From $\mathcal{N} = 4$ Galilean superparticle to three-dimensional non-relativistic $\mathcal{N} = 4$ superfields

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ABSTRACT: We consider the general $\mathcal{N} = 4$, $d = 3$ Galilean superalgebra with arbitrary central charges and study its dynamical realizations. Using the nonlinear realization techniques, we introduce a class of actions for $\mathcal{N} = 4$ three-dimensional non-relativistic superparticle, such that they are linear in the central charge Maurer-Cartan one-forms. As a prerequisite to the quantization, we analyze the phase space constraints structure of our model for various choices of the central charges. The first class constraints generate gauge transformations, involving fermionic κ -gauge transformations. The quantization of the model gives rise to the collection of free $\mathcal{N} = 4$, $d = 3$ Galilean superfields, which can be further employed, e.g., for description of three-dimensional non-relativistic $\mathcal{N} = 4$ supersymmetric theories.

KEYWORDS: Extended Supersymmetry, Space-Time Symmetries, Superspaces

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1 Introduction

In recent years, one can observe the growth of interest in the non-relativistic (NR) field-theoretic models, in particular those describing NR gravity and NR supergravity, e.g., in the framework of the so-called Newton-Cartan geometry [1–5]. Until present, the NR supersymmetric framework [3–5] has been basically developed for $D = 2+1$ -dimensional case,¹ which corresponds to the exotic version of Galilean symmetry with two central charges [6–10]. In this paper we address the next physically interesting case of $\mathcal{N} = 4$, $D = 3+1$ supersymmetric extension of Galilean symmetries. Due to the distinguished role of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (see e.g. [11]), this kind of extended supersymmetry merits as well an attention in the NR case.

¹We denote by D the number of space and time dimensions, and by d the number of space dimensions, i.e. $D = d + 1$.

Similarly as in the relativistic case, one can study the NR $\mathcal{N} = 4$, $D = 3+1$ supersymmetric theories by following several paths:

- i) One can start with NR $\mathcal{N} = 4$, $d = 3$ Galilean superalgebras (for arbitrary $\mathcal{N}$ see [12]) and then construct their superspace and superfield realizations. In this way we obtain the universal tool for constructing NR supersymmetric field theories.
- ii) The NR field theories can be reproduced by performing the non-relativistic contraction $c \rightarrow \infty$ (c is the speed of light) in the known relativistic non-supersymmetric, as well as supersymmetric field theory models (see, e.g., [4, 13, 14]). One of the advantages of such a method is the possibility to derive the proper NR contractions of relativistic action integrals.
- iii) For a definite type of (super)symmetric framework one can consider the dynamics associated with particles, fields, string, p -branes, etc. An important role in such a list is played by the free classical and first-quantized (super)particle models, with the property that their first quantization leads to the classical (super)field realizations (see, e.g., [15, 16]). In our case, we will look for the free superparticle models invariant under $\mathcal{N} = 4$, $d = 3$ Galilean supersymmetry. One can mention that in the relativistic case this way of deriving free superfields from the classical and first-quantized superparticles with extended $\mathcal{N} = 4$, $D = 4$ Poincaré supersymmetry was already proposed in [17, 18].²

In this paper we will follow the path **iii**). We will consider the most general NR $\mathcal{N} = 4$, $d = 3$ Galilean superalgebra, introduce the corresponding $\mathcal{N} = 4$ Galilean supergroup and its cosets, construct the relevant nonlinear realizations and use the associated Maurer-Cartan (MC) one-forms to build NR $\mathcal{N} = 4$ superparticle models. They will be subsequently quantized to obtain the NR superfields providing realizations of $\mathcal{N} = 4$ Galilean supersymmetries. Note that in such a setting the original coset parameters are treated as the $D=1$ world-line fields. However, the whole formalism could be equally applied along the lines of path **i**), with the coset parameters treated as independent NR superspace coordinates.

As a prelude to our considerations, we will describe the Galilean symmetries and their supersymmetrizations in a short historical survey.

The Galilean theories describe the low energy, non-relativistic dynamical systems,³ which can also be obtained as non-relativistic limit ($c \rightarrow \infty$) of the corresponding relativistic theories (see, e.g., [28–32]). Such a contraction limit, applied to $D = 4$ Poincaré algebra ($P_\mu, M_{\mu\nu}; P_\mu = (P_0, P_i)$, $M_{\mu\nu} = (M_i = \frac{1}{2}\varepsilon_{ijk}M_{jk}, N_i = M_{i0})$), after shifting and rescaling

$$P_0 = m_0 c + \frac{\mathbf{H}}{c}, \quad P_i = \mathbf{P}_i, \quad N_i = c \mathbf{B}_i, \quad M_i = \mathbf{J}_i, \quad (1.1)$$

²Here we will deal only with those Galilean supersymmetries which, after quantization, leads to the superfields depending on spinorial Grassmann variables. One could introduce as well relativistic [19–22] and Galilean [23, 24] world-line supersymmetries with vectorial Grassmann variables. However, this option is out of the scope of our study.

³The Galilean symmetries are used also in the description of light cone quantization of relativistic theories [25–27]. In this paper, we shall not deal with this application of Galilean symmetries.

where $\mathbf{H}$ stands for non-relativistic Hamiltonian and $\mathbf{B}_i$ for the Galilean boosts, yields “quantum” $d = 3$ Galilean algebra [33]⁴

$$\begin{aligned} [\mathbf{J}_i, \mathbf{J}_j] &= i\varepsilon_{ijk} \mathbf{J}_k, \\ [\mathbf{J}_i, \mathbf{P}_j] &= i\varepsilon_{ijk} \mathbf{P}_k, & [\mathbf{J}_i, \mathbf{B}_j] &= i\varepsilon_{ijk} \mathbf{B}_k, \\ [\mathbf{H}, \mathbf{B}_i] &= -i\mathbf{P}_i, & [\mathbf{P}_i, \mathbf{B}_j] &= -i\mathbf{M} \delta_{ij}. \end{aligned} \quad (1.2)$$

The central charge $\mathbf{M} = m_0$ describes a non-relativistic mass which can be identified with the relativistic rest mass.

Because bosons and fermions occur in both relativistic and non-relativistic settings, one can consider the non-relativistic supersymmetry as well. The first proposal for supersymmetrization of Galilei algebra (1.2) was given in [34], where $\mathcal{N} = 1$ and $\mathcal{N} = 2$, $d = 3$ Galilean superalgebras were presented. The $\mathcal{N} = 1$, $d = 3$ Galilean superalgebra is an extension of relations (1.2) by complex NR $\text{USp}(2) \simeq \text{SU}(2)$ supercharges $\mathbf{S}_\alpha, \bar{\mathbf{S}}^\alpha := (\mathbf{S}_\alpha)^\dagger$ ($A \rightarrow A^\dagger$ denotes Hermitian conjugation) which satisfy the relations⁵

$$\{\mathbf{S}_\alpha, \bar{\mathbf{S}}^\beta\} = \mathbf{M} \delta_\alpha^\beta, \quad [\mathbf{J}_i, \mathbf{S}_\alpha] = (\sigma_i)_\alpha^\beta \mathbf{S}_\beta, \quad [\mathbf{J}_i, \bar{\mathbf{S}}^\alpha] = -\bar{\mathbf{S}}^\beta (\sigma_i)_\beta^\alpha. \quad (1.3)$$

Passing to the $\mathcal{N} = 2$ $d = 3$ Galilean superalgebra [34] is accomplished by adding to the $\mathcal{N} = 1$ Galilean superalgebra generators $(\mathbf{J}_i, \mathbf{P}_i, \mathbf{B}_i, \mathbf{H}; \mathbf{S}_\alpha, \bar{\mathbf{S}}^\alpha, \mathbf{M})$ the second pair of complex $\text{SU}(2)$ supercharges $\mathbf{Q}_\alpha, \bar{\mathbf{Q}}^\alpha := (\mathbf{Q}_\alpha)^\dagger$, subject to the following relations:

$$\begin{aligned} \{\mathbf{Q}_\alpha, \bar{\mathbf{Q}}^\beta\} &= 2\delta_\alpha^\beta \mathbf{H}, \\ \{\mathbf{Q}_\alpha, \bar{\mathbf{S}}^\beta\} &= 2(\sigma_i)_\alpha^\beta \mathbf{P}_i + 2i\mathbf{Y} \delta_\alpha^\beta, \\ [\mathbf{J}_i, \mathbf{Q}_\alpha] &= (\sigma_i)_\alpha^\beta \mathbf{Q}_\beta, & [\mathbf{B}_i, \mathbf{Q}_\alpha] &= (\sigma_i)_\alpha^\beta \mathbf{S}_\beta, \\ [\mathbf{J}_i, \bar{\mathbf{Q}}^\alpha] &= -\bar{\mathbf{Q}}^\beta (\sigma_i)_\beta^\alpha, & [\mathbf{B}_i, \bar{\mathbf{Q}}^\alpha] &= -\bar{\mathbf{S}}^\beta (\sigma_i)_\beta^\alpha. \end{aligned} \quad (1.4)$$

In the relations (1.4), (1.3), besides the central charge $\mathbf{M}$, there appears the new central charge $\mathbf{Y}$. The $\mathcal{N} = 2$, $d = 3$ Galilean superalgebra can be derived from $\mathcal{N} = 2$, $D = 4$ Poincaré superalgebra ($a = 1, 2$)

$$\begin{aligned} \{Q_\alpha^a, \bar{Q}_{\dot{\beta}b}\} &= 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu \delta_b^a, \\ \{Q_\alpha^a, Q_\beta^b\} &= 2\epsilon^{ab} \epsilon_{\alpha\beta} Z, \\ \{\bar{Q}_{\dot{\alpha}a}, \bar{Q}_{\dot{\beta}b}\} &= -2\epsilon_{ab} \epsilon_{\dot{\alpha}\dot{\beta}} \bar{Z} \end{aligned} \quad (1.5)$$

(plus the commutation relations with Poincaré and internal R-symmetry $\text{U}(2)$ generators) by taking the $c \rightarrow \infty$ contraction limit with $\mathbf{M} = m_0$. In general, the $\mathcal{N} = 2$, $D = 4$ Poincaré superalgebra is endowed with one complex central charge Z or, equivalently, two

⁴The Galilean algebra (1.2) with central charge $\mathbf{M} \neq 0$ is called Bargmann algebra. In $d = 2$ one can introduce also second central charge through the modified commutator $[B_i, B_j] = \epsilon_{ij} \rho$ [6–9].

⁵We list only non-vanishing (anti)commutators.

real central charges, $Z = X + iY \dots$ ⁶ Before taking the NR limit $c \rightarrow \infty$, the Galilean supercharges $\mathbf{Q}_\alpha$ and $\mathbf{S}^\alpha$ (see (1.3), (1.4)) should be identified with the following linear combinations of two $\mathcal{N} = 2$ Weyl supercharges in (1.5)

$$\mathbf{Q}_\alpha = c^{1/2} Q_\alpha^+, \quad \bar{\mathbf{Q}}^\alpha = c^{1/2} \bar{Q}_{\dot{\alpha}}^+, \quad \mathbf{S}_\alpha = c^{-1/2} Q_\alpha^-, \quad \bar{\mathbf{S}}_\alpha = c^{-1/2} \bar{Q}_{\dot{\alpha}}^-, \quad (1.6)$$

where

$$Q_\alpha^\pm = \frac{1}{\sqrt{2}} \left(Q_\alpha^1 \mp \epsilon_{\alpha\beta} \bar{Q}_{\dot{\beta}2} \right), \quad \bar{Q}_{\dot{\alpha}}^\pm = \frac{1}{\sqrt{2}} \left(\bar{Q}_{\dot{\alpha}1} \mp \epsilon_{\dot{\alpha}\dot{\beta}} Q_\beta^2 \right) \quad (1.7)$$

and $\bar{Q}_{\dot{\alpha}}^\pm = (Q_\alpha^\pm)^\dagger$. Also, we should postulate the following c -dependence of the central charges in (1.5)

$$X = -m_0 c + \frac{\mathbf{X}}{c} + \mathcal{O}\left(\frac{1}{c^2}\right), \quad Y = \mathbf{Y} + \mathcal{O}\left(\frac{1}{c}\right). \quad (1.8)$$

If $\mathbf{X}$ is finite in the contraction limit, it merely generates the shift $\mathbf{H} \rightarrow \mathbf{H} + \mathbf{X}$ in the relations of the $\mathcal{N} = 2$, $d = 3$ Galilean superalgebra (see the first relation in (1.4)).

The Galilean $\mathcal{N} = 2$ superalgebra and its dynamical realizations were studied in several papers, but mostly for the case of two ($d = 2$) space dimensions [2–4, 10, 37, 38].

In the present paper we consider $\mathcal{N} = 4$, $d = 3$ Galilean superalgebra with all possible central charges. It will be obtained by the $c \rightarrow \infty$ contraction procedure from the general $\mathcal{N} = 4$, $D = 4$ relativistic Poincaré superalgebra [39] which involves 6 complex central charges $Z^{AB} = -Z^{BA}$ ($A, B = 1, 2, 3, 4$). Correspondingly, the $\mathcal{N} = 4$, $d = 3$ Galilean superalgebra obtained in the $c \rightarrow \infty$ limit involves 12 real central charges.⁷ If these central charges are numerical, then, using a suitable redefinition of supercharges by an unitary 4×4 matrix, one can cast the antisymmetric 4×4 complex matrix of six central charges $Z^{AB} = -Z^{BA}$ ($A, B = 1, 2, 3, 4$) into a quasi-diagonal Jordanian form [40, 41]

$$Z^{AB} = \begin{pmatrix} 0 & -Z_1 & & 0 \\ Z_1 & 0 & & \\ & & 0 & -Z_2 \\ 0 & & Z_2 & 0 \end{pmatrix} \longleftrightarrow Z^{AB} = \text{diag}(\epsilon Z_1, \epsilon Z_2), \quad (1.9)$$

⁶Since $\mathcal{N} = 2$ $D = 4$ Poincaré superalgebra is covariant under the phase transformation of Weyl supercharges ($Q_\alpha^r, \bar{Q}_{\dot{\alpha}}^r; r = 1, 2$)

$$Q_\alpha^r \rightarrow e^{\frac{i}{2}\alpha} Q_\alpha^r \quad Z \rightarrow e^{i\alpha} Z \quad (Z = e^{-i\alpha} |z|)$$

one could think that one real central charge is enough in $\mathcal{N} = 2$ case. However, as was found by studying concrete dynamical models [35, 36], it is the complex $\mathcal{N} = 2$ central charge $Z = X_1 + iX_2$ what actually matters. It amounts to two physical real central charges: the topological magnetic charge X_1 and the non-topological electric charge X_2 . Only if these charges take constant eigenvalues, i.e. are numerical, they can be rotated to the single central charge by the phase transformations just mentioned.

⁷In fact, the NR $\mathcal{N} = 4$ Galilean superalgebra involves 13 central charges if we take into account the Bargmann central charge $\mathbf{M} = m_0$ obtained from the leading terms in the asymptotic expansion of P_0 and X in c (see (1.1) and (1.8)).

where $(\epsilon^{ab}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_2$ is antisymmetric matrix.⁸ The choice (1.9) breaks $U(4)$ internal symmetry of $\mathcal{N} = 4$, $D = 4$ Poincaré superalgebra down to $USp(2) \otimes USp(2) \simeq SU(2) \otimes SU(2) \simeq O(4)$;⁹ if we further put $Z = Z_1 = Z_2$ we obtain $\mathcal{N} = 4$, $D = 4$ Poincaré superalgebra with one complex central charge and unbroken $USp(4) \simeq O(5)$ internal symmetry.

Such a structure of the internal sectors survives in the non-relativistic limit; one can therefore consider $\mathcal{N} = 4$, $d = 3$ Galilean supersymmetric theories with the internal sectors $USp(2) \otimes USp(2)$ (four real Galilean central charges) or $USp(4)$ (a pair of real Galilean central charges).¹⁰

In the most general $\mathcal{N} = 4$ case, when we deal with six complex central charges, the central charge 4×4 matrix can be written as follows

$$Z^{AB} = \begin{pmatrix} Z^{ab} & Z^{\tilde{a}\tilde{b}} \\ Z^{\tilde{a}b} & Z^{\tilde{a}\tilde{b}} \end{pmatrix}, \quad Z^{ab} = \epsilon^{ab} Z_1, \quad Z^{\tilde{a}\tilde{b}} = \epsilon^{\tilde{a}\tilde{b}} Z_2, \quad Z^{\tilde{a}b} = -Z^{b\tilde{a}}, \quad (1.10)$$

where $a, b = 1, 2$ ($\tilde{a}, \tilde{b} = 1, 2$) are the left (right) $USp(2) \simeq SU(2)$ spinor indices. The four complex central charges Z^{ab} constitute complex $O(4, \mathbb{C})$ isovector $Z_M = \frac{1}{2i} (\sigma_M)_{\tilde{a}b} Z^{\tilde{a}b}$, where $(\sigma_M)_{\tilde{a}b}$ are $D = 4$ Euclidean Pauli matrices $\sigma_M = (\sigma_i, i\mathbb{1}_2)$. If $Z_M = 0$ (i.e., the central charge matrix is reduced to (1.10)) we deal with the decomposition of $\mathcal{N} = 4$ Galilean superalgebra into the direct sum of two $\mathcal{N} = 2$ Galilean superalgebras, each possessing $USp(2)$ automorphism; if $Z_M \neq 0$ the decomposition of $\mathcal{N} = 4$ Galilean supersymmetry into such a sum of two $\mathcal{N} = 2$ superalgebras is not possible. As we will see, in the absence of central charges the full compact internal R-symmetry in the NR case is $U(1) \otimes USp(4)$ as opposed to $U(4)$ of the relativistic $\mathcal{N} = 4$, $D = 4$ superalgebra. If the central charges take numerical values, the presence of off-diagonal supercharges (1.10) provides the breaking of $USp(2) \otimes USp(2) \simeq O(4) \subset USp(4)$ internal symmetry (still preserved by the diagonal central charges) down to the exact $O(3)$ or $O(2)$ internal symmetries which form diagonal subgroups in the product $O(3) \otimes O(3) = O(4)$.

The central charges, besides bringing in the mass parameters, are also capable to simplify the formulation of $\mathcal{N} \geq 2$ supersymmetric gauge theories. In particular, recall that $\mathcal{N} = 4$, $D = 4$ Yang-Mills theory with one central charge and internal symmetry broken to $O(5)$, contrary to $\mathcal{N} = 4$, $D = 4$ supersymmetric Yang-Mills theory with $SU(4)$ R-symmetry and without central charges, permits an off-shell superspace formulation which does not require harmonic variables [42, 43].

The plan of the paper is as follows. In section 2, following [12], we derive the general $\mathcal{N} = 4$, $d = 3$ Galilean superalgebra, which contains 12 independent real central charges and

⁸The transformation of the general case with 6 complex central charges Z^{AB} to the case with central charges given by matrix (1.9) is straightforward if Z^{AB} encompass constant mass-like parameters. If Z^{AB} are operators, the map (1.9) is valid only if the real operators $\hat{\alpha}_{AB}$ and $|Z_{A',B'}|$ defined as $Z_{AB} = \exp(i\hat{\alpha}_{AB})|Z_{AB}|$ mutually commute.

⁹Modulo chiral $U(1)$, see below.

¹⁰Further we denote these two non-relativistic superalgebras as $S\hat{g}(3; 4|4)$ and $S\hat{g}(3; 4|2)$, where $S\hat{g}(d; N|n)$ stands for N-extended d-dimensional Galilean superalgebra with n real central charges. The corresponding supergroups will be denoted $SG(d; N|n)$.

the additional thirteenth Bargmann central charge describing the rest mass. As in [28–31], in this derivation we employ the NR contraction $c \rightarrow \infty$ of relativistic $\mathcal{N} = 4$, $D = 4$ Poincaré superalgebra. In section 3 we calculate the MC one-forms on the coset $\mathcal{G}/\mathcal{H}$, where $\mathcal{G} = \text{SG}(3; 4|12)$ (see footnote 9) and stability subgroup $\mathcal{H}$ is given by $\text{SU}(2) \simeq \text{O}(3)$ and $\text{USp}(4)$ generators. In section 4 we study the $\mathcal{G}$ -invariant actions linear in MC one-forms associated with central charges. For different choices of the central charges these actions describe various models of $\mathcal{N} = 4$, $d = 3$ Galilean superparticles. We consider the phase superspace formulation of these superparticle models and present complete set of first and second class constraints. The first class fermionic constraints generate the non-relativistic $\mathcal{N} = 4$ κ -gauge transformations which act in the non-physical part of the Grassmann coordinate sector. In section 5 we quantize the model. Using super Schrödinger realization of quantum phase superspace algebra, we obtain as the quantum solutions of the model a set of free $\mathcal{N} = 4$, $d = 3$ Galilean superfields. In section 6 we present an outlook, in particular, we describe briefly the alternative ways of constructing the $\mathcal{N} = 4$ Galilean superparticle models. Concluding, we hope that our paper will contribute to the issue of superfield description of the interacting non-relativistic $\mathcal{N} = 4$, $d = 3$ supersymmetric field theories.¹¹

2 General Galilean $\mathcal{N} = 4$, $d = 3$ superalgebra with central charges

The $\mathcal{N} = 4$, $D = 4$ Poincaré superalgebra is spanned by the following generators¹²

- i) 4 two-component complex Weyl supercharges Q_α^A , $\bar{Q}_{\dot{\alpha}A} = (Q_\alpha^A)^\dagger$ ($\alpha = 1, 2$, $A = 1, \dots, 4$);
- ii) Poincaré algebra generators $P_\mu = (P_0, P_i)$, $M_{\mu\nu} = (M_i = \frac{1}{2} \epsilon_{ijk} M_{jk}, N_i = M_{i0})$, $i = 1, 2, 3$;
- iii) 16 internal Hermitian $\text{U}(4)$ R-symmetry generators G^A_B , $(G^A_B)^\dagger = G^B_A$, where antisymmetric 6 generators $G^A_{(a)B} = \frac{1}{2} (G^A_B - G^B_A)$ are $\text{O}(4)$ generators and 10 symmetric ones $G^A_{(s)B} = \frac{1}{2} (G^A_B + G^B_A)$ describe the coset $\text{U}(4)/\text{O}(4)$. The axial $\text{U}(1)$ generator $A = G^A_{(s)A}$ can be separated out, i.e. $\text{U}(4) = \text{SU}(4) \otimes \text{U}(1)$, where $\text{SU}(4)$ generators $T^A_B = G^A_B - \frac{1}{4} \delta^A_B A$ are traceless, $T^A_A = 0$, and satisfy the relation

$$[T^A_B, T^C_D] = \delta^C_B T^A_D - \delta^A_D T^C_B; \quad (2.1)$$

- iv) The set of 6 complex central charges $Z^{AB} = -Z^{BA}$, $\bar{Z}_{AB} = (Z^{AB})^\dagger$, or equivalently the set of 12 real central charges $\mathcal{X}^{AB} = -\mathcal{X}^{BA}$, $\mathcal{Y}^{AB} = -\mathcal{Y}^{BA}$, where

$$Z^{AB} = \mathcal{X}^{AB} + i \mathcal{Y}^{AB}. \quad (2.2)$$

¹¹For examples of supersymmetric extensions of QED and Yang-Mills Galilean theories see [44–46].

¹²We define $D = 4$ sigma-matrices as follows: $(\sigma^\mu)_{\alpha\dot{\alpha}} = (1_2, \vec{\sigma})_{\alpha\dot{\alpha}}$, $(\tilde{\sigma}^\mu)^{\dot{\alpha}\alpha} = \epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}} (\sigma^\mu)_{\beta\dot{\beta}} = (1_2, -\vec{\sigma})^{\dot{\alpha}\alpha}$, $\sigma^{\mu\nu} = i \sigma^{[\mu} \tilde{\sigma}^{\nu]}$, $\tilde{\sigma}^{\mu\nu} = i \tilde{\sigma}^{[\mu} \sigma^{\nu]}$. Always in this paper we use weight coefficient in (anti)symmetrization: $A_{(\mu} B_{\nu)} = \frac{1}{2} (A_\mu B_\nu + A_\nu B_\mu)$, $A_{[\mu} B_{\nu]} = \frac{1}{2} (A_\mu B_\nu - A_\nu B_\mu)$. The $D = 4$ metric tensor is taken as $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$.

Each of the real central charges $\mathcal{X}^{AB}$ and $\mathcal{Y}^{AB}$ transform in the adjoint of $O(4) \propto G_{(a)B}^A$. The $U(4)/O(4)$ coset generators properly mix $\mathcal{X}^{AB}$ and $\mathcal{Y}^{AB}$.

