

Emergent space-time and the supersymmetric index

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ABSTRACT: It is of interest to find criteria on a 2d CFT which indicate that it gives rise to emergent gravity in a macroscopic 3d AdS space via holography. Symmetric orbifolds in the large N limit have partition functions which are consistent with an emergent space-time string theory with $L_{\text{string}} \sim L_{\text{AdS}}$. For supersymmetric CFTs, the elliptic genus can serve as a sensitive probe of whether the SCFT admits a large radius gravity description with $L_{\text{string}} \ll L_{\text{AdS}}$ after one deforms away from the symmetric orbifold point in moduli space. We discuss several classes of constructions whose elliptic genera strongly hint that gravity with $L_{\text{Planck}} \ll L_{\text{string}} \ll L_{\text{AdS}}$ can emerge at suitable points in moduli space.

KEYWORDS: AdS-CFT Correspondence, Conformal and W Symmetry, Field Theories in Lower Dimensions

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Contents

1	Introduction	1
2	Modular kindergarten	4
2.1	The space of weak Jacobi forms	4
2.2	NS sector genus	5
2.3	The DMVV formula	6
3	Symmetric orbifolds with slow growth	7
3.1	Basic derivation	7
3.2	Example: $\text{Sym}^N(K3)$	8
3.3	Results for $f(m)$ for other seed Jacobi forms	9
3.4	‘Very special’ weak Jacobi forms	11
3.5	Hecke operators	13
4	Some simple wreath products	14
4.1	Symmetric orbifolds of families of CFTs	14
4.2	K3-based theories	15
5	Discussion	17
A	Derivation of K3 growth	18
B	Hecke transforms of K3	19

1 Introduction

A central question in quantum gravity, after the advent of holography, has been “Which quantum field theories give rise to emergent gravity?” While the answer to this question far outruns present understanding, a logical place to start is with the class of dualities relating 2d CFTs to 3d AdS space-times. The Brown-Henneaux formula [1]

$$c = \frac{3L_{\text{AdS}}}{2G} \tag{1.1}$$

immediately suggests that a weakly curved emergent space-time will require large central charge $c \gg 1$ of the dual CFT. However, this is far from a sufficient condition. For instance, for a variety of reasons, no one expects N copies of the 2d Ising CFT with $N \gg 1$ to be dual to weakly curved, conventional gravity.

Basic conditions going beyond $c \gg 1$ have indeed been derived in recent literature. For instance, given a sequence of CFTs \mathcal{C}_N of central charges c_N asymptoting to $c \rightarrow \infty$

at large N , dual to AdS gravity theories with sequentially smaller curvatures, the simple condition that the KK spectrum converge at large N (so that at the n^{th} mass level, the number of KK modes has stabilized to a fixed, finite number) is already nontrivial. For instance, taking tensor products of a ‘seed CFT’ X already fails this test. One way to eliminate the majority of states in this example is to take an orbifold, for instance by a series \mathcal{G}_N of subgroups of the symmetric group on N objects. Sequences of this form have a convergent spectrum only if \mathcal{G}_N asymptotes to an oligomorphic group [2, 3]. As symmetric orbifolds constitute the most celebrated and widely studied set of examples of AdS₃/CFT₂ duality, results derived in this comparatively simple setting are already useful.

Beyond convergence, one can also derive density of states bounds on the resulting large c CFTs. By requiring that the number of CFT states with $\Delta > \Delta_{BH}$, with Δ_{BH} the dimension corresponding to the mass of the lightest BTZ black hole, reproduce the Bekenstein-Hawking entropies, Hartman-Keller-Stoica were able to derive a bound on the low-energy density of states of a holographic CFT [4]. The connection between the high energy (black hole) states and the low-energy theory comes about via modularity of the torus partition function. The result, roughly, is that Bekenstein-Hawking entropy is reproduced for CFTs whose low-lying spectrum of states is sparse enough, growing at most with the Hagedorn density characteristic of string theory. That is, if the partition function is of the form

$$Z = \sum_n c_n q^n, \quad q = \exp(2\pi i\tau) \tag{1.2}$$

then

$$c_n \leq e^{2\pi n}, \quad n < \frac{c}{12}. \tag{1.3}$$

The most famous examples of AdS₃/CFT₂, like the D1-D5 system on $K3$, are most easily studied at a symmetric orbifold point in moduli space. With Q_1 D1’s and Q_5 D5’s, the resulting dual CFT is a sigma model with target $\text{Sym}^{Q_1 Q_5}(K3)$. Intuitively, we expect this to describe a point far from the large radius gravity solution on $AdS_3 \times S^3 \times K3$, and this is correct. Inspired by the success of this system, it is natural to consider variations of it. Instead of taking the symmetric orbifold of $K3$ or T^4 , one can take an arbitrary seed theory \mathcal{C} . It turns out that to leading order, the spectrum is completely universal, i.e. does not depend on the details of \mathcal{C} , and is given by

$$c_n \sim \begin{cases} e^{2\pi n} & : \quad c \ll n \ll cN/6 \\ e^{2\pi\sqrt{cNn/6}} & : \quad cN/6 \ll n \end{cases} \tag{1.4}$$

More generally, one can consider permutation orbifolds, that is orbifolds by subgroups of S_N . The situation is more complicated, but the results suggest that the spectrum is again more or less independent of the details of \mathcal{C} . In fact [5] established a universal lower bound

$$c_n \sim e^{\frac{2\pi n}{\log n}} \tag{1.5}$$

for n below the black hole bound, at sufficiently large N for arbitrary \mathcal{C} . In summary, the spectrum at the orbifold point is always universal, to leading order independent of the seed theory \mathcal{C} , and grows faster than supergravity.

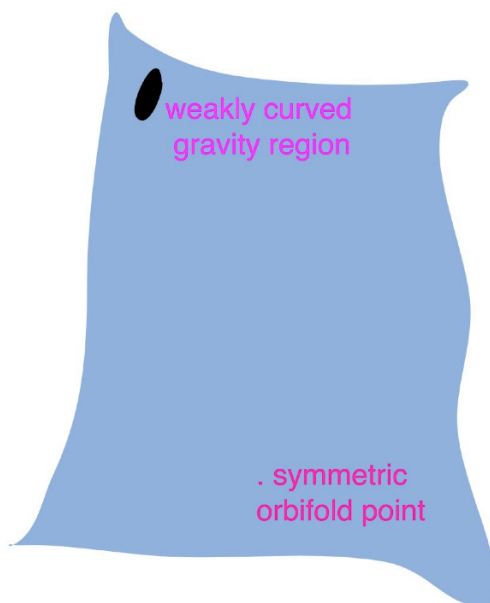


Figure 1. The moduli space of an SCFT, with the symmetric orbifold point and the small region dual to large radius gravity labeled. The elliptic genus computed at any point in the moduli space will match the sparsest spectrum arising anywhere, and so is sensitive to existence of a weakly curved gravity region.

