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Tinkertoys for the E_7 theory

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Abstract: We classify the class S theories of type E_7. These are four-dimensional $\mathcal{N} = 2$ superconformal field theories arising from the compactification of the E_7 $(2, 0)$ theory on a punctured Riemann surface, C. The classification is given by listing all 3-punctured spheres ("fixtures"), and connecting cylinders, which can arise in a pants-decomposition of C. We find exactly 11,000 fixtures with three regular punctures, and an additional 48 with one "irregular puncture" (in the sense used in our previous works). To organize this large number of theories, we have created a web application at https://golem.ph.utexas.edu/class-S/E7/ . Among these theories, we find 10 new ones with a simple exceptional global symmetry group, as well as a new rank-2 SCFT and several new rank-3 SCFTs. As an application, we study the strong-coupling limit of the E_7 gauge theory with 3 hypermultiplets in the 56. Using our results, we also verify recent conjectures that the T^2 compactification of certain 6d $(1, 0)$ theories can alternatively be realized in class S as fixtures in the E_7 or E_8 theories.

Keywords: Conformal Field Models in String Theory, Conformal Field Theory, Supersymmetric Gauge Theory, Supersymmetry and Duality

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Contents

1 Introduction

Class S theories are a large class of four-dimensional $\mathcal{N} = 2$ superconformal field theories arising from the partially-twisted compactification of a six-dimensional $(2, 0)$ theory on a punctured Riemann surface [1, 2]. Along with Lagrangian $\mathcal{N} = 2$ SCFTs of vector and hypermultiplets, class S contains many strongly-interacting SCFTs which have no known Lagrangian description [3, 4]. Nevertheless, the six-dimensional construction gives rise to powerful tools to study their properties (for an extensive review of recent progress see, e.g., the collection [5]).

As the $6d$ $(2,0)$ theories have an ADE classification, the corresponding four-dimensional theories resulting from their compactification also come in ADE type. A program to classify these theories was initiated in [6, 7], where we provided a method for classifying the A and D series, carrying out this classification explicitly for low ranks, before moving on to the E_6 theory in [8].[1] In this work, we classify the E_7-type class S theories. We find 11,000 fixtures with three regular punctures. Of these, 962 have enhanced global symmetries or additional free hypermultiplets. It would be formidable to list all of these here; instead, we have created a web application at https://golem.ph.utexas.edu/class-S/E7/ where the interested reader can explore them. A description and instructions are given in section 3.4. Among the theories on our list, we find a new rank-2 and several new rank-3 interacting SCFTs.[2] Additionally, we find several new SCFTs with a simple exceptional global symmetry group.

Using our results, we construct the E_7 gauge theory with matter in the $3(56)$. We determine its S-duality frames and provide the k-differentials specifying its Seiberg-Witten solution. Additionally, we confirm predictions in [21–23] that the T^2 compactifications of the worldvolume theories on $M5$ branes probing ALE singularities of type E have class S realizations.

2 The E_7 theory

2.1 Coulomb branch geometry

The Coulomb branch geometry for our theories can be realized either by studying parabolic Hitchin systems on the punctured Riemann surface, C, or by studying the Calabi-Yau integrable system for a certain family of non-compact Calabi-Yaus fibered over C.

In the former description, the Seiberg-Witten curve $\Sigma \to C$ is spectral curve

$$\Sigma = \{\mathrm{Det}(\lambda \mathbb{1} - \Phi(z)) = 0\} \subset \mathrm{Tot}(K_C)$$

where (for definiteness) the determinant is taken in the adjoint representation and λ is the Seiberg-Witten differential.

In the latter description, the noncompact Calabi-Yau is the hypersurface

$$\begin{aligned} X_{\vec{u}} = \big\{ 0 = &-w^2 - x^3 + 16xy^3 + \phi_2(z)y^4 + \phi_6(z)y^3 + \phi_8(z)xy \\ &+ \phi_{10}(z)y^2 + \phi_{12}(z)x + \phi_{14}(z)y + \phi_{18}(z) \big\} \subset \mathrm{Tot}(K_C^9 \oplus K_C^6 \oplus K_C^4) \end{aligned}$$

In both cases, the Seiberg-Witten geometry is expressed in terms of meromorphic k-differentials, $\phi_k(z)$, on C, which have poles of various orders at the punctures [24]. It

[1] This class of theories can be enlarged for types A, D, and E_6 by twisting the $(2,0)$ theory by an outer-automorphism when traversing a nontrivial cycle on the punctured Riemann surface. This construction gives rise to a sector of twisted punctures, leading to many new SCFTs. A classification for the twisted theories of type A_{2N-1}, D_N, and E_6 was given in [9–13]. Though a complete classification of the theories of type A_{2N} is still lacking, twists of this type were utilized [14] to construct the $R_{2,2N}$ family of SCFTs. Similarly, a full classification for the S_3-twisted D_4 theory has not yet been carried out, but these twists were used to construct additional $4d$ theories and study their S-duality frames in [15].

[2] In [16–20] a proposed classification of four-dimensional rank-1 $\mathcal{N}=2$ SCFTs was given by constructing the rigid special Kähler geometries consistent with the interpretation as the Coulomb branch of an $\mathcal{N}=2$ SCFT. A natural follow up would be to extend these works to rank-2 and higher.

is most convenient to work in the Katz-Morrison basis [25], where the $\phi_k(z)$ are related to the invariant traces, $P_k = Tr(\Phi^k)$, by

$$
\begin{aligned}
\phi_2 &= \frac{1}{18}P_2\\
\phi_6 &= -\frac{2}{3}P_6+\frac{1}{2916}P_2^3\\
\phi_8 &= -\frac{4}{25}P_8-\frac{22}{675}P_6P_2-\frac{7}{524880}P_2^4\\
\phi_{10} &= -\frac{32}{315}P_{10}-\frac{1}{175}P_8P_2+\frac{17}{36450}P_6P_2^2-\frac{1}{9447840}P_2^5\\
\phi_{12} &= \frac{128}{225}P_{12}-\frac{4096}{42525}P_{10}P_2+\frac{737}{127575}P_8P_2^2-\frac{992}{2025}P_6^2+\frac{167}{492075}P_6P_2^3-\frac{149}{1530550080P_2^6}\\
\phi_{14} &= \frac{1024}{20867}P_{14}-\frac{140864}{18109575}P_{12}P_2+\frac{132848}{311155425}P_{10}P_2^2-\frac{992}{60975}P_8P_6-\frac{1289}{1866932550}P_8P_2^3\\
&\quad+\frac{5648}{2963385}P_6^2P_2-\frac{11609}{7201025550}P_6P_2^4+\frac{11083}{31357297819008}P_2^7\\
\phi_{18} &= -\frac{8192}{167487}P_{18}+\frac{78810880}{94363683183}P_{14}P_2^2+\frac{308224}{12561525}P_{12}P_6-\frac{871487200}{9827303577201}P_{12}P_2^3\\
&\quad+\frac{7553024}{439653375}P_{10}P_8-\frac{72249472}{11870641125}P_{10}P_6P_2+\frac{24365269174}{4221273582024975}P_{10}P_2^4\\
&\quad-\frac{619144}{732755625}P_8^2P_2+\frac{18510930376}{48254156173125}P_8P_6P_2^2-\frac{14715122551}{63319103730374625}P_8P_2^5\\
&\quad-\frac{1921408}{339161175}P_6^3-\frac{4632094024}{5025325692886875}P_6^2P_2^3-\frac{886993691}{8508142378495752491520}P_2^9
\end{aligned}
$$

At a puncture, $\Phi(z)$ has a simple pole with nilpotent residue,

$$\Phi(z) = \frac{N}{z} + \text{regular}$$

where N is a representative of the "Hitchin" Nilpotent orbit which is Spaltenstein-dual [26] to the Nahm orbit (which we use to label our punctures)

$$O_H = d(O_N)$$

Taking traces, one finds an elaborate set of constraints on the coefficients of the polar parts of the $\phi_k(z) = \sum_j \frac{c_j^{(k)}}{z^j} + \text{regular}$. When the special piece of O_N has more than one element, we have an additional quotient by a finite group (the "Sommers-Achar group") acting on the coefficients [26].

2.2 Puncture properties

Here we review the puncture properties listed in our table below, leaving most of the details to [8].

As in our previous works, we use Bala-Carter notation [27, 28] to label the nilpotent orbits, where $O_N = 0$ is the full puncture and $O_N = E_7(a_1)$ is the simple puncture. The flavour symmetry algebra, $\mathfrak{f}$, associated to a puncture is the centralizer of $\rho_N(\mathfrak{su}(2))$ inside $\mathfrak{e}_7$. For the distinguished orbits ($E_7(a_i)$, $i = 1, \ldots, 5$), $\mathfrak{f}$ is trivial, whereas for the 0 orbit

$\mathfrak{f}$ is all of $\mathfrak{e}_7$. The level of each factor $\mathfrak{f}_i \subset \mathfrak{f}$ is determined from the decomposition of the adjoint representation under the embedding $\mathfrak{e}_7 \supset \mathfrak{su}(2) \times \mathfrak{f}_i$ as

$$\mathfrak{e}_7 = \bigoplus_n V_n \otimes R_{n,i} \tag{2.1}$$

where V_n denotes the n-dimensional irreducible representation of $\mathfrak{su}(2)$ and $R_{n,i}$ the corresponding representation of $\mathfrak{f}_i$, which is in general reducible.[3] The level of $\mathfrak{f}_i$ is then given by [26, 29]

$$k_i = \sum_n l_{n,i} \tag{2.2}$$

where $l_{n,i}$ is the Dynkin index of the representation $R_{n,i}$. For $\mathfrak{f}_i = \mathfrak{u}(1)$, $l_{n,i}$ is the $\mathfrak{u}(1)$ charge squared. In the table below, we normalize the $\mathfrak{u}(1)$ generators so that the free hypermultiplets in the mixed fixtures have charge 1.

The $\phi_k(z)$ have poles at the punctures of order at most p_k:

$$\phi_k(z) = \sum_{j=1}^{p_k} \frac{c_j^{(k)}}{z^j} + \text{regular}$$

where the set $\{p_k\}$ is called the *pole structure* of the puncture. The coefficients, $c_j^{(k)}$, typically are not all independent, but instead obey certain polynomial relations, which we list below.

Finally, for each puncture we also list its contribution to the effective number of vector and hypermultiplets (n_h, n_v), which are given in terms of the conformal central charges a and c by $n_v = 4(2a - c)$ and $n_h = 4(5c - 4a)$.

2.3 Regular punctures

The pole structure of an E_7 puncture at $z = 0$ is denoted $\{p_2, p_6, p_8, p_{10}, p_{12}, p_{14}, p_{18}\}$, and is defined to be the set of *leading* pole orders in z of the differentials ϕ_k, for $k = 2, 6, 8, 10, 12, 14, 18$. As discussed above, for certain punctures, there are constraints among *leading* coefficients, and sometimes even for *subleading* ones, in the expansion of the ϕ_k in z.

[3] We list this decomposition for each puncture in appendix A.

Nahm B-C label	Hitchin B-C label	Pole structure	Constraints	Flavour group	$(\delta n_h, \delta n_v)$
0	E_7	$\{1,5,7,9,11,13,17\}$	$-$	$(E_7)_{36}$	$(1596, 1533)$
A_1	$E_7(a_1)$	$\{1,5,7,9,11,13,16\}$	$-$	$\mathrm{Spin}(12)_{28}$	$(1544, 1498)$
$2A_1$	$E_7(a_2)$	$\{1,5,7,9,11,12,16\}$	$-$	$\mathrm{Spin}(9)_{24} \times \mathrm{SU}(2)_{20}$	$(1508, 1471)$
$(3A_1)''$	E_6	$\{1,5,7,8,11,12,16\}$	$-$	$(F_4)_{24}$	$(1488, 1452)$
$(3A_1)'$ ($\underline{ns}$)	$(E_7(a_3), \mathbb{Z}_2)$	$\{1,5,7,9,10,12,16\}$	$-$	$\mathrm{Sp}(3)_{20} \times \mathrm{SU}(2)_{19}$	$(1479, 1448)$
A_2	$E_7(a_3)$	$\{1,5,7,9,10,12,16\}$	$c_{16}^{(18)} = \left(a_8^{(9)}\right)^2$	$\mathrm{SU}(6)_{20}$	$(1460, 1430)$
$4A_1$ ($\underline{ns}$)	$(E_6(a_1), \mathbb{Z}_2)$	$\{1,5,7,8,10,12,16\}$	$-$	$\mathrm{Sp}(3)_{19}$	$(1457, 1429)$
$A_2 + A_1$	$E_6(a_1)$	$\{1,5,7,8,10,12,16\}$	$c_{16}^{(18)} = \left(a_8^{(9)}\right)^2$	$\mathrm{SU}(4)_{18} \times \mathrm{U}(1)_{42}$	$(1436, 1411)$
$A_2 + 2A_1$	$E_7(a_4)$	$\{1,5,7,8,10,12,15\}$	$-$	$\mathrm{SU}(2)_{16} \times \mathrm{SU}(2)_{28} \times \mathrm{SU}(2)_{84}$	$(1416, 1394)$
A_3	$D_6(a_1)$	$\{1,5,7,8,10,12,15\}$	$c_{15}^{(18)} = \left(\frac{1}{3}c_{10}^{(12)} - 3\,c_5^{(6)}a_5^{(6)} - 9\left(a_5^{(6)}\right)^2\right)a_5^{(6)}$ $c_{12}^{(14)} = c_7^{(8)}a_5^{(6)}$	$\mathrm{Spin}(7)_{16} \times \mathrm{SU}(2)_{12}$	$(1364, 1343)$
$2A_2$	$D_5 + A_1$	$\{1,5,7,8,10,11,15\}$	$-$	$(G_2)_{16} \times \mathrm{SU}(2)_{36}$	$(1388, 1367)$
$A_2 + 3A_1$	A_6	$\{1,5,6,8,10,12,15\}$	$-$	$(G_2)_{28}$	$(1400, 1379)$
$(A_3 + A_1)''$	D_5	$\{1,5,7,8,10,11,14\}$	$-$	$\mathrm{Spin}(7)_{16}$	$(1352, 1332)$
$2A_2 + A_1$ ($\underline{ns}$)	$(E_7(a_5), S_3)$	$\{1,5,6,8,10,11,15\}$	$-$	$\mathrm{SU}(2)_{36} \times \mathrm{SU}(2)_{38}$	$(1370, 1352)$
$(A_3 + A_1)'$ ($\underline{ns}$)	$(E_7(a_5), \mathbb{Z}_2)$	$\{1,5,6,8,10,11,15\}$	$c_{15}^{(18)} = -\frac{2}{81}\left(c_5^{(6)}\right)^3 - \frac{1}{27}c_{10}^{(12)}c_5^{(6)}$ $+\frac{2}{81}a_5^{(6)}\left(4\left(a_5^{(6)}\right)^2 - 3\left(c_{10}^{(12)} + \left(c_5^{(6)}\right)^2\right)\right)$	$\mathrm{SU}(2)_{13} \times \mathrm{SU}(2)_{24} \times \mathrm{SU}(2)_{12}$	$(1345, 1328)$

Nahm B-C label	Hitchin B-C label	Pole structure	Constraints	Flavour group	$(\delta n_h, \delta n_v)$
$D_4(a_1)$	$E_7(a_5)$	$\{1,5,6,8,10,11,15\}$	$c_{10}^{(12)} = -(c_5^{(6)})^2 + (a_5^{(6)})^2 - (a'^{(6)}_5)^2$ $c_{15}^{(18)} = -\frac{2}{81}(c_5^{(6)})^3 - \frac{1}{27}c_{10}^{(12)}c_5^{(6)} + \frac{2}{81}a_5^{(6)}\left((a_5^{(6)})^2 + 3(a'^{(6)}_5)^2\right)$	$\mathrm{SU}(2)^3_{12}$	$(1332, 1316)$
$A_3 + 2A_1$ ($\underline{ns}$)	$(E_6(a_3), \mathbb{Z}_2)$	$\{1,5,6,8,10,11,14\}$	–	$\mathrm{SU}(2)_{13} \times \mathrm{SU}(2)_{24}$	$(1333, 1317)$
$D_4(a_1) + A_1$	$E_6(a_3)$	$\{1,5,6,8,10,11,14\}$	$c_{10}^{(12)} = -\frac{3}{4}\left(c_5^{(6)}\right)^2 + 3\left(a_5^{(6)}\right)^2$	$\mathrm{SU}(2)^2_{12}$	$(1320, 1305)$
D_4	$(A_5)''$	$\{1,5,6,8,10,11,15\}$	$c_{10}^{(12)} = -\left(c_5^{(6)}\right)^2$ $c_9^{(12)} = -2c_5^{(6)}c_4^{(6)} - 3c_6^{(8)}a_3^{(4)} - 2c_5^{(6)}c_1^{(2)}a_3^{(4)} - \left(a_3^{(4)}\right)^3$ $c_{11}^{(14)} = -2c_8^{(10)}a_3^{(4)} - \frac{1}{27}\left(3c_6^{(8)}c_5^{(6)} + \left(c_5^{(6)}\right)^2 c_1^{(2)} + 3c_5^{(6)}\left(a_3^{(4)}\right)^2\right)$ $c_{15}^{(18)} = \frac{1}{81}\left(c_5^{(6)}\right)^3$ $c_{14}^{(18)} = c_8^{(10)}\left(a_3^{(4)}\right)^2 + \frac{1}{27}\left(3c_6^{(8)}c_5^{(6)}a_3^{(4)} + \left(c_5^{(6)}\right)^2 c_4^{(6)} + 2\left(c_5^{(6)}\right)^2 c_1^{(2)}a_3^{(4)} + 3c_5^{(6)}\left(a_3^{(4)}\right)^3\right)$ $c_{13}^{(18)} = -c_{10}^{(14)}a_3^{(4)} - c_7^{(10)}\left(a_3^{(4)}\right)^2 - \frac{1}{27}\left(c_8^{(12)}c_5^{(6)} + \left(c_5^{(6)}\right)^2 c_0^{(2)}a_3^{(4)} + \left(c_5^{(6)}\right)^2 c_3^{(6)} + 3c_5^{(8)}c_5^{(6)}a_3^{(4)}\right)$	$\mathrm{Sp}(3)_{12}$	$(1196, 1181)$
$A_3 + A_2$	$D_5(a_1) + A_1$	$\{1,5,6,8,9,11,14\}$	–	$\mathrm{SU}(2)_{12} \times \mathrm{U}(1)_{112}$	$(1308, 1294)$
A_4	$D_5(a_1)$	$\{1,5,6,8,9,11,14\}$	$c_8^{(10)} = \left(a_4^{(5)}\right)^2$ $c_{11}^{(14)} = -2\,a_4^{(5)}a_7^{(9)}$ $c_{14}^{(18)} = \left(a_7^{(9)}\right)^2$	$\mathrm{SU}(3)_{12} \times \mathrm{U}(1)_{24}$	$(1252, 1239)$
$A_3 + A_2 + A_1$	$A_4 + A_2$	$\{1,4,6,8,9,11,14\}$	–	$\mathrm{SU}(2)_{224}$	$(1296, 1283)$
$(A_5)''$	D_4	$\{1,5,6,6,9,9,12\}$	–	$(G_2)_{12}$	$(1144, 1132)$

