Worldsheet instantons and the amplitude for string pair production in an external field as a WKB exact functional integral

James Gordon and Gordon W. Semenoff

*Department of Physics and Astronomy, University of British Columbia,
6224 Agricultural Road, Vancouver, BC V6T 1Z1, Canada*

E-mail: jbgordon@phas.ubc.ca, gordonws@phas.ubc.ca

ABSTRACT: We revisit the problem of charged string pair creation in a constant external electric field. The string states are massive and creation of pairs from the vacuum is a tunnelling process, analogous to the Schwinger process where charged particle-anti-particle pairs are created by an electric field. We find the instantons in the worldsheet sigma model which are responsible for the tunnelling events. We evaluate the sigma model partition function in the multi-instanton sector in the WKB approximation which keeps the classical action and integrates the quadratic fluctuations about the solution. We find that the summation of the result over all multi-instanton sectors reproduces the known amplitude. This suggests that corrections to the WKB limit must cancel. To show that they indeed cancel, we identify a fermionic symmetry of the sigma model which occurs in the instanton sectors and which is associated with collective coordinates. We demonstrate that the action is symmetric and that the interaction action is an exact form. These conditions are sufficient for localization of the worldsheet functional integral onto its WKB limit.

KEYWORDS: Bosonic Strings, D-branes

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1 Introduction

Schwinger’s famous formula [1] for the rate of vacuum decay, per unit volume per unit time, by the production of charged particle-anti-particle pairs in a constant electric field is given by

$$\Gamma_{\text{particle}} = (2J + 1)\pi \sum_{k=1}^{\infty} (-1)^{(k+1)(2J+1)} \left[\frac{E}{4\pi^2 k} \right]^{\frac{D+1}{2}} e^{-\frac{\pi m^2}{E} k} \quad (1.1)$$

where E is the strength of the electric field and J and m are the spin and mass of the particle. We have generalized the usual formula to D space dimensions and we have absorbed the electric charge into the field E . This is one of the few non-perturbative

formulae describing a quantum field theory process which can be obtained exactly, normally by computing the imaginary part of the vacuum energy, and thereby the vacuum decay rate, when a constant electric field is present.

There exists a formula analogous to equation (1.1) for the pair production of electrically charged strings by a constant electric field. In that case, the charges reside on the endpoints of open strings which are in turn confined to travel on D-branes. When the internal electric field on an infinite, flat D-brane has strength E , and assuming that the other end of the string goes to a parallel D-brane with no electric field, the formula for the rate of vacuum decay due to string pair creation, per unit volume per unit time is [2, 3]

$$\Gamma_{\text{string}} = \sum_S \frac{E}{\mathcal{E}} \pi \sum_{k=1}^{\infty} (-1)^{k+1} \left[\frac{\mathcal{E}}{4\pi^2 k} \right]^{\frac{D+1}{2}} e^{-\frac{\pi m_S^2}{\mathcal{E}} k - 2\pi\alpha' \mathcal{E} k} \quad (1.2)$$

where

$$\mathcal{E} = \frac{\text{arctanh} 2\pi\alpha' E}{2\pi\alpha'} \sim E [1 + \mathcal{O}(2\pi\alpha' E)^2] \quad (1.3)$$

The sum over S is the sum over the particle states in the string spectrum. The m_S are the masses of the particles and the multiplicity of the states with each mass is analogous to the $(2J+1)$ factor in the Schwinger formula (1.1).

The string formula (1.2) reduces to the particle formula (1.1), summed over the particles in the string spectrum, in the limit where the electric field is much smaller than the string tension, $2\pi\alpha' E \ll 1$ so that $\mathcal{E} \approx E$. Away from that limit, the string formula (1.2) is not simply identical to the particle formula (1.1) summed over the particle states in the string spectrum. The main (but not only) difference is the replacement of the electric field E by the parameter $\mathcal{E}$, which can be thought of as a type of screening. Moreover, since $\text{arctanh} 2\pi\alpha' E$ becomes complex if $2\pi\alpha' E > 1$, there is an upper critical electric field,

$$E_{\text{crit.}} \equiv \frac{1}{2\pi\alpha'} \quad (1.4)$$

which agrees with other derivations of the upper critical electric field, being simply the value of the electric field which balances the string tension on flat space. It can already be seen to be the singular point in the Born-Infeld action¹ which is contained in the disc amplitude for the string sigma model [4–6]. Equation (1.2) arises from the next order in the string loop expansion, the cylinder amplitude.

There is thus something that is intrinsically stringy about the Schwinger process for strings. One might speculate that, since we are discussing a transient state of the string theory — it is a state which is decaying — it is not an on-shell solution of string theory. In this sense, the interesting formula (1.2) could well be a simple probe of off-shell string theory.

¹The leading terms in an expansion in derivatives of $F_{\mu\nu}$ of the disc amplitude is the Born-Infeld effective action

$$S_{DBI} = \frac{1}{(2\pi)^D \alpha'^{(D+1)/2} g_s} \int d^{D+1} \xi \sqrt{\det(g_{\mu\nu}(\xi) + 2\pi\alpha' F_{\mu\nu}(\xi))}$$

integrated over the D-brane world-volume. For a flat brane, $g_{\mu\nu} = \delta_{\mu\nu}$ and constant electric field, where the non-zero components of $F_{\mu\nu}$ are $F_{01} = -F_{10} = E$, the Born-Infeld action does not have an imaginary part, as long as the electric field is less than the critical field $E \leq \frac{1}{2\pi\alpha'}$. In string perturbation theory about flat space, and with $E \leq \frac{1}{2\pi\alpha'}$, the imaginary part of the vacuum energy first appears in the cylinder amplitude. As we see in equations (1.2), (1.4), the upper critical electric field also appears in the cylinder amplitude.

The weak field limit of equation (1.2) was first discussed by Burgess [7]. Bachas and Porrati [2] derived the full expression by finding an operator solution of the string sigma model with an electric field. The solution was confirmed using the boundary state technique [8]. In this paper, we shall discuss how it can also be obtained by integrating the functional integral for the bosonic string sigma model which describes the appropriate configuration of open strings, the cylinder amplitude²

$$\Gamma_{\text{string}} = \frac{2}{V} \Im \int_0^\infty \frac{dT}{2T} \int [dX^\mu(\sigma, \tau)] [\text{ghost}] e^{-S[X, T]} \quad (1.5)$$

where the Polyakov action in the conformal gauge is

$$S = \frac{1}{4\pi\alpha'} \int_0^1 d\tau \int_0^1 d\sigma \left[T \dot{X}_\mu(\sigma, \tau)^2 + \frac{1}{T} X'_\mu(\sigma, \tau)^2 \right] - \frac{E}{2} \int_0^1 d\tau \left[X_0(0, \tau) \dot{X}_1(0, \tau) - X_1(0, \tau) \dot{X}_0(0, \tau) \right] \quad (1.6)$$

and the metrics of both spacetime and the string worldsheet are Euclidean. As usual, we denote $\dot{x} \equiv \frac{\partial}{\partial \tau} x$ and $x' \equiv \frac{\partial}{\partial \sigma} x$. Here, T is the modular parameter of the cylinder. The factor of $1/2$ in the measure reflects the symmetry under time reversal on the annular worldsheet. (For the charged scalar particle treated in [9, 10] this factor is absent as the scalar field is complex). The ghost determinant is (see subsection 2.2.3)

$$[\text{ghost}] = \det \left[-\frac{1}{T} \partial_\sigma^2 - T \partial_\tau^2 \right] \quad (1.7)$$

In appendix B, we will review how the imaginary part of the cylinder amplitude can be computed, and equation (1.2) obtained directly by first performing the Gaussian integral over the embedding coordinates of the string and then finding the imaginary contributions of some poles in the modular parameter integral. We can consider this a confirmation of equation (1.2) which, as we have discussed above, was found by other techniques. One interesting point is that, when zeta-function regularization is used in order to define the various infinite products and summations which are encountered in taking the functional integral, the result turns out to reproduce (1.2) in every detail, including the overall normalization.

Here, we wish to emphasize an alternative approach which, at the outset, appears less efficient. It is a true semiclassical computation of the partition function where we begin with a classical instanton solution of the equations of motion which are obtained by treating both the embedding coordinates and the modular parameter T as dynamical variables. The action evaluated on such classical instantons has already been seen to produce the large m_S^2 limit of the exponent in equation (1.2) [11]. We will then perform a detailed analysis of the fluctuations about the classical solution, using the Gelfand-Yaglom approach to computing functional determinants. We will find that, with zeta-function regularization, we produce the full expression in equation (1.2), complete with the prefactors. This suggests that this semi-classical calculation is giving us the exact result. We then fashion a proof that the functional integral in an instanton sector is indeed given exactly by the semi-classical,

²Throughout this paper, we will use the symbols $\Re$ and $\Im$ for the real and imaginary parts, respectively.

WKB limit. This proceeds by identifying an interesting nilpotent fermionic transformation of the dynamical variables which is a symmetry of the functional integral. This symmetry is fermionic in that it uses the Fadeev-Popov ghost variables which appear due to a certain gauge fixing, but it differs from the usual BRST supersymmetry. Then we show that the interaction terms in the action in the multi-instanton sector, as well as being closed forms are also exact forms and can therefore be deformed to zero. The WKB approximation then gives the exact result.

Although our calculation does not provide any new information beyond equation (1.2) which is already known, we consider it worth presenting nonetheless. It is one of the few explicit examples of localization of a functional integral and the mechanism for this localization is interesting and new. It becomes one of a short list of instanton computations that can be done exactly and the sum over all instanton sectors indeed reproduces (1.2), which confirms its validity from a fourth point of view.

Also, this semiclassical approach can be a starting point for other interesting calculations where the other approaches do not work, for example, where the spacetime of the D-brane world-volume is curved or where the electromagnetic field is not constant and the integral over embedding coordinates is not Gaussian. Then the perturbative approach which we espouse would be the conventional starting point. Although we shall not explore these issues here, it could be that our result — the knowledge that the flat space, constant field limit is WKB exact — would be of value in understanding corrections to that limit when space-time is not flat and gauge fields are not constant or where the gauge fields are dynamical.³

The rest of the paper is structured as follows. In section 2, we perform a semi-classical computation of the cylinder amplitude in the bosonic open string sigma model. We discuss the instanton solution and we consider the fluctuations of the dynamical variables of the sigma model about the classical solution in the Gaussian approximation. We find that these, subsequently summed over all instanton numbers, gives the exact result, the formula in equation (1.2). In section 3 we shall find a fermionic symmetry of the theory in the multi-instanton sector. We then use this fermionic symmetry to demonstrate that all perturbative corrections to the WKB limit indeed cancel so that the WKB approximation is exact.

In appendix A we examine, as a toy model, a simple integral which has features similar to the string functional integral that we evaluate, and which illustrates the technique of evaluating the instanton amplitude, including exactness of the WKB approximation. The supersymmetry that is identified and which we use there is a close analog of one that we have previously found for the particle in an electric field in [9, 10] and the string in an electric field in this paper. In appendix B we evaluate the imaginary part of the cylinder amplitude (1.5) by first integrating over the string embedding coordinates and subsequently picking up the imaginary contributions of poles in the integral over the modular parameter T . In appendix C we carry out the computation of section 2 using an explicit mode expansion instead of the Gelfand-Yaglom method. In subsequent appendices, we collect

³Some exact results for pair-production in non-constant background fields have been obtained in the case of QED; see [12] and references therein. For the QED analog of the present analysis, including of our localization calculation, see [9].

some properties of the Dedekind eta-, Jacobi theta and Riemann zeta-functions and their transformations which we make use of in the computations in section 2.

2 Semiclassical evaluation of the cylinder amplitude

Now let us consider the case of a bosonic string in an electric field. The scenario we are interested in has an open string that is suspended between two parallel D-branes which both have flat geometry. We turn on a constant U(1) electric field in one of the D-branes. We are interested in the amplitude for the creation of pairs of charged states of the string by the same tunnelling process as the Schwinger process for particles. We will discuss a semiclassical computation which takes into account worldsheet instantons.

The instanton solution of the open string theory has already been found and shown to produce the classical limit of the amplitude [11]. In this section, we shall expand on that calculation. In particular, we will include fluctuations about the classical instanton.

To evaluate the amplitude for the Schwinger process for charged strings in an electric field, we shall look for an imaginary part of the cylinder amplitude. We begin with the amplitude in the conformal gauge appearing in equations (1.5), (1.6) and (2.2.3). For the coordinates which are affected by the electric field, it is convenient to use the complex combination

$$z(\sigma, \tau) = \frac{1}{\sqrt{2}} [X_0(\sigma, \tau) + iX_1(\sigma, \tau)] \quad (2.1)$$

In this notation, the action becomes

$$\begin{aligned} S = & \frac{1}{2\pi\alpha'} \int_0^1 d\tau \int_0^1 d\sigma \left[T|\dot{z}(\sigma, \tau)|^2 + \frac{1}{T}|z'(\sigma, \tau)|^2 \right] + iE \int_0^1 d\tau \bar{z}(0, \tau) \dot{z}(0, \tau) \\ & + \sum_{a=2}^D \frac{1}{4\pi\alpha'} \int_0^1 d\tau \int_0^1 d\sigma \left[T\dot{x}_a(\sigma, \tau)^2 + \frac{1}{T}x'_a(\sigma, \tau)^2 \right] \\ & + \sum_{A=D+1}^{25} \frac{1}{4\pi\alpha'} \int_0^1 d\tau \int_0^1 d\sigma \left[T\dot{x}_A(\sigma, \tau)^2 + \frac{1}{T}x'_A(\sigma, \tau)^2 \right] \end{aligned} \quad (2.2)$$

where we label the coordinates which have Neumann boundary conditions as $x_a(\sigma, \tau)$, with $a = 2, \dots, D$ and those which have Dirichlet boundary conditions as $x_A(\sigma, \tau)$ with $A = D+1, \dots, 25$. The D-brane is flat and infinite, filling the spacetime coordinates $X_0, \dots, X_D$. The boundary conditions are periodic in worldsheet time, $X_\mu(\sigma, \tau + 1) = X_\mu(\sigma, \tau)$ and

$$z'(\tau, \sigma = 1) = 0, \quad z'(\tau, \sigma = 0) = 2\pi\alpha' iET \dot{z}(\tau, \sigma = 0) \quad (2.3)$$

$$x'_a(\tau, \sigma = 1) = 0, \quad x'_a(\tau, \sigma = 0) = 0, \quad a = 2, \dots, D \quad (2.4)$$

$$x_A(\tau, \sigma = 1) = d_A, \quad x_A(\tau, \sigma = 0) = 0, \quad A = D + 1, \dots, 25 \quad (2.5)$$

As usual, the presence of the electric field, which will not appear in the equations of motion, is in the boundary condition (2.3).

2.1 Worldsheet instantons

We will treat the string coordinates $z(\sigma, \tau)$, $x_a(\sigma, \tau)$, $x_A(\sigma, \tau)$ and the modular parameter T as dynamical variables. We will begin by finding the saddle points of the integrand in the functional integral by solving the classical equations of motion. Those equations are obtained by applying the variational principle to the action (2.2). The equations are

$$z''(\sigma, \tau) + T^2 \ddot{z}(\sigma, \tau) = 0 \quad (2.6)$$

$$x''(\sigma, \tau) + T^2 \ddot{x}(\sigma, \tau) = 0 \quad (2.7)$$

$$\int d\tau d\sigma \left[T^2 \left(|\dot{z}(\sigma, \tau)|^2 + \frac{1}{2} \dot{x}(\sigma, \tau)^2 \right) - \left(|z'(\sigma, \tau)|^2 + \frac{1}{2} x'(\sigma, \tau)^2 \right) \right] = 0 \quad (2.8)$$

The solutions must also obey the boundary conditions (2.3)–(2.5). With these boundary conditions, the solutions of the above equations are

$$z_0(\sigma, \tau) = \frac{1}{\sqrt{2}} |\vec{d}| e^{2\pi i k \tau} \frac{\cosh(2\pi \alpha' \mathcal{E}(\sigma - 1))}{2\pi \alpha' \mathcal{E}}, \quad k = 1, 2, \dots \quad (2.9a)$$

$$x_{0a}(\sigma, \tau) = 0, \quad x_{0A}(\sigma, \tau) = d_A \sigma, \quad T_0 = \frac{2\pi \alpha' \mathcal{E}}{2\pi k} \quad (2.9b)$$

where $2\pi \alpha' \mathcal{E} = \operatorname{arctanh} 2\pi \alpha' E$. The solution is a worldsheet with cylindrical topology which intersects the two parallel D-branes between which it is suspended on circles. The positive integer k is the instanton number. It is the number of times that the embedding wraps the cylinder.