The difficult point in finding theories dual to conventional Einstein gravity coupled to a finite tower of low-energy fields — i.e., a system with $L_{\text{Planck}} \ll L_{\text{string}} \ll L_{\text{AdS}}$ — is that the dual CFTs will certainly not live at simple points in their moduli space, like symmetric orbifold points. (This is a specific instantiation of the much more general principle of ‘conservation of misery’ which finds broad applicability in physics and life.) It would be nice to find a criterion that could detect, already at the symmetric orbifold point, the presence of a sparse spectrum — indicative of supergravity as opposed to low-tension string theory growth in the low-energy density of states. Such a criterion for 2d SCFTs (with $\mathcal{N} = (2, 2)$ supersymmetry) was proposed by Benjamin-Cheng-Kachru-Moore-Paquette [6]. The elliptic genus of a SCFT is constant on the moduli space, and therefore gives a lower bound for the number of states at each mass level valid at any point in moduli space.

The criterion proposed in [6] was that if the coefficients in the NS sector elliptic genus exhibit a growth

$$c_n \sim \exp(2\pi n^p) \tag{1.6}$$

for any $p < 1$, then the theory should admit a point in moduli space where the dual has $L_{\text{string}} \ll L_{\text{AdS}}$ (subject to a non-cancellation hypothesis discussed in detail in that reference). The D1-D5 system on K3 was checked and found to satisfy this condition.

In fact we will find that for the elliptic genus the result does depend on the seed theory \mathcal{C} . In particular, it singles out the K3 case $c = 6$. This is not altogether surprising. For a supergravity point to exist in the first place, the theory has to have an interesting moduli

space. Such a modulus has to come from the twisted sector, since untwisted sector moduli only change the seed theory, but do not move away from the orbifold point. The weight of the ground state in a twisted sector of length n is

$$\Delta = \frac{c}{24} \left(n - \frac{1}{n} \right). \tag{1.7}$$

If $c = 12$ or higher, then it is impossible to obtain twist fields with dimension $\frac{1}{2}$, which would be needed for them to be moduli of the $\mathcal{N} = (2, 2)$ theory.

A natural question is: can we give general other constraints on (or examples of) constructions which will satisfy (1.6)? That is the burden that this note undertakes. In section 2, we review salient aspects of weak Jacobi forms (which arise as elliptic genera of $(2, 2)$ SCFTs), and of the Dijkgraaf-Moore-Verlinde-Verlinde formula determining the genus of a symmetric product in terms of that of the seed CFT. In section 3, we describe examples of seed Jacobi forms which, processed via the DMVV formula, satisfy our slow-growth condition. In section 4, we discuss a specific class of constructions (labelled by a single integer a) which yields sequences of CFTs running off to infinite central charge with a spectrum (in the elliptic genus) consistent with an interpretation involving supergravity modes up to a string scale $M_{\text{string}} \ll M_{\text{Planck}}$, at which point the spectrum turns Hagedorn. Finally, we conclude in section 5 with thoughts about future directions. Some detailed derivations are relegated to appendices.

2 Modular kindergarten

In this section, we discuss some basic facts about elliptic genera of 2d SCFTs that we will use throughout the rest of the paper. We will focus on theories with $(2, 2)$ supersymmetry and central charge $c = 6M$ (with integral index M) for ease of presentation and to keep the technicalities to a minimum. Many of our considerations could be extended to theories with half-integral index, or with only $(0, 2)$ supersymmetry.

2.1 The space of weak Jacobi forms

The elliptic genus of a $(2, 2)$ SCFT is defined as

$$Z_{EG}(\tau, z) = \text{Tr}_{RR} \left((-1)^{J_0} e^{2\pi i z J_0} q^{L_0} (-1)^{\bar{J}_0} \bar{q}^{\bar{L}_0} \right). \tag{2.1}$$

In a unitary theory with discrete spectrum and spectral flow symmetry, it is a weak Jacobi form (and in particular, it is a holomorphic object — all dependence on \bar{q} cancels in the trace) [7]. For a thorough introduction to weak Jacobi forms, see e.g. [8]. A weak Jacobi form of *weight* w and *index* M satisfies the modular and elliptic transformation laws

$$\phi \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) = (c\tau + d)^w e^{2\pi i M \frac{cz^2}{c\tau + d}} \phi(\tau, z) \tag{2.2}$$

and

$$\phi(\tau, z + \ell\tau + \ell'z) = e^{-2\pi i M(\ell^2\tau + 2\ell z)} \phi(\tau, z), \quad \ell, \ell' \in \mathbb{Z}. \tag{2.3}$$

The elliptic genera we will study have vanishing weight, and their index is related to the central charge of the CFT by $c = 6M$. Weak Jacobi forms have Fourier expansions

$$\phi(\tau, z) = \sum_{m \geq 0, \ell} c(m, \ell) q^m y^\ell \quad q = e^{2\pi i \tau}, \quad y = e^{2\pi i z}. \quad (2.4)$$

We call the *polarity* of a term $4Mm - \ell^2$, and we have

$$c(m, \ell) = 0 \quad \text{if} \quad 4Mm - \ell^2 < -M^2. \quad (2.5)$$

It is an important fact that weak Jacobi terms are determined by the terms of negative polarity, which implies that for a given weight and index the space of such objects is finite dimensional. In fact, the space of weak Jacobi forms of even weight and integral index is a bi-graded ring with four generators. Two of these are the Eisenstein series E_4

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \quad (2.6)$$

and E_6

$$E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n, \quad (2.7)$$

with index 0. The others are the forms

$$\phi_{0,1}(\tau, z) = 4 \left(\frac{\theta_2(\tau, z)^2}{\theta_2(\tau, 0)^2} + \frac{\theta_3(\tau, z)^2}{\theta_3(\tau, 0)^2} + \frac{\theta_4(\tau, z)^2}{\theta_4(\tau, 0)^2} \right), \quad (2.8)$$

and

$$\phi_{-2,1}(\tau, z) = -\frac{\theta_1(\tau, z)^2}{\eta(\tau)^6} \quad (2.9)$$

with $(w, M) = (0, 1)$ and $(-2, 1)$ respectively. Given these generators, it is a simple matter to write down a basis of weight 0 and index M for any desired M .

2.2 NS sector genus

Our basic criteria for supergravity growth are most easily understood in the NS sector. Starting from the R sector elliptic genus $\phi(\tau, z)$ of index M , we can define by spectral flow the NS sector elliptic genus as

$$\chi(\tau, z) := \exp \left(2\pi i M \left(\frac{\tau}{4} + z + \frac{1}{2} \right) \right) \phi \left(\tau, z + \frac{\tau}{2} + \frac{1}{2} \right). \quad (2.10)$$

This object has the transformation properties

$$\begin{aligned} \chi(-1/\tau, z/\tau) &= (-1)^M \exp(2\pi i M z^2/\tau) \chi(\tau, z) \\ \chi(\tau + 2, z) &= (-1)^M \chi(\tau, z). \end{aligned} \quad (2.11)$$

We will be mostly interested in the specialization $\chi(\tau) := \chi(\tau, 0)$. The $\chi(\tau)$ is then invariant under a subgroup of $\Gamma = \text{SL}(2, \mathbb{Z})$. Namely let S and T denote the standard generators of Γ

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (2.12)$$

Then the congruence subgroup $\tilde{\Gamma}_\theta$ is defined as the subgroup of $\Gamma \tilde{\Gamma}_\theta \equiv \langle T^4, ST^2 \rangle$. We see that the NS sector elliptic genus is invariant under $\tilde{\Gamma}_\theta$.

