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Higher derivative corrections to charged fluids in 2n dimensions

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Abstract: We study anomalous charged fluid in $2n$-dimensions ($n \geq 2$) up to sub-leading derivative order. Only the effect of gauge anomaly is important at this order. Using the Euclidean partition function formalism, we find the constraints on different sub-leading order transport coefficients appearing in parity-even and odd sectors of the fluid. We introduce a new mechanism to count different fluid data at arbitrary derivative order. We show that only the knowledge of independent scalar-data is sufficient to find the constraints. In appendix we further extend this analysis to obtain fluid data at sub-sub-leading order (where both gauge and gravitational anomaly contribute) for parity-odd fluid.

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JHEP05(2015)010

Contents

JHEP05(2015)010

1 Introduction and summary

In past few years there has been much interest and progress in further understanding of relativistic, charged, dissipative fluid in presence of some global anomalies. Presence of quantum anomalies play a crucial role in transport properties of fluid. The first evidence of quantum anomaly in fluid transport was holographically observed in [1, 2]. The authors found a new parity-odd term (and hence a new transport coefficient) in the charge flavour current. The origin of this new term can be traced back to gauge Chern-Simons term in the dual supergravity theory. Soon after these results were published, it was shown that the new parity-odd term in the charge current is essential because of the triangle flavour anomalies and the second law of thermodynamics [3]. In general the second law of thermodynamics (or equivalently the positivity of divergence of entropy current) imposes constraints on different transport coefficients. The same constraint can also be obtained from the equilibrium partition function of fluid [4, 5]. Equilibrium partition function provides an alternate and a microscopically more transparent way to derive the constraints on these transport coefficients. A generalization of this approach for charged U(1) anomalous fluid in arbitrary even dimensions up to leading order has been considered in [6].

In [7] Bhattacharyya et al. studied parity odd transport for a four dimensional non-conformal charged fluid at second order in derivative expansion. In four spacetime dimensions the effect of anomaly appears at one derivative order and the parity-odd transport coefficients at this order are determined in terms of anomaly coefficient. In this paper the authors studied the transport properties at second order and found that out of 27 transport coefficients 7 are fixed in terms of anomaly and lower order transport coefficients. The goal of our current paper is to generalize this work to arbitrary even dimensions. In $2n$ spacetime dimensions the leading effect of anomaly appears at $(n-1)$ derivative order. Hence the subleading corrections appear at nth derivative order. The aim of this paper is to study the constraints on transport coefficients appearing at subleading order. We innovate a systematic mechanism to compute different fluid data at arbitrary derivative order (parity odd or even). We list all possible scalars, vectors and tensors at any arbitrary derivative order in this paper. It seems to be rather difficult to find the independent sets. However, we argue that it is possible to get the correct constraint relations between transport coefficients even without knowing the independent sets of fluid data.

Our analysis is not valid in two spacetime dimensions. In two dimensions the parity odd terms appear at zero derivative order itself, and hence parity-odd and parity even sectors are not independent at any arbitrary order. Independence of these two sectors is important in our computation.

In the parity-even sector, the leading correction appears at first order in derivative expansion, e.g. shear viscosity and bulk viscosity terms in energy momentum tensor etc. In this paper we have extended our calculation to include the sub-leading order correction (i.e. second order corrections) to parity-even sector in constitutive relations in arbitrary even dimensions in presence of U(1) gauge anomaly. This completes the description of fluid dynamics up to sub-leading order in derivative expansion (both in parity-odd and even sectors) in arbitrary even dimensions with *abelian gauge anomaly*.

JHEP05(2015)010

The organization of our paper is as following. In section 2 we explain our notation and perturbation scheme which we use in this paper. In section 3 we construct the partition function for both gauge invariant and non-invariant sectors and compute the constitutive relations from the partition function. We also describe the construction of the anomalous entropy current. Section 4 is the most important section of this paper. Here we first describe how to construct fluid data at arbitrary derivative order. Next, we list all the leading and sub-leading order scalars, vectors and tensors which may appear in constitutive relations up to sub-leading order in derivative expansion both in parity-even and odd sectors. Although, we have not been able to find the *'independent'* parity-odd vectors and tensors at sub-leading order, this does not inhibit us from finding the constraints on the transport coefficients. We elaborate this issue in section 4.4. Finally, in section 5 we list the constraint on the transport coefficients up to sub-leading order. In appendices we explain the Kaluza-Klien decomposition (appendix A) and sub-sub-leading order counting (appendix B).

2 Scheme and the perturbative expansion

We consider a $2n$-dimensional spacetime manifold $\mathcal{M}_{(2n)}$ with metric $\mathrm{d}s^2 = G_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$ and gauge field 1-form $\mathcal{A} = \mathcal{A}_\mu \mathrm{d}x^\mu$. We want to study fluid dynamics in this background. A fluid is a statistical system in local thermodynamic equilibrium, which is generally characterized in terms of (covariant) *energy-momentum tensor* $\bar{\mathcal{T}}^{\mu\nu}$, (covariant) *charge current* $\bar{\mathcal{J}}^\mu$ and their constitutive equations

$$\hat{\nabla}_\mu \bar{\mathcal{T}}^{\mu\nu} = \mathcal{F}^{\nu\rho}\bar{\mathcal{J}}_\rho + \underset{\sim}{\mathfrak{T}}^\nu, \qquad \hat{\nabla}_\mu \bar{\mathcal{J}}^\mu = \underset{\sim}{\mathfrak{J}}. \tag{2.1}$$

$\mathcal{F} = \mathrm{d}\mathcal{A}$ is field strength for $\mathcal{A}$. Here we have introduced a U(1) anomaly $\underset{\sim}{\mathfrak{J}}$ and a gravitational anomaly $\underset{\sim}{\mathfrak{T}}^\nu$. The form of these anomalies is well known in literature [8]. Most of our work here will be concentrated on fluid upto subleading derivative order, where only U(1) anomalies contribute:[1]

$$\underset{\sim}{\mathfrak{J}} = (n+1)C^{(2n)} \star \mathcal{F}^{\wedge n} = (n+1)C^{(2n)} \frac{1}{2^n} \epsilon^{\mu_1\nu_1\cdots\mu_n\nu_n} \mathcal{F}_{\mu_1\nu_1}\ldots\mathcal{F}_{\mu_n\nu_n}. \tag{2.2}$$

$\underset{\sim}{\mathfrak{T}}^\nu$ only starts getting values at subsubleading derivative order. Let us explain our notation here.

- All the fluid quantities (like currents, transport coefficients, independent terms etc.) appearing in parity-odd sector, are denoted by 'tilde' (e.g. $\tilde{A}$). On the other hand we use no special notation for parity-even sector (e.g. A). Wherever applicable, $\bar{A} = A + \tilde{A}$ denotes the total quantity (parity-odd and parity-even).

[1] This is true in derivative counting, which we are interested in. It was shown in [11] that the naive derivative expansion breaks down on a cone, and hence the effects of gravitational and mixed anomaly can be seen at first derivative order itself. In the derivative expansion however, the corresponding coefficients appear as undermined constants, as we shall see in section 5.2.1.

JHEP05(2015)010

- $\hat{\nabla}$ and ∇ denote the covariant derivative and on $\mathcal{M}_{(2n)}$ and the equilibrium manifold $\mathcal{M}_{(d-1)}$ respectively. We use $\wedge$ and $\star$ as wedge product and Hodge Dual on all manifolds, as no confusion is possible.

Due to dissipative nature of fluid, it is not possible to write an exact generating functional W (or action) for fluids from which one can derive the energy-momentum tensor $\bar{\mathcal{T}}^{\mu\nu}$ and charge current $\bar{\mathcal{J}}^{\mu}$. Therefore we write their most generic forms, allowed by symmetries, in terms of fundamental fluid variables and their derivatives in a particular thermodynamic ensemble. In our analysis we consider the fluid variables to be *temperature* ϑ, *chemical potential*[2] ν and *fluid four-velocity* u^{μ} with $u^{\mu}u_{\mu} = -1$.

We prefer to work in *Landau Frame*, where all the dissipation terms are transverse to the direction of the fluid flow. Hence, we can decompose $\bar{\mathcal{T}}^{\mu\nu}$ and $\bar{\mathcal{J}}^{\mu}$ as

$$\bar{\mathcal{T}}^{\mu\nu} = E(\vartheta,\nu)u^{\mu}u^{\nu} + \bar{\Pi}^{\mu\nu}, \quad \bar{\mathcal{J}}^{\mu} = Q(\vartheta,\nu)u^{\mu} + \bar{\Upsilon}^{\mu}, \tag{2.3}$$

where $\bar{\Pi}^{\mu\nu}$ and $\bar{\Upsilon}^{\mu}$ are the most generic symmetric tensor and vector made out of fluid variables. In the Landau frame

$$u_{\mu}\bar{\Pi}^{\mu\nu} = 0, \qquad u_{\mu}\bar{\Upsilon}^{\mu} = 0. \tag{2.4}$$

The easiest way to implement this is to project all vectors or tensors appearing in $\bar{\Upsilon}^{\mu}$ or $\bar{\Pi}^{\mu\nu}$, transverse to u^{μ} using the projection operator

$$P^{\mu\nu} = G^{\mu\nu} + u^{\mu}u^{\nu}.$$

Since fluid is a low energy fluctuation about the local thermodynamic equilibrium, $\bar{\Pi}^{\mu\nu}$ and $\bar{\Upsilon}^{\mu}$ can be expanded in derivatives of fundamental fluid variables $(\vartheta, \nu, u^{\mu})$:

$$\bar{\Pi}^{\mu\nu} = \bar{\Pi}^{\mu\nu}_{(0)} + \bar{\Pi}^{\mu\nu}_{(1)} + \bar{\Pi}^{\mu\nu}_{(2)} \ldots, \qquad \bar{\Upsilon}^{\mu} = \bar{\Upsilon}^{\mu}_{(0)} + \bar{\Upsilon}^{\mu}_{(1)} + \bar{\Upsilon}^{\mu}_{(2)} \ldots, \tag{2.5}$$

where $\bar{\Pi}^{\mu\nu}_{(N)}$ and $\bar{\Upsilon}^{\mu}_{(N)}$ involves N number of derivatives on fluid variables. The terms on r.h.s. can have the most generic form as,

$$\begin{aligned}
\bar{\Pi}^{\mu\nu}_{(N)} &= \sum_t \tau_{(N)t}(\vartheta,\nu)\mathbf{T}^{\mu\nu}_{(N)t} + P^{\mu\nu}\sum_t \sigma_{(N)t}(\vartheta,\nu)\mathbf{S}_{(N)t}, \\
\bar{\Upsilon}^{\mu}_{(N)} &= \frac{\vartheta_o}{\epsilon + P_o}\sum_t \nu_{(N)t}(\vartheta,\nu)\mathbf{V}^{\mu}_{(N)t},
\end{aligned} \tag{2.6}$$

where $\mathbf{S}_{(N)t}$, $\mathbf{V}^{\mu}_{(N)t}$ and $\mathbf{T}^{\mu\nu}_{(N)t}$ are a collection of all possible gauge invariant scalars, vectors and symmetric traceless tensors (collectively known as *data*) respectively, made out of fluid variables and source fields at N derivative order. $\sum_t$ corresponds to sum over independent terms at any particular derivative order. The data which is required for our computation has been enlisted in section 4.

In eq. (2.6), the expression for $\bar{\Pi}^{\mu\nu}_{(N)}$ and $\bar{\Upsilon}^{\mu}_{(N)}$ are fixed up to some undetermined coefficients appearing at each derivative order. Therefore, a fluid is characterized by an infinite

[2]Actually $\nu = \mu/\vartheta$, where μ is the chemical potential.

JHEP05(2015)010

set of such unknown functions $(\tau_{(N)t}, \sigma_{(N)t}, \nu_{(N)t})$, known as *transport coefficients*. Fluid up to a particular derivative order is characterized by a finite number of such transport coefficients. In general, these transport coefficients are not all independent. The second law of thermodynamics (or equivalently, positivity of local entropy current) imposes restrictions on different transport coefficients[3] [9]. Such relations among various transport coefficients are known as *constraints*.

