

Manifesting enhanced cancellations in supergravity: integrand versus integrals

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ABSTRACT: Examples of ‘enhanced ultraviolet cancellations’ with no known standard-symmetry explanation have been found in a variety of supergravity theories. By examining one- and two-loop examples in four- and five-dimensional half-maximal supergravity, we argue that enhanced cancellations in general cannot be exhibited prior to integration. In light of this, we explore reorganizations of integrands into parts that are manifestly finite and parts that have poor power counting but integrate to zero due to integral identities. At two loops we find that in the large loop-momentum limit the required integral identities follow from Lorentz and $SL(2)$ relabeling symmetry. We carry out a nontrivial check at four loops showing that the identities generated in this way are a complete set. We propose that at L loops the combination of Lorentz and $SL(L)$ symmetry is sufficient for displaying enhanced cancellations when they happen, whenever the theory is known to be ultraviolet finite up to $(L - 1)$ loops.

KEYWORDS: Scattering Amplitudes, Models of Quantum Gravity, Supergravity Models

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1 Introduction

The study of ultraviolet properties of four-dimensional gravity theories has a long history, starting from the seminal work of 't Hooft and Veltman [1]. Despite this we do not know the answer to the basic question of at which loop order various gravity theories actually diverge. In addition, when divergences occur in graviton amplitudes we now know that they have unusual properties, including dependence on evanescent effects [2, 3] and suspected links to anomalies [4–7]. Even more interesting are indications in certain supergravity theories that the loop order where the first divergence occurs is higher than previous expectations [8–10]. This renews the possibility that certain theories, such as $\mathcal{N} = 8$ supergravity, are ultraviolet finite at any order in perturbation theory. No known symmetry is powerful enough to render a four-dimensional quantum gravity theory ultraviolet finite, so if this were true it would be extraordinary.

Certain cancellations in gravity theories are different from those in supersymmetric gauge theories in that they cannot be made manifest for ordinary local representations. When such cancellations happen they are dubbed ‘enhanced cancellations’ [9]. In simple cases, these enhanced cancellations can be understood through conventional means by constraining the set of available counterterms from symmetry considerations. For example, at one loop, a well-known counterterm argument [1] explains that the n graviton amplitudes are finite even though the diagrams scale poorly in the ultraviolet. On the other hand,

recent examples of enhanced cancellations have as yet no standard symmetry explanation, despite attempts [11–14] and insight from string theory [15]. These examples include $\mathcal{N} = 5$ supergravity at four loops in $D = 4$ [9], $\mathcal{N} = 4$ supergravity at three loops in $D = 4$ [8], and half-maximal supergravity at two loops in $D = 5$ [10]. In the relatively simple case of half-maximal supergravity at two loops the cancellations have been understood using the double-copy structure that allows the amplitudes to be built from gauge-theory ones [10]. Unfortunately, it is not clear how to generalize this understanding to higher loops.

In light of the difficulties in trying to develop a comprehensive explanation for enhanced cancellations, we should consider alternative approaches. For instance one could try to mimic diagram-based proofs of finiteness that were successfully carried out for $\mathcal{N} = 4$ super-Yang-Mills theory (see for example refs. [16–19]). These were achieved by finding representations of the integrand where every term is ultraviolet finite by power counting. However, enhanced cancellations are different: by definition they cannot be made manifest diagram by diagram at the integrand level, using only standard Feynman propagators. But one can still wonder if some kind of integrand-level reorganization could be found that makes large loop-momentum cancellations manifest or at least clarifies how the cancellations occur.

An obstruction to pursuing these ideas is that we lack a good definition of global variables for all diagrams of a multiloop amplitude including nonplanar diagrams. One way to approach this difficulty is to use unitarity cuts. At one loop, a systematic program was successfully followed for all one-loop (super)gravity amplitudes in ref. [20] using a formalism [21] based on generalized unitarity [22, 23]. This was used to demonstrate the existence of nontrivial cancellations between diagrams as the number of external legs increases. However, a general extension of the one-loop analysis to higher loops remains a challenge.

In this paper instead of attempting a general argument we turn to specific examples in half-maximal supergravity, which we study in some detail. We construct the examples using the Bern-Carrasco-Johansson (BCJ) double-copy construction of gravity loop integrands in terms of gauge-theory ones [24, 25]. These examples are based on the one- and two-loop $\mathcal{N} = 4$ supergravity amplitudes previously obtained in refs. [26–29].

We first show that at one loop it is not possible to construct integrands where cancellations are manifest in general dimensions. In particular, we identify cancellations in $D = 4$ that require integration identities. At two loops we use unitarity cuts to argue that cancellations cannot be made manifest at the integrand level. To further investigate this case, we use integration-by-parts (IBP) technology [30–43] to reorganize the integrand into pieces that are finite by power counting and pieces that are divergent by power counting, yet integrate to zero. Although this re-arrangement of the complete integrand is successful, it requires detailed knowledge of the specific integrals and their relations, making it difficult to generalize to higher loops.

To deal with this, we then turn to a simpler approach by giving up on trying to make the full integrand display enhanced ultraviolet cancellations. Instead we series expand in large loop momenta in order to focus on the ultraviolet behavior. We show that at least in the two loop examples we study the integral identities necessary for exposing the en-

hanced cancellations follow from only Lorentz and $SL(2)$ relabeling invariance. These ideas continue to higher loops, and as a nontrivial confirmation we found that these principles generate all required integral identities for exposing the ultraviolet behavior of maximal supergravity at four loops in the critical dimension where the divergences first occur [44]. Based on these results, we conjecture that at L loops the IBP identities generated by Lorentz and $SL(L)$ relabeling symmetry are sufficient for revealing the enhanced cancellations, when they exist. The principles are generic and present in all amplitudes in the large loop-momentum limit.

This paper is organized as follows. In section 2, we present one- and two-loop examples showing the lack of integrand-level cancellations. In section 3 we outline how one can arrange complete integrands so that they are manifestly finite by power counting up to terms that integrate to zero. In section 4 we then analyze the large loop-momentum limit, bringing us to a conjecture on symmetries of the integrals responsible for making enhanced cancellations visible. We give our conclusions in section 5. We also include an appendix on subtleties regarding boundary terms in integration-by-parts identities.

2 Absence of enhanced cancellations in the integrand

Enhanced cancellations are a recently identified type of ultraviolet cancellation that can occur in gravity theories [8–10]. These cancellations are defined as follows: start with an amplitude organized in terms of diagrams whose denominators are only the usual Feynman propagators $i/(p^2 + i\epsilon)$. Suppose this amplitude is ultraviolet finite, yet there are terms that are divergent by power counting and cannot be re-assigned to other diagrams without introducing additional spurious denominators in other diagrams. This implies nontrivial cancellations that cannot be manifest in the integrand of each diagram. We would then say there is an enhanced cancellation.

This notion is distinct from the question of whether it is possible to exhibit the cancellations at the integrand level; one might imagine that with careful choices of loop variables in each diagram, one might be able to align the loop momenta in just the right way so that poor behavior cancels algebraically between diagrams prior to integration. Here we show that this does not happen.

We present examples of enhanced cancellations to illustrate that it is only after integration that divergences cancel. We focus on the relatively simple cases of 16-supercharge half-maximal supergravity at one and two loops in $D = 4$ and $D = 5$. In $D = 4$ this theory is just $\mathcal{N} = 4$ supergravity [45]. Even though the one-loop $D = 4$ cancellation is a well-known consequence of supersymmetry [46], it provides a relatively simple concrete example of cancellations that do not arise at the integrand level, but can be exposed using Lorentz invariance. We then turn to the more interesting case of two-loop half-maximal supergravity in $D = 5$. In this case no known standard-symmetry argument invalidates [11, 12, 14, 47] the potential R^4 divergence. However, both field theory [10] and string theory [48] calculations show that there are no R^4 terms in the effective action.

In order to construct the integrands we use the BCJ double-copy construction [24, 25], which we review briefly. The double-copy construction is useful because it directly gives us

gravity loop integrands from corresponding gauge-theory ones. In this construction, one of the two gauge-theory amplitudes is first reorganized into diagrams with only cubic vertices,

$$\mathcal{A}_m^{L\text{-loop}} = i^L g^{m-2+2L} \sum_{\mathcal{S}_m} \sum_j \int \prod_{l=1}^L \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{c_j n_j}{\prod_{\alpha_j} D_{\alpha_j}}, \quad (2.1)$$

where the D_{α_j} are the propagators of the j^{th} diagram, L is the number of loops, m is the number of external legs and g is the gauge coupling. The first sum runs over the $m!$ permutations of external legs, denoted by \mathcal{S}_m , while the second sum over j runs over the distinct cubic graphs. The product in the denominator runs over all Feynman propagators. The symmetry factor S_n accounts for any overcounts and internal automorphisms. The c_j are the color factors associated with the diagrams and the n_j are kinematic numerators.

