Surprises from the resummation of ladders in the ABJ(M) cusp anomalous dimension

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ABSTRACT: We study the cusp anomalous dimension in $\mathcal{N} = 6$ ABJ(M) theory, identifying a scaling limit in which the ladder diagrams dominate. The resummation is encoded into a Bethe-Salpeter equation that is mapped to a Schroedinger problem, exactly solvable due to the surprising supersymmetry of the effective Hamiltonian. In the ABJ case the solution implies the diagonalization of the U(N) and U(M) building blocks, suggesting the existence of two independent cusp anomalous dimensions and an unexpected exponentiation structure for the related Wilson loops. While consistent with previous perturbative analysis, the strong coupling limit of our result does not agree with the string theory computation, emphasizing a difference with the analogous resummation in the $\mathcal{N} = 4$ case.

Contents

1 Introduction and summary of the results

The duality between $\mathcal{N} = 6$ Super Chern-Simons theory with matter (also known as ABJ(M) theory [1, 2]) and string theory on $AdS_4 \times CP_3$ represents one the most interesting possibilities to explore AdS/CFT correspondence beyond the original paradigm. Although it seems to share many similarities with the cousin $\mathcal{N} = 4$ Super Yang-Mills theory in four dimensions, there are still many aspects calling for a better comprehension. Supersymmetric Wilson loops, in particular, provide a rich class of BPS observables [3]–[9] that can be computed exactly through localization technique in the simplest situations [10]. While their quantum behavior is still rather mysterious in the general case [11], the well-understood 1/2 BPS and 1/6 BPS circles exhibit an intriguing non-trivial interpolation between weak and strong coupling regime [12]–[15]. A careful study of the relevant matrix models [15] has also unveiled a variety of phenomena of string/M-theory origin [16]–[18]. This contrasts with the relative simplicity of $\mathcal{N} = 4$ SYM, where a gaussian matrix model [19, 20] describes exactly the dynamics of 1/2 BPS Wilson loops [21].

Wilson loops are also relevant in gauge theories (supersymmetric or not) because they encode important properties of scattering amplitudes and infrared radiation: the cusp anomalous dimension $\Gamma(\varphi)$, originally introduced in [22] as the ultraviolet divergence of a Wilson loop with Euclidean cusp angle φ, appears in fact in many interesting physical situations. It was computed in QCD to two-loop order in [23] and, in light-like limit $\varphi \to i\infty$, provides crucial universal informations [24]. In supersymmetric theories $\Gamma(\varphi)$ is not a BPS observable, making its exact computation in $\mathcal{N} = 4$ SYM a difficult challenge. In the light-like limit [25] integrability controls its all-order behavior through an integral equation, whose solution matches both weak coupling expansions [26] and string computations at the strong coupling [27]. In the general case a strategy [28, 29] for computing $\Gamma(\varphi)$ was later proposed (see [30–32] for the original introduction of the related TBA approach in the case of local operators). The cusp anomalous dimension can be generalized including an R-symmetry angle θ that distinguishes the coupling of the scalars to the two halves of the cusp [38]. The new observable $\Gamma(\varphi, \theta)$ interpolates between BPS configurations and generalized quark-antiquark potentials. Exact equations can be written applying integrability and they have been checked successfully at three loops [28, 39]. In the near-BPS limit it is possible to use localization to obtain the exact form of the so-called Bremsstrahlung function [40, 41], that has been later directly recovered from the TBA equations [42, 43]. More recently localization has been used to derive the Bremsstrahlung function in presence of local operator insertions [44] and in more general superconformal field theories [45].

As anticipated before, even for the simple circular 1/2 BPS Wilson loop the weak/strong interpolation for ABJ(M) theory is not non-trivial. Integrability itself has been explored here in a somehow limited range of situations [46]–[50]: when established it still depends on an elusive interpolating function, $h(\lambda)$ [51]–[53]. Recently a proposal for the functional form of $h(\lambda)$ has been advanced [54] and checked at two-loop level in string theory [55], under suitable assumptions. Alternatively $h(\lambda)$ could be determined by computing exactly some quantity by integrability and confronting with the same calculation by localization (or by other QFT techniques in which unknown functions are absent). A natural candidate would be the ABJM Bremsstrahlung function, for which all order proposals exist [56], but integrability has not been yet applied to its determination.

For this reason we think it is important to study $\Gamma(\varphi, \theta)$ in $\mathcal{N} = 6$ Super Chern-Simons theory: it would be useful to obtain exact QFT results to be compared with the integrability approach and, at strong coupling, with string theory. $\Gamma(\varphi, \theta)$ has been introduced in ABJ(M) theory in [57], where its computation at two-loop has been presented and its exponentiation properties discussed. The two halves of the cusp are locally 1/2 BPS and therefore couple also directly to the fermions of the theory, as originally discovered in [7], and not only to the gauge connections and scalars. The resulting cusped Wilson loop is not globally supersymmetric and its exact evaluation seems very challenging. Fortunately, the analogous system in $\mathcal{N} = 4$ SYM can be tackled in a particular limit through Feynman diagrams resummation. In [39] a new scaling limit involving the complexified angle θ was introduced,

$$i\theta \gg 1, \quad \lambda \ll 1, \quad \hat{\lambda} = \lambda \exp(i\theta/4) \text{ fixed.} \tag{1.1}$$

Here $\lambda = g^2 N$ is the 't Hooft coupling of $\mathcal{N} = 4$ SYM. In this limit the leading order contribution is simply given by ladder diagrams, where the rungs are made by scalar exchanges. The ladder diagrams can be summed up efficiently using a Bethe-Salpeter equation, solved exactly in the small φ limit. The strong coupling behavior has been also examined, finding agreement with the corresponding string theory calculation [39]. Later it was also performed an analysis at next-to-leading order, generalizing the original Bethe-Salpeter equation and computing the relevant corrections at strong coupling [58]. Remarkably these corrections have been also obtained from string theory and successfully compared each other. As repeatedly stressed in the original analysis [39], the matching of the strong coupling limit of the Bethe-Salpeter solution with the string theory computation is quite surprising. The ladders limit, $\lambda \to 0$ with $\hat{\lambda}$ fixed, is different from the strong coupling limit $\lambda \to \infty$ with $i\theta \gg 1$ fixed and the result could, in principle, depend on their order: nevertheless they agree at leading and subleading level.

In this paper we consider a similar limit in three-dimensional ABJ(M) theory,

$$i\theta \gg 1, \quad \lambda_r \ll 1, \quad \hat{\lambda}_r = \lambda_r \cos\frac{\theta}{2}, \quad (r = 1, 2) \tag{1.2}$$

obtaining some exact results for $\Gamma(\varphi, \hat{\lambda}_i)$. Here $\lambda_1 = \frac{N}{k}, \lambda_2 = \frac{M}{k}$ are the 't Hooft couplings of the ABJ(M) theory with gauge group $\mathrm{U}(N) \otimes \mathrm{U}(M)$, while k is the Chern-Simons level. The presence of fermionic couplings to the cusped loop inherits a surprising supersymmetric structure in the relevant Bethe-Salpeter equation. More precisely, the effective Schroedinger problem, associated to the integral equation that resums planar diagrams in $\mathcal{N} = 6$ Super Chern-Simons, enjoys an unexpected quantum mechanical supersymmetry. Because only the ground state matters in determining $\Gamma(\varphi, \hat{\lambda}_i)$ [39], supersymmetry produces an exact expression for *any* value of the opening angle φ. This is in sharp contrast with the $\mathcal{N} = 4$ case, where an analytic solution for the Bethe-Salpeter equation exists only at $\varphi = 0$ (that in this case is the only supersymmetric point of the associated Schroedinger equation [39]).

In the ABJM case ($N = M$) we get a very simple solution: $\Gamma(\varphi, \hat{\lambda})$ is exact at one-loop level, as in an abelian theory. The delicate balance between bosonic and fermionic contributions, encoded into the effective supersymmetric quantum mechanics, exponentiates without non-abelian correction the one-loop term. As a matter of fact we do not observe any transition between a weak-coupling and a strong-coupling regime and we cannot match our result with semiclassical computations in string theory, suggesting that it should exist a problem with the order of limits in this case.

In the ABJ case ($N \neq M$) the story is even more intriguing: the Bethe-Salpeter equation reduces to two coupled integral equations, resumming contributions from the upper ($N \times N$) and the lower ($M \times M$) blocks of the holonomy of the $\mathrm{U}(N|M)$ superconnection defining the Wilson loop [7]. By diagonalizing the system we end up with a two-dimensional supersymmetric Schroedinger equation. From the knowledge of its ground state energy and wave-function we reconstruct the original cusped Wilson loops. This works in full generality in three dimensions with explicit UV an IR cut-offs on the lines, while using dimensional regularization we get an explicit solution only for $\varphi = 0$. The main result of our investigation is that, in the ABJ case, the original cusped Wilson loop, the trace of the holonomy

of the superconnection defined in [7], does not renormalize multiplicatively but it mixes, at quantum level, with the supertrace. As a consequence we end up with two independent cusp anomalous dimensions, related to the renormalization constants of the operator eigenstates (with respect to the mixing). On the other hand we could have expected this fact from the very beginning: taking the trace of the holonomy of the superconnection is required to preserve the global supersymmetry of the 1/2 BPS Wilson loop [7], In our case supersymmetry is generically broken and we observe a mixing between the trace and (say) the supertrace. Again the cusp anomalous dimensions are exact at one-loop level

$$
\begin{aligned}
\Gamma_{\text{cusp}}^{(1)}(\varphi) &= \frac{\sqrt{\hat{\lambda}_1 \hat{\lambda}_2}}{\cos \frac{\varphi}{2}} \,, \\
\Gamma_{\text{cusp}}^{(2)}(\varphi) &= -\frac{\sqrt{\hat{\lambda}_1 \hat{\lambda}_2}}{\cos \frac{\varphi}{2}} \,.
\end{aligned}
\tag{1.3}
$$

We expect that subleading corrections should change the simple pattern $\Gamma_{\text{cusp}}^{(2)} = -\Gamma_{\text{cusp}}^{(1)}$: we will come back on this point in the conclusions.