Nahm B-C label	Hitchin B-C label	Pole structure	Constraints	Flavour group	$(\delta n_h, \delta n_v)$
A_4+A_1	A_4+A_1	$\{1,4,6,8,9,11,14\}$	$c_8^{(10)} = \left(a_4^{(5)}\right)^2$ $c_{11}^{(14)} = -2\,a_4^{(5)}a_7^{(9)}$ $c_{14}^{(18)} = \left(a_7^{(9)}\right)^2$	$\mathrm{U}(1)_{54}\times\mathrm{U}(1)_{24}$	$(1239,1228)$
D_4+A_1 ($\underline{ns}$)	$(A_4,\mathbb{Z}_2)$	$\{1,4,6,8,9,11,14\}$	$c_9^{(12)} = \left(a_3^{(4)}\right)^3 + 3c_6^{(8)}a_3^{(4)}$ $c_{11}^{(14)} = 2\,c_8^{(10)}a_3^{(4)}$ $c_{14}^{(18)} = c_8^{(10)}\left(a_3^{(4)}\right)^2$ $c_{13}^{(18)} = c_{10}^{(14)}a_3^{(4)} - c_7^{(10)}\left(a_3^{(4)}\right)^2$	$\mathrm{Sp}(2)_{11}$	$(1182,1170)$
$D_5(a_1)$	A_4	$\{1,4,6,8,9,11,14\}$	$c_8^{(10)} = \left(a_4^{(5)}\right)^2$ $c_9^{(12)} = \left(a_3^{(4)}\right)^3 + 3c_6^{(8)}a_3^{(4)}$ $c_{11}^{(14)} = 2\left(a_4^{(5)}\right)^2 a_3^{(4)}$ $c_{14}^{(18)} = \left(a_4^{(5)}a_3^{(4)}\right)^2$ $c_{13}^{(18)} = c_{10}^{(14)}a_3^{(4)} - c_7^{(10)}\left(a_3^{(4)}\right)^2$	$\mathrm{SU}(2)_{10}\times\mathrm{U}(1)_{28}$	$(1170,1160)$
A_4+A_2	$A_3+A_2+A_1$	$\{1,4,6,7,9,10,13\}$	$-$	$\mathrm{SU}(2)_{108}$	$(1212,1202)$
$D_5(a_1)+A_1$	A_3+A_2	$\{1,4,6,7,9,10,13\}$	$c_9^{(12)} = 3c_6^{(8)}a_3^{(4)} + \left(a_3^{(4)}\right)^3$ $c_{13}^{(18)} = c_{10}^{(14)}a_3^{(4)} - c_7^{(10)}\left(a_3^{(4)}\right)^2$	$\mathrm{SU}(2)_{56}$	$(1160,1151)$
$(A_5)'$ ($\underline{ns}$)	$(D_4(a_1)+A_1,\mathbb{Z}_2)$	$\{1,4,6,7,9,10,13\}$	$c_9^{(12)} = 2a_3^{(4)}\left(4\left(a_3^{(4)}\right)^2 + 3c_6^{(8)}\right)$ $c_{10}^{(14)} = -2c_7^{(10)}a_3^{(4)}$ $c_{13}^{(18)} = c_7^{(10)}\left(4\left(a_3^{(4)}\right)^2 + 3c_6^{(8)}\right)$	$\mathrm{SU}(2)_9\times\mathrm{SU}(2)_{20}$	$(1133,1124)$

Nahm B-C label	Hitchin B-C label	Pole structure	Constraints	Flavour group	$(\delta n_h, \delta n_v)$
$E_6(a_3)$	$D_4(a_1) + A_1$	$\{1,4,6,7,9,10,13\}$	$c_6^{(8)} = -\left((a_3^{(4)})^2 + 3(a'^{(4)}_3)^2\right)$ $c_9^{(12)} = 2a_3^{(4)}\left((a_3^{(4)})^2 - 9(a'^{(4)}_3)^2\right)$ $c_{10}^{(14)} = -2c_7^{(10)} a_3^{(4)}$ $c_{13}^{(18)} = c_7^{(10)}\left((a_3^{(4)})^2 - 9(a'^{(4)}_3)^2\right)$	$\mathrm{SU}(2)_{20}$	$(1124, 1116)$
$A_5 + A_1$ ($\underline{ns}$)	$(D_4(a_1), S_3)$	$\{1,4,6,6,9,9,12\}$	–	$\mathrm{SU}(2)_{26}$	$(1130, 1121)$
$D_6(a_2)$ ($\underline{ns}$)	$(D_4(a_1), \mathbb{Z}_2)$	$\{1,4,6,6,9,9,12\}$	$c_9^{(12)} = 2a_3^{(4)}\left(4(a_3^{(4)})^2 + 3c_6^{(8)}\right)$	$\mathrm{SU}(2)_9$	$(1113, 1105)$
$E_7(a_5)$	$D_4(a_1)$	$\{1,4,6,6,9,9,12\}$	$c_6^{(8)} = -\left((a_3^{(4)})^2 + 3(a'^{(4)}_3)^2\right)$ $c_9^{(12)} = 2a_3^{(4)}\left((a_3^{(4)}))^2 - 9(a'^{(4)}_3)^2\right)$	none	$(1104, 1097)$
D_5	$(A_3 + A_1)''$	$\{1,4,6,7,9,10,13\}$	$c_6^{(8)} = -(a_3^{(4)})^2$ $c_7^{(10)} = a_4^{(6)} a_3^{(4)}$ $c_9^{(12)} = 2(a_3^{(4)})^3$ $c_8^{(12)} = -3\left(c_5^{(8)} a_3^{(4)} - 6c_4^{(6)} a_4^{(6)} - 6a_4^{(6)} a_3^{(4)} c_1^{(2)} - 27(a_4^{(6)})^2\right)$ $c_{10}^{(14)} = -2a_4^{(6)}(a_3^{(4)})^2$ $c_9^{(14)} = -2c_6^{(10)} a_3^{(4)} + c_5^{(8)} a_4^{(6)} - 3(a_4^{(6)})^2 c_1^{(2)}$ $c_{13}^{(18)} = a_4^{(6)}(a_3^{(4)})^3$ $c_{12}^{(18)} = c_6^{(10)}(a_3^{(4)})^2 - c_5^{(8)} a_4^{(6)} a_3^{(4)} + 3(a_4^{(6)})^2 c_4^{(6)}$ $\quad + 6(a_4^{(6)})^2 a_3^{(4)} c_1^{(2)} + 18(a_4^{(6)})^3$ $c_{11}^{(18)} = -c_8^{(14)} a_3^{(4)} + \frac{1}{3} c_7^{(12)} a_4^{(6)} - c_5^{(10)}(a_3^{(4)})^2 + c_4^{(8)} a_4^{(6)} a_3^{(4)}$ $\quad - 3(a_4^{(6)})^2 c_3^{(6)} - 3(a_4^{(6)})^2 a_3^{(4)} c_0^{(2)}$	$\mathrm{SU}(2)_8 \times \mathrm{SU}(2)_{12}$	$(988, 981)$
A_6	$A_2 + 3A_1$	$\{1,4,5,6,7,8,11\}$	–	$\mathrm{SU}(2)_{36}$	$(1004, 998)$

Nahm B-C label	Hitchin B-C label	Pole structure	Constraints	Flavour group	$(\delta n_h, \delta n_v)$
D_5+A_1	$2A_2$	$\{1,4,5,6,8,9,12\}$	$c_8^{(12)} = -4a_4^{(6)}\left(c_4^{(6)}-a_4^{(6)}\right)$ $c_9^{(14)} = -\frac{2}{9}a_4^{(6)}\left(c_5^{(8)}+\frac{2}{3}c_1^{(2)}a_4^{(6)}\right)$ $c_{12}^{(18)} = \frac{4}{27}(a_4^{(6)})^2\left(c_4^{(6)}-\frac{4}{3}a_4^{(6)}\right)$ $c_{11}^{(18)} = -\frac{2}{27}a_4^{(6)}\left(c_7^{(12)}+2a_4^{(6)}c_3^{(6)}\right)$	$\mathrm{SU}(2)_{12}$	$(980,974)$
$D_6(a_1)$	A_3	$\{1,4,6,6,9,9,12\}$	$c_6^{(8)} = -\left(a_3^{(4)}\right)^2$ $c_9^{(12)} = -2\left(a_3^{(4)}\right)^3$ $c_8^{(12)} = 3\,a_3^{(4)}c_5^{(8)}$ $c_9^{(14)} = 2\,a_3^{(4)}c_6^{(10)}$ $c_{12}^{(18)} = \left(a_3^{(4)}\right)^2 c_6^{(10)}$ $c_{11}^{(18)} = c_8^{(14)}a_3^{(4)} - c_5^{(10)}\left(a_3^{(4)}\right)^2$	$\mathrm{SU}(2)_8$	$(976,970)$
$E_7(a_4)$	A_2+2A_1	$\{1,4,5,6,7,8,10\}$	–	none	$(968,963)$
$E_6(a_1)$	A_2+A_1	$\{1,4,5,6,7,8,10\}$	$c_4^{(6)} = 3\left(a_2^{(3)}\right)^2$ $c_5^{(8)} = -6\,a_3^{(5)}a_2^{(3)}$ $c_6^{(10)} = \left(a_3^{(5)}\right)^2$ $c_7^{(12)} = -18\,a_5^{(9)}a_2^{(3)}$ $c_8^{(14)} = 2\,a_5^{(9)}a_3^{(5)}$ $c_{10}^{(18)} = \left(a_5^{(9)}\right)^2$	$\mathrm{U}(1)_{24}$	$(868,864)$
D_6 ($\underline{ns}$)	$(A_2,\mathbb{Z}_2)$	$\{1,4,4,4,6,6,8\}$	–	$\mathrm{SU}(2)_7$	$(767,763)$
$E_7(a_3)$	A_2	$\{1,4,4,4,6,6,8\}$	$c_4^{(6)} = 3\left(a_2^{(3)}\right)^2$	none	$(760,757)$

Nahm B-C label	Hitchin B-C label	Pole structure	Constraints	Flavour group	$(\delta n_h, \delta n_v)$
E_6	$(3A_1)''$	$\{1,3,4,5,6,7,9\}$	$c_4^{(8)} = -\frac{1}{3}\left(2c_3^{(6)}c_1^{(2)} + 2\left(c_1^{(2)}\right)^2 a_2^{(4)} + 3\left(a_2^{(4)}\right)^2\right)$ $c_5^{(10)} = -\frac{1}{9}\left(c_3^{(6)} + c_1^{(2)}a_2^{(4)}\right)a_2^{(4)}$ $c_6^{(12)} = -\left(c_3^{(6)}\right)^2 + \left(\left(c_1^{(2)}\right)^2 + 2a_2^{(4)}\right)\left(a_2^{(4)}\right)^2$ $c_5^{(12)} = -2c_3^{(6)}c_2^{(6)} - a_2^{(4)}\left(3c_3^{(8)} + 2c_3^{(6)}c_0^{(2)}\right.$ $\left.+2c_2^{(6)}c_1^{(2)} + 2a_2^{(4)}c_1^{(2)}c_0^{(2)}\right)$ $c_7^{(14)} = \frac{1}{27}\left(\left(c_3^{(6)}\right)^2 c_1^{(2)} + a_2^{(4)}\left(2c_3^{(6)}\left(c_1^{(2)}\right)^2\right.\right.$ $\left.\left.+6c_3^{(6)}a_2^{(4)} + a_2^{(4)}\left(c_1^{(2)}\right)^3 + 6\left(a_2^{(4)}\right)^2 c_1^{(2)}\right)\right)$ $c_6^{(14)} = -\frac{1}{27}\left(3c_3^{(8)}c_3^{(6)} + \left(c_3^{(6)}\right)^2 c_0^{(2)}\right.$ $\left.+a_2^{(4)}\left(54c_4^{(10)} + 3c_3^{(8)}c_1^{(2)} + 2c_3^{(6)}c_1^{(2)}c_0^{(2)} + a_2^{(4)}\left(c_1^{(2)}\right)^2 c_0^{(2)}\right)\right)$ $c_9^{(18)} = \frac{1}{81}\left(\left(c_3^{(6)}\right)^3 - \left(a_2^{(4)}\right)^2\left(3c_3^{(6)}\left(c_1^{(2)}\right)^2\right.\right.$ $\left.\left.+a_2^{(4)}\left(9c_3^{(6)} + 2\left(c_1^{(2)}\right)^3\right) + 9\left(a_2^{(4)}\right)^2 c_1^{(2)}\right)\right)$ $c_8^{(18)} = \frac{1}{27}\left(\left(c_3^{(6)}\right)^2 c_2^{(6)} + a_2^{(4)}\left(3c_3^{(8)}c_3^{(6)} + 2c_3^{(6)}c_2^{(6)}c_1^{(2)}\right.\right.$ $+2\left(c_3^{(6)}\right)^2 c_0^{(2)} + a_2^{(4)}\left(27c_4^{(10)} + 3c_3^{(8)}c_1^{(2)} + c_2^{(6)}\left(c_1^{(2)}\right)^2\right.$ $\left.\left.\left.+4c_3^{(6)}c_1^{(2)}c_0^{(2)}\right) + 2\left(a_2^{(4)}\right)^2\left(c_1^{(2)}\right)^2 c_0^{(2)}\right)\right)$ $c_7^{(18)} = -\frac{1}{27}\left(c_4^{(12)}c_3^{(6)} + \left(c_3^{(6)}\right)^2 c_1^{(6)}\right.$ $+a_2^{(4)}\left(27c_5^{(14)} + c_4^{(12)}c_1^{(2)} + 3c_2^{(8)}c_3^{(6)} + 2c_3^{(6)}c_1^{(6)}c_1^{(2)}\right)$ $\left.+\left(a_2^{(4)}\right)^2\left(27c_3^{(10)} + 3c_2^{(8)}c_1^{(2)} + c_1^{(6)}\left(c_1^{(2)}\right)^2\right)\right)$	$SU(2)_{12}$	$(604, 601)$

Nahm B-C label	Hitchin B-C label	Pole structure	Constraints	Flavour group	$(\delta n_h, \delta n_v)$
$E_7(a_2)$	$2A_1$	$\{1,3,4,4,6,6,8\}$	$c_3^{(6)} = a_2^{(4)} c_1^{(2)}$ $c_4^{(8)} = -\left(a_2^{(4)}\right)^2$ $c_6^{(12)} = -2\left(a_2^{(4)}\right)^3$ $c_5^{(12)} = 3c_3^{(8)} a_2^{(4)}$ $c_6^{(14)} = 2c_4^{(10)} a_2^{(4)}$ $c_8^{(18)} = c_4^{(10)} \left(a_2^{(4)}\right)^2$ $c_7^{(18)} = c_5^{(14)} a_2^{(4)} - c_3^{(10)} \left(a_2^{(4)}\right)^2$	none	$(592, 590)$
$E_7(a_1)$	A_1	$\{1,2,2,2,3,3,4\}$	–	none	$(384, 383)$

2.4 Cataloging fixtures using the superconformal index

There are 45 nilpotent orbits in $\mathfrak{e}_7$. Excluding the regular orbit (which corresponds to the trivial defect), this yields 44 codimension-2 defects ("punctures"). A 3-punctured sphere is specified by choosing a triple of such defects. There are 15,180 such triples, but 4,180 of them are "bad" (do not lead to well-defined 4D SCFTs[4]). Of the remaining[5] 11,000, one is a free-field fixture (corresponding to three half-hypermultiplets in the 56 of E_7), 262 are "mixed" fixtures (consisting of some number of hypermultiplets plus an interacting SCFT), and the remaining 10,737 are isolated interacting SCFTs. Of these, 654 have "enhanced" global symmetry groups: the global symmetry group of the SCFT is larger than the "manifest" global symmetry associated to the three punctures.