It will be useful in the following to know that $\tilde{\Gamma}_\theta$ is in fact a genus zero subgroup of $SL(2, \mathbb{Z})$, and its Hauptmodul is given by

$$\kappa(\tau) = \left(\frac{2\theta_4}{\theta_2}\right)^2 - \left(\frac{2\theta_2}{\theta_4}\right)^2 = q^{-1/4} - 20q^{1/4} - 62q^{3/4} + \dots \quad (2.13)$$

Therefore, any NS elliptic genus can be expressed as a polynomial in κ . The degree of this polynomial is given by M , and it is an even or odd polynomial if M is even or odd respectively. Note in particular that

$$\kappa(-1/\tau) = -\kappa(\tau), \quad \kappa(\tau + 2) = -\kappa(\tau), \quad (2.14)$$

so that $\chi(\tau)$ picks up similar signs under S and T^2 transformations.

2.3 The DMVV formula

In what follows we will need to know the elliptic genus of symmetric orbifolds. There is a beautiful formula for the generating function of symmetric orbifold elliptic genera derived by Dijkgraaf, Moore, Verlinde and Verlinde [9]. If theory X has elliptic genus

$$Z_{EG}^X(\tau, z) = \sum_{m, \ell} c(m, \ell) q^m y^\ell, \quad (2.15)$$

then the generating function

$$\mathcal{Z}^X(p, \tau, z) = \sum_{N \geq 0} p^N Z_{EG}^{\text{Sym}^N(X)}(\tau, z) \quad (2.16)$$

is given by

$$\mathcal{Z}^X(p, \tau, z) = \prod_{n > 0, m, \ell} \frac{1}{(1 - p^n q^m y^\ell)^{c(nm, \ell)}}. \quad (2.17)$$

That is, there is a simple expression for the generating functional of the elliptic genera of symmetric products, in terms of the coefficients $c(m, \ell)$ of the seed elliptic genus.

There is an alternative way of stating this formula in terms of Hecke operators, which in particular makes manifest the modular transformation properties of $Z_{EG}^{\text{Sym}^N(X)}$. Recall that the Hecke operators T_L act on the elliptic genus via

$$T_L \phi(\tau, z) = \sum_{\substack{ad=L \\ b \text{ mod } d}} \frac{1}{L} \phi\left(\frac{a\tau + b}{d}, az\right) = \sum_{ad=L} \frac{1}{a} \sum_{m \geq 0, \ell} c(md, \ell) q^{am} y^{a\ell}. \quad (2.18)$$

In terms of the Hecke operators, the generating functional \mathcal{Z} takes the simple and elegant form

$$\mathcal{Z}(p, \tau, z) = \exp\left(\sum_L p^L T_L \phi(\tau, z)\right). \quad (2.19)$$

3 Symmetric orbifolds with slow growth

In this section, we show that combining knowledge of the DMVV formula with facts about the growth properties of the coefficients of basis Jacobi forms at each value of the index, we can find strong restrictions on the elliptic genera of a seed X if $\text{Sym}^N(X)$ is to have sub-Hagedorn growth.

3.1 Basic derivation

In this section, we consider symmetric products of seed (2,2) theory X . Define

$$M = \frac{c_X}{6} . \tag{3.1}$$

We are interested in the NS sector elliptic genus of the symmetric orbifold of this theory. Unless the vacuum contribution cancels somehow, χ^X will start out as $q^{-M/4}$. It is thus natural to multiply by $q^{M/4}$ to put the leading term in the NS sector elliptic genus at $\mathcal{O}(q^0)$. Using spectral flow we find from the DMVV formula (2.17) that

$$\begin{aligned} \sum_{N=1}^{\infty} p^N Z_{NS}^{\text{Sym}^N(X)}(\tau, z) &= \sum_{N=1}^{\infty} p^N Z_{EG,R}^{\text{Sym}^N(X)}(q, y\sqrt{q}) y^{NM} q^{NM/2} \\ &= \prod_{\substack{n \geq 1, n \in \mathbb{Z} \\ m \geq 0, m \in \mathbb{Z} \\ \ell \in \mathbb{Z}}} \frac{1}{(1 - p^n q^{m+\ell/2+Mn/2} y^{\ell+Mn})^{c(nm,\ell)}} . \end{aligned} \tag{3.2}$$

Relabeling indices for convenience as

$$\begin{aligned} m' &= m + \frac{\ell}{2} + \frac{Mn}{2} \\ \ell' &= \ell + Mn \end{aligned} \tag{3.3}$$

and dropping primes gives

$$\sum_{N=1}^{\infty} p^N Z_{NS}^{\text{Sym}^N(X)}(\tau, z) = \prod_{\substack{n \geq 1, n \in \mathbb{Z} \\ m \geq \frac{\ell}{2}, 2m \in \mathbb{Z} \\ \ell \in \mathbb{Z}, m - \frac{\ell}{2} \in \mathbb{Z}}} \frac{1}{(1 - p^n q^m y^{\ell})^{c(nm - \frac{n\ell}{2}, \ell - Mn)}} . \tag{3.4}$$

We would like to determine the coefficient of $p^N q^x$ with $x \ll N$ in (3.4), setting $y = 1$. This determines the contribution from CFT states with $\Delta = x$ in the NS sector. A term that will always be able to contribute, regardless of the power of q , is

$$\frac{1}{(1 - p)^{c(0, -M)}} \tag{3.5}$$

assuming $c(0, -M)$ does not vanish. No other term with $m = 0$ can contribute, since the relevant c exponent in the denominator of (3.4) would vanish. The remaining terms in (3.4) must contribute q^x . Suppose some combination of terms contributes $p^a q^x$. Note that a cannot be arbitrarily large due to nonvanishing of $c(nm - \frac{n\ell}{2}, \ell - Mn)$'s. Thus, in the large

N limit, we can take the remaining p^{N-a} from (3.5), and to leading order in N , we can ignore the other contributions to p .

Using the Taylor expansion of (3.5), we see that the coefficient of $q^x p^N$ with $x \ll N$ is

$$\frac{N^{c(0,-M)-1}}{(c(0,-M)-1)!} h(x) + \mathcal{O}(N^{c(0,-M)-2}) \tag{3.6}$$

where $h(x)$ is the coefficient of q^x in the expansion of

$$\prod_{\substack{m>0 \\ 2m \in \mathbb{Z}}} \frac{1}{(1-q^m)^{f(m)}}. \tag{3.7}$$

Here $f(m)$ is given by

$$f(m) := \sum_{n=1}^{\infty} \sum_{\substack{\ell \leq 2m \\ \ell \equiv 2m \pmod{2}}} c\left(nm - \frac{n\ell}{2}, \ell - Mn\right). \tag{3.8}$$

Because $c(n, \ell)$ vanishes for $4Mn - \ell^2 < -M^2$, we can write this as

$$f(m) = \sum_{n=1}^{\lceil \frac{4m}{M} \rceil} \sum_{\substack{\ell = -2m - M \\ \ell \equiv 2m \pmod{2}}}^{2m} c\left(nm - \frac{n\ell}{2}, \ell - Mn\right). \tag{3.9}$$

3.2 Example: $\text{Sym}^N(K3)$

We can use the equations (3.6), (3.7), and (3.9) to determine the growth rate of the coefficients rather efficiently. Let us first illustrate this with the case of the symmetric product K3 CFTs.

