Conformal field theories and compact curves in moduli spaces

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ABSTRACT: We show that there are many compact subsets of the moduli space M_g of Riemann surfaces of genus g that do not intersect any symmetry locus. This has interesting implications for $\mathcal{N} = 2$ supersymmetric conformal field theories in four dimensions.

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1 Introduction

An interesting analysis of conformal manifolds for conformal field theories in even spacetime dimension was presented in [1], using properties of the trace anomaly. For supersymmetric theories whose conformal manifold is Kähler, including the cases of $\mathcal{N} = (2, 2)$ supersymmetry in two dimensions¹ and $\mathcal{N} = 2$ supersymmetry in four dimensions, it was predicted in [1] that the Zamolodchikov metric on the conformal manifold (i.e., the metric determined by the two-point functions of the conformal theory) should have a globally-defined Kähler potential, closely related to the partition function on the sphere as conjectured in [2]. The mathematical implications of this prediction are quite remarkable. In particular, in this case the conformal manifolds in question could not contain compact holomorphic curves.

In two dimensions, a familiar class of examples of $\mathcal{N} = (2, 2)$ theories is given by sigma models whose target is a Calabi-Yau manifold X of fixed topological type, and one “half” of the conformal manifold is represented by the moduli space $\mathcal{M}_X$ of complex structures on the Calabi-Yau manifold. The prediction that the Kähler potential on this conformal manifold is globally well-defined is equivalent to claiming that the Hodge line bundle on $\mathcal{M}_X$, whose fiber at a given X is the one-dimensional space of holomorphic 3-forms on X , admits constant, $U(1)$ -valued transition functions. For several families of Calabi-Yau manifolds, this prediction has been verified in a recent paper [3]. Other aspects of this prediction are analyzed in [4].

In four dimensions, the $SU(2)$ theories of class S with no punctures [5, 6], given by compactifying the $(2, 0)$ theory in six dimensions of type A_1 on a Riemann surface of some fixed genus g , form a collection of $\mathcal{N} = 2$ supersymmetric conformal field theories whose conformal manifold precisely corresponds to the moduli space M_g of Riemann surfaces of genus g . This $3g - 3$ dimensional space is non compact. The prediction from [1] that the Zamolodchikov metric on M_g has a globally well-defined Kähler potential implies that M_g

¹In the $\mathcal{N} = (2, 2)$ case, the argument applies only to one “half” of the conformal manifold at a time: either the half whose tangent space consists of chiral marginal operators, or the half whose tangent space consists of twisted chiral marginal operators.

contains no compact analytic subspaces. As we shall show, this is false. In fact, no Kähler metric on M_g can have a globally well-defined Kähler potential.

In conversations with some of the authors of [1], it was realized that at points of the conformal manifold at which the Riemann surface (and hence the conformal field theory) has extra symmetries, the extra symmetries might interfere with the original prediction. One can instead consider the open subset $M_g^0 \subset M_g$ parametrizing those Riemann surfaces without extra symmetries, i.e., those that do not admit any holomorphic automorphisms. The prediction arising from [1] would then be that this space M_g^0 contains no compact analytic subspaces of positive dimension. In this note we show that, on the contrary, M_g^0 for $g > 3$ does contain lots of compact subspaces. In fact, through every point of M_g^0 , or even any finite collection of its points, there is such a compact subspace.

There were two developments after the first version of this paper was circulated. First, Joerg Teschner pointed out to us that the generalization of [7, 8] from linear quivers to general UV curves proposed in [9] identifies the Kähler potential of the Zamolodchikov metric² with the Liouville partition function on Riemann surfaces of genus g . The Liouville partition function is not globally well-defined on M_g , but is rather the Kähler potential of a hermitian metric on a non-trivial projective line bundle over M_g when $g > 1$, in much the same way as first argued by Friedan and Shenker for rational CFTs [10].³ The result of the present paper shows that this Kähler potential cannot be well-defined even over the subset M_g^0 of M_g when $g > 3$.

Second, motivated by the example which this paper contains, Tachikawa and Yonekura [12] showed that in four dimensions, the trace anomaly mentioned above contains previously-overlooked terms of the form $c_2(R) [\omega^Z]/2\pi$ and $p_1 [\omega^Z]/2\pi$, where $c_2(R)$ is the second Chern class of the background $SU(2)_R$ gauge field, p_1 is the first Pontryagin class of the spacetime metric, and $[\omega^Z]$ is the cohomology class of the Zamolodchikov metric on the conformal manifold. The existence of these terms resolves the apparent conflict between [1] and the result we present below. Further details will appear in [13].