Of the "good" fixtures, we will need to determine which are "mixed" (i.e., include free hypermultiplets) and which have enhanced global symmetries. To carry out this classification, we make recourse to the Hall-Littlewood limit of the superconformal index as we did in [8] for the E_6 theory. This method is a generalization of the work of [30–34] to type E theories. Here, we briefly summarize our procedure in [8].

We assume the Hall-Littlewood index for a fixture in the E_7 theory takes the form

$$\mathcal{I} = \sum_\lambda \frac{\prod_{i=1}^3 \mathcal{K}(\mathbf{a}_i) P^\lambda(\mathbf{a}_i|\tau)}{\mathcal{K}(\{\tau\}) P^\lambda(\{\tau\}|\tau)} \tag{2.3}$$

where

- The sum is over partitions λ labeling the highest weights of finite-dimensional irreducible representations of $\mathfrak{e}_7$.
- The $P^\lambda(\mathbf{a}_i|\tau)$ are Hall-Littlewood polynomials, defined for general $\mathfrak{g}$ by

$$P^\lambda = W^{-1}(\tau) \sum_{w\in W} w \left(e^\lambda \prod_{\alpha\in R^+} \frac{1-\tau^2 e^{-\alpha}}{1-e^{-\alpha}} \right)$$
$$W(\tau) = \sqrt{\sum_{\substack{w\in W \\ w\lambda=\lambda}} \tau^{2\ell(w)}}$$

  where R^+ is the set of positive roots, W the Weyl group, and $\ell(w)$ the length of the Weyl group element w.

- $\mathbf{a}_i \equiv \{e^\alpha\}_{\alpha\in R^+}$ denotes a set of flavor fugacities for the flavor symmetry of the i^{th} puncture. The set $\{\tau\}$ is the set of fugacities for the trivial puncture.

[4]The simplest diagnostic for when an n-punctured sphere is "bad" is that the Riemann-Roch index predicts a negative number for one or more of the graded Coulomb branch dimensions. Equivalently, the Hall-Littlewood index (2.3) diverges.

[5]There are, in addition, 48 fixtures with an "irregular" puncture. These arise when the collision of two punctures *would* have resulted in bubbling-off one of the 4,180 bad 3-punctured spheres. Of the 48, 36 are free-field fixtures, 10 are interacting fixtures and 2 are mixed. They are listed in the tables below.

- To compute the $\mathcal{K}$ factors, first decompose the adjoint representation of $\mathfrak{g}$ as in (2.1). The $\mathcal{K}$ factors are then given by

$$\mathcal{K}(\mathbf{a}) = \mathrm{PE}\left[\sum_n \tau^{n+1}\chi_{\mathfrak{f}}^{R_n}(\mathbf{a})\right].$$

We classify each fixture using the Hall-Littlewood superconformal index following [35]. For a "good" fixture, expanding the index in the superconformal fugacity τ gives

$$\mathcal{I} = 1 + \chi_{\mathrm{F}}^{R}\tau + \chi_{G_{\mathrm{fixt}}}^{adj}\tau^2 + \cdots \tag{2.4}$$

The coefficient of τ signals the presence of free hypermultiplets transforming in the representation R of flavor symmetry F, while the coefficient of τ^2 is the character of the adjoint representation of the global symmetry of the fixture, which is a product $G_{\mathrm{fixt}} = G_{\mathrm{SCFT}} \times F$ of the global symmetry of the SCFT and the global symmetry of the free hypers.

Expanding the index $\mathcal{I}_{\mathrm{free}} = PE[\tau\chi_{\mathrm{F}}^{R}]$ of the free hypers and removing their contribution from (2.4), we arrive at

$$\mathcal{I}_{\mathrm{SCFT}} = \mathcal{I}/\mathcal{I}_{\mathrm{free}} = 1 + \chi_{G_{\mathrm{SCFT}}}^{\mathrm{adj}}\tau^2 + \cdots$$

from which we read off the global symmetry of the SCFT.

To determine when a fixture has an enhanced global symmetry, we note that in (2.3) the first term in the sum over representations (coming from the trivial representation of $\mathfrak{e}_7$) gives, to second order in τ [35],

$$\mathcal{I} = 1 + \chi_{G_{\mathrm{manifest}}}^{\mathrm{adj}}\tau^2 + \cdots$$

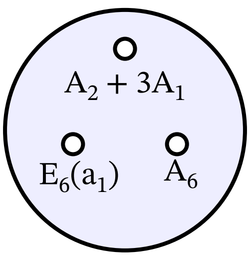
encoding the manifest global symmetry group. The fixture has an enhanced global symmetry if there are terms contributing at order τ^2 coming from the sum over $\lambda > 0$.

As explained in [8], to order τ^2 (2.3) simplifies to

$$\mathcal{I} = 1 + \chi_{G_{\mathrm{manifest}}}^{\mathrm{adj}}\tau^2 + \left[\sum_{\lambda>0} \frac{\prod_{i=1}^{3}\chi^{\lambda}(\mathbf{a}_i|\tau)}{\chi^{\lambda}(\{\tau\}|\tau)}\right]_{\mathcal{O}(\tau^2)} \tag{2.5}$$

To compute (2.5), we consider each $\mathfrak{e}_7$ representation in the sum to be a reducible representation of $\mathfrak{su}(2)\times\mathfrak{f}$ and plug in the corresponding character expansion, where the embedded $\mathfrak{su}(2)$ has fugacity τ. The decomposition of any $\mathfrak{e}_7$ representation in terms of $\mathfrak{su}(2)\times\mathfrak{f}$ representations can be obtained using the projection matrices listed in appendix B.

As an example of such a calculation, the fixture



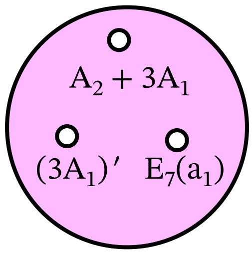
has manifest global symmetry $(G_2)_{28} \times \mathrm{SU}(2)_{36} \times \mathrm{U}(1)$. Its superconformal index has the expansion:

$$I = 1 + \big[(14,1)_0 + (1,3)_0 + (1,1)_0 \\ + \underbrace{(1,2)_1 + (1,2)_{-1}}_{56} + \underbrace{(7,3)_0}_{133} + \underbrace{(7,2)_1 + (7,2)_{-1}}_{912} + \underbrace{(14,1)_0 + (7,1)_0}_{1539}\big]\tau^2 + \dots$$

where we've noted the representations in the sum in (2.3) which make additional contributions to the index at this order. Putting together these contributions, the global symmetry is enhanced to $(E_6)_{18} \times (G_2)_{10}$.

In computing the expansion of (2.3) to order τ^2 we truncate the sum over representations. Knowing exactly at which representation we should truncate the sum for each fixture is tedious to determine due to the complicated Weyl group of $\mathfrak{e}_7$, so in practice we truncate the sum at a very large dimensional representation and check that our results are consistent with various S-dualities. Here, we summed over all irreducible representations of $\mathfrak{e}_7$ up to the $980,343$ dimensional irrep.

The largest representation of $\mathfrak{e}_7$ contributing at order τ^2 was the $253,935$. This occurred for two fixtures:

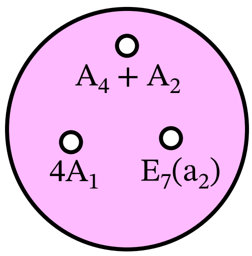


has manifest global symmetry $\mathrm{Sp}(3)_{20} \times \mathrm{SU}(2)_{19} \times (G_2)_{28}$. Its superconformal index picks up contributions at $O(\tau)$ from the 56, 133 and 912 representations, indicating hypermultiplets transforming as the $\frac{1}{2}(6,1,7) + \frac{1}{2}(1,2,7) + \frac{1}{2}(6,1,1)$ of the manifest global symmetry. As shown below, the full index receives contributions from representations up to the $253,935$:

$$I = 1 + \big[\underbrace{(6,1,7)}_{56} + \underbrace{(1,2,7)}_{133} + \underbrace{(6,1,1)}_{912}\big]\tau \\ + \big[\cdots + (1,3,1) + (1,1,14) + \underbrace{(14',2,1)}_{912} + \underbrace{(14,1,7)}_{40{,}755} + \underbrace{(6,2,7)}_{86{,}184} + \underbrace{(21,1,1)}_{253{,}935}\big]\tau^2$$

where the $\dots$ indicate the contribution to $O(\tau^2)$ from the free hypermultiplets. This last contribution completes the enhancement of the global symmetry of the interacting SCFT to $(E_8)_{12}$, and this mixed fixture is the $(E_8)_{12}$ SCFT with 31 hypermultiplets.

The fixture



has hypermultiplets in the $\frac{1}{2}(6,3)+\frac{1}{2}(1,2)+\frac{1}{2}(1,4)$ and has manifest global symmetry $\mathrm{Sp}(3)_{19}\times\mathrm{SU}(2)_{108}$ enhanced to $(E_7)_{16}\times\mathrm{SU}(2)_9$ with, again, the final enhancement coming from the $253,935$.

3 Tinkertoys

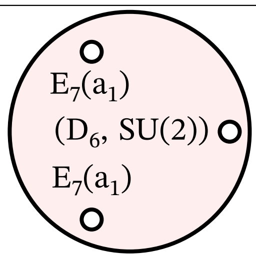
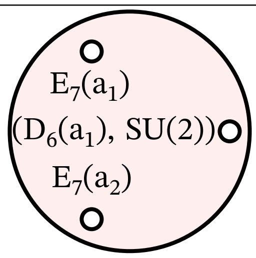
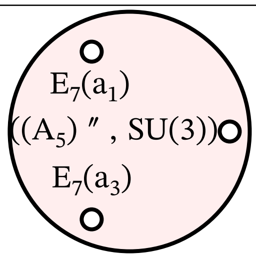
3.1 Free-field fixtures

We indicate a 3-punctured sphere, in the tables below, by listing the Bala-Carter labels of the three punctures. For all but one of the free-field fixtures, one of the punctures is an irregular puncture (in the sense used in our previous papers), which we denote by the pair $(\mathcal{O},G)$, where $\mathcal{O}$ is the regular puncture obtained as the OPE of the two regular punctures which collide. This fixture is attached to the rest of the surface via a cylinder

$$(\mathcal{O},G)\xleftrightarrow{\;G\;}\mathcal{O}$$

with gauge group $G\subset E_7$. The exception is #22, which consists of three regular punctures, and was first discussed in [26].

For each of the free-field fixtures, we indicate how the hypermultiplets transform under the manifest global symmetry of the fixture.

#	Fixture	n_h	Representation
1		1	$\frac{1}{2}(2)$
2		0	empty
3		0	empty

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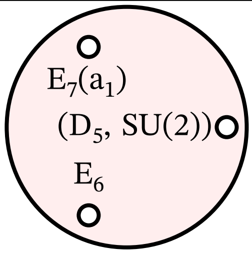
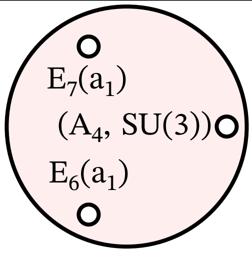
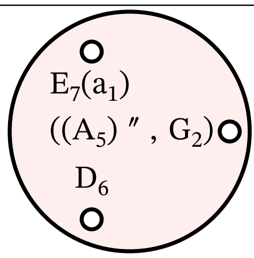
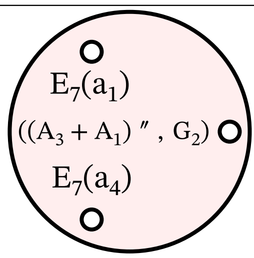
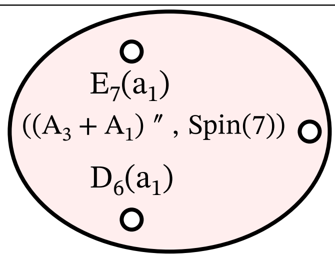
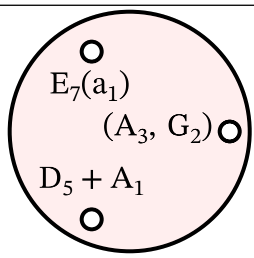
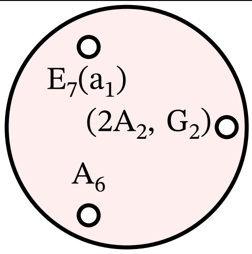
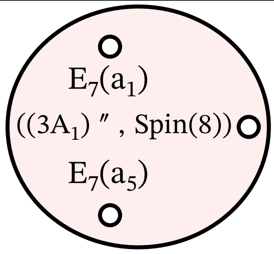
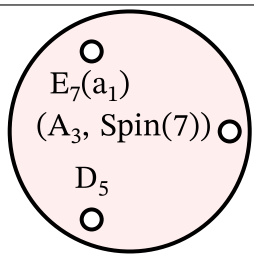
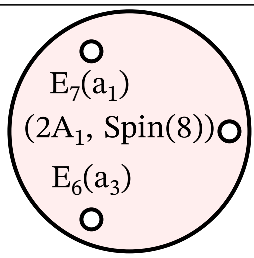
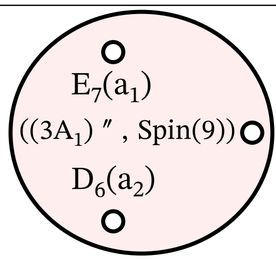
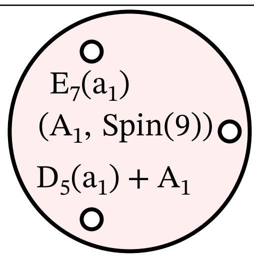
#	Fixture	n_h	Representation
4		0	empty
5		0	empty
6		7	$\frac{1}{2}(2,7)$
7		0	empty
8		8	$\frac{1}{2}(2,8)$
9		0	empty

#	Fixture	n_h	Representation
10		0	empty
11		0	empty
12		8	$\frac{1}{2}(2, 8)$
13		0	empty
14		9	$\frac{1}{2}(2, 9)$
15		0	empty

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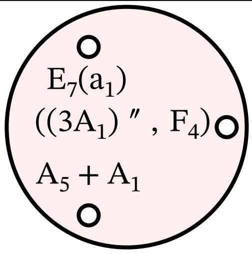
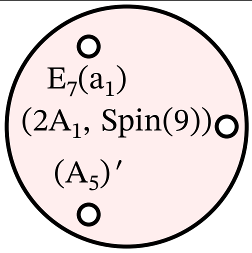
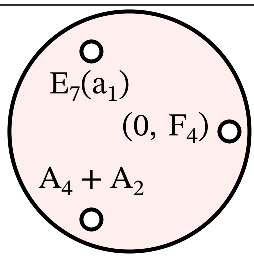
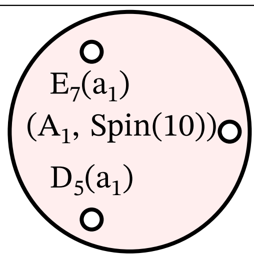
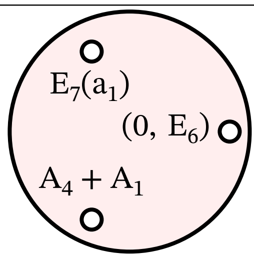
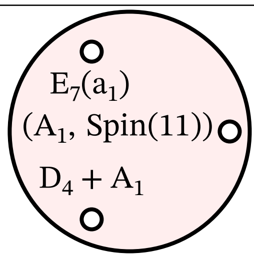
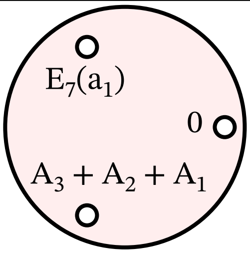
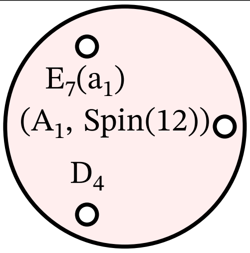
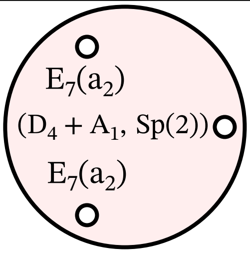
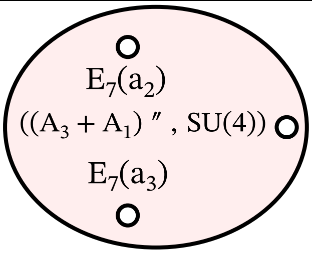
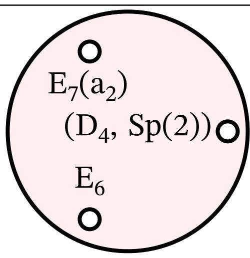
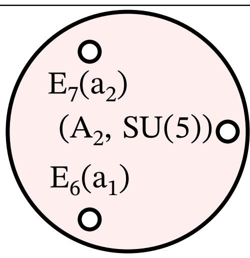
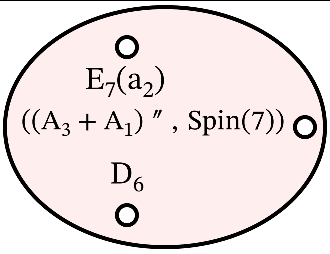
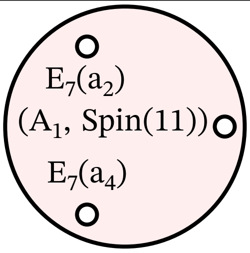
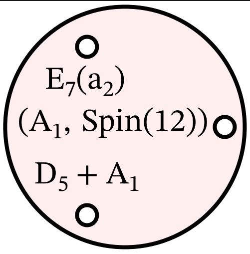
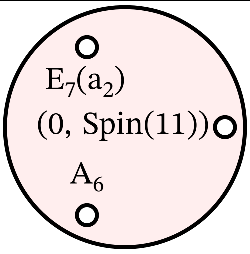
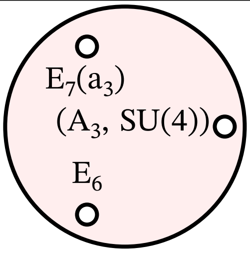
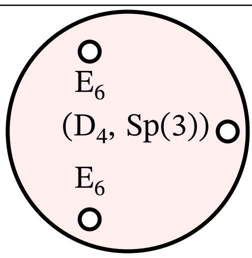
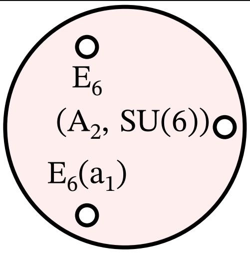
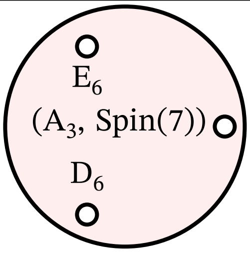
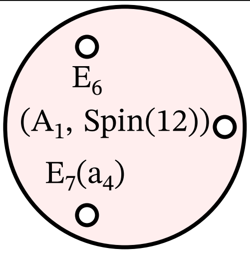
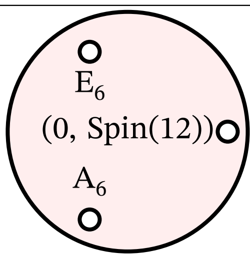
#	Fixture	n_h	Representation
16		26	$\frac{1}{2}(2, 26)$
17		9	$\frac{1}{2}(2, 9)$
18		0	empty
19		10	$\frac{1}{2}(2, 10)$
20		27	(27)
21		22	$\frac{1}{2}(4, 11)$