The general $\mathcal{N} = 4$, $D = 4$ super Poincaré anticommutation relations are

$$\{Q_\alpha^A, \bar{Q}_{\dot{\beta}B}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu \delta_B^A, \quad (2.3)$$

$$\begin{aligned} \{Q_\alpha^A, Q_\beta^B\} &= 2\epsilon_{\alpha\beta} Z^{AB} = 2\epsilon_{\alpha\beta} (\mathcal{X}^{AB} + i\mathcal{Y}^{AB}), \\ [6pt] \{\bar{Q}_{\dot{\alpha}A}, \bar{Q}_{\dot{\beta}B}\} &= 2\epsilon_{\dot{\alpha}\dot{\beta}} \bar{Z}_{AB} = 2\epsilon_{\dot{\alpha}\dot{\beta}} (\mathcal{X}^{AB} - i\mathcal{Y}^{AB}), \end{aligned} \quad (2.4)$$

and the remaining non-zero commutation relations read

$$[M_{\mu\nu}, Q_\alpha^A] = -\frac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta^A, \quad [M_{\mu\nu}, \bar{Q}_{\dot{\alpha}A}] = \frac{1}{2}(\tilde{\sigma}_{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}} \bar{Q}_{\dot{\beta}A}, \quad (2.5)$$

$$[P_\mu, Q_\alpha^A] = [P_\mu, \bar{Q}_{\dot{\alpha}A}] = 0, \quad (2.6)$$

$$[T_B^A, Q_\alpha^C] = \delta_B^C Q_\alpha^A - \frac{1}{4}\delta_B^A Q_\alpha^C, \quad [T_B^A, \bar{Q}_{\dot{\alpha}C}] = -\delta_C^A \bar{Q}_{\dot{\alpha}B} + \frac{1}{4}\delta_B^A \bar{Q}_{\dot{\alpha}C}, \quad (2.7)$$

$$[A, Q_\alpha^A] = \alpha Q_\alpha^A, \quad [A, \bar{Q}_{\dot{\alpha}A}] = -\alpha \bar{Q}_{\dot{\alpha}A}, \quad (2.8)$$

$$[A, Z^{AB}] = 2\alpha Z^{AB}, \quad [A, \bar{Z}_{AB}] = -2\alpha \bar{Z}_{AB}. \quad (2.9)$$

Here α is some real parameter. If we choose $\alpha = 1$, it defines the chirality of supercharges (see (2.8)) and so identifies A as the generator of axial symmetry.¹³

In order to perform the non-relativistic contraction of $\mathcal{N} = 4$, $D = 4$ Poincaré superalgebra to the limit describing $\mathcal{N} = 4$, $d = 3$ Galilean superalgebra one should rewrite the superalgebra (2.3)–(2.8) in the new fermionic Weyl basis¹⁴

$$Q_\alpha^{\pm A} = \frac{1}{\sqrt{2}} \left(Q_\alpha^A \pm \epsilon_{\alpha\beta} \Omega^{AB} \bar{Q}_{\dot{\beta}B} \right), \quad \bar{Q}_{\dot{\alpha}A}^\pm = \frac{1}{\sqrt{2}} \left(\bar{Q}_{\dot{\alpha}A} \mp \epsilon_{\dot{\alpha}\dot{\beta}} \Omega_{AB} Q_\beta^B \right), \quad (2.10)$$

where the real 4×4 matrix $\Omega^{AB} = -\Omega^{BA}$ is the symplectic metric, $\Omega_{AB} = -\Omega^{AB}$, $\Omega^{AC} \Omega_{CB} = \delta_B^A$. In this paper we choose the following explicit form of Ω :

$$\Omega_{AB} = \begin{pmatrix} \epsilon_{ab} & 0 \\ 0 & \epsilon_{\tilde{a}\tilde{b}} \end{pmatrix}, \quad \Omega^{AB} = \begin{pmatrix} \epsilon^{ab} & 0 \\ 0 & \epsilon^{\tilde{a}\tilde{b}} \end{pmatrix}, \quad (2.11)$$

where $\epsilon_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\epsilon^{ab} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $a = 1, 2$, $\tilde{a} = 1, 2$.

The relations (2.10) break manifest Lorentz symmetry $O(3, 1)$ to $O(3)$ (spinorial scalar product $a_\alpha b_{\dot{\alpha}}$ is $U(2)$ -invariant) and the internal symmetry $U(4)$ is broken to its subgroups which depend on the choice of central charges [41].

The supercharges (2.10) by definition satisfy the subsidiary symplectic-Majorana conditions [49]

$$(Q_\alpha^{\pm A})^\dagger = \bar{Q}_{\dot{\alpha}A}^\pm = \mp \epsilon_{\dot{\alpha}\dot{\beta}} \Omega_{AB} Q_\beta^{\pm B}. \quad (2.12)$$

¹³In the cases of $\mathcal{N} = 4$, $D = 4$ superconformal symmetry and the corresponding $d = 3$ Galilean superconformal algebra the generator A fully decouples, so implying $\alpha = 0$ (see [47]).

¹⁴The basis (2.10) was introduced in [48] and further used in [28, 30, 47] in order to obtain Galilean limit for $D = 4$ Majorana supercharges.

The full set of supercharges $(Q_{\alpha}^{\pm a}, \bar{Q}_{\dot{\alpha}a}^{\pm}; Q_{\alpha}^{\pm \tilde{a}}, \bar{Q}_{\dot{\alpha}\tilde{a}}^{\pm})$ can be split into the holomorphic sector $(Q_{\alpha}^{\pm a}, Q_{\alpha}^{\pm \tilde{a}})$ and the antiholomorphic one $(\bar{Q}_{\dot{\alpha}a}^{\pm}, \bar{Q}_{\dot{\alpha}\tilde{a}}^{\pm})$; these both sectors are related by the subsidiary conditions (2.12), thus revealing the quaternionic structure of the pairs of complex supercharges related by Hermitian conjugation (see [47, 50]). Due to the constraints (2.12) one can choose as unconstrained sets of linearly independent supercharges the generators from either holomorphic or antiholomorphic sectors. The $\mathcal{N} = 4$ superalgebra spanned by the generators from holomorphic sector is however not self-conjugate. In order to define the complete self-conjugated Hermitian basis one should choose the full set of pairs of supercharges, which are related by Hermitian conjugation. For the choice (2.11) of the matrix Ω these self-conjugated pairs are

$$Q_{\alpha}^{\pm} = Q_{\alpha}^{\pm 1}, \quad \bar{Q}_{\dot{\alpha}}^{\pm} = \bar{Q}_{\dot{\alpha}1}^{\pm}; \quad \tilde{Q}_{\alpha}^{\pm} = Q_{\alpha}^{\pm \tilde{1}}, \quad \bar{\tilde{Q}}_{\dot{\alpha}}^{\pm} = \bar{Q}_{\dot{\alpha}\tilde{1}}^{\pm}. \quad (2.13)$$

In this paper we will use the supercharges belonging to the holomorphic sector, i.e. $Q_{\alpha}^{\pm A}$ ($A = (a, \tilde{a}) = 1, \dots, 4$). They transform linearly under the $\text{USp}(4) \simeq \text{O}(5)$ R-symmetry, which defines the compact R-symmetry sector of $\mathcal{N} = 4$, $d = 3$ supersymmetry with one central charge $\mathcal{Z}$ corresponding to the following choice of 4×4 central charge matrix (1.10)

$$Z^{AB} = \mathcal{Z} \Omega^{AB}. \quad (2.14)$$

In the holomorphic basis the non-vanishing relations (2.3), (2.4) can be represented as

$$\{Q_{\alpha}^{\pm A}, Q_{\beta}^{\pm B}\} = 2\epsilon_{\alpha\beta} \left(\pm P_0 \Omega^{AB} + X^{AB} \right), \quad (2.15)$$

$$\{Q_{\alpha}^{+A}, Q_{\beta}^{-B}\} = 2 \left(P_i (\sigma_i)_{\alpha\beta} \Omega^{AB} + i\epsilon_{\alpha\beta} Y^{AB} \right), \quad (2.16)$$

where $(\sigma_i)_{\alpha\beta} = (\sigma_i)_{\beta\alpha} = \epsilon_{\alpha\gamma} (\sigma_i)_{\beta\gamma}$, $i = 1, 2, 3$ and

$$X^{AB} := \frac{1}{2} (Z^{AB} - \Omega^{AC} \bar{Z}_{CD} \Omega^{DB}), \quad Y^{AB} := \frac{1}{2i} (Z^{AB} + \Omega^{AC} \bar{Z}_{CD} \Omega^{DB}), \quad (2.17)$$

$$(X^{AB})^{\dagger} = \Omega_{AC} \Omega_{BD} X^{CD}, \quad (Y^{AB})^{\dagger} = \Omega_{AC} \Omega_{BD} Y^{CD}. \quad (2.18)$$

The relations inverse to (2.17) are

$$Z^{AB} = X^{AB} + iY^{AB}, \quad \bar{Z}_{AB} = X_{AB} - iY_{AB}. \quad (2.19)$$

Note that X^{AB} and Y^{AB} do not coincide with $\mathcal{X}^{AB}$ and $\mathcal{Y}^{AB}$ defined in (2.2): while the latter are real in the ordinary sense, the former are subject to the pseudo-reality conditions (2.18). The first kind of reality is preserved by the subgroup $\text{O}(4) \subset \text{SU}(4)$ with both $\mathcal{X}^{AB}$ and $\mathcal{Y}^{AB}$ in the adjoint representation of $\text{O}(4)$; the pseudoreality is consistent with the subgroup $\text{USp}(4) \simeq \text{O}(5) \subset \text{SU}(4)$, such that each of X^{AB} and Y^{AB} encompasses an $\text{O}(5)$ vector and an $\text{O}(5)$ scalar.

If we choose the central charges in accord with (1.10) the anticommutators (2.15), (2.16) take the form

$$\begin{aligned}\{Q_\alpha^{\pm a}, Q_\beta^{\pm b}\} &= 2\epsilon^{ab}\epsilon_{\alpha\beta}\left(\pm P_0 + \text{Re}(Z_1)\right), \\ \{Q_\alpha^{\pm \tilde{a}}, Q_\beta^{\pm \tilde{b}}\} &= 2\epsilon^{\tilde{a}\tilde{b}}\epsilon_{\alpha\beta}\left(\pm P_0 + \text{Re}(Z_2)\right), \\ \{Q_\alpha^{\pm a}, Q_\beta^{\pm \tilde{b}}\} &= 2\epsilon_{\alpha\beta}X^{a\tilde{b}},\end{aligned}\tag{2.20}$$

$$\begin{aligned}\{Q_\alpha^{+a}, Q_\beta^{-b}\} &= 2\epsilon^{ab}\left((\sigma_i)_{\alpha\beta}P_i + i\epsilon_{\alpha\beta}\text{Im}(Z_1)\right), \\ \{Q_\alpha^{+\tilde{a}}, Q_\beta^{-\tilde{b}}\} &= 2\epsilon^{\tilde{a}\tilde{b}}\left((\sigma_i)_{\alpha\beta}P_i + i\epsilon_{\alpha\beta}\text{Im}(Z_2)\right), \\ \{Q_\alpha^{+a}, Q_\beta^{-\tilde{b}}\} &= -\{Q_\alpha^{+\tilde{a}}, Q_\beta^{-b}\} = 2i\epsilon_{\alpha\beta}Y^{a\tilde{b}},\end{aligned}\tag{2.21}$$

where

$$X^{a\tilde{b}} := \frac{1}{2}\left(Z^{a\tilde{b}} + \epsilon^{ac}\epsilon^{\tilde{c}\tilde{d}}\bar{Z}_{c\tilde{d}}\right), \quad Y^{a\tilde{b}} := \frac{1}{2i}\left(Z^{a\tilde{b}} - \epsilon^{ac}\epsilon^{\tilde{c}\tilde{d}}\bar{Z}_{c\tilde{d}}\right).\tag{2.22}$$

It can be pointed out that $X^{a\tilde{b}}$ and $Y^{a\tilde{b}}$ are “real” with respect to the symplectic pseudoreality conditions similar to (2.12) and following from (2.18), $(X^{a\tilde{b}})^\dagger = \epsilon_{ac}\epsilon_{\tilde{b}\tilde{d}}X^{c\tilde{d}}$, $(Y^{a\tilde{b}})^\dagger = \epsilon_{ac}\epsilon_{\tilde{b}\tilde{d}}Y^{c\tilde{d}}$.

The commutators (2.5) for the generators (2.10) can be rewritten as follows

$$[M_{ij}, Q_\alpha^{\pm A}] = -\frac{1}{2}\epsilon_{ijk}(\sigma_k)_\alpha{}^\beta Q_\beta^{\pm A}, \quad [M_{i0}, Q_\alpha^{\pm A}] = \frac{i}{2}(\sigma_i)_\alpha{}^\beta Q_\beta^{\mp A}.\tag{2.23}$$

Further, let us decompose the generators of internal symmetry $\text{SU}(4)$ as

$$T_B^{\pm A} \equiv T_B^A \mp \Omega^{AC}T_C^D\Omega_{DB}.\tag{2.24}$$

The projections (2.24) of T_B^A satisfy the relations

$$T_A^{+C}\Omega_{BC} = 2T_{[A}^C\Omega_{B]C}, \quad T_A^{-C}\Omega_{BC} = 2T_{[A}^C\Omega_{B]C},\tag{2.25}$$

$$T_B^{\pm A} = \mp \Omega^{AC}T_C^{\pm D}\Omega_{DB}.\tag{2.26}$$

The constraint (2.26) amounts to the conditions for the generators $T_B^{\pm A}$,

$$T_{[A}^{+C}\Omega_{B]C} = 0, \quad T_{(A}^{-C}\Omega_{B)C} = 0.\tag{2.27}$$

So, the set of generators T^+ contains 10 independent generators which are symmetric in their indices

$$T_{AB}^+ = \Omega_{AC}T_B^{+C} = T_{(AB)}^+.\tag{2.28}$$

The set T^- involves 5 independent operators forming a traceless antisymmetric matrix

$$T_{AB}^- = \Omega_{AC}T_B^{-C} = T_{[AB]}^-, \quad \Omega^{AB}T_{AB}^- = 0.\tag{2.29}$$

Using (2.1), we find that operators (2.28), (2.29) satisfy the following algebra

$$[T_{AB}^\pm, T_{CD}^\pm] = -(\Omega_{BC}T_{AD}^\pm + \Omega_{AD}T_{BC}^\pm \pm \Omega_{AC}T_{BD}^\pm \pm \Omega_{BD}T_{AC}^\pm),\tag{2.30}$$

$$[T_{AB}^+, T_{CD}^-] = -(\Omega_{BC}T_{AD}^- - \Omega_{AD}T_{BC}^- + \Omega_{AC}T_{BD}^- - \Omega_{BD}T_{AC}^-).\tag{2.31}$$

Thus the original internal $SU(4)$ symmetry generators T_B^A , decomposed according to the relations (2.24), do split into the ones generating $USp(4)$ and the coset $SU(4)/USp(4)$:

$$h^{(3)} = \left(T_B^{+A} \right) \in usp(4), \quad k^{(3)} = \left(T_B^{-A} \right) \in su(4) \ominus usp(4). \quad (2.32)$$

This decomposition of the $su(4)$ algebra provides an example of symmetric Riemannian pair $(h^{(3)}, k^{(3)})$:

$$[h^{(3)}, h^{(3)}] \subset h^{(3)}, \quad [h^{(3)}, k^{(3)}] \subset k^{(3)}, \quad [k^{(3)}, k^{(3)}] \subset h^{(3)}. \quad (2.33)$$

The commutators (2.7) are rewritten in the new basis as

$$[T_B^{+A}, Q_\alpha^{\pm C}] = (\mathcal{U}_B^A)^C_D Q_\alpha^{\pm D}, \quad [T_B^{-A}, Q_\alpha^{\pm C}] = (\tau_B^A)^C_D Q_\alpha^{\mp D}, \quad (2.34)$$

where the 4×4 matrix $(\mathcal{U}_B^A)$,

$$(\mathcal{U}_B^A)^C_D = \delta_D^A \delta_B^C + \Omega^{AC} \Omega_{BD}, \quad (2.35)$$

defines the fundamental 4×4 representations of the $USp(4)$ algebra given by the supercharges $Q_\alpha^{\pm C}$. The matrices

$$(\tau_B^A)^C_D = \delta_D^A \delta_B^C - \Omega^{AC} \Omega_{BD} - \frac{1}{2} \delta_B^A \delta_D^C \quad (2.36)$$

enlarge the matrices $(\mathcal{U}_B^A)$ to the fundamental representations of $SU(4)$ algebra which interchange the $+$ and $-$ projections.

Let us make a comment on the case of $\alpha \neq 0$ in (2.8), (2.9). Choosing $\alpha = 1$, one finds

$$[A, Q_\alpha^{\pm B}] = Q_\alpha^{\mp B}; \quad [A, X^{AB}] = 2i Y^{AB}, \quad [A, Y^{AB}] = -2i X^{AB}. \quad (2.37)$$

Now we are prepared to define the $\mathcal{N} = 4$, $d = 3$ Galilean superalgebra by making use of the NR contraction procedure. One rescales the relativistic supercharges as

$$Q_\alpha^{+A} = c^{-1/2} \mathbf{Q}_\alpha^A, \quad Q_\alpha^{-A} = c^{1/2} \mathbf{S}_\alpha^A. \quad (2.38)$$

The physical rescaling of the bosonic generators of the algebra $\mathfrak{o}(1,3) \oplus \mathfrak{u}(4)$, where $(P_\mu, M_{\mu\nu}) \in \mathfrak{o}(1,3)$ and $(T_B^{\pm A}, A) \in \mathfrak{u}(4)$, is performed as follows

$$\begin{aligned} P_0 &= m_0 c + c^{-1} \mathbf{H}, & P_i &= \mathbf{P}_i, \\ M_{ij} &= \mathbf{J}_{ij}, & M_{i0} &= c \mathbf{B}_i, \\ A &= c \mathbf{A}_0, & T_B^{+A} &= \mathbf{T}_B^{+A}, & T_B^{-A} &= c \mathbf{T}_B^{-A}. \end{aligned} \quad (2.39)$$

where m_0 is the relativistic rest mass. The rescaling of the central charges is given by the formulas (see also (1.8))

$$X^{AB} = -m_0 c \Omega^{AB} + c^{-1} \mathbf{X}^{AB}, \quad Y^{AB} = \mathbf{Y}^{AB}, \quad (2.40)$$

where X^{AB} , Y^{AB} are defined in (2.17) and the operators $\mathbf{X}^{AB} = -\mathbf{X}^{BA}$, $\mathbf{Y}^{AB} = -\mathbf{Y}^{BA}$ satisfy the symplectic pseudoreality conditions.

We will firstly perform the $c \rightarrow \infty$ contraction for a simple choice of the central charge matrix.

i) Jordanian quasi-diagonal form of central charge matrix. Let us consider the special case with central charge matrix in the reduced form (1.9)

$$Z^{AB} = \begin{pmatrix} Z_1 \epsilon^{ab} & 0 \\ 0 & Z_2 \epsilon^{\tilde{a}\tilde{b}} \end{pmatrix}, \quad \bar{Z}_{AB} = \begin{pmatrix} -\bar{Z}_1 \epsilon_{ab} & 0 \\ 0 & -\bar{Z}_2 \epsilon_{\tilde{a}\tilde{b}} \end{pmatrix}, \quad (2.41)$$

where the central charge matrix (2.14) is recovered at $\mathcal{Z} = Z_1 = Z_2$. The rescaling (2.40) takes the more explicit form for this choice

$$\text{Re}(Z_1) = -m_0 c + c^{-1} \mathbf{X}_1, \quad \text{Re}(Z_2) = -m_0 c + c^{-1} \mathbf{X}_2, \quad (2.42)$$

$$\text{Im}(Z_1) = \mathbf{Y}_1, \quad \text{Im}(Z_2) = \mathbf{Y}_2. \quad (2.43)$$

Substituting these expressions, as well as (2.38) and (2.39), into the superalgebra relations (2.20), (2.21) with $Z^{a\tilde{b}} = Z^{\tilde{a}b} = 0$, and making there the $c \rightarrow \infty$ contraction, we obtain

$$\begin{aligned} \{\mathbf{Q}_\alpha^a, \mathbf{Q}_\beta^b\} &= 2 \epsilon^{ab} \epsilon_{\alpha\beta} (\mathbf{H} + \mathbf{X}_1), \\ \{\mathbf{Q}_\alpha^{\tilde{a}}, \mathbf{Q}_\beta^{\tilde{b}}\} &= 2 \epsilon^{\tilde{a}\tilde{b}} \epsilon_{\alpha\beta} (\mathbf{H} + \mathbf{X}_2), \\ \{\mathbf{Q}_\alpha^a, \mathbf{Q}_\beta^{\tilde{b}}\} &= 0, \end{aligned} \quad (2.44)$$

$$\begin{aligned} \{\mathbf{S}_\alpha^a, \mathbf{S}_\beta^b\} &= -4 \epsilon^{ab} \epsilon_{\alpha\beta} m_0, \\ \{\mathbf{S}_\alpha^{\tilde{a}}, \mathbf{S}_\beta^{\tilde{b}}\} &= -4 \epsilon^{\tilde{a}\tilde{b}} \epsilon_{\alpha\beta} m_0, \\ \{\mathbf{S}_\alpha^a, \mathbf{S}_\beta^{\tilde{b}}\} &= 0, \end{aligned} \quad (2.45)$$

$$\begin{aligned} \{\mathbf{Q}_\alpha^a, \mathbf{S}_\beta^b\} &= 2 \epsilon^{ab} \left((\sigma_i)_{\alpha\beta} \mathbf{P}_i + i \epsilon_{\alpha\beta} \mathbf{Y}_1 \right), \\ \{\mathbf{Q}_\alpha^{\tilde{a}}, \mathbf{S}_\beta^{\tilde{b}}\} &= 2 \epsilon^{\tilde{a}\tilde{b}} \left((\sigma_i)_{\alpha\beta} \mathbf{P}_i + i \epsilon_{\alpha\beta} \mathbf{Y}_2 \right), \\ \{\mathbf{Q}_\alpha^a, \mathbf{S}_\beta^{\tilde{b}}\} &= \{\mathbf{Q}_\alpha^{\tilde{a}}, \mathbf{S}_\beta^b\} = 0. \end{aligned} \quad (2.46)$$

Here $\mathbf{Q}_\alpha^A = (\mathbf{Q}_\alpha^a, \mathbf{Q}_\alpha^{\tilde{a}})$, $\mathbf{S}_\alpha^A = (\mathbf{S}_\alpha^a, \mathbf{S}_\alpha^{\tilde{a}})$, and the indices are chosen so that $a = 1, 2$ correspond to $A = 1, 2$ and $\tilde{a} = 1, 2$ to $A = 3, 4$.

ii) General central charge matrix. In the general case with non-zero off-diagonal central charges $\mathbf{X}^{a\tilde{b}} = -\mathbf{X}^{\tilde{b}a}$ and $\mathbf{Y}^{a\tilde{b}} = -\mathbf{Y}^{\tilde{b}a}$, the last lines in (2.44) and (2.46) are replaced, respectively, by the relations

$$\begin{aligned} \{\mathbf{Q}_\alpha^a, \mathbf{Q}_\beta^{\tilde{b}}\} &= 2 \epsilon_{\alpha\beta} \mathbf{X}^{a\tilde{b}}, \\ \{\mathbf{Q}_\alpha^a, \mathbf{S}_\beta^{\tilde{b}}\} &= \{\mathbf{Q}_\beta^{\tilde{b}}, \mathbf{S}_\alpha^a\} = 2i \epsilon_{\alpha\beta} \mathbf{Y}^{a\tilde{b}}. \end{aligned} \quad (2.47)$$

It is easy to check that the rescaling (2.38) preserves the symplectic-Majorana conditions (2.10) and in the limit $c \rightarrow \infty$ one obtains the following Galilean form of $\mathcal{N} = 4$

symplectic-Majorana conditions¹⁵

$$\begin{aligned} (\mathbf{Q}_\alpha^a)^\dagger &\equiv \bar{\mathbf{Q}}_{\dot{\alpha}a} = -\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{ab}\mathbf{Q}_\beta^b, & (\mathbf{Q}_\alpha^{\tilde{a}})^\dagger &\equiv \bar{\mathbf{Q}}_{\dot{\alpha}\tilde{a}} = -\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\tilde{a}\tilde{b}}\mathbf{Q}_\beta^{\tilde{b}}, \\ (\mathbf{S}_\alpha^a)^\dagger &\equiv \bar{\mathbf{S}}_{\dot{\alpha}a} = \epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{ab}\mathbf{S}_\beta^b, & (\mathbf{S}_\alpha^{\tilde{a}})^\dagger &\equiv \bar{\mathbf{S}}_{\dot{\alpha}\tilde{a}} = \epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{\tilde{a}\tilde{b}}\mathbf{S}_\beta^{\tilde{b}}. \end{aligned} \quad (2.48)$$