The circle which the endpoint of the open string traces on the D-brane with the electric field can be thought of as a cyclotron orbit. In Euclidean space, an electric field behaves as a magnetic field and the charged particle residing at the end of the string follows a cyclotron orbit. The radius of the circle is related to the magnitude of the electric field and the separation of the D-branes. Since the minimum (classical) mass of the classical string state is given by the string tension times the string length,

$$\tilde{m}_0 = \sqrt{\left(\frac{|\vec{d}|}{2\pi \alpha'} \right)^2 - \frac{1}{\alpha'}} \approx \frac{|\vec{d}|}{2\pi \alpha'}$$

the radius of the orbit is

$$\text{radius}_E = m_0 \frac{\cosh(2\pi \alpha' \mathcal{E})}{\mathcal{E}} = \frac{1}{\sqrt{1 - \frac{E^2}{(2\pi \alpha')^2}}} \frac{m_0}{\operatorname{arctanh}(2\pi \alpha' E)/2\pi \alpha'} \quad (2.10)$$

which is equal to the cyclotron radius of a relativistic particle, $\frac{m_0}{E}$ when $E \ll 2\pi \alpha'$ but gets very large as E approaches the string scale $2\pi \alpha'$, going to infinity at the critical field. The string endpoint on the D-brane with no electric field also gets dragged in a circle which has a smaller radius

$$\text{radius}_0 = m_0 \frac{1}{\mathcal{E}} = \frac{m_0}{\operatorname{arctanh}(2\pi \alpha' E)/2\pi \alpha'} = \sqrt{1 - \frac{E^2}{(2\pi \alpha')^2}} \cdot \text{radius}_E$$

The classical action in this k -instanton sector is given by

$$S_{\text{classical}} = \frac{\pi k}{\mathcal{E}} \frac{d^2}{(2\pi\alpha')^2} = \frac{\pi m_0^2}{\mathcal{E}} k \quad (2.11)$$

which matches the first, dominant term in the exponent in the string amplitude in equation (1.2). This formula is known from previous work [11]. It also approaches the classical action for the cyclotron orbit of a relativistic particle, $\frac{\pi m_0^2}{E} k$, when $E \ll \frac{1}{2\pi\alpha'}$. When E is of order the string scale, the result is much smaller than that for a particle, going to zero at the critical $E \rightarrow \frac{1}{2\pi\alpha'}$.

For the remaining sections we define

$$\varepsilon \equiv 2\alpha'\mathcal{E} \quad (2.12)$$

to bring our formulae in line with the notation of e.g. [2, 11, 13].

2.2 Fluctuations about the instanton

We now expand the dynamical variables about the classical solution obeying these boundary conditions as

$$\begin{aligned} z(\sigma, \tau) &= z_0(\sigma, \tau) + \delta z(\sigma, \tau), & x_a(\sigma, \tau) &= \delta x_a(\sigma, \tau), \\ x_A(\sigma, \tau) &= d_A\sigma + \delta x_A(\sigma, \tau), & T &= T_0 + \delta T, \end{aligned} \quad (2.13)$$

where the boundary conditions for the fluctuations are

$$\delta z'(\tau, \sigma = 1) = 0, \quad \delta z'(\tau, \sigma = 0) = 2\pi\alpha' iET \delta \dot{z}(\tau, \sigma = 0) \quad (2.14a)$$

$$\delta x'_a(\tau, \sigma = 1) = 0, \quad \delta x'_a(\tau, \sigma = 0) = 0, \quad a = 2, \dots, D \quad (2.14b)$$

$$\delta x_A(\tau, \sigma = 1) = 0, \quad \delta x_A(\tau, \sigma = 0) = 0, \quad A = D + 1, \dots, 25 \quad (2.14c)$$

The expansion of the action to quadratic order in the fluctuations is

$$\begin{aligned} S &= km_0^2 \frac{2\pi\alpha'}{\varepsilon} + \frac{1}{2\pi\alpha'} \frac{\delta T^2}{T_0^3} \int_0^1 d\sigma d\tau \left[|z'_0|^2 + \frac{1}{2} x_{0A}'^2 \right] + iE \int_0^1 d\tau \delta \bar{z} \delta \dot{z} \Big|_{\sigma=0} \\ &+ \frac{1}{2\pi\alpha'} \frac{\delta T}{T_0^2} \int_0^1 d\sigma d\tau \left[T_0^2 (\dot{z}_0 \delta \dot{z} + \delta \dot{z} \dot{z}_0) - (\dot{z}_0' \delta z' + \delta \dot{z}' \dot{z}_0') \right] \\ &+ \frac{1}{2\pi\alpha'} \int_0^1 d\sigma d\tau \left[T_0 |\delta \dot{z}|^2 + \frac{1}{T_0} |\delta z'|^2 + \frac{T_0}{2} \delta \dot{x}^2 + \frac{1}{2T_0} \delta x'^2 \right] + \dots \end{aligned} \quad (2.15)$$

where T_0, z_0, x_{0A} are the classical solutions given in (2.9a), (2.9b), and we defined $\delta \vec{x} \equiv (\frac{\delta \vec{x}_a}{\delta x_A})$. Terms of cubic and higher order in fluctuations will be studied in section 3. This stringy fluctuation path integral has a similar structure to that of the worldline fluctuation integral mediating the Schwinger effect in scalar QED [9]. Accordingly, the mechanism by which it reduces to its semiclassical approximation will largely parallel that of [9], up to some technical complications.

2.2.1 Structure of fluctuation integral

Let us set up the evaluation of the above fluctuation integral with care, as it will streamline the calculation considerably. Ignoring for the moment the second line of (2.15), we have a quadratic form in each of δx_a , δx_A , δz , with identical fluctuation operator

$$\hat{L} = \frac{1}{2\pi\alpha'T_0} (-\partial_\sigma^2 - T_0^2 \partial_\tau^2) \quad (2.16)$$

but different boundary conditions. The τ -dependence is trivially diagonalized by Fourier transforming,

$$\delta z(\sigma, \tau) = \sum_{n=-\infty}^{\infty} e^{2\pi i n \tau} \delta z_n(\sigma), \quad \delta x_i(\sigma, \tau) = \sum_{n=-\infty}^{\infty} e^{2\pi i n \tau} \delta x_{i,n}(\sigma) \quad (2.17)$$

thereby reducing the 2- d spectral problem to 1- d , with Robin, Neumann and Dirichlet boundary conditions for δz , δx_a , and δx_A respectively. Gaussian integration then generates a product over n of functional determinants for the 1- d operator

$$\hat{L}_n = \frac{1}{2\pi\alpha'T_0} (-\partial_\sigma^2 + \Omega_n^2), \quad \Omega_n \equiv 2\pi n T_0. \quad (2.18)$$

Both the determinants themselves and their product over n are formally infinite and will require regularization.

At this stage we note a couple of complications. Firstly, we have ignored the coupling between δT and δz in the second line of (2.15). δT couples to an infinite number of $n = k$ modes of $\Re \delta z$.⁴ Second is the issue of zero modes. Both z_0 and $\dot{z}_0$ are non-constant eigenfunctions of $\hat{L}_k$ with zero eigenvalue.⁵

Let us label the normalized eigenmodes of $\hat{L}_k$ with Robin boundary conditions, and their respective Fourier coefficients, as follows

$$\begin{aligned} \text{Zero modes :} \quad & \hat{z}_0(\sigma) \equiv \frac{z_0(0, \sigma)}{\|z_0\|}, & \mathfrak{v} \\ & \hat{\dot{z}}_0(\sigma) \equiv \frac{\dot{z}_0(0, \sigma)}{\|\dot{z}_0\|}, & \mathfrak{z} \\ \text{Non-zero modes :} \quad & y_i(\sigma) \quad (i > 0), & \mathfrak{y}_i \end{aligned} \quad (2.19)$$

For later convenience we introduce the following notation for the inner product and norm, respectively, on the Hilbert space $\mathcal{L}^2([0, 1] \times [0, 1])$:

$$\langle z | w \rangle \equiv \int_0^1 d\tau \int_0^1 d\sigma [\bar{z}(\sigma, \tau) w(\sigma, \tau) + \bar{w}(\sigma, \tau) z(\sigma, \tau)] \quad (2.20)$$

$$\|z(\sigma, \tau)\| \equiv \sqrt{\langle z | z \rangle} \quad (2.21)$$

⁴Unlike the analogous worldline path integral for scalar QED [9], where δT coupled only to the single mode z_0 .

⁵There is also a constant zero mode for the $0, \dots, (25-D)$ directions, which generates a factor of the brane worldvolume.

Similarly, for the real coordinates X_μ ($\mu = 2, \dots, 25$) we define

$$\langle X|Y \rangle \equiv \int_0^1 d\tau \int_0^1 d\sigma \vec{X}(\sigma, \tau) \cdot \vec{Y}(\sigma, \tau) \quad (2.22)$$

$$\|X(\sigma, \tau)\| \equiv \sqrt{\langle X|X \rangle}. \quad (2.23)$$

$\hat{z}_0(\sigma)$ is a genuine zero mode of the quadratic fluctuation action (2.15), i.e. the action is independent of $\mathfrak{z}$. This is a familiar consequence of expanding a τ -translationally invariant action about a τ -dependent solution, $z_0(\sigma, \tau)$. This will have to be gauge-fixed and the Faddeev-Popov jacobian J_{FP} accounted for. We do this in subsection 2.2.2. On the other hand, $z_0(\sigma, \tau)$ couples linearly to δT , and therefore is not a genuine zero mode of the action.

In fact these comments lead to a key simplifying observation regarding the combined $\delta z_k(\sigma)$, δT quadratic form. This part of the fluctuation action (2.15) can be written, schematically, as

$$S_{\delta z, \delta T} \propto \frac{1}{2} a \delta T^2 + \delta T (b \mathfrak{v} + \vec{c} \cdot \vec{\mathfrak{v}}) + \frac{1}{2} \vec{\mathfrak{v}}^T D \vec{\mathfrak{v}} \quad (2.24)$$

where the values of the constants a , b , $\vec{c}$ and matrix D can be read off from (2.15). Now this quadratic form is not positive definite, but must be defined by analytic continuation, $\mathfrak{v} \rightarrow \pm i \mathfrak{v}$. It is in this manner that the path integral obtains an imaginary part, and thus a non-zero tunnelling probability. The advertised simplification is that all dependence on both $\vec{c}$ and a cancels out once we integrate out δT , $\mathfrak{v}$, and $\vec{\mathfrak{v}}$:

$$\int d\delta T d\mathfrak{v} \prod_{i=1}^N d\mathfrak{v}_i e^{-S_{\delta z, \delta T}} = \pm \sqrt{-1} \frac{(2\pi)^{N+1}}{b \det D} \quad (2.25)$$

We have left implicit the limit $N \rightarrow \infty$. The upshot is that we can as well make the replacement $\delta z(\sigma, \tau) \rightarrow \hat{z}_0(\sigma, \tau) \mathfrak{v}$ in the second line of (2.15), to determine the tachyonic contribution denoted b in (2.25), and we simply omit the zero eigenvalue in our evaluation of $\det \hat{L}_k$.

The semiclassical approximation to (1.5) can then be summarized by the following expression:

$$\begin{aligned} \Gamma_{\text{string}} \simeq & \frac{2}{V} \mathfrak{S} \frac{V}{2T_0} \left(\prod \sqrt{2\pi} \right) e^{-S_{\text{classical}}} [\text{ghost}]|_{T_0} \left[\left(\det'_R \hat{L}_0 \right) \left(\det'_R \hat{L}_k \right) \prod_{n \neq 0, k} \det_R \hat{L}_n \right]^{-1} \\ & \times \left[\det'_N \hat{L}_0 \prod_{n \neq 0} \det_N \hat{L}_n \right]^{-\frac{D-1}{2}} \left[\prod_{n=-\infty}^{\infty} \det_D \hat{L}_n \right]^{\frac{D-25}{2}} \cdot J_{FP}^{\text{classical}} \cdot (\text{tachyon}) \end{aligned} \quad (2.26)$$

Here J_{FP}^{cl} is the leading-order (in fluctuations) part of the Faddeev-Popov determinant. By “tachyon” we mean the imaginary contribution of the $\delta T, \delta z$ form (this is described above and indicated by $\pm i/b$ in equation (2.25)). The expression $(\prod \sqrt{2\pi})$ indicates the Gaussian integration normalization, and will be carefully accounted for later. The operator $\hat{L}_n$ is defined in (2.18). The subscripts $R(\text{obin})$, $N(\text{eumann})$ and $D(\text{irichlet})$ refer to the

relevant boundary conditions:

$$R: \quad \delta z'_n(\sigma=1)=0, \quad \delta z'_n(\sigma=0)+2\pi\alpha'E\Omega_nT_0\delta z_n(\sigma=0)=0 \quad (2.27a)$$

$$N: \quad \delta x'_a(\sigma=1)=0, \quad \delta x'_a(\sigma=0)=0, \quad a=2,\dots,D \quad (2.27b)$$

$$D: \quad \delta x_A(\sigma=1)=0, \quad \delta x_A(\sigma=0)=0, \quad A=D+1,\dots,25 \quad (2.27c)$$

We now proceed to determine each of the above components. Evaluation of the functional determinants is straightforward if one knows the eigenvalues. For the Neumann/Dirichlet cases this is not a problem, but for Robin boundary conditions the eigenvalues are not explicitly known, being determined by a transcendental equation. One option then is to expand the fluctuations δz_n in modes satisfying incorrect boundary conditions, say $\delta z' = 0$. The quadratic form will not be diagonal in this basis, so that more work is required to evaluate its determinant, but we show in appendix C that this can be done in detail and it obtains the same result as the one which we derive below.

A more elegant approach which does not rely on knowledge of the eigenvalues, and which will generalize readily to more complicated setups (e.g. non-constant background fields), is the method of Gelfand-Yaglom. We employ this method in subsection 2.3 to evaluate all determinants in (2.26).