The double-copy construction relies on BCJ duality [24, 25] where triplets of diagram numerators satisfy equations in one-to-one correspondence with the Jacobi identities of the color factors of each diagram,

$$c_i + c_j + c_k = 0 \Rightarrow n_i + n_j + n_k = 0. \quad (2.2)$$

The indices i, j, k label the diagram to which the color factors and numerators belong. If the diagram numerators satisfy the same algebraic properties as the color factors, we can obtain corresponding gravity amplitudes simply replacing the color factors of a second gauge theory with numerator factors where the duality holds:

$$c_i \rightarrow n_i. \quad (2.3)$$

The gauge-theory coupling constant is also replaced by the gravitational one: $g \rightarrow (\kappa/2)$. In this construction the duality (2.2) needs to be manifest in only one of the two gauge theories [25, 49]. This construction also extends to cases where the gauge theory includes fundamental-representation matter particles [50, 51].

2.1 One-loop example

We start with the one-loop amplitude of pure half-maximal $\mathcal{N} = 4$ supergravity in four dimensions [45]. This amplitude is well studied and has been computed in refs. [26–28]. The double-copy construction of this amplitude is particularly simple. We start from the corresponding $\mathcal{N} = 4$ super-Yang-Mills and pure Yang-Mills amplitudes.

The one-loop four-point $\mathcal{N} = 4$ super-Yang-Mills amplitude was first obtained from the low-energy limit of a Type I superstring amplitude [52]. This amplitude is particularly simple and the only nonzero kinematic numerators are those of the box diagrams in figure 1,

$$n_{\mathcal{N}=4}^{\text{box}} = st A_{\mathcal{N}=4}^{\text{tree}}(1, 2, 3, 4), \quad (2.4)$$

where $s = (k_1 + k_2)^2$ and $t = (k_2 + k_3)^2$ are the usual Mandelstam invariants and $A_{\mathcal{N}=4}^{\text{tree}}(1, 2, 3, 4)$ is the color-ordered tree superamplitude. The combination $st A_{\mathcal{N}=4}^{\text{tree}}(1, 2, 3, 4)$ is crossing symmetric, so the three box diagrams have identical numerators. It is easy to check that this representation of the amplitude satisfies the color-kinematics duality (2.2).

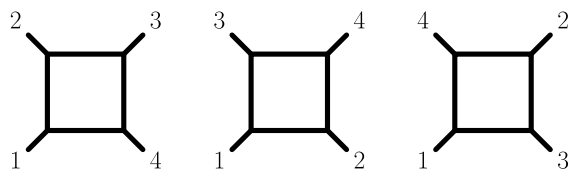


Figure 1. The three box diagrams contributing to the one-loop four-point amplitude of maximal $\mathcal{N} = 4$ super-Yang-Mills theory and half-maximal supergravity.

Replacing the color factors in the pure Yang-Mills box contributions given in ref. [53] with the $\mathcal{N} = 4$ super-Yang-Mills numerators (2.4), we obtain the $\mathcal{N} = 4$ supergravity amplitude as a sum over box diagrams,

$$\mathcal{M}_{\mathcal{N}=4}^{\text{one-loop}} = -\left(\frac{\kappa}{2}\right)^4 st A_{\mathcal{N}=4}^{\text{tree}}(1, 2, 3, 4) \left(I_{1234}[n_{1234,p}] + I_{1324}[n_{1324,p}] + I_{1423}[n_{1423,p}] \right), \quad (2.5)$$

where

$$I_{1234}[n_{1234,p}] \equiv \int \frac{d^D p}{(2\pi)^D} \frac{n_{1234,p}}{p^2(p-k_1)^2(p-k_1-k_2)^2(p+k_4)^2}, \quad (2.6)$$

is the first box integral in figure 1 and $n_{1234,p}$ is the expression defined in eq. (3.5) of ref. [53]. The triangle and bubble contributions from the pure Yang-Mills amplitude are simply set to zero because the corresponding $\mathcal{N} = 4$ SYM numerators vanish. As dictated by the double-copy construction, the supergravity states are given by the tensor product of pure Yang-Mills gluon states with the states of $\mathcal{N} = 4$ super-Yang-Mills theory.

The case of $D = 4$ is an example of enhanced cancellations because the three box diagrams are each logarithmically divergent, yet the sum over diagrams is finite. We can see this by finding power-counting divergent terms in each diagram that cannot be moved to other diagrams without introducing nonlocalities in the diagram numerators. An example is the term,

$$n_{1234,p} \sim p_{\mu_1} p_{\mu_2} p_{\mu_3} p_{\mu_4} \varepsilon_1^{\mu_1} \varepsilon_2^{\mu_2} \varepsilon_3^{\mu_3} \varepsilon_4^{\mu_4} + \dots, \quad (2.7)$$

where $\varepsilon_i^{\mu_i}$ is the gluon polarization of leg i on the pure Yang-Mills side of the double copy.

The cancellations between the diagrams are nontrivial. To see the cancellation of the logarithmic divergences, we expand in large loop momentum or equivalently small external momenta k_i^μ . Because the integrals are only logarithmically divergent in $D = 4$, this amounts to simply setting all k_i^μ to zero in the integrand (keeping the overall prefactor fixed). In this limit, the propagator of each graph become identical, and the resulting graph effectively becomes a scaleless vacuum integral. Such scaleless integrals vanish in dimensional regularization, but we can introduce a mass for each propagator to separate out the infrared divergences without affecting the ultraviolet divergence. Starting with the pure Yang-Mills numerators, keeping only the leading terms in all three box diagrams

results in an integrand proportional to

$$\begin{aligned}
 -i st A_{\mathcal{N}=4}^{\text{tree}}(D_s - 2) \frac{\varepsilon_1^{\mu_1} \varepsilon_2^{\mu_2} \varepsilon_3^{\mu_3} \varepsilon_4^{\mu_4}}{2(p^2 - m^2)^4} & \left[(p^2)^2 (\eta_{\mu_1 \mu_4} \eta_{\mu_2 \mu_3} + \eta_{\mu_1 \mu_3} \eta_{\mu_2 \mu_4} + \eta_{\mu_1 \mu_2} \eta_{\mu_3 \mu_4}) \right. \\
 & - 4p^2 (\eta_{\mu_1 \mu_2} p_{\mu_3} p_{\mu_4} + \eta_{\mu_1 \mu_3} p_{\mu_2} p_{\mu_4} + \eta_{\mu_1 \mu_4} p_{\mu_2} p_{\mu_3} + \eta_{\mu_2 \mu_3} p_{\mu_1} p_{\mu_4} \\
 & \left. + \eta_{\mu_2 \mu_4} p_{\mu_1} p_{\mu_3} + \eta_{\mu_3 \mu_4} p_{\mu_1} p_{\mu_2}) + 24 p_{\mu_1} p_{\mu_2} p_{\mu_3} p_{\mu_4} \right], \quad (2.8)
 \end{aligned}$$

where $A_{\mathcal{N}=4}^{\text{tree}} = A_{\mathcal{N}=4}^{\text{tree}}(1, 2, 3, 4)$ and D_s is a state-counting parameter coming from contractions $\eta_\mu^\mu = D_s$. (In conventional dimensional regularization [54] $D_s = 4 - 2\epsilon$, but in other schemes, such as the four-dimensional helicity scheme [55], $D_s = 4$.) In the expression above we see explicitly that the amplitude is logarithmically divergent by power counting and that no purely algebraic manipulations can expose the cancellation of the divergence. What makes this case particularly simple is that in the large loop-momentum limit all diagrams degenerate to a single vacuum integral, avoiding loop-momentum labeling ambiguities in different terms that plague higher loops.

This example provides a clear demonstration that even after summing over diagrams, enhanced cancellations are not visible prior to using properties of integrals. To expose the ultraviolet cancellation we use Lorentz invariance in the form of integration identities:

$$\int d^D p \frac{p_\mu p_\nu}{(p^2 - m^2)^4} = \int d^D p \frac{1}{D} \frac{\eta_{\mu\nu} p^2}{(p^2 - m^2)^4}, \quad (2.9)$$

$$\int d^D p \frac{p_\mu p_\nu p_\rho p_\sigma}{(p^2 - m^2)^4} = \int d^D p \frac{1}{D(D+2)} \frac{(\eta_{\mu\nu} \eta_{\rho\sigma} + \eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\rho\nu})(p^2)^2}{(p^2 - m^2)^4}. \quad (2.10)$$

With these identities, we find that the integral of eq. (2.8) is equal to the integral of

$$\begin{aligned}
 -i st A_{\mathcal{N}=4}^{\text{tree}}(D_s - 2) \frac{(p^2)^2}{2(p^2 - m^2)^4} \frac{(D-2)(D-4)}{D(D+2)} \\
 \times \varepsilon_1^{\mu_1} \varepsilon_2^{\mu_2} \varepsilon_3^{\mu_3} \varepsilon_4^{\mu_4} (\eta_{\mu_1 \mu_2} \eta_{\mu_3 \mu_4} + \eta_{\mu_1 \mu_3} \eta_{\mu_2 \mu_4} + \eta_{\mu_1 \mu_4} \eta_{\mu_2 \mu_3}), \quad (2.11)
 \end{aligned}$$

which vanishes in $D = 4$. While in this case, the cancellation is understood to be a consequence of supersymmetry [46], it does provide a robust example illustrating that enhanced cancellations become visible in the amplitudes only after making use of integral identities.