Due to (2.18), off-diagonal central charges satisfy the following pseudo-reality conditions

$$\left(\mathbf{X}^{ab}\right)^\dagger = \epsilon_{ac}\epsilon_{\tilde{b}\tilde{d}}\mathbf{X}^{cd}, \quad \left(\mathbf{Y}^{ab}\right)^\dagger = \epsilon_{ac}\epsilon_{\tilde{b}\tilde{d}}\mathbf{Y}^{cd}. \quad (2.49)$$

We point out that the constant m_0 can be considered as an additional thirteenth central charge, i.e. in fact the superalgebra (2.44)–(2.46) contains thirteen central charges $(m_0, \mathbf{X}_1, \mathbf{X}_2, \mathbf{Y}_1, \mathbf{Y}_2, \mathbf{X}^{ab}, \mathbf{Y}^{ab})$ as compared to twelve in $D = 4$ relativistic case.

iii) Internal symmetry sectors. After $c \rightarrow \infty$ contraction (2.23), the covariance relations of the supercharges with respect to NR $O(3)$ rotations $\mathbf{J}_{ij}$ and Galilean boosts $\mathbf{B}_i$ are written as

$$[\mathbf{J}_{ij}, \mathbf{Q}_\alpha^A] = -\frac{1}{2}\epsilon_{ijk}(\sigma_k)_\alpha{}^\beta \mathbf{Q}_\beta^A, \quad [\mathbf{J}_{ij}, \mathbf{S}_\alpha^A] = -\frac{1}{2}\epsilon_{ijk}(\sigma_k)_\alpha{}^\beta \mathbf{S}_\beta^A, \quad (2.50)$$

$$[\mathbf{B}_i, \mathbf{Q}_\alpha^A] = \frac{i}{2}(\sigma_i)_\alpha{}^\beta \mathbf{S}_\beta^A, \quad [\mathbf{B}_i, \mathbf{S}_\alpha^A] = 0. \quad (2.51)$$

Using substitutions (2.38), (2.39), the contraction of the relations (2.34) leads also to the covariance relations of supercharges with respect to the internal symmetry generators $\mathbf{h}^{(3)} = (\mathbf{T}^{+A}_B)$, $\mathbf{k}^{(3)} = (\mathbf{T}^{-A}_B)$ and axial charge $\mathbf{A} = \mathbf{A}_0|_{\alpha=1}$

$$[\mathbf{T}^{+A}_B, \mathbf{Q}_\alpha^C] = (\mathcal{U}^A_B)^C{}_D \mathbf{Q}_\alpha^D, \quad [\mathbf{T}^{+A}_B, \mathbf{S}_\alpha^C] = (\mathcal{U}^A_B)^C{}_D \mathbf{S}_\alpha^D, \quad (2.52)$$

$$[\mathbf{T}^{-A}_B, \mathbf{Q}_\alpha^C] = (\tau^A_B)^C{}_D \mathbf{S}_\alpha^D, \quad [\mathbf{T}^{-A}_B, \mathbf{S}_\alpha^C] = 0, \quad (2.53)$$

$$[\mathbf{A}, \mathbf{Q}_\alpha^A] = -\mathbf{S}_\alpha^A, \quad [\mathbf{A}, \mathbf{S}_\alpha^A] = 0. \quad (2.54)$$

For what follows, it will be useful to have the generators $\mathbf{T}^{+A}_B \in \mathfrak{usp}(4)$ in the splitting $\mathfrak{usp}(2) \oplus \mathfrak{usp}(2)$ basis. This notation corresponds to the following coset decomposition

$$\mathrm{USp}(4) \simeq \frac{\mathrm{USp}(4)}{\mathrm{USp}(2) \otimes \mathrm{USp}(2)} \cdot \left[\mathrm{USp}(2) \otimes \mathrm{USp}(2) \right], \quad (2.55)$$

such that

$$\left(\mathbf{T}^{+a}_b, \mathbf{T}^{+\tilde{a}}_{\tilde{b}}\right) \in \mathfrak{usp}(2) \oplus \mathfrak{usp}(2), \quad (2.56)$$

$$\left(\mathbf{T}^{+\tilde{a}}_b, \mathbf{T}^{+a}_{\tilde{b}}\right) \in \mathfrak{usp}(4) \ominus [\mathfrak{usp}(2) \oplus \mathfrak{usp}(2)]. \quad (2.57)$$

From (2.39) it follows that the coset generators $k^{(3)}$ are rescaled ($h^{(3)} = \mathbf{h}^{(3)}$, $k^{(3)} = c\mathbf{k}^{(3)}$) and in the limit $c \rightarrow \infty$ we get the inhomogeneous extension of $\mathfrak{usp}(4) \cong \mathfrak{o}(5)$ internal algebra

$$[\mathbf{h}^{(3)}, \mathbf{h}^{(3)}] \subset \mathbf{h}^{(3)}, \quad [\mathbf{h}^{(3)}, \mathbf{k}^{(3)}] \subset \mathbf{k}^{(3)}, \quad [\mathbf{k}^{(3)}, \mathbf{k}^{(3)}] = 0. \quad (2.58)$$

¹⁵We recall that product $A_\alpha B_{\dot{\alpha}}$ of two NR spinors A_α , $B_{\dot{\alpha}} \equiv (\overline{B_\alpha})$ is $U(2)$ invariant, i.e. $\delta_{\alpha\dot{\beta}}$ is the $U(2)$ -invariant metric.

The five commuting generators $\mathbf{T}_B^{-A}$ of $\mathbf{k}^{(3)}$ describe a kind of curved internal momenta. Thus in the contraction limit $c \rightarrow \infty$ one gets the following $\mathcal{N} = 4$ Galilean internal inhomogeneous symmetry algebra $\mathbf{T}$

$$\mathbf{T} = \mathbf{h}^{(3)} \oplus \mathbf{k}^{(3)}, \quad \begin{aligned} \mathbf{h}^{(3)} &= u(2; \mathbb{H}) = usp(4), \\ \mathbf{k}^{(3)} &= \mathcal{A}^5 \quad (\text{Abelian}). \end{aligned} \quad (2.59)$$

We will denote the corresponding inhomogeneous group by $\text{IUSp}(4)$. The abelian generator $\mathbf{A}$ can be added to the ideal $\mathbf{k}^{(3)}$, so extending it to six-dimensional one.¹⁶

The action of the $\text{IUSp}(4)$ generators in the $\text{USp}(2) \otimes \text{USp}(2)$ splitting basis, $\mathbf{T}_B^{+A} = (\mathbf{T}_b^{+a}, \mathbf{T}_{\tilde{b}}^{+\tilde{a}}, \mathbf{T}_{\tilde{b}}^{+a}, \mathbf{T}_b^{+\tilde{a}})$ on the supercharges in the same basis, $\mathbf{Q}_\alpha^A = (\mathbf{Q}_\alpha^a, \mathbf{Q}_\alpha^{\tilde{a}})$, $\mathbf{S}_\alpha^A = (\mathbf{S}_\alpha^a, \mathbf{S}_\alpha^{\tilde{a}})$, as well as on the central charges, can be easily found from the relations (2.52). For instance, the commutation relations between $\mathbf{T}_{\tilde{b}}^{+a}$ and central charges are given by

$$[\mathbf{T}_{\tilde{c}}^{+d}, \mathbf{X}^{a\tilde{b}}] = \delta_{\tilde{c}}^{\tilde{b}} \epsilon^{da} (\mathbf{X}_2 - \mathbf{X}_1), \quad [\mathbf{T}_{\tilde{c}}^{+d}, \mathbf{X}_1] = -[\mathbf{T}_{\tilde{c}}^{+d}, \mathbf{X}_2] = \epsilon_{\tilde{c}\tilde{b}} \mathbf{X}^{d\tilde{b}}, \quad (2.60)$$

and by similar formulas for $\mathbf{Y}^{a\tilde{b}}$, $\mathbf{Y}_1, \mathbf{Y}_2$. It follows from these relations that the full set of central charges splits into two $\text{USp}(4) \cong \text{O}(5)$ five-vectors $(\mathbf{X}^{a\tilde{b}}, \mathbf{X}_1 - \mathbf{X}_2)$ and $(\mathbf{Y}^{a\tilde{b}}, \mathbf{Y}_1 - \mathbf{Y}_2)$ and two $\text{O}(5)$ singlets (isoscalsars) $(\mathbf{X}_1 + \mathbf{X}_2)$, $(\mathbf{Y}_1 + \mathbf{Y}_2)$.

The first relation in (2.53) amounts to the following set of relations in the splitting basis

$$\begin{aligned} [\mathbf{T}_b^{-a}, \mathbf{Q}_\alpha^c] &= \frac{1}{2} \delta_b^a \mathbf{S}_\alpha^c, & [\mathbf{T}_{\tilde{b}}^{-\tilde{a}}, \mathbf{Q}_\alpha^c] &= -\frac{1}{2} \delta_{\tilde{b}}^{\tilde{a}} \mathbf{S}_\alpha^c, & [\mathbf{T}_{\tilde{b}}^{-a}, \mathbf{Q}_\alpha^c] &= -\epsilon^{ac} \epsilon_{\tilde{b}\tilde{d}} \mathbf{S}_\alpha^{\tilde{d}} \\ [\mathbf{T}_b^{-a}, \mathbf{Q}_\alpha^{\tilde{c}}] &= -\frac{1}{2} \delta_b^a \mathbf{S}_\alpha^{\tilde{c}}, & [\mathbf{T}_{\tilde{b}}^{-\tilde{a}}, \mathbf{Q}_\alpha^{\tilde{c}}] &= \frac{1}{2} \delta_{\tilde{b}}^{\tilde{a}} \mathbf{S}_\alpha^{\tilde{c}}, & [\mathbf{T}_{\tilde{b}}^{-a}, \mathbf{Q}_\alpha^{\tilde{c}}] &= \delta_{\tilde{b}}^{\tilde{c}} \mathbf{S}_\alpha^a. \end{aligned} \quad (2.61)$$

The commutation relations between $\mathbf{T}_B^{-A}$ and the central charges $\mathbf{X}^{AB}$ and $\mathbf{Y}^{AB}$ read

$$\begin{aligned} [\mathbf{T}_D^{-C}, \mathbf{X}^{AB}] &= -i(\tau_D^C)_E^A \mathbf{Y}^{BE} + i(\tau_D^C)_E^B \mathbf{Y}^{AE}, \\ [\mathbf{T}_D^{-C}, \mathbf{Y}^{AB}] &= 2im_0 \left(\delta_D^A \Omega^{CB} - \delta_D^B \Omega^{CA} - \frac{1}{2} \delta_D^C \Omega^{AB} \right), \end{aligned} \quad (2.62)$$

or, in the splitting basis,

$$\begin{aligned} [\mathbf{T}_g^{-f}, \mathbf{X}_1] &= i\delta_g^f \mathbf{Y}_1, & [\mathbf{T}_g^{-f}, \mathbf{X}_2] &= -i\delta_g^f \mathbf{Y}_2, & [\mathbf{T}_g^{-f}, \mathbf{X}^{a\tilde{b}}] &= 2i(\delta_g^a \mathbf{Y}^{f\tilde{b}} - \delta_g^{\tilde{b}} \mathbf{Y}^{af}), \\ [\mathbf{T}_{\tilde{g}}^{-\tilde{f}}, \mathbf{X}_1] &= -i\delta_{\tilde{g}}^{\tilde{f}} \mathbf{Y}_1, & [\mathbf{T}_{\tilde{g}}^{-\tilde{f}}, \mathbf{X}_2] &= i\delta_{\tilde{g}}^{\tilde{f}} \mathbf{Y}_2, & [\mathbf{T}_{\tilde{g}}^{-\tilde{f}}, \mathbf{X}^{a\tilde{b}}] &= 2i(\delta_{\tilde{g}}^{\tilde{b}} \mathbf{Y}^{a\tilde{f}} - \delta_{\tilde{g}}^{\tilde{f}} \mathbf{Y}^{a\tilde{b}}), \\ [\mathbf{T}_{\tilde{g}}^{-f}, \mathbf{X}_1] &= -i\epsilon_{\tilde{g}\tilde{c}} \mathbf{Y}^{f\tilde{c}}, & [\mathbf{T}_{\tilde{g}}^{-f}, \mathbf{X}_2] &= -i\epsilon_{\tilde{g}\tilde{c}} \mathbf{Y}^{f\tilde{c}}, & [\mathbf{T}_{\tilde{g}}^{-f}, \mathbf{X}^{a\tilde{b}}] &= -i\delta_{\tilde{g}}^{\tilde{b}} \epsilon^{fa} (\mathbf{Y}_1 + \mathbf{Y}_2), \end{aligned} \quad (2.63)$$

and

$$\begin{aligned} [\mathbf{T}_g^{-f}, \mathbf{Y}_1] &= im_0 \delta_g^f, & [\mathbf{T}_g^{-f}, \mathbf{Y}_2] &= -im_0 \delta_g^f, & [\mathbf{T}_g^{-f}, \mathbf{Y}^{a\tilde{b}}] &= 0, \\ [\mathbf{T}_{\tilde{g}}^{-\tilde{f}}, \mathbf{Y}_1] &= -im_0 \delta_{\tilde{g}}^{\tilde{f}}, & [\mathbf{T}_{\tilde{g}}^{-\tilde{f}}, \mathbf{Y}_2] &= im_0 \delta_{\tilde{g}}^{\tilde{f}}, & [\mathbf{T}_{\tilde{g}}^{-\tilde{f}}, \mathbf{Y}^{a\tilde{b}}] &= 0, \\ [\mathbf{T}_{\tilde{g}}^{-f}, \mathbf{Y}_1] &= 0, & [\mathbf{T}_{\tilde{g}}^{-f}, \mathbf{Y}_2] &= 0, & [\mathbf{T}_{\tilde{g}}^{-f}, \mathbf{Y}^{a\tilde{b}}] &= 2im_0 \delta_{\tilde{g}}^{\tilde{b}} \epsilon^{af}. \end{aligned} \quad (2.64)$$

¹⁶We recall here that the replacement of $\text{SU}(2N)$ internal symmetry by $\text{IUSp}(2N)$ in the presence of NR contraction occurred as well in the derivation of Galilean conformal superalgebra [30, 31, 47] from extended relativistic conformal superalgebra $\text{SU}(2, 2|2N)$.

The commutation relations between the U(1) axial generator $\mathbf{A}$ and the central charges have a similar structure.:

$$\begin{aligned} [\mathbf{A}, \mathbf{X}_1] &= -2i \mathbf{Y}_1, & [\mathbf{A}, \mathbf{X}_2] &= -2i \mathbf{Y}_2, & [\mathbf{A}, \mathbf{X}^{a\tilde{b}}] &= -2i \mathbf{Y}^{a\tilde{b}}, \\ [\mathbf{A}, \mathbf{Y}_1] &= -2i m_0, & [\mathbf{A}_0, \mathbf{Y}_2] &= -2i m_0, & [\mathbf{A}, \mathbf{Y}^{a\tilde{b}}] &= 0. \end{aligned} \quad (2.65)$$

Our last remark concerns the $\mathcal{N} = 4$ Galilean algebra with the diagonal choice (1.9), (2.41) for the central charge matrix. Recalling (2.44)–(2.46), we observe that in this case $\mathcal{N} = 4$ Galilean algebra (with suitable restriction of R-symmetry algebras taken into account) reduces to the sum of two $\mathcal{N} = 2$ Galilean superalgebras spanned by the supercharge pairs $(\mathbf{Q}_\alpha^a, \mathbf{S}_\alpha^a), (\mathbf{Q}_\alpha^{\tilde{a}}, \mathbf{S}_\alpha^{\tilde{a}})$, with common generators $\mathbf{H}$ and $\mathbf{P}_i$. The only way to avoid such a splitting is to switch on the off-diagonal central charges as in (2.47). If we consider an extended $\mathcal{N} = 4$ Galilean algebra, with the R-symmetry generators T_B^{+A} included, the $\mathcal{N} = 2$ subsectors in (2.44)–(2.46) will be intertwined by the generators T_b^{+a} , e.g., $[T_b^{+a}, \mathbf{Q}_\alpha^c] = \delta_b^c \mathbf{Q}_\alpha^{\tilde{a}}, [T_b^{+a}, \mathbf{Q}_\alpha^{\tilde{c}}] = \epsilon^{\tilde{a}\tilde{c}} \mathbf{Q}_{b\alpha}$. In this case, the splitting into two $\mathcal{N} = 2$ algebras arises only when we eliminate the generators $\mathbf{T}_b^{+a}$ from the R-symmetry algebra.

To avoid a possible confusion, note that the R-symmetry is described by the group of outer automorphisms of superalgebras and its generators do not appear in the r.h.s. of the (anti)commutators (distinctly from central charges). Therefore, when constructing the specific models, we can restrict the R-symmetry group to some of its subgroup. The maximal R-symmetry group $\text{USp}(4) \sim \text{O}(5)$ can be ensured in two distinct cases: either for the choice (2.14) with $\mathcal{Z}$ being $\text{USp}(4) \sim \text{O}(5)$ invariant (the same if $\mathcal{Z}$ is an operator or a number), or for the generic choice (1.10), with $\mathbf{X}_1 - \mathbf{X}_2, \mathbf{X}^{a\tilde{b}}$ and $\mathbf{Y}_1 - \mathbf{Y}_2, \mathbf{Y}^{a\tilde{b}}$ forming two independent $\text{O}(5)$ vectors (see (2.60)), and with two $\text{O}(5)$ singlets $(\mathbf{X}_1 + \mathbf{X}_2), (\mathbf{Y}_1 + \mathbf{Y}_2)$ accommodating the remaining two central charges.

In the second case one has an additional freedom to eliminate, without breaking $\text{O}(5)$ covariance, either all $\mathbf{Y}$ central charges or all $\mathbf{X}$ central charges, and further choose, e.g., $\mathbf{X}_2 + \mathbf{X}_1 = 0$ or $\mathbf{Y}_2 + \mathbf{Y}_1 = 0$. As was already mentioned, with the general option (1.10) the choice of numerical central charges necessarily breaks $\text{O}(5)$ R-symmetry down to $\text{O}(3)$.

iv) Hermitian basis. One can alternatively formulate NR $\mathcal{N} = 4, d = 3$ superalgebra by using NR contraction of Hermitian pairs of supercharges which are self-conjugate with respect to Hermitian conjugation (see (2.13)). We define the set of unconstrained independent supercharges spanning $\mathcal{N} = 4$ Galilean Hermitian superalgebra as follows

$$\begin{aligned} \mathbf{Q}_\alpha &\equiv \mathbf{Q}_\alpha^1, & \bar{\mathbf{Q}}_{\dot{\alpha}} &\equiv \bar{\mathbf{Q}}_{\dot{\alpha}1}, & \tilde{\mathbf{Q}}_\alpha &\equiv \mathbf{Q}_\alpha^{\tilde{1}}, & \bar{\tilde{\mathbf{Q}}}_{\dot{\alpha}} &\equiv \bar{\mathbf{Q}}_{\dot{\alpha}\tilde{1}}, \\ \mathbf{S}_\alpha &\equiv \mathbf{S}_\alpha^1, & \bar{\mathbf{S}}_{\dot{\alpha}} &\equiv \bar{\mathbf{S}}_{\dot{\alpha}1}, & \tilde{\mathbf{S}}_\alpha &\equiv \mathbf{S}_\alpha^{\tilde{1}}, & \bar{\tilde{\mathbf{S}}}_{\dot{\alpha}} &\equiv \bar{\mathbf{S}}_{\dot{\alpha}\tilde{1}}. \end{aligned} \quad (2.66)$$

One gets

$$\begin{aligned} \{\mathbf{Q}_\alpha, \bar{\mathbf{Q}}_{\dot{\beta}}\} &= 2\delta_{\alpha\dot{\beta}} (\mathbf{H} + \mathbf{X}_1), \\ \{\tilde{\mathbf{Q}}_\alpha, \bar{\tilde{\mathbf{Q}}}_{\dot{\beta}}\} &= 2\delta_{\alpha\dot{\beta}} (\mathbf{H} + \mathbf{X}_2), \\ \{\mathbf{Q}_\alpha, \tilde{\mathbf{Q}}_{\dot{\beta}}\} &= 2\epsilon_{\alpha\dot{\beta}} \mathbf{X}^{1\tilde{1}}, & \{\bar{\mathbf{Q}}_{\dot{\alpha}}, \bar{\tilde{\mathbf{Q}}}_{\dot{\beta}}\} &= 2\epsilon_{\dot{\alpha}\dot{\beta}} \mathbf{X}^{2\tilde{2}}, \\ \{\mathbf{Q}_\alpha, \bar{\tilde{\mathbf{Q}}}_{\dot{\beta}}\} &= -2\delta_{\alpha\dot{\beta}} \mathbf{X}^{1\tilde{2}}, & \{\tilde{\mathbf{Q}}_\alpha, \bar{\mathbf{Q}}_{\dot{\beta}}\} &= 2\delta_{\alpha\dot{\beta}} \mathbf{X}^{2\tilde{1}}, \end{aligned} \quad (2.67)$$

$$\begin{aligned}
 \{\mathbf{S}_\alpha, \bar{\mathbf{S}}_{\dot{\beta}}\} &= 4\delta_{\alpha\dot{\beta}} m_0, \\
 \{\tilde{\mathbf{S}}_\alpha, \bar{\tilde{\mathbf{S}}}_{\dot{\beta}}\} &= 4\delta_{\alpha\dot{\beta}} m_0, \\
 \{\mathbf{S}_\alpha, \bar{\tilde{\mathbf{S}}}_{\dot{\beta}}\} &= \{\tilde{\mathbf{S}}_\alpha, \bar{\mathbf{S}}_{\dot{\beta}}\} = 0,
 \end{aligned} \tag{2.68}$$

$$\begin{aligned}
 \{\mathbf{Q}_\alpha, \bar{\mathbf{S}}_{\dot{\beta}}\} &= 2\left((\sigma_i)_{\alpha\dot{\beta}} \mathbf{P}_i - i\delta_{\alpha\dot{\beta}} \mathbf{Y}_1\right), \\
 \{\tilde{\mathbf{S}}_\alpha, \bar{\tilde{\mathbf{Q}}}_{\dot{\beta}}\} &= -2\left((\sigma_i)_{\alpha\dot{\beta}} \mathbf{P}_i - i\delta_{\alpha\dot{\beta}} \mathbf{Y}_2\right), \\
 \{\mathbf{Q}_\alpha, \bar{\tilde{\mathbf{S}}}_{\dot{\beta}}\} &= 2i\epsilon_{\alpha\dot{\beta}} \mathbf{Y}^{1\bar{1}}, & \{\bar{\mathbf{Q}}_{\dot{\alpha}}, \bar{\tilde{\mathbf{S}}}_{\dot{\beta}}\} &= -2i\epsilon_{\dot{\alpha}\dot{\beta}} \mathbf{Y}^{2\bar{2}}, \\
 \{\mathbf{Q}_\alpha, \bar{\mathbf{S}}_{\dot{\beta}}\} &= 2i\delta_{\alpha\dot{\beta}} \mathbf{Y}^{1\bar{2}}, & \{\tilde{\mathbf{S}}_\alpha, \bar{\mathbf{Q}}_{\dot{\beta}}\} &= 2i\delta_{\alpha\dot{\beta}} \mathbf{Y}^{2\bar{1}}.
 \end{aligned} \tag{2.69}$$

The Hermitian form of $\mathcal{N} = 4$, $d = 3$ Galilean superalgebra (2.67)–(2.69) permits to obtain the generalized positivity conditions for the Hamiltonian $\mathbf{H}$. From first two formulas in (2.67) one derives that for any normalized state $|\Psi\rangle$ belonging to the Hilbert space of physical states of the models the following conditions hold

$$\begin{aligned}
 \frac{1}{4} \sum_{\alpha} \langle \Psi | \{\mathbf{Q}_\alpha, \bar{\mathbf{Q}}_{\dot{\alpha}}\} | \Psi \rangle &= \frac{1}{4} \sum_{|\Phi\rangle} \sum_{\alpha} \left| \langle \Psi | \mathbf{Q}_\alpha | \Phi \rangle \right|^2 = \langle \Psi | (\mathbf{H} + \mathbf{X}_1) | \Psi \rangle \geq 0, \\
 \frac{1}{4} \sum_{\alpha} \langle \Psi | \{\tilde{\mathbf{Q}}_\alpha, \bar{\tilde{\mathbf{Q}}}_{\dot{\alpha}}\} | \Psi \rangle &= \frac{1}{4} \sum_{|\Phi\rangle} \sum_{\alpha} \left| \langle \Psi | \tilde{\mathbf{Q}}_\alpha | \Phi \rangle \right|^2 = \langle \Psi | (\mathbf{H} + \mathbf{X}_2) | \Psi \rangle \geq 0.
 \end{aligned} \tag{2.70}$$

In dynamical models (like those of section 4) the central charges $\mathbf{X}_1$, $\mathbf{X}_2$ are represented on the normalized states $|\Psi\rangle$ by the mass-like parameters m_1 , m_2 , so from (2.70) one gets the lower bound on the energy values $E_\Psi = \langle \Psi | \mathbf{H} | \Psi \rangle \geq -\min(m_1, m_2)$.