2.2.2 Zero mode

Since $Im[\delta z]$ does not couple to δT , in this case the zero eigenvalue of $\hat{L}_k$ corresponds to a genuine zero-mode of the quadratic action. This is expected, and results from proper-time-translation invariance and the fact that the instanton depends on the world-sheet time. Denoting the gauge transformation parameter by t as follows

$$z_t(\sigma, \tau) = z(\sigma, \tau + t) \quad (2.28)$$

one has that $\dot{z}_{0,t}(\tau)$ is a zero mode of the quadratic action:

$$\left. \frac{\delta^2 L}{\delta z \delta \bar{z}} \right|_{z_{0,t}} \frac{d}{dt} z_{0,t} = \frac{d}{dt} \left(\frac{\delta L}{\delta \bar{z}} \right) \Big|_{z_{0,t}} = 0 \quad (2.29)$$

We gauge fix by introducing a collective coordinate. The Faddeev-Popov trick begins by introducing unity into the path integral in the form

$$1 = \frac{1}{\omega} \int_0^1 dt \, \delta(g(t)) \cdot \frac{d}{dt} g(t) \quad (2.30)$$

The gauge-fixing function $g(t)$ is chosen so as to render the integration over the zero mode well-defined. The “Gribov” factor ω is the number solutions of $g(t) = 0$ in the interval $0 < t < 1$. We choose

$$g(t) = \frac{1}{\|\dot{z}_0\|} \langle z_t | \dot{z}_0 \rangle, \quad (2.31)$$

for which

$$\omega = 1/2k. \quad (2.32)$$

The classical solution $z_0(\sigma, \tau)$ was defined in (2.9). Then the time translation symmetry of the path integral is used to translate the argument t to zero. This procedure then amounts to inserting

$$\delta(\mathfrak{z}) \frac{\dot{g}(0)}{2k} \quad (2.33)$$

where

$$\dot{g}(0) \equiv \left. \frac{d}{dt} g(t) \right|_{t=0} = \frac{1}{\|\dot{z}_0\|} \langle \dot{z} | \dot{z}_0 \rangle \quad (2.34)$$

$$= \|\dot{z}_0\| + \frac{1}{\|\dot{z}_0\|} \langle \delta \dot{z} | \dot{z}_0 \rangle \quad (2.35)$$

This allows the integration over $\mathfrak{z}$ to be done using the delta function, and leaves a Faddeev-Popov jacobian whose classical contribution is the first, constant term above. The second term is a correction which we ignore in the present semi-classical computation. The net result then is the insertion of

$$J_{FP}^{\text{classical}} = \pi \|\dot{z}_0\| \quad (2.36)$$

2.2.3 Ghosts

For completeness we briefly review here the contribution of reparametrization ghosts to the path integral. These arise from gauge-fixing of the Diffeomorphism $\otimes$ Weyl symmetry on the worldsheet, as explained in detail in e.g. [14]. An arbitrary, infinitesimal such symmetry variation is given by

$$\delta g_{ab} = 2\omega g_{ab} - \nabla_a \delta \sigma_b - \nabla_b \delta \sigma_a \quad (2.37)$$

where $(\sigma^0, \sigma^1) \equiv (\sigma, \tau)$, g_{ab} is the worldsheet metric, and ω parametrizes an infinitesimal Weyl transformation. The Faddeev-Popov procedure applied to this invariance leads to the following ghost action in conformal gauge:⁶

$$S_{gh} = \frac{1}{2} \int d^2 z \left(b \partial_{\bar{z}} c + \tilde{b} \partial_z \tilde{c} \right) \quad (2.38)$$

$$\equiv \frac{1}{2} \int_0^{1/T} d\tau \int_0^1 d\sigma \left\{ b(\partial_\sigma + i\partial_\tau) c + \tilde{b}(\partial_\sigma - i\partial_\tau) \tilde{c} \right\} \quad (2.39)$$

The integration limit $1/T$ follows from our definition of T in (2.2) (usually the modular parameter is taken as the reciprocal of this). $b, \tilde{b}, c$ and $\tilde{c}$ are ghost fields whose boundary conditions

$$c = \tilde{c}, \quad b = \tilde{b}, \quad \text{on boundary}, \quad (2.40)$$

⁶In this subsection we use the following notation:

$$\begin{aligned} z &= \sigma + i\tau \\ \partial_z, \partial_{\bar{z}} &= \frac{1}{2}(\partial_\sigma \mp i\partial_\tau) \\ d^2 z &= 2d\sigma d\tau. \end{aligned}$$

are inherited from the worldsheet reparametrizations. These conditions are easily implemented using the so-called doubling trick. Define $B(\sigma, \tau)$, $C(\sigma, \tau)$ on the doubled domain $\sigma \in [-1, 1]$ by

$$C(0 < \sigma < 1) \equiv c(\sigma), \quad C(-1 < \sigma < 0) \equiv \tilde{c}(-\sigma) \quad (2.41)$$

$$B(0 < \sigma < 1) \equiv b(\sigma), \quad B(-1 < \sigma < 0) \equiv \tilde{b}(-\sigma) \quad (2.42)$$

with periodic boundary conditions on B , C . The ghost action is then

$$S_{gh} = \int_0^{1/T} d\tau \int_{-1}^1 d\sigma B(\sigma, \tau) \partial_{\bar{z}} C(\sigma, \tau) \quad (2.43)$$

and the expansion

$$B(\sigma, \tau) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{2\pi i n \tau T + \pi i m \sigma} B_{mn}, \quad C(\sigma, \tau) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{2\pi i n \tau T + \pi i m \sigma} C_{mn} \quad (2.44)$$

leads to (we exclude the simultaneous zero mode $m = n = 0$)

$$[\text{ghost}] = \left(\prod_{n \neq 0} (2\pi n) \right) \prod_n \prod_{m \neq 0} [2\pi n - i\pi m/T] \quad (2.45)$$

$$= \eta^2 (i/2T) \quad (2.46)$$

As usual we have used ζ -regularization - see appendix F. For the semiclassical approximation we approximate $T \simeq T_0$ to get

$$[\text{ghost}]|_{T_0} = \eta^2 (ik/\epsilon). \quad (2.47)$$

2.2.4 Tachyon

As described in section 2.2.1, the variables δT and $\mathbf{v}$ couple in the following quadratic form

$$(\delta T, \mathbf{v}) \begin{pmatrix} a & b/2 \\ b/2 & 0 \end{pmatrix} \begin{pmatrix} \delta T \\ \mathbf{v} \end{pmatrix} \quad (2.48)$$

and we can ignore the coupling of δT to the other modes. b is simply the coefficient of $\mathbf{v} \cdot \delta T$ obtained by substituting $\delta z(\sigma, \tau) \rightarrow \hat{z}_0(\sigma, \tau) \mathbf{v}$ in (2.15), and we find

$$(\text{tachyon}) = \pm i \frac{\pi}{b} = \pm i \frac{\alpha' \|z_0\|}{k^2 R^2} \quad (2.49)$$

where $R \equiv \frac{d}{\pi \epsilon}$.

2.3 Determinants à la Gelfand-Yaglom from contour integration

The technique of evaluating functional determinants by relating them to the solution of an initial value problem is originally due to Gelfand and Yaglom [15]. It has since been extended in various directions, including more general boundary conditions, operators with

zero modes and partial differential operators, [16–21]. See [22] for a review. Here we briefly review an elegant and simple derivation based on contour integration [18], for ordinary differential operators with quite general boundary conditions.

Given an ordinary differential operator $\hat{L}$ and boundary conditions on $[0, 1] \subset \mathbb{R}$ we can associate to the spectral problem $\hat{L}\psi_i(x) = \lambda_i\psi_i(x)$ a generalized zeta function

$$\zeta(s) \equiv \sum_{i=1}^{\infty} \lambda_i^{-s}. \quad (2.50)$$

For simplicity we will assume positive eigenvalues, $\lambda_i > 0$. The above sum is only convergent for sufficiently large s ; however, $\zeta(s)$ can be analytically continued to give a meromorphic function on $\mathbb{C}$. The determinant of $\hat{L}$, which nominally is given by the divergent infinite product $\prod_i \lambda_{i>0}$, is then *defined* in zeta function regularization as

$$\det \hat{L} \equiv \exp(-\zeta'(0)) \quad (2.51)$$

Now suppose we have a function $\mathcal{U}(\lambda)$ with zeros at precisely the eigenvalues $\lambda = \lambda_i$ of $\hat{L}$ (without necessarily knowing the values of these eigenvalues). We will come to the problem of constructing such a function presently. Then the logarithmic derivative of $\mathcal{U}(\lambda)$ has poles of unit residue at each λ_i , and we therefore have the following contour integral representation of the sum (2.50):

$$\zeta(s) = \frac{1}{2\pi i} \int_{\gamma} d\lambda \lambda^{-s} \frac{d}{d\lambda} \ln \mathcal{U}(\lambda) \quad (2.52)$$

The contour γ encloses all poles $\lambda_i \in \mathbb{R}^+$ in a counterclockwise sense. Deforming it to the negative real axis we then obtain

$$\zeta(s) = \frac{\sin \pi s}{\pi} \int_0^{\infty} d\lambda \lambda^{-s} \frac{d}{d\lambda} \ln \mathcal{U}(-\lambda) \quad (2.53)$$

Clearly the above expressions are not convergent for all s , (and in particular not for $s = 0$). The usual way of handling this is to compute the ratio of determinants (with respect to the same boundary conditions) of $\hat{L}$ and some simpler normalizing operator $\hat{L}_0$ whose spectrum is known. This improves the convergence; in particular, if the coefficient of the derivative of highest degree in $\hat{L}_0$ is the same as for $\hat{L}$, the ratio of their determinants is finite. Since the large- λ asymptotics of $\mathcal{U}$ and $\mathcal{U}_0$ are the same, the representation

$$\zeta_1(s) - \zeta_2(s) = \frac{\sin \pi s}{\pi} \int_0^{\infty} d\lambda \lambda^{-s} \frac{d}{d\lambda} \ln \frac{\mathcal{U}_1(-\lambda)}{\mathcal{U}_2(-\lambda)} \quad (2.54)$$

is valid around $s = 0$, and one has $\det \hat{L} / \det \hat{L}_0 = \exp \{ - [\zeta'(0) - \zeta'_1(0)] \}$.

For the fluctuation problem in this paper, the main difficulty stems from the Robin boundary conditions. Even for the “massless” operator $-\partial^2$ we do not know the spectrum explicitly. Therefore we will study the “absolute” determinant directly, performing an explicit analytic continuation.

The idea then is to improve the large- λ behavior. By splitting the integration range and subtracting off the leading asymptotic form

$$\mathcal{U}(-\lambda) \stackrel{\lambda \rightarrow \infty}{\sim} \mathcal{U}_\infty(-\lambda) \quad (2.55)$$

of $\mathcal{U}$ we have [20, 21]

$$\zeta(s) = \zeta_{fin}(s) + \zeta_\infty(s) \quad (2.56)$$

where the finite and asymptotic contributions are respectively

$$\zeta_{fin}(s) = \frac{\sin \pi s}{\pi} \left(\int_0^1 d\lambda \lambda^{-s} \frac{d}{d\lambda} \ln \mathcal{U}(-\lambda) + \int_1^\infty d\lambda \lambda^{-s} \frac{d}{d\lambda} \ln \frac{\mathcal{U}(-\lambda)}{\mathcal{U}_\infty(-\lambda)} \right) \quad (2.57)$$

$$\zeta_\infty(s) = \frac{\sin \pi s}{\pi} \int_1^\infty d\lambda \lambda^{-s} \frac{d}{d\lambda} \ln \mathcal{U}_\infty(-\lambda) \quad (2.58)$$

Since $\mathcal{U}_\infty$ is a much simpler function, it will be possible to evaluate the integral (2.58) explicitly (assuming large s), and the resulting meromorphic expression taken to define the analytic continuation of ζ_∞ by allowing $s \in \mathbb{C}$. Meanwhile $\zeta_{fin}(s)$ is now well-defined at $s = 0$, and we have

$$\zeta'_{fin}(0) = -\ln \left[\frac{\mathcal{U}(0)}{\mathcal{U}_\infty(-1)} \right] \quad (2.59)$$

In conclusion,

$$\det \hat{L} = \mathcal{U}(0) \mathcal{U}_\infty^{-1}(-1) e^{-\zeta'_\infty(0)} \quad (2.60)$$

At this stage the problem boils down to constructing a function $\mathcal{U}(\lambda)$ with the desired arrangement of zeros. To this end, consider again the eigenvalue equation

$$(\hat{L} - \lambda) u_\lambda(\sigma) = 0 \quad (2.61)$$

with associated boundary conditions expressed in the general form

$$M \begin{pmatrix} u_\lambda(0) \\ u'_\lambda(0) \end{pmatrix} + N \begin{pmatrix} u_\lambda(1) \\ u'_\lambda(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (2.62)$$

for some constant matrices M, N . Suppose for now, however, we instead impose some arbitrary initial ($\sigma = 0$) conditions on (2.61), leaving the right-hand boundary condition free. This uniquely fixes two independent solutions⁷ $u_\lambda^{(1,2)}(\sigma)$ of (2.61), from which we can construct a general solution $u_\lambda(\sigma) = \alpha u_\lambda^{(1)}(\sigma) + \beta u_\lambda^{(2)}(\sigma)$. Such a solution evaluated at the boundary, $\sigma = 1$, defines a function of λ . It is then straightforward to show that the condition for the existence of a solution $u_\lambda(\sigma)$ satisfying the boundary conditions (2.62) is [18]

$$\det (M + N H_\lambda(1) H_\lambda^{-1}(0)) = 0 \quad (2.63)$$

where $H_\lambda(\sigma)$ is the fundamental matrix defined by

$$H_\lambda(\sigma) \equiv \begin{pmatrix} u_\lambda^{(1)}(\sigma) & u_\lambda^{(2)}(\sigma) \\ u_\lambda^{(1)'}(\sigma) & u_\lambda^{(2)'}(\sigma) \end{pmatrix}. \quad (2.64)$$

⁷Assuming $\hat{L}$ is of second order.

It proves convenient to impose the particular initial condition

$$H_\lambda(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.65)$$

Then the function we are after is given by

$$\mathcal{U}(\lambda) = \det(M + NH_\lambda(1)) \quad (2.66)$$

2.3.1 Evaluation of the functional determinants

It is now a simple matter to apply the algorithm of section 2.3 to the evaluation of $\det_\mu \hat{L}_n$, for $(\mu = R, N, D)$. We first normalize the “kinetic” term to unity, as is customary. That is, we will compute the determinants of the operator

$$\mathcal{L}_n = -\partial_\sigma^2 - \Omega_n^2, \quad \Omega_n \equiv 2\pi n T_0 \quad (2.67)$$

This normalization introduces a factor of $\mathcal{N}$ into the integral form of the zeta function as follows:

$$\zeta(s) \rightarrow \tilde{\zeta}(s) = \frac{\sin \pi s}{\pi} \int_0^\infty d\lambda (\mathcal{N}\lambda)^{-s} \partial_\lambda \ln \mathcal{U}(-\lambda) \quad (2.68)$$

Clearly $\tilde{\zeta}'_{fin}(0) = \zeta'_{fin}(0)$, but the asymptotic piece gives a non-trivial contribution:

$$\tilde{\zeta}'_\infty(0) = \left. \frac{d}{ds} (\mathcal{N}^{-s} \zeta_\infty(s)) \right|_{s=0} \quad (2.69)$$

$$= (-\log \mathcal{N}) \zeta_\infty(0) + \zeta'_\infty(0) \quad (2.70)$$

Consequently, the relation between the determinants of $\hat{L}$ and $\mathcal{L}$ is

$$\det_\mu \hat{L}_n = \mathcal{N}^{\zeta_\infty^{(\mu)}(0)} \det_\mu \mathcal{L}_n, \quad \mu = R, N, D \quad (2.71)$$

However, the (Riemann-zeta-regularized) product over all $n \in \mathbb{Z}$ erases this dependence on $\mathcal{N}$, since $\prod_{n=-\infty}^\infty (const) = 1$, and $\zeta_\infty(s)$ does not depend on n . The only such contribution then is from the excluded zero eigenvalues, so effectively each “prime” on a determinant is accompanied by $\mathcal{N}^{-1}$.

