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Some exact Bradlow vortex solutions

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Abstract: We consider the Bradlow equation for vortices which was recently found by Manton and find a two-parameter class of analytic solutions in closed form on nontrivial geometries with non-constant curvature. The general solution to our class of metrics is given by a hypergeometric function and the area of the vortex domain by the Gaussian hypergeometric function.

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1 Introduction

Vortices are codimension-two solitons and are most famous for their existence in type II superconductors in the presence of an external magnetic field, where they carry quantized fluxes through the superconducting material. The effective theory for the latter effect is the Ginzburg-Landau theory or — from the point of view of the soliton — equivalently the Abelian Higgs model. The type II vortices are described by two partial differential equations (PDEs) — one for the scalar field (order parameter) and one for the gauge field — as well as a parameter $\beta = m_h/m_\gamma > 1$, which is the ratio of the scalar mass to the photon mass. Critically coupled vortices, also called BPS vortices, have $\beta = 1$ and their two PDEs can be reduced to a single PDE which is known as the Taubes equation [1]

$$-\nabla^2 u = 1 - e^{2u} - (\text{delta functions}),$$

where $u = \log|\phi|$ is the logarithm of the modulus of the order parameter. Although the existence and uniqueness have been proven for the Taubes equation [1], no analytic solutions are known on the flat plane, $\mathbb{R}^2$. On the hyperbolic plane, $\mathbb{H}^2$, however, Witten found exact vortex solutions [2][1] by adjusting the constant negative curvature of the hyperbolic plane so as to effectively cancel the constant (vacuum expectation value of the scalar field) in the Taubes equation. This way both the hyperbolic plane and the vortex scalar field are described by Liouville's equation. Only a few other analytic vortex solutions are known

[1] Witten considered instanton solutions by splitting $\mathbb{R}^4$ into $\mathbb{H}^2 \times S^2$ and constructed a nontrivial flux on the hyperbolic plane.

(see e.g. ref. [3]) and one property they have in common — to the best of our knowledge — is that they exist on manifolds of constant Gaussian curvature.

Other vortex equations of similar type to Taubes equation are the Jackiw-Pi equation [4, 5] and the Popov equation [6]; both of which possess exact analytic solutions and again on manifolds of constant Gaussian curvature. The Jackiw-Pi equation comes about naturally in some non-relativistic Chern-Simons theories [4, 5, 7], whereas the Popov equation describes vortices on a 2-sphere that are realized as Yang-Mills instantons on a manifold $S^2 \times \mathbb{H}^2$ (the vortices here are situated on the 2-sphere, whereas in Witten's solution they are situated on the hyperbolic plane). In a recent paper, Manton considered extending the vortex equations to nine different types — five of which can possess vortices — and thereby found two new ones [8]; one of them was dubbed the Bradlow equation and the other remained an unnamed equation. The unnamed equation is, however, not really new as it was found by Ambjørn and Olesen in refs. [9, 10][2] and it describes W-condensation giving rise to a periodic vortex lattice in the electroweak theory. We will thus call the equation the Ambjørn-Olesen-Manton equation. The fact that it resembles the normal vortex equation but with opposite sign of the last term can be interpreted as the effect of anti-screening of the W-boson as opposed to the normal screening effect in Abelian theories [9, 10]. In the flat plane, $\mathbb{R}^2$, the Ambjørn-Olesen-Manton equation does not have a single vortex solution, but instead a lattice of vortices. This can also be seen from the fact that the equation does not possess a (constant) vacuum state, i.e. a value of the field where the Laplacian vanishes. The equation does however have fixed points which are described by analytic solutions presented in ref. [8]. Finally, the Ambjørn-Olesen-Manton equation has also been suggested recently to play a role in a non-Abelian vector bootstrap mechanism generating a primordial magnetic field [11].

The other new vortex equation — the Bradlow equation — is remarkably simple as it simply equates the magnetic flux with a constant. The vortex field is thus energetically absent, but its zeros still specify the positions of the vortices inside this constant magnetic field. The name of the Bradlow equation was coined in ref. [8] due to the similarity of the equation with the Bradlow bound [12], which however was formulated for the Taubes equation and limits the number of vortices that can exist on a compact manifold. Although applications for the Bradlow equation may not seem immediate, we would like to consider this as an approximation to a physical system. Bose-Einstein condensates with constant magnetic fields exist experimentally [13] and vortices — global vortices (i.e. ungauged vortices) — are created in such a way that the magnetic field is constant even around the vortices, and so perhaps the Bradlow equation can be used as a rough description in this case.

In this paper, we consider the Bradlow equation and construct analytic solutions in closed form for a two-parameter family of metrics with non-constant Gaussian curvature; these are the first analytic solutions on manifolds with a non-constant curvature. The solutions are rather simple due to the fact that the Bradlow equation is also rather simple and they consist of the vortex positions, an overall normalization factor and a nontrivial function whose solution is given in terms of a hypergeometric function. We also tie together

[2] We thank P. Olesen for pointing this out.

C_0	C	name	analytic solutions on
-1	-1	Taubes	$\mathbb{H}^2$
0	1	Jackiw-Pi	$\mathbb{R}^2$, T^2
1	1	Popov	S^2
-1	0	Bradlow	$\mathbb{H}^2$
-1	1	Ambjørn-Olesen-Manton	$\mathbb{H}^2$

Table 1. Vortex equation constants C_0 and C for five different theories.

a physical aspect of the system, namely matching the magnetic flux with the vortex number. In the case of the Taubes equation, this is automatic, but for the Bradlow equation, a restriction on the domain (area) is necessary for the double integral of the equation to hold.

