = 0. \quad (\text{G.6})$$

The leading contributions to RG flows are given by regular solutions to these equations subject to the boundary conditions:

$$\varphi(-c) = \varphi_0, \quad \dot{\varphi}(-c) = 0. \quad (\text{G.7})$$

These boundary conditions ensure that flows end at $u_{\text{IR}} = -c$, the value at which the scale factor $e^{A(u)}$ vanishes.

We can discuss both equations (G.5) and (G.6) in a unified way if we define a new coordinate U as follows:

$$U \equiv \begin{cases} -\coth \frac{u+c}{\ell}, & S^d/dS_d \\ -\tanh \frac{u+c}{\ell}, & \text{AdS}_d \end{cases}. \quad (\text{G.8})$$

Equations (G.5) and (G.6) then become

$$(1 - U^2)^2 \varphi'' + (d - 2)U(1 - U^2) \varphi' + \Delta_-(d - \Delta_-) \varphi = 0, \quad (\text{G.9})$$

where we defined $' \equiv \frac{d}{dU}$. The general solution is given by:

$$\begin{aligned} \varphi(U) = & \left[\tilde{C}_1 (U - 1)^{\frac{\Delta_-}{2}} (U + 1)^{\frac{d-\Delta_-}{2}} {}_2F_1 \left(\frac{2-d}{2}, \frac{d}{2}, \frac{2-d+2\Delta_-}{2}, \frac{1-U}{2} \right) \right. \\ & \left. - \tilde{C}_2 (U - 1)^{\frac{\Delta_+}{2}} (U + 1)^{\frac{d-\Delta_+}{2}} {}_2F_1 \left(\frac{2-d}{2}, \frac{d}{2}, \frac{2-d+2\Delta_+}{2}, \frac{1-U}{2} \right) \right], \end{aligned} \quad (\text{G.10})$$

where $\tilde{C}_{1,2}$ are integration constants.

So far this discussion is valid for general d . In the following, we will find it more instructive to restrict our attention to the case with $d = 4$, as this will be most relevant for applications. An additional benefit is that this will make the expressions more manageable. For $d = 4$, the general solution (G.10) reduces to

$$S^4/\text{dS}_4: \varphi(u) = \frac{1}{\sinh^2 \frac{u+c}{\ell}} \left[e^{\delta u/\ell} C_1 \left(\delta - \coth \frac{u+c}{\ell} \right) + e^{-\delta u/\ell} C_2 \left(\delta + \coth \frac{u+c}{\ell} \right) \right], \quad (\text{G.11})$$

$$\text{AdS}_4: \varphi(u) = \frac{1}{\cosh^2 \frac{u+c}{\ell}} \left[e^{\delta u/\ell} C_1 \left(\delta - \tanh \frac{u+c}{\ell} \right) + e^{-\delta u/\ell} C_2 \left(\delta + \tanh \frac{u+c}{\ell} \right) \right], \quad (\text{G.12})$$

where $C_{1,2}$ are integration constants and we have defined $\delta \equiv \sqrt{4 - m^2 \ell^2}$. Implementing the boundary conditions (G.7) we obtain:

$$S^4/\text{dS}_4: \varphi(u) = \frac{3\varphi_0}{2\delta(\delta^2 - 1)} \frac{1}{\sinh^2 \frac{u+c}{\ell}} \left[e^{\delta \frac{u+c}{\ell}} \left(\delta - \coth \frac{u+c}{\ell} \right) + e^{-\delta \frac{u+c}{\ell}} \left(\delta + \coth \frac{u+c}{\ell} \right) \right], \quad (\text{G.13})$$

$$\text{AdS}_4: \varphi(u) = \frac{\varphi_0}{2\delta} \frac{1}{\cosh^2 \frac{u+c}{\ell}} \left[e^{\delta \frac{u+c}{\ell}} \left(\delta - \tanh \frac{u+c}{\ell} \right) + e^{-\delta \frac{u+c}{\ell}} \left(\delta + \tanh \frac{u+c}{\ell} \right) \right]. \quad (\text{G.14})$$

Having an analytic expression for a complete flow will allow us to relate IR quantities such as φ_0 to UV data such as $\mathcal{R}$ and C , which we will do in the next subsection.

Before embarking on this, we need to check the validity of our results by examining how the existence of a non-vanishing flow backreacts on the metric. To this end we substitute our solution for $\varphi(u)$ back into (2.6) and (2.7) and determine how $A(u)$ is corrected. The important observation is that since $\varphi(u) \sim \mathcal{O}(\varphi_0)$, the correction to $A(u)$ arises at $\mathcal{O}(\varphi_0^2)$. Thus, we find that in the regime of interest, i.e. the regime of small φ_0 , our perturbative ansatz is self-consistent.