The paper is structured in the following way: in section 2 we introduce the cusped Wilson loop in ABJ(M) theory as the trace of a superconnection, following [57]. The limit in which the ladder diagrams dominate is described in section 3, where we also discuss how to derive the cusp anomalous dimensions both in the explicit cut-off scheme and in dimensional regularization. In section 4 we derive the relevant Bethe-Salpeter equation and, after diagonalization, we obtain the associated Schroedinger equation. We solve the supersymmetric Schroedinger equations for generic opening angle φ, in the cut-off scheme, and for $\varphi = 0$ in dimensional regularization. In section 5 we obtain the cusp anomalous dimensions at leading order and discuss the operator mixing for the cusped Wilson loops. Our conclusions and the future directions to improve our results appear in section 6. Two appendices complete our presentation.

2 The 1/2 BPS generalized cusped Wilson line in ABJ(M) theory

In this section we review the construction of supersymmetric Wilson lines in ABJ(M) theory [7]. In $d = 3$ the generalized gauge connection can be defined in two different ways according to the degree of preserved supersymmetry. Indeed, we can consider a purely bosonic gauge connection whose holonomy is, for a suitable choice of the path, 1/6 BPS. On the other hand, adding on the lines local couplings to the fermions, we can interpret the Wilson operator as the holonomy of a U($N|M$) superconnection obtaining, for the infinite straight line, a 1/2 BPS operator. Its peculiar structure was also investigated via the so-called Higgsing procedure which gives a physical explanation for the appearance of the superconnection [8].

2.1 The 1/2 BPS Wilson line

In ABJ(M) theory the gauge sector consists of two gauge fields $(A_\mu)_i{}^j$ and $(\hat{A}_\mu)_{\hat{i}}{}^{\hat{j}}$ belonging respectively to the adjoint of U(N) and U(M). The matter sector contains the complex

fields $(C_I)_i{}^{\hat{i}}$ and $(\bar{C}^I)_{\hat{i}}{}^i$ as well as the fermions $(\psi_I)_{\hat{i}}{}^i$ and $(\bar{\psi}^I)_i{}^{\hat{i}}$. The fields $(C, \bar{\psi})$ transform in the $(\mathbf{N}, \bar{\mathbf{M}})$ of the gauge group $\mathrm{U}(N) \times \mathrm{U}(M)$ while the fields $(\bar{C}, \psi)$ live in the $(\bar{\mathbf{N}}, \mathbf{M})$. The additional capital index $I = 1, 2, 3, 4$ belongs to the R-symmetry group $\mathrm{SU}(4)$.

The central idea of [7] is to replace the $\mathrm{U}(N) \times \mathrm{U}(M)$ gauge connection with the super-connection

$$\mathcal{L}(\tau) \equiv -i \begin{pmatrix} i\mathcal{A} & \sqrt{\frac{2\pi}{\kappa}} |\dot{x}| \eta_I \bar{\psi}^I \\ \sqrt{\frac{2\pi}{\kappa}} |\dot{x}| \psi_I \bar{\eta}^I & i\hat{\mathcal{A}} \end{pmatrix} \quad \text{with} \quad \begin{cases} \mathcal{A} \equiv A_\mu \dot{x}^\mu - \frac{2\pi i}{\kappa} |\dot{x}| M_J{}^I C_I \bar{C}^J \\ \hat{\mathcal{A}} \equiv \hat{A}_\mu \dot{x}^\mu - \frac{2\pi i}{\kappa} |\dot{x}| \hat{M}_J{}^I \bar{C}^J C_I, \end{cases} \tag{2.1}$$

belonging to the super-algebra of $\mathrm{U}(N|M)$. In (2.1) the coordinates $x^\mu(\tau)$ define the contour of the loop operator, while $M_J{}^I$, $\hat{M}_J{}^I$ are the scalar couplings and η_I^α, $\bar{\eta}_\alpha^I$ are fermionic (2-components Grassmann even quantities) ones.

The form of $\mathcal{L}$ is determined mainly by dimensional analysis and symmetry properties of the fields. In $d = 3$ the scalars have classical dimension $1/2$, so they could only appear as bilinears, which are in the adjoint and therefore enter in the diagonal blocks together with the gauge fields. Instead the fermions have dimension 1 and should appear linearly. Since they transform in the bifundamental, they are naturally placed in the off-diagonal entries of the matrix.

For a given a path $\mathcal{C}$, it is possible to compute the holonomy of the superconnection (2.1)

$$W[\mathcal{C}] \equiv \mathcal{P} \exp\left(-i \int_{\mathcal{C}} \mathcal{L}(\tau) d\tau \right). \tag{2.2}$$

When the contour is a straight-line S, all the couplings can be chosen to be independent of τ, i.e. constant, in order to preserve the invariance under translations along the line. Further restrictions on scalar and fermionic couplings follow from $R-$symmetry and supersymmetry requirements. Actually, imposing $\delta_{\text{susy}} \mathcal{L}(\tau) = 0$ gives rise to loop operators which are merely bosonic ($\eta = \bar{\eta} = 0$) and at most $1/6$ BPS [11]. The weaker condition of invariance under supersymmetry up to a super-gauge transformation brings out the $1/2$ BPS solution. For (finite) closed path one has to carefully consider the boundary conditions obeyed by the gauge functions to obtain a (super-)gauge invariant object. For instance, in the circle case one has to take the trace of (2.2). For an infinite open circuit, such as the straight line, the naive statement that the fields vanish when $\tau = \pm\infty$ allows two possible supersymmetric operators

$$\begin{aligned} \mathcal{W}_- &= \frac{1}{N - M} \text{Str}\left[\text{Pexp}\left(-i \int d\tau \mathcal{L}(\tau) \right) \right], \\ \mathcal{W}_+ &= \frac{1}{N + M} \text{Tr}\left[\text{Pexp}\left(-i \int d\tau \mathcal{L}(\tau) \right) \right]. \end{aligned} \tag{2.3}$$

Usually one mainly considers the second possibility, since, for particular angles, it is connected through a conformal transformation to a BPS closed loop [57]. Nevertheless the supertraced holonomy has a crucial role as well.
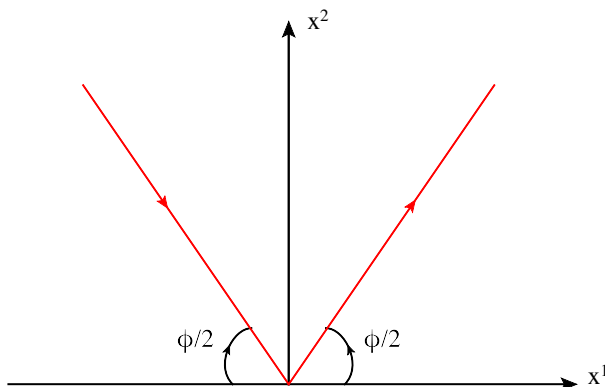
Figure 1. The planar cusp.

2.2 The generalized cusp

We consider the theory on the Euclidean space-time and take the contour depicted in figure 1. The two rays are in the plane $(1, 2)$, intersect at the origin and are given by

$$x^\mu = \left\{ 0, \tau \cos \frac{\varphi}{2}, |\tau| \sin \frac{\varphi}{2} \right\} \qquad -\infty \geq \tau \leq \infty \,. \tag{2.4}$$

The angle between the rays is $\pi - \varphi$, thus $\varphi = 0$ gives the continuous straight line.

It is natural to consider scalar and fermionic couplings different on the two segments of the cusp (but constant on each segment). The fermionic couplings have the factorized form

$$\eta^\alpha_{iM} = n_{iM} \eta^\alpha_i \qquad \text{and} \qquad \bar{\eta}^M_{i\alpha} = n^M_i \bar{\eta}_{i\alpha} \,, \tag{2.5}$$

where the index $i = 1, 2$ specifies which edge of the cusp we are considering. As discussed in [57] we can take:

$$\eta_{1\alpha} = \left(e^{-i\frac{\varphi}{4}} \;\; e^{i\frac{\varphi}{4}} \right) , \qquad \bar{\eta}_{1\alpha} = i \begin{pmatrix} e^{i\frac{\varphi}{4}} \\ e^{-i\frac{\varphi}{4}} \end{pmatrix} , \tag{2.6}$$

and

$$\eta_{2\alpha} = \left(e^{i\frac{\varphi}{4}} \;\; e^{-i\frac{\varphi}{4}} \right) , \qquad \bar{\eta}_{2\alpha} = i \begin{pmatrix} e^{-i\frac{\varphi}{4}} \\ e^{i\frac{\varphi}{4}} \end{pmatrix} . \tag{2.7}$$

The R−symmetry part of the couplings is totally unconstrained and we choose

$$n_{1M} = \left(\cos \frac{\theta}{4} \;\; \sin \frac{\theta}{4} \;\; 0 \;\; 0 \right) \qquad \text{and} \qquad n_{2M} = \left(\cos \frac{\theta}{4} \;\; -\sin \frac{\theta}{4} \;\; 0 \;\; 0 \right), \tag{2.8}$$

(and we denote by $\bar{n}^M_i$ the transpose of n_{iM}). The matrices which couple the scalars on the two edges are

$$M_{1J}{}^I = \hat{M}_{1J}{}^I = \begin{pmatrix} -\cos \frac{\theta}{2} & -\sin \frac{\theta}{2} & 0 & 0 \\ -\sin \frac{\theta}{2} & \cos \frac{\theta}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad M_{2J}{}^I = \hat{M}_{2J}{}^I = \begin{pmatrix} -\cos \frac{\theta}{2} & \sin \frac{\theta}{2} & 0 & 0 \\ \sin \frac{\theta}{2} & \cos \frac{\theta}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.9}$$

The quantum holonomy of the super-connection $\mathcal{L}$ in a representation $\mathcal{R}$ of the supergroup $U(N|M)$ is by definition

$$\langle \mathcal{W}_{\mathcal{R}} \rangle = \frac{1}{\dim_{\mathcal{R}}} \int \mathcal{D}[A, \hat{A}, C, \bar{C}, \psi, \bar{\psi}] \, e^{-S_{\text{ABJ}}} \, \text{Tr}_{\mathcal{R}} \left[\text{P} \exp \left(-i \int_{\Gamma} d\tau \, \mathcal{L}(\tau) \right) \right], \qquad (2.10)$$

where S_{ABJ} is the Euclidean action for $ABJ(M)$ theory (see appendix A). In the following $\mathcal{R}$ will be chosen to be the fundamental representation of $U(N|M)$: this implies that the trace is taken in the fundamental representation $\mathbf{N}$ or $\mathbf{M}$ of the two gauge groups.