The paper is organized as follows. In section 2 we introduce the Bradlow equation in perspective to the other possibilities of the same type of equation. In section 3 we introduce a toy model giving rise to the Bradlow equation and discuss the boundary conditions. Then in section 4 we first solve the Bradlow equation on a flat disc and then present the main result of the paper, namely the solutions on curved manifolds with non-constant curvature. Section 5 concludes with a discussion and outlook. Finally, in the appendix, we contemplate the uniqueness in the cases of the modified vortex equations, useful also for the Bradlow equation.

2 The Bradlow equation

We start by introducing the class of vortex equations to which the Bradlow equation belongs and from which it was discovered. They are given by [8],

$$-\frac{1}{\Omega_0}\nabla^2 u = -C_0 + Ce^{2u} - \frac{2\pi}{\Omega_0}\sum_{i=1}^{N}\delta^{(2)}(z-z_i), \tag{2.1}$$

where the constants C_0 and C are given in table 1, $\{z_i\}$, $i = 1, \ldots, N$, are vortex positions, and N is the total number of vortices. The manifold on which the vortices and the above equations are defined is denoted by M_0 and has the metric factor $\Omega_0(z, \bar{z})$, where the metric is defined by

$$ds^2 = dt^2 - \Omega_0\left[(dx^1)^2 + (dx^2)^2\right] = dt^2 - \Omega_0 dz d\bar{z}, \tag{2.2}$$

the complex coordinate is defined as $z = x^1 + ix^2$, $\Omega_0^{-1}\nabla^2$ is the covariant Laplacian on M_0 and $\nabla^2 = \partial_{x^1}^2 + \partial_{x^2}^2 = 4\partial_{\bar{z}}\partial_z$. As explained in ref. [8], an overall factor of $\{C_0, C\}$ can be scaled away in eq. (2.1) by rescaling the metric factor Ω_0 while C can independently be scaled by a shift in u and thus nine distinct equations are given by C_0, C taking a value in $\{-1, 0, 1\}$. Of the nine possibilities, four do not allow for a vortex with zeroes and a positive magnetic flux.

We will now review the Bradlow integral relation for the generic case [8] by integrating eq. (2.1) to

$$2\pi N = -C_0 A + C \int_{M_0} d^2x\, \Omega_0 e^{2u}, \tag{2.3}$$

where N is the vortex number and A is the area of M_0. The left-hand side of eq. (2.3) is the topological number, which we will take to be finite here; therefore two situations arise: if the area A is infinite, this puts a constraint on the field u to reach the vacuum

$$u \to u_\infty = \frac{1}{2} \log \frac{C_0}{C}, \tag{2.4}$$

at a specific rate (as does the equation of motion of course); this equation is not valid for the Ambjørn-Olesen-Manton equation, but it is for all the other equations. In the cases where either C_0 or C vanishes, the above vacuum expectation value should be taken as the limit of the latter constant being sent to zero.

If, on the other hand, the area A is finite (M_0 is compact), then three cases arise:

- $C_0 = -1$: this is the normal Bradlow bound for the Taubes case: we get

$$N \leq \frac{A}{2\pi}, \qquad N = \frac{A}{2\pi}, \qquad N \geq \frac{A}{2\pi}, \tag{2.5}$$

  for the Taubes, the Bradlow and the Ambjørn-Olesen-Manton equation, respectively, as C is $-1, 0, 1$, respectively.

- $C_0 = 0$: in this case, the equation directly relates the vortex number and the integral of the scalar field (squared)

$$N = \frac{1}{2\pi} \int_{M_0} d^2x\, \Omega_0 e^{2u}. \tag{2.6}$$

  Note: in this case, C cannot vanish and since we take the vortex number to be positive, $C = 1$ is the only possibility: i.e. the Jackiw-Pi equation.

- $C_0 = 1$: in this case the area cannot be too large and $C = 1$ is required to get a positive vorticity; we can write an upper bound for the area

$$A < \int_{M_0} d^2x\, \Omega_0 e^{2u}. \tag{2.7}$$

  This upper bound for the area is valid for the Popov equation.

In the remainder of the paper we will concentrate on the Bradlow equation

$$-\nabla^2 u = \Omega_0 - 2\pi \sum_{i=1}^{N} \delta^{(2)}(z - z_i), \tag{2.8}$$

for which the topological vortex number, N, is related to the area of the manifold or integration domain as

$$N = \frac{A}{2\pi}. \tag{2.9}$$

In the next section we will illustrate a toy model and set up the boundary conditions.