We could continue and solve for $A(u)$ and $\varphi(u)$ to higher orders in φ_0 . The pattern that emerges is that $A(u)$ only contains even powers of φ_0 while only odd powers of φ_0 appear in φ . To summarize, we collect our results for $d = 4$ in the following:

$$S^4/\text{dS}_4: A(u) = \log \left(-\frac{\ell}{\alpha} \sinh \frac{u+c}{\ell} \right) + \mathcal{O}(\varphi_0^2), \quad (\text{G.15})$$

$$\begin{aligned} \varphi(u) = \frac{3\varphi_0}{2\delta(\delta^2 - 1)} \frac{1}{\sinh^2 \frac{u+c}{\ell}} & \left[e^{\delta \frac{u+c}{\ell}} \left(\delta - \coth \frac{u+c}{\ell} \right) \right. \\ & \left. + e^{-\delta \frac{u+c}{\ell}} \left(\delta + \coth \frac{u+c}{\ell} \right) \right] + \mathcal{O}(\varphi_0^3), \end{aligned} \quad (\text{G.16})$$

$$\text{AdS}_4: A(u) = \log \left(\frac{\ell}{\alpha} \cosh \frac{u+c}{\ell} \right) + \mathcal{O}(\varphi_0^2), \quad (\text{G.17})$$

$$\varphi(u) = \frac{\varphi_0}{2\delta} \frac{1}{\cosh^2 \frac{u+c}{\ell}} \left[e^{\delta \frac{u+c}{\ell}} \left(\delta - \tanh \frac{u+c}{\ell} \right) + e^{-\delta \frac{u+c}{\ell}} \left(\delta + \tanh \frac{u+c}{\ell} \right) \right] + \mathcal{O}(\varphi_0^3), \quad (\text{G.18})$$

$$\text{with} \quad \delta \equiv \sqrt{4 - m^2 \ell^2} \quad \text{and} \quad \varphi_0 \ll \sqrt{\frac{|V_0|}{m^2}}.$$

Before moving on, there is one interesting feature which we wish to point out. One can check that for S^4/dS_4 slicings $\varphi(u)$ grows strictly monotonically along a flow from UV to IR. For the AdS_4 case, this is only true as long as $\Delta_- < 1$. For $\Delta_- > 1$ we find that $\varphi(u)$ changes direction along the flow: the flow leaves the UV point at $\varphi = 0$ in one direction, then turns around, before terminating on the other side of $\varphi = 0$. This is what was referred to as a ‘bounce’. Interestingly, we find that in the regime of small φ_0 bounces are generic for $\Delta_- > 1$.

G.2 Relating IR quantities and UV data

The analytical expressions (G.16) and (G.18) will allow us to relate the IR quantity φ_0 to UV data such as $\mathcal{R}$ and C . Thus we will be able to show that our perturbative solution in φ_0 is in fact a large-curvature expansion in $\mathcal{R}^{-\Delta_-/2}$. In addition, we will be able to derive an analytical relation between C and $\mathcal{R}$ valid at large curvatures.

The main observation is that, when approaching the boundary $u \rightarrow -\infty$, a solution $\varphi(u)$ corresponding to an RG flow takes the form given by (3.10). This depends on two integration constants φ_- and C . In particular, the UV limit of $\varphi(u)$ in $d = 4$ is given by (see (3.10))

$$\varphi(u) \xrightarrow{u \rightarrow -\infty} \varphi_- \ell^{\Delta_-} e^{\Delta_- u/\ell} + \frac{4C \varphi_-^{\frac{\Delta_+}{\Delta_-}}}{\Delta_- (4 - 2\Delta_-)} \ell^{\Delta_+} e^{\Delta_+ u/\ell} + \dots \quad (\text{G.19})$$

Given our analytical solutions (G.16) and (G.18), we can arrive at an alternative expression in the limit $u \rightarrow -\infty$ that will involve φ_0 . Comparing the coefficients of the terms $e^{\Delta_{\pm} u/\ell}$ will then allow us to relate UV to IR data. We will discuss the results for S^4/dS_4 slicings and AdS_4 slicings separately.

dS_4/S^4 slicing. In the following, we focus on the example with S^4/dS_4 slices. Expanding (G.16) for $u \rightarrow -\infty$ we find:

$$\begin{aligned} \varphi(u) \xrightarrow{u \rightarrow -\infty} & \frac{6 \varphi_0 e^{\Delta_- c/\ell}}{(2 - \Delta_-)(3 - \Delta_-)} e^{\Delta_- u/\ell} \left(1 + \mathcal{O}(e^{2u/\ell}) \right) \\ & + \frac{6 \varphi_0 e^{\Delta_+ c/\ell}}{(2 - \Delta_-)(1 - \Delta_-)} e^{\Delta_+ u/\ell} \left(1 + \mathcal{O}(e^{2u/\ell}) \right) + \mathcal{O}(\varphi_0^3). \end{aligned} \quad (\text{G.20})$$