Ref. [4] uses a different mechanism to find 'some' of these constraints. The idea is to write an equilibrium partition function for the fluid and derive the energy-momentum tensor and charge current from that partition function. Because of dissipation it is not possible to write a generating functional (W) for the fluid. However, one can still write a generating functional in equilibrium configuration, which we denote by $W^{\rm eqb}$. Using $W^{\rm eqb}$ one can find all the constraint relations involving transport coefficients which comes with data that survives at equilibrium.

More precisely, if the theory has a timelike Killing vector ω^μ, we can write an Euclidean generating functional using the background fields and Killing equation on the decomposed manifold $S^1 \times \mathcal{M}_{(d-1)}$. Here S^1 is the euclidean time circle along ω^μ with time period $\tilde{\beta}$, and $\mathcal{M}_{(d-1)}$ is the spacetime transverse to ω^μ. [4] has conveniently chosen $\omega^\mu = \partial_0$. Therefore, one can decompose the background in Kaluza-Klein form,

$$\begin{aligned} {\rm d}s^2 &= G_{\mu\nu}{\rm d}x^\mu {\rm d}x^\nu = -{\rm e}^{2\sigma}\left({\rm d}t + a_i {\rm d}x^i\right)^2 + g_{ij}{\rm d}x^i{\rm d}x^j, \\ \mathcal{A} &= A(dt + a_i dx^i) + A_i dx^i. \end{aligned} \tag{2.7}$$

For more details please refer appendix A. Using this choice along with the Landau Gauge conditions and velocity normalization, the most-generic energy-momentum tensor and charge current in eq. (2.3) on $\mathcal{M}_{(d)}$ can be decomposed into scalars, vectors and tensors on $S^1 \times \mathcal{M}_{(d-1)}$:

$$\begin{aligned} \bar{T}^{ij} &= E(\vartheta,\nu)v^i v^j + \bar{\pi}^{ij}, \\ \bar{T}^i &= -{\rm e}^\sigma\left(E(\vartheta,\nu)v^i\sqrt{1+v_i v^i} + \frac{v_j\bar{\pi}^{ij}}{\sqrt{1+v_i v^i}}\right), \\ \bar{T} &= {\rm e}^{2\sigma}\left(E(\vartheta,\nu)(1+v_i v^i) + \frac{v_i v_j\bar{\pi}^{ij}}{(1+v_i v^i)}\right), \\ \bar{J}^i &= Q(\vartheta,\nu)v^i + \bar{\varsigma}^i, \\ \bar{J} &= -{\rm e}^\sigma\left(Q(\vartheta,\nu)\sqrt{1+v_i v^i} + \frac{v_i\bar{\varsigma}^i}{\sqrt{1+v_i v^i}}\right), \end{aligned} \tag{2.8}$$

where

$$\bar{T} = \bar{\mathcal{T}}_{00},\ \bar{T}^i = \bar{\mathcal{T}}^i_{\ 0},\ \bar{T}^{ij} = \bar{\mathcal{T}}^{ij}; \quad \bar{J} = \bar{\mathcal{J}}_0,\ \bar{J}^i = \bar{\mathcal{J}}^i,$$

and

$$\bar{\pi} = \bar{\Pi}_{00},\ \bar{\pi}^i = \bar{\Pi}^i_{\ 0},\ \bar{\pi}^{ij} = \bar{\Pi}^{ij}, \quad \bar{\varsigma} = \bar{\Upsilon}_0,\ \bar{\varsigma}^i = \bar{\Upsilon}^i, \quad v = u_0, v^i = u^i.$$

[3] Similar restrictions are also applicable to non-relativistic fluids and has recently been addressed for charged non-relativistic fluids in [10].

JHEP05(2015)010

Indices on $\mathcal{M}_{(d-1)}$ are raised and lowered using g^{ij}. Details of Kaluza-Klein decomposition of fluid variables and background fields have been given in appendix A.

Since the fluid we are considering is in local thermodynamic equilibrium, we can write the fluid variables as a spatial derivative expansion about their equilibrium values

$$\begin{aligned}\vartheta &= \vartheta_o + \bar{\Delta}^{(1)}\vartheta + \bar{\Delta}^{(2)}\vartheta + \dots \\ \nu &= \nu_o + \bar{\Delta}^{(1)}\nu + \bar{\Delta}^{(2)}\nu + \dots \\ v^i &= v^i_o + \bar{\Delta}^{(1)}v^i + \bar{\Delta}^{(2)}v^i + \dots .\end{aligned} \tag{2.9}$$

The terms with subscript 'o' are the equilibrium values, while $\bar{\Delta}^{(N)}$ designates the N^{th} derivative corrections.[4] The zeroth component of fluid velocity $u_0 = v$ also gets derivative corrections which are determined by the corrections to v^i using the four-velocity normalization. Similarly all the transport coefficients can also be expanded using the Taylor Series expansion

$$\alpha(\vartheta,\nu) = \alpha_o(\vartheta_o,\nu_o) + \bar{\Delta}^{(1)}\alpha + \bar{\Delta}^{(2)}\alpha + \dots . \tag{2.10}$$

Therefore the energy-momentum tensor and charge current receive two fold derivative corrections. First of all we write these expressions as a derivative expansion in terms of fluid variables in eq. (2.5). Secondly, each term in that expansion can be further expanded around the equilibrium values of fluid variables according to eq. (2.9). Thus we finally get

$$\begin{aligned}\bar{\pi}^{ij} &= \left[\bar{\pi}^{ij}_{o(0)}\right] + \left[\bar{\Delta}^{(1)}\bar{\pi}^{ij}_{(0)} + \bar{\pi}^{ij}_{o(1)}\right] + \left[\bar{\Delta}^{(2)}\bar{\pi}^{ij}_{(0)} + \bar{\Delta}^{(1)}\bar{\pi}^{ij}_{(1)} + \bar{\pi}^{ij}_{o(2)}\right]\dots, \\ \bar{\varsigma}^{i} &= \left[\bar{\varsigma}^{i}_{o(0)}\right] + \left[\bar{\Delta}^{(1)}\bar{\varsigma}^{i}_{(0)} + \bar{\varsigma}^{i}_{o(1)}\right] + \left[\bar{\Delta}^{(2)}\bar{\varsigma}^{i}_{(0)} + \bar{\Delta}^{(1)}\bar{\varsigma}^{i}_{(1)} + \bar{\varsigma}^{i}_{o(2)}\right]\dots .\end{aligned} \tag{2.11}$$

Expansion of time components can be determined from these using Landau gauge condition eq. (2.4).

We choose the equilibrium convention for ϑ and ν by identifying their equilibrium values to be the red-shifted temperature and Wilson loop in the lower dimensional theory

$$\frac{1}{\vartheta_o} = \beta_o = \tilde{\beta}\sqrt{-G_{00}} = \tilde{\beta}\mathrm{e}^{\sigma}, \qquad \nu_o = \tilde{\beta}\mathcal{A}_0. \tag{2.12}$$

In the next section we construct the equilibrium partition function and obtain energy-momentum tensor and charge current in terms of background data following [4]. After that, we compare these stress tensor and current with the fluid stress tensor and current order by order in derivative expansion to find the constraints among the transport coefficients at any particular derivative order. A typical constraint will connect transport coefficients at equilibrium $\{\alpha_o(\vartheta_o,\nu_o)\}$ and their derivatives with respect to ϑ_o and ν_o (up to a particular derivative order)

$$\mathfrak{C}\Big(\{\alpha_o(\vartheta_o,\nu_o)\}, \{\partial\alpha_o(\vartheta_o,\nu_o)\}\Big) = 0. \tag{2.13}$$

[4] In this paper, $\tilde{\Delta}^{(n)}A$ denotes parity-odd n^{th} derivative corrections to a fluid quantity A, while $\Delta^{(n)}A$ represents the parity-even n^{th} derivative corrections. Entire derivative correction is denoted by $\bar{\Delta}^{(n)}A = \Delta^{(n)}A + \tilde{\Delta}^{(n)}A$.

JHEP05(2015)010

We can extrapolate this constraint to non-equilibrium configurations:

$$\mathfrak{C}\Big(\{\alpha(\vartheta,\nu)\},\{\partial\alpha(\vartheta,\nu)\}\Big)=0, \tag{2.14}$$

while doing this, we are making an error of at least one derivative order higher, which will be compensated at next derivative order computation. This is how we find the generic constraints among fluid transport coefficients. Please note that while the equality constraints determined by this procedure are generic, the inequality constraints are not determined by this method.

3 Equilibrium partition function

The equilibrium partition function[5] $W^{\rm eqb}$ of the theory can generally be disintegrated into two parts:

$$W^{\rm eqb}=W^{\rm eqb}_{(C)}+W^{\rm eqb}_{(A)}. \tag{3.1}$$

The first one is the 'conserved' partition function which is gauge and diffeomorphism invariant, and generates conserved part of currents denoted by $\bar{\mathcal{T}}^{\mu\nu}_{(C)},\bar{\mathcal{J}}^{\mu}_{(C)}$. The other piece is not gauge-invariant and is referred to be 'anomalous' partition function. It generates anomalous piece of 'consistent currents' which will not be gauge-invariant in general. By defining a consistent subtraction scheme (Bardeen-Zumino currents), we can make these anomalous currents gauge invariant (see [11] for details) which we denote by $\bar{\mathcal{T}}^{\mu\nu}_{(A)},\bar{\mathcal{J}}^{\mu}_{(A)}$. Their value at equilibrium is fixed by anomaly, and upto subleading order is given by:

$$\begin{aligned}\bar{\mathcal{T}}^{\mu\nu}_{(A)}&=-2C^{(2n)}\sum_{m=1}^{n}{}^{n+1}\mathbf{C}_{m+1}\vartheta_o^2\nu_o^{m+1}\star\Big(u_o\wedge\mathcal{X}_{o1}^{\wedge(m-1)}\wedge\mathcal{X}_{o2}^{\wedge(n-m)}\Big)^{(\mu}u_o^{\nu)}\\&\equiv-2C^{(2n)}\sum_{m=1}^{n}{}^{n+1}\mathbf{C}_{m+1}\vartheta_o^2\nu_o^{m+1}l^{(\mu}_{o(m)}u_o^{\nu)},\end{aligned} \tag{3.2}$$

$$\begin{aligned}\bar{\mathcal{J}}^{\mu}_{(A)}&=-C^{(2n)}\sum_{m=1}^{n}(n+1)\,{}^{n}\mathbf{C}_{m}\vartheta_o\nu_o^{m}\star\Big(u_o\wedge\mathcal{X}_{o1}^{\wedge(m-1)}\wedge\mathcal{X}_{o2}^{\wedge(n-m)}\Big)^{\mu}\\&\equiv-C^{(2n)}\sum_{m=1}^{n}(n+1)\,{}^{n}\mathbf{C}_{m}\vartheta_o\nu_o^{m}l^{\mu}_{o(m)}.\end{aligned} \tag{3.3}$$

Here $\{\mathcal{X}_1,\mathcal{X}_2\}$ are $\{-\vartheta\mathrm{d}u,\mathrm{d}\mathcal{A}+\vartheta\nu\mathrm{d}u\}$ projected transverse to u^μ, and their equilibrium values upon KK reduction reduce to: $\{f_1,f_2\}=\Big\{\tilde{\vartheta}\mathrm{d}a,\mathrm{d}A\Big\}$. Hence we have:[6]

$$\bar{\mathcal{T}}^{i}_{(A)}=C^{(2n)}\sum_{m=1}^{n}{}^{n+1}\mathbf{C}_{m+1}\mathrm{e}^{\sigma}\vartheta_o^2\nu_o^{m+1}l^{i}_{o(m)},\qquad \bar{\mathcal{T}}^{ij}_{(A)}=\bar{\mathcal{T}}_{(A)}=0, \tag{3.4}$$

$$\bar{J}^{i}_{(A)}=-C^{(2n)}\sum_{m=1}^{n}(n+1)\,{}^{n}\mathbf{C}_{m}\vartheta_o\nu_o^{m}l^{i}_{o(m)},\qquad \bar{J}_{(A)}=0. \tag{3.5}$$

[5] The partition function may be thought of as the Euclidean action for the fluid living on the background with coordinate time t compactified on a circle of length $\tilde{\beta}$.

[6] To get these and some further results we have to used the ideal order results $v_o=-\mathrm{e}^{\sigma},v_o^i=0$, which we will derive in section 5. We use it here to simplify the notation.