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3 Toy model and boundary conditions

It will prove instructive to define a prototype theory giving rise to the Bradlow equation. Let us define the following static energy

$$\begin{aligned} E &= \int_{M_0} d^2x\ \Omega_0 \left\{ \frac{1}{2e^2\Omega_0^2} F_{12}^2 + \Omega_0^{-1}|D_a\phi|^2 + \Omega_0^{-1}|\phi|^2 F_{12} + \frac{1}{2}e^2v^4 \right\} \\ &= \int_{M_0} d^2x\ \Omega_0 \left\{ \frac{1}{2e^2}\left(\Omega_0^{-1}F_{12} + e^2v^2\right)^2 + \Omega_0^{-1}|D_1\phi + iD_2\phi|^2 - i\Omega_0^{-1}\epsilon^{ab}\partial_a(\bar{\phi}D_b\phi) \right\} \\ &\quad - v^2 \int_{M_0} d^2x\ F_{12}, \end{aligned} \tag{3.1}$$

where $F_{12} = \partial_1 A_2 - \partial_2 A_1 = B$ is the magnetic field in the plane, $D_a = \partial_a + iA_a$, is the (gauge) covariant derivative, A_a is the Abelian (U(1)) gauge field, the indices a, b run over $1, 2$; ϕ is a complex scalar field and $e > 0$, $v > 0$ are constants, respectively, gauge coupling constant and vacuum energy (cosmological constant).

Let us emphasize that this theory is just a toy model and we take the Bradlow equation as the defining equation. Therefore any other theory giving rise to the Bradlow equation can be considered equally valid.

Working with the above stated theory, we can read off the Bogomol'nyi equations as

$$D_1\phi + iD_2\phi \equiv 2D_{\bar{z}}\phi = 0, \tag{3.2}$$

$$-\frac{1}{\Omega_0}F_{12} = m^2, \tag{3.3}$$

where we have defined $m \equiv ev > 0$. In addition to the above equations, if we impose that $D_a\phi \to 0$ at the boundary of M_0 such that the total derivative in the energy vanishes, then the total energy is

$$E = v^2m^2A = e^2v^4A, \tag{3.4}$$

where A is the total area of M_0: if M_0 is compact, then the total energy is finite, if not then it is infinite.

Let us start by solving the first BPS equation, i.e. eq. (3.2),

$$D_{\bar{z}}\phi = \partial_{\bar{z}}\phi + iA_{\bar{z}}\phi = \big(-\partial_{\bar{z}}\log s + iA_{\bar{z}}\big)s^{-1}\phi_0, \qquad \Rightarrow A_{\bar{z}} = -i\partial_{\bar{z}}\log s, \tag{3.5}$$

where we have defined

$$\phi(z,\bar{z}) \equiv s^{-1}(z,\bar{z})\phi_0(z), \tag{3.6}$$

where $s(z,\bar{z})$ is everywhere regular and $\phi_0(z)$ is holomorphic and contains all zeros of the field ϕ. Calculating now the field strength, we get

$$F_{12} = 2iF_{\bar{z}z} = -2\partial_{\bar{z}}\partial_z \log|s|^2 = 2\partial_{\bar{z}}\partial_z\log|\phi|^2 - 2\pi\sum_{i=1}^N \delta^{(2)}(z - z_i), \tag{3.7}$$

which by insertion into the other BPS equation (3.3) yields the Bradlow equation

$$-\frac{1}{\Omega_0}\nabla^2 u = 1 - \frac{2\pi}{\Omega_0}\sum_{i=1}^N \delta^{(2)}(z - z_i), \tag{3.8}$$

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where $\nabla^2 = 4\partial_{\bar{z}}\partial_z$, we have defined $u \equiv \frac{1}{2}\log|\phi|^2$, and rescaled the mass squared, m^2, into the conformal factor of the metric, $\Omega_0 \to \Omega_0/m^2$.

Let us consider carefully what assumptions go into the statement that the vortex number is proportional to the area of M_0, see eq. (2.9). Inserting eq. (3.6) into eq. (3.8), we get

$$\frac{1}{2\Omega_0}\nabla^2 \log|s|^2 = 1, \tag{3.9}$$

where $s(z,\bar{z})$ is everywhere regular. If we now integrate the above equation over M_0, we get

$$\begin{aligned} A &= \frac{1}{2}\int_{M_0} d^2x\ \nabla^2 \log|s|^2 = i\int_{M_0} dz \wedge d\bar{z}\ \partial_{\bar{z}}\partial_z \log|s|^2 = -i\oint_{\partial M_0} dz\ \log|s|^2 \\ &= -i\oint_{\partial M_0} dz\ \log|z|^{2k} = 2\pi k, \end{aligned} \tag{3.10}$$

where we have used Green's theorem and in the second line we have assumed the boundary condition

$$\lim_{z\to z_{\partial M_0}} |s|^2 = |z|^{2k}, \tag{3.11}$$

where $z_{\partial M_0}$ is the coordinate lying on the boundary of M_0. Hence we see that the area of the manifold M_0 is related to the winding of the gauge field A_a.

Let us first discuss the case of M_0 being noncompact; in particular take $M_0 = \mathbb{R}^2$. We will assume that the vortex under study is topological, which means that

$$\lim_{|z|\to\infty} |\phi| = \mathrm{const} > 0, \tag{3.12}$$

and hence we get that $|s^{-1}\phi_0| \sim |z|^{N-k}$ forces $N = k$.[3] We will then consider the contribution to the energy (3.1) from the total derivative term. In this case ($M_0 = \mathbb{R}^2$) we need to impose the condition that $D_a\phi$ goes to zero at the boundary of M_0 so that the total derivative term in the energy (3.1) vanishes. Since $D_{\bar{z}}\phi = 0$ everywhere, this means that we need to impose the condition

$$0 = \lim_{|z|\to\infty} D_z\phi = \left(-\partial_z \log|s|^2 + \partial_z \log\phi_0\right) s^{-1}\phi_0 \sim \left(-\frac{k}{z} + \frac{N}{z}\right) s^{-1}\phi_0. \tag{3.13}$$