Comparing with (G.19) and matching the coefficients of $e^{\Delta_- u/\ell}$ we find

$$\left(\frac{\mathcal{R}}{48} \right)^{-\frac{\Delta_-}{2}} = \frac{6}{(2 - \Delta_-)(3 - \Delta_-)} \varphi_0 \left(1 + \mathcal{O}(\varphi_0^2) \right), \quad (\text{G.21})$$

where we used that $c = -\frac{\ell}{2} \log\left(\frac{48}{\ell^2 R^{uv}}\right)$. Similarly, by comparing the coefficients of $e^{\Delta+u/\ell}$ we obtain

$$C = \frac{\Delta_-(2-\Delta_-)}{2} \frac{(3-\Delta_-)}{(1-\Delta_-)} \left(\frac{\mathcal{R}}{48}\right)^{2-\Delta_-} (1 + \mathcal{O}(\mathcal{R}^{-\Delta_-})) , \quad (\text{G.22})$$

where we also used (G.21). We learn the following from these results, (which will also hold for the AdS₄ slicing).

- We found that $\varphi_0 \sim |\mathcal{R}|^{-\Delta_-/2}$. We hence confirm that our expansion in small φ_0 indeed corresponds to an expansion in the large (dimensionless) curvature.
- Therefore, in the regime of validity of our solution, we find that the larger the (dimensionless) curvature $\mathcal{R}$, the closer to the UV point RG flows end. The numerical examples of section 5 show that this observation holds more widely, also outside of the regime of large curvature.
- Last, as the quantity C determines the vev of the operator dual to φ , we find

$$\begin{aligned} \langle \mathcal{O} \rangle_- &\equiv \frac{4}{\Delta_-} C |\varphi_-|^{\frac{\Delta_+}{\Delta_-}} \\ &= \frac{2(2-\Delta_-)(3-\Delta_-)}{(1-\Delta_-)} \left(\frac{\mathcal{R}}{48}\right)^{2-\Delta_-} |\varphi_-|^{\frac{\Delta_+}{\Delta_-}} (1 + \mathcal{O}(\mathcal{R}^{-\Delta_-})) . \end{aligned} \quad (\text{G.23})$$

AdS₄ slicing. We can repeat the analysis for the case with AdS₄ slicing. While we find the same qualitative behavior as for the S^4/dS_4 slicing, there are some quantitative differences. Expanding the solution (G.18) for $u \rightarrow -\infty$ we find:

$$\begin{aligned} \varphi(u) \xrightarrow{u \rightarrow -\infty} & \frac{2\varphi_0(1-\Delta_-)}{(2-\Delta_-)} e^{\Delta_-c/\ell} e^{\Delta_-u/\ell} \left(1 + \mathcal{O}(e^{2u/\ell})\right) \\ & + \frac{2\varphi_0(3-\Delta_-)}{(2-\Delta_-)} e^{\Delta_+c/\ell} e^{\Delta_+u/\ell} \left(1 + \mathcal{O}(e^{2u/\ell})\right) + \mathcal{O}(\varphi_0^3) . \end{aligned} \quad (\text{G.24})$$

Comparing with (G.19) we hence obtain:

$$\left(-\frac{\mathcal{R}}{48}\right)^{-\frac{\Delta_-}{2}} = \frac{2(1-\Delta_-)}{(2-\Delta_-)} \varphi_0 (1 + \mathcal{O}(\varphi_0^2)) . \quad (\text{G.25})$$

In addition, we find

$$C = \frac{\Delta_-(2-\Delta_-)}{2} \frac{(3-\Delta_-)}{(1-\Delta_-)} \left(-\frac{\mathcal{R}}{48}\right)^{2-\Delta_-} (1 + \mathcal{O}(|\mathcal{R}|^{-\Delta_-})) . \quad (\text{G.26})$$

This results in the following expression for the vev $\langle \mathcal{O} \rangle_-$:

$$\begin{aligned} \langle \mathcal{O} \rangle_- &\equiv \frac{4}{\Delta_-} C |\varphi_-|^{\frac{\Delta_+}{\Delta_-}} \\ &= \frac{2(2-\Delta_-)(3-\Delta_-)}{(1-\Delta_-)} \left(-\frac{\mathcal{R}}{48}\right)^{2-\Delta_-} |\varphi_-|^{\frac{\Delta_+}{\Delta_-}} (1 + \mathcal{O}(|\mathcal{R}|^{-\Delta_-})) . \end{aligned} \quad (\text{G.27})$$

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