2.2 Two-loop example

Enhanced cancellations become more interesting beyond one loop where they correspond to a variety of ultraviolet cancellations for which standard-symmetry explanations are not known [11–14]. We therefore turn to half-maximal supergravity at two loops. In $D = 4$ the cancellations are well understood to be a consequence of supersymmetry [56, 57], but in $D = 5$ no such explanation is known [10].

In $D = 4$ we can enormously simplify the integrand by using helicity states. A simple trick that helps us simplify the analysis in higher dimensions as well is to start with the higher-dimensional theory but to restrict the external states and momenta to live in a four-dimensional subspace. In this way we can use four-dimensional helicity methods to

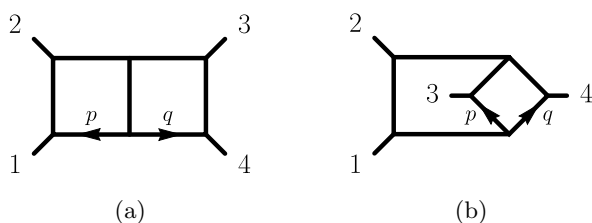


Figure 2. The planar and nonplanar double-box diagrams that contribute to the four-point amplitudes of $\mathcal{N} = 4$ supergravity.

enormously simplify higher-dimensional integrands as well. This trick, of course, does not work for all states in the higher-dimensional theory, but is sufficient for our purpose of illustrating the difficulty of exposing enhanced cancellations at the integrand level.

Consider the four-point two-loop amplitude of $\mathcal{N} = 4$ supergravity. This amplitude has already been discussed in some detail in ref. [29]. The double-copy construction of the two-loop integrand is rather straightforward. We start from the dimensionally-regularized $D = 4$ all-plus helicity ($++++$) pure Yang-Mills amplitude in the form given in ref. [53]. (An earlier form of the integrand may be found in ref. [58].) In this representation the kinematic numerators of the planar and nonplanar double-box diagrams shown in figure 2 are

$$n_{1234}^{\text{PYM}} = \mathcal{T} \left((D_s - 2)s (\lambda_p^2 \lambda_q^2 + \lambda_p^2 \lambda_{p+q}^2 + \lambda_q^2 \lambda_{p+q}^2) + 16s ((\lambda_p \cdot \lambda_q)^2 - \lambda_p^2 \lambda_q^2) + \frac{1}{2}(D_s - 2)(p + q)^2 ((D_s - 2)\lambda_p^2 \lambda_q^2 + 8(\lambda_p^2 + \lambda_q^2)(\lambda_p \cdot \lambda_q)) \right), \quad (2.12)$$

$$n_{1234}^{\text{NPYM}} = \mathcal{T} \left((D_s - 2)s (\lambda_p^2 \lambda_q^2 + \lambda_p^2 \lambda_{p+q}^2 + \lambda_q^2 \lambda_{p+q}^2) + 16s ((\lambda_p \cdot \lambda_q)^2 - \lambda_p^2 \lambda_q^2) \right), \quad (2.13)$$

where D_s is the state-counting parameter similar to that at one loop and the subscript ‘1234’ refers to the diagram external leg labeling as in figure 2. The momenta p and q are the momenta carried by the propagators indicated in figure 2, while λ_p and λ_q are their (-2ϵ) components, where $\epsilon = (4 - D)/2$. We use λ_{p+q} as a shorthand for $\lambda_p + \lambda_q$. The crossing symmetric prefactor

$$\mathcal{T} = \frac{[12][34]}{\langle 12 \rangle \langle 34 \rangle}, \quad (2.14)$$

is defined in terms of spinor inner products, following the notation of ref. [59]. The remaining planar and nonplanar double-box numerators are given by relabeling these. There are contributions to the Yang-Mills integrand from other types of diagrams as well, but we will not need them for the double-copy procedure.

To obtain half-maximal supergravity we then take the pure-Yang-Mills amplitude and replace the color factors with $\mathcal{N} = 4$ super-Yang-Mills numerators that satisfy BCJ duality using eq. (2.2). For the two-loop four-point amplitude of $\mathcal{N} = 4$ SYM a representation that satisfies the duality happens to match the original construction [60]. The only non-vanishing diagrams are the planar and nonplanar double boxes shown in figure 2. The substitution (2.3) is simply

$$\begin{aligned} c_{1234}^{\text{P}} &\rightarrow n_{1234}^{\text{PN}^4} = s^2 t A_{\mathcal{N}=4}^{\text{tree}}(1, 2, 3, 4), \\ c_{1234}^{\text{NP}} &\rightarrow n_{1234}^{\text{NPN}^4} = s^2 t A_{\mathcal{N}=4}^{\text{tree}}(1, 2, 3, 4), \end{aligned} \quad (2.15)$$

where numerators other than the planar and nonplanar ones vanish. As for the one-loop case, we package the $\mathcal{N} = 4$ super-Yang-Mills tree amplitude for all states into a single superamplitude. The half-maximal supergravity amplitude is then obtained by summing over the planar and nonplanar double boxes in figure 2, with kinematics numerators given by the product of pure Yang-Mills and $\mathcal{N} = 4$ super-Yang-Mills numerators,

$$\begin{aligned} N_{1234}^{\text{P half-max. sugra}} &= s^2 t A_{\mathcal{N}=4}^{\text{tree}}(1, 2, 3, 4) \times n_{1234}^{\text{PYM}}, \\ N_{1234}^{\text{NP half-max. sugra}} &= s^2 t A_{\mathcal{N}=4}^{\text{tree}}(1, 2, 3, 4) \times n_{1234}^{\text{NP YM}}. \end{aligned} \tag{2.16}$$

The remaining supergravity planar and nonplanar double-box numerators are given by simple relabelings. Diagrams other than the planar and non-planar double boxes vanish.

This construction is also valid for the $D = 5$ theory with the external states restricted to a $D = 4$ subspace. We simply take $\epsilon \rightarrow -1/2 + \epsilon$ and accordingly the λ_p and λ_q become one dimensional up to $\mathcal{O}(\epsilon)$ corrections. Similarly the state-counting parameter should be shifted, $D_s \rightarrow D_s + 1$. With these modifications, the simple integrand in eq. (2.16) is valid for the $D = 5$ theory as well.

As terminology for the rest of the paper, when we label an amplitude by its external helicity, we are not referring to the helicities of the supergravity theory, but to the helicities of the pure Yang-Mills theory comprising one side of the double-copy supergravity theory.

2.2.1 Cuts and labels for nonplanar amplitudes

Enhanced cancellations generally occur between diagrams of different topologies. A difficulty for exposing the cancellations at the integrand level beyond one loop is that there is no unique and well-defined notion of an integrand involving nonplanar diagrams. Nor is it clear in general how one should choose momentum labels in each diagram that would allow cancellations between diagrams of various topologies to occur. For planar diagrams there is a canonical choice of global variables for all diagrams based on dual variables [61], but no analogous notion is known in the nonplanar case. As a simple example consider the planar and nonplanar double-box diagrams in figure 2. Fundamentally, the propagator structure is different, making it unclear how one might be able show the cancellation without integration.

A way to sidestep the labeling issue is to focus on unitarity cuts. Generalized unitarity cuts that place at least one line on-shell in every loop impose global momentum labels on the cut. We can then ask whether we can find nontrivial cancellations in the cut linked to enhanced cancellations. If such cancellations happen at the level of the integrand, one should expect an improvement in the overall power counting after summing over all contributions to the cuts compared to individual terms. Some care is required because cuts can also obscure cancellations by restricting the diagrams that appear. The more legs that are cut, the fewer diagrams are included, since only those diagrams that contain propagators corresponding to the cut ones will be included. Because of this, it is best to focus on cuts where only a few legs are placed on shell.

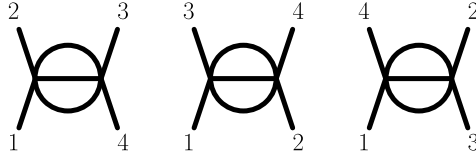


Figure 3. The three sunset integrals. These are ultraviolet divergent in $D = 4$ and $D = 5$.

2.2.2 Absence of cancellations in a three-particle cut

The three-particle cut in figure 4 is useful for studying enhanced cancellations. In the following section, using integration-by-parts technology we describe an arrangement of the integrand where potential divergences are pushed into sunset diagrams, illustrated in figure 3. This suggests that the three-particle cut, where the cut lines correspond to the three propagators of a sunset diagram, is a natural one for studying enhanced cancellations. In addition, this cut fixes all loop momentum labels in this amplitude in terms of the momenta of the cut lines. An obvious guess is that if we apply the three-particle cut corresponding to the internal lines of the sunset diagram, we should find improved power counting in the full sum over terms compared to individual contributions.

The $(+++)$ amplitude has a number of special features that simplify the analysis of the cut, making it easier to find ultraviolet cancellations if they exist. On the three-particle cut, the terms in the numerator proportional to $(p+q)^2$ in eq. (2.12) are set to zero because they corresponds to one of the on-shell inverse propagators ℓ_1^2 , ℓ_2^2 or ℓ_3^2 , as can be seen in figure 4, making the form of the planar and nonplanar numerators identical in the three-particle cut. A useful feature of the remaining numerator terms that we exploit is that they are invariant under relabelings: the expression is the same under any mapping of the p and q propagator labels to any two of the three ℓ_1 , ℓ_2 and ℓ_3 . In addition, up to prefactors depending on external momenta, the dependence of the numerators is only on the components outside the four-dimensional subspace where the external momenta and helicities live. These features enormously simplify the analysis of the cut because most of the numerator factors out and is independent of permutations of external or internal legs.