Let us now concentrate on $W^{\rm eqb}_{(C)}$. It's variation on background (2.7) will determine the conserved currents:

$$\delta W^{\rm eqb}_{(C)} = \int d^{2n}x\ \sqrt{G}\left[-\frac{1}{2}\bar{\mathcal{T}}^{\mu\nu}_{(C)}\delta G_{\mu\nu} + \bar{\mathcal{J}}^{\mu}_{(C)}\delta\mathcal{A}_{\mu}\right]. \tag{3.6}$$

And hence,

$$\bar{\mathcal{T}}^{\mu\nu}_{(C)} = 2\frac{\delta W^{\rm eqb}_{(C)}}{\delta G_{\mu\nu}}, \qquad \bar{\mathcal{J}}^{\mu}_{(C)} = \frac{\delta W^{\rm eqb}_{(C)}}{\delta\mathcal{A}_{\mu}}. \tag{3.7}$$

Kaluza-Klein decomposition of eq. (3.7) gives,

$$\begin{aligned}
&\bar{T}^{ij}_{(C)} = 2\vartheta_o\frac{\delta W^{\rm eqb}_{(C)}}{\delta g_{ij}}, \qquad \bar{T}^{i}_{(C)} + \mathrm{e}^{\sigma}\vartheta_o\nu_o\bar{\mathcal{J}}^{i}_{(C)} = \vartheta_o\frac{\delta W^{\rm eqb}_{(C)}}{\delta a_i}, \qquad \bar{T}_{(C)} = \mathrm{e}^{2\sigma}\vartheta_o^2\frac{\delta W^{\rm eqb}_{(C)}}{\delta\vartheta_o},\\
&\bar{J}^{i}_{(C)} = \vartheta_o\frac{\delta W^{\rm eqb}_{(C)}}{\delta A_i}, \qquad\qquad \bar{J}_{(C)} = -\mathrm{e}^{\sigma}\frac{\delta W^{\rm eqb}_{(C)}}{\delta\nu_o}.
\end{aligned} \tag{3.8}$$

Here we have switched the basis to $\vartheta_o = \mathrm{e}^{-\sigma}/\tilde{\beta}$ and $\nu_o = \tilde{\beta}A$ for later convenience. a^i is the Kaluza-Klein gauge field. Note that while $W^{\rm eqb}_{(C)}$ is gauge invariant, its integrand does not need to be. We can include a typical Chern-Simons term to it, which is defined such that its integral is gauge invariant:[7]

$$\int_{\mathcal{M}_{(2n-1)}} I^{2n-1} = -\int d^{2n-1}x\sqrt{g}\left\{\sum_{m=1}^{n}{}^{n}\mathbf{C}_{m-1}C_{m-1}A_i l^i_{o(m)} + \tilde{\vartheta}C_n a_i l^i_{o(n)}\right\}. \tag{3.9}$$

Here C_m's are constants. This is indeed a valid Chern-Simons form as at equilibrium $l^i_{o(m)}$ is just made of Chern classes of f_1 and f_2:

$$l^i_{o(m)} = \star\left(f_1^{\wedge(m-1)}\wedge f_2^{\wedge(n-m)}\right)^i. \tag{3.10}$$

For the gauge-invariant integrand, we assume that curvature scales of the background $\mathcal{M}_{(d-1)}$ is much much larger than the mean free path of the fluid, therefore the whole manifold can be thought of as union of various flat patches. The system can be thought of in thermal equilibrium in each local patch. On each patch we can define the euclidean partition function locally, hence giving us

$$W^{\rm eqb}_{(C)} = \int d^{2n-1}x\sqrt{g}\ \beta(\vec{x})P(\vec{x}) + \int d^{2n-1}x\sqrt{g}\star I^{2n-1}, \tag{3.11}$$

where $P(\vec{x})$ is local thermodynamic *pressure* and $\beta(\vec{x})$ is local thermodynamic temperature. Given pressure, we can use the thermodynamic relations in local patch

$$\mathrm{d}P = \frac{\epsilon+P}{\vartheta}\mathrm{d}\vartheta + \vartheta q\mathrm{d}\nu, \quad \epsilon + P = \vartheta s + \nu q, \tag{3.12}$$

to define *energy density* ϵ, *entropy density* s and *charge density* q of the fluid. All are functions of ϑ and ν. We can expand $W^{\rm eqb}_{(C)}$ around its equilibrium value as

$$W^{\rm eqb}_{(C)} = \int d^{2n-1}x\sqrt{g}\ \beta_o P_o + \bar{\Delta}W^{\rm eqb}_{(C)}. \tag{3.13}$$

[7] We have left the terms in I^{2n-1} which can be related to others upto a total derivative.

JHEP05(2015)010

Derivative correction to the ideal fluid partition function is denoted by $\bar{\Delta}W^{\rm eqb}_{(C)}$, which will contain all the possible gauge invariant scalars made out of background metric and gauge field components at a particular derivative order. We have computed these scalars (till the derivative level of our interest) in section 4.

Collating together the conserved currents in eq. (3.8) and the anomalous pieces in eqs. (3.5), (3.4), and varying the Chern-Simons terms in $W^{\rm eqb}_{(C)}$ i.e. eq. (3.9), we can finally write:

$$\begin{aligned}
\bar{T}^{ij} &= 2\vartheta_o \frac{\delta W^{\rm eqb}_{(C)}}{\delta g_{ij}}, \\
\bar{T}^i + \mathrm{e}^{\sigma}\vartheta_o\nu_o\bar{J}^i &= \vartheta_o \frac{\delta W^{\rm eqb}_{(C)}}{\delta a_i} \\
&\quad -\vartheta_o \sum_{m=1}^{n} {}^{n-1}\mathbf{C}_{m-1}\left\{\frac{n(n+1)}{(m+1)}C^{(2n)}\mathrm{e}^{\sigma}\vartheta_o\nu_o\nu_o^m l^i_{o(m)} + nC_m\tilde{\vartheta}l^i_{o(m)}\right\}, \\
\bar{T} &= \mathrm{e}^{2\sigma}\vartheta_o^2 \frac{\delta W^{\rm eqb}_{(C)}}{\delta \vartheta_o}, \\
\bar{J}^i &= \vartheta_o \frac{\delta W^{\rm eqb}_{(C)}}{\delta A_i} \\
&\quad -\vartheta_o \sum_{m=1}^{n} {}^{n-1}\mathbf{C}_{m-1}\left\{\frac{n(n+1)}{m}C^{(2n)}\nu_o^m l^i_{o(m)} + nC_{m-1}l^i_{o(m)}\right\}, \\
\bar{J} &= -\mathrm{e}^{\sigma}\frac{\delta W^{\rm eqb}_{(C)}}{\delta \nu_o}.
\end{aligned} \tag{3.14}$$

Comparing these to the most generic fluid expressions in eq. (2.8) we can compute the constraints. Thus, we see that it is only the gauge invariant part $W^{\rm eqb}_{(C)}$ of the partition function that we need to evaluate at any desired order.

3.1 Anomalous entropy current

In last section we reviewed a procedure to get equality type constraints among fluid transport coefficients. It is generally known that these very constraints can also be get by demanding existence of an entropy current whose divergence is positive semi-definite. The most generic Entropy Current can be written as:

$$\bar{\mathcal{J}}^{\mu}_S = \bar{\mathcal{J}}^{\mu}_{S(C)} + \bar{\mathcal{J}}^{\mu}_{S(A)}, \tag{3.15}$$

where $\bar{\mathcal{J}}^{\mu}_{S(A)}$ is the part which captures the explicit dependence on anomaly coefficients. However, the other piece $\bar{\mathcal{J}}^{\mu}_{S(C)}$ can get implicit dependence on the anomaly coefficients through the fluid equations of motion. We need to demand this current to be positive semi-definite,

$$\hat{\nabla}_{\mu}\bar{\mathcal{J}}^{\mu}_S = \hat{\nabla}_{\mu}\bar{\mathcal{J}}^{\mu}_{S(C)} + \hat{\nabla}_{\mu}\bar{\mathcal{J}}^{\mu}_{S(A)} \geq 0, \tag{3.16}$$

whenever EOM are satisfied. For equilibrium fluid configuration, both the pieces can be demanded to be positive semi-definite separately. Such decoupling is not always possible,

JHEP05(2015)010

as the fluid equations of motion depend on anomaly coefficients, which can induce some implicit anomaly dependence in $\bar{\mathcal{J}}^{\mu}_{S(C)}$. However, for equilibrium fluid configurations, the equations of motion are trivially satisfied and thus entire information of anomaly can be incorporated in $\bar{\mathcal{J}}^{\mu}_{S(A)}$. Hence, if any part of $\hat{\nabla}_{\mu}\bar{\mathcal{J}}^{\mu}_{S(C)}$ couple to $\hat{\nabla}_{\mu}\bar{\mathcal{J}}^{\mu}_{S(A)}$, the respective transport coefficients will be determined in terms of anomaly coefficients, and hence will be present in $\bar{\mathcal{J}}^{\mu}_{S(A)}$ at the first place. Therefore all the information about constraints among fluid transport coefficients is encoded in the existence of $\mathcal{J}^{\mu}_{S(C)}$. In [12, 13] the author gives an explicit construction of entropy current from Eqb. Partition Function.

Now concentrating on the second term: at equilibrium, $\hat{\nabla}_{\mu}\bar{\mathcal{J}}^{\mu}_{S(A)} \geq 0$, since it does not have any independent coefficients, just constants, one cannot apply any constraints for it to be satisfied. Therefore $\mathcal{J}^{\mu}_{S(A)}$ must be exact. But any current is always ambiguous upto some exact terms, and hence we can choose $\mathcal{J}^{\mu}_{S(A)} = 0$ equally well. We can hence write in a generic hydrodynamic frame:[8]

$$\bar{\mathcal{T}}^{\mu\nu}_{(A)} = 2\epsilon u^{(\mu}_{(A)} u^{\nu)}_{(C)} + 2\bar{q}^{(\mu}_{(A)} u^{\nu)}_{(C)} + 2\bar{q}^{(\mu}_{(C)} u^{\nu)}_{(A)} + \tilde{\Pi}^{\mu\nu}_{(A)}, \tag{3.17}$$

$$\bar{\mathcal{J}}^{\mu}_{(A)} = q u^{\mu}_{(A)} + \bar{\Upsilon}^{\mu}_{(A)}, \tag{3.18}$$

$$0 = s u^{\mu}_{(A)} + \bar{\Upsilon}^{\mu}_{S(A)}. \tag{3.19}$$

Note that in the expression for $\bar{\mathcal{T}}^{\mu\nu}_{(A)}$ we have used the fact that anomalies are parity-odd. Now depending on the choice of hydrodynamic frame, these conditions can be used to determine anomalous dissipative parts of the various currents. For example, if we define u^{μ} such that it does not contribute to anomaly, i.e. $u^{\mu}_{(A)} = 0$, we will get:

$$\tilde{\Upsilon}^{\mu}_{(A)} = \bar{\mathcal{J}}^{\mu}_{(A)}, \quad q^{\mu}_{(A)} = -\bar{\mathcal{T}}^{\mu\nu}_{(A)} u_{\nu(C)}, \quad \tilde{\Pi}^{\mu\nu}_{(A)} = 2\bar{\mathcal{T}}^{(\mu\alpha}_{(A)} \left(\delta^{\nu)}_{\alpha} + u^{\nu)}_{(C)} u_{\alpha(C)}\right), \quad \tilde{\Upsilon}^{\mu}_{S(A)} = 0. \tag{3.20}$$

This is the neatest frame for anomalies. Similar results for U(1) anomaly were derived in [14], however these expressions are also applicable to gravitational anomalies.[9] Here we present explicit expressions for the anomalous parts of currents, in presence of both U(1) and gravitational anomaly. Following the generic expressions given in [8], these can be computed directly from the anomaly polynomial. The anomaly polynomial in $2n$ dimensions up to $(n+1)$ derivative order is given as [15],

$$\mathcal{P} = C^{(2n)} \mathcal{F}^{\wedge(n+1)} + c_m \mathcal{F}^{\wedge(n-1)} \wedge \mathrm{Tr}[\mathfrak{R} \wedge \mathfrak{R}], \tag{3.21}$$

where, $C^{(2n)}$ is gauge anomaly coefficient which we have already introduced in the last section and c_m is gravitational anomaly coefficient. The two form $\mathfrak{R}$ is defined in terms of the Riemann tensor as,

$$\mathfrak{R}^{\alpha}_{\ \beta} = \mathcal{R}^{\alpha}_{\ \beta\gamma\delta} \mathrm{d}x^{\gamma} \wedge \mathrm{d}x^{\delta}. \tag{3.22}$$

Taking appropriate derivative of the above, one can find explicit expressions for anomalous parts of the currents. The leading part of the currents proportional to the gauge anomaly

[8] We have used the thermodynamic functions ϵ, q, s here, which will be explicitly proved in section 5.