For convenience, we will absorb the conventional powers of $\sqrt{2\pi}$ accompanying each Gaussian integration (and hence eigenvalue) into $\mathcal{N}$, defining

$$\mathcal{N} \equiv \frac{1}{2\pi\alpha'T_0} \frac{1}{2\pi} = \frac{k}{2\pi^2\alpha'\epsilon}. \quad (2.72)$$

With these notations, we write (2.26) as

$$\begin{aligned} \Gamma_{\text{string}} \simeq & \Im \frac{2}{V} \frac{V}{2T_0} e^{-S_{\text{classical}}} [\text{ghost}]|_{T_0} \left[\mathcal{N}^{-2} (\det'_R \mathcal{L}_0) (\det'_R \mathcal{L}_k) \prod_{n \neq 0, k} \det_R \mathcal{L}_n \right]^{-1} \\ & \times \left[\frac{\det'_N \mathcal{L}_0}{\mathcal{N}} \prod_{n \neq 0} \det_N \mathcal{L}_n \right]^{-\frac{D-1}{2}} \left[\prod_{n=-\infty}^\infty \det_D \mathcal{L}_n \right]^{\frac{D-25}{2}} \cdot J_{FP}^{cl} \cdot (\text{tachyon}) \end{aligned} \quad (2.73)$$

Without further ado, let us compute the determinants. The fundamental matrix (2.64) satisfying (2.65) is

$$H_\lambda(\sigma) = \begin{pmatrix} \cos\left(\sigma\sqrt{\lambda - \Omega_n^2}\right) & \frac{\sin\left(\sigma\sqrt{\lambda - \Omega_n^2}\right)}{\sqrt{\lambda - \Omega_n^2}} \\ -\sqrt{\lambda - \Omega_n^2} \sin\left(\sigma\sqrt{\lambda - \Omega_n^2}\right) & \cos\left(\sigma\sqrt{\lambda - \Omega_n^2}\right) \end{pmatrix} \quad (2.74)$$

• **Robin boundary conditions.**

In the notation of (2.62) we have

$$M = \begin{pmatrix} 2\pi\alpha' E\Omega_n & 1 \\ 0 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.75)$$

Thus

$$\mathcal{U}(-\lambda) = 2\pi\alpha' E\Omega_n \cosh\sqrt{\lambda + \Omega_n^2} - \sqrt{\lambda + \Omega_n^2} \sinh\sqrt{\lambda + \Omega_n^2}, \quad (2.76)$$

$$\mathcal{U}_\infty(-\lambda) = -\frac{1}{2}\sqrt{\lambda}e^{\sqrt{\lambda}}$$

$$\zeta_\infty(s) = \frac{\sin\pi s}{2\pi} \left(\frac{1}{s} + \frac{1}{s - \frac{1}{2}} \right), \quad \lim_{s \rightarrow 0} \zeta'_\infty(0) = -1 \quad (2.77)$$

Therefore recalling (2.60)

$$\det \mathcal{L} = \mathcal{U}(0) \mathcal{U}_\infty^{-1}(-1) e^{-\zeta'_\infty(0)} \quad (2.78)$$

we have

$$\det_R \mathcal{L}_n = 2\Omega_n \sinh \Omega_n [1 - 2\pi\alpha' E \coth \Omega_n] \quad (2.79)$$

Using the infinite product formulae derived in appendix F, we obtain

$$\prod_{n \neq 0, k} \det_R \mathcal{L}_n = \left[\frac{(-1)^{k+1} e^{-\pi k \epsilon}}{\cosh^2 \pi \epsilon} \right]^{-1} \frac{\eta^2(ik/\epsilon)}{2\pi \epsilon \sinh \pi \epsilon} \frac{2k^2}{\epsilon^2} \quad (2.80)$$

To handle the zero modes, we choose to modify the operators with a regulator $\beta \ll 1$, leaving the boundary conditions unchanged:

$n = k$: take

$$\mathcal{L}_k \rightarrow \tilde{\mathcal{L}}_k = -\partial_\sigma^2 - \Omega_k^2(1 + \beta)^2 \quad (2.81)$$

Note that $\dot{z}_0$ (z_0) is still an eigenfunction, but with eigenvalue $\lambda_0 = 2\beta(\pi\epsilon)^2$, and the determinant is⁸

$$\det'_R \mathcal{L}_k = \lim_{\beta \rightarrow 0} \frac{\det_R \tilde{\mathcal{L}}_k}{\tilde{\lambda}_0} = \frac{2\pi\epsilon + \sinh 2\pi\epsilon}{2\pi\epsilon} \operatorname{sech} \pi\epsilon = \frac{4||z_0||^2}{R_0^2 \cosh \pi\epsilon} \quad (2.82)$$

⁸Alternatively, the result (E.5) gives

$$\det'_R \mathcal{L}_k = -2\mathcal{B}|y(\sigma)|^2,$$

with $y(\sigma) = \tanh(\pi\epsilon) \sinh(\Omega_k \sigma) - \cosh(\Omega_k \sigma)$ and $\mathcal{B} = \cosh \pi\epsilon$, in agreement with (2.82).

$n = 0$: the boundary conditions reduce to pure Neumann. Taking $\tilde{\mathcal{L}}_0 = -\partial_\sigma^2 + \beta^2$, the zero mode $y_0(\sigma) = 1$ is unchanged but the eigenvalue shifts to β^2 , giving

$$\det'_R \mathcal{L}_0 = 2 \quad (2.83)$$

Therefore the ‘‘Robin’’ contribution $[\mathcal{N}^{-2} (\det'_R \mathcal{L}_0) (\det'_R \mathcal{L}_k) \prod_{n \neq 0, k} \det_R \mathcal{L}_n]^{-1}$ is

$$\text{Robin} = \mathcal{N}^2 \left[\frac{(-1)^{k+1} e^{-\pi k \epsilon}}{\cosh^2 \pi \epsilon} \right] \frac{(2\pi \epsilon) \sinh \pi \epsilon}{\eta^2(ik/\epsilon)} \left(\frac{\epsilon}{2k} \right)^2 \frac{R^2 \cosh \pi \epsilon}{2||z_0||^2} \quad (2.84)$$

- **Neumann boundary conditions.**

The Neumann result is obtained by setting $E = 0$ in (2.79), (2.83), and using the infinite product formulae in appendix F:

$$\text{Neumann} = \left[\mathcal{N}^{-1} \det'_N \mathcal{L}_0 \prod_{n \neq 0} \det_N \mathcal{L}_n \right]^{-\frac{D-1}{2}} \quad (2.85)$$

$$= \mathcal{N}^{\frac{D-1}{2}} \left[2 \prod_{n \neq 0} 2\Omega_n \sinh \Omega_n \right]^{-\frac{D-1}{2}} \quad (2.86)$$

$$= \left[\frac{\epsilon}{8\pi^2 \alpha' k} \right]^{\frac{D-1}{2}} \eta^{1-D}(ik/\epsilon) \quad (2.87)$$

- **Dirichlet boundary conditions.**

We have $M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, therefore

$$\begin{aligned} \mathcal{U}(-\lambda) &= \frac{\sinh \sqrt{\lambda + \Omega_n^2}}{\sqrt{\lambda + \Omega_n^2}}, \quad \mathcal{U}_\infty(-\lambda) = \frac{e^{\sqrt{\lambda}}}{2\sqrt{\lambda}} \\ \zeta_\infty(s) &= \frac{\sin \pi s}{2\pi} \left(\frac{1}{s - \frac{1}{2}} - \frac{1}{s} \right), \quad \lim_{s \rightarrow 0} \zeta'_\infty(0) = -1 \end{aligned} \quad (2.88)$$

and

$$\det_D \mathcal{L}_n = \frac{2 \sinh \Omega_n}{\Omega_n}. \quad (2.89)$$

The net contribution is thus

$$\text{Dirichlet} = \prod_{n=-\infty}^{\infty} (\det_D \mathcal{L}_n)^{\frac{D-25}{2}} = [\eta(ik/\epsilon)]^{D-25} \quad (2.90)$$

2.4 Result

Gathering all factors in (2.26), we find that the semi-classical approximation to the annular string partition function with action (2.2) yields

$$\Gamma^{\text{semicl.}} = \frac{\tanh \pi \epsilon}{\epsilon} \frac{(-1)^{k+1} e^{-\pi k \epsilon}}{\eta^{24}(ik/\epsilon)} \left[\frac{\epsilon}{8\pi^2 \alpha' k} \right]^{\frac{D+1}{2}} e^{-2\pi \alpha' k m_0^2 / \epsilon}, \quad (2.91)$$

where we recall that $m_0 = \frac{d}{2\pi\alpha'}$ and $\epsilon = \frac{1}{\pi} \text{arctanh}(2\pi\alpha' E)$. This is identical to the result (equation (25)) of [2]. To see this, it is useful to recall the mass-shell relation of the open string,

$$m_S^2 = \frac{d^2}{(2\pi\alpha')^2} + \frac{1}{\alpha'} (\mathcal{N} - 1) \quad (2.92)$$

where $\mathcal{N}$ is the level number, from which one obtains

$$\sum_S e^{-(2\pi\alpha') k m_S^2 / \epsilon} = e^{-(2\pi\alpha') k m_0^2 / \epsilon} \eta^{-24} (ik / \epsilon). \quad (2.93)$$

(Note that in contrast to our setup, theirs allows for both string endpoints to be charged, but is specialized to a spacetime-filling D-brane, $D = 25$.)

3 Exactness of semiclassical approximation: proof by localization

It is remarkable that our semi-classical computation has produced the *exact* amplitude for pair production. It implies that all higher-order corrections must find a way to cancel. In this section we will study the full path integral and construct a proof that it localizes onto its semi-classical approximation.

There is an analogous localization of the worldline path integral of scalar QED in a constant electric field, which we addressed in [9] (and which inspired the present investigation). While the underlying mechanism by which the two path integrals localize is essentially the same, the 2d worldsheet with boundary does introduce some complications relative to the particle worldline, and in particular the manipulations of this section become significantly more cumbersome.

The cleanest approach proceeds by identifying a fermionic symmetry of the gauge-fixed action, mixing ghost and bosonic variables (but distinct from the usual BRST symmetry). Recall that the quadratic action had a zero-mode associated with proper time translation invariance. The gauge-fixing factor introduced in equation (2.30) can be represented as follows

$$\frac{1}{\omega} \int_0^1 dt \delta(g(t)) \frac{d}{dt} g(t) = \frac{1}{\omega} \int dt \int [dB d\mathfrak{c} d\bar{\mathfrak{c}}] e^{-[2\pi i B \cdot g(t) + \bar{\mathfrak{c}} \mathfrak{c} \cdot \frac{d}{dt} g(t)]}, \quad (3.1)$$

where $\mathfrak{c}$ and $\bar{\mathfrak{c}}$ are constant, anti-commuting Faddeev-Popov ghosts, and B is a Lagrange multiplier.⁹ Thus after using the gauge-invariance of the path integral to translate t to

⁹The gauge-fixed action S_{gf} , which now includes the exponent in (3.1), enjoys as usual a BRST symmetry

$$\hat{\delta} S_{gf} = 0, \quad (3.2)$$

where

$$\hat{\delta} t = -\mathfrak{c}, \quad \hat{\delta} \bar{\mathfrak{c}} = 2\pi i B, \quad \hat{\delta} \mathfrak{c} = \hat{\delta} B = \hat{\delta}(\delta T) = 0. \quad (3.3)$$

Recall that t parametrizes the gauge transformation, $\delta z_t(\sigma, \tau) \equiv \delta z(\sigma, \tau + t)$. The gauge-fixing part introduced in (3.1) can be written as the following BRST exact expression:

$$\hat{\delta} [\bar{\mathfrak{c}} g(t)].$$

zero, we can write the full path integral as follows

$$\Gamma = \frac{2}{\omega V} \Im \sum_{k=1}^{\infty} \int [d(\delta X_{\mu})] d(\delta T) dB d\mathbf{c} d\bar{\mathbf{c}} \frac{\eta^2 \left(\frac{i/2}{T_0^{(k)} + \delta T} \right)}{T_0^{(k)} + \delta T} e^{-S(X_0^{(k)} + \delta X, T_0^{(k)} + \delta T) - [2\pi i B \cdot g(0) + \bar{\mathbf{c}} \cdot \dot{g}(0)]} \quad (3.4)$$

In order to construct the sought-after transformation, let us specify our gauge-fixing function. For present purposes, a judicious choice will be

$$g(t) = \frac{2\pi k}{\kappa} \left[\langle z_t | \dot{z}_0 \rangle - \frac{1}{(2\pi k)^2} \frac{1}{T_0 T} \langle z'_t | \dot{z}'_0 \rangle \right] \quad (3.5)$$

where $z_t \equiv z(\sigma, \tau + t)$ and κ is defined as

$$\kappa \equiv \frac{1}{\|\dot{z}_0\|} \left(\|\dot{z}_0\|^2 - \frac{1}{T_0^2} \|\dot{z}'_0\|^2 \right) = \frac{(2\pi k R)^2}{\|\dot{z}_0\|}, \quad R \equiv d/\pi\epsilon. \quad (3.6)$$

This differs from our earlier choice (2.31) by the addition of the second term, but it is normalized such that the leading order Jacobian $J_{FP}^{\text{classical}}$ is the same as before. The Gribov factor is still $\omega = 2k$. We will argue that the fluctuation part can ultimately be set to zero. The constraint function becomes

$$\begin{aligned} g(0) &= \frac{2\pi k}{\kappa} \left[\langle \delta z | \dot{z}_0 \rangle - \frac{1}{(2\pi k)^2} \frac{\langle \delta z' | \dot{z}'_0 \rangle}{T_0 T} \right] \\ &= \frac{2\pi k}{\kappa} \left[\frac{\mathfrak{z}}{\|\dot{z}_0\|} \left(\|\dot{z}_0\|^2 - \frac{\|\dot{z}'_0\|^2}{T_0 T} \right) \right] - f(\delta T, \{\mathfrak{y}_i\}) \\ &= \mathfrak{z} \left[1 - \frac{\kappa}{4\pi\alpha'} \delta T h(\delta T) \right] - f(\delta T, \{\mathfrak{y}_i\}) \end{aligned} \quad (3.7)$$

where the function f depends on all modes except $\mathfrak{v}$ and $\mathfrak{z}$, while h depends only on δT . Our symmetry argument will eliminate the second term in square brackets, so that the constraint will reduce to $\delta(\mathfrak{z} - f)$ with unit coefficient in front of $\mathfrak{z}$. Similarly, the Faddeev-Popov jacobian becomes¹⁰

$$J_{FP}(z, T) \equiv \dot{g}(0) = \frac{2\pi k}{\kappa} \left[\langle \dot{z} | \dot{z}_0 \rangle - \frac{\langle \dot{z}' | \dot{z}'_0 \rangle}{T_0 T} \right] \quad (3.8a)$$

$$= \|\dot{z}_0\| + \dots \quad (3.8b)$$

Expanding the full gauge-fixed action S_{gf} about the k 'th instanton, we have

$$S_{gf}^{(k)} = S_{\text{classical}}^{(k)} + S_{\text{quad}}^{(k)} + S_{\text{int}}^{(k)}, \quad (3.9)$$

¹⁰Here we exclude the Gribov factor $1/\omega$ from our definition of J_{FP} , unlike in the previous section.