Using these observations, after inserting the numerators into the planar and nonplanar double-box diagrams and taking the three-particle cut shown in figure 4, we obtain the expression:

$$\begin{aligned}
 \mathcal{I}^{\text{cut}} = & \mathcal{P}(\ell_1, \ell_2, \ell_3) \\
 & \times \left[\left(\frac{1}{2} \frac{t^2}{(\ell_1+k_1)^2(\ell_3+k_2)^2(\ell_3-k_3)^2(\ell_1-k_4)^2} + \frac{t^2}{(\ell_2+k_1)^2(\ell_3+k_2)^2(\ell_3-k_3)^2(\ell_1-k_4)^2} \right. \right. \\
 & + \frac{s^2}{(\ell_1+\ell_2)^2(\ell_2+\ell_3)^2(\ell_3+k_2)^2(\ell_1-k_4)^2} + \frac{1}{2} \frac{s^2}{(\ell_2+\ell_3)^2(\ell_3+k_1)^2(\ell_2+k_2)^2(\ell_1-k_4)^2} \\
 & \left. \left. + \frac{1}{2} \frac{s^2}{(\ell_1+\ell_2)^2(\ell_3+k_2)^2(\ell_2-k_3)^2(\ell_1-k_4)^2} \right) + \text{perms}(\ell_1, \ell_2, \ell_3) \right. \\
 & \left. + (1 \leftrightarrow 2) + (3 \leftrightarrow 4) + (1 \leftrightarrow 2, 3 \leftrightarrow 4) \right], \tag{2.17}
 \end{aligned}$$

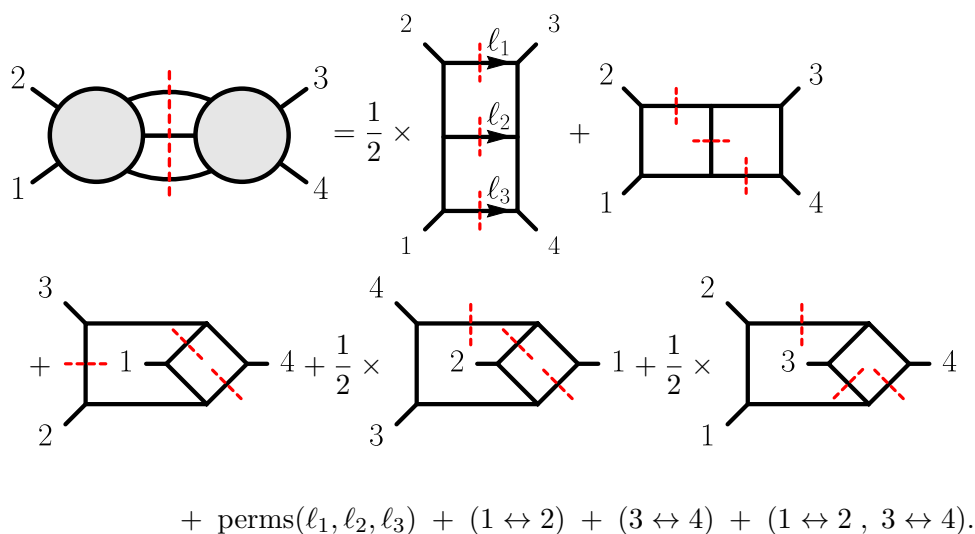


Figure 4. The contributing diagrams of the three-particle cut of the half-maximal supergravity two-loop four-point amplitude. The shaded (red) dashed lines indicated the legs which are cut.

where the on-shell conditions $\ell_1^2 = \ell_2^2 = \ell_3^2 = 0$ are imposed. The prefactor $\mathcal{P}(\ell_1, \ell_2, \ell_3)$ is

$$\mathcal{P}(\ell_1, \ell_2, \ell_3) = -i(D_s - 2)stA_{\mathcal{N}=4}^{\text{tree}}(1, 2, 3, 4) \mathcal{T} \times \left((\lambda_{\ell_1}^2 \lambda_{\ell_2}^2 + \lambda_{\ell_1}^2 \lambda_{\ell_3}^2 + \lambda_{\ell_2}^2 \lambda_{\ell_3}^2) + 16s((\lambda_{\ell_1} \cdot \lambda_{\ell_2})^2 - \lambda_{\ell_1}^2 \lambda_{\ell_2}^2) \right), \quad (2.18)$$

which is invariant under the permutations of external and internal cut legs indicated in eq. (2.17). We have analyzed eq. (2.17) both analytically and numerically and we find that for $\ell_i \rightarrow \infty$ there is no improvement in the large loop-momentum behavior after summing over all terms, compared to the behavior of a single term. In fact, this is no surprise because other than the overall prefactor (2.18), this sum over terms is *precisely* the same one that appears in the three-particle cut of the two-loop four-point amplitude of $\mathcal{N} = 8$ supergravity given in eq. (5.15) of ref. [62]. In $\mathcal{N} = 8$ supergravity we know there are no further cancellations arising from the sum over diagrams. This can be seen as follows: the only nonvanishing diagrams in $\mathcal{N} = 8$ supergravity are the planar and nonplanar double boxes of figure 2, but with no loop momenta in the numerators [62]. Simple power counting shows that each diagram of $\mathcal{N} = 8$ supergravity is ultraviolet divergent in dimensions $D \geq 7$. This divergence does not cancel in the sum over diagrams, leading to a divergence of the four-point amplitude of $\mathcal{N} = 8$ supergravity [62]:

$$\mathcal{M}_4^{\text{two-loop}, D=7-2\epsilon} \Big|_{\text{UV div.}} = \frac{1}{2\epsilon(4\pi)^7} \frac{\pi}{3} (s^2 + t^2 + u^2) \times \left(\frac{\kappa}{2} \right)^6 stuM_{\mathcal{N}=8}^{\text{tree}}(1, 2, 3, 4), \quad (2.19)$$

where we have stripped the coupling constant and $M_{\mathcal{N}=8}^{\text{tree}}$ is the supergravity tree amplitude. The fact that there are no further cancellations in $\mathcal{N} = 8$ supergravity implies that no integrand-level cancellation is possible in our $\mathcal{N} = 4$ supergravity three-particle cut (2.17).

One might imagine trying to include relabelings $\ell_i \rightarrow -\ell_i$ in the spirit of ref. [63] or other relabelings in order to try to expose cancellations. However, because of the link to the $\mathcal{N} = 8$ supergravity cut, it is clear there are no further cancellations to be found.

In summary, we see no evidence of cancellations at the integrand level. The usual supergraph Feynman rules or amplitudes-based proofs of ultraviolet finiteness in gauge theory (see for example, refs. [16–18]) rely on the ability to make the integrand manifestly ultraviolet finite by power counting. The difficulty in finding a standard-symmetry based explanation for enhanced cancellations [11–14] in gravity theories is presumably tied to our difficulty in identifying the cancellations at the integrand level. This greatly complicates any all-order understanding of the divergence properties of supergravity theories. If we are to unravel enhanced cancellations, we need to turn to the systematics of cancellations from integral identities.

3 Rearranging the integrand to show finiteness

As discussed in the previous section, it does not appear possible to expose enhanced cancellations purely at the integrand level. In this section we show how one can rearrange integrands into a form where all terms are manifestly finite by power counting, except those that integrate to zero. We do so using modern integration-by-parts (IBP) technology [30–43]. In our discussion we will be using the language of integrands and integrals interchangeably. This is because the modern approaches to integration by parts can be used to track terms in the integrand that integrate to zero, in a manner analogous to the one-loop technology of refs. [21, 64].

We first outline how IBP relations can be used to reorganize integrands with enhanced cancellations so that all terms that are naively ultraviolet divergent by power counting integrate to zero. We start from a given integrand that has the schematic structure

$$\mathcal{I}^{\text{total}} = \sum_i \mathcal{I}_i^{\text{fin.}} + \sum_j \mathcal{I}_j^{\text{div.}}. \tag{3.1}$$

The sum runs over the various pieces of the integrand, denoted by $\mathcal{I}_i^{\text{fin.}}$, which are finite by power counting, and $\mathcal{I}_j^{\text{div.}}$ which are divergent by power counting. After integration, however, the total may be finite. The idea is to reorganize this integrand into the form

$$\mathcal{I}^{\text{total}} = \sum_i \tilde{\mathcal{I}}_i^{\text{fin.}} + \sum_j \tilde{\mathcal{I}}_j^{\text{van.}}, \tag{3.2}$$

where $\tilde{\mathcal{I}}_i^{\text{fin.}}$ is another set of integrands that are finite after integration and $\tilde{\mathcal{I}}_j^{\text{van.}}$ can be divergent by power counting but integrate to zero,

$$\int \tilde{\mathcal{I}}_j^{\text{van.}} = 0, \tag{3.3}$$

thus making the finiteness manifest. The reorganization is accomplished by writing the sum over power-counting divergent integrals as

$$\sum_j \mathcal{I}_j^{\text{div.}} = \sum_j \mathcal{I}'_j^{\text{fin.}} + \sum_j \left(\mathcal{I}_j^{\text{div.}} - \mathcal{I}'_j^{\text{fin.}} \right), \tag{3.4}$$

where the terms in parentheses integrates to zero and the finite integrals $\mathcal{I}_j^{\text{fin.}}$ are included with the finite ones in eq. (3.2).