[9] Restricted to equilibrium configurations.

JHEP05(2015)010

coefficient $C^{(2n)}$ have already been given in eqs. (3.2), (3.3). Here we present the subleading order contributions to currents coming due to the gravitational anomaly,

$$\bar{\mathcal{J}}^{\mu}_{(A)} = c_m(n-1)\Bigg[\star\left(u_o \wedge \mathcal{F}^{\wedge(n-2)} \wedge \Lambda_{\alpha\beta}\left(\Lambda^{\alpha\beta}U - 2\mathfrak{R}^{\alpha\beta}\right)\right)^{\mu} + \sum_{m=1}^{n-2} {}^{n-2}\mathbf{C}_m(\vartheta_o\nu_o)^m \star\Big(u_o \wedge U^{\wedge(m-1)} \wedge \mathcal{F}^{\wedge(n-2-m)} \wedge\left(\mathfrak{R}^{\alpha\beta} - \Lambda^{\alpha\beta}U\right) \wedge \left(\mathfrak{R}_{\alpha\beta} - \Lambda_{\alpha\beta}U\right)\Big)^{\mu}\Bigg], \tag{3.23}$$

where,

$$\Lambda_{\mu\nu} = \frac{1}{2}\left(U_{\mu\nu} - 4\frac{1}{\vartheta_o}u_{o[\nu}P_{\mu]\alpha}\hat{\nabla}^{\alpha}\vartheta_o\right), \quad U_{\mu\nu} = 2P_{[\mu\alpha}P_{\nu]\beta}\hat{\nabla}^{\alpha}u^{\beta}. \tag{3.24}$$

The heat current has the form,

$$\begin{aligned}\mathfrak{q}^{\mu}_{(A)} = -c_m\frac{1}{\vartheta_o}\Bigg[&\star\left(u_o \wedge \mathcal{F}^{\wedge(n-1)}\right)^{\mu}\Lambda^{\alpha\beta}\Lambda_{\alpha\beta} \\ &+ \sum_{m=2}^{n-1} {}^{n-1}\mathbf{C}_m(m-1)(\vartheta_o\nu_o)^m \star\Big(u_o \wedge U^{\wedge(m-2)} \wedge \mathcal{F}^{\wedge(n-1-m)} \\ &\qquad \wedge\left(\mathfrak{R}^{\alpha\beta} - \Lambda^{\alpha\beta}U\right) \wedge \left(\mathfrak{R}_{\alpha\beta} - \Lambda_{\alpha\beta}U\right)\Big)^{\mu} \\ &-2\sum_{m=1}^{n-1} {}^{n-1}\mathbf{C}_m(\vartheta_o\nu_o)^m \star\Big(u_o \wedge U^{\wedge(m-1)} \wedge \mathcal{F}^{\wedge(n-1-m)} \\ &\qquad \wedge\Lambda^{\alpha\beta}\left(\mathfrak{R}_{\alpha\beta} - \Lambda_{\alpha\beta}U\right)\Big)^{\mu}\Bigg].\end{aligned} \tag{3.25}$$

Finally, the stress tensor looks like,

$$\begin{aligned}\bar{\mathcal{T}}^{\mu\nu}_{(A)} = 4c_m\hat{\nabla}_{\rho}\Bigg[\sum_{m=1}^{n-1} {}^{n-1}\mathbf{C}_m(\vartheta_o\nu_o)^m \star\left(u_o \wedge U^{\wedge(m-1)} \wedge \mathcal{F}^{\wedge(n-m-1)} \wedge \left(\mathfrak{R}^{\rho(\nu} - \Lambda^{\rho(\nu}U\right)\right)^{\mu)} \\ - \star\left(u_o \wedge \mathcal{F}^{\wedge(n-1)}\Lambda^{\rho(\nu}\right)^{\mu)}\Bigg] - 2\vartheta_o u^{(\mu}\mathfrak{q}^{\nu)}_{(A)}.\end{aligned} \tag{3.26}$$

Instead if we are working in Landau Frame, where $\bar{q}^{\mu}_{(A)} = \bar{q}^{\mu}_{(C)} = 0$, we will getcondition:

$$-\bar{\mathcal{T}}^{\mu\nu}_{(A)}u_{\nu(C)} = \left(\epsilon G^{\mu\nu} + \bar{\mathcal{T}}^{\mu\nu}_{(C)}\right)u_{\nu(A)} = \left(\epsilon G^{\mu\nu} + \bar{\Pi}^{\mu\nu}_{(C)}\right)u_{\nu(A)}. \tag{3.27}$$

We need to invert $\left(\epsilon G^{\mu\nu} + \bar{\Pi}^{\mu\nu}_{(C)}\right)$, which can be done perturbatively in derivatives. To leading order:

$$u^{\mu}_{(A)} = -\frac{1}{\epsilon + P}\bar{\mathcal{T}}^{\mu\nu}_{(A)}u_{\nu(C)} + \ldots, \tag{3.28}$$

JHEP05(2015)010

and hence

$$\tilde{\Upsilon}^{\mu}_{(A)} = \bar{\mathcal{J}}^{\mu}_{(A)} + \frac{q}{\epsilon+P}\bar{\mathcal{T}}^{\mu\nu}_{(A)}u_{\nu(C)} + \ldots, \quad \tilde{\Pi}^{\mu\nu}_{(A)} = 2\bar{\mathcal{T}}^{(\mu\alpha}_{(A)}\left(\delta^{\nu)}_{\alpha} + \frac{\epsilon}{\epsilon+P}u^{\nu)}_{(C)}u_{\alpha(C)}\right) + \ldots, \tag{3.29}$$

$$\tilde{\Upsilon}^{\mu}_{S(A)} = \frac{s}{\epsilon+P}\bar{\mathcal{T}}^{\mu\nu}_{(A)}u_{\nu(C)} + \ldots = \frac{1}{\vartheta}\bar{\mathcal{T}}^{\mu\nu}_{(A)}u_{\nu(C)} + \nu\bar{\mathcal{J}}^{\mu}_{(A)} - \nu\tilde{\Upsilon}^{\mu}_{(A)} + \ldots. \tag{3.30}$$

As showed by [14], in presence of just U(1) anomaly, it gives the exact result of Son-Sorowka [3]. To write a similar expression for gravitational anomaly in Landau Frame, one will need to find anomalous velocity to subsubleading order, which might be non-trivial.

4 Counting of independent terms

This section is dedicated to develop a systematic procedure to compute independent fluid data (vectors, tensors transverse to velocity and scalars). First we will review the counting in parity-even sector in generic dimensions. Then we will extend this idea to parity-odd sector in generic dimensions at arbitrary derivative order through a procedure we call *'derivative counting'*.

After describing the generic procedure, we explicitly construct leading and sub-leading order parity-odd and even terms which are important for our current work. Many of these terms vanish in equilibrium. In tables 2, 1, 5 we list all the leading and sub-leading terms both parity-odd and even and check if they survive at equilibrium. Further in appendix B we extend this counting procedure to parity-odd subsubleading derivative order fluid. For this reason we will keep our illustrations in the construction explicit to subsubleading order.

4.1 Parity-even counting

In this subsection we present the parity-even counting in generic dimensions. One can always count independent data in the local rest frame (LRF) of the fluid, which turns out to be easier. We can later covariantize the terms to a generic reference frame by following simple (and generic) rules.[10] In LRF, the fundamental quantities are

- Temperature — ϑ, Chemical Potential — ν.
- Derivatives of fluid velocity[11] — $\partial_0 u^i$, $\partial^j u^i$.
- Field Tensor — $\mathcal{F}^{ij}$, $\mathcal{E}^i = \mathcal{F}^{i\nu}u_\nu$.
- Curvature — $\mathcal{R}^{ijkl}$, $\mathcal{R}^{ijk0}$, $\mathcal{R}^{i0k0}$.

All other quantities are merely derivatives of these fundamental quantities. Since LRF is locally flat, we are using the coordinate derivatives ∂_o and ∂_i. We introduce a notation for

[10] The rules can be summarized as: replace 1) '0' index with contraction with u^μ, 2) 'i' indices with a projection along $P^{\mu\nu}$, 3) ∂ with $\hat{\nabla}$, 4) $\epsilon^{ki_2j_2\ldots i_nj_n}$ with $\epsilon^{\mu_1\nu_1\mu_2\nu_2\ldots\mu_n\nu_n}u_{\mu_1}$, and finally 5) put all extra factors of projectors and velocities on left-most, so no derivatives act on them.

[11] $u^\mu u_\mu = -1$ would imply $u^\mu\partial u_\mu = 0$ and hence in local rest frame $\partial u_0 = 0$.

JHEP05(2015)010

parity-even terms which will be useful later in parity-odd counting. Terms with d derivatives and i indices will be denoted collectively as $(\frac{i}{d}, i, d)$. When working at equilibrium, it is also convenient to define:[12]

$$S^{\mu\nu} = 2\nabla^{(\mu}u^{\nu)}, \qquad U^{\mu\nu} = 2\nabla^{[\mu}u^{\nu]}, \tag{4.1}$$

$$\mathcal{X}^{\mu\nu}_{\Lambda} = \left\{-\vartheta U^{\mu\nu}, P^{\mu\alpha}P^{\nu\beta}\mathcal{F}_{\alpha\beta} + \vartheta\nu U^{\mu\nu}\right\}, \quad \Lambda = 1, 2. \tag{4.2}$$

The purpose of above notation is revealed in Kaluza Klein formalism: at equilibrium only spatial components of $\mathcal{X}^{\mu\nu}_{\Lambda}$ survive which land exactly to f^{ij}_{Λ} defined by:

$$a_{\Lambda} = \left\{\tilde{\vartheta}a^i, A^i\right\}, \qquad f^{ij}_{\Lambda} = \nabla^i a^j_{\Lambda} - \nabla^j a^i_{\Lambda}. \tag{4.3}$$

In the same spirit we define

$$\mathcal{K}^{\mu\nu\rho\sigma} = P^{\mu\alpha}P^{\nu\beta}P^{\rho\gamma}P^{\sigma\delta}\mathcal{R}_{\alpha\beta\gamma\delta} - \left(U^{\mu\nu}U^{\rho\sigma} + \frac{1}{2}U^{\mu\rho}U^{\nu\sigma} - \frac{1}{2}U^{\nu\rho}U^{\mu\sigma}\right). \tag{4.4}$$

Only spatial components of $\mathcal{K}^{\mu\nu\rho\sigma}$ survive at equilibrium, and they exactly match R^{ijkl}. The usage of index 'Λ' is purely to facilitate counting and computations. Similarly we define $\vartheta_{\Lambda} = \{\vartheta, \nu\}$.

Bianchi identity: in counting, we will extensively use the Bianchi identity to get rid of many terms, so it would be worth to spend some time on it. The Bianchi Identities for Field Tensor, Vorticity and Riemann Tensor take the form:

$$\hat{\nabla}_{[\mu}\mathcal{F}_{\nu\rho]} = \hat{\nabla}_{[\mu}\hat{\nabla}_{\nu}u_{\rho]} = \hat{\nabla}_{[\mu}\mathcal{R}_{\nu\rho]\sigma\delta} = \mathcal{R}_{[\mu\nu\rho]\sigma} = 0. \tag{4.5}$$

However our redefined variables $\mathcal{X}_{\Lambda}$ and $\mathcal{K}$ do not satisfy Bianchi Identities. But nevertheless we can always use these identities to relate

$$\hat{\nabla}_{[\mu}\mathcal{X}_{\Lambda\nu\rho]}, \qquad \hat{\nabla}_{[\mu}\mathcal{K}_{\nu\rho]\sigma\delta}, \qquad \mathcal{K}_{[\mu\nu\rho]\sigma}, \tag{4.6}$$

to other terms, and hence we can safely get rid of these in the following computation. In rest frame especially (or at equilibrium in any generic frame), one can check that $\mathcal{X}_{\Lambda}$ and $\mathcal{K}$ also satisfy Bianchi Identities.