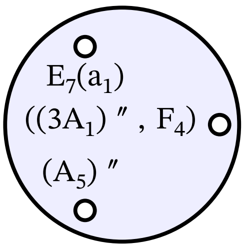
#	Fixture	n_h	Representation
22		84	$\frac{1}{2}(3, 56)$
23		36	$\frac{1}{2}(6, 12)$
24		2	$\frac{1}{2}(4)$
25		0	empty
26		0	empty
27		0	empty

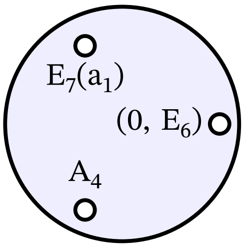
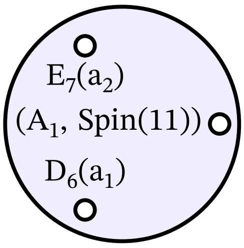
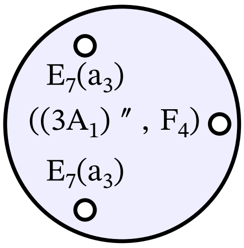
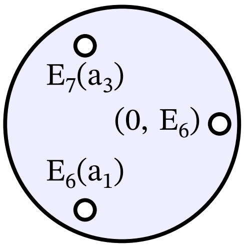
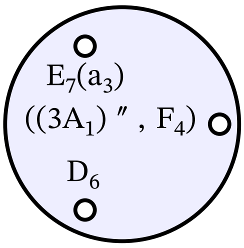
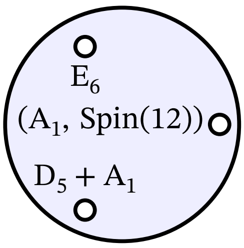
#	Fixture	n_h	Representation
28		7	$\frac{1}{2}(2, 7)$
29		16	$\frac{1}{2}(32)$
30		28	$\frac{1}{2}(2, 12) + \frac{1}{2}(1, 32)$
31		0	empty
32		0	empty
33		12	$\frac{1}{2}(2, 2, 6)$

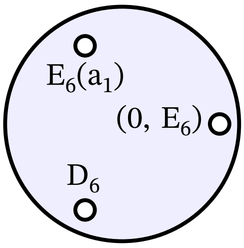
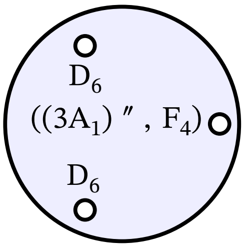
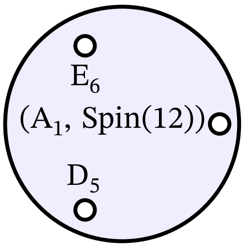
#	Fixture	n_h	Representation
34		12	$(2,6)$
35		7	$\frac{1}{2}(2,7)$
36		28	$\frac{1}{2}(2,12)+\frac{1}{2}(1,32)$
37		12	$\frac{1}{2}(2,12)$

3.2 Interacting fixtures with one irregular puncture

There are 10 interacting fixtures involving two regular and one irregular puncture. They are all generalized Minahan-Nemeschansky theories (whose Higgs branches are (multi-)instanton moduli spaces for $E_{6,7,8}$) or products thereof, except for #9 and #10. The former is the $(F_4)_{12} \times \mathrm{SU}(2)^2_7$ theory which first appeared in [7] as a fixture in the untwisted D_4 theory. The latter is new.

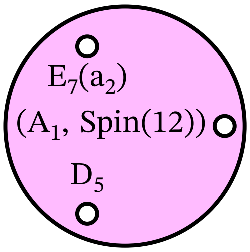
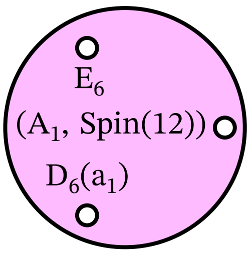
#	Fixture	$(n_2,n_3,n_4,n_6,n_8,n_{10},n_{12},n_{14},n_{18})$	(n_h,n_v)	Theory
1		$(0,0,0,1,0,0,0,0,0)$	$(40,11)$	$(E_8)_{12}$ SCFT

#	Fixture	$(n_2, n_3, n_4, n_6, n_8, n_{10}, n_{12}, n_{14}, n_{18})$	(n_h, n_v)	Theory
2		$(0, 0, 0, 1, 0, 0, 0, 0, 0)$	$(40, 11)$	$(E_8)_{12}$ SCFT
3		$(0, 0, 1, 0, 0, 0, 0, 0, 0)$	$(24, 7)$	$(E_7)_8$ SCFT
4		$(0, 2, 0, 0, 0, 0, 0, 0, 0)$	$(32, 10)$	$\left[(E_6)_6 \text{ SCFT}\right]^2$
5		$(0, 2, 0, 0, 0, 0, 0, 0, 0)$	$(32, 10)$	$\left[(E_6)_6 \text{ SCFT}\right]^2$
6		$(0, 1, 0, 1, 0, 0, 0, 0, 0)$	$(39, 16)$	$(E_6)_{12} \times SU(2)_7$ SCFT
7		$(0, 0, 0, 1, 0, 0, 0, 0, 0)$	$(40, 11)$	$(E_8)_{12}$ SCFT

#	Fixture	$(n_2, n_3, n_4, n_6, n_8, n_{10}, n_{12}, n_{14}, n_{18})$	(n_h, n_v)	Theory
8		$(0,1,0,1,0,0,0,0,0)$	$(39,16)$	$(E_6)_{12} \times \mathrm{SU}(2)_7$ SCFT
9		$(0,0,0,2,0,0,0,0,0)$	$(46,22)$	$(F_4)_{12} \times \mathrm{SU}(2)_7^2$ SCFT
10		$(0,0,1,1,0,0,0,0,0)$	$(48,18)$	$(E_8)_{12} \times \mathrm{SU}(2)_8$ SCFT

3.3 Mixed fixtures with one irregular puncture

There are two mixed fixtures with two regular and one irregular puncture. The value of n_h listed below is the one associated to the SCFT, *after* subtracting the contribution of the free hypermultiplets.

#	Fixture	$(n_2, n_3, n_4, n_6, n_8, n_{10}, n_{12}, n_{14}, n_{18})$	(n_h, n_v)	Theory
1		$(0,0,1,0,0,0,0,0,0)$	$(24,7)$	$(E_7)_8$ SCFT $+ \frac{1}{2}(2,12)$
2		$(0,0,1,0,0,0,0,0,0)$	$(24,7)$	$(E_7)_8$ SCFT $+ \frac{1}{2}(2,12)$

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3.4 Interacting and mixed fixtures

There are exactly 11,000 fixtures with three regular punctures. Of these, 654 have enhanced global symmetry, 262 are mixed, and 1 is free.

Rather than listing all of these, we have created a web application where the interested reader can explore these theories for him or herself. The website, https://golem.ph.utexas.edu/class-S/E7/ , has three sections:

- A compendium of the 44 regular punctures and their properties:
  https://golem.ph.utexas.edu/class-S/E7/punctures/
- A compendium of the 11,000 3-punctured spheres:
  https://golem.ph.utexas.edu/class-S/E7/fixtures/
- A compendium of the 178,365 4-punctured spheres and their S-duality frames:
  https://golem.ph.utexas.edu/class-S/E7/four_punctured_sphere/

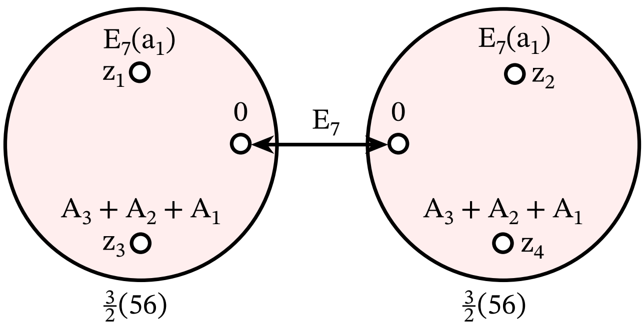
For each S-duality frame, clicking on a fixture brings up its properties. When viewing a fixture, clicking on a puncture brings up the latter's properties.

If you find the data on the website useful in your own work, please cite *this* paper instead.

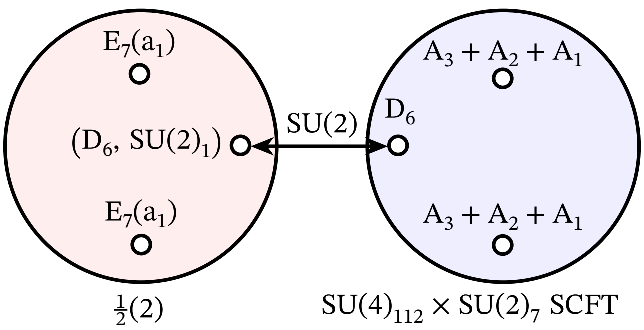
4 Applications

4.1 $E_7 + 3(56)$

E_7 gauge theory, with three fundamental hypermultiplets, is superconformal. It is realized as the 4-punctured sphere



The S-dual theory is an SU(2) gauging of the $SU(4)_{112} \times SU(2)_7$ SCFT, with an additional half-hypermultiplet in the fundamental.



The k-differentials, which determine the Seiberg-Witten solution, are

$$\begin{aligned}
\phi_2(z) &= \frac{u_2\, z_{12} z_{34}\, (dz)^2}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)} \\
\phi_6(z) &= \frac{u_6\, z_{12}^2 z_{34}^4 (dz)^6}{(z-z_1)^2(z-z_2)^2(z-z_3)^4(z-z_4)^4} \\
\phi_8(z) &= \frac{u_8\, z_{12}^2 z_{34}^6 (dz)^8}{(z-z_1)^2(z-z_2)^2(z-z_3)^6(z-z_4)^6} \\
\phi_{10}(z) &= \frac{u_{10}\, z_{12}^2 z_{34}^8 (dz)^{10}}{(z-z_1)^2(z-z_2)^2(z-z_3)^8(z-z_4)^8} \\
\phi_{12}(z) &= \frac{u_{12}\, z_{12}^3 z_{34}^9 (dz)^{12}}{(z-z_1)^3(z-z_2)^3(z-z_3)^9(z-z_4)^9} \\
\phi_{14}(z) &= \frac{u_{14}\, z_{12}^3 z_{34}^{11} (dz)^{14}}{(z-z_1)^3(z-z_2)^3(z-z_3)^{11}(z-z_4)^{11}} \\
\phi_{18}(z) &= \frac{u_{18}\, z_{12}^4 z_{34}^{14} (dz)^{18}}{(z-z_1)^4(z-z_2)^4(z-z_3)^{14}(z-z_4)^{14}}
\end{aligned}$$

4.2 Adding $(E_8)_{12}$ SCFTs

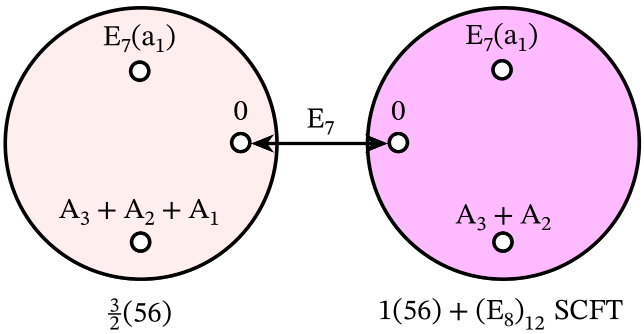
Since the index of the 56 of E_7 is 12, we can start with the $E_7 + 3(56)$ gauge theory and trade half-hypermultiplets in the 56 for copies of the $(E_8)_{12}$ SCFT. A similar analysis was carried out for the $E_6 + 4(27)$ gauge theory in [8].

For n half-hypermultiplets in the 56 and $6-n$ copies of the $(E_8)_{12}$ SCFT, the theory has flavor symmetry

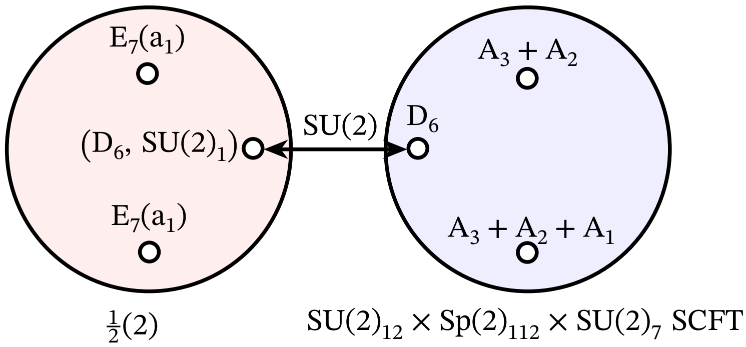
$$F = \mathrm{SU}(2)_{12}^{6-n} \times \mathrm{SO}(n)_k,$$

where $k = 112$ for $n \neq 3$, and $k = 224$ for $n = 3$. Each of these theories has an S-dual description as an SU(2) gauging of the $\mathrm{SU}(2)_{12}^{6-n} \times \mathrm{SO}(n)_k \times \mathrm{SU}(2)_7$ SCFT, with an additional half-hypermultiplet in the fundamental.