In general this Wilson loop operator is not supersymmetric unless $\theta = \pm\varphi$: in this case, having chosen the trace in its definition, it can be mapped by a suitable conformal transformation to a closed 1/6 BPS Wilson loop [57].

3 The cusp anomalous dimension in ABJ(M) theory and its computation through ladder diagrams

In this section we discuss the definition of the cusp anomalous dimension in ABJ(M) theories and its computation in a limit in which ladder diagrams dominate. New features in the $N \neq M$ will emerge due to the exponentiation properties of the cusped loops.

3.1 The cusp anomalous dimension in ABJ(M) theory

We start by recalling the $\mathcal{N} = 4$ SYM case: the generalized cusp anomalous dimension is defined by the logarithmic divergent behaviour of a cusped Wilson loop [22]

$$\langle \mathcal{W}_{\text{cusp}} \rangle \simeq e^{-\Gamma_{\text{cusp}}(\varphi,\theta) \log \frac{\Lambda_{\text{UV}}}{\Lambda_{\text{IR}}}}. \qquad (3.1)$$

Here Λ_{UV} and Λ_{IR} stand for the ultraviolet and infrared cut-offs respectively, that regularize the specific divergences associated to the cusp angle and the infinite extension of the lines. Typically one takes $\Lambda_{\text{IR}} = 1/L$, where L measures the (finite) length of the two edges and $\Lambda_{\text{UV}} = 1/\delta$, with δ being a short-length scale either cutting or smoothing out the the cusp singularity. Alternatively one can use dimensional regularization [23] and the logarithm is replaced by a simple pole $1/\epsilon$. After the usual renormalization of the gauge theory,[1] the relation between the bare and renormalized Wilson loop operator, for closed contours, is

$$\mathcal{W}_{\text{cusp}}^{B} = Z_{\text{cusp}} \mathcal{W}_{\text{cusp}}^{R}, \qquad (3.2)$$

where the renormalization constant Z_{cusp} depends on the dimensional regularization parameter ϵ and the subtraction point μ (in addition to the coupling constant λ). When considering the standard cusp built out of two straight lines of length L, the relation (3.2) must be, in general, corrected as follows [33–36]

$$\mathcal{W}_{\text{cusp}}^{B} = Z_{\text{cusp}} Z_{\text{open}} \mathcal{W}_{\text{cusp}}^{R}. \qquad (3.3)$$

The second renormalization constant cancels spurious divergences due to the fact that we are dealing with an open (and then non gauge-invariant) loop operator. The separation

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[1]In $\mathcal{N} = 4$ SYM this step is superfluous, being the β-function vanishing.

between the two contributions is fixed by the renormalization condition $Z_{\text{cusp}}|_{\varphi=\theta=0} = 1$. The cusp anomalous dimension is defined as[2]

$$\Gamma_{\text{cusp}}(\varphi, \theta) = \mu \frac{d}{d\mu} \log Z_{\text{cusp}}, \tag{3.4}$$

and plays the role of the anomalous dimension of a (non-local) quantum operator. Although the open cusped Wilson loop is not gauge invariant, the cusp anomalous dimension Γ_{cusp} turns out to be so. At the perturbative level the cusp divergence comes from diagrams with propagators connecting both rays of the cusp and its exponentiation is governed by their maximal non-abelian part [23]. In the $\mathcal{N} = 4$ SYM case, the perfect balance between the gauge and the scalars contributions cancels, in the Feynman gauge, all the infinities related to integrations along the smooth part of the contour ($Z_{\text{open}} = 1$). Thus only the singularities associated to the discontinuous behavior at the cusp appear and one immediately singles out the relevant diagrams to be computed [37, 38].

In the ABJ(M) case the situation is more subtle. First of all the presence of the fermionic contributions breaks that balance and divergences persist even in the straight-line limit, at least in dimensional regularization and in the Landau gauge [57]. Moreover there are two gauge groups and the fermionic interactions, that live in the off-diagonal sector of the super-connection, mix non-trivially the U(N) and U(M) structures. Thus a non-standard form of exponentiation for the divergence of the cusped Wilson loop is expected. Actually in the ABJM case ($N = M$) the renormalization properties should be similar to the $\mathcal{N} = 4$ SYM case, as explicitly checked at two-loop in [57]. In particular we will expect a single exponential behavior for the vacuum expectation value of the cusped loop operator.

In the ABJ case ($N \neq M$), as usual in the theory of renormalization of composed local operator, we would expect instead the arising of a matrix-valued set of renormalization constants:

$$\mathcal{W}_a^B = \tilde{Z}_{ab} \mathcal{W}_b^R, \tag{3.5}$$

where $a, b = \pm$ refers to the traced and supertraced operators.[3] Similarly to (3.4), the anomalous dimensions matrix is

$$(\Gamma_{\text{cusp}})_{ab} = \left[\mu \frac{\partial}{\partial \mu} \log \tilde{Z}_{\text{cusp}} \right]_{ab}. \tag{3.6}$$

In general the matrix $\tilde{Z}_{\text{cusp}}$ in (3.6) is obtained by rewriting the matrix $\tilde{Z}_{ab}$ in (3.5) as $(Z_{\text{open}} \tilde{Z}_{\text{cusp}})_{ab}$ with normalization condition $(\tilde{Z}_{\text{cusp}})_{ab}|_{\varphi=\theta=0} = \delta_{ab}$. The scaling limit (1.2) considered in this paper selects only diagrams connecting the two halves of the loop, then $Z_{\text{open}} = 1$ and $\tilde{Z} = \tilde{Z}_{\text{cusp}}$.
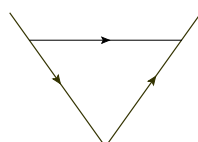
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[2]In the definition of the anomalous dimension, the limit $\epsilon \to 0$ is understood.

[3]These two combinations provides a natural gauge invariant basis for the information contained in the superholonomy defined by $\mathcal{L}(\tau)$. The presence of these two different possibilities is the source of all the main differences with the $\mathcal{N} = 4$ case, where only the trace makes sense.

3.2 The scaling limit selecting ladder diagrams

In [39] it was considered the scaling limit (1.1) in $\mathcal{N} = 4$ SYM: there pure scalar exchanges between the rungs of the cusped Wilson loop become dominant and can be resummed by means of a Bethe-Salpeter equation [39]. Subleading corrections can be also systematically included in this scheme and consistency at strong coupling with semiclassical string computations has been found [58].

In our case, we want to consider a similar limit: upon a quick inspection of the perturbative diagrams we recognize two types of relevant contributions (we refer to [57] for details on the perturbative expansion and related computations). At one-loop we have the single fermionic exchange (that is the same for the up and down diagonal blocks)

$$\text{(diagram)} = \left(\frac{2\pi}{\kappa}\right) \frac{MN}{M+N} \frac{\Gamma\left(\frac{1}{2}-\epsilon\right)}{4\pi^{3/2-\epsilon}} (\mu L)^{2\epsilon} \frac{1}{\epsilon} \frac{\cos\frac{\theta}{2}}{\cos\frac{\varphi}{2}} .$$
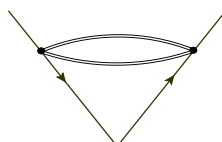
We have considered here the appropriate normalization of the trace. We notice that for $N = M$ this contribution is proportional to $\lambda \cos\frac{\theta}{2}$, suggesting to perform the scaling limit

$$i\theta \gg 1, \quad \lambda \ll 1, \quad \hat{\lambda} = \lambda \cos\frac{\theta}{2} \text{ fixed.} \tag{3.7}$$

The natural generalization to the case $N \neq M$ is therefore

$$i\theta \gg 1, \quad \lambda_{1,2} \ll 1, \quad \hat{\lambda}_{1,2} = \lambda_{1,2} \cos\frac{\theta}{2} \text{ fixed.} \tag{3.8}$$

At two-loop we observe that the above limit suppresses all the diagrams in which interactions are present. Obviously the double-fermionic exchange survives but also a pure scalar exchange comes into the game

$$\text{(diagram)} = -\left(\frac{2\pi}{\kappa}\right)^2 MN \frac{\Gamma^2\left(\frac{1}{2}-\epsilon\right)}{16\pi^{3-2\epsilon}} (\mu L)^{4\epsilon} \cos^2\frac{\theta}{2} \frac{1}{\epsilon} \frac{\varphi}{\sin\varphi} .$$

This last contribution has exactly the same form of the one-loop scalar exchange in $\mathcal{N} = 4$ SYM, except that here it appears at two-loop and the scaling behavior is different.

It is not difficult to realize that, at leading order, the generic diagrams surviving the limit consist of ladders made by fermionic and scalar exchanges, that should therefore summed up to obtain the complete result. We remark that the contributions coming from diagrams ending on a single line, and so leading to divergences not related to the cusp renormalization constant, are automatically suppressed in our limit. In the next section we will derive an efficient way to sum up all the relevant ladder diagrams.