3 Nonlinear realizations of $\mathcal{N} = 4$, $d = 3$ Galilean supersymmetries

In the nonlinear realization of $\mathcal{N} = 4$, $d = 3$ Galilean supersymmetries we will assume that the linearization subgroup H involves the 3-dimensional space rotations generators $\mathbf{J}^{ik}$, the internal symmetry $\text{USp}(4)$ generators $\mathbf{T}_B^{+A}$ and the abelian generator $\mathbf{A}_0$. All other generators are placed in the coset $G = SG(3; 4|12)/H$. Some of the parameters belonging to G can be relocated into the linearization subgroup H just by nullifying the respective coset parameters. The coset element G can be written explicitly as

$$G = G_{(1)} G_{(2)} G_{(3)} G_{(4)} G_{(5)} G_{(6)} \equiv \hat{G} G_{(6)}, \tag{3.1}$$

where

$$\begin{aligned}
 G_{(1)} &= \exp i\{t\mathbf{H} + x^i \mathbf{P}^i\}, \\
 G_{(2)} &= \exp i\{\xi_a^\alpha \mathbf{Q}_\alpha^a + \xi_{\bar{a}}^\alpha \mathbf{Q}_{\dot{\alpha}}^{\bar{a}}\}, \\
 G_{(3)} &= \exp i\{\theta_a^\alpha \mathbf{S}_\alpha^a + \theta_{\bar{a}}^\alpha \mathbf{S}_{\dot{\alpha}}^{\bar{a}}\}, \\
 G_{(4)} &= \exp i\{k^i \mathbf{B}^i\}, \\
 G_{(5)} &= \exp i\{s\mathbf{M} + h_1 \mathbf{X}_1 + h_2 \mathbf{X}_2 + h_{a\bar{b}} \mathbf{X}^{a\bar{b}} + f_1 \mathbf{Y}_1 + f_2 \mathbf{Y}_2 + f_{a\bar{b}} \mathbf{Y}^{a\bar{b}}\}, \\
 G_{(6)} &= \exp i\{u_a^b \mathbf{T}_b^{-a} + u_{\bar{a}}^{\bar{b}} \mathbf{T}_{\bar{b}}^{-\bar{a}} + u_a^{\bar{b}} \mathbf{T}_{\bar{b}}^{-a}\}.
 \end{aligned} \tag{3.2}$$

The factors $G_{(1)}$, $G_{(4)}$ are parametrized by $d = 3$ Galilei group parameters (see (1.1)), $G_{(5)}$ by the central charge parameters dual to the Galilean central charges, $G_{(6)}$ represents the abelian 5-dimensional coset $\text{IUSp}(4)/\text{USp}(4)$ and $G_{(2)}$, $G_{(3)}$ collect the parameters of the fermionic (odd) sector of $\text{SG}(3; 4|12)$.

The odd generators satisfy the symplectic-Majorana conditions (see also (2.12))

$$(\mathbf{Q}_\alpha^a)^\dagger = \epsilon^{\alpha\beta} \epsilon_{ab} \mathbf{Q}_\beta^b, \quad (\mathbf{Q}_{\tilde{\alpha}}^{\tilde{a}})^\dagger = \epsilon^{\alpha\beta} \epsilon_{\tilde{a}\tilde{b}} \mathbf{Q}_{\tilde{\beta}}^{\tilde{b}}, \quad (\mathbf{S}_\alpha^a)^\dagger = -\epsilon^{\alpha\beta} \epsilon_{ab} \mathbf{S}_\beta^b, \quad (\mathbf{S}_{\tilde{\alpha}}^{\tilde{a}})^\dagger = -\epsilon^{\alpha\beta} \epsilon_{\tilde{a}\tilde{b}} \mathbf{S}_{\tilde{\beta}}^{\tilde{b}}. \quad (3.3)$$

The Grassmann coordinates dual to these odd generators satisfy similar pseudo-reality conditions

$$(\xi_\alpha^\alpha)^* = \epsilon_{\alpha\beta} \epsilon^{ab} \xi_b^\beta, \quad (\xi_{\tilde{\alpha}}^{\tilde{\alpha}})^* = \epsilon_{\alpha\beta} \epsilon^{\tilde{a}\tilde{b}} \xi_{\tilde{b}}^{\tilde{\beta}}, \quad (\theta_a^\alpha)^* = -\epsilon_{\alpha\beta} \epsilon^{ab} \theta_b^\beta, \quad (\theta_{\tilde{a}}^{\tilde{\alpha}})^* = -\epsilon_{\alpha\beta} \epsilon^{\tilde{a}\tilde{b}} \theta_{\tilde{b}}^{\tilde{\beta}}. \quad (3.4)$$

Being dual to the relations (2.49), the reality conditions for the tensorial central charge coordinates read

$$(h_{a\tilde{b}})^* = \epsilon^{ac} \epsilon^{\tilde{b}\tilde{d}} h_{c\tilde{d}}, \quad (f_{a\tilde{b}})^* = \epsilon^{ac} \epsilon^{\tilde{b}\tilde{d}} f_{c\tilde{d}}. \quad (3.5)$$

The full set of the left-covariant MC one-forms is given by

$$G^{-1}dG = G_{(6)}^{-1}(\hat{G}^{-1}d\hat{G})G_{(6)} + G_{(6)}^{-1}dG_{(6)}, \quad (3.6)$$

where

$$\begin{aligned} \hat{G}^{-1}d\hat{G} &= G_{(4)}^{-1}G_{(3)}^{-1}G_{(2)}^{-1}\left(G_{(1)}^{-1}dG_{(1)}\right)G_{(2)}G_{(3)}G_{(4)} + G_{(4)}^{-1}G_{(3)}^{-1}\left(G_{(2)}^{-1}dG_{(2)}\right)G_{(3)}G_{(4)} \\ &\quad + G_{(4)}^{-1}\left(G_{(3)}^{-1}dG_{(3)}\right)G_{(4)} + G_{(4)}^{-1}dG_{(4)} + G_{(5)}^{-1}dG_{(5)} \\ &:= (I) + (II) + (III) + (IV) + (V). \end{aligned} \quad (3.7)$$

A straightforward calculation yields

$$\begin{aligned} G_{(1)}^{-1}dG_{(1)} &= i\left(dt\mathbf{H} + dx_i\mathbf{P}_i\right), \\ G_{(2)}^{-1}dG_{(2)} &= i\left(d\xi_\alpha^\alpha\mathbf{Q}_\alpha^a + d\xi_{\tilde{\alpha}}^{\tilde{\alpha}}\mathbf{Q}_{\tilde{\alpha}}^{\tilde{a}}\right) - \left(\xi_\alpha^\alpha d\xi_\alpha^a + \xi_{\tilde{\alpha}}^{\tilde{\alpha}} d\xi_{\tilde{\alpha}}^{\tilde{a}}\right)\mathbf{H}, \\ &\quad - \left(\xi_\alpha^\alpha d\xi_\alpha^a\right)\mathbf{X}_1 - \left(\xi_{\tilde{\alpha}}^{\tilde{\alpha}} d\xi_{\tilde{\alpha}}^{\tilde{a}}\right)\mathbf{X}_2 - \left(\xi_\alpha^\alpha d\xi_{\alpha\tilde{b}} - \xi_{\tilde{b}}^{\tilde{\alpha}} d\xi_{\alpha\alpha}\right)\mathbf{X}^{a\tilde{b}}, \\ (III) &= G_{(3)}^{-1}dG_{(3)} = i\left(d\theta_a^\alpha\mathbf{S}_\alpha^a + d\theta_{\tilde{a}}^{\tilde{\alpha}}\mathbf{S}_{\tilde{\alpha}}^{\tilde{a}}\right) + 2\left(\theta_a^\alpha d\theta_a^a + \theta_{\tilde{a}}^{\tilde{\alpha}} d\theta_{\tilde{\alpha}}^{\tilde{a}}\right)\mathbf{M}, \\ (IV) &= G_{(4)}^{-1}dG_{(4)} = i dk^i\mathbf{B}^i, \\ (V) &= G_{(5)}^{-1}dG_{(5)} = i\left(ds\mathbf{M} + dh_1\mathbf{X}_1 + dh_2\mathbf{X}_2 + dh_{a\tilde{b}}\mathbf{X}^{a\tilde{b}} + df_1\mathbf{Y}_1 + df_2\mathbf{Y}_2 + df_{a\tilde{b}}\mathbf{Y}^{a\tilde{b}}\right). \end{aligned} \quad (3.8)$$

The remaining part of (3.7) is as follows

$$(I) = i\left[dt\mathbf{H} + (dx_i + k_i dt)\mathbf{P}_i + \left(k_i dx_i + \frac{1}{2}k^2 dt\right)\mathbf{M}\right], \quad (3.9)$$

$$\begin{aligned}
 (II) = & G_{(2)}^{-1} dG_{(2)} - 2 \left[(\sigma_i)_{\alpha\beta} \left(\theta^{b\alpha} d\xi_b^\beta + \theta^{\tilde{b}\alpha} d\xi_{\tilde{b}}^\beta \right) + \frac{1}{2} k_i \left(\xi_a^\alpha d\xi_\alpha^a + \xi_{\tilde{a}}^\alpha d\xi_{\tilde{\alpha}}^{\tilde{a}} \right) \right] \mathbf{P}_i \\
 & + \frac{i}{2} k_i (\sigma_i)_\alpha{}^\beta \left(d\xi_a^\alpha \mathbf{S}_\beta^a + d\xi_{\tilde{a}}^\alpha \mathbf{S}_{\tilde{\beta}}^{\tilde{a}} \right) \\
 & - \left[2k_i (\sigma_i)_{\alpha\beta} \left(\theta^{b\alpha} d\xi_b^\beta + \theta^{\tilde{b}\alpha} d\xi_{\tilde{b}}^\beta \right) + \frac{1}{2} k^2 \left(\xi_a^\alpha d\xi_\alpha^a + \xi_{\tilde{a}}^\alpha d\xi_{\tilde{\alpha}}^{\tilde{a}} \right) \right] \mathbf{M} \\
 & + 2i \left(\theta^{\alpha b} d\xi_{\alpha b} \right) \mathbf{Y}_1 + 2i \left(\theta^{\alpha \tilde{b}} d\xi_{\alpha \tilde{b}} \right) \mathbf{Y}_2 - 2i \left(\theta_a^\alpha d\xi_{\alpha \tilde{b}} - \theta_{\tilde{b}}^\alpha d\xi_{\alpha a} \right) \mathbf{Y}^{a\tilde{b}}. \quad (3.10)
 \end{aligned}$$

We can write the formula (3.7) in the following way

$$\hat{G}^{-1} d\hat{G} := i \sum_K \hat{\omega}_{(K)} T_{(K)}, \quad (3.11)$$

where $T_{(K)}$ stand for all coset G generators, and $\hat{\omega}_{(K)}$ denote the corresponding MC one-forms. We obtain

$$\begin{aligned}
 \hat{\omega}_{(Q)a}^\alpha &= d\xi_a^\alpha, \\
 \hat{\omega}_{(Q)\tilde{a}}^\alpha &= d\xi_{\tilde{a}}^\alpha, \\
 \hat{\omega}_{(S)a}^\alpha &= d\theta_a^\alpha + \frac{1}{2} k_i (\sigma_i)_\beta{}^\alpha d\xi_a^\beta, \\
 \hat{\omega}_{(S)\tilde{a}}^\alpha &= d\theta_{\tilde{a}}^\alpha + \frac{1}{2} k_i (\sigma_i)_\beta{}^\alpha d\xi_{\tilde{a}}^\beta, \\
 \hat{\omega}_{(H)} &= dt + i \left(\xi_a^\alpha d\xi_\alpha^a + \xi_{\tilde{a}}^\alpha d\xi_{\tilde{\alpha}}^{\tilde{a}} \right), \\
 \hat{\omega}_{(B)i} &= dk_i, \\
 \hat{\omega}_{(P)i} &= \left[dx_i + 2i (\sigma_i)_{\alpha\beta} \left(\theta^{b\alpha} d\xi_b^\beta + \theta^{\tilde{b}\alpha} d\xi_{\tilde{b}}^\beta \right) \right] + k_i \hat{\omega}_{(H)}, \\
 \hat{\omega}_{(M)} &= ds + k_i \hat{\omega}_{(P)i} - \frac{1}{2} k^2 \hat{\omega}_{(H)} - 2i \left(\theta_a^\alpha d\theta_\alpha^a + \theta_{\tilde{a}}^\alpha d\theta_{\tilde{\alpha}}^{\tilde{a}} \right), \\
 \hat{\omega}_{(X)1} &= dh_1 + i \xi_a^\alpha d\xi_\alpha^a, \\
 \hat{\omega}_{(X)2} &= dh_2 + i \xi_{\tilde{a}}^\alpha d\xi_{\tilde{\alpha}}^{\tilde{a}}, \\
 \hat{\omega}_{(X)a\tilde{b}} &= dh_{a\tilde{b}} + i \left(\xi_a^\alpha d\xi_{\alpha \tilde{b}} - \xi_{\tilde{b}}^\alpha d\xi_{\alpha a} \right), \\
 \hat{\omega}_{(Y)1} &= df_1 + 2\theta^{\alpha a} d\xi_{\alpha a}, \\
 \hat{\omega}_{(Y)2} &= df_2 + 2\theta^{\alpha \tilde{a}} d\xi_{\alpha \tilde{a}}, \\
 \hat{\omega}_{(Y)a\tilde{b}} &= df_{a\tilde{b}} - 2 \left(\theta_a^\alpha d\xi_{\alpha \tilde{b}} - \theta_{\tilde{b}}^\alpha d\xi_{\alpha a} \right),
 \end{aligned} \quad (3.12)$$

where $k^2 := k_i k_i$. The MC one-forms describing the whole coset G are defined as follows

$$G^{-1} dG := i \sum_K \omega_{(K)} T_{(K)} \quad (3.13)$$

and can be calculated by the formulas (3.2) and (3.6). We observe that

$$G_{(6)}^{-1} dG_{(6)} = i \left(du_a^b \mathbf{T}_b^{-a} + du_{\tilde{a}}^{\tilde{b}} \mathbf{T}_{\tilde{b}}^{-\tilde{a}} + du_a^{\tilde{b}} \mathbf{T}_{\tilde{b}}^{-a} \right) \quad (3.14)$$

and, further,

$$\begin{aligned}
 G_{(6)}^{-1} \mathbf{Q}_\alpha^a G_{(6)} &= \mathbf{Q}_\alpha^a + \frac{i}{2} (u_b^{\tilde{b}} - u_d^{\tilde{d}}) \mathbf{S}_\alpha^a - i u^{a\tilde{b}} \mathbf{S}_{\alpha\tilde{b}}, \\
 G_{(6)}^{-1} \mathbf{Q}_\alpha^{\tilde{a}} G_{(6)} &= \mathbf{Q}_\alpha^{\tilde{a}} - \frac{i}{2} (u_b^{\tilde{b}} - u_d^{\tilde{d}}) \mathbf{S}_\alpha^{\tilde{a}} - i u_b^{\tilde{a}} \mathbf{S}_\alpha^b, \\
 G_{(6)}^{-1} \mathbf{Y}_1 G_{(6)} &= \mathbf{Y}_1 + (u_a^a - u_{\tilde{a}}^{\tilde{a}}) \mathbf{M}, \\
 G_{(6)}^{-1} \mathbf{Y}_2 G_{(6)} &= \mathbf{Y}_2 - (u_a^a - u_{\tilde{a}}^{\tilde{a}}) \mathbf{M}, \\
 G_{(6)}^{-1} \mathbf{Y}^{a\tilde{b}} G_{(6)} &= \mathbf{Y}^{a\tilde{b}} + 2 u^{a\tilde{b}} \mathbf{M}, \\
 G_{(6)}^{-1} \mathbf{X}_1 G_{(6)} &= \mathbf{X}_1 + (u_a^a - u_{\tilde{a}}^{\tilde{a}}) \mathbf{Y}_1 + u_{a\tilde{c}} \mathbf{Y}^{a\tilde{c}} + \frac{1}{2} \left[(u_a^a - u_{\tilde{a}}^{\tilde{a}})^2 + 2 u_{a\tilde{c}} u^{a\tilde{c}} \right] \mathbf{M}, \\
 G_{(6)}^{-1} \mathbf{X}_2 G_{(6)} &= \mathbf{X}_2 - (u_a^a - u_{\tilde{a}}^{\tilde{a}}) \mathbf{Y}_2 + u_{a\tilde{c}} \mathbf{Y}^{a\tilde{c}} + \frac{1}{2} \left[(u_a^a - u_{\tilde{a}}^{\tilde{a}})^2 + 2 u_{a\tilde{c}} u^{a\tilde{c}} \right] \mathbf{M}, \\
 G_{(6)}^{-1} \mathbf{X}^{a\tilde{b}} G_{(6)} &= \mathbf{X}^{a\tilde{b}} + 2 u_d^a \mathbf{Y}^{d\tilde{b}} + 2 u_{\tilde{d}}^{\tilde{b}} \mathbf{Y}^{a\tilde{d}} - 2 (u_c^c + u_{\tilde{c}}^{\tilde{c}}) \mathbf{Y}^{a\tilde{b}} + u^{a\tilde{b}} (\mathbf{Y}_1 + \mathbf{Y}_2) \\
 &\quad - 2 \left[(u_c^c + u_{\tilde{c}}^{\tilde{c}}) u^{a\tilde{b}} - u^{a\tilde{d}} u_{\tilde{d}}^{\tilde{b}} - u_d^a u^{\tilde{d}\tilde{b}} \right] \mathbf{M}.
 \end{aligned} \tag{3.15}$$

We see, in particular, that

$$\omega_{(H)} = \hat{\omega}_{(H)}, \quad \omega_{(P)} i = \hat{\omega}_{(P)} i. \tag{3.16}$$

Let us find supersymmetry transformations of the coset coordinates. For this purpose we will use the well known formula $iG^{-1}(\varepsilon \cdot T)G = G^{-1}\delta G + \delta h$, where T denotes the collection of coset generators and δh defines induced transformations of the stability subgroup $h_{ind} = 1 + \delta h$ (see, e.g., [51]).

Supersymmetry transformations generated by the left action of generators $\mathbf{Q}_\alpha^a, \mathbf{Q}_\alpha^{\tilde{a}}$

$$\exp i \{ \varepsilon_a^\alpha \mathbf{Q}_\alpha^a + \varepsilon_{\tilde{a}}^\alpha \mathbf{Q}_\alpha^{\tilde{a}} \} G = G' h, \tag{3.17}$$

where $\varepsilon_a^\alpha, \varepsilon_{\tilde{a}}^\alpha$ are odd constant parameters, lead to the following transformations of the coset coordinates:

$$\begin{aligned}
 \delta_\varepsilon \xi_a^\alpha &= \varepsilon_a^\alpha, & \delta_\varepsilon \xi_{\tilde{a}}^\alpha &= \varepsilon_{\tilde{a}}^\alpha, & \delta_\varepsilon t &= -i (\varepsilon_a^\alpha \xi_\alpha^a + \varepsilon_{\tilde{a}}^\alpha \xi_\alpha^{\tilde{a}}), \\
 \delta_\varepsilon h_1 &= -i \varepsilon_a^\alpha \xi_\alpha^a, & \delta_\varepsilon h_2 &= -i \varepsilon_{\tilde{a}}^\alpha \xi_\alpha^{\tilde{a}}, & \delta_\varepsilon h_{a\tilde{b}} &= -i (\varepsilon_a^\alpha \xi_{\alpha\tilde{b}} - \varepsilon_{\tilde{b}}^\alpha \xi_{\alpha a}), \\
 \delta_\varepsilon x_i &= 0, & \delta_\varepsilon k_i &= 0, \\
 \delta_\varepsilon \theta_a^\alpha &= 0, & \delta_\varepsilon \theta_{\tilde{a}}^\alpha &= 0, & \delta_\varepsilon s &= 0, \\
 \delta_\varepsilon f_1 &= 0, & \delta_\varepsilon f_2 &= 0, & \delta_\varepsilon f_{a\tilde{b}} &= 0, \\
 \delta_\varepsilon u_a^b &= 0, & \delta_\varepsilon u_{\tilde{a}}^{\tilde{b}} &= 0, & \delta_\varepsilon u_a^{\tilde{b}} &= 0.
 \end{aligned} \tag{3.18}$$

The second half of the odd left shifts, those generated by $\mathbf{S}_\alpha^a, \mathbf{S}_\alpha^{\tilde{a}}$,

$$\exp i \{ \eta_a^\alpha \mathbf{S}_\alpha^a + \eta_{\tilde{a}}^\alpha \mathbf{S}_\alpha^{\tilde{a}} \} G = G' h, \tag{3.19}$$

lead to the transformations

$$\begin{aligned}
 \delta_\eta \xi_a^\alpha &= 0, & \delta_\eta \xi_{\tilde{a}}^\alpha &= 0, & \delta_\eta t &= 0, \\
 \delta_\eta h_1 &= 0, & \delta_\eta h_2 &= 0, & \delta_\eta h_{a\tilde{b}} &= 0, \\
 \delta_\eta x_i &= 2i \left(\eta_a^\alpha \xi^{\beta a} + \eta_{\tilde{a}}^\alpha \xi^{\beta \tilde{a}} \right) (\sigma_i)_{\alpha\beta}, & \delta_\eta k_i &= 0, \\
 \delta_\eta \theta_a^\alpha &= \eta_a^\alpha, & \delta_\eta \theta_{\tilde{a}}^\alpha &= \eta_{\tilde{a}}^\alpha, & \delta_\eta s &= 2i \left(\eta_a^\alpha \theta_\alpha^a + \eta_{\tilde{a}}^\alpha \theta_{\tilde{\alpha}}^{\tilde{a}} \right), \\
 \delta_\eta f_1 &= 2\eta_a^\alpha \xi_\alpha^a, & \delta_\eta f_2 &= 2\eta_{\tilde{a}}^\alpha \xi_{\tilde{\alpha}}^{\tilde{a}}, & \delta_\eta f_{a\tilde{b}} &= 2 \left(\eta_a^\alpha \xi_{\alpha\tilde{b}} - \eta_{\tilde{b}}^\alpha \xi_{\alpha a} \right), \\
 \delta_\eta u_a^b &= 0, & \delta_\eta u_{\tilde{a}}^{\tilde{b}} &= 0, & \delta_\eta u_a^{\tilde{b}} &= 0,
 \end{aligned} \tag{3.20}$$

where $\eta_a^\alpha, \eta_{\tilde{a}}^\alpha$ are the appropriate odd parameters. It follows from (3.18) and (3.20) that the three-vector k_i and “harmonic variables” u are inert under all supersymmetry transformations.

From the form of the supersymmetry transformations (3.18) follows that the set of coordinates $(t; \xi_a^\alpha, \xi_{\tilde{a}}^\alpha; h_1, h_2, h_{a\tilde{b}})$ is closed under the action of $\mathbf{Q}$ - supersymmetry, while this supersymmetry does not act on the remaining coordinates $(x_i; \theta_a^\alpha, \theta_{\tilde{a}}^\alpha; s, f_1, f_2, f_{a\tilde{b}})$. Alternatively, $\mathbf{S}$ - supersymmetry (3.20) leaves inert the subset $(t; \xi_a^\alpha, \xi_{\tilde{a}}^\alpha; h_1, h_2, h_{a\tilde{b}})$ and transforms the remaining coordinates $(x_i; \theta_a^\alpha, \theta_{\tilde{a}}^\alpha; s, f_1, f_2, f_{a\tilde{b}})$ (however with dependence of $\delta_\eta x_i, \delta_\eta f_1, \delta_\eta f_2, \delta_\eta f_{a\tilde{b}}$ on ξ -variables). This split of the full set of coset parameters into two subsets, each closed under the action of one half of the supersymmetries and inert under another half, is due to the choice of coset parametrization (3.1), (3.2) with the particular order of the factors G_2 and G_3 , $G = \dots G_2 G_3 \dots$. In [38], there was used another parametrization, $G = \dots G_3 G_2 \dots$, and the separation of $\mathbf{Q}$ - and $\mathbf{S}$ - transformations into two sectors could not be seen.