IBP technology offers a systematic means for accomplishing this. We briefly review this. The IBP method [30–35] takes advantage of the fact that in dimensional regularization a total derivative vanishes:

$$\int \prod_i d^D \ell_i \frac{\partial}{\partial \ell_j^\mu} \left(\frac{v_j^\mu}{\prod_k D_k} \right) = 0, \tag{3.5}$$

where $1/D_k$ are propagators and v_j^μ are arbitrary functions of loop momenta as well as external kinematics or other vectors in the problem. Evaluating the derivatives gives a sum of terms, and the vanishing of the integral therefore implies a relation amongst the integrals corresponding to each term. By exhausting all such independent relations one can choose a basis of integrals in terms of which to express a given amplitude. The standard basis choice at one loop is a combination of boxes, triangles, and bubbles [65], but at higher loops there is no canonical choice. In general, different bases might be used to manifest different aspects of the amplitude, such as its symmetries and/or behavior on certain unitarity cuts.

Generically, when applying integration-by-parts identities, there is no natural separation of the type in eq. (3.2). In general, the coefficients of individual terms can develop $1/\epsilon$ singularities, and divergences cancel in complicated ways, making the finiteness unclear. To avoid this, some care is required to pick integral bases that (a) do not introduce divergences in integral coefficients and (b) contain a minimal number of divergent integrals. Usually, one picks a linearly independent set of integrals, because this minimizes the number of objects that need to be computed. But, even for an ultraviolet finite amplitude, a general choice of basis will likely have explicit ultraviolet divergences either in basis integrals or in their coefficients. The finiteness is thus obscured because the divergence cancels only in the full sum over contributions. A way to avoid this problem and express the amplitude in the form of eq. (3.2) is to use an overcomplete set of integrals. The overcompleteness gives sufficient freedom that we can exploit to make the finiteness manifest.

We illustrate this procedure with a simple example. Suppose our expression is given as the sum of integrals:

$$A = \frac{1}{70} \text{Box} - \frac{1}{2s^2} \text{Bubble}_1 - \frac{1}{2t^2} \text{Bubble}_2. \tag{3.6}$$

Each of these integrals are ultraviolet divergent in five dimensions with the following leading divergences (omitting an overall $\pi/32$):

$$\text{Box} \Big|_{\text{UV div.}} = \frac{1}{3\epsilon}, \quad \text{Bubble}_1 \Big|_{\text{UV div.}} = \frac{s^2}{210\epsilon}, \quad \text{Bubble}_2 \Big|_{\text{UV div.}} = \frac{t^2}{210\epsilon}. \tag{3.7}$$

Evaluating the divergence shows that eq. (3.6) is finite, but this is not manifest in the above representation. Now consider the following IBP identities

$$\begin{aligned}
 d\omega_1 &= \text{Box}(1,2,3,4) - \frac{70}{s^2} \text{Sunset}(1,2,3,4) - \frac{1}{3s^2} \text{Box}(1,2,3,4)_{\text{doubled}}, \\
 d\omega_2 &= \text{Box}(1,2,3,4) + \frac{70}{su} \text{Sunset}(1,2,3,4) + \frac{70}{tu} \text{Sunset}(1,2,3,4)_{\text{doubled}} - \frac{st}{3u} \text{Box}(1,2,3,4)_{\text{doubled}},
 \end{aligned} \tag{3.8}$$

where $d\omega_1$ and $d\omega_2$ are appropriate total derivatives; their precise form is not important for our purposes. The dot placed on a propagator indicates that the propagator is doubled, i.e., squared. This choice is convenient because the two integrals with doubled propagators are both ultraviolet finite in $D = 5$.

For this simple example, one can solve this system of equations for two of the three ultraviolet-divergent integrals. Plugging in the solution leaves only a single ultraviolet-divergent integral whose coefficient must vanish, if the amplitude is finite. However, the ability to express A in eq. (3.6) in terms of a basis of manifestly finite integrals is a consequence of the simplicity of this example, and for more complicated amplitudes this straightforward approach will not suffice. We will therefore take a more general approach for this example. In particular, we can use eq. (3.8) to rewrite the crossed box integral as

$$\begin{aligned}
 \text{Box}(1,2,3,4) &= \alpha \left(-\frac{70}{su} \text{Sunset}(1,2,3,4) - \frac{70}{tu} \text{Sunset}(1,2,3,4)_{\text{doubled}} + \frac{st}{3u} \text{Box}(1,2,3,4)_{\text{doubled}} \right) \\
 &+ (1 - \alpha) \left(\frac{70}{s^2} \text{Sunset}(1,2,3,4) + \frac{1}{3s^2} \text{Box}(1,2,3,4)_{\text{doubled}} \right) + d\left((1 - \alpha)\omega_1 + \alpha\omega_2 \right),
 \end{aligned} \tag{3.9}$$

where α is a free parameter. In this way we traded one ultraviolet-divergent integral for two ultraviolet-divergent sunset integrals which were already in the basis, plus two other finite integrals and a collection of integrals that vanish (i.e., are total derivatives). Plugging this back into the original expression for A gives

$$\begin{aligned}
 A &= \left(\frac{1 - \alpha}{s^2} - \frac{\alpha}{su} - \frac{1}{2s^2} \right) \text{Sunset}(1,2,3,4) - \left(\frac{\alpha}{tu} + \frac{1}{2t^2} \right) \text{Sunset}(1,2,3,4)_{\text{doubled}} \\
 &+ \text{finite} + \frac{1}{70} d\left((1 - \alpha)\omega_1 + \alpha\omega_2 \right),
 \end{aligned} \tag{3.10}$$

where “finite” corresponds to integrals that are manifestly ultraviolet finite with finite coefficients and the term $\frac{1}{70}d(\dots)$ vanishes upon integration. For general α this form of A

is still not manifestly finite, but since α is arbitrary we can take it to be $\alpha = -u/2t$, in which case the coefficients to the two sunsets both vanish and A is then manifestly a sum of finite integrals and integrals that vanish. In general, one free parameter will not be enough to tune away two coefficients of ultraviolet-divergent integrals. For more complicated examples one needs to generate more IBP relations and introduce more tunable parameters, and in general each parameter can be used to set one coefficient to an ultraviolet-divergent integral to zero.

As a nontrivial example, we carried out this procedure for the $(-+++)$ two-loop amplitude of half-maximal supergravity in $D = 5$. (Recall that the helicity labels refer to the helicities of the pure Yang-Mills side of the double copy, with the external states restricted to live in a four-dimensional subspace.) The structure of this amplitude is much more complicated than the $(++++)$ case and more representative of generic cases. In the first step we reduce the full integrand to a basis of master integrals using Larsen and Zhang’s method [43]. After this procedure the only contributing ultraviolet-divergent integrals are the three different labels of the sunsets and a few others. We then used these types of over-complete relations to express all of the (non-sunset) ultraviolet-divergent integrals in terms of ultraviolet-divergent sunset integrals, finite integrals and total derivatives that integrate to zero. The tunable parameters are solved so that coefficients of the three sunsets vanish separately, while maintaining finiteness of the coefficients of all finite integrals. Therefore, by allowing for an over-complete basis and tuning the parameters that keep track of this over-completeness, we are able to write the amplitude in the desired form, eq. (3.2).

We note that unless special care is taken, an IBP identity in general involves doubled propagators, as in eq. (3.9). This has the unwanted side effect of introducing spurious infrared singularities even in $D = 5$. With more modern approaches [36–43] we can avoid the appearance of such integrals. This is achieved by imposing

$$\sum_j v_j^\mu \frac{\partial}{\partial \ell_j^\mu} D_k = f_k D_k, \tag{3.11}$$

on the v_j^μ and where f_k has polynomial dependence on Lorentz-invariant dot products of momenta. We have also applied the more modern approach and find similar results.

The procedure sketched above shows that the $D = 5$ two-loop four-point integrand of half-maximal supergravity can be rewritten in a form that is manifestly finite, up to terms that integrate to zero. However, this procedure relies on the specific details of the integrand and corresponding IBP relations. It is also computationally difficult to extend to higher loops. Clearly, we need an approach where the necessary identities can be derived from generic properties of loop integrals. We will describe such an approach in the next section.