4 Bethe-Salpeter equation for the generalized cusp in ABJ(M) at leading order in the scaling limit

The general strategy to sum up ladder diagrams for Wilson loops constructed by straight-lines was introduced in [59] and used in [39] to derive the leading behavior of the $\mathcal{N} = 4$ SYM cusp anomaly in the scaling limit. We adapt their technique to our case.
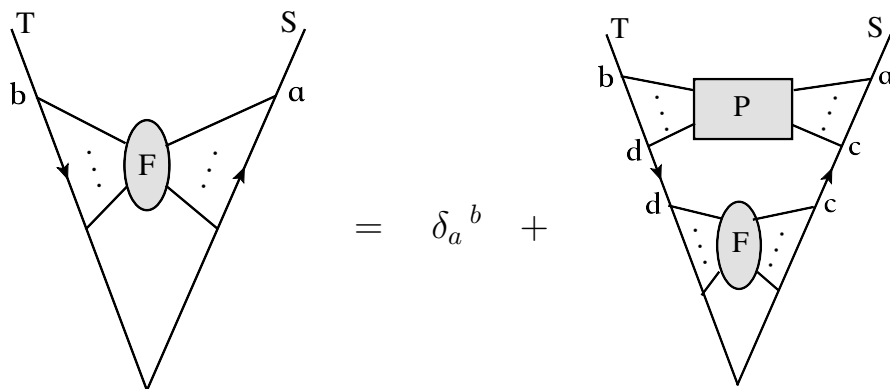
Figure 2. Bethe-Salpeter equation at leading order

We denote the sum of the ladder diagrams with end-points within the intervals $(0, S)$ and $(0, T)$ of the cusp by $F_a{}^b(S, T)$, where (a, b) are group indices that can be $(i, j) \in \mathrm{U}(N)$ or $(\hat{i}, \hat{j}) \in \mathrm{U}(M)$, respectively. $F_a{}^b(S, T)$ satisfies a Bethe-Salpeter equation

$$F_a{}^b(S, T) = \delta_a{}^b + \int_0^S ds \int_0^T dt \, F_c{}^d(s, t) P_a{}^c{}_d{}^b(s, t) \,, \tag{4.1}$$

that is shown schematically in figure 2. In the scaling limit (1.2) the scalar and the fermionic couplings of the loop dominate and one has to consider only exchanges of these fields between the two segments of the Wilson loop. The indices sequence follows the path-ordering of the Wilson loop and the scalar and fermionic propagators fix the kernel indices as follow:
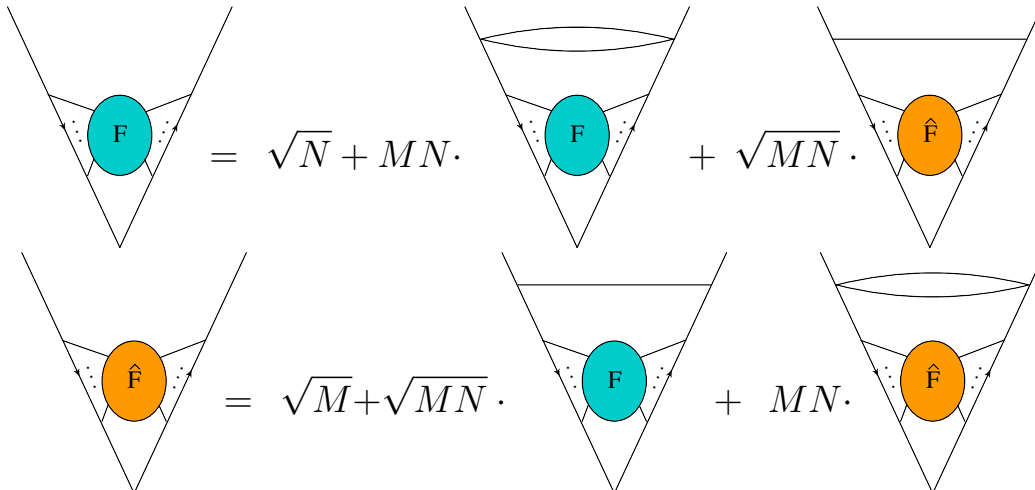
$$P_a{}^c{}_d{}^b(s, t) \simeq \delta_a{}^b \delta^c{}_d \times (\text{a "scalar" function of } s \text{ and } t), \tag{4.2}$$

in particular

$$
\begin{aligned}
P_i{}^k{}_l{}^j(s, t) &= M \delta_i{}^j \delta^k{}_l P^{(B)}(s, t) \,, \\
P_i{}^{\hat{k}}{}_{\hat{l}}{}^j(s, t) &= \delta_i{}^j \delta^{\hat{k}}{}_{\hat{l}} P^{(F)}(s, t) \,, \\
P_{\hat{i}}{}^k{}_l{}^{\hat{j}}(s, t) &= \delta_{\hat{i}}{}^{\hat{j}} \delta^k{}_l P^{(F)}(s, t) \,, \\
P_{\hat{i}}{}^{\hat{k}}{}_{\hat{l}}{}^{\hat{j}}(s, t) &= N \delta_{\hat{i}}{}^{\hat{j}} \delta^{\hat{k}}{}_{\hat{l}} P^{(B)}(s, t) \,,
\end{aligned}
\tag{4.3}
$$

where $P^{(F)}(s, t)$ is the fermionic effective propagator and $P^{(B)}(s, t)$ the scalar effective propagator (double exchange) defined by [57]:

$$
\begin{aligned}
P^{(F)}(s, t) &= -\left(\frac{2\pi}{k}\right) \frac{\Gamma(1/2 - \epsilon)\mu^{2\epsilon}}{4\pi^{3/2-\epsilon}} \frac{\cos\theta/2}{\cos\varphi/2} (\partial_s + \partial_t) \frac{1}{(s^2 + t^2 + 2st\cos\varphi)^{\frac{1}{2}-\epsilon}} \,, \\
P^{(B)}(s, t) &= \left(\frac{2\pi}{k}\right)^2 \frac{\Gamma^2(1/2 - \epsilon)\mu^{4\epsilon}}{4\pi^{3-2\epsilon}} \frac{\cos^2\theta/2}{\cos^2\varphi/2} \frac{\cos^2\varphi/2}{(s^2 + t^2 + 2st\cos\varphi)^{1-2\epsilon}} \,.
\end{aligned}
\tag{4.4}
$$

Figure 3. Bethe-Salpeter equation at leading order.

According to U(N) or U(M) indices (4.1) splits into

$$F_i{}^j(S,T) = \delta_i{}^j + \int_0^S ds \int_0^T dt \left(M F_k{}^l(s,t) \delta_i{}^j \delta^k{}_l P^{(B)}(s,t) + F_{\hat{k}}{}^{\hat{l}}(s,t) \delta_i{}^j \delta^{\hat{k}}{}_{\hat{l}} P^{(F)}(s,t), \right),$$
$$F_{\hat{i}}{}^{\hat{j}}(S,T) = \delta_{\hat{i}}{}^{\hat{j}} + \int_0^S ds \int_0^T dt \left(F_k{}^l(s,t) \delta_{\hat{i}}{}^{\hat{j}} \delta^k{}_l P^{(F)}(s,t) + N F_{\hat{k}}{}^{\hat{l}}(s,t) \delta_{\hat{i}}{}^{\hat{j}} \delta^{\hat{k}}{}_{\hat{l}} P^{(B)}(s,t) \right).$$

(4.5)

Thus, defining

$$F(S,T) = \frac{1}{\sqrt{N}} \mathrm{Tr}_{\mathbf{N}}[F_i{}^j(S,T)],$$
$$\hat{F}(S,T) = \frac{1}{\sqrt{M}} \mathrm{Tr}_{\mathbf{M}}[F_{\hat{i}}{}^{\hat{j}}(S,T)],$$

(4.6)

we get

$$F(S,T) = \sqrt{N} + \int_0^S ds \int_0^T dt \left(MN F(s,t) P^{(B)}(s,t) + \sqrt{MN} \hat{F}(s,t) P^{(F)}(s,t) \right),$$
$$\hat{F}(S,T) = \sqrt{M} + \int_0^S ds \int_0^T dt \left(\sqrt{MN} F(s,t) P^{(F)}(s,t) + MN \hat{F}(s,t) P^{(B)}(s,t) \right).$$

(4.7)

Changing variables according to $s = Le^{\sigma'}$, $t = Le^{\tau'}$, where L is an arbitrary length scale, we get

$$F(\sigma,\tau) = \sqrt{N} + \int_{-\infty}^{\sigma} d\sigma' \int_{-\infty}^{\tau} d\tau' \left(MN F(\sigma',\tau') P^{(B)}(\sigma',\tau') + \sqrt{MN} \hat{F}(\sigma',\tau') P^{(F)}(\sigma',\tau') \right),$$
$$\hat{F}(\sigma,\tau) = \sqrt{M} + \int_{-\infty}^{\sigma} d\sigma' \int_{-\infty}^{\tau} d\tau' \left(\sqrt{MN} F(\sigma',\tau') P^{(F)}(\sigma',\tau') + MN \hat{F}(\sigma',\tau') P^{(B)}(\sigma',\tau') \right).$$

(4.8)

with

$$P^{(F)}(\sigma,\tau) = -\left(\frac{2\pi}{k}\right)\frac{\Gamma(1/2-\epsilon)(\mu L)^{2\epsilon}}{2^{5/2-\epsilon}\pi^{3/2-\epsilon}}\frac{\cos\theta/2}{\cos\varphi/2}\,(e^\tau\partial_\sigma + e^\sigma\partial_\tau)\frac{e^{-\frac{1}{2}(\sigma+\tau)}e^{\epsilon(\sigma+\tau)}}{(\cosh(\sigma-\tau)+\cos\varphi)^{\frac{1}{2}-\epsilon}}\,,$$

$$P^{(B)}(\sigma,\tau) = \left(\frac{2\pi}{k}\right)^2\frac{\Gamma^2(1/2-\epsilon)(\mu L)^{4\epsilon}}{2^{3-2\epsilon}\pi^{3-2\epsilon}}\frac{\cos^2\theta/2}{\cos^2\varphi/2}\frac{\cos^2\varphi/2\;e^{2\epsilon(\sigma+\tau)}}{(\cosh(\sigma-\tau)+\cos\varphi)^{1-2\epsilon}}. \tag{4.9}$$