The closure of the transformations (3.18) and (3.20) generates all the bosonic transformations of G which do not belong to the stability subgroup H . The transformations of subgroup H are realized as some linear homogeneous maps of the coset fields and MC 1-forms. The abelian generators $\mathbf{T}^{-A}_B$ do not appear in the closure of fermionic generators, so the left shifts by these generators should be considered separately. The corresponding transformations of the coset parameters can be found explicitly, using the formulas (3.15). The coset parameters u_B^A are changed only by the pure shifts. Actually, in this paper we will not make use of these $\mathbf{T}^{-A}_B$ transformations.

We also observe that all MC forms $\hat{\omega}_{(K)}$ (see (3.12)) transform linearly under H transformations and are inert with respect to the odd transformations (3.18) and (3.20).

4 The phase-space formulation of $\mathcal{N} = 4, d = 3$ Galilean superparticle model and κ -gauge freedom

Let us describe the mechanical system on the coset G with evolution parameter τ and with all parameters of G promoted to the $d = 1$ fields: $t = t(\tau)$, $x_i = x_i(\tau)$, $k_i = k_i(\tau)$, $\xi_a^\alpha = \xi_a^\alpha(\tau)$, $\xi_{\tilde{a}}^\alpha = \xi_{\tilde{a}}^\alpha(\tau)$, $\theta_a^\alpha = \theta_a^\alpha(\tau)$, $\theta_{\tilde{a}}^\alpha = \theta_{\tilde{a}}^\alpha(\tau)$, *etc.* We shall deal with the simplified situation, with all internal coordinates u_B^A being suppressed, which means that we transfer the generators $\mathbf{T}^{-A}_B$ into the stability subgroup and use the “truncated” MC one-forms

$\hat{\omega}_{(K)}$. We will not employ the strict invariance of the superparticle actions under these abelian outer automorphisms, as well under the full compact R-symmetry $\text{USp}(4)$. Only the symmetries under some particular subgroups of the latter, as well as the $\text{O}(3)$ space symmetry generated by $\mathbf{J}_{ij}$, will be respected.

We can covariantly eliminate the fields $k_i(t)$ by imposing the following algebraic inverse Higgs constraints [52]

$$\omega_{(P) i} = \hat{\omega}_{(P) i} = 0. \quad (4.1)$$

Taking into account that

$$\omega_{(P) i} = d\tau (\pi_i + k_i \pi_0), \quad (4.2)$$

where

$$\pi_i := \dot{x}_i + 2i(\sigma_i)_{\alpha\beta} \left(\theta^{b\alpha} \dot{\xi}_b^\beta + \theta^{\tilde{b}\alpha} \dot{\xi}_{\tilde{b}}^\beta \right), \quad \pi_0 := \dot{t} + i \left(\xi_a^\alpha \dot{\xi}_\alpha^a + \xi_{\tilde{a}}^\alpha \dot{\xi}_\alpha^{\tilde{a}} \right), \quad (4.3)$$

the constraints (4.1) are solved by

$$k_i = -\frac{\pi_i}{\pi_0}, \quad (4.4)$$

where $\dot{x}_i = dx_i/d\tau$, $\dot{t} = dt/d\tau$, etc.

Using (4.1), one obtains that

$$\hat{\omega}_{(M)} = d\tau \left[\dot{s} - \frac{\pi_i \pi_i}{2\pi_0} - 2i \left(\theta_a^\alpha \dot{\theta}_\alpha^a + \theta_{\tilde{a}}^\alpha \dot{\theta}_\alpha^{\tilde{a}} \right) \right]. \quad (4.5)$$

4.1 Simplest bosonic case: Schrödinger NR particle

As the instructive step we consider the standard bosonic Schrödinger particle. We recall how to derive the action of non-relativistic massive particle, which, after quantization, leads to the non-relativistic Schrödinger equation.

Such an action is obtained from the MC one-form (4.5), which in the pure bosonic case is given by

$$\hat{\omega}_M = d\tau \left(\dot{s} - \frac{\dot{x}_i \dot{x}_i}{2\dot{t}} \right). \quad (4.6)$$

Selecting the rest mass as the normalization factor and omitting a total τ derivative, we obtain

$$S_0 = -m_0 \int \hat{\omega}_{(M)} = \int d\tau L_0 = m_0 \int d\tau \frac{\dot{x}_i \dot{x}_i}{2\dot{t}}. \quad (4.7)$$

It leads to NR particle model studied in [28, 38].

The action (4.7) provides the canonical momenta

$$p_t = -m_0 \frac{\dot{x}_i \dot{x}_i}{2\dot{t}^2}, \quad p_{xi} = m_0 \frac{\dot{x}_i}{\dot{t}} \quad (4.8)$$

and the vanishing canonical Hamiltonian:

$$H_0 = p_t \dot{t} + p_{xi} \dot{x}_i - L_0 = 0. \quad (4.9)$$

The expressions (4.8) imply the first-class constraint defining free NR energy-momentum dispersion relation called free Schrödinger constraint

$$p_t + \frac{p_{xi} p_{xi}}{2m_0} \approx 0, \quad (4.10)$$

which, after quantization in the Schrödinger realization,

$$p_t = -i\hbar \frac{\partial}{\partial t}, \quad p_{xi} = -i\hbar \frac{\partial}{\partial x_i}, \quad (4.11)$$

gives the non-relativistic Schrödinger equation for a free NR particle of mass m_0 .

4.2 The superparticle model with vanishing off-diagonal central charges

As the next step, we consider the action with the Lagrangian density taken as a linear combination of the MC one-forms associated with central charges described by Jordanian quasi-diagonal form of the central charge matrix

$$S_1 = -m_0 \int \hat{\omega}_{(M)} + \int \left(a \hat{\omega}_{(H)} + m_1 \hat{\omega}_{(X)_1} + m_2 \hat{\omega}_{(X)_2} + \mu_1 \hat{\omega}_{(Y)_1} + \mu_2 \hat{\omega}_{(Y)_2} \right), \quad (4.12)$$

where $a, m_1, m_2, \mu_1, \mu_2$ are real constants. The choice of these parameters specifies the explicit form of odd constraints, including the first class ones generating local κ -symmetries.

Using the expressions of the MC forms (3.12), (4.5) and omitting total derivative terms, we get from (4.12) the Lagrangian

$$\begin{aligned} L_1 = m_0 \frac{\pi_i \pi_i}{2\pi_0} + 2i m_0 (\theta_a^\alpha \dot{\theta}_\alpha^a + \theta_{\tilde{a}}^\alpha \dot{\theta}_\alpha^{\tilde{a}}) + i a (\xi_a^\alpha \dot{\xi}_\alpha^a + \xi_{\tilde{a}}^\alpha \dot{\xi}_\alpha^{\tilde{a}}) \\ + i m_1 \xi_a^\alpha \dot{\xi}_\alpha^a + i m_2 \xi_{\tilde{a}}^\alpha \dot{\xi}_\alpha^{\tilde{a}} + 2\mu_1 \theta^{\alpha a} \dot{\xi}_{a\alpha} + 2\mu_2 \theta^{\alpha \tilde{a}} \dot{\xi}_{\tilde{a}\alpha}, \end{aligned} \quad (4.13)$$

where $\pi_i = \dot{x}_i + 2i(\sigma_i)_{\alpha\beta} (\theta^{b\alpha} \dot{\xi}_b^\beta + \theta^{\tilde{b}\alpha} \dot{\xi}_{\tilde{b}}^\beta)$ and $\pi_0 = \dot{t} + i (\xi_a^\alpha \dot{\xi}_\alpha^a + \xi_{\tilde{a}}^\alpha \dot{\xi}_\alpha^{\tilde{a}})$ were defined in (4.3).

Without the loss of generality, the terms proportional to a in (4.13) can be omitted because they can be re-absorbed by the redefinition of m_1 and m_2 . Therefore we will put $a = 0$ (see the same condition in [38], assumed, however, for another reason). Then the Lagrangian (4.13) produces the following bosonic momenta

$$p_t = -m_0 \frac{\pi_i \pi_i}{2(\pi_0)^2}, \quad p_{xi} = m_0 \frac{\pi_i}{\pi_0}, \quad (4.14)$$

and the fermionic ones

$$\begin{aligned} p_{\xi_\alpha}^a &= 2m_0 i \frac{\pi_i}{\pi_0} (\sigma_i)_{\alpha\beta} \theta^{\beta b} + i \left(m_1 - m_0 \frac{\pi_i \pi_i}{2(\pi_0)^2} \right) \xi_\alpha^a - 2\mu_1 \theta_\alpha^a, \\ p_{\xi_\alpha}^{\tilde{a}} &= 2m_0 i \frac{\pi_i}{\pi_0} (\sigma_i)_{\alpha\beta} \theta^{\beta \tilde{b}} + i \left(m_2 - m_0 \frac{\pi_i \pi_i}{2(\pi_0)^2} \right) \xi_\alpha^{\tilde{a}} - 2\mu_2 \theta_\alpha^{\tilde{a}}, \\ p_{\theta_\alpha}^a &= 2m_0 i \theta_\alpha^a, \\ p_{\theta_\alpha}^{\tilde{a}} &= 2m_0 i \theta_\alpha^{\tilde{a}}. \end{aligned} \quad (4.15)$$

In accordance with (3.3) and (3.4), the odd momenta satisfy the symplectic-Majorana conditions

$$\begin{aligned} (p_{\xi_\alpha}^a)^* &= -\epsilon^{\alpha\beta} \epsilon_{ab} p_{\xi_\beta}^b, & (p_{\xi_\alpha}^{\tilde{a}})^* &= -\epsilon^{\alpha\beta} \epsilon_{\tilde{a}\tilde{b}} p_{\xi_\beta}^{\tilde{b}}, \\ (p_{\theta_\alpha}^a)^* &= \epsilon^{\alpha\beta} \epsilon_{ab} p_{\theta_\beta}^b, & (p_{\theta_\alpha}^{\tilde{a}})^* &= \epsilon^{\alpha\beta} \epsilon_{\tilde{a}\tilde{b}} p_{\theta_\beta}^{\tilde{b}}. \end{aligned} \quad (4.16)$$

The model with the Lagrangian (4.13) is invariant under the NR supersymmetry transformations (3.18) and (3.20). Note that all terms in (4.13), except the first one, are of the Wess-Zumino type. Noether charges which generate the supersymmetry transformations (3.18) and (3.20) are as follows

$$\begin{aligned} \mathbf{Q}_\alpha^a &= p_{\xi_\alpha^a} + i(p_t + m_1)\xi_\alpha^a, \\ \mathbf{Q}_\alpha^{\tilde{a}} &= p_{\xi_\alpha^{\tilde{a}}} + i(p_t + m_2)\xi_\alpha^{\tilde{a}}, \\ \mathbf{S}_\alpha^a &= p_{\theta_\alpha^a} + 2m_0 i \theta_\alpha^a - 2i p_{x\alpha\beta} \xi^{\beta a} - 2\mu_1 \xi_\alpha^a, \\ \mathbf{S}_\alpha^{\tilde{a}} &= p_{\theta_\alpha^{\tilde{a}}} + 2m_0 i \theta_\alpha^{\tilde{a}} - 2i p_{x\alpha\beta} \xi^{\beta \tilde{a}} - 2\mu_2 \xi_\alpha^{\tilde{a}}, \end{aligned} \quad (4.17)$$

where $p_{x\alpha\beta} := p_{xi}(\sigma_i)_{\alpha\beta}$. Using the canonical Poisson brackets,

$$\begin{aligned} \{t, p_t\} &= 1, & \{x_i, p_{xj}\} &= \delta_{ij}, \\ \{\xi_\alpha^a, p_{\xi_\beta^b}\} &= \delta_\beta^\alpha \delta_a^b, & \{\xi_\alpha^{\tilde{a}}, p_{\xi_\beta^{\tilde{b}}}\} &= \delta_\beta^\alpha \delta_{\tilde{a}}^{\tilde{b}}, & \{\theta_\alpha^a, p_{\theta_\beta^b}\} &= \delta_\beta^\alpha \delta_a^b, & \{\theta_\alpha^{\tilde{a}}, p_{\theta_\beta^{\tilde{b}}}\} &= \delta_\beta^\alpha \delta_{\tilde{a}}^{\tilde{b}}, \end{aligned} \quad (4.18)$$

we find the non-vanishing Poisson brackets of the classical NR supersymmetry generators (4.17):

$$\begin{aligned} \{\mathbf{Q}_\alpha^a, \mathbf{Q}_\beta^b\} &= 2i \epsilon^{ab} \epsilon_{\alpha\beta} (p_t + m_1), & \{\mathbf{Q}_\alpha^{\tilde{a}}, \mathbf{Q}_\beta^{\tilde{b}}\} &= 2i \epsilon^{\tilde{a}\tilde{b}} \epsilon_{\alpha\beta} (p_t + m_2), \\ \{\mathbf{S}_\alpha^a, \mathbf{S}_\beta^b\} &= 4i \epsilon^{ab} \epsilon_{\alpha\beta} m_0, & \{\mathbf{S}_\alpha^{\tilde{a}}, \mathbf{S}_\beta^{\tilde{b}}\} &= 4i \epsilon^{\tilde{a}\tilde{b}} \epsilon_{\alpha\beta} m_0, \\ \{\mathbf{Q}_\alpha^a, \mathbf{S}_\beta^b\} &= 2i \epsilon^{ab} (p_{x\alpha\beta} + i\mu_1 \epsilon_{\alpha\beta}), & \{\mathbf{Q}_\alpha^{\tilde{a}}, \mathbf{S}_\beta^{\tilde{b}}\} &= 2i \epsilon^{\tilde{a}\tilde{b}} (p_{x\alpha\beta} + i\mu_2 \epsilon_{\alpha\beta}). \end{aligned} \quad (4.19)$$

The Poisson brackets (4.19) are the classical counterparts of the anticommutators (2.44)–(2.46). We see that in the model (4.13) the parameters m_1 , m_2 and μ_1 , μ_2 generate the constant central charges $\mathbf{X}_1$, $\mathbf{X}_2$ and $\mathbf{Y}_1$, $\mathbf{Y}_2$.¹⁷

Canonical Hamiltonian of the model (4.13) is vanishing as in the bosonic case

$$H_1 = p_t \dot{t} + p_{xi} \dot{x}_i + p_{\xi_\alpha^a} \dot{\xi}_\alpha^a + p_{\xi_\alpha^{\tilde{a}}} \dot{\xi}_\alpha^{\tilde{a}} + p_{\theta_\alpha^a} \dot{\theta}_\alpha^a + p_{\theta_\alpha^{\tilde{a}}} \dot{\theta}_\alpha^{\tilde{a}} - L_1 = 0, \quad (4.20)$$

which indicates the reparametrization invariance of the model.

The expressions for the bosonic momenta (4.14) lead again to the free Schrödinger constraint (4.10):

$$T := p_t + \frac{p_{xi} p_{xi}}{2m_0} \approx 0. \quad (4.21)$$

The definitions (4.15) of fermionic momenta lead to the constraints:

$$\begin{aligned} D_{\xi_\alpha^a} &:= p_{\xi_\alpha^a} - i(p_t + m_1)\xi_\alpha^a - 2i p_{x\alpha\beta} \theta^{\beta a} + 2\mu_1 \theta_\alpha^a \approx 0, \\ D_{\xi_\alpha^{\tilde{a}}} &:= p_{\xi_\alpha^{\tilde{a}}} - i(p_t + m_2)\xi_\alpha^{\tilde{a}} - 2i p_{x\alpha\beta} \theta^{\beta \tilde{a}} + 2\mu_2 \theta_\alpha^{\tilde{a}} \approx 0, \\ D_{\theta_\alpha^a} &:= p_{\theta_\alpha^a} - 2m_0 i \theta_\alpha^a \approx 0, \\ D_{\theta_\alpha^{\tilde{a}}} &:= p_{\theta_\alpha^{\tilde{a}}} - 2m_0 i \theta_\alpha^{\tilde{a}} \approx 0. \end{aligned} \quad (4.22)$$

¹⁷Note that the value of central charge $\mathbf{M}$ in the model (4.13) is equal to $-m_0$ as we see from the comparison of (4.19) and (2.44)–(2.46). The appearance of the sign minus here is the consequence of the choice of the sign minus in front of the one-form $\omega_{(M)}$ in action (4.12), what leads to the Lagrangian (4.13) with standard Lagrangian in the bosonic limit.

Using (4.18), we obtain the non-vanishing Poisson brackets for the system of constraints (4.21) and (4.22)

$$\begin{aligned}
 \{D_{\xi\alpha}^a, D_{\xi\beta}^b\} &= -2i(p_t + m_1)\epsilon_{\alpha\beta}\epsilon^{ab}, & \{D_{\xi\alpha}^{\tilde{a}}, D_{\xi\beta}^{\tilde{b}}\} &= -2i(p_t + m_2)\epsilon_{\alpha\beta}\epsilon^{\tilde{a}\tilde{b}}, \\
 \{D_{\theta\alpha}^a, D_{\theta\beta}^b\} &= -4m_0i\epsilon_{\alpha\beta}\epsilon^{ab}, & \{D_{\theta\alpha}^{\tilde{a}}, D_{\theta\beta}^{\tilde{b}}\} &= -4m_0i\epsilon_{\alpha\beta}\epsilon^{\tilde{a}\tilde{b}}, \\
 \{D_{\xi\alpha}^a, D_{\theta\beta}^b\} &= -2i(p_{x\alpha\beta} + \mu_1i\epsilon_{\alpha\beta})\epsilon^{ab}, \\
 \{D_{\xi\alpha}^{\tilde{a}}, D_{\theta\beta}^{\tilde{b}}\} &= -2i(p_{x\alpha\beta} + \mu_2i\epsilon_{\alpha\beta})\epsilon^{\tilde{a}\tilde{b}}.
 \end{aligned} \tag{4.23}$$

We will be interested in the NR superparticle models possessing local fermionic κ symmetry [53, 54], after imposing suitable relations between the parameters of the model (see, e.g., (4.26), (4.27) below). In the $\mathcal{N} = 2, d = 2$ case this kind of NR superparticles was considered in [38]. We recall that in the phase space formulation, κ symmetry is generated by the first class odd constraints.

Let us determine the values of central charges in the model which imply the first class odd constraints. For that purpose we should calculate the determinant of the 16-dimensional matrix of the Poisson brackets of fermionic constraints (4.23), in the presence of the bosonic constraint (4.21), and assume that this determinant becomes zero.

Defining $D_{\mathcal{A}} := (D_{\xi\alpha}^a, D_{\xi\alpha}^{\tilde{a}}, D_{\theta\alpha}^a, D_{\theta\alpha}^{\tilde{a}})$ and

$$\{D_{\mathcal{A}}, D^{\mathcal{B}}\} = 2i\mathcal{P}_{\mathcal{A}}^{\mathcal{B}}, \tag{4.24}$$

we find that the determinant of the matrix $\mathcal{P}$ of the fermionic constraints (4.23) is given, modulo a multiplicative constant, by the expression

$$(m_0)^8 \left[p_t + \frac{p_{xi}p_{xi}}{2m_0} + m_1 + \frac{(\mu_1)^2}{2m_0} \right]^4 \left[p_t + \frac{p_{xi}p_{xi}}{2m_0} + m_2 + \frac{(\mu_2)^2}{2m_0} \right]^4. \tag{4.25}$$

Thus, first class odd constraints are present at least under one of the following two conditions

$$m_1 = -\frac{(\mu_1)^2}{2m_0}, \tag{4.26}$$

$$m_2 = -\frac{(\mu_2)^2}{2m_0}. \tag{4.27}$$

If the condition (4.26) is valid, half of the odd constraints linear in $D_{\xi\alpha}^a, D_{\theta\alpha}^a$ are first class. Explicitly, these constraints are

$$F_{\xi\alpha}^a := D_{\xi\alpha}^a + \frac{1}{2m_0}(p_{x\alpha}{}^{\beta} - i\mu_1\delta_{\alpha}^{\beta})D_{\theta\beta}^a \approx 0 \tag{4.28}$$

The Poisson brackets involving the constraints (4.28) form the following set

$$\begin{aligned}
 \{F_{\xi\alpha}^a, D_{\xi\beta}^b\} &= -2i\epsilon_{\alpha\beta}\epsilon^{ab}T - 2i\epsilon_{\alpha\beta}\epsilon^{ab}\left(m_1 + \frac{(\mu_1)^2}{2m_0}\right), & \{F_{\xi\alpha}^a, D_{\theta\beta}^b\} &= 0, \\
 \{F_{\xi\alpha}^a, F_{\xi\beta}^b\} &= -2i\epsilon_{\alpha\beta}\epsilon^{ab}T - 2i\epsilon_{\alpha\beta}\epsilon^{ab}\left(m_1 + \frac{(\mu_1)^2}{2m_0}\right).
 \end{aligned} \tag{4.29}$$

We see that the full set of the original constraints $(D_{\xi\alpha}^a, D_{\theta\alpha}^a)$ is equivalent to the set $(F_{\xi\alpha}^a, D_{\theta\alpha}^a)$, where $F_{\xi\alpha}^a$ are first class, and $D_{\theta\alpha}^a$ are second class.

The analysis of the second half of the odd constraints (with tilded indices) is performed quite analogously. If the condition (4.27) holds, the constraints

$$F_{\xi\alpha}^{\tilde{a}} := D_{\xi\alpha}^{\tilde{a}} + \frac{1}{2m_0} \left(p_x \alpha^\beta - i\mu_2 \delta_\alpha^\beta \right) D_{\theta\beta}^{\tilde{a}} \approx 0. \quad (4.30)$$

are first class, with the following Poisson brackets with all odd constraints

$$\begin{aligned} \{F_{\xi\alpha}^{\tilde{a}}, D_{\xi\beta}^{\tilde{b}}\} &= -2i\epsilon_{\alpha\beta}\epsilon^{\tilde{a}\tilde{b}} T - 2i\epsilon_{\alpha\beta}\epsilon^{\tilde{a}\tilde{b}} \left(m_2 + \frac{(\mu_2)^2}{2m_0} \right), & \{F_{\xi\alpha}^{\tilde{a}}, D_{\theta\beta}^{\tilde{b}}\} &= 0, \\ \{F_{\xi\alpha}^{\tilde{a}}, F_{\xi\beta}^{\tilde{b}}\} &= -2i\epsilon_{\alpha\beta}\epsilon^{\tilde{a}\tilde{b}} T - 2i\epsilon_{\alpha\beta}\epsilon^{\tilde{a}\tilde{b}} \left(m_2 + \frac{(\mu_2)^2}{2m_0} \right). \end{aligned} \quad (4.31)$$

The set of constraints $(D_{\xi\alpha}^{\tilde{a}}, D_{\theta\alpha}^{\tilde{a}})$ is therefore equivalent to the set $(F_{\xi\alpha}^{\tilde{a}}, D_{\theta\alpha}^{\tilde{a}})$, with $F_{\xi\alpha}^{\tilde{a}}$ being first class and $D_{\theta\alpha}^{\tilde{a}}$ second class.

In section 5 we will study the quantization of the superparticle model defined by the action (4.12), (4.13) possessing κ -symmetries due to the presence of conditions (4.26) and (4.27). Using (4.22), we obtain the following explicit form of the first class constraints (4.28) and (4.30) generating κ -symmetries

$$\begin{aligned} F_{\xi\alpha}^a &= p_{\xi\alpha}^a - i(p_t + m_1) \xi_\alpha^a + \frac{1}{2m_0} \left(p_x \alpha^\beta - i\mu_1 \delta_\alpha^\beta \right) \left(p_{\theta\beta}^a + 2m_0 i \theta_\beta^a \right) \approx 0, \\ F_{\xi\alpha}^{\tilde{a}} &= p_{\xi\alpha}^{\tilde{a}} - i(p_t + m_2) \xi_\alpha^{\tilde{a}} + \frac{1}{2m_0} \left(p_x \alpha^\beta - i\mu_2 \delta_\alpha^\beta \right) \left(p_{\theta\beta}^{\tilde{a}} + 2m_0 i \theta_\beta^{\tilde{a}} \right) \approx 0. \end{aligned} \quad (4.32)$$

We notice that, if we specialize our discussion of constraints to one sector only, with either index a or index $\tilde{a}$, we recover the model with smaller $\mathcal{N} = 2$ Galilean supersymmetry. Such models in $d = 2$ case have been studied in [38] for the case of $\mathcal{N} = 2$, $d = 2$ Galilean supersymmetry with only one central charge. The $d = 2$ models of [38] can be obtained as a special case of our model with only one of the $\mathcal{N} = 2$ sectors retained.