Killing equation: if the theory has a unit Killing direction ω^{μ} we have the following Killing equation for a general tensor

$$\pounds_{\omega}T^{\alpha_1\alpha_2\cdots} = 0 \implies \omega^{\mu}\hat{\nabla}_{\mu}T^{\alpha_1\alpha_2\cdots} = \sum_k T^{\alpha_1\ldots\alpha_{k-1}\sigma\alpha_{k+1}\cdots}\hat{\nabla}_{\sigma}\omega^{\alpha_k}, \tag{4.7}$$

[12] Our conventions are:

$$\mathcal{A}^{[\mu\nu]} = \frac{1}{2}P^{\mu}_{\ \alpha}P^{\nu}_{\ \beta}\left(\mathcal{A}^{\alpha\beta} - \mathcal{A}^{\beta\alpha}\right), \quad \mathcal{A}^{(\mu\nu)} = \frac{1}{2}P^{\mu}_{\ \alpha}P^{\nu}_{\ \beta}\left(\mathcal{A}^{\alpha\beta} + \mathcal{A}^{\beta\alpha}\right), \quad \mathcal{A}^{\langle\mu\nu\rangle} = \mathcal{A}^{(\mu\nu)} - \frac{P^{\mu\nu}}{d-1}P_{\alpha\beta}\mathcal{A}^{\alpha\beta},$$

$$A^{[ij]} = \frac{1}{2}\left(A^{ij} - A^{ji}\right), \qquad A^{(ij)} = \frac{1}{2}\left(A^{ij} + A^{ji}\right), \qquad A^{\langle ij\rangle} = A^{(ij)} - \frac{g^{ij}}{d-1}g_{ij}A^{ij}.$$

JHEP05(2015)010

which in local rest frame becomes

$$\partial_0 T^{\alpha\beta\gamma\cdots} = 0. \tag{4.8}$$

Therefore if we are considering a theory at equilibrium, we do not have to consider the ∂_0 derivatives. Secondly, the Killing equation for metric $G^{\mu\nu}$ is given by

$$\hat{\nabla}^\beta \omega^\alpha + \hat{\nabla}^\alpha \omega^\beta = 0. \tag{4.9}$$

Taking $\frac{\omega^\mu}{\sqrt{-\omega^2}} = u^\mu$ and using Killing Equation for scalars this translates to:

$$\hat{\nabla}^\beta u^\alpha + \hat{\nabla}^\alpha u^\beta = 0. \tag{4.10}$$

Hence in local rest frame $S^{ij} = \partial^i u^j + \partial^j u^i = 0$.

4.1.1 First derivative order

Below, we compute all possible terms at first derivative order in LRF.

1. $(2,2,1)$: S^{ij}, $\mathcal{X}^{ij}_\Lambda$
2. $(1,1,1)$: $\partial^i \vartheta_\Lambda$, $\boxed{\partial_0 u^i}$, $\mathcal{E}^i$
3. $(0,0,1)$: $S^k{}_k$, $\boxed{\partial_0 \vartheta_\Lambda}$

However all these first derivative terms are not independent on-shell. Using first order equations of motion one can eliminate some of them. The equations of motion are given by eq. (2.1) (at equilibrium)

1. $(1,1,1)$: $\partial_\mu \bar{\mathcal{T}}^{\mu i} = \mathcal{F}^{i\alpha} \bar{\mathcal{J}}_\alpha + \underset{\approx}{\mathfrak{T}}^\nu$
2. $(0,0,1)$: $\partial_\mu \bar{\mathcal{T}}^\mu{}_0 = -\mathcal{E}^\alpha \bar{\mathcal{J}}_\alpha + \underset{\approx}{\mathfrak{T}}^\nu u_\nu, \quad \partial_\mu \bar{\mathcal{J}}^\mu = \underset{\sim}{\mathfrak{J}}.$

Using these equations we have killed the $\boxed{\text{boxed}}$ terms in the counting.

4.1.2 Second derivative order

Below we list all possible *pure* second derivative terms. By pure we mean they are not product of two first derivative terms. Product of two lower derivative terms are called *composite* terms.

1. $(2,4,2)$: $\mathcal{K}^{ijkl}$
2. $(\frac{3}{2},3,2)$: $\partial^i S^{jk}$, $\partial^i \mathcal{X}^{jk}_\Lambda$, $\mathcal{R}^{ijk}{}_0$
3. $(1,2,2)$: $\partial^i\partial^j \vartheta_\Lambda$, $\partial^i \mathcal{E}^j$, $\boxed{\partial_0 S^{ij}}$, $\boxed{\partial_0 \mathcal{X}^{ij}_1}$, $\partial_0 \mathcal{X}^{ij}_2$, $\mathcal{R}^i{}_0{}^k{}_0$, $\mathcal{K}^{iaj}{}_a$
4. $(\frac{1}{2},1,2)$: $\boxed{\partial^i \partial_0 \vartheta_\Lambda}$, $\boxed{\partial_0 \partial_0 u^i}$, $\partial_0 \mathcal{E}^i$, $\partial_i S^{ij}$, $\partial_i \mathcal{X}^{ij}_\Lambda$, $\mathcal{R}^{ia}{}_{a0}$
5. $(0,0,2)$: $\boxed{\partial_0 S^k{}_k}$, $\boxed{\partial_0 \partial_0 \vartheta_\Lambda}$, $\partial_i \partial^i \vartheta_\Lambda$, $\partial_i \mathcal{E}^i$, $\mathcal{K}^{ab}{}_{ab}$, $\mathcal{R}^a{}_{0a0}$

JHEP05(2015)010

Name	LRF	Covariant	Equilibrium
Θ	$\frac{1}{2}S^i{}_i$	$\frac{1}{2}S^\mu{}_\mu$	0
V^μ_Λ	$\partial^i\vartheta_\Lambda$	$P^{\mu\alpha}\hat{\nabla}_\alpha\vartheta_\Lambda$	$\nabla^i\vartheta_{\Lambda o}$
V^μ_3	$\mathcal{E}^i-\vartheta_1 V^i_2$	$\mathcal{E}^\mu-\vartheta_1 V^\mu_2$	0
$\sigma^{\mu\nu}$	$\frac{1}{2}S^{\langle ij\rangle}$	$\frac{1}{2}S^{\langle\mu\nu\rangle}$	0

Table 1. Independent leading order parity-even data.

Here also all the terms are not independent because of equations of motion. The second order equations of motion are given by,

1. $(1,2,2)$: $\partial^{(k}\partial_\mu\mathcal{T}^{\mu i)}=\partial^{(k}\left(\mathcal{F}^{i)\alpha}\mathcal{J}_\alpha+\underset{\approx}{\mathfrak{T}}^{i)}\right)$, $\quad\partial^{[k}\partial_\mu\mathcal{T}^{\mu i]}=\partial^{[k}\left(\mathcal{F}^{i]\alpha}\mathcal{J}_\alpha+\underset{\approx}{\mathfrak{T}}^{i]}\right)$
2. $(\frac{1}{2},1,2)$: $\partial_0\partial_\mu\mathcal{T}^{\mu i}=\partial_0\left(\mathcal{F}^{i\alpha}\mathcal{J}_\alpha+\underset{\approx}{\mathfrak{T}}^{i}\right)$, $\quad\partial^i\partial_\mu\mathcal{T}^\mu{}_0=-\partial^i\left(\mathcal{E}^\alpha\mathcal{J}_\alpha-\underset{\approx}{\mathfrak{T}}^\alpha u_\alpha\right)$, $\partial^i\partial_\mu\mathcal{J}^\mu=\partial^i\underset{\sim}{\mathfrak{J}}$
3. $(0,0,2)$: $\partial_i\partial_\mu\mathcal{T}^{\mu i}=\partial_i\left(\mathcal{F}^{i\alpha}\mathcal{J}_\alpha+\underset{\approx}{\mathfrak{T}}^{i}\right)$, $\quad\partial_0\partial_\mu\mathcal{T}^\mu{}_0=-\partial_0\left(\mathcal{E}^\alpha\mathcal{J}_\alpha-\underset{\approx}{\mathfrak{T}}^\alpha u_\alpha\right)$, $\partial_0\partial_\mu\mathcal{J}^\mu=\partial_0\underset{\sim}{\mathfrak{J}}$

Again we have killed $\boxed{\text{boxed}}$ terms in the counting using equations of motion. We have provided a list of all terms till second order (also composites) in covariant form and their equilibrium values in tables 1, 2. We can iterate this procedure to further derivative orders as required by the cause. Note that, for a pure term at Nth derivative order, the maximum number of indices possible are $N+2$; we will need it later.

4.2 Parity-odd counting

In this section we shall compute the parity-odd leading and sub-leading derivative fluid data. Calculation in parity-odd sector is a lot more cumbersome, even in LRF. We introduce here a scheme called 'derivative counting' to compute these terms step by step. Any parity-odd term in $(2n)$-dimension must have a $(2n-1)$-dim Levi-Civita involved in LRF

$$\epsilon^{ii_2j_2\ldots i_nj_n}. \tag{4.11}$$

We are interested in constructing all possible scalars, vectors and symmetric tensors using it. A bit of thinking will reveal that one needs at least $(2n-2)$-rank parity-even tensors to be combined with $\epsilon^{ii_2j_2\ldots i_nj_n}$ for this purpose. One can subsequently form a list of *parity-odd data types*:

1. $\mathbf{V}_\epsilon$: vectors with free index on ϵ ($2n-2$ rank parity-even tensor contracted with ϵ).
2. $\mathbf{S}$: scalars with all indices contracted with ϵ ($2n-1$ rank parity-even tensor contracted with ϵ).
3. $\mathbf{T}_\epsilon$: tensors with one free index on ϵ ($2n-1$ rank parity-even tensor contracted with ϵ).