F and $\hat{F}$ obey the differential equations

$$\partial_\sigma\partial_\tau F(\sigma,\tau) = MNF(\sigma,\tau)P^{(B)}(\sigma,\tau) + \sqrt{MN}\hat{F}(\sigma,\tau)P^{(F)}(\sigma,\tau)\,,$$
$$\partial_\sigma\partial_\tau \hat{F}(\sigma,\tau) = \sqrt{MN}F(\sigma,\tau)P^{(F)}(\sigma,\tau) + MN\hat{F}(\sigma,\tau)P^{(B)}(\sigma,\tau)\,, \tag{4.10}$$

with boundary conditions $F(-\infty,\tau)=F(\sigma,-\infty)=\sqrt{N}$ and $\hat{F}(-\infty,\tau)=\hat{F}(\sigma,-\infty)=\sqrt{M}$. Then we write $x=\sigma-\tau$ and $y=(\sigma+\tau)/2$ and obtain

$$\left(\frac{1}{4}\partial_y^2 - \partial_x^2\right)F(x,y) = MNF(x,y)\tilde{P}^{(B)}(x,y) + \sqrt{MN}\hat{F}(x,y)\tilde{P}^{(F)}(x,y)\,,$$
$$\left(\frac{1}{4}\partial_y^2 - \partial_x^2\right)\hat{F}(x,y) = \sqrt{MN}F(x,y)\tilde{P}^{(F)}(x,y) + MN\hat{F}(x,y)\tilde{P}^{(B)}(x,y)\,, \tag{4.11}$$

with

$$\tilde{P}^{(F)}(x,y) = \left(\frac{2\pi}{k}\right)\frac{\Gamma\left(\frac{1}{2}-\epsilon\right)(\mu L)^{2\epsilon}}{(2\pi)^{3/2-\epsilon}}\frac{\cos\frac{\theta}{2}}{\cos\frac{\varphi}{2}}e^{2\epsilon y}$$
$$\times\left\{\frac{d}{dx}\left[\frac{\sinh x/2}{(\cosh x + \cos\varphi)^{\frac{1}{2}-\epsilon}}\right] - \frac{\epsilon\cosh x/2}{(\cosh x + \cos\varphi)^{\frac{1}{2}-\epsilon}}\right\}\,,$$

$$\tilde{P}^{(B)}(x,y) = \left(\frac{2\pi}{k}\right)^2\frac{\Gamma^2\left(\frac{1}{2}-\epsilon\right)(\mu L)^{4\epsilon}\cos^2\frac{\theta}{2}}{(2\pi)^{3-2\epsilon}}\frac{\cos^2\frac{\theta}{2}}{\cos^2\frac{\varphi}{2}}e^{4\epsilon y}$$
$$\times\left\{\frac{(\cosh x + \cos\varphi)^{2\epsilon}}{2} - \frac{\sinh^2 x/2}{(\cosh x + \cos\varphi)^{1-2\epsilon}}\right\}\,. \tag{4.12}$$

4.1 General solution in $d=3$

For $\epsilon=0$ equations (4.11) can be decoupled easily since the kernels (4.12) are independent of y. Indeed, by introducing

$$\mathcal{H}(x,y) = F(x,y) + \hat{F}(x,y)\,, \qquad \mathcal{K}(x,y) = F(x,y) - \hat{F}(x,y)\,, \tag{4.13}$$

eqs. (4.11) are equivalent to

$$\left(\frac{1}{4}\partial_y^2 - \partial_x^2\right)\mathcal{H}(x,y) = \left(aW'(x) - a^2W^2(x) + \frac{a^2}{2}\right)\mathcal{H}(x,y)\,,$$
$$\left(\frac{1}{4}\partial_y^2 - \partial_x^2\right)\mathcal{K}(x,y) = \left(-aW'(x) - a^2W^2(x) + \frac{a^2}{2}\right)\mathcal{K}(x,y)\,. \tag{4.14}$$

with

$$a = \left(\frac{2\pi}{k}\right)\frac{\sqrt{MN}}{2^{3/2}\pi}\frac{\cos\theta/2}{\cos\varphi/2}\,, \tag{4.15}$$

and

$$W(x) = \frac{\sinh x/2}{(\cosh x + \cos\varphi)^{\frac{1}{2}}}\,. \tag{4.16}$$

These equations can be solved using the separation variable method. Setting

$$\mathcal{H}(x,y) = h(y)\psi_+(x)\,, \qquad \mathcal{K}(x,y) = k(y)\psi_-(x)\,, \tag{4.17}$$

we get

$$\partial_y^2 h(y) = 4\left(-E + \frac{a^2}{2}\right)h(y)\,, \tag{4.18}$$
$$\partial_y^2 k(y) = 4\left(-\tilde{E} + \frac{a^2}{2}\right)k(y)\,,$$

and

$$\left(-\partial_x^2 + a^2 W^2(x) - a W'(x)\right)\psi_+(x) = E\psi_+(x)\,, \tag{4.19}$$
$$\left(-\partial_x^2 + a^2 W^2(x) + a W'(x)\right)\psi_-(x) = \tilde{E}\psi_-(x)\,.$$

The solution of the y dependent equations is simply

$$h(y) = C_1 e^{2\sqrt{-E+\frac{a^2}{2}}\,y} + C_2 e^{-2\sqrt{-E+\frac{a^2}{2}}\,y}\,, \tag{4.20}$$
$$k(y) = C_3 e^{2\sqrt{-\tilde{E}+\frac{a^2}{2}}\,y} + C_4 e^{-2\sqrt{-\tilde{E}+\frac{a^2}{2}}\,y}\,,$$

with $C_{1,2,3,4}$ constants which have to be fixed by imposing the boundary conditions as we will discuss in the following. The $x-$dependent equations (4.19) can be seen as the two Schroedinger equations of a supersymmetric quantum mechanical system [60], therefore E and $\tilde{E}$ are non-negative. In principle one could solve for these equations. However we are only interested to consider the case in which the edges of the cusp extend to infinity, i.e. S and T very large. In this limit $x \sim 0$ and y is very large, thus we make the ansatz $E = \tilde{E} = 0$,[4] and we set $\psi_+(0) = \psi_-(0) = 1$ since they can be reabsorbed in the normalization constants C_1, C_2, C_3, C_4.

Using (4.17), (4.20) and (4.13) we get

$$F(0,y) = \frac{C_1 + C_3}{2}e^{\sqrt{2}ay} + \frac{C_2 + C_4}{2}e^{-\sqrt{2}ay}\,, \tag{4.21}$$
$$\hat{F}(0,y) = \frac{C_1 - C_3}{2}e^{\sqrt{2}ay} + \frac{C_2 - C_4}{2}e^{-\sqrt{2}ay}\,.$$

We fix the constants $C_1, \ldots, C_4$ by matching the perturbative result. In $d = 3$ there are UV divergences coming from the integration regions close to the cusps. To isolate this divergence we set $s_{\min} = t_{\min} = \delta$ which means $y_{\min} = \ln\frac{\delta}{L} \equiv -L_0$. At tree level, obviously

$$F^{(0)}(0,-L_0) = \sqrt{N}\,, \qquad \text{and} \qquad \hat{F}^{(0)}(0,-L_0) = \sqrt{M}\,. \tag{4.22}$$

---

[4]In the weak coupling limit for positive energy values the solutions of (4.20) become oscillatory.

Inserting these conditions in (4.21) we obtain

$$F(0, y) = \sqrt{N} e^{\sqrt{2}a(y+L_0)} - \frac{C_2 + C_4}{2} \sinh \sqrt{2}a(y + L_0) e^{\sqrt{2}aL_0} \,,$$

$$\hat{F}(0, y) = \sqrt{M} e^{\sqrt{2}a(y+L_0)} - \frac{C_2 - C_4}{2} \sinh \sqrt{2}a(y + L_0) e^{\sqrt{2}aL_0} \,. \tag{4.23}$$

The two remaining constants are determined by matching the first order in the coupling (which is contained in a) of our solution (4.23) with the first iteration of the Bethe-Salpeter equations (4.8) at the same order

$$F^{(1)}(\tau, \tau) = M\sqrt{N} \int_{-L_0}^{\tau} d\sigma' \int_{-L_0}^{\tau} d\tau' P^F(\sigma', \tau') \,,$$

$$\hat{F}^{(1)}(\tau, \tau) = N\sqrt{M} \int_{-L_0}^{\tau} d\sigma' \int_{-L_0}^{\tau} d\tau' P^F(\sigma', \tau') \,, \tag{4.24}$$

which gives

$$C_2 = \frac{\sqrt{N} + \sqrt{M}}{2} (1 - A) \,,$$

$$C_4 = \frac{\sqrt{N} - \sqrt{M}}{2} (1 + A) \,, \tag{4.25}$$

where

$$A = \lim_{\substack{y \to \infty \\ L_0 \to \infty}} \frac{\sqrt{MN}}{\sqrt{2}a(y + L_0)} \int_{-L_0}^{y} d\sigma' \int_{-L_0}^{y} d\tau' P^F(\sigma', \tau') \,. \tag{4.26}$$

In the appendix B we compute this integral and we find $A = 1$, thus $C_2 = 0$ and $C_4 = \sqrt{N} - \sqrt{M}$. Inserting this result in (4.23) we finally obtain

$$\langle \mathcal{W}_+ \rangle = \frac{\sqrt{N}F + \sqrt{M}\hat{F}}{N + M} = \cosh \sqrt{2}a(y + L_0) + \frac{2\sqrt{MN}}{N + M} \sinh \sqrt{2}a(y + L_0) \,.$$

$$\langle \mathcal{W}_- \rangle = \frac{\sqrt{N}F - \sqrt{M}\hat{F}}{N - M} = \cosh \sqrt{2}a(y + L_0) \,. \tag{4.27}$$

In order to extract the cusp anomalous dimension at arbitrary φ from the result (4.27), we have simply to recast them in a suitable form to single out the logarithmic divergence. Going back to the original (dimensionful) variables, we define

$$T = S = \Lambda_{\text{IR}}^{-1} = Le^y \quad \to \quad y = -\log L\Lambda_{\text{IR}} = \log \frac{T}{L};$$

$$\delta = \Lambda_{\text{UV}}^{-1} = Le^{-L_0} \quad \to \quad L_0 = \log L\Lambda_{\text{UV}} = \log \frac{L}{\delta} \,, \tag{4.28}$$

where Λ_{IR} is the natural IR cut-off (associated to the length of the cusp) and $\Lambda_{\text{UV}} = \frac{1}{\delta}$ is the UV cut-off (cutting-off the cusp, where ladders collapse). The length scale L, introduced previously for dimensional reason, will not play any role in the following. With the above definitions we get

$$(y + L_0) = \log \frac{T}{\delta} = \log \frac{\Lambda_{\text{UV}}}{\Lambda_{\text{IR}}} \,. \tag{4.29}$$