Note that from eq. (3.10), we get that for an infinite area, k is also infinite and therefore, for the above covariant derivative to vanish at the boundary of M_0, we need $N = k$. Thus, we again get $k = N$ and hence we come to the conclusion that

$$2\pi N = A, \tag{3.14}$$

for M_0 being the flat infinite plane, $\mathbb{R}^2$. Note that N is required to be infinite due to the infinite area. The fact that the gauge field (formally) winds N times and the scalar field has exactly N zeros (counted with multiplicity, i.e. they may be coincident) is in fact very natural for vortices. Nevertheless, depending on the underlying theory, this is not strictly necessary in the case of the Bradlow vortices, whereas in the case of the Taubes

[3] If $N < k$, $\lim_{|z|\to\infty}|\phi| = 0$ and this is called a non-topological vortex.

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equation, the vacuum of the scalar field also forces $|s| = |\phi_0|$ at the boundary and hence $k = N$. Therefore in the case of the Bradlow vortex, one could in principle contemplate the situation where $k \neq N$, but that would imply that the covariant derivative of the scalar field cannot go to zero at the boundary of M_0 (for $N > k$, and the scalar field will diverge at infinity) or that the vortices are not topological (for $N < k$). In particular for $M_0 = \mathbb{R}^2$ and $N > k$, that would imply two separate diverging contributions to the energy in the toy model; one from the magnetic field and one from the total derivative term.

Let us now discuss a compact case, in particular the flat disc, $M_0 = \mathbb{D}^2$ with radius R. If we still demand that $D_z\phi = 0$ for $|z| = R$, then we get

$$-\partial_z \log |s|^2 + \partial_z \log \phi_0(z)\big|_{|z|=R} = -\partial_z \log |s|^2 + \partial_z \log |\phi_0(z)|^2\big|_{|z|=R} = \partial_z u|_{|z|=R} = 0, \tag{3.15}$$

where in the second equality we have used the holomorphicity of $\phi_0(z)$ to add $\log \overline{\phi_0(z)}$ in the derivative. This condition, however, cannot in general be satisfied by solutions to the Bradlow equation. Nevertheless, in the case of compact M_0, the boundary term in the energy (3.1) does not give a diverging contribution due to the finite size of M_0 and thus finite circumference. Therefore one boundary condition we can choose to impose on the disc is $u(R) = 0$. However, other boundary conditions may be equally reasonable, depending on the desired behavior of the solution on the boundary ∂M_0.

In view of the above discussion, we will restrict the rest of the paper to cases where M_0 is a compact manifold.

4 Bradlow vortex solutions

4.1 Bradlow vortices on a flat disc

In this section, we will gain some intuition by studying the Bradlow equation on the flat disc. On a disc with vanishing curvature ($\Omega_0 = 1$) and radius R, the Bradlow vortex is the solution to the Bradlow equation (2.8) and it has the analytic solution with axial symmetry

$$u = -\frac{r^2}{4} + u_0 + \frac{N}{2} \log r^2, \tag{4.1}$$

where $r = |z|$ is the radial coordinate, u_0 is a constant and N is the vortex number. Let us impose the boundary condition that $u(R) = 0$, yielding

$$u = \frac{R^2 - r^2}{4} + \frac{N}{2} \log \frac{r^2}{R^2}, \tag{4.2}$$

where we have adjusted u_0 to make u match its boundary conditions.

Allowing for the vortices to have generic positions, the most general solution is

$$u = -\frac{|z|^2}{4} + u_0 + \frac{1}{2}\sum_{i=1}^{N} \log |z - z_i|^2 + g(z) + \overline{g(z)}, \tag{4.3}$$

which reduces to eq. (4.1) when all $z_i = 0$ and g are set to zero. Imposing the boundary condition $u(R) = 0$ is however highly nontrivial in the this general case. The constant

u_0 can still be fixed to $R^2/4$ so that $u(R) = 0$ except for the contribution due to the logarithms. Naively to cancel the contribution coming from the logarithms, one would guess that subtracting

$$\frac{1}{2}\sum_{i=1}^{N}\log|Re^{i\theta}-z_i|^2 = \frac{1}{2}\sum_{i=1}^{N}\log\left|\frac{Rz}{|z|}-z_i\right|^2, \tag{4.4}$$

from the solution would make it satisfy the boundary condition $u(R) = 0$. However, the above term does not vanish when acted upon by the Laplacian due to the non-holomorphic $1/|z|$ necessary for giving the phase factor $e^{i\theta}$. As we can see from the general solution (4.3), we can adjust u_0 and $g(z)$ to make the solution satisfy the boundary condition $u(R) = 0$. However, a holomorphic function which cancels the contribution due to the logarithms at $|z| = R$ will also cancel the logarithmic singularity defining the vortex center. Hence we make the following conjecture.