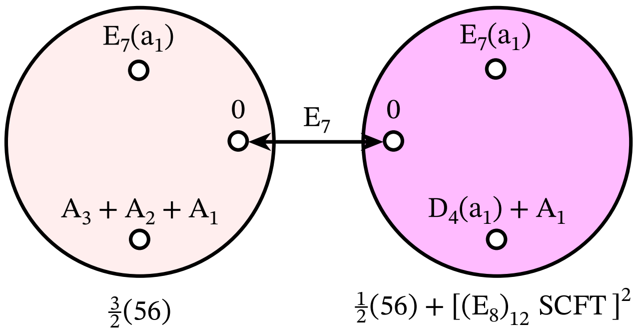
$n = 5$. With one copy of the $(E_8)_{12}$ SCFT,



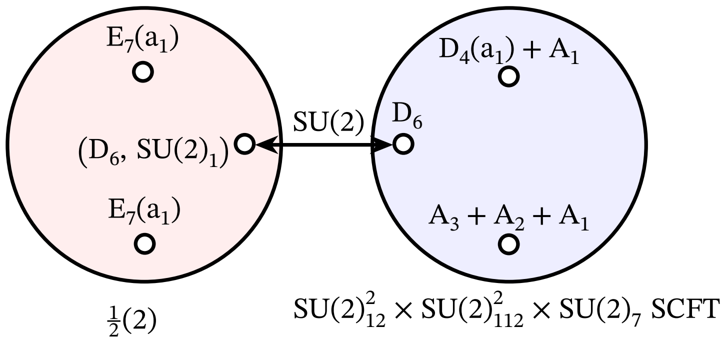
is dual to



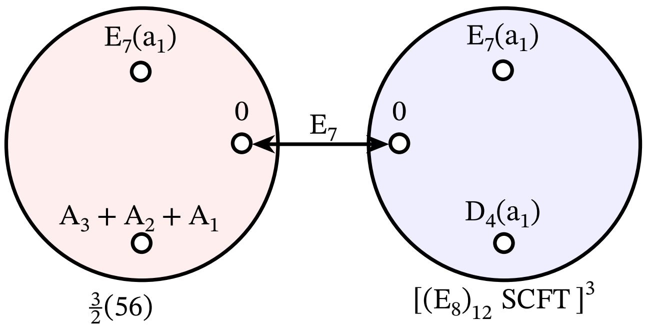
$\boldsymbol{n = 4.}$ With two copies of the $(E_8)_{12}$ SCFT,



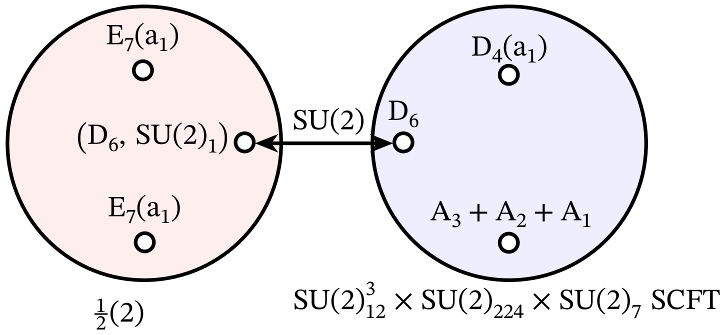
is dual to



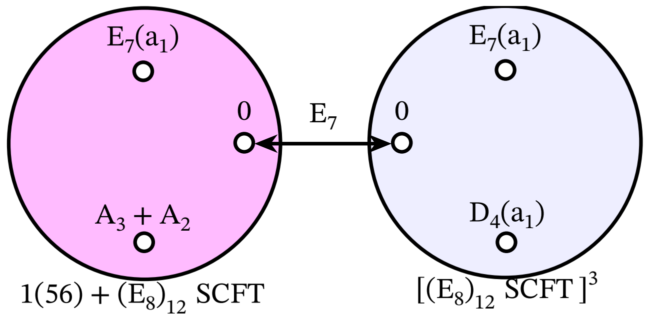
$\boldsymbol{n = 3.}$ With three copies of the $(E_8)_{12}$ SCFT,



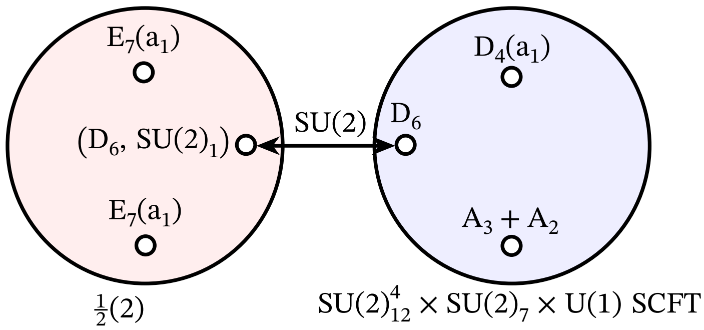
is dual to



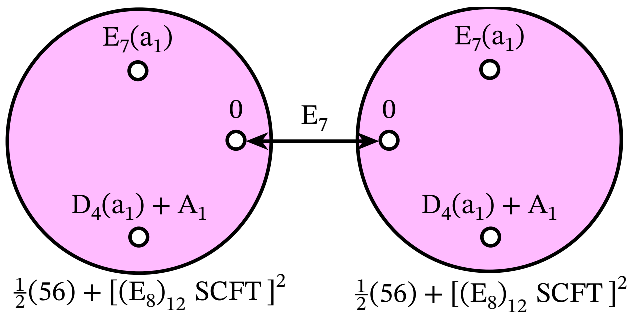
$\boldsymbol{n = 2.}$ With four copies of the $(E_8)_{12}$ SCFT, we have two possible realizations. Either,



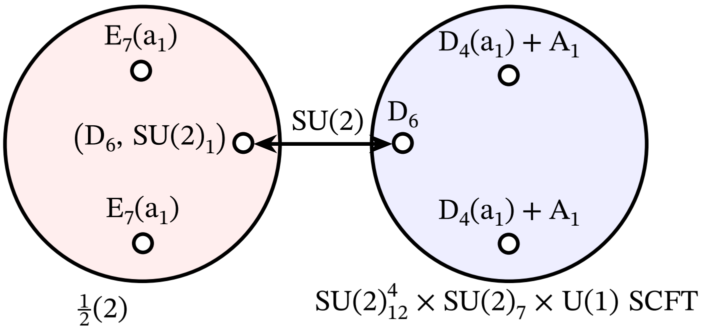
is dual to



or

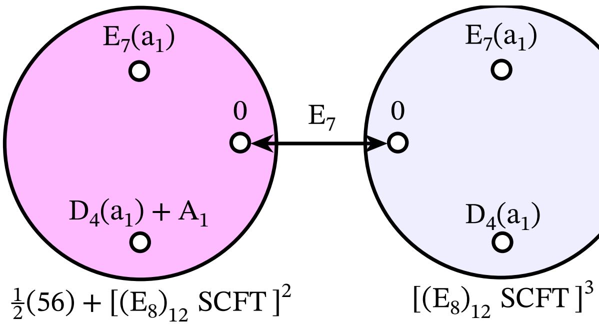


is dual to

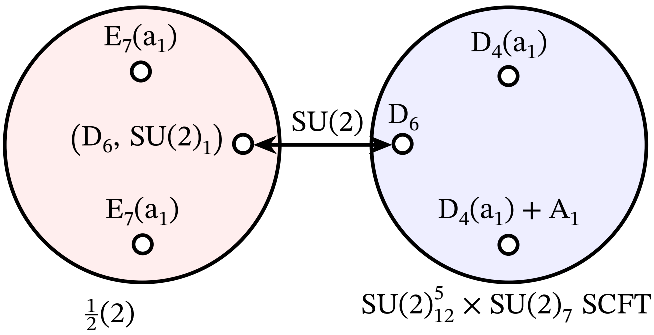


This gives two distinct realizations of the $SU(2)^4_{12} \times SU(2)_7 \times U(1)$ SCFT.

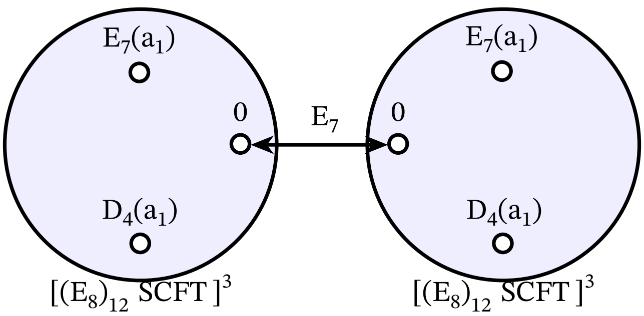
$\boldsymbol{n = 1.}$ With five copies of the $(E_8)_{12}$ SCFT,



is dual to



$\boldsymbol{n = 0.}$ Finally, the E_7 gauging of six copies of the $(E_8)_{12}$ SCFT,



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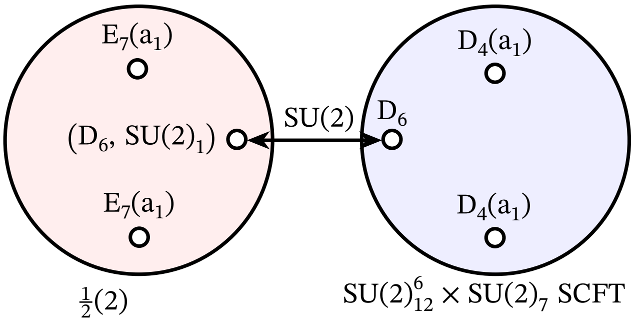
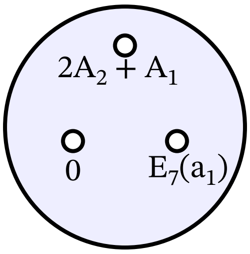
is dual to



4.3 New 6d realizations of SCFTs

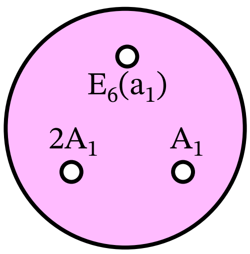
4.3.1 Higher-rank Minahan-Nemeschansky SCFTs

The rank-n Minahan-Nemeschansky theories have Higgs branches which are the n-instanton moduli space for $E_{6,7,8}$. They are realized in F-theory as the SCFT on n D3-branes probing a IV^*, III^* or II^* singularity. For small values of n, they appear ubiquitously among our fixtures. Here, we find our first realization, in the E-series, of the $(E_8)_{36} \times \mathrm{SU}(2)_{38}$ SCFT, which is the theory on $n = 3$ D3 branes probing a II^* singularity in F-theory. This is realized on the fixture

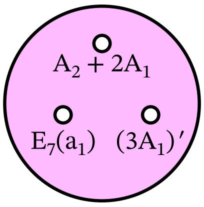
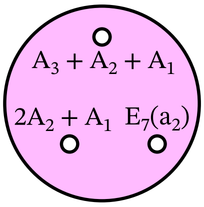


4.3.2 Other low-rank SCFTs

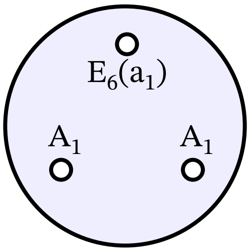
In addition to various Minahan-Nemeschansky theories, the $(F_4)_{12} \times \mathrm{SU}(2)^2_7$ theory and the $(E_8)_{12} \times \mathrm{SU}(2)_8$ theory (see section 3.2), we find two additional rank-2 SCFTs. The $\mathrm{Spin}(20)_{16}$ SCFT, for which we find a new realization, appeared previously as the mixed fixture



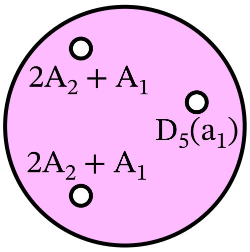
in the E_6 theory. The $\mathrm{Sp}(6)_{11}$ theory is new.

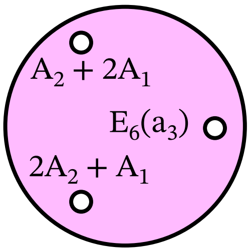
Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{10}, n_{12}, n_{14}, n_{18})$	(n_h, n_v)	Global Symmetry	Free Hypers
	$(0, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0)$	$(72, 26)$	$\mathrm{Spin}(20)_{16}$	15
	$(0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0)$	$(58, 26)$	$\mathrm{Sp}(6)_{11}$	8

We also find several new rank-3 SCFTs. The $\mathrm{SU}(12)_{18}$ theory first appeared as the interacting fixture

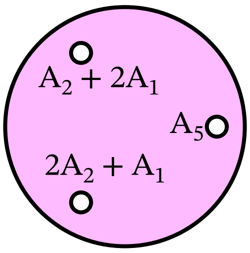


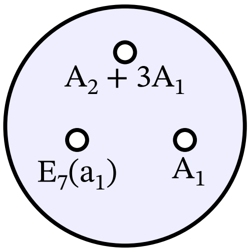
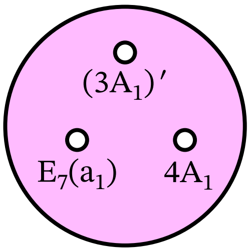
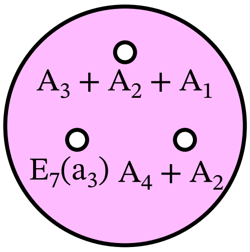
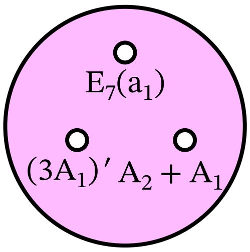
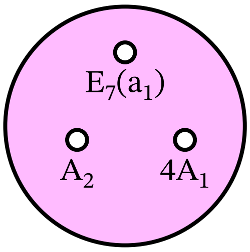
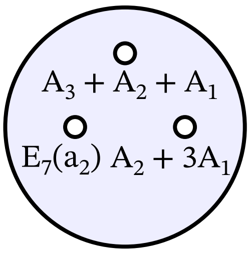
in the E_6 theory. Here, it has two distinct realizations as a mixed fixture. The $\mathrm{Sp}(3)_{26}$ SCFT also appeared in the E_6 theory, as the mixed fixtures

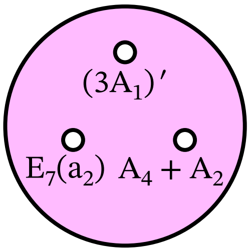
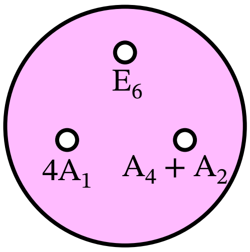
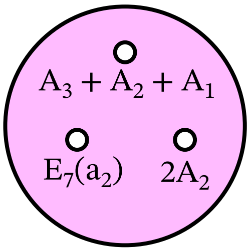
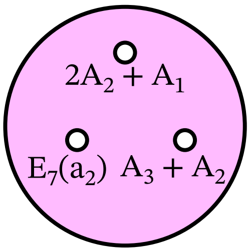
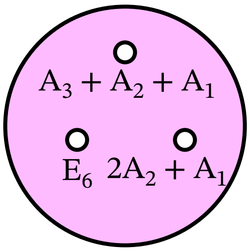
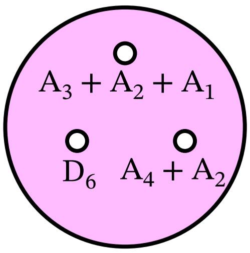


,


and the $\mathrm{Sp}(3)_{26} \times \mathrm{SU}(2)_7$ SCFT appeared as



Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{10}, n_{12}, n_{14}, n_{18})$	(n_h, n_v)	Global Symmetry	Free Hypers
	$(0,0,0,0,1,0,0,0,1,1,0)$	$(136,61)$	$\mathrm{Spin}(19)_{28}$	0
	$(0,0,0,0,1,1,0,0,0,0,1)$	$(125,61)$	$\mathrm{Sp}(7)_{19}$	3
	$(0,1,0,0,0,1,0,0,1,0,0)$	$(70,43)$	$\mathrm{Sp}(3)_{26}$	6
 	$(0,0,0,0,1,1,1,0,0,0,0)$	$(100,43)$	$\mathrm{SU}(12)_{18}$	7 9
	$(0,0,1,0,0,0,0,1,0,1,0)$	$(96,53)$	$(E_6)_{28}$	0

Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{10}, n_{12}, n_{14}, n_{18})$	(n_h, n_v)	Global Symmetry	Free Hypers
	$(0,0,1,0,0,1,0,1,0,0,0)$	$(88,41)$	$\mathrm{Spin}(15)_{20} \times \mathrm{SU}(2)_{16}$	3
	$(0,0,1,0,1,1,0,0,0,0,0)$	$(72,33)$	$\mathrm{Spin}(12)_{16} \times \mathrm{Spin}(7)_{12}$	9
	$(0,0,1,0,0,1,0,1,0,0,0)$	$(81,41)$	$(F_4)_{16} \times \mathrm{Sp}(3)_{11}$	3
 	$(0,0,1,0,1,0,0,1,0,0,0)$	$(73,37)$	$\mathrm{Sp}(4)_{12} \times \mathrm{Sp}(3)_{11}$	5
	$(0,0,0,0,1,1,0,0,1,0,0)$	$(77,49)$	$\mathrm{Sp}(3)_{26} \times \mathrm{SU}(2)_{7}$	6

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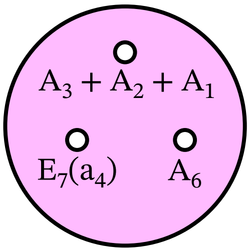
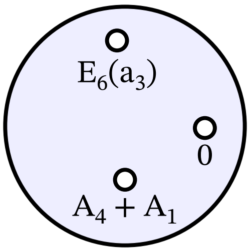
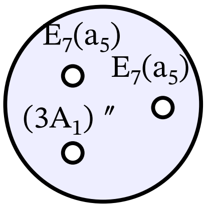
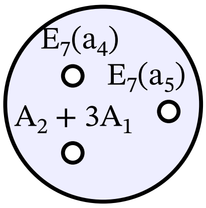
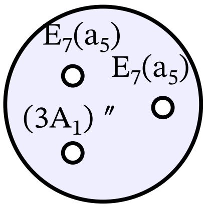
Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{10}, n_{12}, n_{14}, n_{18})$	(n_h, n_v)	Global Symmetry	Free Hypers
	(0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 0)	(73, 45)	$SU(2)_{128-k}$ $\times SU(2)_k$ $\times Sp(3)_{11}$	3

4.3.3 New SCFTs with exceptional global symmetry

In our list, we find 10 new SCFTs whose global symmetry is a simple exceptional group. One of these is the rank-3 $(E_6)_{28}$ example listed in section 4.3.2. We also find two new realizations of the $(E_7)_{24}$ SCFT, which was first found in [8] as the interacting fixture



in the E_6 theory. Here, it has two realizations, both as mixed fixtures.

Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{10}, n_{12}, n_{14}, n_{18})$	(n_h, n_v)	Global Symmetry
	(0, 0, 4, 0, 2, 1, 0, 1, 3, 3, 4)	(416, 374)	$(G_2)_{28}$
	(0, 0, 2, 0, 2, 1, 0, 1, 2, 2, 2)	(280, 240)	$(G_2)_{28}$
	(0, 0, 4, 0, 2, 2, 0, 1, 4, 3, 5)	(504, 447)	$(F_4)_{24}$

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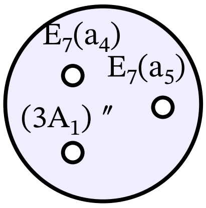
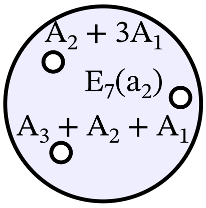
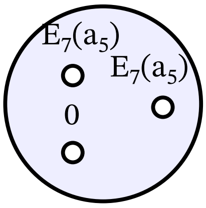
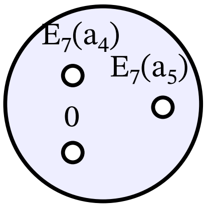
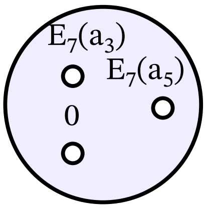
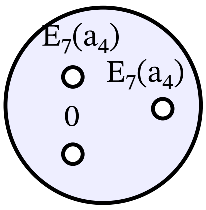
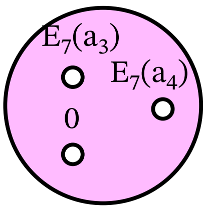
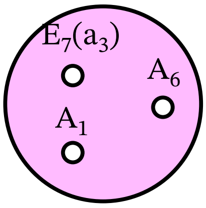
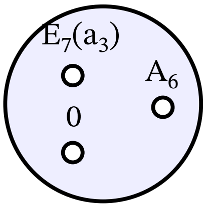
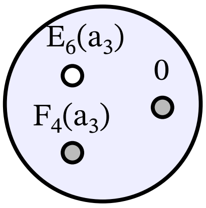
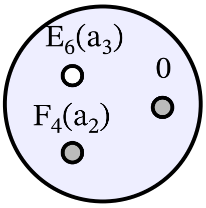
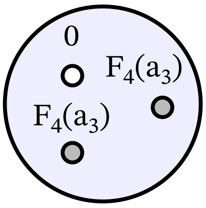
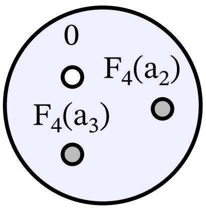
Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{10}, n_{12}, n_{14}, n_{18})$	(n_h, n_v)	Global Symmetry
	$(0, 0, 2, 0, 2, 2, 0, 1, 3, 2, 3)$	$(368, 313)$	$(F_4)_{24}$
	$(0, 0, 1, 0, 0, 0, 0, 1, 0, 1, 0)$	$(96, 53)$	$(E_6)_{28}$
	$(0, 0, 4, 0, 2, 2, 0, 2, 4, 4, 6)$	$(612, 528)$	$(E_7)_{36}$
	$(0, 0, 2, 0, 2, 2, 0, 2, 3, 3, 4)$	$(476, 394)$	$(E_7)_{36}$
	$(0, 1, 2, 0, 1, 1, 0, 0, 2, 1, 2)$	$(268, 188)$	$(E_7)_{36}$
	$(0, 0, 0, 0, 2, 2, 0, 2, 2, 2, 2)$	$(340, 260)$	$(E_7)_{36}$
	$(0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 0)$	$(104, 54)$	$(E_7)_{24} + \frac{1}{2}(56)$

Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{10}, n_{12}, n_{14}, n_{18})$	(n_h, n_v)	Global Symmetry
	$(0,1,0,0,1,1,0,0,1,0,0)$	$(104, 54)$	$(E_7)_{24} + \frac{1}{2}(12,2)$
	$(0,1,0,0,1,1,0,0,1,0,1)$	$(168, 89)$	$(E_8)_{36}$

In the E_6 theory, fixtures of the form $(0, D, D)$ or $(2A_2, D, D)$, where D is either $E_6(a_3)$ or $E_6(a_1)$, are all bad, so we do not get additional SCFTs with simple exceptional global symmetries in this way. However, we can construct 5 more of these in the twisted sector:

Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{12})$	(n_h, n_v)	Global Symmetry
	$(0,4,0,1,1,2,2,3)$	$(208, 173)$	$(F_4)_{18}$
	$(0,1,1,1,1,1,1,1)$	$(120, 87)$	$(F_4)_{18}$
	$(0,6,0,2,2,4,4,6)$	$(384, 336)$	$(E_6)_{24}$
	$(0,3,1,2,2,3,3,4)$	$(296, 250)$	$(E_6)_{24}$

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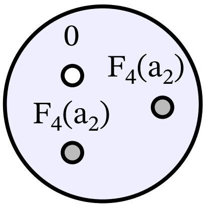
Fixture	$(n_2, n_3, n_4, n_5, n_6, n_8, n_9, n_{12})$	(n_h, n_v)	Global Symmetry
	$(0, 0, 2, 2, 2, 2, 2, 2)$	$(208, 164)$	$(E_6)_{24}$

4.3.4 Enhanced global symmetries and Sommers-Achar group action on the Higgs branch

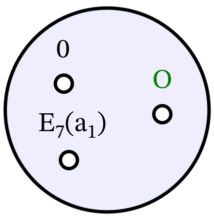
As in section 5 of [12], we can consider families of fixtures where we fix two punctures and let the third vary over a special piece, $\{O\}$. We denote by O_s the special puncture in this special piece and by O_m the puncture with the maximal Sommers-Achar group, whose Hitchin pole is $(d(O), S_n)$ [26]. It is often the case that, when $O = O_s$, a simple factor in the manifest global symmetry group associated to one of the two fixed punctures becomes enhanced as

$$F_{kn} \to (F_k)^n$$

When this happens, then, for $O = O_m$, the F_{kn} is unenhanced and, as O varies over the special piece, the enhancement is the subgroup of $(F_k)^n$ which is invariant under $C(O)$ acting by permutations of the n copies of F_k.

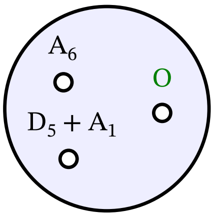
We found numerous examples of this in [8] and [12], and were able to verify, using various S-dualities (see, e.g., section 4 of [8]) that the levels of the factors of F in the global symmetry behaved as predicted by this permutation action.

The E_7 theory provides further examples of this phenomenon. One interesting example is given by fixtures with 0 and $E_7(a_1)$ punctures and the third puncture O coming from the special piece $\{D_4(a_1), (A_3 + A_1)', 2A_2 + A_1\}$. The $(E_7)_{36}$ of puncture 0 is enhanced to the subgroup of $(E_7)^3_{12}$ that is invariant under $C(O)$. With certain SU(2) groups coming from O, the enhanced E_7 groups are further enhanced to E_8 groups. The resulting theories are E_8 Minahan-Nemeschansky SCFTs of various rank l whose Higgs branches are the moduli space of l E_8 instantons, denoted by $M(E_8, l)$.



O	$C(O)$	Theory	Higgs Branch	$\dim_{\mathbb{H}} \mathcal{H}$	(n_h, n_v)
$D_4(a_1)$	1	$[(E_8)_{12}\ \text{SCFT}]^3$	$M(E_8, 1)^3$	87	$(120, 33)$
$(A_3 + A_1)'$	$\mathbb{Z}_2$	$[(E_8)_{12}\,\text{SCFT}]$ $\times[(E_8)_{24} \times \text{SU}(2)_{13}\,\text{SCFT}]$	$M(E_8, 1)$ $\times M(E_8, 2)$	88	$(133, 45)$
$2A_2 + A_1$	S_3	$[(E_8)_{36} \times \text{SU}(2)_{38}\ \ \text{SCFT}]$	$M(E_8, 3)$	89	$(158, 69)$

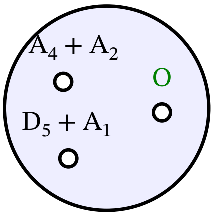
We can use this behavior to fill in some of the missing levels which cannot be determined from the information extracted from superconformal index. For example, still from the special piece $\{D_4(a_1), (A_3+A_1)', 2A_2+A_1\}$, we find another sequence, given by the fixtures with punctures (A_6, D_5+A_1, O):



O	$C(O)$	Global symmetry
$D_4(a_1)$	1	$\mathrm{SU}(2)_{12}^4 \times \mathrm{SU}(2)_{k_1} \times \mathrm{SU}(2)_{k_2} \times \mathrm{SU}(2)_{36-k_1-k_2}$
$(A_3+A_1)'$	$\mathbf{Z}_2$	$\mathrm{SU}(2)_{13} \times \mathrm{SU}(2)_{24} \times \mathrm{SU}(2)_{12}^2 \times \mathrm{SU}(2)_k \times \mathrm{SU}(2)_{36-k}$
$2A_2+A_1$	S_3	$\mathrm{SU}(2)_{36} \times \mathrm{SU}(2)_{38} \times \mathrm{SU}(2)_{12} \times \mathrm{SU}(2)_{36}$

The $\mathrm{SU}(2)_{36}$ from the A_6 puncture is enhanced to subgroups of $\mathrm{SU}(2)_{12}^3$. The Sommers-Achar group action tells us $k_1 = k_2 = k = 12$.

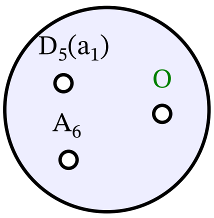
For another example, let's look at the special piece $\{E_7(a_5), D_6(a_2), A_5+A_1\}$. Consider the fixture with punctures (A_4+A_2, D_5+A_1, O):



O	$C(O)$	Global symmetry
$E_7(a_5)$	1	$(G_2)_{12} \times \mathrm{SU}(2)_{k_1} \times \mathrm{SU}(2)_{k_2} \times \mathrm{SU}(2)_{72-k_1-k_2}$
$D_6(a_2)$	$\mathbf{Z}_2$	$(G_2)_{12} \times \mathrm{SU}(2)_9 \times \mathrm{SU}(2)_{72-k} \times \mathrm{SU}(2)_k$
A_5+A_1	S_3	$(G_2)_{12} \times \mathrm{SU}(2)_{26} \times \mathrm{SU}(2)_{72}$

Similar to the previous examples, we can dertermine that $k_1 = k_2 = k = 24$.

For fixture $(A_6, D_5(a_1), O)$ where O belongs to the special piece $\{E_6(a_3), A_5\prime\}$



O	$C(O)$	Global symmetry
$E_6(a_3)$	1	$\mathrm{SU}(2)_{10} \times \mathrm{SU}(4)_{20} \times \mathrm{U}(1) \times \mathrm{SU}(2)_{16-k} \times \mathrm{SU}(2)_k$
A_5'	$\mathbf{Z}_2$	$\mathrm{SU}(2)_{10} \times \mathrm{SU}(2)_9 \times \mathrm{SU}(4)_{20} \times \mathrm{U}(1) \times \mathrm{SU}(2)_{16}$

from which we conclude that $k = 8$.

4.4 Connections with 6d (1,0) SCFTs on T^2

Another large class of $4d$ $\mathcal{N}=2$ SCFTs arises from compactifications of $6d$ $(1,0)$ SCFTs on T^2. Following the recent classification of $6d$ $(1,0)$ SCFTs [36–38], the study of their T^2 compactifications was initiated in [21–23]. In those papers, various T^2 compactifications of $(1,0)$ theories were found to also have class S realizations. Here, we comment on the models which were conjectured to have a class S realization in either the E_7 or E_8 theories.

4.4.1 Very Higgsable theories on T^2

In [21], a subset of the $6d$ $(1,0)$ SCFTs of [37] was singled out by the authors, which they termed "very Higgsable". These SCFTs are those which have a Higgs branch with no tensor multiplet degrees of freedom. In their F-theory realization, these are the theories for which successive blow-downs of -1 curves in the base of the elliptically-fibered Calabi-Yau threefold removes (after a further complex structure deformation) the singularity in the base completely. They found that the central charges of the $4d$ $\mathcal{N}=2$ SCFT resulting from the T^2 compactification of a very Higgsable $6d$ $(1,0)$ theory are given by

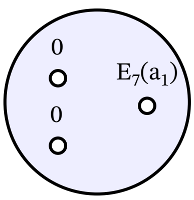
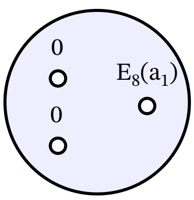
$$\begin{aligned} a &= 24\alpha - 12\beta - 18\gamma, \\ c &= 64\alpha - 12\beta - 8\gamma, \\ k_i &= 192\kappa_i, \end{aligned} \tag{4.1}$$

where α, β, and κ_i are the coefficients appearing in the anomaly 8-form of the $6d$ theory

$$I_8 \supset \alpha p_1(T)^2 + \beta p_1(T) c_2(R) + \gamma p_2(T) + \sum_i \kappa_i p_1(T) \mathrm{Tr}\, F_i^2,$$

which can be computed following [39, 40]; see also [41].

Using this formula, the authors argued that the minimal "conformal matter" theory, $\mathcal{T}(G,1)$ (the theory on a single M5-brane at a $G = ADE$-type singularity), on T^2 coincides with the class S theory of type G on a fixture with two full punctures and one minimal puncture. For $G = E_7, E_8$, these fixtures are

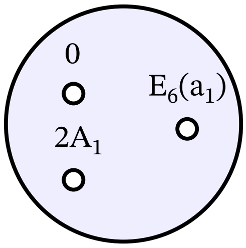
G	Fixture	$(n_2,n_3,n_4,n_5,n_6,n_8,n_9,n_{10},n_{12},$ $n_{14},n_{18},n_{20},n_{24},n_{30})$	(n_h,n_v)	Global Symmetry
E_7		$(0,0,0,0,1,1,0,1,2,2,3,0,0,0)$	$(384,250)$	$(E_7)_{36}\times(E_7)_{36}$
E_8		$(0,0,0,0,1,1,0,0,2,2,3,3,4,5)$	$(1080,831)$	$(E_8)_{60}\times(E_8)_{60}$

The graded Coulomb branch dimensions for these two fixtures are in agreement with those computed from the mirror geometry of the corresponding 6d (1,0) theories on T^2 in [23] and the central charges agree with those obtained in [21].

We can also realize some of the (G, G') conformal matter theories of [42] on E_7 and E_8 fixtures. These conformal matter theories correspond to fractional M5-branes on an ALE singularity:

Global Symmetry	# M5	ALE type
$(E_7, \mathrm{SO}(7))$	$\frac{1}{2}$	E_7
(E_8, G_2)	$\frac{1}{3}$	E_8
(E_8, F_4)	$\frac{1}{2}$	E_8

In [21], the first of these was identified with the E_6 fixture

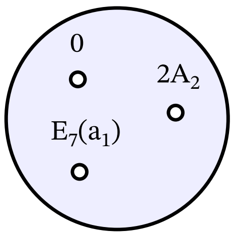


which appears as the third entry in table 3.4 of [8].

Computing the anomaly polynomial of the other two theories following [39, 40] and plugging into (4.1), we find that the T^2 compactified (E_8, G_2) conformal matter theory has central charges

$$a = \frac{149}{6}, c = \frac{86}{3}, k_{E_8} = 36, k_{G_2} = 16.$$

These are the central charges of the class S theory realized by compactifying the E_7 (2,0) theory on



In this realization, only a $(E_7)_{36} \times \mathrm{SU}(2)_{36} \times (G_2)_{16}$ subgroup of the global symmetry group is manifest. We can check the enhancement to $(E_8)_{36} \times (G_2)_{16}$ by computing the order τ^2 expansion of the superconformal index, which is given by

$$\mathcal{I} = 1 + (\chi^{\mathbf{248}}_{E_8} + \chi^{\mathbf{14}}_{G_2})\tau^2 + \ldots$$

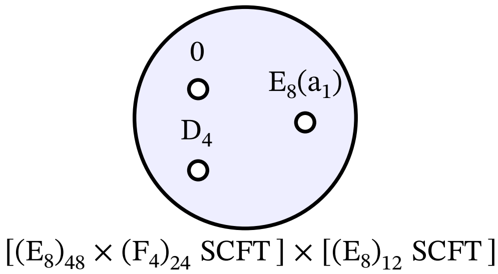
where

$$\chi^{\mathbf{248}}_{E_8} = \chi^{\mathbf{133}}_{E_7} + \chi^{\mathbf{3}}_{\mathrm{SU}(2)} + \chi^{\mathbf{56}}_{E_7}\chi^{\mathbf{2}}_{\mathrm{SU}(2)}.$$

Similarly, we find the T^2 compactified (E_8, F_4) conformal matter theory has central charges

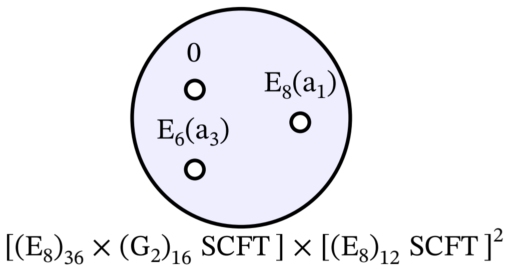
$$a = \frac{179}{3}, c = \frac{196}{3}, k_{E_8} = 48, k_{F_4} = 24$$

Comparing with the E_7 and E_8 tinkertoys [43], we do not find a direct realization in class S. The closest one can come[6] is the fixture



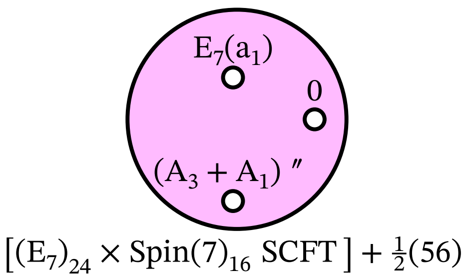
in the E_8 theory. This is a product of the desired SCFT (whose global symmetry is $(E_8)_{48} \times (F_4)_{24}$ and $(n_h, n_v) = (352, 216)$) with the $(E_8)_{12}$ SCFT $(n_h, n_v) = (40, 11)$).