Since the K3 seed gives an elliptic genus which has index $M = 1$, (3.9) becomes

$$f(m) = \sum_{n=1}^{4m} \sum_{\substack{\ell = -2m \\ \ell \equiv 2m \pmod{2}}}^{2m} c\left(nm - \frac{n\ell}{2}, \ell - n\right). \tag{3.10}$$

We prove in appendix A that in fact, in this case

$$f(m) = \begin{cases} 28, & m \in \mathbb{Z} \\ 44, & m \in \mathbb{Z} + \frac{1}{2}. \end{cases} \tag{3.11}$$

Then following the reasoning of section 3.1, and using the fact that $c(0, -1) = 2$ for K3, the growth of the q^x term in $\text{Sym}^N(K3)$ (for $x \ll N$) goes as

$$c_{\text{Sym}^N(K3)}(x) = Nh(x) + \mathcal{O}(1), \tag{3.12}$$

where $h(x)$ is the coefficient of q^x in

$$\prod_{m=1}^{\infty} \left(\frac{1}{(1-q^m)^{28}} \frac{1}{(1-q^{m-\frac{1}{2}})^{44}} \right). \tag{3.13}$$

Now, we are in business. As far as asymptotics are concerned, for $1 \ll x \ll N$, (3.13) is equivalent to the partition function of 72 free bosons (up to subleading corrections in powers of x). So the asymptotic growth at large x is

$$h(x) \sim \exp(\sqrt{48\pi}\sqrt{x}) \tag{3.14}$$

and hence

$$c_{\text{Sym}^N(K3)}(x) \sim N \exp(\sqrt{48\pi}\sqrt{x}) . \tag{3.15}$$

This gives a sharper result than the one in [6], and indeed has parametrically slower than Hagedorn growth for states of $\Delta \ll c$.

For the symmetric orbifold of K3, the elliptic genus was analyzed carefully in [10, 11] and matched to the expected spectrum of the supergravity states.¹ Naively one might have expected a growth as $\sim e^{x^{5/6}}$ from the modes coming from $AdS_3 \times S^3$. We can obtain a slightly less naive count by considering the growth of quarter-BPS states, as opposed to any arbitrary states. This would predict a growth of $\sim e^{x^{3/4}}$, coming from two constraints: imposing that the right-movers are chiral primary (linking the right-moving weight and U(1) charge), and imposing that the supergravity modes have spin at most two (linking the left and right U(1) charges). For the elliptic genus however there are sufficiently many cancellations amongst the BPS states that reduce this growth to $\sim e^{\sqrt{x}}$. Note that the behavior of the elliptic genus is drastically different from the growth of the ordinary partition function, which exhibits Hagedorn growth [13]. There are enough cancellations to change the behavior.

3.3 Results for $f(m)$ for other seed Jacobi forms

In the previous section we discussed the case $M = 1$. Let us now investigate more general theories. We found that the properties of $f(m)$ determine the growth of the symmetric product elliptic genus. If $f(m)$ is a constant, one will obtain $e^{\sqrt{x}}$ type growth for states of energy x far below the Planck scale. On the other hand if $f(m) \sim m^\alpha$, one can write (3.7) with $q \rightarrow 1 - \epsilon$ as

$$\begin{aligned} \prod_m \frac{1}{(1 - q^m)^{m^\alpha}} &= e^{-\sum_m m^\alpha \log(1 - q^m)} \\ &= e^{\sum_m \sum_{n=1}^\infty \frac{m^\alpha}{n} q^{mn}} \\ &\sim e^{\sum_{n=1}^\infty \frac{1}{n} \frac{1}{(1 - q^n)^{\alpha+1}}} \\ &\sim e^{\sum_{n=1}^\infty \frac{1}{n} \frac{1}{(n\epsilon)^{\alpha+1}}} \\ &\sim e^{\frac{1}{(\log q)^{\alpha+1}}} . \end{aligned} \tag{3.16}$$

We estimate the q^x coefficient of (3.16) by

$$c_N(x) \sim \frac{1}{2\pi i} \oint dq e^{\frac{1}{(\log q)^{\alpha+1}} + x \log q} \tag{3.17}$$

¹A more general discussion of the matching between elliptic genera and supergravity BPS spectra in AdS_3/CFT_2 can be found in [12].

Seed	$f(\frac{1}{2})$	$f(1)$	$f(\frac{3}{2})$	$f(2)$	$f(\frac{5}{2})$	$f(3)$	$f(\frac{7}{2})$	$f(4)$	$f(\frac{9}{2})$	$f(5)$
$2\phi_{0,1}$	44	28	44	28	44	28	44	28	44	28
$\phi_{0,1}^2$	40	380	-4056	71660	-1327064	25594236	-507617240	10278253676	-211419320280	4403151249788
$\phi_{-2,1}^2 E_4$	-8	284	-4104	71564	-1327112	25594140	-507617288	10278253580	-211419320328	4403151249692
$\phi_{0,1}^3$	60	1014	1816	-14294	-644356	17951660	-147123588	-2426621198	83845804120	-729784346858
$\phi_{0,1}\phi_{-2,1}^2 E_4$	12	54	4120	-30806	-561460	17523212	-144884148	-2438633486	83911341400	-730146191018
$\phi_{-2,1}^3 E_6$	-12	-426	5704	-39062	-520012	17309420	-143764428	-2444639630	83944110472	-730327113098
$\phi_{0,1}^4$	80	2152	17984	-52844	-594784	21495912	-541151200	11318888452	-213084244240	4272269158504
$\phi_{0,1}^2\phi_{-2,1}^2 E_4$	32	232	4736	33556	-1034176	21493992	-521030848	11119909252	-212085289888	4272269156584
$\phi_{0,1}\phi_{-2,1}^3 E_6$	8	-728	-1888	81940	-1253872	21493032	-510970672	11020424836	-211585812712	4272269155624
$\phi_{-2,1}^4 E_4^2$	-16	616	-8512	131476	-1473568	21494376	-500910496	10920941572	-211086335536	4272269156968

Table 1. List of $f(m)$ for various seed theories.

which has a saddle at

$$\log q \sim x^{-\frac{1}{\alpha+2}}. \tag{3.18}$$

When plugged back into (3.17) this finally gives a degeneracy of

$$c_N(x) \sim \exp(x^{\frac{\alpha+1}{\alpha+2}}), \quad 1 \ll x \ll N. \tag{3.19}$$

Exponential growth of $f(m)$ with m likewise translates into Hagedorn (or faster) growth of the density of states, for states far below the energy of the lightest black holes.