Conjecture 1 *The only solution satisfying the Bradlow equation (2.8) on the flat disc, $\mathbb{D}^2$, with a finite radius $R < \infty$ and the boundary condition $u(R) = 0$, is the axially symmetric solution (4.2) where all $z_i = 0$, $\forall i$.*

Of course other boundary conditions than $u(R) = 0$ may be chosen. An alternative, is to choose the boundary conditions such that the contribution due to the metric in the Bradlow equation vanishes at the boundary of the disc, but neglecting the nontrivial function due to the logarithms. From this point of view, one can choose the positions of the vortices, but the solution on the boundary does not satisfy any simple boundary conditions. We can write such a solution as

$$u = \frac{R^2-|z|^2}{4} + \frac{1}{2}\sum_{i=1}^{N}\log|z-z_i|^2. \tag{4.5}$$

One can come arbitrarily close to the boundary condition $u(R) = 0$ if the size of the disc is parametrically larger than the distance from the vortices to the center of the disc, in particular, $R \gg |z_i|$, $\forall i$. In this case, we can fix u_0 as follows

$$u = \frac{R^2-|z|^2}{4} + \frac{1}{2}\sum_{i=1}^{N}\log\frac{|z-z_i|^2}{R^2}. \tag{4.6}$$

In this case at the boundary, we have

$$u(|z|=R) = \frac{1}{2}\sum_{i=1}^{N}\log\left|\frac{z}{|z|}-\frac{z_i}{R}\right|^2 \simeq -\frac{1}{2}\sum_{i=1}^{N}\left[\frac{z_i|z|}{Rz}+\frac{\bar{z}_i|z|}{R\bar{z}}+\mathcal{O}\left(\frac{|z_i|^2}{R^2}\right)\right], \tag{4.7}$$

and hence for R parametrically bigger than $|z_i|$ the discrepancy between the above solution with the boundary condition $u(R)$ can become arbitrarily small. Nevertheless, for any finite size disc, the only solution strictly satisfying $u(R) = 0$ is conjectured to be eq. (4.2), see Conjecture 1.

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Since the vortex number is related to the area of the disc by the Bradlow equation (2.8), the radius of the disc is determined as

$$N = -\frac{1}{2\pi}\int_{\mathbb{D}^2} d^2x\left(\nabla^2 u - 2\pi\sum_{i=1}^{N}\delta^{(2)}(z-z_i)\right) = \frac{1}{2\pi}\int_{\mathbb{D}^2} d^2x = \frac{1}{2}R^2, \tag{4.8}$$

and so we have

$$R = \sqrt{2N}. \tag{4.9}$$

For completeness, we evaluate the boundary term in the toy model (3.1) for the solution (4.6)

$$\begin{aligned} -i\int_D d^2x\, \epsilon^{ab}\partial_a(\bar{\phi}D_b\phi) &= 2\int_D d^2x\, \partial_{\bar{z}}(e^{2u}\partial_z u) \\ &= -i\oint_{\partial D} dz\, e^{\frac{R^2-|z|^2}{2}}\prod_{j=1}^{N}\frac{|z-z_j|^2}{R^2}\left(-\frac{\bar{z}}{4}+\sum_{i=1}^{N}\frac{1}{2(z-z_i)}\right). \end{aligned} \tag{4.10}$$

Notice that the conformal factor of the metric drops out and the result does not depend on the metric (but only on the shape of the integration domain). For $N=1$, the expression is fairly simple and yields

$$2\pi\left[\left(1+\frac{|z_1|^2}{R^2}\right)\left(-\frac{R^2}{4}+\frac{1}{2}\right)-\frac{|z_1|^2}{2R^2}\right] = -\frac{\pi|z_1|^2}{2}, \tag{4.11}$$

where we have used the Bradlow integral relation $R=\sqrt{2N}$ in the last equality. One could interpret this result as the boundary attracting the single vortex and only at the center of the disc the attraction cancels out (i.e. it becomes isotropic). For general vortex positions, the contribution (4.10) is in general a complicated function. However, setting $z_i = 0$, we get the axially symmetric solution (4.2) for which the boundary contribution reduces to

$$-i\oint_{\partial D} dz\left(-\frac{R^2}{4z}+\frac{N}{2z}\right) = 2\pi\left(-\frac{R^2}{4}+\frac{N}{2}\right) = 0, \tag{4.12}$$

where we have used the Bradlow integral relation $R=\sqrt{2N}$. Let us emphasize that the boundary term depends on the explicit choice of underlying theory and this is just a concrete example of the toy model studied in section 3.

4.2 Bradlow vortices on M_0 with nontrivial metric

In this section we extend the Bradlow vortex solutions to nontrivial manifolds of nonvanishing and non-constant curvature. Let us consider metrics of the form

$$ds^2 = dt^2 - \Omega_0(|z|^2)dzd\bar{z}, \tag{4.13}$$

where the conformal factor, Ω_0 depends only on the modulus (squared) of the complex coordinate $|z|^2$. From the Bradlow equation (2.8) we can see that the solution u has two contributions; one is the logarithmic terms corresponding to the vortex positions (zeros of

e^{2u}) and the other is the inverse Laplacian of the conformal factor Ω_0. Formally, we can write the solutions as

$$u = u_0 - F(|z|^2) + \frac{1}{2}\sum_{i=1}^{N} \log|z - z_i|^2 + g(z) + \overline{g(z)}, \tag{4.14}$$

where the function F is the solution to

$$\nabla^2 F = \Omega_0. \tag{4.15}$$

Now imposing the boundary condition $u(z_{\partial M_0}) = 0$, i.e. u vanishes at the boundary of M_0 is highly nontrivial. If we choose M_0 to be a disc with conformal factor Ω_0 and radius R, then we can again use Conjecture 1 to set all $z_i = 0$ and hence the solution satisfying the boundary condition $u(R) = 0$ reads