Similarly, [23] also conjectured that the T^2-compactification of the (E_8, G_2) theory is realized in Class-S as the fixture



in the E_8 theory. In fact, this fixture is a product of the desired SCFT with *two* copies of the $(E_8)_{12}$ SCFT.

The fact that these fixtures yield not the desired SCFT, but rather its product with some additional decoupled degrees of freedom, is not unheralded. Already in the case of the T^2 compactification of the $(E_7, \mathrm{SO}(7))$ conformal matter theory, [23] noticed that their class-S realization, the fixture



in the E_7 theory, yields not the $(E_7)_{24} \times \mathrm{Spin}(7)_{16}$ SCFT, but rather the desired SCFT with additional hypermultiplets in the $\frac{1}{2}(56)$ of E_7.

[6]This fixture was conjectured in [23] to realize the SCFT we are seeking. We see here that it realizes, instead, a *product* of the desired SCFT and the Minahan-Nemeschansky $(E_8)_{12}$ SCFT.

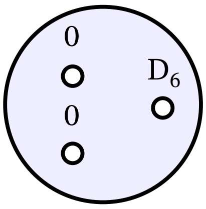
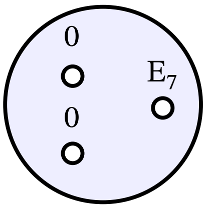
4.4.2 N $M5$ branes probing an ADE singularity on T^2

The T^2 compactification of the $(1,0)$ theory on the worldvolume of $N > 1$ M5-branes on an ALE singularity was studied in [22, 23]. In the F-theory realization of these theories, after successively blowing down all (-1)-curves, one reaches an endpoint which is a chain of (-2)-curves, intersecting as an A_{N-1} Dynkin diagram. Thus, these theories are not in the class of very Higgsable SCFTs, but are instead Higgsable to a $(2,0)$ theory. The T^2 compactifications of such $(1,0)$ theories were systematically studied in [22]. They found that, in general, the T^2 compactification of a $(1,0)$ SCFT Higgsable to a $(2,0)$ SCFT of type G does not give an SCFT, but rather has following structure (following the notation of [22]):

$$\mathcal{T}^{6d}\langle T^2_\tau\rangle = \frac{\mathcal{U}^{4d}\{G,H\} \times \mathcal{V}^{4d}\{H\}}{G_\tau \times H}$$

where $\mathcal{U}^{4d}\{G,H\}$ is a $4d$ $\mathcal{N}=2$ SCFT with $G\times H$ global symmetry and $\mathcal{V}^{4d}\{H\}$ is a $4d$ $\mathcal{N}=2$ SCFT with H global symmetry. These two SCFTs are coupled by $G\times H$ gauge fields, where the gauge coupling for G is exactly marginal and can be identified with the complex structure parameter τ of the torus. The gauge coupling for H is IR free.

For $N = 2$ M5-branes at an ALE singularity of type $\mathfrak{g}$, for each singularity type the authors of [22] found that the theory $\mathcal{U}$ is a free hypermultiplet in the $\frac{1}{2}(3,2)$ of $SU(2)_u \times SU(2)_v$ while the theory $\mathcal{V}$ is a class S theory of type $\mathfrak{g}$.[7] Using our results, we can construct this theory for $\mathfrak{g} = E_7$ and E_8:

Singularity Type	Fixture	$\dim_{\text{Coul}}$	(n_h, n_v)	Global Symmetry
E_7		26	$(767, 630)$	$(E_7)_{36}\times(E_7)_{36}\times SU(2)_7$
E_8		49	$(2159, 1907)$	$(E_8)_{60}\times(E_8)_{60}\times SU(2)_7$

The $SU(2)_v$ factor weakly gauges the $SU(2)_7$ flavor symmetry carried by the non-maximal puncture of each fixture listed above. Since this $SU(2)$ is coupled to an additional three fundamental half-hypermultiplets, it is infrared free.

For a general number N of M5 branes probing an ALE singularity of type $\mathfrak{g}$, the theory $\mathcal{V}$ is the class $\mathcal{S}$ theory of type $\mathfrak{g}$ on a fixture with three full punctures, i.e., the $T_{\mathfrak{g}}$ theory. The theory $\mathcal{U}$ is given by a $4d$ SCFT $\mathcal{S}^{4d}_{(\varnothing,\mathfrak{g}),N}\{SU(N),\mathfrak{g}\}$, which is the T^2 compactification of the $6d$ $(1,0)$ SCFT living on N M5-branes at the intersection of the Hořava-Witten

[7] For $\mathfrak{g} = A_{k-1}$ there is an additional fundamental hypermultiplet of $SU(2)_v$.

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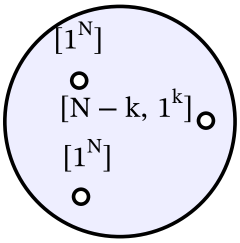
E_8-wall and an ALE singularity of type $\mathfrak{g}$. It was calculated in [22] that this $4d$ SCFT has flavor symmetry $\mathrm{SU}(N)_{4N} \times \mathfrak{g}_{2h^\vee(\mathfrak{g})+2}$. For $\mathfrak{g} = A_{k-1}$, they identified this theory as the class S theory on the fixture[8]

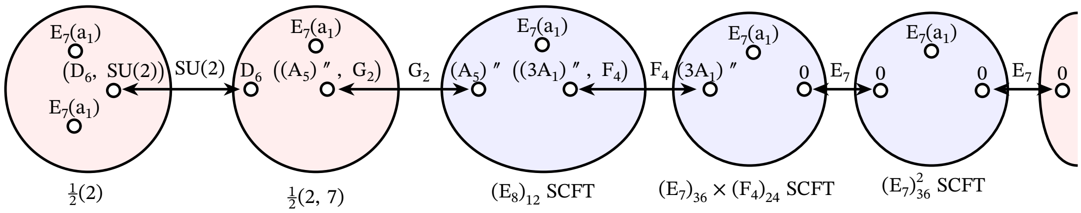


where the $\mathrm{SU}(N)_{4N}$ global symmetry is realized as the diagonal subgroup of the $\mathrm{SU}(N)_{2N}$ flavor symmetries of the two full punctures.

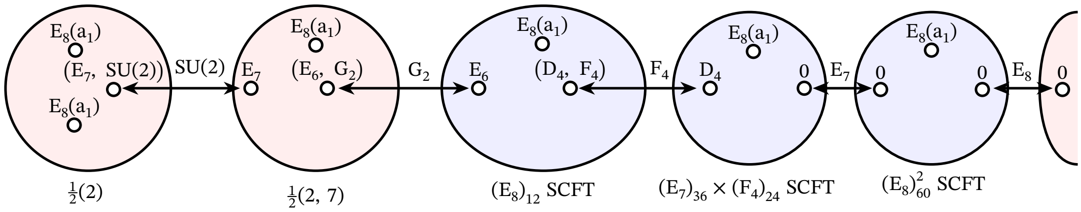
For $\mathfrak{g} \neq A$, they were not able to identify this SCFT with other known $4d$ SCFTs. We also do not find this theory for any $\mathfrak{g}, N$ on any E_7 fixture. We have not yet checked if it appears in the list of E_8 fixtures, which is work in progress [43].

Mass deforming $\mathcal{S}^{4d}_{(\varnothing,\mathfrak{g}),N}\{\mathrm{SU}(N), \mathfrak{g}\}$ by the $\mathrm{SU}(N)$ mass parameter, one obtains the generalized quiver tail produced by colliding $N-1$ minimal punctures on a sphere. For $\mathfrak{g} = A_{k-1}$, the class S realization of this quiver tail is well-known [2]. In [22], the authors worked out the quiver tails for $\mathfrak{g} = D_k, E_6$, and, from the structure of the E_6 quiver tail, conjectured the answer for $\mathfrak{g} = E_7$ and E_8 as well. Using our results, we can confirm their prediction for E_7 and E_8:

For E_7, the quiver is given by



while for E_8, the quiver is



Here, the $(E_7)_{36} \times (F_4)_{24}$ SCFT has $(n_h, n_v) = (276, 169)$ and graded Coulomb branch dimensions $(d_2, d_6, d_8, d_{10}, d_{12}, d_{14}, d_{18}) = (0, 1, 1, 0, 2, 1, 2)$. Colliding additional minimal punctures gives additional copies of the $(E_7)^2_{36}$ ($(E_8)^2_{60}$) SCFT (the T^2 compactification of the E_7 (E_8) minimal conformal matter theory), whose properties were discussed in section 4.4.1.

[8]We quote the result here for $N > k$. The theory for $k < N$ is obtained by exchanging $k \leftrightarrow N$. For $k = N$, they identified $S^{4d}_{(\varnothing,\mathfrak{su}(k)),N}$ with the T_N theory with an additional free hypermultiplet in the fundamental of $\mathrm{SU}(N)$.

Acknowledgments

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A Embeddings of SU(2) in E_7

Bala-Carter	$\mathfrak{f}$	56	133
A_1	$\mathfrak{so}(12)$	$(1,32)+(2,12)$	$(1,66)+(2,\overline{32})+(3,1)$
$2A_1$	$\mathfrak{so}(9)\times\mathfrak{su}(2)$	$(1;9,2)+(2;16,1)$ $+(3;2,1)$	$(1;1,3)+(1;36,1)+(2;16,2)$ $+(3;1,1)+(3;9,1)$
$(3A_1)''$	$\mathfrak{f}_4$	$(2;26)+(4;1)$	$(1;52)+(3;1)+(3;26)$
$(3A_1)'$	$\mathfrak{sp}(3)\times\mathfrak{su}(2)$	$(1;14',1)+(2;6,2)$ $+(3;6,1)$	$(1;1,3)+(1;21,1)+(2;14,2)$ $+(3;1,1)+(3;14,1)+(4;1,2)$
A_2	$\mathfrak{su}(6)$	$(1;20)+(3;6)+(3;\overline{6})$	$(1;35)+(3;1)+(3;15)$ $+(3;\overline{15})+(5;1)$
$4A_1$	$\mathfrak{sp}(3)$	$(1;6)+(2;14)$ $+(3;6)+(4;1)$	$(1;21)+(2;6)+(2;14')$ $+2(3;1)+(3;14)+(4;6)$
A_2+A_1	$\mathfrak{su}(4)\times\mathfrak{u}(1)$	$(1;4)_{-3/2}+(1;\overline{4})_{3/2}$ $+(2;1)_1+(2;1)_{-1}$ $+(2;6)_0+(3;4)_{1/2}$ $+(3;\overline{4})_{-1/2}+(4;1)_1$ $+(4;1)_{-1}$	$(1,1)_0+(1;15)_0+(2;4)_{3/2}$ $+(2;4)_{-1/2}+(2;\overline{4})_{1/2}+(2;\overline{4})_{-3/2}$ $+(3;1)_2+2(3;1)_0+(3;1)_{-2}$ $+(3;6)_1+(3;6)_{-1}+(4;4)_{-1/2}$ $+(4;\overline{4})_{1/2}+(5;1)_0$
A_2+2A_1	$\mathfrak{su}(2)^3$	$(1;2,3,1)+(2;1,2,4)$ $+(3;2,1,3)+(4;1,2,2)$	$(1;3,1,1)+(1;1,3,1)$ $+(1;1,1,3)+(2;2,2,4)$ $+(3;1,1,1)+(3;1,3,3)$ $+(3;1,1,5)+(4;2,2,2)$ $+(5;1,1,3)$
A_3	$\mathfrak{so}(7)\times\mathfrak{su}(2)$	$(1;7,2)+(4;8,1)+(5;1,2)$	$(1;1,3)+(1;21,1)+(3;1,1)$ $+(4;8,2)+(5;7,1)+(7;1,1)$
$2A_2$	$\mathfrak{g}_2\times\mathfrak{su}(2)$	$(1;1,4)+(3;7,2)+(5;1,2)$	$(1;1,3)+(1;14,1)+(3;1,1)$ $+(3;7,3)+(5;1,3)+(5;7,1)$

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Bala-Carter	$\mathfrak{f}$	56	133
A_2+3A_1	$\mathfrak{g}_2$	$(2;14)+(4;7)$	$(1;14)+(3;1)+(3;27)+(5;7)$
$(A_3+A_1)''$	$\mathfrak{so}(7)$	$(2;7)+(4;1)+(4;8)+(6;1)$	$(1;21)+2(3;1)+(3;8)+(5;7)$ $+(5;8)+(7;1)$
$2A_2+A_1$	$\mathfrak{su}(2)^2$	$(1;4,1)+(2;2,2)$ $+(3;2,3)+(4;2,2)$ $+(5;2,1)$	$(1;3,1)+(1;1,3)+(2;3,2)$ $+(2;1,4)+2(3;1,1)+(3;3,3)$ $+(4;1,2)+(4;3,2)+(5;3,1)$ $+(5;1,3)+(6;1,2)$
$(A_3+A_1)'$	$\mathfrak{su}(2)^3$	$(1;1,3,2)+(2;2,1,2)$ $+(3;1,2,1)+(4;2,2,1)$ $+(5;1,2,1)+(5;1,1,2)$	$(1;3,1,1)+(1;1,3,1)$ $+(1;1,1,3)+(2;2,3,1)$ $+2(3;1,1,1)+(3;1,2,2)$ $+(4;2,1,1)+(4;2,2,2)$ $+(5;1,2,2)+(5;1,3,1)$ $+(6;2,1,1)+(7;1,1,1)$
$D_4(a_1)$	$\mathfrak{su}(2)^3$	$(1;2,2,2)+(3;2,1,1)$ $+(3;1,2,1)+(3;1,1,2)$ $+(5;2,1,1)+(5;1,2,1)$ $+(5;1,1,2)$	$(1;3,1,1)+(1;1,3,1)$ $+(1;1,1,3)+3(3;1,1,1)$ $+(3;2,2,1)+(3;2,1,2)$ $+(3;1,2,2)+(5;1,1,1)$ $+(5;2,2,1)+(5;2,1,2)$ $+(5;1,2,2)+2(7;1,1,1)$
A_3+2A_1	$\mathfrak{su}(2)^2$	$(1;2,1)+(2;1,3)$ $+(3;1,2)+(3;2,1)$ $+(4;1,1)+(4;2,2)$ $+(5;1,2)+(6;1,1)$	$(1;3,1)+(1;1,3)+(2;1,2)$ $+(2;2,3)+3(3;1,1)+(3;2,2)$ $+(4;2,1)+2(4;1,2)+(5;2,2)$ $+(5;1,3)+(6;2,1)+(6;1,2)$ $+(7;1,1)$
D_4	$\mathfrak{sp}(3)$	$(1;14')+(7;6)$	$(1;21)+(3;1)+(7;14)+(11;1)$
$D_4(a_1)+A_1$	$\mathfrak{su}(2)^2$	$(2;1,1)+(2;2,2)$ $+(3;1,2)+(3;2,1)$ $+2(4;1,1)+(5;1,2)$ $+(5;2,1)+(6;1,1)$	$(1;3,1)+(1;1,3)+(2;2,1)$ $+(2;1,2)+4(3;1,1)+(3;2,2)$ $+2(4;2,1)+2(4;1,2)+(5;1,1)$ $+(5;2,2)+(6;2,1)+(6;1,2)$ $+2(7;1,1)$
A_3+A_2	$\mathfrak{su}(2)\times\mathfrak{u}(1)$	$(1;2)_0+(2;1)_1$ $+(2;1)_{-1}+(3;2)_2$ $+(3;2)_{-2}+(4;1)_3$ $+(4;1)_1+(4;1)_{-1}$ $+(4;1)_{-3}+(5;2)_0$ $+(6;1)_1+(6;1)_{-1}$	$(1;1)_0+(1;3)_0+(2;2)_1$ $+(2;2)_{-1}+(3;1)_4+2(3;1)_2$ $+2(3;1)_0+2(3;1)_{-2}+(3;1)_{-4}$ $+(4;2)_3+(4;2)_1+(4;2)_{-1}$ $+(4;2)_{-3}+(5;1)_2+2(5;1)_0$ $+(5;1)_{-2}+(6;2)_1+(6;2)_{-1}$ $+(7;1)_2+(7;1)_0+(7;1)_{-2}$
A_4	$\mathfrak{su}(3)\times\mathfrak{u}(1)$	$(1;3)_{-5/3}+(1;\overline{3})_{5/3}$ $+(3;1)_{-1}+(3;1)_1$ $+(5;3)_{1/3}+(5;\overline{3})_{-1/3}$ $+(7;1)_{-1}+(7;1)_1$	$(1;1)_0+(1;8)_0+(3;1)_0$ $+(3;3)_{-2/3}+(3;\overline{3})_{2/3}+(5;1)_2$ $+(5;1)_0+(5;1)_{-2}+(5;3)_{4/3}$ $+(5;\overline{3})_{-4/3}+(7;1)_0+(7;3)_{-2/3}$ $+(7;\overline{3})_{2/3}+(9;1)_0$