We can finally rewrite the expectation value of the cusped Wilson loop in the suggestive form as follows:

$$\langle \mathcal{W}_+ \rangle = \frac{(\sqrt{M}+\sqrt{N})^2}{2(M+N)} e^{\sqrt{2}a \log \frac{\Lambda_{\mathrm{UV}}}{\Lambda_{\mathrm{IR}}}} + \frac{(\sqrt{M}-\sqrt{N})^2}{2(M+N)} e^{-\sqrt{2}a \log \frac{\Lambda_{\mathrm{UV}}}{\Lambda_{\mathrm{IR}}}} ,$$

$$\langle \mathcal{W}_- \rangle = \frac{1}{2} e^{\sqrt{2}a \log \frac{\Lambda_{\mathrm{UV}}}{\Lambda_{\mathrm{IR}}}} + \frac{1}{2} e^{-\sqrt{2}a \log \frac{\Lambda_{\mathrm{UV}}}{\Lambda_{\mathrm{IR}}}} . \tag{4.30}$$

We have exactly reproduced the double-exponential structure found at two-loop in [57]: it comes from an all-order computation, in a particular limit, strongly supporting the mixing picture. In section 5 we will discuss how the cusp anomalous dimension is related to this result.

4.2 The straight-line limit: the solution for $\epsilon \neq 0$

We consider here the straight-line limit $\varphi = 0$: remarkably the system enjoys supersymmetry for any value of ϵ and the cusp anomalous dimension can be computed exactly. To show this fact we first perform the change of variable $x \to ix'$ and $y \to \frac{1}{2}y'$. Then using (4.13), eqs. (4.11) become

$$\Box\mathcal{H}(x',y') = \left[\vec{\nabla}W(x',y') \cdot \vec{\nabla}W(x',y') - \Box W(x',y') \right] \mathcal{H}(x',y') ,$$

$$\Box\mathcal{K}(x',y') = \left[\vec{\nabla}W(x',y') \cdot \vec{\nabla}W(x',y') + \Box W(x',y') \right] \mathcal{K}(x',y') , \tag{4.31}$$

with $\vec{\nabla} = (\partial_{x'}, \partial_{y'})$, $\Box = \partial_{x'}^2 + \partial_{y'}^2$ and

$$W(x',y') = \frac{2^{\epsilon-1/2}a_\epsilon}{\epsilon} e^{\epsilon y'} \cos^{2\epsilon}\frac{x'}{2} , \qquad a_\epsilon = \left(\frac{2\pi}{k}\right)\sqrt{MN}\frac{\Gamma(1/2-\epsilon)(\mu L)^{2\epsilon}}{(2\pi)^{3/2-\epsilon}}\cos\theta/2 . \tag{4.32}$$

These equations are the Schroedinger equations of the two bosonic sectors of a two-dimensional $\mathcal{N}=2$ supersymmetric quantum mechanics. The wave function of the ground state can be exactly found and gives

$$\mathcal{H}(x,y) = C_1 e^{-W(-ix',2y')} = C_1 e^{-\frac{2^{\epsilon-1/2}a_\epsilon}{\epsilon}e^{2\epsilon y}\cosh^{2\epsilon}\frac{x}{2}} ,$$

$$\mathcal{K}(x,y) = C_2 e^{W(-ix',2y')} = C_2 e^{\frac{2^{\epsilon-1/2}a_\epsilon}{\epsilon}e^{2\epsilon y}\cosh^{2\epsilon}\frac{x}{2}} , \tag{4.33}$$

with C_1 and C_2 normalization constants. Thus, using (4.13), one finds

$$F(x,y) = \frac{C_1}{2}e^{-\frac{2^{\epsilon-1/2}a_\epsilon}{\epsilon}e^{2\epsilon y}\cosh^{2\epsilon}\frac{x}{2}} + \frac{C_2}{2}e^{\frac{2^{\epsilon-1/2}a_\epsilon}{\epsilon}e^{2\epsilon y}\cosh^{2\epsilon}\frac{x}{2}} ,$$

$$\hat{F}(x,y) = \frac{C_1}{2}e^{-\frac{2^{\epsilon-1/2}a_\epsilon}{\epsilon}e^{2\epsilon y}\cosh^{2\epsilon}\frac{x}{2}} - \frac{C_2}{2}e^{\frac{2^{\epsilon-1/2}a_\epsilon}{\epsilon}e^{2\epsilon y}\cosh^{2\epsilon}\frac{x}{2}} . \tag{4.34}$$

Here too we use the boundary conditions

$$F(0,-\infty) = \sqrt{N} , \quad \text{and} \quad \hat{F}(0,-\infty) = \sqrt{M} , \tag{4.35}$$

to fix the constants in (4.34), getting

$$C_1 = \sqrt{N} + \sqrt{M} ,$$

$$C_2 = \sqrt{N} - \sqrt{M} . \tag{4.36}$$

The traced and supertraced operators for any x and y are

$$\langle \mathcal{W}_+^{\varphi=0} \rangle = \frac{\sqrt{N}F + \sqrt{M}\hat{F}}{N + M} = \cosh V_\epsilon(x,y) + \frac{2\sqrt{MN}}{M + N}\sinh V_\epsilon(x,y),$$

$$\langle \mathcal{W}_-^{\varphi=0} \rangle = \frac{\sqrt{N}F - \sqrt{M}\hat{F}}{N - M} = \cosh V_\epsilon(x,y),$$

(4.37)

where

$$V_\epsilon(x,y) = -W(-ix, 2y) = -\frac{2^{\epsilon-1/2}a_\epsilon}{\epsilon}e^{2\epsilon y}\cosh^{2\epsilon}\frac{x}{2}. \qquad (4.38)$$

In order to consider the infinite cusped Wilson loop, recalling the original variables of the Bethe-Salpeter integrals, we have to set $x, y = 0$, then we have

$$\langle \mathcal{W}_+^{\varphi=0} \rangle = \frac{(\sqrt{M} + \sqrt{N})^2}{2(M + N)}e^{-\frac{2^{\epsilon-1/2}}{\epsilon}a_\epsilon} + \frac{(\sqrt{M} - \sqrt{N})^2}{2(M + N)}e^{\frac{2^{\epsilon-1/2}}{\epsilon}a_\epsilon},$$

$$\langle \mathcal{W}_-^{\varphi=0} \rangle = \frac{1}{2}e^{-\frac{2^{\epsilon-1/2}}{\epsilon}a_\epsilon} + \frac{1}{2}e^{\frac{2^{\epsilon-1/2}}{\epsilon}a_\epsilon}.$$

(4.39)

5 The determination of $\Gamma_{\text{cusp}}(\varphi)$

Above we have seen that the quantum expectation value of our cusped Wilson loop organizes itself as a double exponential in the limit (1.2) (see (4.30) or (4.39)). Here we want to extract the cusp anomalous dimension from the Bethe-Salpeter results. We find convenient to express the traced and supertraced operators $\mathcal{W}_a^B$, $a = \pm$, on a basis $\mathcal{W}_i^B$, $i = 1, 2$, whose elements renormalize multiplicatively, i.e.

$$\mathcal{W}_i^B = Z_{\text{cusp}}^{(i)}\,\mathcal{W}_i^R. \qquad (5.1)$$

The two sets of bare operators are related by a linear transformation depending only on the ranks of the gauge groups

$$\mathcal{W}_a^B = \mathcal{A}_{ai}\mathcal{W}_i^B. \qquad (5.2)$$

From the explicit solution (4.30) or (4.39) of the Bethe-Salpeter equation and (3.5) one reads

$$\mathcal{A} = \frac{1}{2(M + N)}\left(\begin{array}{cc}(\sqrt{M} + \sqrt{N})^2 & (\sqrt{M} - \sqrt{N})^2 \\ M + N & M + N\end{array}\right), \qquad (5.3)$$

while the regulator dependent parts enter in the $Z_{\text{cusp}}^{(i)}$ (in this discussion we can neglect the additional divergences subtracted by Z_{open} since they are suppressed in the limit of large imaginary θ).

In the $\varphi = 0$ case one has

$$Z_{\text{cusp}}^{(1)} = e^{-\frac{2^{\epsilon-1/2}}{\epsilon}a_\epsilon}, \qquad Z_{\text{cusp}}^{(2)} = e^{\frac{2^{\epsilon-1/2}}{\epsilon}a_\epsilon}, \qquad (5.4)$$

thus

$$\Gamma_{\text{cusp}}^{(1)} = \mu \frac{\partial}{\partial \mu} \log(Z_{\text{cusp}}^{(1)}) = -\frac{\sqrt{MN}}{\kappa} \cos \frac{\theta}{2} = -\sqrt{\hat{\lambda}_1 \hat{\lambda}_2} \,,$$

$$\Gamma_{\text{cusp}}^{(2)} = \mu \frac{\partial}{\partial \mu} \log(Z_{\text{cusp}}^{(2)}) = \frac{\sqrt{MN}}{\kappa} \cos \frac{\theta}{2} = \sqrt{\hat{\lambda}_1 \hat{\lambda}_2} \,,$$

(5.5)

where the definition (4.32) for a_ϵ has been used and $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are the two effective coupling constants introduced in (1.2). In the case of ABJM, where $N = M$, the two coupling constants of course coincide: $\hat{\lambda}_1 = \hat{\lambda}_2 \equiv \hat{\lambda}$.