$S_{\text{classical}}^{(k)}$ is given by (2.11), and using (2.15), (3.1), (3.7) and (3.8), we have¹¹

$$S_{\text{quad}}^{(k)} = \frac{1}{4\pi\alpha'} \left[T_0 \|\delta\dot{X}\|^2 + \frac{1}{T_0} \|\delta X'\|^2 \right] + \frac{\delta T}{2\pi\alpha'} \left[\langle \dot{z}_0 | \delta \dot{z} \rangle - \frac{1}{T_0^2} \langle z'_0 | \delta z' \rangle \right] + iE \int_0^1 d\tau \delta \bar{z} \delta \dot{z} \Big|_{\sigma=0} \\ + \frac{1}{2} a \delta T^2 + 2\pi i B (\mathfrak{z} - f) + \bar{\mathfrak{c}} \mathfrak{c} J_{FP}^{\text{classical}} \\ S_{\text{int}}^{(k)} = \frac{\delta T}{4\pi\alpha'} \left[\|\delta\dot{X}\|^2 - \frac{1}{T_0^2} \|\delta X'\|^2 \right] - \frac{1}{4\pi\alpha'} \sum_{j=2}^{\infty} \frac{(-\delta T)^j}{T_0^{j+1}} \left[2 \langle X'_0 | \delta X' \rangle + \|\delta X'\|^2 \right] \quad (3.10)$$

$$- \frac{\kappa}{4\pi\alpha'} 2\pi i B \mathfrak{z} \delta T \cdot h(\delta T) + \bar{\mathfrak{c}} \mathfrak{c} [J_{FP}(z, T) - \|\dot{z}_0\|] \quad (3.11)$$

(Although f is not quadratic, we include it in S_{quad} for convenience since the final result will be independent of f). Similarly, we expand the measure around $T = T_0^{(k)}$:

$$\frac{1}{T} \eta^2 [i/2T] = \frac{1}{T_0^{(k)}} \eta^2 \left[i/2T_0^{(k)} \right] e^{-F^{(k)}(\delta T)} = \frac{1}{T_0^{(k)}} \eta^2 \left[i/2T_0^{(k)} \right] (1 + \mathcal{O}(\delta T)) \quad (3.12)$$

where the fluctuation factor $F^{(k)}$, given by

$$F^{(k)}(\delta T) = \log \left[\frac{\eta^2(i/2T)}{(1 + \delta T/T_0^{(k)}) \eta^2(i/2T_0^{(k)})} \right] \quad (3.13)$$

$$= \sum_{j=1}^{\infty} F_j^{(k)} \delta T^j, \quad (3.14)$$

contains all corrections to the measure, including the reparametrization ghost contribution. The Taylor expansion starts at δT^1 , and is well-defined since $\eta(x)$ is a holomorphic function in the upper half-plane. The amplitude is therefore

$$\Gamma(\lambda) \equiv \frac{2}{\omega V} \Im \sum_{k=1}^{\infty} e^{-S_{\text{classical}}^{(k)}} \int_{-\infty}^{\infty} d(\delta T) \int [d(\delta X_{\mu})] dB d\mathfrak{c} d\bar{\mathfrak{c}} \frac{\eta^2 [i/2T_0^{(k)}]}{T_0^{(k)}} e^{-S_{\text{quad}}^{(k)} - \lambda (S_{\text{int}}^{(k)} + F^{(k)}(\delta T))} \quad (3.15)$$

evaluated at $\lambda = 1$. Moreover, the semiclassical approximation to Γ that we computed in section 2 is given by $\Gamma(0)$.

In what follows, we will demonstrate that $\frac{\partial}{\partial \lambda} \Gamma(\lambda) = 0$ and therefore λ can be deformed to zero without altering the value of Γ . From now on we drop the superscript (k) .

Define the nilpotent fermionic transformation Δ by

$$\Delta \bar{\mathfrak{c}} = \frac{\kappa}{4\pi\alpha'} \delta T, \quad \Delta \delta z(\sigma, \tau) = -\frac{1}{2} (2\pi k) z_0(\sigma, \tau) \cdot \mathfrak{c}, \quad \Delta(\text{other}) = 0. \quad (3.16)$$

with κ given by (3.6). It leaves the quadratic action invariant:

$$\Delta S_{\text{quad}}^{(k)} = 0. \quad (3.17)$$

¹¹Recall that uppercase X_{μ} stands for all spacetime components, $\mu = 0, \dots, 25$. Lowercase x_{μ} has $\mu = 2, \dots, 25$, i.e.

$$\|\delta\dot{X}\|^2 = \|\delta\dot{z}\|^2 + \|\delta\dot{x}\|^2.$$

We will show that S_{int} is Δ -exact (and therefore in particular it is Δ -closed: $\Delta S_{\text{int}} = 0$). In a given (k -)instanton sector, all bosonic interaction terms are contained in the combination

$$\mathcal{T} \equiv S(T, X) - S(T_0^{(k)}, X) + F^{(k)}(\delta T) \quad (3.18)$$

$$= \frac{\kappa}{4\pi\alpha'} \delta T \cdot \xi [T, X] \quad (3.19)$$

where we have defined

$$\xi \equiv \frac{1}{\kappa} \left(\|\dot{X}\|^2 - \frac{1}{T_0 T} \|X'\|^2 \right) + \frac{4\pi\alpha'}{\kappa} F^{(k)}(\delta T) / \delta T \quad (3.20)$$

We showed earlier that in the absence of interaction terms, the path integral does not depend on the δT^2 term or the $\delta T \cdot \mathfrak{y}_i$ cross terms. Therefore the semi-classical approximation corresponds, in the bosonic sector, to replacing $\xi \rightarrow 2\mathfrak{v}$.

It is now easy to see that $\mathcal{T}$ and $\bar{\mathfrak{c}}\dot{g}(0)$ are generated by Δ as follows:

$$\Delta(\bar{\mathfrak{c}}\xi) = \mathcal{T} + \bar{\mathfrak{c}}\mathfrak{c} J_{FP}, \quad (3.21)$$

while the correction to the “constraint” term can be written

$$\Delta(-\bar{\mathfrak{c}} \cdot 2\pi i B_3 h(\delta T)) = -\frac{\kappa}{4\pi\alpha'} 2\pi i B_3 \delta T \cdot h(\delta T). \quad (3.22)$$

We have thus written a part of the action, including S_{int} , as a Δ -exact quantity. However, we cannot simply set this to zero, as all $\mathfrak{v}$ and $\bar{\mathfrak{c}}\mathfrak{c}$ dependence would be eliminated, i.e. we would get $\Gamma \sim 0 \cdot \infty$. Before localizing we must separate out the quadratic part ($\bar{\mathfrak{c}}\mathfrak{c} J_{FP}^{\text{classical}} + \frac{\kappa}{4\pi\alpha'} \delta T (\mathfrak{v} + \vec{c} \cdot \vec{\eta})$) of (3.21) by subtracting off its preimage under Δ on the left hand side, namely

$$\xi_0 \equiv \frac{2}{\kappa} \left(\langle \delta \dot{X} | \dot{X}^{cl} \rangle - \frac{1}{T_0^2} \langle \delta X' | X'^{cl} \rangle \right); \quad \Delta \xi_0 = \bar{\mathfrak{c}}\mathfrak{c} J_{FP}^{\text{classical}} + \frac{\kappa}{4\pi\alpha'} \delta T (\mathfrak{v} + \vec{c} \cdot \vec{\eta}). \quad (3.23)$$

(The quantities $\vec{c}$ and $\vec{\eta}$ were introduced in equation (2.24)). Finally, we have

$$\begin{aligned} S_{\text{int}}(\delta T, \delta X) + F^{(k)}(\delta T) &= \left(\mathcal{T} - \frac{\kappa}{4\pi\alpha'} \delta T \xi_0 \right) + \bar{\mathfrak{c}}\mathfrak{c} \left(J_{FP} - J_{FP}^{cl} \right) + 2\pi i B_3 \frac{\kappa}{4\pi\alpha'} \delta T \cdot h(\delta T) \\ &= \Delta \psi \end{aligned} \quad (3.24)$$

where

$$\psi \equiv \bar{\mathfrak{c}} (\xi - \xi_0 - 2\pi i B_3 h(\delta T)), \quad (3.25)$$

and ξ , ξ_0 and $h(\delta T)$ were defined in (3.20), (3.23) and (3.7) respectively. Consequently,

$$\begin{aligned} \frac{\partial}{\partial \lambda} \Gamma(\lambda) &= \frac{2}{\omega V} \Im \sum_{k=1}^{\infty} \int [d(\delta X_\mu)] d(\delta T) dB d\mathfrak{c} d\bar{\mathfrak{c}} \frac{\eta^2 \left[i/2T_0^{(k)} \right]}{T_0^{(k)}} (\Delta \psi) e^{-S_{\text{classical}}^{(k)} - S_{\text{quad}}^{(k)} - \lambda \Delta \psi} \\ &= \frac{2}{\omega V} \Im \sum_{k=1}^{\infty} \int [d(\delta X_\mu)] d(\delta T) dB d\mathfrak{c} d\bar{\mathfrak{c}} \Delta \left\{ \frac{\eta^2 \left[i/2T_0^{(k)} \right]}{T_0^{(k)}} \psi e^{-S_{\text{classical}}^{(k)} - S_{\text{quad}}^{(k)} - \lambda \Delta \psi} \right\} \\ &= 0, \end{aligned} \quad (3.26)$$

where the last equality follows from the fact that we are integrating a total derivative. We have thus proven that the full, interacting path integral (1.5) is given exactly by its semi-classical approximation.

4 Discussion

In this paper we have studied the open bosonic string where the D-brane on which the string ends contains a constant electric field. We have examined the amplitude for string pair creation by tunnelling, the analog of the Schwinger effect for charged particle-antiparticle pairs in an electric field. The string theory tunnelling process is mediated by instantons of the string sigma model which computes the open string annulus amplitude. We have analyzed fluctuations about the classical multi-instanton solutions and we found that integrating the Gaussian fluctuations and summing over all possible multi-instanton configurations obtains the known formula for the amplitude. This can be regarded as another confirmation of that formula. We call the approximation which retains the classical instanton action and the determinants due to the quadratic fluctuations the WKB limit and our first result suggests that the WKB limit is exact. We have then fashioned a localization argument to prove that it is indeed exact. The cohomology used for the localization of the functional integral utilizes the Fadeev-Popov ghosts which arise from the introduction of a particular collective coordinate, but it differs in form from the usual BRST cohomology.

The fermionic symmetry which we find could be useful in computing correlation functions, for example, even in integrals which are not WKB exact, it could be used for reorganizing the integral to a more convenient form. We leave a detailed study of whether this can indeed be used to simplify perturbative computations of more complicated scenarios such as those with non-constant electric fields, to future study. The problem of string pair production in non-homogeneous background fields has, to our knowledge, only been addressed in a few works to date [23–25] and demands further exploration.

Finally let us point out some appealing connections with the literature. Under a T-duality transformation in the $\vec{E}$ direction, our Dp -brane setup is mapped to a pair of $D(p-1)$ -branes with constant relative velocity (but zero extension) in the $\vec{E}$ direction. This establishes a connection to questions of D-brane dynamics, see for example [26–28]. From this perspective, pair production occurs due to time dependence of the brane separation and, consequently, the open string spectrum. The critical electric field (1.4) manifests itself as a limiting velocity for the relativistic mechanics of D-branes, namely the speed of light.

Our work here may serve as a useful toy model for related calculations in curved space, in particular in the holographic context, for example involving the instanton fluctuation prefactor in the holographic Schwinger effect [29], and perhaps even the meson decay process analyzed in [30].¹² Another intriguing application is to pomeron physics. In [31], high energy, inelastic scattering of dipoles in holographic QCD is studied and is found to be well modeled in a certain “soft pomeron” regime by D0-brane scattering in flat space. This is in turn related — via the T-duality just discussed — to Schwinger pair production mediated by worldsheet instantons.

Lastly, it could be interesting to consider a hybrid particle-string version of the model (1.6), wherein one includes a particle-like kinetic term on the boundary in addi-

¹²In the latter, mesons at finite temperature are modeled by strings connected to a D7 “flavor” brane outside of a black hole in $AdS_5 \times S^5$. Dissociation of the mesons is mediated by worldsheet instantons, which allow for leaking of mesons into the black hole.

tion to the gauge field coupling and string bulk terms. On the one hand, such a model provides a kind of interpolation between the string and particle models considered in this section and the previous one (see appendix A of [32] where such a calculation is performed explicitly for the string disk worldsheet). On the other hand, it was shown in [33] by integrating out the free bulk degrees of freedom of the worldsheet that such a setup is equivalent to the dissipative quantum mechanics of Caldeira and Leggett [34, 35]. This permits one to study, for example, pair nucleation of particles in a dissipative setting [36].

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A A toy model

Let us consider the simple model with an ordinary integral where we want to compute the imaginary part. Consider the integral

$$Z = \Im \int_0^\infty \frac{1}{T} \int d^2 z e^{-(T-T_0)\bar{z}z - M^2/T} \quad (\text{A.1})$$

The integral measure and the integrand are real and positive, so the reader might wonder where it gets an imaginary part. To understand this, notice that, when T_0 is real and positive, the coefficient of the quadratic form $\bar{z}z$ in the integrand is not positive for all values of T , if $T < T_0$ it is negative and the z -integration would diverge. To define the integral, we first assume that T_0 is negative, perform the integration which is then well-defined, and obtain a function of T_0 which can be analytically continued to the complex T_0 plane. It is this analytic continuation which produces an imaginary part for the integral when T_0 is positive.

Explicitly, we first do the Gaussian integral over z to get

$$Z = \Im \int_0^\infty \frac{1}{T} \frac{\pi}{T - T_0 - i\varepsilon} e^{-M^2/T} = \frac{\pi^2}{T_0} e^{-M^2/T_0} \quad (\text{A.2})$$

Then, if T_0 is real and positive, we define the distribution in the integrand by taking T_0 to the real axis from the upper half-plane by replacing $\frac{\pi}{T-T_0}$ with $\frac{\pi}{T-T_0-i\varepsilon}$. We can then use the identity $\Im \frac{1}{T-T_0-i\varepsilon} = \pi \delta(T-T_0)$. This allows us to find the exact imaginary part

$$Z = \frac{\pi^2}{T_0} e^{-M^2/T_0} \quad (\text{A.3})$$

Alternatively, we could consider an approximate evaluation of the integral by saddle point technique. Such an approximation should be accurate when the “action”

$$S = (T - T_0)\bar{z}z + M^2/T \quad (\text{A.4})$$

is large, that is then $\frac{M^2}{T_0} \gg 1$. To implement the saddle-point technique, we consider the classical equations of motion,

$$\bar{z}z - M^2/T^2 = 0, \quad (T - T_0)z = 0 \quad (\text{A.5})$$

and we find the classical solutions $z = \frac{M}{T_0}$, $T = T_0$ and we expand the integration variables as the classical solutions plus “fluctuations”,

$$z = \frac{M}{T} + \delta z, \quad T = T_0 + \delta T \quad (\text{A.6})$$

Before we proceed, we notice that the action has a flat direction, it is invariant under the phase transformation $z \rightarrow e^{-i\theta}z$, $\bar{z} \rightarrow e^{i\theta}\bar{z}$. Such a symmetry will lead to a zero mode in the fluctuations about the classical solution. We must take care of this symmetry, and degeneracy of the solution by gauge fixing. Most convenient is the Fadeev-Popov trick of inserting unity into the integration measure in equation (A.1) in the following form,

$$1 = \frac{1}{2} \int_0^{2\pi} d\theta \delta \left(\frac{1}{2i} (e^{i\theta}\bar{z} - e^{-i\theta}z) \right) \left| \frac{d}{d\theta} \left(\frac{1}{2i} (e^{i\theta}\bar{z} - e^{-i\theta}z) \right) \right| \quad (\text{A.7})$$

The $\frac{1}{2}$ which appears in front of the right-hand-side arises from a Gribov copy — there are two solutions of the equation $e^{i\theta}\bar{z} - e^{-i\theta}z = 0$ in the interval $\theta \in [0, 2\pi)$. One then removes θ by a symmetry transformation. The upshot is the insertion of the following into the integral in equation (A.1),

$$\pi \delta(\Im z) |\Re z| = \int d\bar{c}dcdb e^{-\pi \Re z \bar{c}c - 2\pi i b \Im z}$$

where c and $\bar{c}$ are Fadeev-Popov ghosts. It now takes the form

$$Z = \Im \int [dT d^2z d\bar{c}dcdb] \frac{1}{T} e^{-(T-T_0)\bar{z}z - M^2/T - \pi \Re z \bar{c}c - 2\pi i b \Im z} \quad (\text{A.8})$$

Now, we expand the action about the classical solution (with the classical part of the ghosts vanishing). It becomes

$$S = S_0 + S_{\text{quad}} + S_{\text{int}} \quad (\text{A.9})$$

$$S_0 = \frac{M^2}{T_0} \quad (\text{A.10})$$

$$S_{\text{quad}} = \frac{M}{T_0} \delta T (\delta z + \delta \bar{z}) + \frac{M^2}{T_0^3} \delta T^2 + \frac{M}{T_0} \pi \bar{c}c + 2\pi i \Im z b \quad (\text{A.10})$$

$$S_{\text{int}} = \delta T \delta \bar{z} \delta z + \pi \Re z \delta \bar{c}c + \sum_{k=3}^{\infty} \frac{M^2}{T_0^{k+1}} (-\delta T)^k \quad (\text{A.11})$$

The classical part of the action $S_0 = \frac{M^2}{T_0}$ matches the exponent in equation (A.3). The corrections to this classical limit begin with dropping all terms of order higher than quadratic in the fluctuations, that is, dropping S_{int} , and doing the remaining Gaussian integral over the fluctuations. The resulting integral is complex, due to the fact that the determinant of bosonic quadratic form is negative

$$\det \begin{bmatrix} 0 & \frac{M}{T_0} \\ \frac{M}{T_0} & 2\frac{M^2}{T_0^3} \end{bmatrix} = - \left(\frac{M}{T_0} \right)^2 \quad (\text{A.12})$$

and the gaussian integration contributes the inverse of the square root of this determinant. This produces a factor of $i\pi\frac{T_0}{M}$ in the measure, where we have chosen an appropriate sign for $i = \sqrt{-1}$. The fermionic quadratic form contributes $\pi\frac{M}{T_0}$ so the net factor from integrating the fluctuations is $i\pi^2$. The remainder of the integrand is evaluated at the classical solution. The result of the WKB approximation to the integral in this “instanton” sector is then purely imaginary

$$Z = \Im \left\{ i \frac{\pi^2}{T_0} e^{-M^2/T_0} \right\}$$

and it appears to get the imaginary part of the integral, compare with (A.3), exactly. However, this should be an approximation. There are higher order than quadratic terms in the action, S_{int} as well as the factor $\frac{1}{T}$ in the measure, and there should be corrections to this result. Apparently, if the two computations are to match, such corrections must cancel.