$$u = F(R^2) - F(|z|^2) + \frac{N}{2}\log\frac{|z|^2}{R^2}. \tag{4.16}$$

4.3 Solutions for a class of metrics

Let us now consider a class of metrics of the form

$$ds^2 = dt^2 - \kappa^{-1}(1 \pm |z|^{2k})^{\ell} dz d\bar{z}, \tag{4.17}$$

with $k \in \mathbb{Z}_{>0}$ a positive definite integer, $\ell \in \mathbb{Z}$ is an integer, $\kappa \in \mathbb{R}_{>0}$ is a real positive-definite constant. For the lower sign, the coordinate z is defined within the unit circle: $|z| < 1$ and for the upper sign $z \in \mathbb{C}$. $\ell = 0$ corresponds to the flat disc $\mathbb{D}^2$. Another special case is $\ell = -2$ and $k = 1$ which is the 2-sphere S^2 for the upper sign and the hyperbolic plane $\mathbb{H}^2$ for the lower sign, both with constant Gaussian curvature. To see this, let us calculate the Gaussian curvature for this manifold

$$K_0 = -\frac{1}{2\Omega_0}\nabla^2 \log\Omega_0 = \mp\frac{2\kappa\ell k^2|z|^{2k-2}}{(1 \pm |z|^{2k})^{\ell+2}}. \tag{4.18}$$

As promised, $\ell = -2$ renders the denominator constant and $k = 1$ the numerator constant, yielding a constant positive (negative) Gaussian curvature of 4κ (-4κ) for the upper (lower) signs. If $k > 1$ then the curvature vanishes at the origin but is non-vanishing away from it. If $k = 1$ then only the denominator influences the curvature and hence increases (decreases) the curvature with increasing radii $|z|$ for $\ell > 0$ or $\ell < -2$ (for $\ell = -1$) for both signs, while the curvature is constant and vanishing for $\ell = 0$ and just constant for $\ell = -2$.

The analytic solutions to the Bradlow equation in this case are thus given by eq. (4.14) with the F-function for this class of metrics, i.e.,

$$F^{(\ell,k)} = \frac{|z|^2}{4\kappa}\,{}_3F_2\left[k^{-1}, -\ell, k^{-1}; 1 + k^{-1}, 1 + k^{-1}; \mp|z|^{2k}\right], \tag{4.19}$$

where ${}_3F_2$ is a hypergeometric function.

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In some cases, we can write F as a (finite) sum of fractions. As a good check, let us first consider the solution for $\ell = -2$ and $k = 1$ for which we know that the Gaussian curvature is constant. In that case, we get

$$F^{(-2,1)} = \pm\frac{1}{4\kappa}\log(1 \pm |z|^2), \tag{4.20}$$

as one would expect [8] (in the latter reference $\kappa = 1/4$). Note that this function does not have any singularities inducing more vortices as $|z| < 1$ for the lower sign.

Another sanity check of the solution (4.19) is $\ell = 0$ for which it reduces to

$$F^{(0,k)} = \frac{|z|^2}{4\kappa}, \tag{4.21}$$

which is the solution for the flat disc $\mathbb{D}^2$, see eq. (4.3) (the latter equation corresponds to $\kappa = 1$).

Other families of solutions that can be written as fractions are, for $\ell = 1$:

$$\kappa F^{(1,k)} = \frac{|z|^2}{4} \pm \frac{|z|^{2k+2}}{4(1+k)^2}, \tag{4.22}$$

and for $\ell = 2$:

$$\kappa F^{(2,k)} = \frac{|z|^2}{4} \pm \frac{|z|^{2k+2}}{2(1+k)^2} + \frac{|z|^{4k+2}}{4(1+2k)^2}, \tag{4.23}$$

and for generic $\ell \geq 1$:

$$F^{(\ell\geq 1,k)} = \frac{|z|^2}{4\kappa}\sum_{p=0}^{\ell}\begin{pmatrix}\ell\\ p\end{pmatrix}\frac{(\pm 1)^p|z|^{2pk}}{(1+pk)^2}. \tag{4.24}$$

Finally, for $\ell = -1$ we can write the solution as

$$F^{(-1,k)} = \frac{|z|^2}{4\kappa k^2}\Phi\left[\mp|z|^{2k}, 2, k^{-1}\right] = \frac{|z|^2}{2\kappa}\sum_{p=0}^{\infty}\frac{(\pm 1)^p|z|^{2pk}}{(1+pk)^2}, \tag{4.25}$$

where Φ is the Hurwitz-Lerch transcendent. The right-most expression is only well defined for $|z| < 1$.

4.4 Flux matching

We have found the analytic solution in closed form for the class of metrics (4.17); however, from a physical point of view, we should still make sure that the number of vortices N specified with positions by the delta functions in eq. (2.8) match the magnetic flux, i.e. that

$$2\pi N = \int_{M_0} d^2x\ \Omega_0 = A, \tag{4.26}$$

holds, which is simply eq. (2.9). By Green's theorem the magnetic flux is giving the vortex number, N, but in order for it to match the area of the manifold, M_0, the above equation needs to be imposed as well. For the upper sign (in the solutions) with $\ell \geq 0$ or for the lower sign with $\ell < 0$ this just restricts the size of the domain where the vortices have support.