In order to extract the cusp anomalous dimension at arbitrary φ, we have to consider the result (4.30). We first consider the ABJM case. As already announced in section 3.1 we observe a drastic simplification of the expectation value of the traced operator with the disappearance of one of the two exponentials. In other words the trace renormalizes multiplicatively. Conversely the supertrace stills mixes with the trace. Therefore, for $M = N$, we can associate a cusp anomalous dimension, $\Gamma_{\text{cusp}}(\varphi, \hat{\lambda})$, directly to $\mathcal{W}_+$ though the usual definition (3.1) and, using (4.15), we find:

$$\langle \mathcal{W}_+ \rangle^{\text{ABJM}} = e^{\sqrt{2}a \log \frac{\Lambda_{\text{UV}}}{\Lambda_{\text{IR}}}} \qquad \Rightarrow \qquad \Gamma_{\text{cusp}}(\varphi) = -\sqrt{2}a = -\frac{N}{\kappa} \frac{\cos \theta/2}{\cos \varphi/2} = -\frac{\hat{\lambda}}{\cos \frac{\varphi}{2}}. \quad (5.6)$$

In the general situation, when $N \neq M$, the expectation value of both $\mathcal{W}_\pm$ contains a double exponential, and the coefficients of the divergent logarithms in the exponentials are identified with $\Gamma_{\text{cusp}}^{(1,2)}(\varphi)$:

$$\Gamma_{\text{cusp}}^{(1)}(\varphi) = -\frac{\sqrt{\hat{\lambda}_1 \hat{\lambda}_2}}{\cos \frac{\varphi}{2}} \,, \qquad \Gamma_{\text{cusp}}^{(2)}(\varphi) = \frac{\sqrt{\hat{\lambda}_1 \hat{\lambda}_2}}{\cos \frac{\varphi}{2}}, \quad (5.7)$$

as explained in more details in the case $\varphi = 0$. Moreover the result (5.7) is perfectly consistent with the limit (5.5) and for $\varphi = \theta = i\infty$, $\Gamma_{\text{cusp}}^{(1,2)}$ vanish as expected!

Some remarks on the results (4.30) and (4.39) and the consequent form of $\Gamma_{cusp}^{(i)}$ are now in order. First we analyze the structure of the exponentiation in the ABJ case: looking at our explicit calculation, one could expect that the positive cusp anomalous dimension dominates, while the negative one gives a subleading contribution. On the other hand they appear on the same footing in our computations and, much more crucially, consistency with perturbative results requires the presence of both of them. However we believe that the simple relation $\Gamma_{\text{cusp}}^{(1)}(\varphi) = -\Gamma_{\text{cusp}}^{(2)}(\varphi)$ implied by (5.7) does not survive when the subleading corrections in θ are included. A second important observation concerns the actual functional form of $\Gamma_{\text{cusp}}(\varphi)$. Let us concentrate for the moment on the $N = M$ case. The final expression (5.6) is just the exponentiation of the one-loop result, namely the leading cusp divergence undergoes to an abelian exponentiation in the ladder limit. This result is completely different from the analogous $\mathcal{N} = 4$ SYM resummation, where an highly non-trivial function appears at this order, even for $\varphi = 0$. The reason relies of course in the supersymmetric structure of the effective Schroedinger equation but it has also a perturbative explanation: fermionic and bosonic diagrams do not exponentiate

in an abelian way by themselves and it is their delicate balance that, order by order in the coupling constant, generates this nice behavior. We have checked explicitly at three-loop in perturbation theory this fact. The $N \neq M$ situation presents instead a slightly more involved structure: we have still an abelian-like exponentiation at this order, but when expressed in terms of the two (scaled) 't Hooft couplings $\hat{\lambda}_1$, $\hat{\lambda}_2$ it appears through a square root of their product. This is a further effect of the diagonalization process and at moment we do not have a satisfying explanation from general principles. We stress that from the point of view of the original CS level k the exponentiation is one-loop as well. A third and, may be, more interesting remark is related to the strong-coupling limit and the connection with string theory. To be concrete, we shall consider the simpler case of ABJM: because of the abelian-like exponentiation we do not have any non-trivial interpolation between weak and strong-coupling and the scaling limit does not match the $\sqrt{\hat{\lambda}}$ behavior of string theory. At variance with $\mathcal{N} = 4$ SYM the scaling limit does not seem to commute with the strong-coupling limit, a fact that in four-dimensions was not expected a priori (see the comments in the original computation [39]). We hope to come back soon on this point when the subleading contributions will be computed.

6 Conclusions

In this paper we have studied a cusped Wilson loop in $\mathcal{N} = 6$ Super Chern-Simons theory, constructed with lines that are 1/2 BPS. We have computed the associated cusp anomalous dimension in a scaling limit in which ladder diagrams dominate: because of the 1/2 BPS character of the two halves, we have both bosonic and fermionic ladder exchanges and their resummation is encoded into a coupled Bethe-Salpeter equation. We have seen that it can be mapped into a supersymmetric Schroedinger equation whose ground state solution provide an exact expression for the cusp anomalous dimensions. Actually we found that, in the general $N \neq M$ case, the traced Wilson loop undergoes through a double-exponentiation, as first observed in [57]. This has been interpreted as an operator mixing under cusp renormalization: we have associated to the eigenvalues of the mixing matrix two independent cusp anomalous dimensions. The final result is very simple and the exponentiations are abelian, the cusp anomalous dimensions are one-loop exact up diagonalization. The strong-coupling limit is therefore trivial and we do not find consistency with string theory computation [61]: we argue that the scaling limit considered here does not commute with the strong-coupling limit. Concerning abelian exponentiation, a similar phenomenon has been observed recently [62] in studying $\mathcal{N} = 4$ SYM cusped Wilson loops in k-symmetric representations: at large N and k planar diagrams dominate and the exponentiation is of abelian type.

The obvious follow-up of the present work is to take into account the subleading corrections to the scaling limit: in [58] a systematic approach to this computation has been developed in the $\mathcal{N} = 4$ SYM case and it should be possible to perform an analogous investigation here. Preliminary results seem promising. It would be interesting to see if the supersymmetric structure we have found is preserved beyond leading order: in any case we expect a non-trivial modification of the relation $\Gamma_{\text{cusp}}^{(1)}(\varphi) = -\Gamma_{\text{cusp}}^{(2)}(\varphi)$. Another
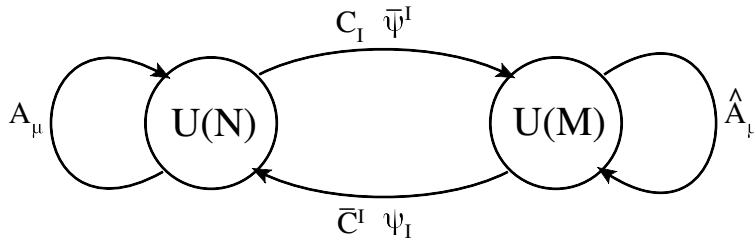
Figure 4. Quiver diagram for ABJ(M) theory.

direction consists in checking the exponential structure at three-loop: the mixing we have observed here prescribes an exponentiation with definite group-dependent coefficients (see eq. (4.30)), that appear to be the same both in the scaling limit and in the general two-loop result [57]. It would be of course nice to have a deeper understanding for the occurrence of the mixing coefficients: a closer look at the supersymmetric quantum mechanics discussed in [8], where the 1/2 BPS line is obtained from a Higgsing procedure, should be probably useful for this task.

More ambitiously, one would like to approach the generalized cusp anomalous dimension in ABJ(M) theory from a general point of view, with the hope to obtain other all-order result by integrability or localization. In four-dimensions a particularly powerful approach has been pushed forward recently [63, 64], applying to cusped Wilson loop the technique of the quantum spectral curve. Beautiful results have been obtained for the Bremsstrahlung function and the quark-anti-quark potential. It would be nice to extend this approach to ABJ(M) case, in which the quantum spectral curve has been already studied [65]. It should be also possible to extend the TBA equations derived in [28, 29] in the three-dimensional context, taking advantage of the investigations presented in [66–68].

Acknowledgments

A $\mathcal{N} = 6$ three-dimensional Chern-Simons-matter theory

The field content of the ABJ(M) theory can be schematically represented in the quiver in figure 4. The gauge sector consists of two gauge fields $(A_\mu)_i{}^j$ and $(\hat{A}_\mu)_{\hat{i}}{}^{\hat{j}}$ belonging respectively to the adjoint of U(N) and U(M). We denote by i, $\hat{i}$ the gauge indices in the fundamental of the first and the second gauge group respectively. The matter sector instead contains the complex fields $(C_I)_i{}^{\hat{j}}$ and $(\bar{C}^I)_{\hat{i}}{}^j$ as well as the fermions $(\psi_I)_{\hat{i}}{}^j$ and $(\bar{\psi}^I)_i{}^{\hat{j}}$. The fields $(C, \bar{\psi})$ transform in the $(\mathbf{N}, \bar{\mathbf{M}})$ of the gauge group U(N) × U(M) while the pair $(\bar{C}, \psi)$ lives in the $(\bar{\mathbf{N}}, \mathbf{M})$. The additional capitol index $I = 1, 2, 3, 4$ belongs to the R-symmetry group SU(4). The ABJ(M) action is

$$S_{\text{ABJ(M)}} = S_{\text{CS}} + S_{\text{gf}} + S_{\text{Matter}} + S_{\text{int}}^F + S_{\text{int}}^B \,, \tag{A.1}$$

where

$$S_{\text{CS}} = -i\frac{\kappa}{4\pi}\int d^3x \epsilon^{\mu\nu\rho}\left[\text{Tr}(A_\mu\partial_\nu A_\rho + \tfrac{2}{3}iA_\mu A_\nu A_\rho) - \text{Tr}(\hat{A}_\mu\partial_\nu\hat{A}_\rho + \tfrac{2}{3}i\hat{A}_\mu\hat{A}_\nu\hat{A}_\rho)\right],$$

$$S_{\text{gf}} = \frac{\kappa}{4\pi}\int d^3x \left[\tfrac{1}{\xi}\text{Tr}\,(\partial_\mu A_\mu)^2 + \text{Tr}\,(\partial_\mu\bar{c}D_\mu c) - \tfrac{1}{\xi}\text{Tr}(\partial_\mu\hat{A}_\mu)^2 + \text{Tr}\,(\partial_\mu\hat{\bar{c}}D_\mu\hat{c})\right],$$