To see how this can happen, we observe that the action in equations (A.9)–(A.11) has a fermionic symmetry under the transformations

$$\Delta \bar{c} = -\delta T, \quad \Delta \delta z = \Delta \delta \bar{z} = \frac{\pi}{2} c, \quad (\text{A.13})$$

$$\Delta c = \Delta \delta T = \Delta b = 0 \quad (\text{A.14})$$

This transformation is nilpotent, $\Delta^2(\text{anything}) = 0$ and also

$$\int d\bar{c}c \Delta(\text{anything even in } c, \bar{c}) = 0$$

since the integral of any quantity which is odd in the Grassman numbers must vanish.

It is easy to check that $\Delta S = 0$. As well, the measure factor, $\frac{1}{T} = \frac{1}{T_0} \frac{1}{1+\delta T/T_0}$, is invariant

$$\Delta \left[\frac{1}{T_0} \frac{1}{1+\delta T/T_0} \right] = 0$$

Moreover, the “interaction” terms in the action can be written as transformations of simple functions,

$$S_{\text{int}} = \delta T \delta \bar{z} \delta z + \pi \delta z \bar{c} c + \sum_{k=3}^{\infty} \frac{M^2}{T_0^{k+1}} (-\delta T)^k = \Delta \left(-\delta \bar{z} \delta z \bar{c} - \sum_{k=3}^{\infty} \frac{M^2}{T_0^{k+1}} (-\delta T)^{k-1} \bar{c} \right) \quad (\text{A.15})$$

In addition the measure factor

$$\frac{1}{T_0} \frac{1}{1+\frac{\delta T}{T_0}} = \frac{1}{T_0} + \Delta \left[\sum_{k=1}^{\infty} \bar{c} \frac{(-\delta T)^{k-1}}{T_0^k} \right]$$

Then,

$$\begin{aligned} Z &= \int [dT d^2 z d\bar{c} d c d b] \frac{1}{T_0} \frac{1}{1+\frac{\delta T}{T_0}} e^{-S_0 - S_{\text{quad}} - S_{\text{int}}} \\ &= \int [dT d^2 z d\bar{c} d c d b] \left[\frac{1}{T_0} e^{-S_0 - S_{\text{quad}}} + \Delta \{ \dots \} \right] = \int [dT d^2 z d\bar{c} d c d b] \frac{1}{T_0} e^{-S_0 - S_{\text{quad}}} \end{aligned} \quad (\text{A.16})$$

which tells us that the saddle point approximation of the integral is equal to the exact result in the “instanton sector”.

B Gaussian integral computation

We can evaluate the partition function with action (2.2) and open string boundary conditions in two different ways. In section 2 we considered the semi-classical limit and integrated both T and the coordinates using a saddlepoint technique. In this appendix we shall do the Gaussian functional integral over the string embedding coordinates (z, x_a, x_A) and then the integral over the modular parameter of the cylinder, T . By picking up the residues at an infinite sequence of poles we shall reproduce the formula (2.91) for the imaginary part of the vacuum energy [2].

For this computation, we introduce the mode expansion

$$z(\sigma, \tau) = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} e^{2\pi i n \tau} \sqrt{2} \cos m\pi\sigma z_{nm} + \sum_{n=-\infty}^{\infty} e^{2\pi i n \tau} z_{n0} \quad (\text{B.1})$$

$$x_a(\sigma, \tau) = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} e^{2\pi i n \tau} \sqrt{2} \cos m\pi\sigma x_{mna} + \sum_{n=-\infty}^{\infty} e^{2\pi i n \tau} x_{an0} \quad (\text{B.2})$$

$$x_A(\sigma, \tau) = \sigma \vec{d} + \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} e^{2\pi i n \tau} \sqrt{2} \sin m\pi\sigma x_{Anm} \quad (\text{B.3})$$

where we have used Neumann boundary conditions for x_a and Dirichlet boundary conditions for x_A . We will also use Neumann boundary conditions for the normal modes in which we expand $z(\sigma, \tau)$. These are not the correct boundary conditions for z . The result of using these boundary conditions is that the quadratic form which we will obtain is not diagonal.

When we plug the mode expansions into the action, it becomes

$$\begin{aligned} S = & \frac{d^2}{4\pi\alpha'T} \\ & + \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \frac{[(2\pi n)^2 T^2 + (\pi m)^2]}{4\pi\alpha'T} |x_{anm}|^2 + \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{[(2\pi n)^2 T^2 + (\pi m)^2]}{4\pi\alpha'T} |x_{Anm}|^2 \\ & + \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \frac{[(2\pi n)^2 T^2 + (\pi m)^2]}{2\pi\alpha'T} |z_{nm}|^2 - \sum_{n=-\infty}^{\infty} (2\pi n E) \left| z_{n0} + \sum_{m=1}^{\infty} \sqrt{2} z_{nm} \right|^2 \end{aligned} \quad (\text{B.4})$$

The path integral measure is

$$\begin{aligned} \int_0^\infty \frac{dT}{2T} \int [dX^\mu(\sigma, \tau)] [\text{ghost}] = & V \int_0^\infty \frac{dT}{2T} \prod_{n=-\infty}^{\infty} \prod_{m=1}^{\infty} dz_{nm} d\bar{z}_{nm} dx_{anm} dx_{Anm} \\ & \cdot \prod_{n=-\infty, n \neq 0}^{\infty} dz_{n0} d\bar{z}_{n0} dx_{an0} \det' \left[-\frac{1}{T} \partial_\sigma^2 - T \partial_\tau^2 \right] \end{aligned} \quad (\text{B.5})$$

where we have included the ghost determinant and the prime on the ghost determinant indicates that simultaneous zero mode of ∂_τ and ∂_σ is omitted. The factor of the D-brane worldvolume V is from the integral over similar zero modes of x_a and z , namely x_{a00} and z_{00} .

The quadratic form in the last line of the action in equation (B.4) is a non-diagonal matrix. This is not surprising, as we have expanded in modes which do not obey the correct

boundary condition. To proceed, we will have to find the eigenvalues of that matrix. We can make this slightly easier to deal with by rescaling the modes by

$$z_{nm} \rightarrow \sqrt{\frac{2\pi\alpha'T}{(2\pi n)^2 T^2 + (\pi m)^2}} z_{nm}, \quad \bar{z}_{nm} \rightarrow \sqrt{\frac{2\pi\alpha'T}{(2\pi n)^2 T^2 + (\pi m)^2}} \bar{z}_{nm} \quad (\text{B.6})$$

The resulting action is

$$\begin{aligned} S = & \frac{\vec{d}^2}{4\pi\alpha'T} + \sum_{m=0}^{\infty} \sum_{n \neq 0} \bar{z}_{nm} [\delta_{mm'} - \mathcal{E}_{mm'}] z_{m'n} + \sum_{m=1}^{\infty} |z_{m0}|^2 \\ & + \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \frac{[(2\pi n)^2 T^2 + (\pi m)^2]}{4\pi\alpha'T} |x_{anm}|^2 + \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{[(2\pi n)^2 T^2 + (\pi m)^2]}{4\pi\alpha'T} |x_{Anm}|^2 \end{aligned} \quad (\text{B.7})$$

where we define the matrix

$$\begin{bmatrix} \mathcal{E}_{00}(n) & \mathcal{E}_{0m'}(n) \\ \mathcal{E}_{m0}(n) & \mathcal{E}_{mm'}(n) \end{bmatrix} = 2\pi n E \begin{bmatrix} g(n, 0) & \sqrt{2g(n, 0)g(n, m')} \\ \sqrt{2g(n, m)g(n, 0)} & 2\sqrt{g(n, m)g(n, m')} \end{bmatrix}, \quad (\text{B.8})$$

$$g(n, m) = \frac{2\pi\alpha'T}{(2\pi n)^2 T^2 + (\pi m)^2}. \quad (\text{B.9})$$

Taking into account the Jacobian in the measure resulting from this rescaling, and then doing the Gaussian integrals over the coordinates will result in the appearance of the determinants in the integrand:

$$\begin{aligned} Z = & V \int_0^{\infty} \frac{dT}{2T} e^{-\frac{\vec{d}^2}{4\pi\alpha'T}} \prod_{n=-\infty}^{\infty} \prod_{m=1}^{\infty} (2\pi)^{12} \left[\frac{[(2\pi n)^2 T^2 + (\pi m)^2]}{2\pi\alpha'T} \right]^{-12} \\ & \cdot \prod_{n \neq 0} (2\pi)^{\frac{D-1}{2}} \left[\frac{(2\pi n)^2 T}{2\pi\alpha'} \right]^{-\frac{D+1}{2}} \cdot \frac{1}{2\pi} \prod_{n \neq 0} [\det [\delta_{mm'} - \mathcal{E}_{mm'}]]^{-1} \end{aligned} \quad (\text{B.10})$$

where the infinite products

$$\prod_{n=-\infty}^{\infty} \prod_{m=1}^{\infty} (2\pi)^{12} \left[\frac{[(2\pi n)^2 T^2 + (\pi m)^2]}{2\pi\alpha'T} \right]^{-12}$$

which appear in the first line are the determinants arising from integrating x_{anm} , a factor from scaling $z_{nm}, \bar{z}_{nm}$ and the ghost determinant. The determinant in the second line is from the integral over z_{nm} and $\bar{z}_{nm}$. When infinite products diverge, we use ζ -function regularization to define them (see appendix F. For example, in finding the determinant in the second line of (B.10), we encounter a product over all of the modes of the factor 2π ,

$$\begin{aligned} \prod_{mn} (2\pi) &= \left(\prod_{m \geq 1} 2\pi \right) \left(\prod_{n \neq 0} \prod_{m \geq 0} 2\pi \right) = (2\pi)^{\zeta(0)} \left(\prod_{n \neq 0} (2\pi) \cdot (2\pi)^{\zeta(0)} \right) = \frac{1}{\sqrt{2\pi}} \left(\prod_{n \neq 0} \sqrt{2\pi} \right) \\ &= \frac{1}{\sqrt{2\pi}} \left(\prod_{n \geq 1} 2\pi \right) = \frac{1}{\sqrt{2\pi}} (2\pi)^{\zeta(0)} = \frac{1}{2\pi}, \end{aligned} \quad (\text{B.11})$$

which is the factor in front of the determinant.

The infinite products in (B.10) can be put in a more convenient form. Using the results of appendix F we have that the infinite product in the first line of (B.10) reduces to the usual modular form

$$\prod_{m=1}^{\infty} \prod_{n=-\infty}^{\infty} \left[\frac{[(2\pi n)^2 T^2 + (\pi m)^2]}{2\pi \alpha' T} \right]^{-12} = \left[e^{-\pi/12T} \prod_{m=1}^{\infty} [1 - e^{-\pi m/T}]^2 \right]^{-12} = \eta^{-24}(i/2T) \quad (\text{B.12})$$

where $\eta(\tau)$ is the Dedekind eta-function (see appendix A). In addition, using zeta-function regularization,

$$\prod_{n=-\infty, n \neq 0}^{\infty} (2\pi)^{\frac{D-1}{2}} \left[\frac{(2\pi n)^2 T}{2\pi \alpha'} \right]^{-\frac{D+1}{2}} = 2\pi \prod_{n=1}^{\infty} \left[\frac{(2\pi n)^2 T}{4\pi^2 \alpha'} \right]^{-(D+1)} = 2\pi \left[\frac{T}{4\pi^2 \alpha'} \right]^{\frac{D+1}{2}}.$$

We are left with the last determinant in the second line of (B.10). Observe that the matrix $\mathcal{E}$ defined in (B.8) is the outer product of a vector and its transpose:

$$\mathcal{E} = \tanh(\pi\epsilon) \coth(2\pi n T) v_m v_{m'}, \quad (\text{B.13})$$

where the normalized vector $\vec{v}$ is given by

$$v_m = \sqrt{\frac{n}{\alpha'} \tanh(2\pi n T) (\sqrt{2})^{1-\delta_{m0}}} \sqrt{g(n, m)}. \quad (\text{B.14})$$

It therefore has only one non-zero eigenvalue, namely

$$2\pi \alpha' E \coth(2\pi n T) = \tanh(\pi\epsilon) \coth(2\pi n T)$$

Inserting this result into the partition function, and after transforming the integration variable $T \rightarrow 1/2T$, we find the expression

$$Z = V \int_0^{\infty} \frac{dT}{T} \frac{e^{-Td^2/2\pi\alpha'} \eta^{-24}(iT) [8\pi^2 \alpha' T]^{-\frac{D+1}{2}}}{\prod_{n=1}^{\infty} [1 - (2\pi \alpha' E)^2 \coth^2(\pi n/T)]} \quad (\text{B.15})$$

When $E \rightarrow 0$ this expression approaches the usual cylinder amplitude of the open bosonic string suspended between two D-branes. Moreover, the integrand now has poles at discrete values of T ,

$$T_k = \frac{k}{\epsilon} \quad (k \in \mathbb{Z}^+), \quad \tanh \pi\epsilon \equiv 2\pi \alpha' E \quad (\text{B.16})$$

To proceed, it is necessary to find the residues of the poles. For this purpose, it is convenient to perform a modular transformation of the infinite product. In order to do this, it is convenient to first write the infinite product as a product representation of Jacobi theta functions¹³ and then to use the known modular transformation property of the theta function. For this purpose, we use the following sequence of manipulations,

$$\begin{aligned} \prod_{n=1}^{\infty} \frac{1}{[1 - \tanh^2 \pi\epsilon \coth^2(\pi n/T)]} &= \prod_{n=1}^{\infty} \frac{\cosh^2 \pi\epsilon \sinh^2(\pi n/T)}{[\sinh(\pi n/T - \pi\epsilon) \sinh(\pi n/T + \pi\epsilon)]} \\ &= \frac{1}{\cosh \pi\epsilon} \prod_{n=1}^{\infty} \frac{[1 - e^{-2\pi n/T}]^3}{[1 - e^{-2\pi n/T}][1 - e^{-2(\pi n/T - \pi\epsilon)}][1 - e^{-2(\pi n/T + \pi\epsilon)}]} = \frac{2}{i \coth \pi\epsilon} \frac{\eta^3(i/T)}{\Theta_{11}(i\epsilon|i/T)} \end{aligned}$$

¹³See appendix A for a definition of the relevant theta function and its modular transformation property.