An important point about using $f(m)$ is that, as is clear from (3.8) and (3.9), it is **linear** in the elliptic genus of the seed theory. This means that for a seed of given index M , we can determine all possible behaviors by determining the f 's associated with a basis of the Jacobi forms of weight 0 and index M . Table 1 illustrates the growth of f for the simplest, low index seed weak Jacobi forms (bases for the seed forms up through $M = 4$).

The data illustrates the difference between $M = 1$ and higher M . For $M > 1$ in the chosen basis, the $f(m)$ grow very rapidly with m . For a generic weak Jacobi form, we expect no significant cancellations. Under this assumption we can then estimate the seed theory coefficients from the Cardy formula

$$c(n, \ell) \sim \exp\left(\pi\sqrt{4Mn - \ell^2}\right). \tag{3.20}$$

In (3.9), the dominant contribution to $f(m)$ comes from the term in the sum with maximal polarity. A simple saddle point computation shows this is at

$$\begin{aligned} n_* &= \frac{2m}{M} \\ \ell_* &= 0. \end{aligned} \tag{3.21}$$

This gives

$$f(m) \sim \exp(2\pi m). \tag{3.22}$$

Note that in table 1, this is roughly the growth we see for arbitrary seed theory. This growth of $f(m)$ gives a Hagedorn growth in the density of states.

However, one can envision choosing linear combinations of the basis forms which enjoy cancellations between the (rapidly growing) $f(m)$ at a given $M > 1$. We discuss this

strategy in the next section, and find what we conjecture to be the full set of special cases where $f(m)$ grows slowly enough that the large N symmetric product exhibits supergravity (as opposed to Hagedorn) growth in some regime beneath the lightest black hole mass.

3.4 ‘Very special’ weak Jacobi forms

In this subsection, we define a special class of weight 0 weak Jacobi forms, the ‘very special’ weak Jacobi forms. These will be precisely the forms at a given index that (we conjecture) give sub-Hagedorn growth of the polar spectrum, when run through the DMVV formula.

To motivate the definition of ‘very special’ Jacobi forms, let us consider the NS sector elliptic genus with $z = 0$, which we can call $Z_{NS}(\tau)$. At index M , $Z_{NS}(\tau)$ can be written as a polynomial of degree M in $\kappa(\tau)$, defined in (2.13). Moreover, if M is odd, the polynomial contains only odd powers of κ ; if M is even, only even powers.

We define a ‘very special’ weak Jacobi form as one which satisfies

$$Z_{NS}(\tau) = v\kappa(\tau) = vq^{-1/4} + \dots \tag{3.23}$$

for some positive integer v . Note that very special weak Jacobi forms only exist for odd M . If there are no cancellations in the genus, at index M , then the NS-sector weak Jacobi form starts off as

$$Z_{NS}(\tau) \sim q^{-M/4} + \dots \tag{3.24}$$

Thus to become a very special weak Jacobi form at index M , many delicate cancellations are needed (from the right-moving Ramond ground states) for the first $\frac{M-1}{2}$ terms. For instance, the elliptic genus of K3, $2\phi_{0,1}$, is a very special weak Jacobi form with $M = 1$, and $v = 2$; the function $\phi_{0,1}^3 - \phi_{0,1}E_4\phi_{-2,1}^2$ is very special with $M = 3$ and $v = 48$.

Now we will derive the analogue of (3.9) for seed theories with vanishing “vacuum” terms. (Recall that in deriving (3.9), we assumed that $c(0, -M)$ was nonzero.) Suppose we are considering a seed theory which has the first a terms in $Z_{NS}(\tau)$ vanishing. In other words,

$$Z_{NS}(\tau) = vq^{\frac{-M+2a}{4}} + \dots \tag{3.25}$$

Let’s again define a generating function for the elliptic genus of $\text{Sym}^N(X)$ in the NS sector shifting energy so that the first nonzero term appears at q^0 . The analog of (3.2) is then

$$\begin{aligned} \sum_{N=1}^{\infty} p^N Z_{NS}^{\text{Sym}^N(X)}(\tau, z) &= \sum_{N=1}^{\infty} p^N Z_{EG,R}^{\text{Sym}^N(X)}(q, y\sqrt{q}) y^{NM} q^{NM-a/2} \\ &= \prod_{\substack{n \geq 1, n \in \mathbb{Z} \\ m \geq 0, m \in \mathbb{Z} \\ \ell \in \mathbb{Z}}} \frac{1}{(1 - p^n q^{m+\ell/2+Mn/2-an/2} y^{\ell+Mn})^{c(nm,\ell)}}. \end{aligned} \tag{3.26}$$

Relabelling

$$\begin{aligned} m' &= m + \frac{\ell}{2} + \frac{nM}{2} - \frac{an}{2} \\ \ell' &= \ell + Mn \end{aligned} \tag{3.27}$$

and dropping primes, we get

$$\sum_{N=1}^{\infty} p^N Z_{NS}^{\text{Sym}^N(X)}(\tau, z) = \prod_{\substack{n \geq 1, n \in \mathbb{Z} \\ m - \frac{\ell}{2} + \frac{an}{2} \in \mathbb{Z} \\ \ell \in \mathbb{Z}}} \frac{1}{(1 - p^n q^m y^\ell)^{c(nm - \frac{\ell n}{2} + \frac{an^2}{2}, \ell - Mn)}}. \quad (3.28)$$

In analogy to (3.9), we see that the growth of this, to leading order in N at large N , is given by the q^x coefficient of

$$\frac{1}{(1 - q^m)^{f(m)}} \quad (3.29)$$

where

$$f(m) = \sum_{n=1}^{\infty} \sum_{\ell}^* c\left(nm - \frac{\ell n}{2} + \frac{an^2}{2}, \ell - Mn\right). \quad (3.30)$$

In (3.30), the $*$ indicates the same limits on ℓ as in (3.28). In particular, if both a and n are odd, ℓ is odd for integer m and even for half-integer m ; if one of a or m is even, ℓ is even for integer m and odd for half-integer m .

Empirically we find that for very special weak Jacobi forms, $f(m)$ does not grow with m , but alternates between two values depending on if m is integral or half-integral. This implies a growth of states as

$$c_{\Delta} \sim e^{\sqrt{\Delta}}. \quad (3.31)$$

In particular, we find for a seed weak Jacobi form of index M that satisfies

$$Z_{NS}(\tau) = v\kappa(\tau), \quad (3.32)$$

we get

$$f(m) = \begin{cases} \frac{\chi}{2} + 8v & m \in \mathbb{Z} \\ \frac{\chi}{2} + 16v & m \in \mathbb{Z} + \frac{1}{2} \end{cases} \quad (3.33)$$

where χ is the Witten index of the seed theory. For K3, $v = 2$ and $\chi = 24$, which indeed reproduces (3.11).

The physical interpretation of the sub-Hagedorn growth of higher index very special weak Jacobi forms is unclear. In particular, for these theories, we are counting states $(\frac{M-1}{4})N$ above the vacuum. The Planck mass is at $\frac{MN}{4}$ above the vacuum, so at large M , we are counting states extremely close to the Planck mass. The fact that all the lower terms vanish simply means that we do not get a meaningful bound for the number of such light states.