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For the upper sign with $\ell < 0$ or for the lower sign with $\ell > 0$ this relation can restrict the number of vortices similarly to the Bradlow bound, depending on the values of ℓ and k.

Let us calculate the area for the metrics (4.17) in the case of an axially symmetric domain (for simplicity)

$$A^{(\ell,k)} = \frac{2\pi}{\kappa}\int_0^R dr\; r(1 \pm r^{2k})^\ell = \frac{\pi R^2}{\kappa}{}_2F_1\left[k^{-1}, -\ell; 1+k^{-1}; \mp R^{2k}\right], \tag{4.27}$$

where ${}_2F_1$ is the Gaussian hypergeometric function. As a consistency check, we can set $\ell = 0$ and verify that

$$A^{(0,k)} = \frac{\pi R^2}{\kappa}, \tag{4.28}$$

as it should for a flat disc, $\mathbb{D}^2$. We can again simplify the hypergeometric function in cases of positive ℓ; in particular for $\ell = 1$:

$$A^{(1,k)} = \frac{\pi R^2}{\kappa}\left(1 \pm \frac{R^{2k}}{1+k}\right), \tag{4.29}$$

and for $\ell = 2$:

$$A^{(2,k)} = \frac{\pi R^2}{\kappa}\left(1 \pm \frac{2R^{2k}}{1+k} + \frac{R^{4k}}{1+2k}\right), \tag{4.30}$$

and for generic $\ell \geq 1$:

$$A^{(\ell,k)} = \frac{\pi R^2}{\kappa}\sum_{p=0}^{\ell}\binom{\ell}{p}\frac{(\pm 1)^p R^{2pk}}{1+pk}. \tag{4.31}$$

As mentioned above, for the upper sign with $\ell \geq 0$, this area just fixes the radius in terms of the vortex number by eq. (4.26), but for the lower sign with $\ell > 0$ it can limit the number of vortices possible.

Let us consider the lower sign with the radius $R = 1 - \epsilon$, where ϵ is an infinitesimal real number. In this case, we can expand the Gaussian hypergeometric function to get

$$A^{(\ell,k)} = -\frac{\pi^2 R^2 \csc(\pi\ell)\Gamma\left(1+k^{-1}\right)}{\kappa\Gamma(-\ell)\Gamma\left(1+k^{-1}+\ell\right)}(1+2\epsilon) + \epsilon^\ell\left(-\frac{2^{1+\ell}k^\ell \pi r^2}{\kappa(1+\ell)}\epsilon + \mathcal{O}(\epsilon^2)\right). \tag{4.32}$$

We can see that if $\ell < -1$ then the second term diverges and thus yields an unlimited area (as expected). In the case of $\ell = -1$, $\csc(-\pi)$ is ill-defined (infinite) and the area is again infinite. We thus confirmed that for the lower sign with $\ell < 0$, the area is unlimited for $R < 1$. However, for $\ell \geq 0$, the area renders finite and as a few examples we get for $R = 1$

$$A^{(0,k)} < \frac{\pi}{\kappa}, \tag{4.33}$$

$$A^{(1,k)} < \frac{\pi}{\kappa}\frac{\Gamma(1+k^{-1})}{\Gamma(2+k^{-1})}, \tag{4.34}$$

$$A^{(2,k)} < \frac{2\pi}{\kappa}\frac{\Gamma(1+k^{-1})}{\Gamma(3+k^{-1})}, \tag{4.35}$$

and for general $\ell \geq 0$:

$$A^{(\ell\geq 0,k)} < \frac{\ell!\pi}{\kappa}\frac{\Gamma(1+k^{-1})}{\Gamma(\ell+1+k^{-1})}. \tag{4.36}$$

Since the right-hand side is a monotonically increasing function with k, the hardest restriction happens when $k = 1$ for which we can write

$$A^{(\ell\geq 0,1)} < \frac{\ell!\pi}{\kappa}\frac{\Gamma(2)}{\Gamma(2+\ell)} = \frac{\pi}{\kappa(1+\ell)}. \tag{4.37}$$

We can write this as a Bradlow bound

$$N < \frac{1}{2\kappa(1+\ell)}, \tag{4.38}$$

which requires $\kappa < 1/2$ to allow for a single vortex and even smaller for $\ell > 0$. The biggest areas we can get from eq. (4.36) is by sending $k \to \infty$ for which we get

$$A^{(\ell\geq 0,\infty)} < \frac{\pi}{\kappa}, \tag{4.39}$$

yielding the Bradlow bound

$$N < \frac{1}{2\kappa}. \tag{4.40}$$

In case of the upper sign and $\ell < 0$, the area is also finite and limits the vortex number. Let us consider $\ell = -2$, for which we get

$$A^{(-2,k)} = \left(1-\frac{1}{k}\right)\frac{\pi}{\kappa}\Gamma\left(1+\frac{1}{k}\right)\Gamma\left(1-\frac{1}{k}\right) = \frac{k-1}{k^2}\pi^2\csc\left(\frac{\pi}{k}\right). \tag{4.41}$$

This area is maximal for the two limits: $k = 1$ and $k \to \infty$: both yielding

$$A^{(-2,1)} = A^{(-2,\infty)} = \frac{\pi}{\kappa}, \tag{4.42}$$

where the first is the area of the 2-sphere ($\kappa = 1/4$ corresponds to the unit 2-sphere) and this in turn gives the Bradlow bound

$$N \leq \frac{1}{2\kappa}. \tag{4.43}$$

The most restricting bound is obtained for the smallest area of the function (4.41), which is for $k = 2$:

$$A^{(-2,2)} = \frac{\pi^2}{4\kappa}, \tag{4.44}$$

and in turn the Bradlow bound

$$N \leq \frac{\pi}{8\kappa}. \tag{4.45}$$

For $k = 2$ to the limit $k \to \infty$, the area (4.41) grows monotonically with k.