4 Vacuum expansion and systematics of ultraviolet cancellations

In this section we describe a systematic approach to understanding enhanced cancellations, in a manner that appears to have an all-loop generalization. We continue to focus on the two-loop amplitudes of half-maximal supergravity. The ultraviolet behavior is determined at the integrand level by large values of loop momenta, or equivalently small external

momenta. It is therefore natural to series expand the integrand in this limit. Although this expansion has the unwanted effect of losing contact with the unitarity cuts and introducing spurious singularities, such as doubled propagators, it does have the important advantage of focusing on the term directly relevant for the ultraviolet behavior. In general, we are interested in the logarithmic divergences, so we series expand to the appropriate order where the integrals become logarithmically divergent in ultraviolet [14, 66, 67]. (We note that while dimensional regularization does not have direct access to power divergences, such divergences become logarithmic simply by lowering the dimension.) This expansion generates a set of vacuum integrals. For example, at two loops these integrals have the form

$$\int d^D p d^D q \frac{\mathcal{N}(p, q, k_i)}{(p^2)^A (q^2)^B ((p+q)^2)^C}, \tag{4.1}$$

where A, B and C denote the powers of the propagators. In addition to being ultraviolet divergent, these vacuum integrals also are infrared divergent. This complicates the extraction of the ultraviolet divergences. For example, in dimensional regularization these integrals are scaleless, and the infrared singularities exactly cancel the ultraviolet ones. This is usually dealt with by introducing a mass regulator or by injecting external momentum into the diagram. (See, for example, refs. [14, 44, 66, 67].) We will avoid this complication by systematically finding relations between the divergences of the integrals using integration by parts.

As noted in the previous section, the simplest example to analyze is the case where the external gluons in the pure Yang-Mills side of the double-copy are restricted to live in four dimensions, and correspond to all-plus helicity (+ + + +). For this helicity configuration on the pure Yang-Mills side of the double copy, we use the spinor-helicity integrands in eq. (2.12) and (2.13). For the remaining helicity configurations we used the pure Yang-Mills integrand from ref. [68]. The only contributions needed are those whose color structure matches those of the planar and nonplanar double-box diagrams. For other helicities we used the gauge-invariant projection method to be described in ref. [69].

In four-dimensions these integrals do not have overall ultraviolet divergences because they are suppressed by the numerators; they are proportional to the (-2ϵ) -dimensional components of loop momenta. (They do however contain subdivergences which cancel.) To have a nontrivial example, we turn to the same integrand but with the internal states in $D = 5$. In this case the numerator is not suppressed because λ_p and λ_q are one-dimensional. (In the context of dimensional regularization in $D = 5 - 2\epsilon$, they are actually $(1 - 2\epsilon)$ dimensional.) Using $D = 5$ properties the integrand simplifies: in $D = 5$ the λ_p and λ_q become one-dimensional so that

$$(\lambda_p \cdot \lambda_q)^2 - \lambda_p^2 \lambda_q^2 \rightarrow \mathcal{O}(\epsilon), \tag{4.2}$$

in eq. (2.12) and (2.13).

In the large loop-momentum limit, the logarithmically divergent terms in $D = 5$ are given by

$$I^{\text{P, NP}} = (D_s - 2)s \int d^D p d^D q \frac{(\lambda_p^2 \lambda_q^2 + \lambda_p^2 \lambda_{p+q}^2 + \lambda_q^2 \lambda_{p+q}^2)}{(p^2)^A (q^2)^B [(p+q)^2]^C} + \text{UV finite}, \tag{4.3}$$

where

$$(A, B, C) = \begin{cases} (3, 3, 1), & \text{P: planar double box,} \\ (3, 2, 2), & \text{NP: nonplanar double box.} \end{cases} \quad (4.4)$$

In the planar case there are power divergences coming from terms proportional to $(p+q)^2$, which removes the middle propagator generating a product of one-loop integrals. Such terms do not give rise to logarithmic divergences. (This is consistent with finiteness of such integrals in dimensional regularization, which is sensitive only to logarithmic divergences.) We may then ignore such terms for the purposes of trying to understand overall two-loop logarithmic divergence.

One way to evaluate eq. (4.3) is to consider vacuum integrals with numerators that are polynomial in $v_j \cdot p$ and $v_j \cdot q$, where the v_j 's are a set of orthonormal basis vectors for the five-dimensional momentum space. We have

$$v_5 \cdot p = \lambda_p, \quad v_5 \cdot q = \lambda_q, \quad \sum_j (v_j \cdot p)(v_j \cdot p) = p^2, \quad \sum_j (v_j \cdot q)(v_j \cdot q) = q^2, \quad (4.5)$$

with appropriate factors of i inserted for the metric signature. Lorentz invariance then implies

$$\text{UV finite} = \int d^D p d^D q v_i^{[\mu} v_j^{\nu]} \left(p_\mu \frac{\partial}{\partial p^\nu} + q_\mu \frac{\partial}{\partial q^\nu} \right) \frac{\mathcal{N}(v_k \cdot p, v_k \cdot q)}{(p^2)^A (q^2)^B [(p+q)^2]^C}, \quad (4.6)$$

where the Lorentz indices μ and ν are antisymmetrized. By replacing \mathcal{N} in the above equation by all possible monomials in $v_i \cdot p$ and $v_i \cdot q$ up to degree four, we generate linear relations between vacuum integrals with different numerators, allowing us to reduce eq. (4.3) to scalar vacuum integrals. The result of this procedure is

$$\begin{aligned} I^{\text{P, NP}} &= \frac{3}{70} (D_s - 2) s \int d^D p d^D q \frac{[(p^2)^2 + (q^2)^2 + ((p+q)^2)^2]}{(p^2)^A (q^2)^B [(p+q)^2]^C} \\ &= \frac{3}{70} (D_s - 2) s (I_{A-2, B, C} + I_{A, B-2, C} + I_{A, B, C-2}), \end{aligned} \quad (4.7)$$

where the scalar vacuum integrals are defined as

$$I_{A, B, C} = \int d^D p d^D q \frac{1}{(p^2)^A (q^2)^B [(p+q)^2]^C}, \quad (4.8)$$

which is invariant under the six permutations of $\{A, B, C\}$. One can also obtain this equation by reducing the implicit tensor integrals in eq. (4.3), using Lorentz invariance in the more traditional way following for example eq. (4.18) of ref. [44]. Alternatively, Mastrolia et. al. recently proposed an efficient algorithm to integrate away loop momentum components orthogonal to all external momenta [70].

For the particular cases of eq. (4.7) we obtain

$$\begin{aligned} I^{\text{P}} &= \frac{3s}{70} (D_s - 2) (I_{1,3,1} + I_{3,1,1} + I_{3,3,-1}) + \text{UV finite} \\ &= \frac{3s}{70} (D_s - 2) (2I_{3,1,1} + I_{3,3,-1}) + \text{UV finite}, \end{aligned} \quad (4.9)$$

$$I^{\text{NP}} = \frac{3s}{70} (D_s - 2) (I_{1,2,2} + I_{3,0,2} + I_{3,2,0}) + \text{UV finite}, \quad (4.10)$$

where we used the fact that the integrals are invariant under the exchange of p and q in the second equality in eq. (4.9). Summing the planar and nonplanar contributions, we conclude that the logarithmic UV divergence is given by

$$(I^P + I^{NP})|_{\log UV} = \frac{3s}{70}(D_s - 2)(2I_{3,1,1} + I_{1,2,2})|_{\log UV}. \quad (4.11)$$

As explained above, the terms with “one-loop squared” propagator structures (e.g., $I_{3,2,0}$ or $I_{3,3,-1}$) do not contain logarithmic UV divergences. Also, it is not surprising that the final result is a linear combination of $I_{3,1,1}$ and $I_{1,2,2}$, as these are the only two possible logarithmically divergent vacuum integrals in $D = 5$.

By explicit evaluation using a uniform internal mass m as an infrared regulator and dimensional regularization in $5 - 2\epsilon$ dimensions as an ultraviolet regulator, we find

$$\begin{aligned} I_{3,1,1}|_{UV \text{ div.}} &= -\frac{\pi}{192\epsilon}, \\ I_{1,2,2}|_{UV \text{ div.}} &= \frac{\pi}{96\epsilon}, \end{aligned} \quad (4.12)$$

so the combination of integrals in eq. (4.11) is ultraviolet finite in $D = 5$. However, in order to understand the general structure of the cancellations, it is illuminating to instead show this using IBP identities.

4.1 Extracting divergences using IBP identities

We recall that the fundamental assumption of the IBP method is that the integral of a total derivative vanishes in dimensional regularization, as shown in eq. (3.5). Obviously, integrals of total derivatives only vanish when boundary contributions vanish. In dimensional regularization however, we can consider the integral in a dimension where the boundary contribution is vanishing and then analytically continue the result (zero) to the original dimension. But in another regularization scheme one has to consider the behavior of boundary terms. In particular, if the boundary term contains ultraviolet or infrared divergences itself, the corresponding IBP identity cannot be used to relate the divergences of the integrals.