Our precise claim is that there are (lots of) compact curves in M_g , for $g > 2$, and that most of those miss the loci of enhanced symmetry, if $g > 3$. The gist of the argument is as follows. Even though we are interested in M_g , or rather its open subset M_g^0 , it will be convenient to work temporarily with a compactification. We construct a $3g - 3$ -dimensional projective variety $\overline{M}_g^S$ that contains M_g as a dense open subset. In it, we need to control two types of loci: the boundary $\overline{M}_g^S \setminus M_g$, and the loci of Riemann surfaces with a symmetry. The technical part of the argument shows that all of these loci have codimension ≥ 2 , so dimension $\leq 3g - 5$. Since $\overline{M}_g^S$ is a projective variety, it can be embedded in projective spaces $\mathbb{P}^N$ for arbitrarily large N . Given a finite set S of points of M_g^0 , choose a large enough N and $3g - 4$ generic hypersurfaces containing the image of S in $\mathbb{P}^N$. The locus B cut out by these hypersurfaces will have codimension $3g - 4$ in $\mathbb{P}^N$. If these hypersurfaces are sufficiently generic, this B will meet the image of $\overline{M}_g^S$ in

²That is, the four-sphere partition function [1].

³In addition, a very direct relation between the classical Liouville partition function and the Weil-Petersson Kähler form was obtained by Takhtajan and Zograf [11]. Yuji Tachikawa has pointed out to us that this should be regarded as a type of “large N limit” of Teschner’s observation.

a 1-dimensional compact submanifold, and will meet neither the boundary $\overline{M}_g^S \setminus M_g$ nor the loci of Riemann surfaces with a symmetry. This produces the desired compact curve in M_g^0 containing the specified set S .

2 The codimension of the boundary

The best known compactification of M_g is the Deligne-Mumford compactification $\overline{M}_g^{\text{DM}}$ of M_g , [14], parametrizing stable curves. Its boundary $\overline{M}_g^{\text{DM}} \setminus M_g$ is too big — each of its components has codimension 1. We need a different compactification in which the boundary is blown down to a smaller dimensional locus. This is provided by the Satake-type compactification $\overline{M}_g^S$ that we describe below.

As is well known, the boundary $\overline{M}_g^{\text{DM}} \setminus M_g$ consists of smooth, normal crossing divisors Δ_i , $i \geq 0$. All the Δ_i , $i \geq 0$, are divisors, i.e. codimension 1 loci, in the Deligne-Mumford compactification $\overline{M}_g^{\text{DM}}$. For $i > 0$, Δ_i parametrizes reducible curves $C = C' \cup C''$. At a generic point of Δ_i , C' is a smooth curve of genus i and C'' is a smooth curve of genus $g-i$, and they intersect transversally at one point, forming a node there. At special points of Δ_i , C' and C'' can themselves be stable curves, still of genera i and $g-i$ respectively. At a generic point of Δ_0 , the curve C is nodal but now irreducible. It can be obtained from a curve C' of genus $g-1$ (called the normalization of C) by gluing two points of C' to each other, transversally. It is easy to count that the dimension of each Δ_i equals $3g-4$.

We let A_g denote the moduli space of g -dimensional, principally polarized abelian varieties. This is a quotient of Siegel's space, parametrizing *period matrices*, or symmetric $g \times g$ complex matrices with positive imaginary part, by the integral symplectic group. Its dimension is $g(g+1)/2$. The moduli space A_g has its own compactification, the Satake compactification $\overline{A}_g^S$. This is a compact complex space, which as a set can be identified with the disjoint union $\cup_{i=0}^g A_i$ of the A_i for $i \leq g$. Its boundary $\overline{A}_g^S \setminus A_g$ is small: it is the disjoint union $\cup_{i=0}^{g-1} A_i$ of the A_i for $i < g$. It has codimension g in $\overline{A}_g^S$.

The Abel-Jacobi map $AJ : M_g \rightarrow A_g$ sends a Riemann surface C to its Jacobian variety $J(C)$, which is the abelian variety specified by the period matrix of C . Torelli's theorem tells us that AJ is injective, so (ignoring some irrelevant minor complication along the hyperelliptic locus), we can identify M_g with its image $AJ(M_g)$. It turns out that the Abel-Jacobi map extends to a map $\overline{AJ} : \overline{M}_g^{\text{DM}} \rightarrow \overline{A}_g^S$ from the Deligne-Mumford compactification $\overline{M}_g^{\text{DM}}$ to the Satake compactification $\overline{A}_g^S$. The Satake compactification $\overline{M}_g^S$ of M_g is the image of $\overline{AJ}$. It is our alternative compactification of M_g .