The constraints (4.32) generate local κ -transformations of an arbitrary phase space function X by the following Poisson bracket

$$\delta X = \{ \kappa_a^\alpha F_{\xi\alpha}^a + \kappa_{\tilde{a}}^\alpha F_{\xi\alpha}^{\tilde{a}}, X \}, \quad (4.33)$$

where $\kappa_a^\alpha(\tau)$ and $\kappa_{\tilde{a}}^\alpha(\tau)$ are local Grassmann parameters. The κ -transformations (4.33) of the variables in the Lagrangian (4.13) with $a = 0$ are as follows

$$\begin{aligned} \delta t &= i \kappa_a^\alpha \xi_\alpha^a + i \kappa_{\tilde{a}}^\alpha \xi_\alpha^{\tilde{a}}, & \delta x_i &= -2i \kappa_a^\alpha (\sigma_i)_\alpha{}^\beta \xi_\beta^a - 2i \kappa_{\tilde{a}}^\alpha (\sigma_i)_\alpha{}^\beta \xi_\beta^{\tilde{a}}, \\ \delta \xi_\alpha^a &= \kappa_a^\alpha, & \delta \xi_\alpha^{\tilde{a}} &= \kappa_{\tilde{a}}^\alpha, \\ \delta \theta_\alpha^a &= \frac{\pi_i}{2\pi_0} \kappa_a^\beta (\sigma_i)_\beta{}^\alpha - \frac{i\mu_1}{2m_0} \kappa_a^\alpha, & \delta \theta_\alpha^{\tilde{a}} &= \frac{\pi_i}{2\pi_0} \kappa_{\tilde{a}}^\beta (\sigma_i)_\beta{}^\alpha - \frac{i\mu_2}{2m_0} \kappa_{\tilde{a}}^\alpha. \end{aligned} \quad (4.34)$$

Under these transformations, the variation of Lagrangian (4.13) (with $a = 0$) is

$$\begin{aligned} \delta L_1 = & 2i \left(m_1 + \frac{(\mu_1)^2}{2m_0} \right) \kappa_a^\alpha \dot{\xi}_\alpha^a + 2i \left(m_2 + \frac{(\mu_2)^2}{2m_0} \right) \kappa_{\tilde{a}}^\alpha \dot{\xi}_\alpha^{\tilde{a}} \\ & - \frac{\partial}{\partial \tau} \left[im_0 \frac{\pi_i}{\pi_0} \kappa_a^\alpha (\sigma_i)_\alpha{}^\beta \theta_\beta^a + im_1 \kappa_a^\alpha \xi_\alpha^a + \mu_1 \kappa_a^\alpha \theta_\alpha^a \right] \\ & - \frac{\partial}{\partial \tau} \left[im_0 \frac{\pi_i}{\pi_0} \kappa_{\tilde{a}}^\alpha (\sigma_i)_\alpha{}^\beta \theta_\beta^{\tilde{a}} + im_2 \kappa_{\tilde{a}}^\alpha \xi_\alpha^{\tilde{a}} + \mu_2 \kappa_{\tilde{a}}^\alpha \theta_\alpha^{\tilde{a}} \right]. \end{aligned} \quad (4.35)$$

It should be pointed out that the variation (4.35) is a total derivative only provided the conditions (4.26) and (4.27) are taken into account. Thus we have shown explicitly that the superparticle model (4.13) is κ -invariant only when the central charge parameters satisfy the conditions (4.26) and (4.27).

Using local transformations (4.34) one can choose the gauge $\xi_\alpha^a = 0$, $\xi_\alpha^{\tilde{a}} = 0$. In such a gauge, the rigid $\mathbf{Q}$ - transformations (3.18) should be accompanied by the appropriate compensating gauge transformations (see, e.g., [38]). In this case, as well as in other cases considered below, we will not impose such gauges, reserving it for more complicated $\mathcal{N} = 4$ NR superparticle models still to be constructed, e.g., those formulated on external electromagnetic background.

4.3 The superparticle model with all central charges incorporated

We can add to the action (4.12) at $a = 0$ the additional terms associated with off-diagonal central charges

$$S_2 = \int L_2 d\tau = \int \left(n^{a\tilde{b}} \hat{\omega}_{(X)a\tilde{b}} + \nu^{a\tilde{b}} \hat{\omega}_{(Y)a\tilde{b}} \right), \quad (4.36)$$

where $n^{a\tilde{b}}$ and $\nu^{a\tilde{b}}$ are constants with the reality conditions as for $\text{USp}(2) \otimes \text{USp}(2) \simeq \text{O}(4)$ bispinors (see (2.49), (3.5))

$$\left(n^{a\tilde{b}} \right)^* = \epsilon_{ac} \epsilon_{\tilde{b}\tilde{d}} n^{c\tilde{d}}, \quad \left(\nu^{a\tilde{b}} \right)^* = \epsilon_{ac} \epsilon_{\tilde{b}\tilde{d}} \nu^{c\tilde{d}}. \quad (4.37)$$

These bi-spinorial constants can be represented as internal four-vectors (isovectors)

$$n^{a\tilde{b}} = i(\tilde{\sigma}_M)^{a\tilde{b}} n_M, \quad \nu^{a\tilde{b}} = i(\tilde{\sigma}_M)^{a\tilde{b}} \nu_M, \quad (4.38)$$

where $(\tilde{\sigma}_M)^{a\tilde{b}}$, $M = 1, 2, 3, 4$, are $\text{O}(4)$ Euclidean σ -matrices and $n_M = (n_M)^*$, $\nu_M = (\nu_M)^*$ are the real internal $\text{O}(4)$ vectors. Below we use the matrix realization $(\tilde{\sigma}_M)^{a\tilde{b}} = (\sigma_i, -i1_2)$, where σ_i , $i = 1, 2, 3$, are the standard Pauli matrices.

The total action can be written as follows

$$S = S_1 + S_2, \quad (4.39)$$

which includes thirteen coupling constants $m_0, m_1, m_2, \mu_1, \mu_2, n_M, \nu_M$. Together with the $\text{O}(4)$ singlets $(m_1 - m_2)$ and $(\mu_1 - \mu_2)$, respectively, the $\text{O}(4)$ vectors n_M and ν_M can be combined in two independent constant $\text{USp}(4) \sim \text{O}(5)$ vectors. The remaining constants $(m_1 + m_2)$ and $(\mu_1 + \mu_2)$ are $\text{O}(5)$ singlets. Constant Euclidean four-vectors n_M and

ν_M break both $O(5)$ and $USp(2) \otimes USp(2) \simeq O(4) \subset O(5)$ internal symmetries to some their subgroups, depending on the concrete choice of the four-vector components. As we will note later, the broken $O(5)$ and $O(4)$ symmetries can be used to bring the constant parameters entering (4.39) to the set with the diagonal unbroken R-symmetry $USp(2) \simeq O(3) \subset USp(2) \otimes USp(2) \simeq O(4)$. With some particular choices of the parameters only $O(2)$ remains as the unbroken internal symmetry.

In the presence of arbitrary central charges the preservation of the total $O(5)$ or $O(4) \subset O(5)$ invariances as Noether symmetries can be restored either by introducing additional harmonic variables or by choosing a different kind of dynamics. Such dynamical model respecting $O(4)$ invariance in the sector described by off-diagonal central charges, will be discussed in section 6.

The Lagrangian in (4.39) is given by the following expression

$$\begin{aligned} L = L_1 + L_2 = m_0 \frac{\pi_i \pi_i}{2\pi_0} + 2m_0 i (\theta_a^\alpha \dot{\theta}_\alpha^a + \theta_a^\alpha \dot{\theta}_{\tilde{\alpha}}^{\tilde{a}}) \\ + i m_1 \xi_\alpha^a \dot{\xi}_a^\alpha + i m_2 \xi_\alpha^{\tilde{a}} \dot{\xi}_{\tilde{a}}^\alpha + 2\mu_1 \theta^{\alpha a} \dot{\xi}_{a\alpha} + 2\mu_2 \theta^{\alpha \tilde{a}} \dot{\xi}_{\alpha \tilde{a}} \\ + i n^{a\tilde{b}} (\xi_{a\tilde{b}} \dot{\xi}_a^\alpha - \xi_{\alpha a} \dot{\xi}_{\tilde{b}}^\alpha) - 2\nu^{a\tilde{b}} (\theta_{\alpha\tilde{b}} \dot{\xi}_a^\alpha - \theta_{\alpha a} \dot{\xi}_{\tilde{b}}^\alpha). \end{aligned} \quad (4.40)$$

In comparison with (4.13), the Lagrangian (4.40) contains two additional terms, which involve only derivatives of ξ 's. Therefore, the momenta p_t , p_{xi} , $p_{\theta_\alpha^a}$, $p_{\theta_\alpha^{\tilde{a}}}$ are the same as in (4.14), (4.15), whereas the momenta $p_{\xi_\alpha^a}$, $p_{\xi_\alpha^{\tilde{a}}}$ acquire additional terms as compared with (4.15). Then it follows that the bosonic constraint $T \approx 0$ (see (4.21)) and the fermionic constraints $D_{\theta_\alpha^a} \approx 0$, $D_{\theta_\alpha^{\tilde{a}}} \approx 0$, defined by (4.22), remain the same, whereas fermionic constraints $D_{\xi_\alpha^a} \approx 0$, $D_{\xi_\alpha^{\tilde{a}}} \approx 0$ will acquire additional terms in comparison with (4.22):

$$\begin{aligned} D_{\xi_\alpha^a} &= p_{\xi_\alpha^a} - 2i p_{x\alpha\beta} \theta^{\beta a} - i(p_t + m_1) \xi_\alpha^a + 2\mu_1 \theta_\alpha^a - i n^{a\tilde{b}} \xi_{\alpha\tilde{b}} + 2\nu^{a\tilde{b}} \theta_{\alpha\tilde{b}} \approx 0, \\ D_{\xi_\alpha^{\tilde{a}}} &= p_{\xi_\alpha^{\tilde{a}}} - 2i p_{x\alpha\beta} \theta^{\beta \tilde{a}} - i(p_t + m_2) \xi_\alpha^{\tilde{a}} + 2\mu_2 \theta_\alpha^{\tilde{a}} + i n^{b\tilde{a}} \xi_{\alpha b} - 2\nu^{b\tilde{a}} \theta_{\alpha b} \approx 0. \end{aligned} \quad (4.41)$$

One can check again that the canonical Hamiltonian is vanishing.

Besides the Poisson brackets (4.23), one gets the following additional non-vanishing Poisson brackets of fermionic constraints $D_{\theta_\alpha^a} \approx 0$, $D_{\theta_\alpha^{\tilde{a}}} \approx 0$ (4.22) and $D_{\xi_\alpha^a} \approx 0$, $D_{\xi_\alpha^{\tilde{a}}} \approx 0$ (4.41):

$$\begin{aligned} \{D_{\xi_\alpha^a}, D_{\xi_\beta^{\tilde{b}}}\} &= -2i \epsilon_{\alpha\beta} n^{a\tilde{b}}, \\ \{D_{\xi_\alpha^a}, D_{\theta_\beta^{\tilde{b}}}\} &= 2\epsilon_{\alpha\beta} \nu^{a\tilde{b}}, \quad \{D_{\xi_\alpha^{\tilde{a}}}, D_{\theta_\beta^b}\} = -2\epsilon_{\alpha\beta} \nu^{b\tilde{a}}. \end{aligned} \quad (4.42)$$

For the model with Lagrangian (4.40) the Noether charges generating the NR supersymmetry transformations (3.18) and (3.20) contain, in comparison with the expressions (4.17), some additional terms and take the form

$$\begin{aligned} \mathbf{Q}_\alpha^a &= p_{\xi_\alpha^a} + i(p_t + m_1) \xi_\alpha^a + i n^{a\tilde{b}} \xi_{\alpha\tilde{b}}, \\ \mathbf{Q}_\alpha^{\tilde{a}} &= p_{\xi_\alpha^{\tilde{a}}} + i(p_t + m_2) \xi_\alpha^{\tilde{a}} - i n^{b\tilde{a}} \xi_{\alpha b}, \\ \mathbf{S}_\alpha^a &= p_{\theta_\alpha^a} + 2m_0 i \theta_\alpha^a - 2i p_{x\alpha\beta} \xi^{\beta a} - 2\mu_1 \xi_\alpha^a - 2\nu^{a\tilde{b}} \xi_{\alpha\tilde{b}}, \\ \mathbf{S}_\alpha^{\tilde{a}} &= p_{\theta_\alpha^{\tilde{a}}} + 2m_0 i \theta_\alpha^{\tilde{a}} - 2i p_{x\alpha\beta} \xi^{\beta \tilde{a}} - 2\mu_2 \xi_\alpha^{\tilde{a}} + 2\nu^{b\tilde{a}} \xi_{\alpha b}. \end{aligned} \quad (4.43)$$

The set of non-vanishing Poisson brackets between the classical supersymmetry generators (4.43) involves the relations (4.19) and, in addition, the following Poisson brackets

$$\begin{aligned}\{\mathbf{Q}_\alpha^a, \mathbf{Q}_\beta^{\tilde{b}}\} &= 2i \epsilon_{\alpha\beta} n^{a\tilde{b}}, \\ \{\mathbf{Q}_\alpha^a, \mathbf{S}_\beta^{\tilde{b}}\} &= \{\mathbf{Q}_\beta^{\tilde{b}}, \mathbf{S}_\alpha^a\} = -2 \epsilon_{\alpha\beta} \nu^{a\tilde{b}}.\end{aligned}\quad (4.44)$$

The Poisson brackets (4.44) are classical counterparts of the anticommutators (2.47). We see that the constants $n^{a\tilde{b}}$ and $\nu^{a\tilde{b}}$ of the general model (4.40) reappear at the level of Poisson brackets in place of the central charges $\mathbf{X}^{a\tilde{b}}$ and $\mathbf{Y}^{a\tilde{b}}$.

4.3.1 Analysis of the constraints

The determinant of the Poisson brackets matrix $\mathcal{P}$ of the fermionic constraints (4.41), (4.22) defined in (4.24) in the case under consideration looks more complicated than (4.25). It is equal, up to a multiplicative constant, to the following expression:

$$\Delta := (m_0)^8 \left[\left(T + m_1 + \frac{(\mu_1)^2 + \hat{\nu}}{2m_0} \right) \left(T + m_2 + \frac{(\mu_2)^2 + \hat{\nu}}{2m_0} \right) - \hat{w} \right]^4, \quad (4.45)$$

where

$$\hat{\nu} := \frac{1}{2} \nu^{a\tilde{b}} \nu_{a\tilde{b}} = \nu_M \nu_M, \quad \hat{w} := \frac{1}{2} w^{a\tilde{b}} w_{a\tilde{b}} = w_M w_M, \quad (4.46)$$

T is the first class constraint (4.21), and the 4-vector $w^{a\tilde{b}} = i(\tilde{\sigma}_M)^{a\tilde{b}} w_M$ is defined by

$$w^{a\tilde{b}} := n^{a\tilde{b}} + \frac{\mu_1 + \mu_2}{2m_0} \nu^{a\tilde{b}} \quad \Leftrightarrow \quad w_M = n_M + \frac{\mu_1 + \mu_2}{2m_0} \nu_M. \quad (4.47)$$

It should be added that $\hat{n} = \frac{1}{2} n^{a\tilde{b}} \nu_{a\tilde{b}} = n_M n_M$ and $\hat{\nu}$, $\hat{w}$ are the length squares of $O(4)$ internal symmetry vectors, i.e. one gets that $\hat{n} \geq 0$, $\hat{\nu} \geq 0$ and $\hat{w} \geq 0$. Moreover, $\hat{n} = 0$ leads to $n^{a\tilde{b}} = 0$ as well as $\hat{\nu} = 0$ leads to $\nu^{a\tilde{b}} = 0$; similarly, $\hat{w} = 0$ implies $w^{a\tilde{b}} = 0$.

The odd first class constraints generating κ -symmetry, are present provided that

$$\det \mathcal{P} \simeq \Delta = 0. \quad (4.48)$$

The constants $n^{a\tilde{b}}$, $\nu^{a\tilde{b}}$ enter the expression (4.45) and the equation (4.48) only through two quantities, $\hat{\nu}$ and $\hat{w}$ defined by (4.46). Since (4.45) is not factorized, in contrast to (4.25), resolving eq. (4.48) is a more complicated task.

The condition (4.48) is necessary for (any number of) odd first class constraint. The full number of such constraints is found by solving the characteristic equation

$$\det (\mathcal{P} - \lambda I) = 0, \quad (4.49)$$

which determines the eigenvalues λ of the matrix $\mathcal{P}$ (see analogous consideration, e.g., in [55, 56]). In (4.49), λ describes the spectral parameter and I is the unit matrix. The number of first class constraints is equal to the number of solutions $\lambda = 0$ of the characteristic equation (4.49). In the presence of k odd first class constraints among sixteen constraints D_A , the equation (4.49) has the form

$$\lambda^k (\lambda - \lambda_1) \dots (\lambda - \lambda_{16-k}) = 0, \quad \text{where } \lambda_1 \neq 0, \quad \dots, \quad \lambda_{16-k} \neq 0. \quad (4.50)$$

In the model considered here the characteristic equation (4.49) coincides with the equation (4.48) in which the substitutions $p_t \rightarrow (p_t - \lambda)$ and $m_0 \rightarrow (m_0 - \lambda)$ are performed. Using the expression (4.45)¹⁸ the characteristic equation (4.49) can be written in the following form

$$\Delta + (m_0)^4 \lambda^4 (A + B\lambda + C\lambda^2 + \lambda^3)^4 = 0, \quad (4.51)$$

where

$$A = -(p_t + m_1 + m_0) \left(T + m_2 + \frac{(\mu_2)^2 + \hat{\nu}}{2m_0} \right) - (p_t + m_2 + m_0) \left(T + m_1 + \frac{(\mu_1)^2 + \hat{\nu}}{2m_0} \right) + 2n^{a\bar{b}} w_{a\bar{b}}, \quad (4.52)$$

$$B = \frac{1}{m_0} (p_t + m_1 + m_0) (p_t + m_2 + m_0) + 2T + m_1 + m_2 + \frac{(\mu_1)^2 + (\mu_2)^2 + 2\hat{\nu} - \hat{n}}{2m_0}, \quad (4.53)$$

$$C = -2 \left(p_t + \frac{m_1 + m_2}{2} + m_0 \right). \quad (4.54)$$

As we see from (4.51), the condition (4.48) implies the presence of at least four odd first class constraints in the total set of sixteen fermionic constraints (4.41), (4.22).

The condition $A = 0$, together with (4.48), lead to the presence of eight odd first class constraints. The condition $A = 0$ requires vanishing those terms in (4.52) which are proportional to p_t :

$$m_1 + \frac{(\mu_1)^2 + \hat{\nu}}{2m_0} + m_2 + \frac{(\mu_2)^2 + \hat{\nu}}{2m_0} = 0. \quad (4.55)$$

An additional condition which stems from $A = 0$ is the vanishing of the remaining constant term:

$$m_2 \left(m_1 + \frac{(\mu_1)^2 + \hat{\nu}}{2m_0} \right) + m_1 \left(m_2 + \frac{(\mu_2)^2 + \hat{\nu}}{2m_0} \right) = n^{a\bar{b}} w_{a\bar{b}}. \quad (4.56)$$

For $m_1 = m_2$ it leads to the condition $n^{a\bar{b}} w_{a\bar{b}} = 2n_M \nu_M = 0$.

Further, one can show that $B \neq 0$ and $C \neq 0$ in (4.51) due to non-vanishing constant coefficients in (4.53) and (4.54) in front of $(p_t)^2$ and p_t .

Thus, by definite choices of central charges, we can recover the cases, when the number of odd first class constraints is quarter or half the total number of odd constraints. Note that, up to a sign, the algebra of the fermionic constraints (4.23), (4.42) coincides with the NR superalgebra (4.19), (4.44) and the number of the first class constraints equals the number of preserved supersymmetries in BPS configurations. Therefore, respective models describe BPS configurations preserving 1/4 or 1/2 of NR supersymmetry.

In the last part of this section, we will consider in detail two special cases.

4.3.2 The case when half of odd constraints is first class

This particular example is specified by the following condition on isotensorial central charges:

$$\hat{w} = 0 \quad (4.57)$$

¹⁸A possible numerical coefficient before Δ can be absorbed into a rescaling of the matrix $\mathcal{P}$ and, further, into the redefinition of the parameter λ in (4.49).

or, equivalently,

$$w^{a\bar{b}} = n^{a\bar{b}} + \frac{\mu_1 + \mu_2}{2m_0} \nu^{a\bar{b}} = 0. \quad (4.58)$$

In this case the vanishing of the quantity (4.45) (with $T \approx 0$) requires that

$$\left[m_1 + \frac{(\mu_1)^2 + \hat{\nu}}{2m_0} \right] \left[m_2 + \frac{(\mu_2)^2 + \hat{\nu}}{2m_0} \right] = 0. \quad (4.59)$$

The relation (4.59) is obeyed provided at least one of two conditions

$$m_1 = -\frac{(\mu_1)^2 + \hat{\nu}}{2m_0}, \quad (4.60)$$

$$m_2 = -\frac{(\mu_2)^2 + \hat{\nu}}{2m_0}, \quad (4.61)$$

or both of them are fulfilled. The conditions (4.60), (4.61) are the obvious generalizations of (4.26) and (4.27).

Now we present the full set of the constraints which occur when the conditions (4.60), (4.61) and (4.58) are valid.

The first class bosonic constraint (4.21) represents the Schrödinger equation as in the previous cases.

Fermionic constraints $(D_{\xi_\alpha}^a, D_{\xi_\alpha}^{\bar{a}}; D_{\theta_\alpha}^a, D_{\theta_\alpha}^{\bar{a}})$ are equivalent to the set $(F_{\xi_\alpha}^a, F_{\xi_\alpha}^{\bar{a}}; D_{\theta_\alpha}^a, D_{\theta_\alpha}^{\bar{a}})$, where $(F_{\xi_\alpha}^a, F_{\xi_\alpha}^{\bar{a}})$ are defined by the following expressions

$$F_{\xi_\alpha}^a = D_{\xi_\alpha}^a + \frac{1}{2m_0} \left(p_x \alpha^\beta - i\mu_1 \delta_\alpha^\beta \right) D_{\theta_\beta}^a + \frac{i}{2m_0} \nu_b^a D_{\theta_\alpha}^{\bar{b}} \approx 0, \quad (4.62)$$

$$F_{\xi_\alpha}^{\bar{a}} = D_{\xi_\alpha}^{\bar{a}} + \frac{1}{2m_0} \left(p_x \alpha^\beta - i\mu_2 \delta_\alpha^\beta \right) D_{\theta_\beta}^{\bar{a}} - \frac{i}{2m_0} \nu_b^{\bar{a}} D_{\theta_\alpha}^b \approx 0. \quad (4.63)$$

The complete set of non-vanishing Poisson brackets for the constraints (4.62), (4.63) reads

$$\begin{aligned} \{F_{\xi_\alpha}^a, D_{\xi_\beta}^b\} &= -2i\epsilon_{\alpha\beta}\epsilon^{ab} \left(T + m_1 + \frac{(\mu_1)^2 + \hat{\nu}}{2m_0} \right), & \{F_{\xi_\alpha}^a, D_{\theta_\beta}^{\bar{b}}\} &= -2i\epsilon_{\alpha\beta}w^{a\bar{b}}, \\ \{F_{\xi_\alpha}^{\bar{a}}, D_{\xi_\beta}^{\bar{b}}\} &= -2i\epsilon_{\alpha\beta}\epsilon^{\bar{a}\bar{b}} \left(T + m_2 + \frac{(\mu_2)^2 + \hat{\nu}}{2m_0} \right), & \{F_{\xi_\alpha}^{\bar{a}}, D_{\theta_\beta}^b\} &= -2i\epsilon_{\alpha\beta}w^{b\bar{a}}, \\ \{F_{\xi_\alpha}^a, F_{\xi_\beta}^b\} &= -2i\epsilon_{\alpha\beta}\epsilon^{ab} \left(T + m_1 + \frac{(\mu_1)^2 + \hat{\nu}}{2m_0} \right), & \{F_{\xi_\alpha}^a, F_{\xi_\beta}^{\bar{b}}\} &= -2i\epsilon_{\alpha\beta}w^{a\bar{b}}, \\ \{F_{\xi_\alpha}^{\bar{a}}, F_{\xi_\beta}^{\bar{b}}\} &= -2i\epsilon_{\alpha\beta}\epsilon^{\bar{a}\bar{b}} \left(T + m_2 + \frac{(\mu_2)^2 + \hat{\nu}}{2m_0} \right). \end{aligned} \quad (4.64)$$

We see that if the conditions (4.60), (4.60) and (4.58) are valid, the constraints $(D_{\theta_\alpha}^a, D_{\theta_\alpha}^{\bar{a}})$ are second class, while the constraints $(F_{\xi_\alpha}^a, F_{\xi_\alpha}^{\bar{a}})$ are first class and generate κ -symmetries.