JHEP05(2015)010

Name	LRF	Covariant	Equilibrium
$\mathbf{S}_{1\Lambda}$	$\partial^i\partial_i\vartheta_\Lambda$	$P^{\alpha\beta}\hat{\nabla}_\alpha\hat{\nabla}_\beta\vartheta_\Lambda$	$\nabla^i\nabla_i\vartheta_{\Lambda o}$
$\mathbf{S}_{2(\Lambda\Gamma)}$	$\partial^i\vartheta_\Lambda\partial_i\vartheta_\Gamma$	$P^{\alpha\beta}\hat{\nabla}_\alpha\vartheta_\Lambda\hat{\nabla}_\beta\vartheta_\Gamma$	$\nabla^i\vartheta_{\Lambda o}\nabla_i\vartheta_{\Gamma o}$
$\mathbf{S}_{3(\Lambda\Gamma)}$	$\mathcal{X}_\Lambda^{ij}\mathcal{X}_{\Gamma ij}$	$\mathcal{X}_\Lambda^{\mu\nu}\mathcal{X}_{\Gamma\mu\nu}$	$f_\Lambda^{ij}f_{\Gamma ij}$
$\mathbf{S}_4$	$\mathcal{K}$	$\mathcal{K}$	R
$\mathbf{S}_5$	$\partial_i V_3^i$	$P_{\mu\nu}\hat{\nabla}^\mu V_3^\nu$	0
$\mathbf{S}_6$	$\mathcal{R}_{00}+\frac{1}{\vartheta}\partial_i\partial^i\vartheta$ $-2\frac{1}{\vartheta^2}\partial_i\vartheta\partial^i\vartheta-\frac{1}{4}\frac{1}{\vartheta^2}\mathcal{X}_1^{ij}\mathcal{X}_{1ij}$	$u^\mu u^\nu\mathcal{R}_{\mu\nu}+\frac{1}{\vartheta}\mathbf{S}_{1,1}$ $-2\frac{1}{\vartheta^2}\mathbf{S}_{2,11}-\frac{1}{4}\frac{1}{\vartheta^2}\mathbf{S}_{3,11}$	0
$\mathbf{S}_{7\Lambda}$	$V_\Lambda^i V_{3i}$	$V_\Lambda^\mu V_{3\mu}$	0
$\mathbf{S}_8$	$V_3^i V_{3i}$	$V_3^\mu V_{3\mu}$	0
$\mathbf{S}_9$	Θ^2	Θ^2	0
$\mathbf{S}_{10}$	$S^{ij}S_{ij}$	$S^{\mu\nu}S_{\mu\nu}$	0
$\mathbf{V}_{1\Lambda}^\mu$	$\partial_k\mathcal{X}_\Lambda^{ki}$	$P^{\mu\gamma}P_{\alpha\beta}\hat{\nabla}^\alpha\mathcal{X}_{\Lambda\gamma}^\beta$	$\nabla_k f_\Lambda^{ki}$
$\mathbf{V}_{2\Lambda\Gamma}^\mu$	$\mathcal{X}_\Lambda^{ik}V_{\Gamma k}$	$\mathcal{X}_\Lambda^{\mu\alpha}V_{\Gamma\alpha}$	$f_\Lambda^{ik}\nabla_k\vartheta_{\Gamma o}$
$\mathbf{V}_3^\mu$	$\partial_0 V_3^i$	$P^{\mu\beta}u^\alpha\hat{\nabla}_\alpha V_{3\beta}$	0
$\mathbf{V}_4^\mu$	$\partial_i S^{ij}$	$P^{\mu\beta}\hat{\nabla}^\alpha S_{\alpha\beta}$	0
$\mathbf{V}_5^\mu$	$\mathcal{R}^i{}_0-\frac{1}{2\vartheta}\partial_k\mathcal{X}_1^{ki}-\frac{3}{2\vartheta^2}\mathcal{X}_1^{ik}\partial_k\vartheta$	$P^{\mu\beta}u^\alpha\mathcal{R}_{\alpha\beta}-\frac{1}{2\vartheta}\mathbf{V}_{1,1}^\mu-\frac{3}{2\vartheta^2}\mathbf{V}_{2,11}^i$	0
$\mathbf{V}_{6\Lambda}^\mu$	ΘV_Λ^i	ΘV_Λ^μ	0
$\mathbf{V}_7^\mu$	ΘV_3^i	ΘV_3^μ	0
$\mathbf{V}_{8\Lambda}^\mu$	$S^{ij}V_{\Lambda j}$	$S^{\mu\nu}V_{\Lambda\nu}$	0
$\mathbf{V}_9^\mu$	$S^{ij}V_{3j}$	$S^{\mu\nu}V_{3\nu}$	0
$\mathbf{V}_{10\Lambda}^\mu$	$\mathcal{X}_\Lambda^{ij}V_{3j}$	$\mathcal{X}_\Lambda^{\mu\nu}V_{3\nu}$	0
$\mathbf{T}_{1\Lambda}^{\mu\nu}$	$\partial^{\langle i}\partial^{j\rangle}\vartheta_\Lambda$	$P^{\langle\mu\alpha}P^{\nu\rangle\beta}\hat{\nabla}_\alpha\hat{\nabla}_\beta\vartheta_\Lambda$	$\nabla^{\langle i}\nabla^{j\rangle}\vartheta_{\Lambda o}$
$\mathbf{T}_{2(\Lambda\Gamma)}^{\mu\nu}$	$\partial^{\langle i}\vartheta_\Lambda\partial^{j\rangle}\vartheta_\Gamma$	$P^{\langle\mu\alpha}P^{\nu\rangle\beta}\hat{\nabla}_\alpha\vartheta_\Lambda\hat{\nabla}_\beta\vartheta_\Gamma$	$\nabla^{\langle i}\vartheta_{\Lambda o}\nabla^{j\rangle}\vartheta_{\Gamma o}$
$\mathbf{T}_{3(\Lambda\Gamma)}^{\mu\nu}$	$\mathcal{X}_\Lambda^{\langle ik}\mathcal{X}_{\Gamma k}^{j\rangle}$	$\mathcal{X}_\Lambda^{\langle\mu\alpha}\mathcal{X}_{\Gamma\alpha}^{\nu\rangle}$	$f_\Lambda^{\langle ik}f_{\Gamma k}^{j\rangle}$
$\mathbf{T}_4^{\mu\nu}$	$\mathcal{K}^{\langle ij\rangle}$	$\mathcal{K}^{\langle\mu\nu\rangle}$	$R^{\langle ij\rangle}$
$\mathbf{T}_5^{\mu\nu}$	$\partial^{\langle i}V_3^{j\rangle}$	$\hat{\nabla}^{\langle\mu}V_3^{\nu\rangle}$	0
$\mathbf{T}_6^{\mu\nu}$	$\mathcal{R}^{\langle i\ j\rangle}_{\ 0\ 0}+\frac{1}{\vartheta}\partial^{\langle i}\partial^{j\rangle}\vartheta$ $-2\frac{1}{\vartheta^2}\partial^{\langle i}\vartheta\partial^{j\rangle}\vartheta-\frac{1}{4\vartheta^2}\mathcal{X}_1^{\langle i}{}_a\mathcal{X}_1^{j\rangle a}$	$P^{\langle\mu\rho}P^{\nu\rangle\sigma}u^\alpha u^\beta\mathcal{R}_{\rho\alpha\sigma\beta}+\frac{1}{\vartheta}\mathbf{T}_{1,1}^{\mu\nu}$ $-2\frac{1}{\vartheta^2}\mathbf{T}_{2,11}^{\mu\nu}-\frac{1}{4\vartheta^2}\mathbf{T}_{3,11}^{\mu\nu}$	0
$\mathbf{T}_{7\Lambda}^{\mu\nu}$	$V_\Lambda^{\langle i}V_3^{j\rangle}$	$V_\Lambda^{\langle\mu}V_3^{\nu\rangle}$	0
$\mathbf{T}_8^{\mu\nu}$	$V_3^{\langle i}V_3^{j\rangle}$	$V_3^{\langle\mu}V_3^{\nu\rangle}$	0
$\mathbf{T}_9^{\mu\nu}$	$\Theta\sigma^{ij}$	$\Theta\sigma^{\mu\nu}$	0
$\mathbf{T}_{10}^{\mu\nu}$	$S^{\langle ik}S^{j\rangle}_{\ k}$	$S^{\langle\mu\alpha}S^{\nu\rangle}_{\ \alpha}$	0
$\mathbf{T}_{11\Lambda}^{\mu\nu}$	$S^{\langle ik}\mathcal{X}_\Lambda^{j\rangle}{}_k$	$S^{\langle\mu\alpha}\mathcal{X}_\Lambda^{\nu\rangle}{}_\alpha$	0

Table 2. Independent subleading order parity-even data.

JHEP05(2015)010

4. $\mathbf{V}_f$: vectors with free index not on ϵ ($2n$ rank parity-even tensor contracted with ϵ).

5. $\mathbf{V}_\epsilon^C$: vectors formed of contraction of two non-ϵ indices with free index on ϵ ($2n$ rank parity-even tensor contracted with ϵ).

6. $\mathbf{T}_f$: tensors with no free index on ϵ ($2n+1$ rank parity-even tensor contracted with ϵ).

7. $\mathbf{S}^C$: scalars formed of contraction of $\mathbf{T}_f$ ($2n+1$ rank parity-even tensor contracted with ϵ).

8. $\mathbf{T}_\epsilon^C$: tensors formed of contraction of two non-ϵ indices with one free index on ϵ ($2n+1$ rank parity-even tensor contracted with ϵ).

9. $\mathbf{V}_f^C$: vectors formed of contraction of two non-ϵ indices with one free index not on ϵ ($2n+2$ rank parity-even tensor contracted with ϵ).

10. $\mathbf{V}_\epsilon^{CC}$: vectors formed of contraction of four non-ϵ indices with free index on ϵ ($2n+2$ rank parity-even tensor contracted with ϵ)

    .

    .

    .

    and so on.

Here we note that given D derivatives, one cannot construct a parity-even term, pure or composite, with more than $2D$ indices, because $(2,2,1)$ and $(2,4,2)$ have the highest index to derivative ratio, which is 2. Therefore, if we are interested in a fluid at $(n-2+s)$ derivative order ($s=1$ corresponds to parity-odd leading order and so on), we can get at most $2(n-2+s)$ indices. The list of parity-odd data types we gave above is complete till subsubleading derivative order ($s=3$).

Independent data types: we should emphasise that not all parity-odd data-types listed above are independent. The dependence comes from that fact that when we are working in $2n-1$ dimensions, any antisymmetrization over $2n$ or more indices will vanish. Given that we are dealing with parity-even tensors of arbitrary rank which are to be contracted with ϵ, there are a whole lot of these antisymmetrizations possible. Hence, to find the independent data-types becomes highly non-trivial.

Let's look at a special case of this dependence. We construct a $2n$-antisymmetrization,

$$\epsilon^{[i_1\ldots i_{2n-1}} A^{k_1]k_2\ldots k_t}{}_{i_1\ldots i_{2n-1}} = 0, \tag{4.12}$$

therefore,

$$\epsilon^{i_1\ldots i_{2n-1}} A^{k_1k_2\ldots k_t}{}_{i_1\ldots i_{2n-1}} = \sum_{a=1}^{2n-1} (-1)^{a+1} \epsilon^{k_1 i_1\ldots i_{2n-2}} A^{xk_2\ldots k_t}{}_{i_1\ldots i_{a-1}xi_a\ldots i_{2n-2}}. \tag{4.13}$$

The consequence of this is that the data types $[\,]_f$ (i.e. ones with a free index not on ϵ) can be expressed in terms of $[\,]_\epsilon^C$ (i.e. the ones with a free index on ϵ and an extra contraction). Hence data-types $[\,]_f$ for example $\mathbf{V}_f, \mathbf{T}_f, \mathbf{V}_f^C$ are not independent.

JHEP05(2015)010

Note that this result is only based on a specific form of $2n$-antisymmetrization (eq. (4.12)). One can in principle go on with any random antisymmetrizations over $2n$ or more indices and find relations among the data, which as it turns out, is not a trivial task to do. We will come back to this issue in section 4.4. For now we continue with the counting.

4.2.1 Derivative counting

We have classified parity-odd terms in data-types based on the number of parity-even indices required. We want to construct all allowed parity odd terms with D derivatives. We observe that it is not required to include all parity-even data type of the form (r, i, d) in this construction. We will show this below.

For a parity-odd fluid at $D = (n - 2 + s)$ derivative order, we need to construct all the D derivative parity-even terms with number of indices ranging from $2D$ (the maximum possible) to $2(D + 1 - s)$ $(= 2n - 2$, the minimum required), i.e.

$$2(D + 1 - s) \leq \text{No of indices of a parity-even D derivative term} \leq 2D.$$

These D-derivative parity-even terms can be constructed out of pure derivative terms. We need not consider pure terms with self contractions in parity-even data types as they have been included in our counting procedure.

We now want to argue that not all parity-even data-types are required for this construction. For a data-type $(\frac{i}{N}, i, N)$ to be included at least once, the following combination with $(2D - 2N + i)$ indices must be included:

$$(D - N) \times (2, 2, 1) \otimes \left(\frac{i}{N}, i, N\right)$$

Since the minimum rank of this term must be $2n-2 = 2(D+1-s)$ and maximum possible rank is $N + 2$, therefore we get,

$$N + 2 \geq i \geq 2(N - s + 1). \tag{4.14}$$

For this equation to have a solution $N \leq 2s$. So we need at max $2s$ derivative order parity-even terms, to construct parity-odd terms till $(n - 2 + s)$ derivative order. For example at leading order, $s = 1$, only pure terms with at max 2 derivatives are required. The parity even terms required till $s = 3$ are enlisted in tables 3, 4. Further, if we were only interested in finding terms that survive at equilibrium, we can use the Killing condition and drop all terms with ∂_0 derivatives.