Note that the left-moving vacuum always leads to a contribution at $q^{-M/4}$. The only way to eliminate this contribution is if the left-moving vacuum does not just couple to the right-moving vacuum, but also to additional primary fields with differing fermion number. This implies that the theory has an enhanced symmetry. A trivial example of this is the case of T^4 , where the entire elliptic genus vanishes. This probably simply implies that the standard elliptic genus is too coarse an index to give any meaningful information on the

number of light states, and that one should consider a more refined index, such as the one introduced in [14] for the T^4 case.

Empirically we find that for any other non-constant $Z_{NS}(\tau)$, the $f(m)$ grow exponentially, giving Hagedorn growth. (The final case, if $Z_{NS}(\tau)$ is a constant, leads to the symmetric product elliptic genus having only one non-vanishing term, corresponding to taking N copies of the non-vanishing term, and thus does not have interesting growth.) In the next subsection we will take some steps towards proving this.

3.5 Hecke operators

To prove this conjecture, it seems natural to use the language of Hecke operators and start from (2.19). First consider the case $M = 1$. The specialization χ_L of the L^{th} Hecke transform of the unique weak Jacobi form $\phi_{0,1}$ is given by

$$\chi_L(q) = \frac{1}{L}q^{-L/4} + \sum_{ad=L} \chi^0(d) + o(q) . \tag{3.34}$$

where

$$\chi^0(d) = \begin{cases} a^{-1}c(0) = \frac{10}{a} & d \equiv 0 \pmod{4} \\ a^{-1}2c(-1) = \frac{2}{a} & d \equiv 2 \pmod{4} \\ 0 & \text{else} \end{cases} \tag{3.35}$$

as is shown in appendix B.

For L even/odd, we can write

$$\chi_L(q) = \sum_{n=-L/4}^{\infty} d(n)q^n \tag{3.36}$$

as an even/odd polynomial in κ . Using (2.14), we can obtain the asymptotic behavior by repeating the Cardy argument. This gives

$$d_L(n) \sim (-1)^L (16n^3L^3)^{-1/4} e^{2\pi\sqrt{Ln}} , \tag{3.37}$$

where the sign $(-1)^L$ comes from the behavior of κ under the S transform. The generating function for the symmetric orbifold elliptic genus is

$$\mathcal{Z} = \exp \sum_L p^L T_L \phi . \tag{3.38}$$

For the shifted NS partition function at $N = \infty$ we thus have

$$\chi_\infty = \exp \sum_L (q^{L/4} \chi_L(q) - 1/L) , \tag{3.39}$$

so that that the n^{th} term of $\log \chi_\infty(q)$ is

$$d(n) = \sum_{L=1}^{4n} d_L(n - L/4) . \tag{3.40}$$

The biggest term in (3.40) is thus

$$d_{2n}(n) \sim 2n^{-3/2} e^{2\pi n} . \tag{3.41}$$

If there were no cancellations, we would thus have Hagedorn growth already in the exponent, so that $\chi_\infty(q)$ would have Hagedorn growth too. Crucially, because of the factor $(-1)^L$, there are a lot of cancellations. In fact

$$\log \chi_\infty(q) = -22\sqrt{q} + 25q - \frac{1}{3}88q^{3/2} + \frac{53q^2}{2} - \frac{132q^{5/2}}{5} + \dots \tag{3.42}$$

so that the coefficients grow very slowly. This explains why we do not get Hagedorn growth in this case.

For $M = 2$, the Hecke transforms give

$$\chi_L(q) = \frac{1}{L} q^{-L/2} + \mathcal{O}(1) . \tag{3.43}$$

Note that χ_L for the different weak Jacobi forms differ only in the constant term. Also note that now for every L we get an even polynomial in κ , which means that all coefficients $d_L(n)$ are positive. This means that there are no cancellations, so that we indeed get Hagedorn growth.

We believe that there should be a similar argument that establishes Hagedorn growth for all weak Jacobi forms that are not very special.

4 Some simple wreath products

4.1 Symmetric orbifolds of families of CFTs

The results of our previous section suggest that the only symmetric orbifolds that can have supergravity growth come from seed theories with $c = 6$. To find other examples, let us therefore generalize the discussion to more general permutation orbifolds.

In particular, consider $\text{Sym}^a(\mathcal{C}_b)$ with a fixed and $b = 1, 2, 3, \dots$ indexing the sequence of CFTs \mathcal{C}_b asymptoting to infinite central charge. Eventually we will be interested in the elliptic genus of $\text{Sym}^a(\text{Sym}^b(K3))$, but let us first discuss some general points.

Let us start by discussing partition functions rather than elliptic genera. This has the advantage that no cancellations can occur. We will see that that behavior of the two is very similar. Take a partition function Z_b of central charge b . If we do not make any additional assumptions, the symmetric orbifold $\text{Sym}^a(Z_b)$ will have the following behavior:

$$\rho_{a,b}(\Delta) \sim \begin{cases} e^{2\pi\sqrt{ab\Delta/6}} & \Delta > ab/6 \\ e^{2\pi\Delta} & ab/6 > \Delta > b \\ \text{non-universal} & b > \Delta \end{cases} \tag{4.1}$$

To say something about the region $\Delta < b$, we need to make additional assumptions on Z_b . Assume that the growth of Z_b is governed by an effective central charge \hat{c} such that

$$\rho_b(\Delta) \sim e^{2\pi\sqrt{\hat{c}\Delta/6}} . \tag{4.2}$$

Examples of such families are for instance the putative extremal CFTs or indeed the holographic dual to the D1-D5 system at the supergravity point. First note that the twisted states of the Sym^a orbifold have weight $\Delta_{tw} > b/16$ and thus do not contribute here. The number of states in the untwisted sector coming from K non-trivial states and $a - K$ vacuum states is

$$\sim \frac{e^{2\pi\sqrt{\hat{c}\Delta K/6}}}{K!} \sim e^{2\pi\sqrt{\hat{c}\Delta K/6} - K \log K + K} \tag{4.3}$$

This is maximized for $K \sim \frac{\pi^2 \hat{c} \Delta / 6}{(\log \pi^2 \hat{c} \Delta / 6)^2}$ and gives

$$\rho_{ab} \sim \exp\left(\frac{\pi^2 \hat{c} \Delta / 6}{\log \pi^2 \hat{c} \Delta / 6}\right). \tag{4.4}$$

This growth is sub-Hagedorn, but it is much faster than supergravity. Note that $K < a$. If $a < b$, we roughly have

$$\rho_{ab}(\Delta) \sim \begin{cases} e^{2\pi\sqrt{\hat{c}a\Delta/6}} & b > \Delta > a \\ e^{\frac{\pi^2 \hat{c} \Delta / 6}{\log \pi^2 \hat{c} \Delta / 6}} & a > \Delta \end{cases} \tag{4.5}$$

This suggests that there is a new phase in the regime $a > \Delta$. Depending whether $a < b$, there is also a supergravity phase for $a < \Delta < b$.