Substituting the expressions (4.41) into the κ -symmetry generators (4.62), (4.63), we obtain them in the following explicit form

$$\begin{aligned}
F_{\xi\alpha}^a &= p_{\xi\alpha}^a - i(p_t + m_1)\xi_\alpha^a - in^{a\tilde{b}}\xi_{\alpha\tilde{b}} \\
&\quad + \frac{1}{2m_0}\left(p_{x\alpha}^\beta - i\mu_1\delta_\alpha^\beta\right)\left(p_{\theta\beta}^a + 2m_0i\theta_\beta^a\right) + \frac{i}{2m_0}\nu_{\tilde{b}}^a\left(p_{\theta\alpha}^{\tilde{b}} + 2m_0i\theta_\alpha^{\tilde{b}}\right) \approx 0, \\
F_{\xi\alpha}^{\tilde{a}} &= p_{\xi\alpha}^{\tilde{a}} - i(p_t + m_2)\xi_\alpha^{\tilde{a}} + in^{b\tilde{a}}\xi_{\alpha b} \\
&\quad + \frac{1}{2m_0}\left(p_{x\alpha}^\beta - i\mu_2\delta_\alpha^\beta\right)\left(p_{\theta\beta}^{\tilde{a}} + 2m_0i\theta_\beta^{\tilde{a}}\right) - \frac{i}{2m_0}\nu_b^{\tilde{a}}\left(p_{\theta\alpha}^b + 2m_0i\theta_\alpha^b\right) \approx 0. \quad (4.65)
\end{aligned}$$

As opposed to the constraints (4.32) in section 4.2, in the considered case the constraints (4.65) mix two $\text{USp}(2)$ sectors characterized by untilded and tilded $\text{USp}(2)$ -indices.

It turns out, however, that this model for $m_1 \neq m_2$ and $\mu_1 \neq \mu_2$ is just the model of section 4.2 in disguise. To show this, recall that the full Lagrangian (4.40) is formally invariant under the simultaneous $\text{O}(5)$ rotation of the $d = 1$ fields and the set of coupling constants $m_1, m_2, \mu_1, \mu_2, n^{a\tilde{b}}, \nu^{a\tilde{b}}$, which gives an opportunity to pass to $\text{O}(5)$ frame where these constants are reduced to some minimal set.¹⁹ It is important that the coupling constants are divided into the $\text{O}(5)$ singlets $m_1 + m_2$, $\mu_1 + \mu_2$ and $\text{O}(5)$ vectors $(n^{a\tilde{b}}, m_1 - m_2)$ and $(\nu^{a\tilde{b}}, \mu_1 - \mu_2)$; then the condition (4.58) means the vanishing of some particular linear combination of the $\text{O}(4)$ vector components of these two $\text{O}(5)$ vectors. The fifth component of the $\text{O}(5)$ vector containing the $\text{O}(4)$ vector (4.58), is

$$m_1 - m_2 + \frac{\mu_1 + \mu_2}{2m_0}(\mu_1 - \mu_2) = m_1 - m_2 + \frac{(\mu_1)^2 - (\mu_2)^2}{2m_0}. \quad (4.66)$$

This quantity is zero just as a difference of the conditions (4.60), and (4.61)! The rest of these conditions, their sum, expresses the particular $\text{O}(5)$ invariant $(m_1 + m_2)$ through other one,

$$m_1 + m_2 = -\frac{1}{2m_0}\left[\mathcal{A} + \frac{1}{2}(\mu_1 + \mu_2)^2\right], \quad \mathcal{A} := \nu^{a\tilde{b}}\nu_{a\tilde{b}} + \frac{1}{2}(\mu_1 - \mu_2)^2. \quad (4.67)$$

Thus it follows that the conditions (4.58) and (4.60), (4.61) lead to the vanishing of one out of two independent $\text{O}(5)$ vectors in the space of coupling constants and as well relate with each other some $\text{O}(5)$ invariant combinations of these constants. Thus these conditions preserve $\text{O}(5)$ covariance, and one can still use the $\text{O}(5)$ rotations in order to choose the frame where $n^{a\tilde{b}}$ (or $\nu^{a\tilde{b}}$) are zero. In such a frame, due to (4.58), both $\text{O}(4)$ vectors are zero and we are left with the constraints (4.26), (4.27) as the only remaining ones. So for the off-diagonal central charges satisfying the conditions (4.58) our model becomes identical to the one considered in section 4.2.

To construct a system with eight first-class constraints, which would be non-equivalent to the system of section 4.2 and involve the constants $n^{a\tilde{b}}, \nu^{a\tilde{b}}$ which cannot be removed, one

¹⁹To avoid a possible confusion, we point out that this “invariance” is provided if we transform as well the constant parameters in the model; it does not give rise to any Noether charges and serves just to choose the convenient $\text{O}(5)$ frame.

needs to break explicitly the $\text{USp}(4) \simeq \text{O}(5)$ covariance in the space of coupling constants. The simplest option is to assume

$$m_1 = m_2 \quad \Leftrightarrow \quad \mu_1 = \pm \mu_2. \quad (4.68)$$

The $\text{USp}(4) \sim \text{O}(5)$ covariance is also explicitly broken in a system with four first-class constraints corresponding to 1/4 BPS states. We will consider it as the second example.

4.3.3 The case when quarter of odd constraints is first class

Our second example is characterized by non-vanishing off-diagonal central charges, with all quasi-diagonal ones vanishing:

$$m_1 = m_2 = 0, \quad \mu_1 = \mu_2 = 0. \quad (4.69)$$

In this case it follows from (4.47) that $w^{ab} = n^{ab}$. Further, vanishing of the quantity (4.45) required for the presence of odd first class constraints leads to the condition

$$(\hat{\nu})^2 = 4(m_0)^2 \hat{n}. \quad (4.70)$$

If we wish to have eight odd first class constraints with the conditions (4.69), the relations (4.55) and (4.70) (as consequences of the condition $A = 0$) are valid, and they imply that $\hat{n} = 0$, $\hat{\nu} = 0$, whence $\hat{w} = 0$ and, further, $n^{ab} = \nu^{ab} = 0$. Thus, if non-vanishing central charges n^{ab} , ν^{ab} are present and $m_0 \neq 0$, we can only obtain four odd first class constraints.

Let us separate now odd first and second class constraints.

The initial fermionic constraints $(D_{\xi_\alpha}^a, D_{\xi_\alpha}^{\tilde{a}}; D_{\theta_\alpha}^a, D_{\theta_\alpha}^{\tilde{a}})$ are equivalent to the set $(G_\alpha^a, F_\alpha^{\tilde{a}}; D_{\theta_\alpha}^a, D_{\theta_\alpha}^{\tilde{a}})$, where $(G_\alpha^a, F_\alpha^{\tilde{a}})$ are defined by the following expressions

$$G_\alpha^a := D_{\xi_\alpha}^a + \frac{1}{2m_0} p_{x\alpha}{}^\beta D_{\theta_\beta}^a + \frac{i}{2m_0} \nu^a{}_{\tilde{b}} D_{\theta_\alpha}^{\tilde{b}} \approx 0, \quad (4.71)$$

$$F_\alpha^{\tilde{a}} := D_{\xi_\alpha}^{\tilde{a}} + \frac{1}{2m_0} p_{x\alpha}{}^\beta D_{\theta_\beta}^{\tilde{a}} - \frac{i}{2m_0} \nu_b{}^{\tilde{a}} D_{\theta_\alpha}^b - \frac{2m_0}{\hat{\nu}} n_b{}^{\tilde{a}} G_\alpha^b \approx 0 \quad (4.72)$$

(cf. (4.62)). The complete set of non-vanishing Poisson brackets for the constraints (4.71), (4.72) and $D_{\theta_\alpha}^a, D_{\theta_\alpha}^{\tilde{a}}$ looks as follows

$$\begin{aligned} \{F_\alpha^{\tilde{a}}, F_\beta^{\tilde{b}}\} &= -2i \left[1 + \hat{n} \left(\frac{2m_0}{\hat{\nu}} \right)^2 \right] \epsilon_{\alpha\beta} \epsilon^{\tilde{a}\tilde{b}} T + \frac{4im_0}{\hat{\nu}} \epsilon_{\alpha\beta} \epsilon^{\tilde{a}\tilde{b}} \left[\hat{n} - \left(\frac{\hat{\nu}}{2m_0} \right)^2 \right], \\ \{F_\alpha^{\tilde{a}}, G_\beta^b\} &= -\frac{4im_0}{\hat{\nu}} \epsilon_{\alpha\beta} n^{b\tilde{a}} T, \\ \{G_\alpha^a, G_\beta^b\} &= -2i \epsilon_{\alpha\beta} \epsilon^{ab} \left(T + \frac{\hat{\nu}}{2m_0} \right), \\ \{D_{\theta_\alpha}^a, D_{\theta_\beta}^b\} &= -4i m_0 \epsilon_{\alpha\beta} \epsilon^{ab}, \quad \{D_{\theta_\alpha}^{\tilde{a}}, D_{\theta_\beta}^{\tilde{b}}\} = -4i m_0 \epsilon_{\alpha\beta} \epsilon^{\tilde{a}\tilde{b}}. \end{aligned} \quad (4.73)$$

Thus, if the condition (4.70) is valid, four constraints $F_\alpha^{\tilde{a}}$, defined in (4.72) are first class. The constraints $D_{\theta_\alpha}^a, D_{\theta_\alpha}^{\tilde{a}}$ and G_α^a , defined in (4.71) are second class.

5 Quantization of the model and $\mathcal{N} = 4$, $d = 3$ Galilean superfields

In this section we present the canonical operator quantization of our model. We will introduce the (super)Schrödinger realization of quantum phase coordinates and obtain the superfield description of $\mathcal{N} = 4$, $d = 3$ Galilean states. For this purpose we will quantize the second class constraints by the Gupta-Bleuler (GB) procedure [57, 58], without introducing for them the Dirac brackets.

We will consider two versions of our model: the first one with eight first class constraints (introducing $\frac{1}{2}$ BPS states or fraction $\frac{1}{2}$ of unbroken supersymmetry) and the second one with four first class constraints (introducing $\frac{1}{4}$ BPS states or fraction $\frac{1}{4}$ of unbroken supersymmetry).

We will use, instead of the symplectic-Majorana real quantities, the complex Hermitian conjugate Grassmann coordinates, which are defined by (3.4) in the following way

$$\begin{aligned} \xi^\alpha &:= \xi_1^\alpha, & \bar{\xi}_\alpha &:= (\xi^\alpha)^\dagger, & \xi_2^\alpha &= -\epsilon^{\alpha\beta} \bar{\xi}_\beta, & \tilde{\xi}^\alpha &:= \xi_1^\alpha, & \bar{\tilde{\xi}}_\alpha &:= (\tilde{\xi}^\alpha)^\dagger, & \xi_2^\alpha &= -\epsilon^{\alpha\beta} \bar{\tilde{\xi}}_\beta, \\ \theta^\alpha &:= \theta_1^\alpha, & \bar{\theta}_\alpha &:= (\theta^\alpha)^\dagger, & \theta_2^\alpha &= \epsilon^{\alpha\beta} \bar{\theta}_\beta, & \tilde{\theta}^\alpha &:= \theta_1^\alpha, & \bar{\tilde{\theta}}_\alpha &:= (\tilde{\theta}^\alpha)^\dagger, & \theta_2^\alpha &= \epsilon^{\alpha\beta} \bar{\tilde{\theta}}_\beta. \end{aligned} \quad (5.1)$$

The corresponding complex momenta, which are explicit solutions of the conditions (4.16), are

$$\begin{aligned} p_{\xi\alpha} &:= p_{\xi_1^\alpha}, & \bar{p}_{\xi}^\alpha &:= -(p_{\xi\alpha})^\dagger, & p_{\xi_2^\alpha} &= \epsilon_{\alpha\beta} \bar{p}_{\xi}^\beta, & p_{\tilde{\xi}\alpha} &:= p_{\xi_1^\alpha}, & \bar{p}_{\tilde{\xi}}^\alpha &:= -(p_{\tilde{\xi}\alpha})^\dagger, & p_{\xi_2^\alpha} &= \epsilon_{\alpha\beta} \bar{p}_{\tilde{\xi}}^\beta, \\ p_{\theta\alpha} &:= p_{\theta_1^\alpha}, & \bar{p}_{\theta}^\alpha &:= -(p_{\theta\alpha})^\dagger, & p_{\theta_2^\alpha} &= -\epsilon_{\alpha\beta} \bar{p}_{\theta}^\beta, & p_{\tilde{\theta}\alpha} &:= p_{\theta_1^\alpha}, & \bar{p}_{\tilde{\theta}}^\alpha &:= -(p_{\tilde{\theta}\alpha})^\dagger, & p_{\theta_2^\alpha} &= -\epsilon_{\alpha\beta} \bar{p}_{\tilde{\theta}}^\beta. \end{aligned} \quad (5.2)$$

Poisson brackets of these phase superspace variables are given by

$$\begin{aligned} \{\xi^\alpha, p_{\xi\beta}\} &= \delta_\beta^\alpha, & \{\tilde{\xi}^\alpha, p_{\tilde{\xi}\beta}\} &= \delta_\beta^\alpha, & \{\theta^\alpha, p_{\theta\beta}\} &= \delta_\beta^\alpha, & \{\tilde{\theta}^\alpha, p_{\tilde{\theta}\beta}\} &= \delta_\beta^\alpha, \\ \{\bar{\xi}_\alpha, \bar{p}_{\xi}^\beta\} &= \delta_\alpha^\beta, & \{\bar{\tilde{\xi}}_\alpha, \bar{p}_{\tilde{\xi}}^\beta\} &= \delta_\alpha^\beta, & \{\bar{\theta}_\alpha, \bar{p}_{\theta}^\beta\} &= \delta_\alpha^\beta, & \{\bar{\tilde{\theta}}_\alpha, \bar{p}_{\tilde{\theta}}^\beta\} &= \delta_\alpha^\beta. \end{aligned} \quad (5.3)$$

In the quantization procedure, we will use the graded coordinate representation with the following super-Schrödinger realization for the momenta:

$$\begin{aligned} p_t &= -i \frac{\partial}{\partial t}, & p_{xi} &= -i \frac{\partial}{\partial x_i}, \\ p_{\xi\alpha} &= i \frac{\partial}{\partial \xi^\alpha}, & p_{\tilde{\xi}\alpha} &= i \frac{\partial}{\partial \tilde{\xi}^\alpha}, & p_{\theta\alpha} &= i \frac{\partial}{\partial \theta^\alpha}, & p_{\tilde{\theta}\alpha} &= i \frac{\partial}{\partial \tilde{\theta}^\alpha}, \\ \bar{p}_{\xi}^\alpha &= i \frac{\partial}{\partial \bar{\xi}_\alpha}, & \bar{p}_{\tilde{\xi}}^\alpha &= i \frac{\partial}{\partial \bar{\tilde{\xi}}_\alpha}, & \bar{p}_{\theta}^\alpha &= i \frac{\partial}{\partial \bar{\theta}_\alpha}, & \bar{p}_{\tilde{\theta}}^\alpha &= i \frac{\partial}{\partial \bar{\tilde{\theta}}_\alpha}. \end{aligned} \quad (5.4)$$

5.1 The model with vanishing off-diagonal central charges

The system is described by even first class constraint (4.21)

$$T = p_t + \frac{p_{xi} p_{xi}}{2m_0} \approx 0, \quad (5.5)$$

odd first class constraints (4.32) and odd second class constraints (4.22). In terms of complex variables (5.1), (5.2) odd first class constraints (4.32) are written as

$$\begin{aligned}
 F_{\xi\alpha} = F_{\xi\alpha}^1 &= p_{\xi\alpha} - i(p_t + m_1)\bar{\xi}_\alpha + \frac{1}{2m_0}(p_{x\alpha}{}^\beta - i\mu_1\delta_\alpha^\beta)(p_{\theta\beta} - 2m_0i\bar{\theta}_\beta) \approx 0, \\
 \bar{F}_\xi{}^\alpha = \epsilon^{\alpha\beta}F_{\xi\beta}^2 &= \bar{p}_\xi{}^\alpha - i(p_t + m_1)\xi^\alpha + \frac{1}{2m_0}(\bar{p}_\theta{}^\beta - 2m_0i\theta^\beta)(p_{x\beta}{}^\alpha + i\mu_1\delta_\beta^\alpha) \approx 0, \\
 F_{\tilde{\xi}\alpha} = F_{\tilde{\xi}\alpha}^{\bar{1}} &= p_{\tilde{\xi}\alpha} - i(p_t + m_2)\bar{\tilde{\xi}}_\alpha + \frac{1}{2m_0}(p_{x\alpha}{}^\beta - i\mu_2\delta_\alpha^\beta)(p_{\bar{\theta}\beta} - 2m_0i\bar{\bar{\theta}}_\beta) \approx 0, \\
 \bar{F}_{\tilde{\xi}}{}^\alpha = \epsilon^{\alpha\beta}F_{\tilde{\xi}\beta}^{\bar{2}} &= \bar{p}_{\tilde{\xi}}{}^\alpha - i(p_t + m_2)\tilde{\xi}^\alpha + \frac{1}{2m_0}(\bar{p}_{\bar{\theta}}{}^\beta - 2m_0i\bar{\bar{\theta}}^\beta)(p_{x\beta}{}^\alpha + i\mu_2\delta_\beta^\alpha) \approx 0. \quad (5.6)
 \end{aligned}$$

Odd second class constraints (4.22) read

$$\begin{aligned}
 D_{\theta\alpha} = D_{\theta\alpha}^1 &= p_{\theta\alpha} + 2m_0i\bar{\theta}_\alpha \approx 0, \\
 \bar{D}_\theta{}^\alpha = -\epsilon^{\alpha\beta}D_{\theta\beta}^2 &= \bar{p}_\theta{}^\alpha + 2m_0i\theta^\alpha \approx 0, \\
 D_{\bar{\theta}\alpha} = D_{\bar{\theta}\alpha}^{\bar{1}} &= p_{\bar{\theta}\alpha} + 2m_0i\bar{\bar{\theta}}_\alpha \approx 0, \\
 \bar{D}_{\bar{\theta}}{}^\alpha = -\epsilon^{\alpha\beta}D_{\bar{\theta}\beta}^{\bar{2}} &= \bar{p}_{\bar{\theta}}{}^\alpha + 2m_0i\bar{\bar{\theta}}^\alpha \approx 0. \quad (5.7)
 \end{aligned}$$

Second class constraints (5.7) form the Hermitian conjugate pairs $(D_{\theta\alpha}, \bar{D}_\theta{}^\alpha)$ and $(D_{\bar{\theta}\alpha}, \bar{D}_{\bar{\theta}}{}^\alpha)$, what permits us to apply Gupta-Bleuler quantization. In accord with this quantization technique we impose on wave function half of second class constraints, i.e., $\bar{D}_\theta{}^\alpha$ and $\bar{D}_{\bar{\theta}}{}^\alpha$.

Thus the wave function is the superfield

$$\Psi = \Psi(t, x_i; \xi^\alpha, \bar{\xi}_\alpha; \tilde{\xi}^\alpha, \bar{\tilde{\xi}}_\alpha; \theta^\alpha, \bar{\theta}_\alpha; \tilde{\theta}^\alpha, \bar{\tilde{\theta}}_\alpha), \quad (5.8)$$

which satisfies the conditions

$$T\Psi = 0, \quad (5.9)$$

$$F_{\xi\alpha}\Psi = 0, \quad \bar{F}_\xi{}^\alpha\Psi = 0, \quad F_{\tilde{\xi}\alpha}\Psi = 0, \quad \bar{F}_{\tilde{\xi}}{}^\alpha\Psi = 0, \quad (5.10)$$

$$\bar{D}_\theta{}^\alpha\Psi = 0, \quad \bar{D}_{\bar{\theta}}{}^\alpha\Psi = 0. \quad (5.11)$$

The solution of second class constraints (5.11) is the following

$$\Psi = \exp\left\{2m_0\left(\theta^\alpha\bar{\theta}_\alpha + \tilde{\theta}^\alpha\bar{\tilde{\theta}}_\alpha\right)\right\}\Phi, \quad (5.12)$$

where the superfield

$$\Phi = \Phi\left(t, x_i; \xi^\alpha, \bar{\xi}_\alpha; \tilde{\xi}^\alpha, \bar{\tilde{\xi}}_\alpha; \theta^\alpha; \tilde{\theta}^\alpha\right) \quad (5.13)$$

is chiral with respect to the variables $(\theta^\alpha, \tilde{\theta}^\alpha)$.

Even first class constraint (5.9) yields the Schrödinger equations for all component fields.

Odd first class constraint (5.10) provide the following four superwave equations for the superfield (5.13)

$$\begin{aligned}
 [\mathcal{D}_{\xi\alpha} - \Delta_{\theta\alpha}] \Phi &:= \left[\mathcal{D}_{\xi\alpha} + \frac{1}{2m_0} \left(p_{x\alpha}{}^\beta - i\mu_1 \delta_\alpha^\beta \right) \frac{\partial}{\partial \theta^\beta} \right] \Phi = 0, \\
 [\bar{\mathcal{D}}_\xi^\alpha - \bar{\Delta}_\theta^\alpha] \Phi &:= \left[\bar{\mathcal{D}}_\xi^\alpha - 2\theta^\beta \left(p_{x\beta}{}^\alpha + i\mu_1 \delta_\beta^\alpha \right) \right] \Phi = 0, \\
 [\mathcal{D}_{\tilde{\xi}\alpha} - \tilde{\Delta}_{\theta\alpha}] \Phi &:= \left[\mathcal{D}_{\tilde{\xi}\alpha} + \frac{1}{2m_0} \left(p_{x\alpha}{}^\beta - i\mu_2 \delta_\alpha^\beta \right) \frac{\partial}{\partial \tilde{\theta}^\beta} \right] \Phi = 0, \\
 [\bar{\mathcal{D}}_{\tilde{\xi}}^\alpha - \bar{\tilde{\Delta}}_\theta^\alpha] \Phi &:= \left[\bar{\mathcal{D}}_{\tilde{\xi}}^\alpha - 2\tilde{\theta}^\beta \left(p_{x\beta}{}^\alpha + i\mu_2 \delta_\beta^\alpha \right) \right] \Phi = 0.
 \end{aligned} \tag{5.14}$$

Here we have introduced the operators $\Delta_{\theta\alpha}$, $\bar{\Delta}_\theta^\alpha$, $\tilde{\Delta}_{\theta\alpha}$, $\bar{\tilde{\Delta}}_\theta^\alpha$ which do not depend on ξ -variables, and the covariant derivatives for $\mathcal{N} = 8$ extended one-dimensional supersymmetry

$$\begin{aligned}
 \mathcal{D}_{\xi\alpha} &:= \frac{\partial}{\partial \xi^\alpha} - (p_t + m_1) \bar{\xi}_\alpha, & \bar{\mathcal{D}}_\xi^\alpha &:= \frac{\partial}{\partial \bar{\xi}_\alpha} - (p_t + m_1) \xi^\alpha, \\
 \mathcal{D}_{\tilde{\xi}\alpha} &:= \frac{\partial}{\partial \tilde{\xi}^\alpha} - (p_t + m_2) \bar{\tilde{\xi}}_\alpha, & \bar{\mathcal{D}}_{\tilde{\xi}}^\alpha &:= \frac{\partial}{\partial \bar{\tilde{\xi}}_\alpha} - (p_t + m_2) \tilde{\xi}^\alpha,
 \end{aligned} \tag{5.15}$$

which form two mutually anticommuting $\mathcal{N} = 4$ $d=1$ superalgebras

$$\{\mathcal{D}_{\xi\alpha}, \bar{\mathcal{D}}_{\tilde{\xi}}^\beta\} = -2(p_t + m_1) \delta_\alpha^\beta, \quad \{\mathcal{D}_{\tilde{\xi}\alpha}, \bar{\mathcal{D}}_\xi^\beta\} = -2(p_t + m_1) \delta_\alpha^\beta, \tag{5.16}$$

with constants m_1, m_2 playing the role of central charges. It is straightforward to check that the integrability condition for the equations (5.14) is just the Schrödinger equations (5.9) (we mention that the conditions (4.26), (4.27) are valid).