Bala-Carter	$\mathfrak{f}$	56	133
$A_3+A_2+A_1$	$\mathfrak{su}(2)$	$(2;5)+(4;7)+(6;3)$	$(1;3)+(3;1)+(3;5)$ $+(3;9)+(5;3)+(5;7)+(7;5)$
$(A_5)''$	$\mathfrak{g}_2$	$(4;1)+(6;7)+(10;1)$	$(1;14)+(3;1)+(5;7)+(7;1)$ $+(9;7)+(11;1)$
D_4+A_1	$\mathfrak{sp}(2)$	$(1;4)+(2;5)+(6;1)$ $+(7;4)+(8;1)$	$(1;10)+(2;4)+2(3;1)+(6;4)$ $+(7;1)+(7;5)+(8;4)+(11;1)$
A_4+A_1	$\mathfrak{u}(1)^2$	$1_{-2,-5/3}+1_{2,5/3}+2_{1,-5/3}$ $+2_{-1,5/3}+3_{0,-1}+3_{0,1}$ $+4_{1,1/3}+4_{-1,-1/3}+5_{-2,1/3}$ $+5_{2,-1/3}+6_{1,1/3}+6_{-1,-1/3}$ $+7_{0,-1}+7_{0,1}$	$2(1_{0,0})+2_{3,0}+2_{-3,0}+2_{1,-2/3}$ $+2_{-1,2/3}+3_{0,0}+3_{0,0}+3_{2,2/3}$ $+3_{-2,-2/3}+4_{-1,2/3}+4_{1,-2/3}+4_{1,4/3}$ $+4_{-1,-4/3}+5_{0,0}+5_{0,2}+5_{0,-2}$ $+5_{2,-4/3}+5_{-2,4/3}+6_{1,4/3}+6_{1,-2/3}$ $+6_{-1,2/3}+6_{-1,-4/3}+7_{2,2/3}+7_{0,0}$ $+7_{-2,-2/3}+8_{1,-2/3}+8_{-1,2/3}+9_{0,0}$
$D_5(a_1)$	$\mathfrak{su}(2)\times\mathfrak{u}(1)$	$(1;2)_2+(1;2)_{-2}$ $+(2;1)_1+(2;1)_{-1}$ $+(3;2)_0+(6;1)_1$ $+(6;1)_{-1}+(7;2)_0$ $+(8;1)_1+(8;1)_{-1}$	$(1;1)_0+(1;3)_0+(2;2)_1$ $+(2;2)_{-1}+(3;1)_2+2(3;1)_0$ $+(3;1)_{-2}+(5;1)_0+(6;2)_1$ $+(6;2)_{-1}+(7;1)_2+(7;1)_{-2}$ $+2(7;1)_0+(8;2)_1+(8;2)_{-1}$ $+(9;1)_0+(11;1)_0$
A_4+A_2	$\mathfrak{su}(2)$	$(3;6)+(5;2)+(7;4)$	$(1;3)+(3;1)+(3;5)+(5;3)$ $+(5;7)+(7;5)+(9;3)$
$(A_5)'$	$\mathfrak{su}(2)^2$	$(1;1,4)+(5;1,2)$ $+(6;2,2)+(9;1,2)$	$(1;3,1)+(1;1,3)+(3;1,1)$ $+(4;2,1)+(5;1,3)+(6;2,3)$ $+(7;1,1)+(9;1,3)+(10;2,1)$ $+(11;1,1)$
A_5+A_1	$\mathfrak{su}(2)$	$(4;1)+(5;2)+(6;3)$ $+(7;2)+(10;1)$	$(1;3)+(2;4)+2(3;1)+(4;2)$ $+(5;3)+(6;2)+(7;1)+(8;2)$ $+(9;3)+(10;2)+(11;1)$
$D_5(a_1)+A_1$	$\mathfrak{su}(2)$	$(2;5)+(4;1)+(6;3)+(8;3)$	$(1;3)+2(3;1)+(3;5)+(5;3)$ $+(7;3)+(7;5)+(9;3)+(11;1)$
$D_6(a_2)$	$\mathfrak{su}(2)$	$2(4;1)+(5;2)+(6;1)$ $+(7;2)+(8;1)+(10;1)$	$(1;3)+3(3;1)+2(4;2)+(5;1)$ $+(6;2)+3(7;1)+(8;2)+(9;1)$ $+(10;2)+2(11;1)$
$E_6(a_3)$	$\mathfrak{su}(2)$	$(1;4)+2(5;2)$ $+(7;2)+(9;2)$	$(1;3)+3(3;1)+(5;1)+2(5;3)$ $+(7;1)+(7;3)+(9;1)+(9;3)$ $+2(11;1)$
D_5	$\mathfrak{su}(2)^2$	$(1;2,3)+(5;1,2)$ $+(9;2,1)+(11;1,2)$	$(1;3,1)+(1;1,3)+(3;1,1)$ $+(5;2,2)+(7;1,1)+(9;1,3)$ $+(11;1,1)+(11;2,2)+(15;1,1)$
$E_7(a_5)$	—	$3(4)+3(6)+2(8)+10$	$6(3)+4(5)+5(7)+3(9)+3(11)$
A_6	$\mathfrak{su}(2)$	$(3;2)+(7;4)+(11;2)$	$(1;3)+(3;1)+(5;3)+(7;5)$ $+(9;3)+(11;1)+(13;3)$

Bala-Carter	$\mathfrak{f}$	56	133
$D_5 + A_1$	$\mathfrak{su}(2)$	$(2;3)+(5;2)+(8;1)$ $+(10;1)+(11;2)$	$(1;3)+2(3;1)+(4;2)+(6;2)$ $+(7;1)+(9;3)+(10;2)+(11;1)$ $+(12;2)+(15;1)$
$D_6(a_1)$	$\mathfrak{su}(2)$	$(3;2)+(4;1)+(6;1)$ $+(9;2)+(10;1)+(12;1)$	$(1;3)+2(3;1)+(4;2)+(6;2)$ $+2(7;1)+(9;1)+(10;2)+2(11;1)$ $+(12;2)+(15;1)$
$E_7(a_4)$	—	$2+2(4)+6$ $+8+2(10)+12$	$4(3)+2(5)+3(7)+2(9)+4(11)$ $+13+15$
D_6	$\mathfrak{su}(2)$	$(1;2)+(6;1)+(10;1)$ $+(11;2)+(16;1)$	$(1;3)+(3;1)+(6;2)+(7;1)$ $+(10;2)+2(11;1)+(15;1)$ $+(16;2)+(19;1)$
$E_6(a_1)$	$\mathfrak{u}(1)$	$1_3+1_{-3}+5_1$ $+5_{-1}+9_1+9_{-1}$ $+13_1+13_{-1}$	$1_0+3_0+5_2+5_0+5_{-2}+7_0$ $+9_2+9_0+9_{-2}+2(11_0)+13_2$ $+13_{-2}+15_0+17_0$
E_6	$\mathfrak{su}(2)$	$(1;4)+(9;2)+(17;2)$	$(1;3)+(3;1)+(9;3)+(11;1)$ $+(15;1)+(17;3)+(23;1)$
$E_7(a_3)$	—	$2+6+2(10)+12+16$	$2(3)+5+2(7)+9+3(11)$ $+2(15)+17+19$
$E_7(a_2)$	—	$4+8+10+16+18$	$2(3)+7+9+2(11)+2(15)+17$ $+19+23$
$E_7(a_1)$	—	$6+12+16+22$	$3+7+2(11)+15+17+19$ $+23+27$
E_7	—	$10+18+28$	$3+11+15+19+23+27+35$

B Projection matrices

Bala-Carter	$\mathfrak{f}$	Projection Matrix
A_1	$\mathfrak{so}(12)$	$\begin{pmatrix} 0&0&0&0&0&1&0 \\ 0&1&2&2&2&1&1 \\ 1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1 \end{pmatrix}$
$2A_1$	$\mathfrak{so}(9)\times\mathfrak{su}(2)$	$\begin{pmatrix} 1&1&2&2&2&1&1 \\ 0&1&1&1&1&0&0 \\ 0&0&0&0&0&0&1 \\ 1&1&1&0&0&0&0 \\ -1&-1&0&1&0&0&0 \\ 1&3&4&3&2&2&2 \end{pmatrix}$

Bala-Carter	$\mathfrak{f}$	Projection Matrix
$(3A_1)''$	$\mathfrak{f}_4$	$\begin{pmatrix} 2&4&6&5&4&3&3 \\ 0&0&0&0&0&0&1 \\ 0&0&1&0&0&0&0 \\ 0&1&0&1&0&0&0 \\ 1&0&0&0&1&0&0 \end{pmatrix}$
$(3A_1)'$	$\mathfrak{sp}(3)\times\mathfrak{su}(2)$	$\begin{pmatrix} 3&6&8&6&4&2&4 \\ 0&0&0&0&0&1&1 \\ 0&0&1&0&1&0&0 \\ 0&0&0&1&0&0&0 \\ 1&0&0&0&0&0&0 \end{pmatrix}$
A_2	$\mathfrak{su}(6)$	$\begin{pmatrix} 4&6&8&6&4&2&4 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1 \end{pmatrix}$
$4A_1$	$\mathfrak{sp}(3)$	$\begin{pmatrix} 3&6&9&7&5&3&4 \\ 1&0&0&0&1&0&0 \\ 0&1&0&1&0&0&0 \\ 0&0&1&0&0&0&1 \end{pmatrix}$
A_2+A_1	$\mathfrak{su}(4)\times\mathfrak{u}(1)$	$\begin{pmatrix} 3&7&10&8&6&3&5 \\ 0&0&0&1&0&0&0 \\ 0&0&1&0&0&0&1 \\ 1&1&0&0&0&0&0 \\ 1/2&1/2&1&1/2&0&1&0 \end{pmatrix}$
A_2+2A_1	$\mathfrak{su}(2)^3$	$\begin{pmatrix} 4&8&12&9&6&3&6 \\ 0&0&0&0&0&0&1 \\ 0&0&0&1&0&1&0 \\ 0&2&0&-1&0&1&0 \end{pmatrix}$
A_3	$\mathfrak{so}(7)\times\mathfrak{su}(2)$	$\begin{pmatrix} 0&1&4&0&0&0&3 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&1&0&0 \\ 0&1&2&2&0&0&1 \\ 2&3&4&3&2&1&2 \end{pmatrix}$

Bala-Carter	$\mathfrak{f}$	Projection Matrix
$2A_2$	$\mathfrak{g}_2 \times \mathfrak{su}(2)$	$\begin{pmatrix} 4 & 8 & 12 & 10 & 8 & 4 & 6 \\ 1 & 0 & 2 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$
$A_2 + 3A_1$	$\mathfrak{g}_2$	$\begin{pmatrix} 4 & 8 & 12 & 9 & 6 & 3 & 5 \\ 1 & 0 & 2 & 2 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$
$(A_3 + A_1)''$	$\mathfrak{so}(7)$	$\begin{pmatrix} 4 & 10 & 14 & 11 & 8 & 5 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$
$2A_2 + A_1$	$\mathfrak{su}(2)^2$	$\begin{pmatrix} 5 & 10 & 14 & 11 & 8 & 4 & 7 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}$
$(A_3 + A_1)'$	$\mathfrak{su}(2)^3$	$\begin{pmatrix} 6 & 11 & 16 & 12 & 8 & 4 & 8 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$
$D_4(a_1)$	$\mathfrak{su}(2)^3$	$\begin{pmatrix} 6 & 12 & 16 & 12 & 8 & 4 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$
$A_3 + 2A_1$	$\mathfrak{su}(2)^2$	$\begin{pmatrix} 6 & 11 & 16 & 13 & 9 & 5 & 8 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$
D_4	$\mathfrak{sp}(3)$	$\begin{pmatrix} 10 & 18 & 24 & 18 & 12 & 6 & 12 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$
$D_4(a_1) + A_1$	$\mathfrak{su}(2)^2$	$\begin{pmatrix} 6 & 12 & 17 & 13 & 9 & 5 & 9 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$

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Bala-Carter	$\mathfrak{f}$	Projection Matrix
A_3+A_2	$\mathfrak{su}(2)\times\mathfrak{u}(1)$	$\begin{pmatrix} 6 & 12 & 18 & 14 & 10 & 5 & 9 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$
A_4	$\mathfrak{su}(3)\times\mathfrak{u}(1)$	$\begin{pmatrix} 6 & 14 & 20 & 16 & 12 & 6 & 10 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2/3 & 1/3 & 0 & 1 & 1 \end{pmatrix}$
$A_3+A_2+A_1$	$\mathfrak{su}(2)$	$\begin{pmatrix} 6 & 12 & 18 & 15 & 10 & 5 & 9 \\ 4 & 6 & 6 & 0 & 2 & 2 & 4 \end{pmatrix}$
$(A_5)''$	$\mathfrak{g}_2$	$\begin{pmatrix} 10 & 18 & 26 & 21 & 16 & 9 & 13 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$
D_4+A_1	$\mathfrak{sp}(2)$	$\begin{pmatrix} 10 & 17 & 25 & 19 & 13 & 7 & 13 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$
A_4+A_1	$\mathfrak{u}(1)^2$	$\begin{pmatrix} 8 & 15 & 22 & 17 & 12 & 6 & 11 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & -2/3 & 0 & 1/3 & 0 & 1 & 1/3 \end{pmatrix}$
$D_5(a_1)$	$\mathfrak{su}(2)\times\mathfrak{u}(1)$	$\begin{pmatrix} 10 & 18 & 26 & 20 & 14 & 7 & 13 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$
A_4+A_2	$\mathfrak{su}(2)$	$\begin{pmatrix} 8 & 16 & 24 & 18 & 12 & 6 & 12 \\ 0 & 2 & 0 & 3 & 4 & 3 & 1 \end{pmatrix}$
$(A_5)'$	$\mathfrak{su}(2)^2$	$\begin{pmatrix} 10 & 19 & 28 & 22 & 16 & 8 & 14 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$
A_5+A_1	$\mathfrak{su}(2)$	$\begin{pmatrix} 10 & 19 & 28 & 22 & 16 & 9 & 14 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$
$D_5(a_1)+A_1$	$\mathfrak{su}(2)$	$\begin{pmatrix} 10 & 18 & 26 & 21 & 14 & 7 & 13 \\ 0 & 0 & 2 & 0 & 2 & 2 & 2 \end{pmatrix}$
$D_6(a_2)$	$\mathfrak{su}(2)$	$\begin{pmatrix} 10 & 20 & 29 & 23 & 16 & 9 & 15 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$

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Bala-Carter	$\mathfrak{f}$	Projection Matrix
$E_6(a_3)$	$\mathfrak{su}(2)$	$\begin{pmatrix} 10 & 20 & 28 & 22 & 16 & 8 & 14 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$
D_5	$\mathfrak{su}(2)^2$	$\begin{pmatrix} 14 & 24 & 36 & 28 & 20 & 10 & 18 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$
$E_7(a_5)$	—	$\begin{pmatrix} 10 & 20 & 30 & 23 & 16 & 9 & 15 \end{pmatrix}$
A_6	$\mathfrak{su}(2)$	$\begin{pmatrix} 12 & 24 & 36 & 28 & 20 & 10 & 18 \\ 2 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$
$D_5 + A_1$	$\mathfrak{su}(2)$	$\begin{pmatrix} 11 & 25 & 37 & 29 & 20 & 10 & 19 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$
$D_6(a_1)$	$\mathfrak{su}(2)$	$\begin{pmatrix} 14 & 26 & 37 & 29 & 20 & 11 & 19 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$
$E_7(a_4)$	—	$\begin{pmatrix} 14 & 26 & 38 & 29 & 20 & 11 & 19 \end{pmatrix}$
D_6	$\mathfrak{su}(2)$	$\begin{pmatrix} 18 & 33 & 48 & 39 & 28 & 15 & 23 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
$E_6(a_1)$	$\mathfrak{u}(1)$	$\begin{pmatrix} 16 & 30 & 44 & 34 & 24 & 12 & 22 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$
E_6	$\mathfrak{su}(2)$	$\begin{pmatrix} 22 & 42 & 60 & 46 & 32 & 16 & 30 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$
$E_7(a_3)$	—	$\begin{pmatrix} 18 & 34 & 50 & 39 & 28 & 15 & 25 \end{pmatrix}$
$E_7(a_2)$	—	$\begin{pmatrix} 22 & 42 & 60 & 47 & 32 & 17 & 31 \end{pmatrix}$
$E_7(a_1)$	—	$\begin{pmatrix} 26 & 50 & 72 & 57 & 40 & 21 & 37 \end{pmatrix}$
E_7	—	$\begin{pmatrix} 34 & 66 & 96 & 75 & 52 & 27 & 49 \end{pmatrix}$

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