We have thus shown that vortices can exist for any finite κ when $\ell < 0$ for the lower sign and for small enough κ and $\ell \geq 0$, again for the lower sign. For the upper sign, there is no restriction on the vortex number when $\ell \geq 0$, but for $\ell < 0$ the Bradlow bound again limits the vortex number; again vortices can only exist for small enough κ.

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5 Discussion

In this paper we have constructed a two-parameter family of new analytic solutions to the newly discovered Bradlow equation for a special kind of vortices. The derivation of the equation relies on the Bogomol'nyi trick and thus gives a single second order PDE for the vortices; this implicitly means that they are critically coupled [8]. From the same action giving rise to said equations, the vortex scalar field does not contribute to the energy; only the magnetic field and the constant corresponding to the vacuum expectation value of the scalar field appear. We would like to think of this as a system in which the magnetic field dominates and in the same time contains vortices that are energetically negligible. If such system — if only approximately — can be realized experimentally, our solutions may find use there. Bose-Einstein condensates (BECs) with constant magnetic fields can be realized experimentally by trapped ultracold atomic gases, for which these magnetic fields are optically synthesized although the trapped atoms are neutral [13]; if the magnetic fields are larger than a critical value, vortices — but global vortices — are created, where the magnetic field remains constant even in the presence of vortices, which is in good agreement with the Bradlow equation. Then, the question is whether the vortices contribute negligibly to the total energy. Although this may not be true for BECs, we hope that it may be described approximately by the Bradlow equation. It is also possible that a potential trapping atoms may be designed to have minima on a curved two-dimensional surface so that a curved space is realized. Finally, it is perhaps possible that the materials in experiment are genuinely curved; for this, some metric can easily be constructed (if not already in our class of metrics) and the Bradlow vortex can probably readily be calculated. Even if the flat metric is the one that finds use in any experimental setup, then we also provide such solution, for the first time.

In some sense the Bradlow vortex is somewhat similar to the interior of the large-winding Bolognesi vortex — up to a constant proportional to the vortex potential times the area [14]. The vortex condensates of these two systems, however, obey different dynamics, of course.

Finally, for the Taubes equation, a non-Abelian extension is possible which is most easily achieved by using the moduli matrix technique [15], for which the so-called the master equation reduces to the Taubes equation for the U(1) case. A natural question is whether there exists a non-Abelian extension for the case of the Bradlow equation or other types of equations mentioned in the introduction.

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A Uniqueness

Let us consider uniqueness of the vortex equations (2.1). Let us start by assuming that two *different* solutions to the same equation exist, $u_{1,2}$ and both having exactly the same vortex positions (same moduli) and both satisfy the same boundary condition (2.4) appropriate for the specific equation (except for the Ambjørn-Olesen-Manton equation, for which another boundary condition should be specified). We define

$$\delta u \equiv u_1 - u_2, \tag{A.1}$$

and subtract their two respective equations of motion which yields

$$-\frac{1}{\Omega_0}\nabla^2 \delta u = C e^{2u_2}\left(e^{2\delta u} - 1\right). \tag{A.2}$$

This equation is independent of C_0 and the delta functions present in eq. (2.1) also canceled out. Since the delta functions are gone, no logarithmic singularities can be present in the solution δu and since both $u_{1,2}$ obey the same boundary condition, $\delta u \to 0$ asymptotically or at the boundary of the manifold M_0.

A key observation is that since $u_2 \in \mathbb{R}$ is a real-valued field, e^{2u_2} is positive semi-definite and vanishes only at the vortex centers.

Let us consider a simplified situation where we locate all vortices at the origin of our manifold M_0 (in some coordinates) and to make sure that the vortices are not destroyed, we impose $\delta u = 0$ at the vortex position. Now it is clear that since eq. (A.2) yields a monotonic behavior for δu; more specifically

$$\operatorname{sign}\left[C\Omega_0 \delta u\right] = \begin{cases} +, & \delta u \text{ monotonically decreasing} \\ -, & \delta u \text{ monotonically increasing} \end{cases} \tag{A.3}$$

Then it is clear that no monotonically behaving function can satisfy $\delta u = 0$ at the boundary of M_0 and simultaneously $\delta u(z_i) = 0$, $\forall i$. A more rigorous proof can be carried out along the lines of Taubes' proof [1], which was made for the case $C_0 = C = -1$.

In the case of the Bradlow equation (2.8), $C = 0$ and hence we have that the covariant Laplacian of the perturbation δu on M_0, vanishes

$$\frac{1}{\Omega_0}\nabla^2 \delta u = 0. \tag{A.4}$$

It is clear that no regular nontrivial solution with $\delta u = 0$ at the boundary exists. Therefore, the Bradlow vortex is unique once the moduli and boundary conditions have been specified (the boundary conditions completely fixes the part of the homogeneous solution).

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