On the other hand, dimensional regularization is known to regulate the ultraviolet and infrared simultaneously. In general this is very convenient, but this fact might obstruct the use of certain IBPs in this scheme for extracting ultraviolet divergences. The reason for this is that IBP identities in dimensional regularization can mix up ultraviolet and infrared poles. To illustrate this consider the following identity that relates bubble and triangle integrals in $D = 4$:

$$d\omega = s\epsilon \times \left(\begin{array}{c} 2 \\ \diagup \quad \diagdown \\ \triangle \\ \diagdown \quad \diagup \\ 1 \end{array} + \begin{array}{c} 2 \quad 3 \\ \diagdown \quad \diagup \\ \circlearrowleft \\ \diagup \quad \diagdown \\ 1 \quad 4 \end{array} \right), \quad (4.13)$$

where ω is not relevant for the discussion. The internal propagators are all massless. The triangle integral has only an infrared divergence with a $1/\epsilon^2$ pole and the bubble has

only an ultraviolet divergence with a $1/\epsilon$ pole. The ϵ dependence in the coefficient of the triangle allows the infrared and ultraviolet divergences to mix. In order to directly extract ultraviolet divergences without introducing an explicit infrared cutoff (such as a mass) we must make sure that the IBPs being used do not mix infrared and ultraviolet poles. These subtleties are pertinent to our discussion since our aim is to extract ultraviolet divergences by focusing on scaleless vacuum integrals, which vanish in dimensional regularization.

However, IBP identities that avoid both of the above complications can be directly used to give relations between the ultraviolet divergences of different dimensionally-regularized vacuum integrals without introducing an additional explicit infrared cutoff. In this way we can demonstrate ultraviolet cancellations without explicitly evaluating any integrals. The situation in the presence of subdivergences is more subtle and outside the scope of our present discussion. We note that our principal aim is to examine the loop order where ultraviolet divergences might first occur, so subdivergences are not of primary concern.

Consider the following identities between two-loop vacuum integrals

$$\begin{aligned}
 \text{UV finite} &= \int d^D p d^D q \left(p^\mu \frac{\partial}{\partial p^\mu} - q^\mu \frac{\partial}{\partial q^\mu} \right) \frac{1}{(p^2)^A (q^2)^B ((p+q)^2)^C} \\
 &= (-2A + 2B) I_{A,B,C} - 2C I_{A-1,B,C+1} + 2C I_{A,B-1,C+1}, \\
 \text{UV finite} &= \int d^D p d^D q \left(p^\mu \frac{\partial}{\partial q^\mu} \right) \frac{1}{(p^2)^A (q^2)^B ((p+q)^2)^C} \\
 &= (-B + C) I_{A,B,C} - B I_{A-1,B+1,C} + B I_{A,B+1,C-1} \\
 &\quad + C I_{A-1,B,C+1} - C I_{A,B-1,C+1}, \\
 \text{UV finite} &= \int d^D p d^D q \left(q^\mu \frac{\partial}{\partial p^\mu} \right) \frac{1}{(p^2)^A (q^2)^B ((p+q)^2)^C} \\
 &= (-A + C) I_{A,B,C} - A I_{A+1,B-1,C} + A I_{A+1,B,C-1} \\
 &\quad + C I_{A,B-1,C+1} - C I_{A-1,B,C+1}.
 \end{aligned} \tag{4.14}$$

In any of the three above identities, we can easily write the integrand as a total derivative because the contributions arising from commuting the loop momenta past the derivatives vanish. As desired there is no explicit dependence on the dimension D . With $A+B+C = 5$, the above IBP identities relate logarithmically divergent integrals in $D = 5$.

With dimensional regularization (and a mass as infrared cutoff) there are no boundary terms, but here we allow more general regularization schemes, in which case there may be a ultraviolet finite boundary term on the left hand side of eqs. (4.14). As elaborated in the appendix, even in such schemes, boundary terms do not contain divergences and do not modify the relations. We therefore use eq. (4.14) as a direct relationship between the ultraviolet divergences of the vacuum integrals.

With $A = 1, B = C = 2$, the first equation in eqs. (4.14) provides the following relation between the leading overall divergences of the integrals

$$(I_{1,2,2} + 2I_{1,1,3} - 2I_{0,2,3}) \Big|_{\log \text{ UV}} = (I_{1,2,2} + 2I_{1,1,3}) \Big|_{\log \text{ UV}} = 0, \tag{4.15}$$

where we used the fact that $I_{0,2,3}$ is a ‘‘one-loop squared’’ integral with power divergences and no logarithmic divergence. This is consistent with the explicit results in eq. (4.12),

while allowing us to expose cancellations in eq. (4.11) without computing divergences of individual integrals or using identities that depend on details of the integrand.

In addition, by starting with the Yang-Mills integrand from ref. [68] to construct the half-maximal supergravity integrand via eq. (2.16), we have checked that for *any* external state, the log divergences in $D = 5$ are always proportional to the same combination as above,

$$(I_{1,2,2} + 2I_{3,1,1}), \tag{4.16}$$

whose leading log divergence vanishes.

While dimensional regularization is not sensitive to the potential quadratic divergences in $D = 5$, we can study these divergences by lowering the dimension to $D = 4$. In $D = 4$ one finds that for any helicity configuration h the expanded amplitude is

$$\mathcal{A}_h = C_h (2I_{3,3,-2} - 11I_{3,2,-1} + 7I_{3,1,0} + 5I_{2,2,0}) + \text{UV finite}, \tag{4.17}$$

for some coefficient C_h depending on the external states and on choices made for reference momenta when choosing external polarizations. We constructed the required integrand by starting from two-loop four-point Feynman diagrams for pure-Yang-Mills and then applied to double-copy procedure to generate the diagrams of half-maximal supergravity. These are then expanded large loop momentum and simplified using Lorentz symmetry to obtained eq. (4.17). We apply the identities (4.14) to the $D = 4$ case, under the logarithmic power-counting requirement $A + B + C = 4$, with A, B, C chosen to be all possible combinations of integers (some of which may be negative) with some cutoff on their absolute values. Dozens of IBP identities are generated, and the resulting linear system relates all integrals to $I_{1,2,2}$. In this way, we obtain cancellation of the divergences of eq. (4.17) for the vacuum expansion of the $\mathcal{N} = 4$ supergravity amplitude.

Thus, we see that the two-loop cancellations in $D = 4$ and $D = 5$ can be understood entirely and systematically using IBP identities.

4.2 Generalizations and an all-loop conjecture

In general, the structure of IBP equations can be rather opaque. Might there be a simple organizing principle that applies to all loop orders? A strong hint is that the subset of IBP identities given in eq. (4.6) follows from Lorentz symmetry. We also saw the key role that Lorentz symmetry played at one loop in section 2. The obvious L -loop extension is

$$\text{UV finite} = \int \left(\prod_{a=1}^L d^D \ell_a \right) v_i^{[\mu} v_j^{\nu]} \sum_{a=1}^L \ell_{a\mu} \frac{\partial}{\partial \ell_a^\nu} \frac{\mathcal{N}(\ell_a \cdot v_b, \ell_a \cdot \ell_b)}{\prod_j D_j^{A_j}}, \tag{4.18}$$

where the ℓ_a are an independent set of loop momenta to be integrated, the v_a a set of external vectors in the problem and the $1/D_j$ the propagators in the diagram. As noted earlier, we can equivalently apply Lorentz invariance following the methods in refs. [44, 70].

What about the identities in eq. (4.14)? These can be understood as belonging to a special class of IBP identities generated by $SL(2)$ transformations of the loop momenta of the form

$$\begin{pmatrix} p \\ q \end{pmatrix} \rightarrow e^\omega \begin{pmatrix} p \\ q \end{pmatrix}, \tag{4.19}$$

$$\text{UV finite} = \frac{1}{2} \left(\text{Diagram 1} + 2 \text{Diagram 2} - \text{Diagram 3} - \text{Diagram 4} \right)$$

Figure 5. A four-loop relation between ultraviolet divergences of vacuum integrals in $D = 11/2$ dimensions, matching identity 22 from table I in appendix C of ref. [44]. Where a black dot appears, the propagator is raised to a squared power.

with some traceless 2×2 matrix ω . Since such an $\text{SL}(2)$ transformation leaves the integration measure $d^D p d^D q$ invariant, we have

$$\text{UV finite} = \int d^D p d^D q \omega_{ab} \ell_a^\mu \frac{\partial}{\partial \ell_b^\mu} \frac{1}{(p^2)^A (q^2)^B [(p+q)^2]^C}, \quad (4.20)$$

where we used the notation $(\ell_1, \ell_2) = (p, q)$. We can rewrite this as an IBP relation,

$$\text{UV finite} = \int d^D p d^D q \frac{\partial}{\partial \ell_b^\mu} \frac{\omega_{ab} \ell_a^\mu}{(p^2)^A (q^2)^B [(p+q)^2]^C}, \quad (4.21)$$

due to ω_{ab} being traceless. This also shows that these relations do not have explicit dependence on the spacetime dimension D .

In particular, the IBP identity which come from the first equation in (4.14) used to exhibit the cancellation of the logarithmic divergence in $D = 5$ is given by the $\text{SL}(2)$ generator,

$$\omega_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (4.22)$$

In fact, the above ideas generalize trivially to the L -loop case by considering generators of $\text{SL}(L)$. In more generality, the combination of Lorentz invariance and $\text{SL}(L)$ transformations gives rise to some subset of $\text{SL}(DL)$ transformations. As a nontrivial check that these ideas provide the key relations between the ultraviolet divergences of vacuum integrals, we have reproduced the relations between ultraviolet divergences of four-loop vacuum integrals in appendix C of ref. [44] in the context of obtaining the four-loop ultraviolet divergence for $\mathcal{N} = 8$ supergravity in the critical dimension, $D = 11/2$. One example of such a relation is given graphically in figure 5. This shows that Lorentz and $\text{SL}(4)$ symmetry generates a complete set of IBP identities necessary for reducing the vacuum integrals encoding the ultraviolet divergence to an independent set. (We know the set is independent from eq. (4.15) of ref. [44].) In this case there were no enhanced cancellations, but had they been present they would have been found after applying the identities.