4.2 K3-based theories

Consider a theory where the seed (NS-sector) elliptic genus has the form

$$Z_{NS}(q) = \sum_{n=0}^{\infty} c(n) q^{n-b/4} \tag{4.6}$$

where

$$c(n) = \begin{cases} e^{\sqrt{n}} & n < b/4 \\ e^{\sqrt{bn}} & n > b/4 \end{cases} \tag{4.7}$$

This is, for instance, the case for $\text{Sym}^b(K3)$ at large b .

Now, consider $\text{Sym}^a(\text{Sym}^b(K3))$, with a fixed and taking b large. The value of a labels which family of CFTs we are considering, and b is taken to infinity in each family. The key point is that by introducing a new parameter a , we hope to introduce a new scale in the problem. With this in mind, we estimate the growth of states in four different regimes: states with dimension $\Delta \ll a$, $a \ll \Delta \ll b$, $b \ll \Delta \ll ab$, and $ab \ll \Delta$.

For states with $\Delta \ll a$, we of course cannot excite states in all a copies of the seed theory. Suppose we excite states in some k copies to the $\frac{\Delta}{k}$ level. This would give us a degeneracy of roughly

$$\sim \frac{\left(e^{\sqrt{\frac{\Delta}{k}}}\right)^k}{k!} \sim \frac{e^{\sqrt{\Delta k}}}{k!} \sim e^{\sqrt{\Delta k} - k \log k + k}. \tag{4.8}$$

This is maximized for $k \sim \frac{\Delta}{(\log \Delta)^2}$ and gives

$$c_{\text{Sym}^a(\text{Sym}^b)}(\Delta) \sim \exp\left(\frac{\Delta}{\log \Delta}\right). \tag{4.9}$$

In the regime of states with $a \ll \Delta \ll b$, one only draws states (to symmetrize) from the upper case in (4.7). The leading growth is easily estimated via saddle point (roughly coming from taking a contribution to Δ of $\mathcal{O}(\Delta/a)$ from the a copies of $\text{Sym}^b(K3)$), and gives growth

$$c_{\text{Sym}^a(\text{Sym}^b)}(\Delta) \sim \exp(\sqrt{a\Delta}). \tag{4.10}$$

In the regime where $b \ll \Delta \ll ab$, we can suddenly draw from black hole states in the seed theory in (4.7). Taking states with energy $\mathcal{O}(b)$ from $\frac{\Delta}{b}$ copies of the seed gives a growth

$$c_{\text{Sym}^a(\text{Sym}^b)}(\Delta) \sim \exp(\Delta). \tag{4.11}$$

This shows a Hagedorn density of states, which hints that stringy states show up in the dual at this energy.

Finally in the regime where $ab \ll \Delta$, we are in the Cardy regime of the full theory, which means

$$c_{\text{Sym}^a(\text{Sym}^b)}(\Delta) \sim \exp(\sqrt{ab\Delta}). \tag{4.12}$$

Thus in total the growth goes as

$$\rho(\Delta) = \begin{cases} \exp\left(\frac{\Delta}{\log \Delta}\right) & \Delta \ll a \\ \exp(\sqrt{a\Delta}) & a \ll \Delta \ll b \\ \exp(\Delta) & b \ll \Delta \ll ab \\ \exp(\sqrt{ab\Delta}) & ab \ll \Delta. \end{cases} \tag{4.13}$$

In our derivation of the first three lines of (4.13), we have only considered states in the untwisted sector of the Sym^a orbifold. However, states in the twisted sector of the Sym^a orbifold will start contributing at $\Delta \sim \mathcal{O}(b)$, so the neglect of the twisted sector states in the first two lines of (4.13) is justified. Moreover, the partition function in the regime $b \ll \Delta \ll ab$ grows exponentially, as seen in e.g. [5, 13], which puts an upper bound on the growth of the genus. Combining this with the lower bound from the untwisted sector, we see that the third line in (4.13) is also accurate. The fourth line comes from Cardy formula, which of course takes into account the twisted sectors.

In figure 2, we show a plot of the growth of states of $\text{Sym}^2(\text{Sym}^{40}(\phi_{0,1}))$, where we see evidence of (4.13) in the different regimes.² This is all compatible with our initial observation about the existence of moduli in the twisted sector: $\text{Sym}^b(K3)$ does have moduli with which we can deform away from the orbifold point. In this way we can move to a point in the moduli space where the growth of states in the full partition function is as in (4.2). By the previous analysis, Sym^a of this will then indeed be as in (4.13).

²We compute this for $\phi_{0,1}$ instead of the elliptic genus of K3; these differ by a factor of 2. This is simply to avoid the large Ramond ground state degeneracy of the K3 case, which shows up in the elliptic genus. One can achieve $\phi_{0,1}$ as the elliptic genus of a known manifold — the Enriques surface [15].

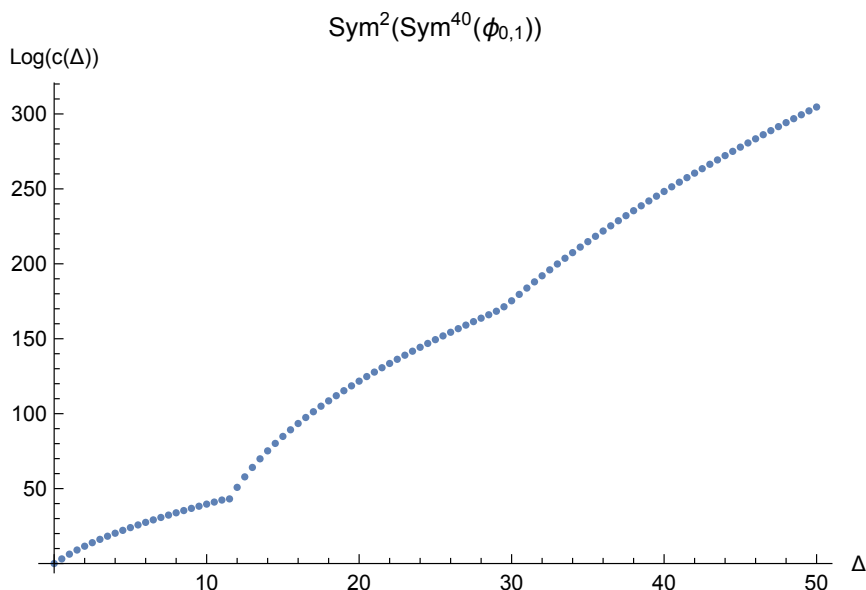


Figure 2. Growth of coefficients in NS sector of $\text{Sym}^2(\text{Sym}^{40}(\phi_{0,1}))$. Note the three very distinctive regions of $a < \Delta < \frac{b}{4}$, $\frac{b}{4} < \Delta < \frac{ab}{4}$, and $\frac{ab}{4} < \Delta$. (The region $\Delta < a$ is too small here to notice.)