$$S_{\text{Matter}} = \int d^3x \left[\text{Tr}\,(D_\mu C_I D^\mu\bar{C}^I) + i\text{Tr}\,(\bar{\psi}^I\slashed{D}\psi_I)\right],$$

$$(A.2)$$

and

$$S_{\text{int}}^F = -\frac{2\pi i}{\kappa}\int d^3x \left[\text{Tr}(\bar{C}^I C_I \psi_J\bar{\psi}^J) - \text{Tr}(C_I\bar{C}^I\bar{\psi}^J\psi_J) + 2\text{Tr}(C_I\bar{C}^J\bar{\psi}^I\psi_J)\right.$$
$$\left. -2\text{Tr}(\bar{C}^I C_J\psi_I\bar{\psi}^J) - \epsilon_{IJKL}\text{Tr}(\bar{C}^I\bar{\psi}^J\bar{C}^K\bar{\psi}^L) + \epsilon^{IJKL}\text{Tr}(C_I\psi_J C_K\psi_L)\right],$$

$$S_{\text{int}}^B = -\frac{4\pi^2}{3\kappa^2}\int d^3x \left[\text{Tr}(C_I\bar{C}^I C_J\bar{C}^J C_K\bar{C}^K) + \text{Tr}(\bar{C}^I C_I\bar{C}^J C_J\bar{C}^K C_K)\right.$$
$$\left. +4\text{Tr}(C_I\bar{C}^J C_K\bar{C}^I C_J\bar{C}^K) - 6\text{Tr}(C_I\bar{C}^J C_J\bar{C}^I C_K\bar{C}^K)\right],$$

$$(A.3)$$

where $\epsilon^{1234} = \epsilon_{1234} = 1$ and κ is the Chern-Simons level. The matter covariant derivatives are defined as

$$D_\mu C_I = \partial_\mu C_I + i(A_\mu C_I - C_I\hat{A}_\mu), \qquad D_\mu\bar{C}^I = \partial_\mu\bar{C}^I - i(\bar{C}^I A_\mu - \hat{A}_\mu\bar{C}^I),$$
$$D_\mu\psi_I = \partial_\mu\psi_I + i(\hat{A}_\mu\psi_I - \psi_I A_\mu), \qquad D_\mu\bar{\psi}^I = \partial_\mu\bar{\psi}^I - i(\bar{\psi}^I\hat{A}_\mu - A_\mu\bar{\psi}^I).$$

$$(A.4)$$

Propagators and bilinears. The position-space propagators are obtained from those in momentum space (see e.g. [4]) by means of the following master integral

$$\int\frac{d^{3-2\epsilon}p}{(2\pi)^{3-2\epsilon}}\frac{e^{ip\cdot x}}{(p^2)^s} = \frac{\Gamma\left(\frac{3}{2}-s-\epsilon\right)}{4^s\pi^{\frac{3}{2}-\epsilon}\Gamma(s)}\frac{1}{(x^2)^{\frac{3}{2}-s-\epsilon}}.$$

$$(A.5)$$

In the Landau gauge, we have the following propagators

$$\langle (A_\mu)_i{}^j(x)(A_\nu)_k{}^l(y)\rangle = \delta_i^l\delta_k^j\left(\frac{2\pi i}{\kappa}\right)\epsilon_{\mu\nu\rho}\partial_x^\rho D(x-y),$$

$$\langle (\hat{A}_\mu)_{\hat{i}}{}^{\hat{j}}(x)(\hat{A}_\nu)_{\hat{k}}{}^{\hat{l}}(y)\rangle = -\delta_{\hat{i}}^{\hat{l}}\delta_{\hat{k}}^{\hat{j}}\left(\frac{2\pi i}{\kappa}\right)\epsilon_{\mu\nu\rho}\partial_x^\rho D(x-y).$$

$$\langle (C_I)_i{}^{\hat{j}}(x)(\bar{C}^J)_{\hat{k}}{}^l(y)\rangle = \delta_I^J\,\delta_i^l\,\delta_{\hat{k}}^{\hat{j}}D(x-y),$$

$$\langle (\psi_I)_{\hat{i}}{}^j(x)(\bar{\psi}^J)_k{}^{\hat{l}}(y)\rangle = \delta_I^J\,\delta_{\hat{i}}^{\hat{l}}\,\delta_k^j i\gamma^\mu\partial_\mu D(x-y),$$

$$(A.6)$$

where

$$D(x-y) \equiv \frac{\Gamma\left(\frac{1}{2}-\epsilon\right)}{4\pi^{\frac{3}{2}-\epsilon}}\frac{1}{((x-y)^2)^{\frac{1}{2}-\epsilon}}.$$

$$(A.7)$$

Computing the fermionic diagram contributing to the Wilson loop we have to deal with the bilinear $\eta\gamma\bar{\eta}$. Its expression in terms of the position along the line is [57]

$$(\eta_2\gamma^\mu\bar{\eta}_1) = -\frac{2}{(\eta_1\bar{\eta}_2)}\left[\frac{\dot{x}_1{}^\mu}{|\dot{x}_1|} + \frac{\dot{x}_2{}^\mu}{|\dot{x}_2|} - i\frac{\dot{x}_2{}^\lambda}{|\dot{x}_2|}\frac{\dot{x}_1{}^\nu}{|\dot{x}_1|}\epsilon_{\lambda\nu}{}^\mu\right],$$

$$(A.8)$$

where 1 and 2 denote two different points of the contour.

For our specific circuit all the possible products between η's are the following

$$\eta_1 \bar{\eta}_1 = \eta_2 \bar{\eta}_2 = 2i, \qquad\qquad \eta_1 \bar{\eta}_2 = \eta_2 \bar{\eta}_1 = 2i \cos \frac{\varphi}{2},$$

$$\eta_2 \eta_1 = -2i \sin \frac{\varphi}{2}, \qquad\qquad \bar{\eta}_1 \bar{\eta}_2 = 2i \sin \frac{\varphi}{2}. \tag{A.9}$$

Here the indices 1 and 2 label the two different edges of the cusp.

B Useful integral

In the following we want to compute the quantity A appearing in the section 4.1 and defined by

$$A = \lim_{\substack{y \to \infty \\ L_0 \to \infty}} \frac{\sqrt{MN}}{\sqrt{2}a(y + L_0)} \int_{-L_0}^{y} d\tau' \int_{-L_0}^{y} d\sigma' P^{(F)}(\sigma', \tau'). \tag{B.1}$$

It is convenient to have different upper bounds in the integrals. Using the definition of $P^{(F)}$ given by (4.9), we have:

$$\int_{-L_0}^{\tau} d\tau' \int_{-L_0}^{\sigma} d\sigma' P^{(F)}(\sigma', \tau')$$

$$= -\frac{\cos \theta/2}{\sqrt{2}\kappa} \frac{1}{\cos \varphi/2} \int_{-L_0}^{\tau} d\tau' \int_{-L_0}^{\sigma} d\sigma' \frac{d}{d\sigma'} \left(\frac{e^{\frac{1}{2}(\tau' - \sigma')}}{(\cosh(\sigma' - \tau') + \cos \varphi)^{\frac{1}{2}}} \right)$$

$$= -\frac{\cos \theta/2}{\sqrt{2}\kappa} \frac{1}{\cos \varphi/2} \left(I(\sigma, \tau) - I(-L_0, \tau) \right), \tag{B.2}$$

where we have computed the first integral using the total derivative and where we have defined:

$$I(\sigma, \tau) \equiv \int_{-L_0}^{\tau} d\tau' \frac{e^{\frac{1}{2}(\tau' - \sigma)}}{(\cosh(\sigma - \tau') + \cos \varphi)^{\frac{1}{2}}}. \tag{B.3}$$

We perform the change of variable $z = e^{\sigma - \tau'}$ and solve the first integral:

$$I(\sigma, \tau) = \sqrt{2} \int_{-L_0}^{\tau} d\tau' \frac{1}{\left[e^{2(\sigma - \tau')} + 2e^{(\sigma - \tau')} \cos \varphi + 1 \right]^{1/2}}$$

$$= -\sqrt{2} \int_{e^{(\sigma + L_0)}}^{e^{(\sigma - \tau)}} \frac{dz}{z} \frac{1}{\left[z^2 + 2z \cos \varphi + 1 \right]^{1/2}} \tag{B.4}$$

$$= -\sqrt{2} \left[-\log \left(\frac{1 + z \cos \varphi + \sqrt{z^2 + 2z \cos \varphi + 1}}{z} \right) \right]_{e^{(\sigma + L_0)}}^{e^{(\sigma - \tau)}}.$$

The second contribution is:

$$I(-L_0, \tau) = -\sqrt{2} \left[-\log \left(\frac{1 + z \cos \varphi + \sqrt{z^2 + 2z \cos \varphi + 1}}{z} \right) \right]_{1}^{e^{-(\tau + L_0)}}. \tag{B.5}$$

Summing up, we obtain:

$$I(\sigma, \tau) - I(-L_0, \tau) = \sqrt{2} \left[G(\sigma - \tau) - G(\sigma + L_0) - G(-\tau - L_0) + G(0) \right], \tag{B.6}$$

where

$$G(x) = \log\left(1 + e^x \cos\varphi + \sqrt{e^{2x} + 2e^x \cos\varphi + 1}\right).$$ (B.7)

Now setting $\tau = \sigma = y$, we have

$$I(y,y) - I(-L_0, y) = \sqrt{2}\left[2G(0) - G(-y - L_0) - G(y + L_0)\right].$$ (B.8)

For large y and L_0, we can write the following expansion:

$$I(y,y) - I(-L_0, y) \simeq -\sqrt{2}(y + L_0) + \text{const} + \mathcal{O}(e^{-(y+L_0)}).$$ (B.9)

Therefore

$$\sqrt{MN}\int_{-L_0}^{y} d\tau' \int_{-L_0}^{y} d\sigma' P^{(F)}(\sigma', \tau') \simeq \frac{\cos\theta/2}{\cos\varphi/2}\frac{\sqrt{MN}}{\kappa}(y + L_0) = \sqrt{2}a(y + L).$$ (B.10)

Recalling the definition (B.1), we obtain:

$$A = 1.$$ (B.11)

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