Using the modular transformation of the theta- and eta-functions, we find

$$\begin{aligned} \prod_{n=1}^{\infty} \frac{1}{[1 - (2\pi\alpha'E)^2 \coth^2(\pi n/T)]} &= 2T \tanh \pi\epsilon \cdot e^{-\pi\epsilon^2 T} \frac{\eta^3(iT)}{\Theta_{11}(-\epsilon T|iT)} \\ &= \tanh \pi\epsilon \frac{T e^{-\pi\epsilon^2 T}}{\sin(\pi\epsilon T)} \prod_{n=1}^{\infty} \frac{[1 - e^{-2\pi T n}]^2}{(1 - e^{-2T\pi(n+i\epsilon)})(1 - e^{-2T\pi(n-i\epsilon)})} \end{aligned} \quad (\text{B.17})$$

Now, the factors in the infinite product are regular for nonzero real values of T . The poles on the real T -axis originate from the factor of the inverse of $\sin(\pi\epsilon T)$ which is outside of the infinite product. The poles occur at $T = k/\epsilon$ and, at each pole, the residue is

$$(-1)^k k e^{-\pi k \epsilon} \frac{\tanh \pi \epsilon}{\pi \epsilon^2}$$

Note that the remaining infinite product is simply equal to one at the position of the pole. The imaginary part of the partition function is then given by a sum over (half-) residues at the poles,

$$Z = \Re(Z) + V \frac{1}{2} 2\pi i \sum_{k=1}^{\infty} \frac{1}{2T_k} (-1)^k k e^{-\pi k \epsilon} \frac{\tanh \pi \epsilon}{\pi \epsilon^2} \frac{e^{-k\bar{d}^2/2\pi\alpha'\epsilon}}{\eta^{24}(ik/\epsilon)} [8\pi^2\alpha'k/\epsilon]^{-\frac{D+1}{2}} \quad (\text{B.18})$$

so that the rate of pair production is

$$\Gamma = \frac{2}{V} \Im(Z) = \frac{\tanh \pi \epsilon}{\epsilon} \sum_{k=1}^{\infty} \frac{(-1)^{k+1} e^{-\pi k M^2 \frac{2\alpha'}{\epsilon} - k\pi\epsilon}}{\eta^{24}(ik/\epsilon)} \left[\frac{\epsilon/2\alpha'}{4\pi^2 k} \right]^{\frac{D+1}{2}}. \quad (\text{B.19})$$

C Fluctuation prefactor from explicit mode expansion

In section 2 we evaluated the quadratic fluctuation prefactor using the Gelfand-Yaglom approach for functional determinants. In this appendix we present, for completeness, a “brute-force” calculation of the same fluctuation integral using an explicit mode expansion. Since the eigenvalues of the $\mu = 0, 1$ fluctuation operator are determined by a transcendental equation, and therefore not known explicitly, we use modes for δz which do not obey the correct boundary condition, but would be appropriate to the same problem with Neumann boundary conditions. This yields a non-diagonal quadratic form. With a little work, and some cavalier manipulations of infinite matrices, we are able to find its determinant, as well as extract the zero- and tachyonic modes, to obtain finally the result (1.2).

Returning therefore to equation (2.15), we now expand the fluctuations in modes as

$$\delta z(\sigma, \tau) = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} e^{2\pi i n \tau} \sqrt{2} \cos \pi m \sigma \delta z_{nm} + \sum_{n=-\infty}^{\infty} e^{2\pi i n \tau} \delta z_{n0} \quad (\text{C.1})$$

$$\delta x_a(\sigma, \tau) = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} e^{2\pi i n \tau} \sqrt{2} \cos \pi m \sigma \delta x_{anm} + \sum_{n=-\infty}^{\infty} e^{2\pi i n \tau} \delta x_{an0} \quad (\text{C.2})$$

$$\delta x_A(\sigma, \tau) = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{\infty} e^{2\pi i n \tau} \sqrt{2} \sin \pi m \sigma \delta x_{Anm} \quad (\text{C.3})$$

Even though δz has the wrong boundary conditions, the linear terms in δz vanish due to the fact that the classical solution obeys the equation of motion and it has the correct boundary conditions. Using this mode expansion and the equations of motion the action becomes

$$\begin{aligned}
 S = & km_0^2 \frac{2\pi\alpha'}{\varepsilon} + a\delta T^2 + \frac{d}{2\pi\alpha'T_0^2} \sinh(\pi\epsilon)\delta T \sum_{m=0}^{\infty} q_m \frac{(\delta z_{km} + \delta \bar{z}_{km})}{\sqrt{2}} \\
 & + \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \frac{|\delta z_{nm}|^2}{2\pi\alpha'T_0} [T_0^2(2\pi n)^2 + (\pi m)^2] - \sum_{n=-\infty}^{\infty} 2\pi n E \left| \delta z_{n0} + \sum_{m=1}^{\infty} \sqrt{2} \delta z_{nm} \right|^2 \\
 & + \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \frac{|\delta x_{anm}|^2}{4\pi\alpha'T_0} [T_0^2(2\pi n)^2 + (\pi m)^2] \\
 & + \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{|\delta x_{Anm}|^2}{4\pi\alpha'T_0} [T_0^2(2\pi n)^2 + (\pi m)^2] + \dots
 \end{aligned} \tag{C.4}$$

where

$$q_m \equiv (\sqrt{2})^{1-\delta_{m0}} \frac{(\pi\epsilon)^2 - \pi^2 m^2}{(\pi\epsilon)^2 + \pi^2 m^2}, \quad a \equiv \frac{d^2}{2\pi\alpha'} \frac{1}{4T_0^3} \left(1 + \frac{\cosh(\pi\epsilon) \sinh(\pi\epsilon)}{\pi\epsilon} \right) \tag{C.5}$$

Transverse fluctuations and ghosts. Performing the Gaussian integral over the coordinates $\delta x_{a,A}$ yields the following factors in the path integral measure, for each of the modes with $m \neq 0$

$$\prod_{n=-\infty}^{\infty} \prod_{m=1}^{\infty} \left[\frac{2\pi\alpha'T_0}{(2\pi n)^2 T_0^2 + (\pi m)^2} \right]^{12} = \eta^{-24}(i/2T_0) \tag{C.6}$$

where $\eta(z)$ is the Dedekind eta-function. For the modes with $m = 0$

$$\begin{aligned}
 \prod_{n \neq 0} (2\pi)^{\frac{D-1}{2}} \left[\frac{2\pi\alpha'}{(2\pi n)^2 T_0} \right]^{\frac{D-1}{2}} &= \prod_{n > 0} \left[\frac{4\pi^2 \alpha'}{(2\pi n)^2 T_0} \right]^{D-1} \\
 &= \left[\frac{4\pi^2 \alpha'}{(2\pi)^2 T_0} \right]^{(D-1)\zeta(0)} e^{2(D-1)\zeta'(0)} = \left[\frac{T_0}{4\pi^2 \alpha'} \right]^{\frac{D-1}{2}}.
 \end{aligned} \tag{C.7}$$

where we have used zeta-function regularization to define the formally divergent infinite product (see appendix F). Evaluating the ghost determinant (see discussion of reparametrization ghosts in section 2) yields the factor

$$\det \left[-\frac{1}{T_0} \partial_\sigma^2 - T_0 \partial_\tau^2 \right] = \prod_{n=-\infty}^{\infty} \prod_{m=1}^{\infty} \frac{(2\pi n)^2 T_0^2 + (\pi m)^2}{2\pi\alpha'T_0} = \eta^{+2}(i/2T_0) \tag{C.8}$$

Lightcone-coordinate fluctuations. Consider the quadratic form containing the variables δz_{nm} . In order to evaluate its determinant, we shall have to find its eigenvalues. In order to define its eigenvalues it is convenient to rescale all δz_{nm} where either m or n is nonzero as

$$\delta z_{nm} \rightarrow \sqrt{g(n, m)} \delta z_{nm}, \quad \text{with} \quad g(n, m) = \frac{2\pi\alpha'T_0}{T_0^2(2\pi n)^2 + (\pi m)^2} \tag{C.9}$$

This rescaling results in a Jacobian in the measure, (where the second product arises from the $m = 0$ modes)

$$\begin{aligned}
 & \prod_{n=-\infty}^{\infty} \prod_{m=1}^{\infty} \left[\frac{2\pi\alpha'T_0}{T_0^2(2\pi n)^2 + (\pi m)^2} \right] \prod_{n=1}^{\infty} \left[\frac{2\pi\alpha'T_0}{T_0^2(2\pi n)^2} \right]^2 \\
 &= \left\{ \prod_{n=-\infty}^{\infty} \prod_{m=1}^{\infty} \left[\frac{2\pi\alpha'T_0}{T_0^2(2\pi n)^2 + (\pi m)^2} \right] \right\} \left[\frac{2\pi\alpha'T_0}{T_0^2(2\pi)^2} \right]^{2\zeta(0)} e^{4\zeta'(0)} \\
 &= \frac{2\pi}{\eta^2(i/2T_0)} \frac{T_0}{4\pi^2\alpha'}
 \end{aligned} \tag{C.10}$$

This leaves the following quadratic form in the action

$$\begin{aligned}
 S_{\text{quadratic}} &= a\delta T^2 + \frac{d}{2\pi\alpha'T_0^2} \sinh(\pi\epsilon)\delta T \sum_{m=0}^{\infty} q_m \sqrt{g(k, m)} \frac{(\delta z_{km} + \delta \bar{z}_{km})}{\sqrt{2}} \\
 &+ \sum_{m=0}^{\infty} \sum_{n \neq 0} \delta \bar{z}_{nm} [\delta_{mm'} - \mathcal{E}_{mm'}(n)] \delta z_{nm'} + \sum_{m=1}^{\infty} |\delta z_{0m}|^2 + \dots
 \end{aligned} \tag{C.11}$$

where

$$\begin{bmatrix} \mathcal{E}_{00}(n) & \mathcal{E}_{0m'}(n) \\ \mathcal{E}_{m0}(n) & \mathcal{E}_{mm'}(n) \end{bmatrix} = 2\pi n E \begin{bmatrix} g(n, 0) & \sqrt{2g(n, 0)g(n, m')} \\ \sqrt{2g(n, m)g(n, 0)} & 2\sqrt{g(n, m)g(n, m')} \end{bmatrix}, \quad m, m' = 0, 1, 2, \dots$$

$$\tag{C.12}$$

$$= \tanh(\pi\epsilon) \coth(\pi\epsilon n/k) v_m v_{m'} \tag{C.13}$$

This is just the matrix in (B.8), evaluated at $T = T_0$. Consequently the quadratic form $(\mathcal{I} - \mathcal{E}(n))$ has eigenvalues 1 and $1 - \tanh(\pi\epsilon) \coth(\pi\epsilon n/k)$. When $n = k$ the latter is zero. The integration of the modes δz_{nm} with $n = k$ excluded produces the factor

$$\begin{aligned}
 \prod_{n \neq 0, k} \frac{1}{\det(\mathcal{I} - \mathcal{E}(n))} &= \frac{1}{\prod_{n \neq 0, k} [1 - \tanh(\pi\epsilon) \coth(\pi\epsilon n/k)]} \\
 &= \frac{(-1)^{k+1} e^{-\pi k \epsilon}}{\cosh^2 \pi \epsilon}
 \end{aligned} \tag{C.14}$$

which is derived in appendix F, equation (F.6).

Tachyon: real $n = k$ modes coupled to δT . Now consider $n = k$, for which $(\mathcal{I} - \mathcal{E})$ has a zero eigenvalue. Note that δT couples only to the real part of δz_k . More precisely, we have that $\frac{\delta z_{km} + \delta \bar{z}_{km}}{\sqrt{2}}$ and δT are coupled in the following quadratic form:

$$\mathcal{M} = \begin{pmatrix} a & \vec{J}^\top \\ \vec{J} & \mathcal{I} - \mathcal{E}_k \end{pmatrix} \tag{C.15}$$

where the δT , δz cross-term is

$$J_m = \frac{d \sinh(\pi\epsilon)}{4\pi\alpha'T_0^2} q_m \sqrt{g(k, m)} \tag{C.16}$$

and q_m is given by (C.5). We found the spectrum of the submatrix $(\mathcal{I} - \mathcal{E})$ above. In the gaussian integration over the quadratic form $\mathcal{M}$, we will find that although δT couples to all eigenmodes of $(\mathcal{I} - \mathcal{E})$, only its coupling to the would-be zero mode $\mathbf{v}$ contributes to the Gaussian integral. We demonstrate this as follows. $(\mathcal{I} - \mathcal{E})$ is diagonalized by its matrix of orthonormal eigenvectors

$$\begin{aligned}\Lambda &\equiv \mathcal{S}^\top (\mathcal{I} - \mathcal{E}) \mathcal{S} \\ &= \text{diag}[0, 1, 1, 1 \dots],\end{aligned}\tag{C.17}$$

$$\mathcal{S} = (\vec{v}, \vec{u}_1, \vec{u}_2, \dots)\tag{C.18}$$

$\vec{v}$ is defined in (B.14), and we will not need the precise form of the $\vec{u}$'s. Under this change of variables, whose jacobian is 1, $\mathcal{M}$ becomes

$$\begin{pmatrix} 1 & \\ & \mathcal{S}^\top \end{pmatrix} \mathcal{M} \begin{pmatrix} 1 & \\ & \mathcal{S} \end{pmatrix} = \begin{pmatrix} a & \vec{J} \cdot \vec{v} & \vec{J} \cdot \vec{u}_1 & \vec{J} \cdot \vec{u}_2 & \dots \\ \vec{J} \cdot \vec{v} & 0 & & & \\ \vec{J} \cdot \vec{u}_1 & & \lambda_1 & & \\ \vec{J} \cdot \vec{u}_2 & & & \lambda_2 & \\ \vdots & & & & \ddots \end{pmatrix}\tag{C.19}$$

We denote our new integration variables $\mathbf{y}_{m'} \equiv (\mathcal{S}^\top)_{m'm} \frac{\delta z_{km} + \delta \bar{z}_{km}}{\sqrt{2}}$, and for the tachyonic mode, $y_0 \equiv \mathbf{v} = \sum_{m=0}^{\infty} v_m \frac{\delta z_{km} + \delta \bar{z}_{km}}{\sqrt{2}}$. We can now write the quadratic form as follows

$$(\delta T, \vec{y}^\top) \mathcal{M} \begin{pmatrix} \delta T \\ \vec{y} \end{pmatrix} = \vec{y}^\top \Lambda \vec{y} + 2\delta T \sum_{i=1}^{\infty} y^i \vec{J} \cdot \vec{u}_i + (\delta T, v_+) \begin{pmatrix} a & \vec{J} \cdot \vec{v} \\ \vec{J} \cdot \vec{v} & 0 \end{pmatrix} \begin{pmatrix} \delta T \\ v_+ \end{pmatrix}\tag{C.20}$$

The first two terms on the r.h.s. are independent of y_0 . Thus completing the square in $\mathbf{y}_i$ ($i > 0$) to eliminate the second term only has the effect of modifying the coefficient a of δT^2 appearing in the last term. But due to the form of the latter, the determinant obtained after integrating out δT , y_0 is independent of a .

Therefore the total contribution from integrating out $Re[\delta z_{km}]$ and δT is

$$\begin{aligned}\det' (\mathcal{I} - \mathcal{E}(k))^{-1/2} \det \left[\begin{pmatrix} a & \vec{J} \cdot \vec{v} \\ \vec{J} \cdot \vec{v} & 0 \end{pmatrix} \right]^{-1/2} &= (\prod' 1) \det \left[\begin{pmatrix} a & \vec{J} \cdot \vec{v} \\ \vec{J} \cdot \vec{v} & 0 \end{pmatrix} \right]^{-1/2} \\ &= \pm i \left| \vec{J} \cdot \vec{v} \right|^{-1}\end{aligned}\tag{C.21}$$

$$= \pm i \sqrt{\frac{4\pi\alpha' T_0^3 \sinh(2\pi\epsilon)}{d^2\pi\epsilon}}\tag{C.22}$$

where the prime means we exclude the zero mode. The square root of a negative determinant gives rise to a factor i . This “tachyonic” mode corresponds to fluctuations in the radius of the instanton, with respect to which it is unstable.