This extended Abel-Jacobi map $\overline{AJ}$ is no longer an embedding: it collapses the boundary divisors. For $i > 1$, the divisor (i.e. codimension 1 locus) Δ_i collapses to a locus of codimension 3 in $\overline{M}_g^S$. This locus is actually in the interior of $\overline{A}_g^S$, not in its boundary: the image points describe compact, i.e. non-singular, abelian varieties, that happen to be products of two lower dimensional abelian varieties, namely the Jacobians of C' , C'' . Equivalently, the period matrix of such an abelian variety is block diagonal, the two blocks being the periods of C' , C'' . The fibers of $\overline{AJ}$ on these Δ_i are two dimensional: the locations of the two glued points are ignored by $\overline{AJ}$.

For $i = 1$, Δ_1 still goes to the locus of products, in the interior of A_g , but its image is larger: each elliptic tail decreases the codimension by 1. When $g > 2$ there can be only one elliptic tail, so the codimension of the image in $\overline{M}_g^S$ is 2, and the fibers are one dimensional: the location of the non elliptic glued point is ignored. When $g = 2$ there are two elliptic tails, and the codimension of the image is 1. This is why we need to assume $g > 2$.

The divisor Δ_0 , on the other hand, goes to a locus in the boundary of $\overline{A}_g^S$: a nodal curve C with normalization C' goes to the Jacobian of C' , which is in the boundary $\overline{A}_g^S \setminus A_g = \cup_{i=0}^{g-1} A_i$. The image of Δ_0 is therefore the locus of genus $g - 1$ Jacobians; its dimension is therefore $3g - 6$, so it has codimension 3 in the Satake compactification $\overline{M}_g^S$.

All in all, we see that for $g > 2$, the Satake compactification $\overline{M}_g^S$ has a boundary of codimension ≥ 2 as desired.

3 The codimension of the symmetry locus

Next, here is the count showing that the codimension in M_g of the symmetry loci is at least 2 whenever $g > 3$.

Let g be the genus of a (smooth) curve C with automorphism $a : C \rightarrow C$ of order d . Let g_0 be the genus of the quotient curve $C_0 := C / \langle a \rangle$, and let b be the number of branch points. The number of moduli of curves with such automorphisms is $3g_0 - 3 + b$, because given C_0 and the branch locus there is a finite number of such C 's.

The Hurwitz relation gives: $2g - 2 \geq d(2g_0 - 2) + db/2$. So the codimension in moduli of the locus of curves C with enhanced symmetry is at least:

$$\begin{aligned} 3g - 3 - (3g_0 - 3 + b) &\geq (3/2)(d(2g_0 - 2) + db/2) - (3g_0 - 3 + b) \\ &= (3d - 3)(g_0 - 1) + (3d/4 - 1)b \end{aligned}$$

If $g_0 > 1$ this codimension is at least $3d - 3 \geq 3$.

If $g_0 = 1$, this is $(3d/4 - 1)b$, which is at least 2 unless $d = 2$ and $b = 2$ so $g = 2$.

If $g_0 = 0$, our lower bound is

$$3bd/4 - b - 3d + 3 = (3/4)[(d - (4/3))(b - 4)] - 1.$$

This can be ≤ 1 only if $[(d - (4/3))(b - 4)] \leq 8/3$. So:

- either $d = 2$ and $b \leq 8$, so $g \leq 3$ (This is the case of hyperelliptic curves, which indeed have codimension 1 in genus 3);
- or $b \leq 5$, so the dimension of the enhanced symmetry locus is at most $5 - 3 = 2$, so the codimension in M_g is at least 4 unless $g \leq 2$.

We thank Richard Hain for pointing out that calculations similar to the content of the present section have appeared elsewhere, e.g. in Proposition 5.1 of [15].

4 Conclusion

We have seen that, as long as $g > 3$, all components of both the boundary and the symmetry loci have codimension at least 2 in $\overline{M}_g^S$, so dimension at most $3g - 5$. As we saw in the introduction, after we embed $\overline{M}_g^S$ in a large projective space $\mathbb{P}^N$, it follows that the intersection of $\overline{M}_g^S$ with $3g - 4$ generic hypersurfaces in $\mathbb{P}^N$ is a smooth, compact 1-dimensional space that avoids both the boundary and the loci of enhanced symmetry. Its inverse image in M_g is then a smooth, compact curve, contained in M_g and avoiding the enhanced symmetry loci. Clearly we can choose our hypersurfaces to pass through any point or finite collection of points of M_g^0 .

As a consequence of this mathematical analysis, we conclude that it is not possible for any Kähler metric on M_g^0 (including the Zamolodchikov metric) to have a globally-defined Kähler potential. This provides additional evidence for the validity of the modification made in [12, 13] to the original analysis of [1].

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