This brings us to a conjecture:

- Given a loop integrand, homogeneous linear transformations of the loop momentum variables with unit Jacobian are sufficient for revealing enhanced cancellations of potential ultraviolet divergences in gravity theories.

Generally, we are interested in the first divergence of a theory in a given dimension so we do not need to concern ourselves with complications due to subdivergences or divergences beyond the logarithmic ones. Even if the cancellations are not complete and an ultraviolet divergence remains we expect these symmetries to generate a complete set of IBP identities for studying logarithmic divergences.

If this conjecture were to hold in general, it would shed light on the mysterious enhanced cancellations that have been observed in various supergravity theories. Furthermore, these transformations can be connected to the labeling difficulty of nonplanar integrands. Remarkably, even though there does not seem to be a single “discrete” relabeling of the integration variables for each diagram that allows us to construct an integrand that would manifest the cancellations, the freedom to change integration variables appears to be at the root of the cancellations.

5 Conclusions

In this paper we took initial steps towards systematically understanding enhanced ultraviolet cancellations in supergravity theories [8–10]. These cancellations go beyond those presently understood from standard-symmetry argumentation [11–14] and therefore appear to require novel explanations.

While a different avenue for understanding enhanced cancellations based on exploiting the double-copy structure of gravity theories has been successful for the special case of half-maximal supergravity in $D = 5$ [10], it is unclear how to extend that argument beyond two loops. In contrast, our large loop-momentum analysis here relies only on generic properties of the integrands and integrals.

In nonabelian gauge theories, standard methods including superspace techniques can be used to expose ultraviolet cancellations at the integrand level. One might have thought that it is possible to similarly find organizations of multi-loop integrands of supergravity theory. However, as we showed via one- and two-loop examples, it does not seem possible to do this without relying also on integration properties.

The simplest example of an enhanced cancellation in a supergravity theory is probably the vanishing of one-loop divergences in pure $\mathcal{N} = 4$ supergravity in four dimensions. While the cancellation of the divergence in $D = 4$ is well understood as a consequence of supersymmetry [46], the pattern of cancellation amongst the diagrams serves as a prototype for enhanced cancellations. The double-copy construction [25] allowed us to obtain the $\mathcal{N} = 4$ supergravity integrand very easily from the corresponding ones of pure-Yang-Mills and $\mathcal{N} = 4$ super-Yang-Mills theory. Even in this relatively simple case where there are no labeling ambiguities, we found that the cancellations cannot be exposed at purely the integrand level. After using integral identities that follow from Lorentz invariance, the cancellations become visible.

We also investigated the more interesting case of half-maximal supergravity at two loops. In $D = 5$, no standard symmetry explanation is known for the cancellation that removes the logarithmic divergence [10, 11]. We showed that the three-particle cuts display no integrand-level cancellations, even though the final integrated expression does display

the cancellations. Based on our considerations, purely integrand-based proofs of the observed enhanced cancellations do not appear to be possible.

In order to systematize ultraviolet cancellations after integration, we used integration-by-parts identities [30–35]. This gives a systematic means for finding all relations between the different integrals. While the machinery of doing so is generally difficult to apply at high loop orders, at two-loops we made use of various advances for controlling the complexity of the identities [36–43]. As an example we showed that one can use these ideas to rearrange the full integrands of amplitudes so that they consist of terms that are manifestly finite as well as terms that integrate to zero. While this construction is a proof of principle and gives some insight into how the cancellations happen, it is too dependent on details of the integrands and the associated identities to be useful for developing an all-orders understanding.

To develop such an understanding, we instead focused on the large loop-momentum behavior of the integrands. For the two-loop $\mathcal{N} = 4$ supergravity amplitude, by series expanding at large loop momentum, we demonstrated that the only identities needed to expose the cancellation are those that follow from Lorentz and an $SL(2)$ symmetry. Using these principles we also reproduced the necessary four-loop identities [44] for extracting the ultraviolet divergence of $\mathcal{N} = 8$ in the critical dimension where it first appears, suggesting that we have identified the key identities.

This led us to conjecture that at L loop order the integral identities generated by Lorentz and $SL(L)$ symmetry are sufficient for exposing the enhanced cancellations of ultraviolet divergences, when they happen. If generally true, it would point towards a symmetry explanation of enhanced cancellations.

There are a number of avenues for further exploration. It would be important to first explicitly confirm our conjecture for the known three- and four-loop examples of enhanced ultraviolet cancellations [8, 9], and to develop an all-loop understanding. It would also be interesting to study whether this set of integral identities is also applicable to more general problems in QCD and other theories that involve extracting ultraviolet divergences. It may also turn out to be helpful for efficiently obtaining the required integration-by-parts identities for analyzing divergences in $\mathcal{N} = 8$ supergravity at five loops and beyond, once the integrands become available [71].

We expect that in the coming years, as new theoretical tools are developed, a complete and satisfactory understanding of enhanced ultraviolet cancellations in gravity theories will follow.

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A Boundary terms in logarithmically divergent IBPs

In section 4 we claimed that for logarithmically divergent integrals even in schemes other than dimensional regularization, the boundary contributions of the IBP relations do not alter the relation between the divergences. Here we demonstrate this. This is relevant to our discussion because it supports the notion that the required IBP relations to obtain the cancellations of the studied logarithmic divergences are robust and do not depend on details of the scheme.

First, recall that the vacuum expansion to logarithmically divergent integrals, the IBPs are of the form,

$$\int \prod_i d^D \ell_i \frac{\partial}{\partial \ell_j^\mu} \left(\frac{\ell_k^\mu \prod_a N_a^{B_a}}{\prod_b D_b^{A_b}} \right), \quad (\text{A.1})$$

where the powers A_b and B_a of the propagators $1/D_b$ and irreducible numerators N_a are such that the integrals are logarithmically divergent. Consider ultraviolet regularization after Wick rotation using a physical cut off Λ , under which the right-hand-side of eq. (A.1), as a total divergence, is turned into a boundary integral at the compact cutoff surface by Stokes' theorem. Since the number of propagators makes the integral logarithmically divergent, the boundary integral also has mass dimension 0. In Wilson's floating cutoff picture, a change in the cutoff Λ does not change the boundary integral, which precludes it from having an ultraviolet divergence. Note that the above argument breaks down if we consider, e.g. quadratically divergent IBP relations. This argument is equivalent to the textbook explanation of the finiteness of anomalies in one-loop diagrams given by a boundary term of a linearly divergent integral [72].

However, there is an extra subtlety at higher loops that does not arise in the study of anomalies. The argument cannot be trivially extended to the case where there are subdivergences because there is no longer just one UV divergence coefficient to be fixed by a single floating cutoff. However, this is of secondary concern because usually we are interested in studying the very first potential divergence of a supergravity theory. (There are some subtleties with evanescent effects feeding into divergences which require some care [2, 3].) The most interesting cases, such as $\mathcal{N} = 8$ supergravity at five loops in $D = 24/5$, automatically have no subdivergences because of a lack of lower-loop divergences. It would be nevertheless interesting to understand the behavior of boundary terms in general and study whether the relations generated by Lorentz and $SL(L)$ symmetry can be applied to more general problems of extracting divergences from vacuum integrals in the presence of subdivergences.

We also comment on the dimensional regularization, which requires a mass regulator to separate out infrared singularities. One might worry that this mass regulator might interfere with the IBP identities. However, it is easy to argue that when there are no subdivergences the mass regulator does not cause any issues. To prevent IBP identities from mixing up ultraviolet and infrared poles, infrared divergences can be regulated by introducing a uniform mass m to every propagator on the right-hand-side of eq. (A.1). It is best to introduce the mass prior to vacuum expansion to retain cancellations of subdivergences [66, 67]. After series expanding in small external momentum, we again obtain a sum

of logarithmically divergent vacuum integrals (whose internal propagators are regulated by the uniform mass), but we also obtain additional vacuum integrals multiplied by factors of m^2 . To have the correct dimensions, these additional integrals must have negative mass dimension and are power-counting finite in the ultraviolet. Assuming there are no one-loop subdivergences, a naive power counting is sufficient for establishing the lack of ultraviolet divergence. Therefore we obtain relations between logarithmic ultraviolet divergences of massive vacuum integrals. Furthermore, there is a smooth limit when the dimension D tends to a fixed integer (or a fractional number in more exotic cases), while the mass m tends to zero, because our special IBP identities have no D dependence and because leading logarithmic ultraviolet divergences are mass-independent. So we end up with relations between logarithmic ultraviolet divergences of massless vacuum integrals. This argument is applicable whenever dimensional regularization rules out lower-loop subdivergences, for example for supergravity calculations in fractional dimensions (see e.g., ref. [44]). We note that ref. [73] also investigated well-defined limits of IBP identities as the dimension tends to an integer, in the different context of studying finite integrals.

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