Imaginary $n = k$ modes & zero mode. Returning to (C.11), we note that $Im[\delta z]$ does not couple to δT , so in this case the zero eigenvalue of $(\mathcal{I} - \mathcal{E}(k))$ corresponds to a

genuine zero-mode of the quadratic action:

$$\mathfrak{z} \equiv \sum_{m=0}^{\infty} v_m \frac{\delta z_{km} - \delta \bar{z}_{km}}{\sqrt{2}i} \quad (\text{C.23})$$

The gauge-fixing procedure is described in subsection 2.2.2. The net result is the removal of the zero eigenvalue, and the introduction of a compensating Faddeev-Popov jacobian,

$$\iint d\sigma d\tau \left[\dot{\bar{z}}(\sigma, \tau) \dot{z}^{cl}(\sigma, \tau) + \text{c.c.} \right] = \|\dot{z}^{cl}\|^2 + \iint d\sigma d\tau \left[\delta \dot{\bar{z}} \dot{z}^{cl}(\sigma, \tau) + \text{c.c.} \right] \quad (\text{C.24})$$

The first term evaluates to

$$2\pi k \frac{d}{\sqrt{2}\pi\epsilon} \sqrt{\frac{2\pi\epsilon + \sinh(2\pi\epsilon)}{4\pi\epsilon}},$$

while the latter term, which is the projection of δz onto the tachyonic mode z^{cl} , is a subleading correction and is to be dropped in the semiclassical approximation.

It is important here to note the following subtlety, that the rescaling of δz_{nm} (equation (C.9)) does not commute with the introduction of our collective coordinate. To account for this, we regulate the determinant, divide out by the (putative) zero eigenvalue, and then take the limit of the regulator going to zero.

We want the determinant, with zero-eigenvalue excluded, of the operator

$$\hat{L} \equiv \frac{1}{2\pi\alpha'T_0} (-\partial_\sigma^2 + (\pi\epsilon)^2)$$

with the appropriate boundary conditions. A possible regularization is

$$\hat{L}^{(\delta)} = \frac{1}{2\pi\alpha'T_0} (-\partial_\sigma^2 + (\pi\epsilon(1+\delta))^2)$$

Then the zero eigenvalue gets shifted to

$$\lambda_0^{(\delta)} = \frac{2\pi k\epsilon}{\alpha'} \delta + \mathcal{O}(\delta^2)$$

and $g(n, m)$ (equation (C.9)) gets modified accordingly. Now rescale as before, $\delta z_{km} \rightarrow \sqrt{g_{km}^{(\delta)}} \delta z_{km}$. The corresponding jacobian J_g has an $\mathcal{O}(\delta)$ correction which will not be important. The resulting quadratic form is now $(\mathcal{I} - \mathcal{E}^{(\delta)}(k))$, where $\mathcal{E}^{(\delta)}$ is defined as in (C.12) but with $g(n, m)$ replaced everywhere by $g^{(\delta)}(n, m)$. This has the same structure as before, with only the pseudo-zero eigenvalue $\tilde{\lambda}_0$ modified to

$$\begin{aligned} \tilde{\lambda}_0^{(\delta)} &= 1 - 2\pi k E \left[g_{k0}^{(\delta)} + 2 \sum_{m=1}^{\infty} g_{km}^{(\delta)} \right] = 1 - \frac{\tanh(\pi\epsilon) \coth(\pi\epsilon(1+\delta))}{(1+\delta)} \\ &= (1 + 2\pi\epsilon \operatorname{csch}(2\pi\epsilon)) \cdot \delta + \mathcal{O}(\delta^2) \end{aligned} \quad (\text{C.25})$$

The determinant is then

$$(\det' \hat{L})^{-1/2} = \lim_{\delta \rightarrow 0} \sqrt{\frac{\lambda_0^{(\delta)}}{\det \hat{L}^{(\delta)}}} = \lim_{\delta \rightarrow 0} J_g^{(\delta)} \sqrt{\lambda_0^{(\delta)} / \tilde{\lambda}_0^{(\delta)}} = J_g \left[\frac{2\pi k\epsilon}{\alpha'} \frac{\sinh(2\pi\epsilon)}{2\pi\epsilon + \sinh(2\pi\epsilon)} \right]^{1/2} \quad (\text{C.26})$$

The factor J_g was already accounted in (C.10), so the net contribution obtained here for the $n = k$ imaginary modes (including the Gribov factor $\omega^{-1} = 1/2k$) is

$$\left[\frac{2\pi k \epsilon}{\alpha'} \frac{\sinh(2\pi\epsilon)}{2\pi\epsilon + \sinh(2\pi\epsilon)} \right]^{1/2} \cdot \frac{1}{2k} \cdot \frac{2\pi k d}{\pi\epsilon} \sqrt{\frac{2\pi\epsilon + \sinh(2\pi\epsilon)}{4\pi\epsilon}} = \frac{d k \sinh(\pi\epsilon)}{\epsilon \sqrt{k\alpha' \tanh(\pi\epsilon)}} \quad (\text{C.27})$$

Final result. Gathering all of the factors, including the product $\prod_{nm}(2\pi)$ evaluated in (B.11), we obtain for the tunnelling amplitude¹⁴

$$\Gamma_{\text{string}} = \frac{2}{V} \Im(Z) = \pm i \frac{\tanh \pi\epsilon}{\epsilon} \sum_{k=1}^{\infty} \frac{(-1)^{k+1} e^{-\frac{2\pi\alpha' k M^2}{\epsilon} - k\pi\epsilon}}{\eta^{24}(ik/\epsilon)} \left[\frac{\epsilon/2\alpha'}{4\pi^2 k} \right]^{\frac{D+1}{2}} \quad (\text{C.29})$$

in agreement with the Gelfand-Yaglom calculation of section 2.

D The Dedekind eta and Jacobi theta functions

The Dedekind eta function is defined by

$$\eta(\tau) = e^{\pi i \tau / 12} \prod_{k=1}^{\infty} (1 - e^{2\pi i k \tau}) \quad (\text{D.1})$$

It has the property that

$$\eta(-1/\tau) = \sqrt{-i\tau} \eta(\tau) \quad (\text{D.2})$$

We shall also make use of the Jacobi theta function

$$\Theta_{11}(\nu|\tau) = -2e^{i\frac{\pi\tau}{4}} \sin \pi\nu \prod_{m=1}^{\infty} (1 - e^{2\pi i m \tau})(1 - e^{2\pi i(m\tau + \nu)})(1 - e^{2\pi i(m\tau - \nu)}) \quad (\text{D.3})$$

This theta function has the modular transformations

$$\Theta_{11}(\nu|\tau + 1) = e^{i\frac{\pi}{4}} \Theta_{11}(\nu|\tau) \quad (\text{D.4})$$

$$\Theta_{11}(\nu/\tau, -1/\tau) = -i(-i\tau)^{\frac{1}{2}} e^{\pi i \nu^2 / \tau} \Theta_{11}(\nu|\tau) \quad (\text{D.5})$$

The definition, further properties and uses of the eta- and theta-functions can be found in string theory textbooks. Here, we use the notational conventions of Polchinski [14].

¹⁴In detail:

$$\begin{aligned} \frac{2}{V} \Im(Z) = & 2 \sum_{k=1}^{\infty} e^{-\frac{k d^2}{2\pi\alpha'\epsilon}} \cdot \underbrace{\frac{1}{2T_0}}_{\text{measure}} \cdot \underbrace{\eta^{-24}(i/2T_0)}_{\delta x_{a/A}^{m>0}} \cdot \underbrace{\left[\frac{T_0}{4\pi^2\alpha'} \right]^{\frac{D-1}{2}}}_{\delta x_{a/A}^{m=0}} \cdot \underbrace{\eta^{+2}(i/2T_0)}_{\text{ghosts}} \cdot \underbrace{\frac{2\pi}{\eta^2(i/2T_0)} \frac{T_0}{4\pi^2\alpha'}}_{\delta z \text{ rescaling}} \\ & \cdot \underbrace{\frac{(-1)^{k+1} e^{-\pi k \epsilon}}{\cosh^2 \pi \epsilon}}_{\delta z_{n \neq k}} \cdot \underbrace{(\pm i) \sqrt{\frac{4\pi\alpha' T_0^3 \sinh(2\pi\epsilon)}{d^2 \pi \epsilon}}}_{\text{tachyon}} \cdot \underbrace{\frac{d k \sinh(\pi\epsilon)}{\epsilon \sqrt{k\alpha' \tanh(\pi\epsilon)}}}_{J_{FP}} \cdot \frac{1}{2\pi}. \end{aligned} \quad (\text{C.28})$$

E Modified ζ -function for excluded zero modes

When $\hat{L}$ has a zero mode, the usual approach is to introduce an ad hoc regulator and divide out the pseudo-zero-eigenvalue by hand. A more rigorous, regulator-independent approach detailed in [18, 19] involves a slight modification to the above zeta-function computation. We briefly summarize their results, as pertaining to the calculation in this paper.

The required determinant $\det' \hat{L}$ with zero eigenvalue excluded follows from (2.51), (2.52) if we modify $\zeta(s)$ appropriately. Namely, we replace $\mathcal{U}(\lambda)$ in (2.52) by a function $f(\lambda)$ with the same positive zeros, but which is non-zero at the origin. This allows for γ to be deformed to $\mathbb{R}^-$ as before, by eliminating the singularity at 0. Such a function is given by

$$f(\lambda) \equiv -\frac{1}{\lambda} \det (M + N H_\lambda(1)) \quad (\text{E.1})$$

Furthermore, if we define

$$\begin{aligned} u_\lambda(x) \equiv & - \left[m_{12} + n_{11} u^{(2)}(1) + n_{12} u'^{(2)}(1) \right] u^{(1)}(x) \\ & + \left[m_{11} + n_{11} u^{(1)}(1) + n_{12} u'^{(1)}(1) \right] u^{(2)}(x) \end{aligned} \quad (\text{E.2})$$

with $u^{(1,2)}(x)$ as before, then we have

$$f(\lambda) = -\mathcal{B} \int dx u_0(x)^* u_\lambda(x) \quad (\text{E.3})$$

where the constant $\mathcal{B}$ is given by

$$\mathcal{B} = \frac{n_{12}}{u'_0(1)^*} \quad \text{if } n_{12} \neq 0; \quad \mathcal{B} = \frac{-n_{22}}{u_0(1)^*} \quad \text{if } n_{22} \neq 0. \quad (\text{E.4})$$

for the general case of “separable” boundary conditions, i.e. $\det M = \det N = 0$. Equation (2.60) is thus replaced with

$$\det' \hat{L} = -\mathcal{B} |y(\sigma)|^2 \frac{e^{-\zeta'_\infty(0)}}{\mathcal{U}_\infty(-1)} \quad (\text{E.5})$$

where, in the notation of [18], $y_1(\sigma) \equiv \lim_{\lambda \rightarrow 0} u_\lambda(0)$.

In subsection 2.3.1 we evaluate several determinants with excluded zero modes using a regulator, but it is satisfying to note that the results are reproduced in each case by the formula (E.5).

F Infinite products and ζ -regularization

We use zeta-function regularization to define all divergent summations and products in this paper. The Riemann zeta-function is given by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

This function is defined as a function of the complex variable which is the analytic continuation of the function of real s in the domain $s > 1$ where it is well-defined. As a function of complex s , it has a pole on the real axis at $s = 1$ and it is finite at negative real values of s . For completeness we give here explicit computations of some of the infinite products occurring throughout the paper. The values of the zeta-function of interest to us are

$$\zeta(0) = -\frac{1}{2}, \quad \zeta(-1) = -\frac{1}{12}, \quad \zeta'(0) = -\frac{1}{2} \ln(2\pi)$$

Examples of infinite sums and products which we encounter in our computations are

$$\sum_{m=1}^{\infty} 1 \equiv \lim_{s \rightarrow 0} \zeta(s) = -\frac{1}{2} \quad (\text{F.1})$$

$$\sum_{m=1}^{\infty} m \equiv \lim_{s \rightarrow -1} \zeta(s) = -\frac{1}{12} \quad (\text{F.2})$$

$$\prod_{n=1}^{\infty} (2\pi n) \equiv \lim_{s \rightarrow 0} (2\pi)^{\zeta(s)} \cdot \lim_{\bar{s} \rightarrow 0} e^{-\frac{d}{d\bar{s}} \zeta(\bar{s})} = 1 \quad (\text{F.3})$$

$$\prod_{n>0} \Omega_n = \prod_{n>0} (2\pi n) \frac{\varepsilon}{2k} = \sqrt{\frac{2k}{\varepsilon}} \quad (\text{F.4})$$

Using the above regularizations, and the product formula

$$\prod_{n=1}^{\infty} \left[1 + \frac{a^2}{n^2} \right] = \frac{\sinh \pi a}{\pi a} \quad (\text{F.5})$$

we find

$$\begin{aligned} \prod_{n=-\infty}^{\infty} [(\pi n)^2 a^2 + (\pi m)^2 b^2] &= (\pi m b)^2 \prod_{n>0} [(\pi n)^2 a^2 + (\pi m)^2 b^2]^2 \\ &= (\pi m b)^2 \prod_{n>0} (\pi n a)^4 \prod_{n>0} \left[1 + \left(\frac{\pi m b}{\pi n a} \right)^2 \right]^2 \\ &= (\pi m b)^2 \left(\frac{2}{a} \right)^2 \frac{\sinh^2(\pi m b/a)}{(\pi m b/a)^2} \\ &= 4 \sinh^2(\pi m b/a) \end{aligned}$$

By (D.1) and (F.2) we further have that

$$\begin{aligned} \prod_{m \geq 1} 2 \sinh(\pi m b/a) &= e^{\pi b/a \sum_{m=1}^{\infty} m} \prod_{m \geq 1} (1 - e^{-2\pi m b a}) \\ &= e^{-\frac{1}{12} \pi b/a} \prod_{m=1}^{\infty} (1 - e^{-2\pi m b/a}) \\ &= \eta(ib/a) = \sqrt{\frac{a}{b}} \eta(ia/b). \end{aligned}$$

In section 2.3.1 and appendix C one encounters the following infinite product:

$$\begin{aligned}
\prod_{n \neq 0, k} \frac{1}{\det(\mathcal{I} - \mathcal{E}(n))} &= \frac{1}{\prod_{n \neq 0, k} [1 - \tanh \pi \varepsilon \coth \pi \varepsilon n / k]} \\
&= \lim_{\varkappa \rightarrow k} \frac{[1 - \tanh \pi \varepsilon \coth \pi \varepsilon k / \varkappa]}{\prod_{n=1}^{\infty} [1 - \tanh^2 \pi \varepsilon \coth^2 \pi \varepsilon n / \varkappa]} \\
&= \lim_{\varkappa \rightarrow k} [1 - \tanh \pi \varepsilon \coth \pi \varepsilon k / \varkappa] \frac{\varkappa}{\varepsilon} \frac{\tanh \pi \varepsilon e^{-\pi \varkappa \varepsilon}}{\sin(\pi \varkappa)} \cdot \\
&\quad \cdot \prod_{n=1}^{\infty} \frac{[1 - e^{-2\pi \varkappa n / \varepsilon}]^2}{(1 - e^{-2\pi \varkappa (n+i\varepsilon)/\varepsilon})(1 - e^{-2\pi \varkappa (n-i\varepsilon)/\varepsilon})} \\
&= \frac{(-1)^{k+1} e^{-\pi k \varepsilon}}{\cosh^2 \pi \varepsilon} \tag{F.6}
\end{aligned}$$

In the third line we performed a modular transformation as in equation (B.17), and for the last step observed that the remaining infinite product is regular for all $\varkappa > 0$.

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