5 Discussion

The main focus of this paper was to provide examples where the elliptic genus, a 2d supersymmetric index, could be a useful indicator of an emergent macroscopic space-time geometry that is computable directly in quantum field theory. The physical criteria which slow-growth of the genus indicate, include a sparse spectrum of low-energy modes, or equivalently (via the modularity arguments in [4, 6]) a good match to Bekenstein-Hawking entropies of BPS black holes. Another equivalent criterion (though not manifestly so) requires the existence of a conventional Hawking-Page phase transition. The main advantages of the genus relative to the full partition function are that it is computable, and (subject to a non-cancellation hypothesis) it captures the growth characteristic of the sparsest regions in moduli space. It thereby serves immediately as an indicator of whether *any* point in the CFT moduli space can be dual to weakly curved gravity.

There are other natural candidates to serve as order parameters for emergent space-time. The entanglement entropy at large central charge enjoys various special properties [16], and matching entanglement computed via the Ryu-Takayanagi formula in suitable geometries to the entanglement in a given quantum field theory could serve as a possible test. More recently, it has been proposed that CFTs whose out-of-time-order four point functions saturate a certain ‘chaos bound’ may be dual to Einstein gravity [17]. It is interesting to consider the precise relationship between these criteria as indicators of emergent space-time geometry; this is the subject of ongoing research [18].

Even within the world of 2d supersymmetric CFTs and density of states bounds, much work remains to be done. We focused here on theories with (2, 2) supersymmetry. While our work could be easily generalized to classes of theories with less supersymmetry, a wide class

of $(0, 4)$ theories which arise on MSW black strings [19] — the strings arising as M5-branes wrapping divisors in Calabi-Yau threefold compactifications of M-theory — do not fall into this category. For these theories, the conventional elliptic genus is uninteresting, and one should compute instead an improved object, the M5-brane elliptic genus of [20]. This object is still modular, but has more involved properties than a weak Jacobi form. Even for a small number of M5-branes, the computations involved become quite sophisticated [21]. It would be interesting to develop a generalization of the criteria derived in [6] and explored in this paper, to that class of theories.

A related issue is the existence of theories with enhanced symmetry, where the elliptic genus fails to capture the BPS information, but a refined index can do so [14, 22]. It seems likely that by suitable refinement, one can define an improved index that could capture more information about the spectra of the theories yielding ‘very special’ Jacobi forms as their elliptic genera.

Finally, we note that the modular properties of the elliptic genus have been crucial in all of our considerations. However, it is clear that the individual terms in the q, y expansion are themselves constants across the moduli space. So, for instance, instead of considering the NS elliptic genus at $y = 1$, one could gain more information by taking a fixed power of q in the NS genus and summing the absolute values of the coefficients of the various powers of y which appear. The resulting counting function would certainly not have any elegant modular properties. But numerical studies of this function could yield more information about the detailed growth of the spectrum in various classes of theories (for instance, those where the coefficients vanish after specialization to $y = 1$).

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A Derivation of K3 growth

In this appendix, we will derive equation (3.11), namely that for seed theory K3,

$$f(m) = \sum_{n=1}^{\infty} \sum_{\substack{\ell=-2m \\ \ell \equiv 2m \pmod{2}}}^{\ell=2m} c\left(nm - \frac{n\ell}{2}, \ell - n\right) = \begin{cases} 28 & m \in \mathbb{Z} \\ 44 & m \in \mathbb{Z} + \frac{1}{2}. \end{cases} \tag{A.1}$$

First note that for K3, from spectral flow $c(n, \ell) = c(4n - \ell^2)$. Thus we can rewrite (A.1) as

$$f(m) = \sum_{\substack{\ell=-2m \\ \ell \equiv 2m \pmod{2}}}^{\ell=2m} \sum_{n=1}^{\infty} c(4mn - n^2 - \ell^2). \tag{A.2}$$

Define $\tilde{n} = n - 2m$ to get

$$\begin{aligned} f(m) &= \sum_{\substack{\ell=-2m \\ \ell \equiv 2m \pmod{2}}}^{\ell=2m} \sum_{\tilde{n}=-2m+1}^{\infty} c(4m^2 - \tilde{n}^2 - \ell^2) \\ &= \sum_{\substack{\ell=-2m \\ \ell \equiv 2m \pmod{2}}}^{\ell=2m} \left(\sum_{\tilde{n}=-\infty}^{\infty} c(4m^2 - \tilde{n}^2 - \ell^2) \right) - c(-\ell^2) \\ &= \sum_{\substack{\ell=-2m \\ \ell \equiv 2m \pmod{2}}}^{\ell=2m} \left(\sum_{\tilde{n}=-\infty}^{\infty} c\left(m^2 - \frac{\ell^2}{4}, \tilde{n}\right) \right) - c(0, \ell). \end{aligned} \tag{A.3}$$

The first term in (A.3) vanishes unless $m^2 - \frac{\ell^2}{4} = 0$ (this is the statement that Witten index has no q -dependence). These are precisely the $\ell = \pm 2m$ terms in the sum, which each contribute $\chi_{K3} = 24$. The second term, if $m \in \mathbb{Z}$, is $c(0, 0) = 20$; if $m \in \mathbb{Z} + \frac{1}{2}$, is $c(0, 1) + c(0, -1) = 4$. Thus we reproduce (A.1).

B Hecke transforms of K3

Let us gather the contributions to the polar and the constant terms of χ . Define the polarity as $A := m + \ell/2 + d/4$ and $d = 4m + d'$ such that $A = d/2 + \ell/2 - d'/4$. We know that $c(m, \ell)$ vanishes if

$$4Mm - \ell^2 < -M^2. \tag{B.1}$$

The condition on ℓ is thus $\ell^2 \leq d^2 - d'd + 1$. We can then estimate

$$A \geq \frac{1}{2} \frac{d'^2/4 - 1}{d - d'/2 + \sqrt{d^2 - d'd + 1}}. \tag{B.2}$$

With $d' \leq d$ it follows automatically that polar terms can only come from $|d'| < 2$, and constant terms from $|d'| \leq 2$. Analyzing case by case we find that the only polar term comes from $d' = 1, d = 1, a = L$ with $m = 0$ and $\ell = -1$. Define $\chi_L(q)$ as the spectrally flowed and specialized Hecke transform of ϕ . We thus have

$$\chi_L(q) = \frac{1}{L} q^{-L/4} + \mathcal{O}(1). \tag{B.3}$$

The constant terms come from $d' = -2, 0, 2$. For $d' = 0$, we have a contribution from $m = d/4, \ell = -d$. For $d' = \pm 2$, we get a contribution from $m = (d \mp 2)/4, \ell = -d \pm 1$.

Since m has to be integer, in total we get

$$\chi^0(d) = \begin{cases} a^{-1}c(0) = \frac{10}{a} & d \equiv 0 \pmod{4} \\ a^{-1}2c(-1) = \frac{2}{a} & d \equiv 2 \pmod{4} \\ 0 & \text{else} \end{cases} \quad (\text{B.4})$$

The total result is thus

$$\chi_L(q) = \frac{1}{L}q^{-L/4} + \sum_{ad=L} \chi^0(d) + o(q) . \quad (\text{B.5})$$

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