It is easy to find the explicit solution of equations (5.14). Consider the expansion of superfield (5.13) over the Grassmann coordinates $\xi^\alpha, \bar{\xi}_\alpha, \tilde{\xi}^\alpha, \bar{\tilde{\xi}}_\alpha$

$$\Phi = \Phi_0 + \xi^\alpha \Phi_\alpha + \bar{\xi}_\alpha \bar{\Phi}^\alpha + \tilde{\xi}^\alpha \tilde{\Phi}_\alpha + \bar{\tilde{\xi}}_\alpha \bar{\tilde{\Phi}}^\alpha + \xi^\alpha \xi^\beta \Phi_{\alpha\beta} + \tilde{\xi}^\alpha \tilde{\xi}^\beta \tilde{\Phi}_{\alpha\beta} + \dots, \tag{5.17}$$

where all components $\Phi_0, \Phi_\alpha, \bar{\Phi}^\alpha$, etc, are “two-chiral” superfields dependent on (t, x_i) and two chiral spinorial coordinates, $(\theta^\alpha, \tilde{\theta}^\alpha)$. The equations (5.14) express all the component superfields in the expansion (5.17) in terms of the first component, the superfield $\Phi_0(t, x_i, \theta^\alpha, \tilde{\theta}^\alpha)$. It follows from (5.14), that all covariant derivatives $\mathcal{D}_{\xi\alpha}\Phi, \bar{\mathcal{D}}_\xi^\alpha\Phi, \mathcal{D}_{\tilde{\xi}\alpha}\Phi, \bar{\mathcal{D}}_{\tilde{\xi}}^\alpha\Phi$ and all their powers are expressed in terms of powers of the Δ -operators acting on Φ : $\mathcal{D}_{\xi\alpha}\bar{\mathcal{D}}_{\tilde{\xi}}^\beta\Phi = -\bar{\Delta}_\theta^\beta\Delta_{\theta\alpha}\Phi$ etc. So, being taken at $\xi^\alpha = \bar{\xi}_\alpha = \tilde{\xi}^\alpha = \bar{\tilde{\xi}}_\alpha = 0$, all superfields in the expansion (5.17) are expressed through derivatives of a single superfield $\Phi_0(t, x_i, \theta^\alpha, \tilde{\theta}^\alpha)$.

To summarize, physical states of the considered model are described by the two-chiral superfield $\Phi_0(t, x_i, \theta^\alpha, \tilde{\theta}^\alpha)$ with the following component expansion

$$\begin{aligned}
 \Phi_0 &= A + \theta^\alpha \Omega_\alpha + \tilde{\theta}^\alpha \tilde{\Omega}_\alpha + \theta^\alpha \theta_\alpha B + \tilde{\theta}^\alpha \tilde{\theta}_\alpha \tilde{B} + \theta^\alpha \tilde{\theta}^\beta B_{\alpha\beta} \\
 &\quad + \tilde{\theta}^\beta \tilde{\theta}_\beta \theta^\alpha \Lambda_\alpha + \theta^\beta \theta_\beta \tilde{\theta}^\alpha \tilde{\Lambda}_\alpha + \theta^\alpha \theta_\alpha \tilde{\theta}^\beta \tilde{\theta}_\beta C.
 \end{aligned} \tag{5.18}$$

In (5.18) all component fields are complex functions of t and x_i which satisfy the Schrödinger equation (5.9).

Thus the presented model results in five complex scalar fields $A(t, x_i)$, $B(t, x_i)$, $\tilde{B}(t, x_i)$, $B_{[\alpha\beta]}(t, x_i)$, $C(t, x_i)$ which describe spin 0 states; one complex vectorial field $B_{(\alpha\beta)}(t, x_i)$, which accommodates spin 1 states; and four spinorial fields $\Omega_\alpha(t, x_i)$, $\tilde{\Omega}_\alpha(t, x_i)$, $\Lambda_\alpha(t, x_i)$, $\tilde{\Lambda}_\alpha(t, x_i)$ corresponding to spin 1/2. It is easy to see that we obtain in such a way equal number of 8 bosonic and 8 fermionic component fields.

It is worth to emphasize that the description of physical states by the two-chiral superfield (5.18) is consistent with the possibility of imposing the gauges $\xi_\alpha^a = 0$, $\tilde{\xi}_\alpha^{\tilde{a}} = 0$ on the local transformations (4.34), as was mentioned in section 4.2.

5.2 The models with non-vanishing off-diagonal central charges

Like in the previous subsection, we will use the complex variables (5.1), (5.2).

5.2.1 The models with eight odd first class constraints

We consider firstly the case with condition (4.58).

The wave function has the form (5.8) and the wave equations are derived from even first class constraint (5.9) as well as odd first and second class constraints, the last ones quantized by the Gupta-Bleuler quantization method.

Second class constraints have the same form as in (5.7), while the solution of the constraints (5.11) leads to the reduced superwave function (5.12), (5.13).

For simplicity we will restrict our study to the case of parallel four-vectors n_M, ν_M

$$n^{a\tilde{b}} = n\epsilon^{a\tilde{b}}, \quad \nu^{a\tilde{b}} = \nu\epsilon^{a\tilde{b}}, \quad \hat{\nu} = \nu^2 \quad (5.19)$$

where (see (4.58))

$$n = -\frac{\mu_1 + \mu_2}{2m_0} \nu. \quad (5.20)$$

Other choices of the constants $n^{a\tilde{b}}$ and $\nu^{a\tilde{b}}$ satisfying (4.58) lead to the same set of physical states.

Using complex variables (5.1), (5.2), the odd first class constraints (4.65) can be brought to the form

$$\begin{aligned} F_{\xi\alpha} &= p_{\xi\alpha} - i(p_t + m_1)\bar{\xi}_\alpha - in\tilde{\xi}_\alpha \\ &\quad + \frac{1}{2m_0} \left(p_{x\alpha}{}^\beta - i\mu_1\delta_\alpha^\beta \right) \left(p_{\theta\beta} - 2m_0i\bar{\theta}_\beta \right) - \frac{i}{2m_0} \nu \left(p_{\bar{\theta}\alpha} - 2m_0i\bar{\tilde{\theta}}_\alpha \right) \approx 0, \\ \bar{F}_\xi{}^\alpha &= \bar{p}_\xi{}^\alpha - i(p_t + m_1)\xi^\alpha - in\tilde{\xi}^\alpha \\ &\quad + \frac{1}{2m_0} \left(\bar{p}_{\theta}{}^\beta - 2m_0i\theta^\beta \right) \left(p_{x\beta}{}^\alpha + i\mu_1\delta_\beta^\alpha \right) + \frac{i}{2m_0} \nu \left(\bar{p}_{\tilde{\theta}}{}^\alpha - 2m_0i\tilde{\theta}^\alpha \right) \approx 0, \end{aligned} \quad (5.21)$$

$$\begin{aligned} F_{\tilde{\xi}\alpha} &= p_{\tilde{\xi}\alpha} - i(p_t + m_2)\bar{\tilde{\xi}}_\alpha - in\tilde{\tilde{\xi}}_\alpha \\ &\quad + \frac{1}{2m_0} \left(p_{x\alpha}{}^\beta - i\mu_2\delta_\alpha^\beta \right) \left(p_{\tilde{\theta}\beta} - 2m_0i\bar{\tilde{\theta}}_\beta \right) - \frac{i}{2m_0} \nu \left(p_{\theta\alpha} - 2m_0i\bar{\theta}_\alpha \right) \approx 0, \\ \bar{F}_{\tilde{\xi}}{}^\alpha &= \bar{p}_{\tilde{\xi}}{}^\alpha - i(p_t + m_2)\tilde{\xi}^\alpha - in\tilde{\tilde{\xi}}^\alpha \\ &\quad + \frac{1}{2m_0} \left(\bar{p}_{\tilde{\theta}}{}^\beta - 2m_0i\tilde{\theta}^\beta \right) \left(p_{x\beta}{}^\alpha + i\mu_2\delta_\beta^\alpha \right) + \frac{i}{2m_0} \nu \left(\bar{p}_\theta{}^\alpha - 2m_0i\theta^\alpha \right) \approx 0. \end{aligned} \quad (5.22)$$

These expressions generalize the ones given in (5.6). Imposing the constraints (5.10) on the superwave function (5.12), we obtain that the superfield (5.13) satisfies the following generalization of four superwave equations (5.14)

$$\begin{aligned} \left[\nabla_{\xi\alpha} + \frac{1}{2m_0} \left(p_{x\alpha}{}^\beta - i\mu_1 \delta_\alpha^\beta \right) \frac{\partial}{\partial \theta^\beta} - \frac{i\nu}{2m_0} \frac{\partial}{\partial \tilde{\theta}^\alpha} \right] \Phi &= 0, \\ \left[\bar{\nabla}_{\xi}{}^\alpha - 2\theta^\beta \left(p_{x\beta}{}^\alpha + i\mu_1 \delta_\beta^\alpha \right) - 2i\nu \tilde{\theta}^\alpha \right] \Phi &= 0, \end{aligned} \quad (5.23)$$

$$\begin{aligned} \left[\nabla_{\tilde{\xi}\alpha} + \frac{1}{2m_0} \left(p_{x\alpha}{}^\beta - i\mu_2 \delta_\alpha^\beta \right) \frac{\partial}{\partial \tilde{\theta}^\beta} - \frac{i\nu}{2m_0} \frac{\partial}{\partial \theta^\alpha} \right] \Phi &= 0, \\ \left[\bar{\nabla}_{\tilde{\xi}}{}^\alpha - 2\tilde{\theta}^\beta \left(p_{x\beta}{}^\alpha + i\mu_2 \delta_\beta^\alpha \right) - 2i\nu \theta^\alpha \right] \Phi &= 0, \end{aligned} \quad (5.24)$$

where

$$\begin{aligned} \nabla_{\xi\alpha} &:= \mathcal{D}_{\xi\alpha} - n_{\tilde{\xi}\alpha}, & \bar{\nabla}_{\xi}{}^\alpha &:= \bar{\mathcal{D}}_{\xi}{}^\alpha - n_{\tilde{\xi}}{}^\alpha, \\ \nabla_{\tilde{\xi}\alpha} &:= \mathcal{D}_{\tilde{\xi}\alpha} - n_{\xi\alpha}, & \bar{\nabla}_{\tilde{\xi}}{}^\alpha &:= \bar{\mathcal{D}}_{\tilde{\xi}}{}^\alpha - n_{\xi}{}^\alpha \end{aligned} \quad (5.25)$$

and $\mathcal{D}_{\xi\alpha}$, $\bar{\mathcal{D}}_{\xi}{}^\alpha$, $\mathcal{D}_{\tilde{\xi}\alpha}$, $\bar{\mathcal{D}}_{\tilde{\xi}}{}^\alpha$ are defined in (5.15). If the conditions (4.60), (4.61) and (4.58) are valid once again, the integrability condition for the system of equations (5.23), (5.24) is the free Schrödinger equations (5.9).

The unconstrained superfield can be found as in the previous subsection: we apply the expansion (5.17) and again find that all superfield components in this expansion are expressed through the derivatives of single two-chiral superfield $\Phi_0(t, x_i, \theta^\alpha, \tilde{\theta}^\alpha)$ (see (5.18)). Therefore, the considered model with off-diagonal central charges has the same physical fields content as the previously studied model with the diagonal central charges only.

5.2.2 The models with four odd first class constraints

Now we will consider the case when central charges take the values (4.69), (4.70). Also, for definiteness, we choose the values (5.19), (5.20) for off-shell central charges. Imposing on the wave function (5.8) the second class constraints (5.7), (5.11) we obtain the same superfield solution (5.12), (5.13) as in the previous cases.

The remaining odd constraints are second class constraints G_α^a , defined in (4.71), and first class constraints $F_\alpha^{\tilde{a}}$, defined in (4.72). The second class constraints (4.71) coincide with the constraints (5.21), (5.23) taken at $m_1 = m_2 = 0$, $\mu_1 = \mu_2 = 0$:

$$G_\alpha^a = (F_{\xi\alpha}, \bar{F}_{\tilde{\xi}}{}^\alpha). \quad (5.26)$$

We quantize these constraints by the Gupta-Bleuler procedure, imposing on the wave function the half of them:

$$\bar{F}_{\xi}{}^\alpha \Psi = 0 \quad \Rightarrow \quad \bar{F}_{\tilde{\xi}}{}^\alpha \Phi = 0. \quad (5.27)$$

Using (5.23) at $m_1 = m_2 = 0$, $\mu_1 = \mu_2 = 0$, we find the solution of (5.27)

$$\Phi = \exp \left\{ -p_t \xi^\alpha \bar{\xi}_\alpha - n_{\tilde{\xi}}{}^\alpha \bar{\xi}_\alpha - 2\theta^\alpha p_{x\alpha}{}^\beta \bar{\xi}_\beta - 2i\nu \tilde{\theta}^\alpha \bar{\xi}_\alpha \right\} \Omega, \quad (5.28)$$

where

$$\Omega = \Omega \left(t, x_i; \xi^\alpha; \tilde{\xi}^\alpha, \bar{\xi}_\alpha; \theta^\alpha; \tilde{\theta}^\alpha \right) \quad (5.29)$$

is a three-chiral superfield with respect to ξ^α and $(\theta^\alpha, \tilde{\theta}^\alpha)$ variables.

The last constraints which we need to solve are the first class constraints rewritten in terms of the operators (5.21), (5.22) and (5.23), (5.24) (at $m_1 = m_2 = 0$, $\mu_1 = \mu_2 = 0$) in the following form

$$F_\alpha \Phi = \left(F_{\tilde{\xi}\alpha} - \frac{2m_0 n}{\hat{\nu}} F_{\xi\alpha} \right) \Phi = 0, \quad (5.30)$$

$$\bar{F}^\alpha \Phi = \left(\bar{F}_{\tilde{\xi}}^\alpha - \frac{2m_0 n}{\hat{\nu}} \bar{F}_{\xi}^\alpha \right) \Phi = \bar{F}_{\tilde{\xi}}^\alpha \Phi = 0. \quad (5.31)$$

Using (5.23), (5.24), we obtain that the equations (5.30), (5.31) amount to the following pair of equations for the superfield (5.29)

$$\left(\mathcal{D}_{\tilde{\xi}\alpha} + \frac{1}{2m_0} p_{x\alpha}{}^\beta \frac{\partial}{\partial \tilde{\theta}^\beta} - \frac{i\nu}{2m_0} \frac{\partial}{\partial \theta^\alpha} - \frac{2m_0 n}{\hat{\nu}} \frac{\partial}{\partial \xi^\alpha} - \frac{n}{\hat{\nu}} p_{x\alpha}{}^\beta \frac{\partial}{\partial \theta^\beta} + \frac{in}{\nu} \frac{\partial}{\partial \tilde{\theta}^\alpha} \right) \Omega = 0, \quad (5.32)$$

$$\left(\bar{\mathcal{D}}_{\tilde{\xi}}^\alpha - n\xi^\alpha - 2\tilde{\theta}^\beta p_{x\beta}{}^\alpha - 2i\nu\theta^\alpha \right) \Omega = 0. \quad (5.33)$$

One can consider its general expansion with respect to Grassmann coordinates $\tilde{\xi}^\alpha, \tilde{\xi}_\alpha$

$$\Omega = \Omega_0 + \tilde{\xi}^\alpha \Omega_\alpha + \tilde{\xi}_\alpha \bar{\Omega}^\alpha + \tilde{\xi}^\alpha \tilde{\xi}_\alpha \Omega_1 + \tilde{\xi}_\alpha \tilde{\xi}^\alpha \bar{\Omega}_1 + \tilde{\xi}^\alpha \tilde{\xi}^\beta \Omega_{\alpha\beta} + \dots, \quad (5.34)$$

where all components $\Omega_0, \Omega_\alpha, \bar{\Omega}^\alpha$, etc., are three-chiral superfields depending on t, x_i and three chiral Weyl spinors $\theta^\alpha, \tilde{\theta}^\alpha, \xi^\alpha$. As in the previous subsections, we obtain that due to the equations (5.32), (5.33), all three-chiral superfields in (5.34) (except Ω_0) are represented as derivatives of this lowest component superfield Ω_0 .

Thus, the solutions of the model in this subsection are parametrized by a three-chiral superfield

$$\Omega_0 = \Omega_0(t, x_i, \theta^\alpha, \tilde{\theta}^\alpha, \xi^\alpha). \quad (5.35)$$

Furthermore we recall that all component fields of the superfield (5.35) satisfy the Schrödinger equation.

Finally, we would like to notice that both types of chiral superfields describing superwave functions in this section are essentially on-shell since they require Schrödinger equation as the integrability condition of the relevant odd first class constraints.

6 Conclusions

In this paper we considered the $\mathcal{N} = 4$, $d = 3$ NR superparticle models with twelve constant central charges transforming in certain representations of $\text{USp}(4) \sim \text{O}(5)$ and $\text{USp}(2) \otimes \text{USp}(2) \simeq \text{O}(4) \subset \text{O}(5)$ internal R-symmetry groups. The maximal $\text{U}(4)$ R-symmetry group of relativistic $\mathcal{N} = 4$, $D = 4$ superalgebra in the NR contraction limit is reduced to a semi-direct product of the compact R-symmetry group $\text{USp}(4) \simeq \text{O}(5)$ and some abelian six-dimensional commutative ideal. In the dynamical framework of our superparticle model, after quantization the central charges are identified with constant parameters of the underlying world-line Lagrangian. Depending on the specific non-vanishing

values of these central charges, we are left, before any Hamiltonian analysis, with different fractions of unbroken internal symmetry, $G_{\text{int}} \subset \text{USp}(4) \simeq \text{O}(5)$, namely

- a) If only one central charge $\mathcal{Z}$ (see (2.14)) is present, the maximal R-symmetry $\text{O}(5)$ of NR $\mathcal{N} = 4$ superalgebra remains in the model; the central charge is $\text{O}(5)$ singlet;
- b) If we have two quasi-diagonal central charges (see (1.9)) the internal symmetry is broken to $G_{\text{int}} = \text{USp}(2) \otimes \text{USp}(2) \simeq \text{SU}(2) \otimes \text{SU}(2) \simeq \text{O}(4)$. The central charges are presented by four $\text{USp}(2)$ singlets m_1, m_2, μ_1 and μ_2 ;
- c) Adding the off-diagonal central charges described by two arbitrary constant $\text{O}(4)$ isovectors $n^{a\tilde{b}}$ and $\nu^{a\tilde{b}}$ (see (4.38)) radically changes the situation. At $m_1 \neq m_2$ and $\mu_1 \neq \mu_2$ the set of twelve constant central charges determines two $\text{O}(5)$ vectors ($n^{a\tilde{b}}, m_1 - m_2$) and ($\nu^{a\tilde{b}}, \mu_1 - \mu_2$) and two $\text{O}(5)$ singlets ($m_1 + m_2$) and ($\mu_1 + \mu_2$). The $\text{O}(5)$ frame can be fixed so that one of these $\text{O}(5)$ vectors carries only one non-zero component, say $m_1 - m_2$. There still remains $\text{O}(4)$ covariance which can be further restricted in such a way that another $\text{O}(4)$ vector $\nu^{a\tilde{b}}$ (see (4.38)) will have only one non-zero component, $\nu^{a\tilde{b}} = \epsilon^{a\tilde{b}} \nu$. Thus in such R-symmetry frame we end up with five independent constant central charges and $\text{O}(3)$ as the residual R-symmetry group;
- d) If $m_1 = m_2$ and/or $\mu_1 = \mu_2$, the $\text{O}(5)$ covariance is reduced to $\text{O}(4)$. Hence we can choose the frame where $\text{O}(4)$ isovector $n^{a\tilde{b}}$ contains only one non-zero component, $n^{a\tilde{b}} = \epsilon^{a\tilde{b}} n$, and the residual R-symmetry group is $\text{O}(3)$. The second non-parallel $\text{O}(4)$ vector $\nu^{a\tilde{b}}$ can be split into $\text{O}(3)$ singlet and vector parts, with only one non-zero vector component, $\nu^{a\tilde{b}} \rightarrow (\epsilon^{a\tilde{b}} \nu, \delta^{a\tilde{b}} \nu_2)$, that reduces R-symmetry $\text{O}(3)$ to the minimally possible one, given by $\text{O}(2)$. Thus in this particular frame we end up with six (or less) independent constant central charges and $\text{O}(2)$ as the minimal exact internal symmetry in our model.

It should be added that rest mass m_0 , describing the Bargmann central charge in Galilean sector, can be treated as the thirteenth central charge, which does not break any R-symmetry.

The superparticle action, constructed in section 3 and section 4 of the present paper, is linear in the MC one-forms associated with central charges. The numerical coefficients in front of the central charge MC one-forms provide the numerical values of central charges. In our further work we plan to consider also alternative action densities as nonlinear functions of MC forms, which would permit, e.g., a generalization of (4.36) to the action manifestly invariant under the $\text{O}(4)$ internal symmetries. In particular, following the construction of the model for free relativistic massive particle,²⁰ one can replace the action (4.36) by the $\text{USp}(2) \otimes \text{USp}(2)$ -invariant action depending on all eight off-diagonal central charges

$$S'_2 = \int \left(k_1 \sqrt{\hat{\omega}_{(X)}^{a\tilde{b}} \hat{\omega}_{(X)} a\tilde{b}} + k_2 \sqrt{\hat{\omega}_{(Y)}^{a\tilde{b}} \hat{\omega}_{(Y)} a\tilde{b}} + k_3 \sqrt{\hat{\omega}_{(X)}^{a\tilde{b}} \hat{\omega}_{(Y)} a\tilde{b}} \right), \quad (6.1)$$

²⁰In the case of relativistic particle in a flat space the action is equal to $m \int (\omega^\mu \omega_\mu)^{1/2}$, where $\omega^\mu = dx^\mu$. In NR theories we are interested in, such type of actions will be employed in the internal symmetry sector only.

where k_1, k_2, k_3 are constant and $\hat{\omega}_{(X)}^{a\tilde{b}}, \hat{\omega}_{(Y)}^{a\tilde{b}}$ are defined in (3.12). The system described by the action $S_1 + S'_2$ produces the same fermionic constraints (4.41) where, however, $n^{a\tilde{b}}$ and $\nu^{a\tilde{b}}$ are not constant anymore: they become the canonical momenta for the tensorial central charge coordinates $h_{a\tilde{b}}$ and $f_{a\tilde{b}}$ (see section 3, eqs. (3.2) and (3.12)). So, although the fermionic constraints have basically the same form in both models (see (6.1) and (4.36)), in the case of the action (6.1) the group parameters $h_{a\tilde{b}}$ and $f_{a\tilde{b}}$ are introduced as the dynamical tensorial central charges coordinates. In such a way we deal with an extension of the bosonic target space sector (t, x_i) describing NR space-time to an extended target space with auxiliary central charge coordinates $(t, x_i; h_{a\tilde{b}}, f_{a\tilde{b}})$. Additional coordinates $h_{a\tilde{b}}, f_{a\tilde{b}}$ enter into new three bosonic constraints which fix $\hat{n} = n_M n_M, \hat{\nu} = \nu_M \nu_M, n_M \nu_M$ (see (4.46)) by $(k_1)^2, (k_2)^2, (k_3)^2$. In such a way we obtain a sort of Kaluza-Klein (KK) extension of the superparticle model, with auxiliary KK bosonic dimensions represented by central charge coordinates. Analysis of this modified $\mathcal{N} = 4$ NR superparticle model will be given elsewhere (for an early attempt in this direction see [59]).

In the future we plan also to examine another way of preserving internal symmetry by using the harmonic type variables u_a^b and $\tilde{u}_{\tilde{a}}^{\tilde{b}}$ which occur in the coset space parametrization (3.2), as well as the “genuine” harmonic variables defined for the R-symmetry group $\text{USp}(4)$.²¹ Further direction for the future study is to couple the NR superparticle presented here to electromagnetic, YM and supergravity backgrounds. It can be important for the following reason. The energy momentum dispersion relations in our model for arbitrary spin states are described by the free Schrödinger equation depending on the same mass parameter m_0 . One can argue that, after switching on the background fields, other central charges will also become dynamically active and will contribute to the modification of Schrödinger equation.

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²¹The description of various harmonic superspaces for the relativistic $\mathcal{N} = 4, D = 4$ superalgebra and their implications in studying the quantum structure of $\mathcal{N} = 4$ SYM theory can be found in a recent review [60].

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