Some of the *combinations* constructed by this procedure using table 3 are:

1. ($2D$ indices): $D(2, 2, 1)$
2. (a) ($2D - 1$ indices): $(D - 1)(2, 2, 1) \oplus (1, 1, 1)$
3. (a) ($2D - 1$ indices): $(D - 2)(2, 2, 1) \oplus (\frac{3}{2}, 3, 2)$
   (b) ($2D - 2$ indices): $(D - 2)(2, 2, 1) \oplus 2(1, 1, 1)$
   (c) ($2D - 2$ indices): $(D - 2)(2, 2, 1) \oplus (1, 2, 2)$

JHEP05(2015)010

Data Type	Decomposition	Local Rest Frame	Equilibrium
$(2,2,1)$		$\mathcal{X}^{ij}_\Lambda$	f^{ij}_Λ
$(1,1,1)$		$\partial^i\vartheta_\Lambda$	$\nabla^i\vartheta_{\Lambda o}$
$(2,4,2)$		$\mathcal{K}^{ijkl}$	R^{ijkl}
$(\frac{3}{2},3,2)$	$\partial^i(2,2,1)$	$\partial^i\mathcal{X}^{jk}_\Lambda$	$\nabla^i f^{jk}_\Lambda$
$(1,2,2)$	$\partial^i(1,1,1)$	$\partial^i\partial^j\vartheta_\Lambda$	$\nabla^i\nabla^j\vartheta_{\Lambda o}$
$(\frac{5}{3},5,3)$	$\partial^i(2,4,2)$	$\partial^i\mathcal{K}^{jklm}$	$\nabla^i R^{jklm}$
$(\frac{4}{3},4,3)$	$\partial^i\partial^j(2,2,1)$	$\partial^i\partial^j\mathcal{X}^{kl}_\Lambda$	$\nabla^i\nabla^j f^{kl}_\Lambda$
$(1,3,3)$	$\partial^i\partial^j(1,1,1)$	$\partial^i\partial^j\partial^k\vartheta_\Lambda$	$\nabla^i\nabla^j\nabla^k\vartheta_{\Lambda o}$
$(\frac{3}{2},6,4)$	$\partial^i\partial^j(2,4,2)$	$\partial^i\partial^j\mathcal{K}^{klmn}$	$\nabla^i\nabla^j R^{klmn}$
$(\frac{5}{4},5,4)$	$\partial^i\partial^j\partial^k(2,2,1)$	$\partial^i\partial^j\partial^k\mathcal{X}^{lm}_\Lambda$	$\nabla^i\nabla^j\nabla^k f^{lm}_\Lambda$
$(1,4,4)$	$\partial^i\partial^j\partial^k(1,1,1)$	$\partial^i\partial^j\partial^k\partial^l\vartheta_\Lambda$	$\nabla^i\nabla^j\nabla^k\nabla^l\vartheta_{\Lambda o}$
$(\frac{7}{5},7,5)$	$\partial^i\partial^j\partial^k(2,4,2)$	$\partial^i\partial^j\partial^k\mathcal{K}^{lmno}$	$\nabla^i\nabla^j\nabla^k R^{lmno}$
$(\frac{6}{5},6,5)$	$\partial^i\partial^j\partial^k\partial^l(2,2,1)$	$\partial^i\partial^j\partial^k\partial^l\mathcal{X}^{mn}_\Lambda$	$\nabla^i\nabla^j\nabla^k\nabla^l f^{mn}_\Lambda$
$(\frac{4}{3},8,6)$	$\partial^i\partial^j\partial^k\partial^l(2,4,2)$	$\partial^i\partial^j\partial^k\partial^l\mathcal{K}^{mnop}$	$\nabla^i\nabla^j\nabla^k\nabla^l R^{mnop}$

Table 3. Parity-even data-types — Surviving at equilibrium.

and so on... The counting can be extended arbitrarily to the derivative order we need. In next section we will construct terms till subleading order, and later in appendix B we will extend it to subsubleading order. We will suppress the usage of data-type $(2,4,2)$ for brevity; combinations involving it can always be reached by exchanging $(2,4,2)$ with two $(2,2,1)$'s.

4.3 Examples of parity-odd counting

4.3.1 Leading order $(D = n-1)$ $(s = 1)$

For $s = 1$, the required indices are merely $2D = 2n-2$ ($\mathbf{V}_\epsilon$), which amounts to the only combination:

$$D(2,2,1), \tag{4.15}$$

along with the terms involving $(2,4,2)$. However in $\mathbf{V}_\epsilon$ all the free indices are contracted with Levi-Civita, which will kill any term involving $(2,4,2)$ due to Bianchi Identity. The only remaining combination is — (n) vectors

$$\left\langle {m-1 \atop n-m} \right\rangle^i \Big|_{m=1}^{n},$$

where we define,

$$\begin{aligned}
\left\langle {m \atop n-a-m} \right\rangle_{\mu_1\nu_1\ldots\mu_a\nu_a} &= \frac{1}{2^{n-a}}\epsilon_{\mu_1\nu_1\ldots\mu_n\nu_n}\prod_{x=a+1}^{m+a}\mathcal{X}_1^{\mu_x\nu_x}\prod_{y=m+a+1}^{n}\mathcal{X}_2^{\mu_y\nu_y},\\
\left\langle {m \atop n-a-m} \right\rangle_{ii_2j_2\ldots i_aj_a} &= \frac{1}{2^{n-a}}\epsilon_{ii_2j_2\ldots i_nj_n}\prod_{x=a+1}^{m+a}f_1^{i_xj_x}\prod_{y=m+a+1}^{n}f_2^{i_yj_y}.
\end{aligned} \tag{4.16}$$

Data Type	Decomposition	Local Rest Frame
$(2,2,1)$	$\sigma^{ij} := \frac{1}{2}S^{\langle ij\rangle}$	
$(1,1,1)$	$V_3^i := \mathcal{E}^i - \vartheta_1 V_2^i$	
$(\frac{3}{2},3,2)$	$\Xi^{ijk} := \mathcal{R}^{ijk}{}_0 - \frac{1}{2\vartheta_o}\nabla^k\mathcal{X}_1^{ij} + \frac{1}{\vartheta_o^2}\left(f_1^{ij}\nabla^k\vartheta_o + \frac{1}{2}\mathcal{X}_1^{ik}\nabla^j\vartheta_o - \frac{1}{2}\mathcal{X}_1^{jk}\nabla^i\vartheta_o\right)$	
	$\partial^i(2,2,1)$	$\partial^i\sigma^{jk}$
$(1,2,2)$	$\Xi^{ij} := \mathcal{R}^{i\ j}{}_{0\ 0} + \frac{1}{\vartheta}\partial^i\partial^j\vartheta - 2\frac{1}{\vartheta^2}\partial^i\vartheta\partial^j\vartheta - \frac{1}{4\vartheta^2}\mathcal{X}_1^i{}_a\mathcal{X}_1^{ja}$	
	$\partial_0(2,2,1)$	$\partial_0\mathcal{X}_\Lambda^{jk}$, $\partial_0\sigma^{ij}$
	$\partial^i(1,1,1)$	$\partial^i V_3^j$
$(\frac{1}{2},1,2)$	$\partial_0(1,1,1)$	$\partial_0\partial_0\vartheta_\Lambda$, $\partial_0 V_3^i$
$(\frac{4}{3},4,3)$	$\partial^i(\frac{3}{2},3,2)$	$\partial^i\Xi^{jkl}$
	$\partial_0(2,4,2)$	$\partial_0\mathcal{K}^{jklm}$
	$\partial^i\partial^j(2,2,1)$	$\partial^i\partial^j\sigma^{kl}$
$(1,3,3)$	$\partial_0(\frac{3}{2},3,2)$	$\partial_0\Xi^{ijk}$
	$\partial^i(1,2,2)$	$\partial^i\Xi^{jk}$
	$\partial_0\partial^j(2,2,1)$	$\partial_0\partial^j\mathcal{X}_\Lambda^{kl}$, $\partial_0\partial^i\sigma^{jk}$
	$\partial^i\partial^j(1,1,1)$	$\partial^i\partial^j V_3^k$
$(\frac{2}{3},2,3)$	$\partial_0(1,2,2)$	$\partial_0\Xi^{ij}$
	$\partial_0\partial_0(2,2,1)$	$\partial_0\partial_0\mathcal{X}_\Lambda^{kl}$, $\partial_0\partial_0\sigma^{ij}$
	$\partial_0\partial^j(1,1,1)$	$\partial_0\partial^j\partial^k\vartheta_\Lambda$, $\partial_0\partial^i V_3^j$
$(\frac{5}{4},5,4)$	$\partial^i\partial^j(\frac{3}{2},3,2)$	$\partial^i\partial^j\Xi^{klm}$
	$\partial_0\partial^j(2,4,2)$	$\partial_0\partial^j\mathcal{K}^{klmn}$
	$\partial^i\partial^j\partial^k(2,2,1)$	$\partial^i\partial^j\partial^k\sigma^{lm}$
$(1,4,4)$	$\partial_0\partial^i(\frac{3}{2},3,2)$	$\partial_0\partial^i\Xi^{jkl}$
	$\partial^i\partial^j(1,2,2)$	$\partial^i\partial^j\Xi^{kl}$
	$\partial_0\partial_0(2,4,2)$	$\partial_0\partial_0\mathcal{K}^{klmn}$
	$\partial_0\partial^j\partial^k(2,2,1)$	$\partial_0\partial^j\partial^k\mathcal{X}_\Lambda^{lm}$, $\partial_0\partial^j\partial^k\sigma^{lm}$
	$\partial^i\partial^j\partial^k(1,1,1)$	$\partial^i\partial^j\partial^k V_3^l$
$(\frac{6}{5},6,5)$	$\partial^i\partial^j\partial^k(\frac{3}{2},3,2)$	$\partial^i\partial^j\partial^k\Xi^{lmn}$
	$\partial_0\partial^j\partial^k(2,4,2)$	$\partial_0\partial^j\partial^k\mathcal{K}^{lmno}$
	$\partial^i\partial^j\partial^k\partial^l(2,2,1)$	$\partial^i\partial^j\partial^k\partial^l\sigma^{mn}$

Table 4. Parity-even data-types — Vanishing at equilibrium.

JHEP05(2015)010

4.3.2 Subleading order ($D = n$, $s = 2$) — Surviving at equilibrium

At subleading order, index families required are: $2D = 2n$ ($\mathbf{V}_\epsilon^C$), $2D-1 = 2n-1$ ($\mathbf{T}_\epsilon$) and $2D-2 = 2n-2$ ($\mathbf{V}_\epsilon$). We only compute terms surviving at equilibrium because that is what we need for the current work.

$2D$ family: $2D$ family was already discussed in section 4.3.1, but this time since two indices are free from ϵ, one $(2,4,2)$ can appear with two antisymmetric indices of R^{ijkl} contracted. However we are supposed to take a contraction on remaining indices, which again due to antisymmetry vanish. Only remaining data are — $(n-1)$ vectors:

$$\left\langle {}^{m-1}_{n-1-m} \right\rangle_{ijk} \mathcal{X}_1^{ja} \mathcal{X}_{2a}^{k} \Big|_{m=1}^{n-1}.$$

$2D-1$ family: combinations in $(2D-1)$ family which survive at equilibrium are:

1. $(D-1)(2,2,1) \oplus (1,1,1)$
2. $(D-2)(2,2,1) \oplus (\frac{3}{2},3,2)$
3. $(D-3)(2,2,1) \oplus (\frac{5}{3},5,3)$

along with the combinations with $(2,4,2)$. In $\mathbf{T}_\epsilon$ only one index stays free from ϵ, hence again $(2,4,2)$ and $(\frac{5}{3},5,3)$ cannot appear. The remaining two combinations will yield:

1. $(n-1)(2,2,1) \oplus (1,1,1)$: 2 possibilities — $(6n-4)$ traceless symmetric tensors and $(2n)$ scalars

$$\left\langle {}^{m-1}_{n-m} \right\rangle^{\langle i} \partial^{j\rangle} \vartheta_\Lambda \Big|_{m=1}^{n}, \left\langle {}^{m-1}_{n-m-1} \right\rangle^{\langle ijk} \partial_j \vartheta_\Lambda \mathcal{X}_{\Gamma k}^{\ \ l\rangle} \Big|_{m=1}^{n-1}.$$

   Scalars:

$$\left\langle {}^{m-1}_{n-m} \right\rangle^{i} \partial_i \vartheta_\Lambda \Big|_{m=1}^{n}.$$

2. $(n-2)(2,2,1) \oplus (\frac{3}{2},3,2)$: 1 possibility — $(2n-2)$ traceless symmetric tensors

$$\left\langle {}^{m-1}_{n-m-1} \right\rangle^{(ijk} \partial^{l)} \mathcal{X}_{\Lambda jk} \Big|_{m=1}^{n-1}.$$

$2D-2$ family: combinations in $(2D-2)$ family which survive at equilibrium are:

1. $(D-2)(2,2,1) \oplus 2(1,1,1)$
2. $(D-2)(2,2,1) \oplus (1,2,2)$
3. $(D-3)(2,2,1) \oplus (\frac{3}{2},3,2) \oplus (1,1,1)$
4. $(D-3)(2,2,1) \oplus (\frac{4}{3},4,3)$
5. $(D-4)(2,2,1) \oplus 2(\frac{3}{2},3,2)$

JHEP05(2015)010

Name	Term	Equilibrium
$l^{\mu}_{m}\big\vert^{n}_{m=1}$	$\left\langle {}^{m-1}_{n-m}\right\rangle^{\mu\nu}u_{\nu}$	$\left\langle {}^{m-1}_{n-m}\right\rangle^{i}$
$\tilde{\mathbf{S}}_{\Lambda m}\big\vert^{n}_{m=1}$	$l^{\mu}_{m}\hat{\nabla}_{\mu}\vartheta_{\Lambda}$	$\left\langle {}^{m-1}_{n-m}\right\rangle^{i}\nabla_{i}\vartheta_{\Lambda o}$
$\tilde{\mathbf{V}}^{\mu}_{1m}\big\vert^{n-1}_{m=1}$	$\left\langle {}^{m-1}_{n-1-m}\right\rangle_{\mu\nu\rho\sigma}u^{\nu}\mathcal{X}_{1}^{\rho\alpha}\mathcal{X}_{2\alpha}^{\sigma}$	$\left\langle {}^{m-1}_{n-1-m}\right\rangle_{kij}f_{1}^{ia}f_{2a}^{j}$
$\tilde{\mathbf{V}}^{\mu}_{2m}\big\vert^{n-1}_{m=1}$	$\left\langle {}^{m-1}_{n-m-1}\right\rangle^{\mu\nu\rho\sigma}u_{\nu}\hat{\nabla}_{\rho}\vartheta_{1}\hat{\nabla}_{\sigma}\vartheta_{2}$	$\left\langle {}^{m-1}_{n-m-1}\right\rangle^{ijk}\nabla_{j}\vartheta_{1o}\nabla_{k}\vartheta_{2o}$
$\tilde{\mathbf{T}}^{\mu\nu}_{1\Lambda m}\big\vert^{n}_{m=1}$	$l^{\langle\mu}_{m}P^{\nu\rangle\alpha}\hat{\nabla}_{\alpha}\vartheta_{\Lambda}$	$\left\langle {}^{m-1}_{n-m}\right\rangle^{\langle i}\hat{\nabla}^{j\rangle}\vartheta_{\Lambda o}$
$\tilde{\mathbf{T}}^{\mu\nu}_{2\Lambda m}\big\vert^{n-1}_{m=1}$	$\left\langle {}^{m-1}_{n-m-1}\right\rangle^{(\mu\nu\rho\sigma}u_{\nu}\hat{\nabla}^{\alpha)}\mathcal{X}_{\Lambda\rho\sigma}$	$\left\langle {}^{m-1}_{n-m-1}\right\rangle^{(ijk}\nabla^{l)}f_{\Lambda jk}$
$\tilde{\mathbf{T}}^{\mu\nu}_{3\Lambda\Gamma m}\big\vert^{n-1}_{m=1}$	$\left\langle {}^{m-1}_{n-m-1}\right\rangle^{(\mu\nu\rho\sigma}u_{\nu}\hat{\nabla}_{\rho}\vartheta_{\Lambda}\mathcal{X}_{\Gamma\sigma}^{\ \ \delta)}$	$\left\langle {}^{m-1}_{n-m-1}\right\rangle^{(ijk}\nabla_{j}\vartheta_{\Lambda o}f_{\Gamma k}^{\ \ l)}$

Table 5. Independent leading and subleading order parity-odd data at equilibrium.

6. $(D-4)(2,2,1)\oplus(\frac{5}{3},5,3)\oplus(1,1,1)$
7. $(D-4)(2,2,1)\oplus(\frac{3}{2},6,4)$
8. $(D-5)(2,2,1)\oplus(\frac{3}{2},3,2)\oplus(\frac{5}{3},5,3)$
9. $(D-6)(2,2,1)\oplus 2(\frac{5}{3},5,3)$

Along with these, we have the combinations with $(2,4,2)$. However, $\mathbf{V}_{\epsilon}$ has no index free from ϵ, and hence Bianchi Identity will not allow $(2,4,2)$, $(\frac{5}{3},5,3)$ and $(\frac{3}{2},6,4)$. Further, $(1,2,2)$, $(\frac{3}{2},3,2)$ and $(\frac{4}{3},4,3)$ will vanish as they cannot be made completely antisymmetric. Finally only one combination will remain, yielding:

1. $(n-2)(2,2,1)\oplus 2(1,1,1)$: 1 possibility — $(n-1)$ vectors

$$\left\langle {}^{m-1}_{n-m-1}\right\rangle^{ijk}\partial_{j}\vartheta_{1}\partial_{k}\vartheta_{2}\Big\vert^{n-1}_{m=1}.$$

At equilibrium we have $(2n)$ scalars, $(2n-2)$ vectors and $(8n-6)$ traceless symmetric tensors. We have tabulated these data and their equilibrium values in table 5.

4.4 The basis of independent data

As we discussed in section 4.2, the data we have enlisted in the preceding sections is a 'complete set' but not independent. There might exist numerous relations among them through antisymmetrizations of $2n$ or more indices. If we look back at section 2, the need of all independent data arose to write down the most generic form of the constitutive relations. We write the energy-momentum tensor and charged current as a combination of all independent tensors and vectors respectively up to some undetermined coefficients which are called transport coefficients. We then determine the same quantities from equilibrium

JHEP05(2015)010

partition function and compare with the fluid results. It turns out that the transport coefficients which destroys the positivity of entropy current divergence are set to zero by this procedure. We call these transport coefficients *unphysical*. Put differently, the partition function generates only the *physical* transport coefficients in the constitutive relations (eq. (3.14)) at equilibrium.

Now if we relax the condition *'independence'* while writing fluid constitutive relations, i.e., add more terms to these relations which could have been determined in terms of others; they can be regarded as redundant transport coefficients in our system. Since the charge current and the energy-momentum tensor we derive from the partition function remain unchanged, we get relations between the transport coefficients (including the redundant coefficients) and the coefficients appearing in partition function. However, we still have our answers — the independent transport coefficients and distinct constitutive relations.

Let us explain with an example. Suppose at some particular derivative order, we have total I number of vectors V_i^μ. We can write charge current at this order as, $J^\mu = \sum_{i=1}^{I} a_i V_i^\mu$, where a_i's are transport coefficients. On the other hand, suppose our partition function has X number of independent coefficients C_j's, and it generates a charge current $J^\mu = \sum_{i=1}^{I} c_i(C_j) V_i^\mu$. $c_i(C_j)$ are some functions of C_j's. By comparison we will get $a_i = c_i(C_j)$. These are I relations with X free parameters, and thus imposes $I - X$ constraints on a_i.

Now let's add to our set K more vectors $V_\alpha^\mu, \alpha = I+1, \ldots, I+K$ which could in principle be determined as: $V_\alpha^\mu = \sum_{i=1}^{I} C_{\alpha i} V_i^\mu$. Then we would have guessed our ansatz to be $J^\mu = \sum_{i=1}^{K+I} b_i V_i^\mu$, and by varying partition function we will get $J^\mu = \sum_{i=1}^{K+I} d_i(C_j) V_i^\mu$. $d_i(C_j)$ are some functions of C_j's determined by relation $c_i = \left(d_i - \sum_{\alpha=I+1}^{I+K} d_\alpha C_{\alpha i}\right)$, as our partition function is still the same. By comparison we will get $b_i = d_i(C_j)$. These are $K + I$ relations with X free parameters, and thus imposes $K + I - X$ constraints on b_i. We hence get exactly K extra constraints, to kill the K extra degrees of freedom we added in the system. But once we have imposed these constraints, we will only be left with X independent transport coefficients.

However, note that we still need independent set of scalars that enters the equilibrium partition function, for our arguments to make sense. We check it here before we proceed. At leading order there are no scalars. At subleading order the scalars do not have enough indices for $2n$ or more antisymmetrizations, as a result all the scalars we get are independent. At higher order however, it may not be so easy to find out all the independent set of scalars.

Lets look at an example of such residual $2n$-antisymmetrization conditions. In eq. (4.13) if we chose B to be of the form $\mathbf{S}g^{ij}$, we will get:

$$\sum_{a=1}^{n-1} (-1)^{a+1} \epsilon^{\langle p i_1 \ldots i_{n-2}} A_{i_1 \ldots i_{a-1}}{}^{q\rangle}{}_{i_a \ldots i_{n-2}} = 0, \tag{4.17}$$

where $\langle\ \rangle$ denotes the traceless symmetric part of a matrix. Hence one of these matrices of type $\mathbf{T}_\epsilon$ (after making traceless) is not independent for a given A. A similar argument is valid on other tensors like $\mathbf{T}_\epsilon^C$ using $\mathbf{S}^C g^{ij}$. But as we are treating all symmetric traceless

tensors (of type $[\,]_\epsilon$) to be independent, this should reflect in our final constraints, and as we will see, it will. It turns out that till subleading order, eq. (4.17) is the only remaining residual constraint, and thus we can construct an independent basis; but this issue might turn more subtle at higher derivative orders. To illustrate the procedure we will not start with the independent basis even for subleading order, and show that we get consistent results at the end.

5 Fluid constitutive relations

Having all the data we require, we are ready to find the constitutive relations for fluid. We start with the results which are already known in literature, i.e. fluid up to leading derivative order. We revisit the results in our notation. Later we consider charged fluid at subleading order in section 5.3. We also set up the notation and architecture for sub-subleading order parity-odd fluid in this formalism in appendix B. However we do not compute the constitutive relations explicitly, as we will discuss, the calculation becomes a lot non-trivial.

5.1 Ideal fluid

At zero derivative order only energy-momentum gets a transverse contribution:

$$\Pi^{\mu\nu}_{(0)} = AP^{\mu\nu}, \tag{5.1}$$

where A is some arbitrary function of ϑ and ν. Now comparing eq. (2.8) with eq. (3.14) we can write at ideal order,

$$\begin{aligned} E_o v_o^i v_o^j + A_o g^{ij} &= g^{ij} P_o, \\ -E_o v_o^i \sqrt{1+v_{io}v_o^i} - \frac{v_o^i}{\sqrt{1+v_{io}v_o^i}} + \vartheta_o \nu_o Q_o v_o^i &= 0, \\ E_o(1+v_{io}v_o^i) + \frac{v_{io}v_o^i}{(1+v_{io}v_o^i)} &= \epsilon_o, \\ Q_o v_o^i &= 0, \\ Q\sqrt{1+v_{io}v_o^i} &= q_o. \end{aligned} \tag{5.2}$$

The identifications will then give

$$v_o = -\mathrm{e}^{\sigma}, \qquad v_o^i = 0, \qquad A = P, \qquad Q = q, \qquad E = \epsilon. \tag{5.3}$$

Note that we have identified A, Q, E exactly, and not just at equilibrium, as we explained in section 2. Therefore, the energy-momentum tensor and charge current for ideal fluid can be written as,

$$\mathcal{T}^{\mu\nu}_{(0)} = \epsilon u^\mu u^\nu + PP^{\mu\nu}, \qquad \mathcal{J}^\mu_{(0)} = qu^\mu. \tag{5.4}$$

5.2 Leading order fluid

One can divide the constitutive relations in hydrodynamics in two different sectors — *parity even* and *parity odd*. Ideal fluid belongs to the first sector (in $d > 2$). The first non trivial derivative corrections in parity-even sector appears at the first derivative order e.g. shear viscosity term in energy-momentum tensor. Whereas in the parity-odd sector, the leading terms appear at $(n-1)$ derivative order for a fluid in $2n$ dimensions. All these terms and the corresponding transport coefficients (at leading order) have already been found in [6]. We shall discuss their result in our notation.

5.2.1 Parity-odd

Since there is no parity odd scalar and transverse symmetric traceless tensor at $(n-1)$ derivative order (see table 5), only charge current gets parity-odd corrections:

$$\tilde{\Upsilon}^{\mu}_{(n-1)} = \sum_{m=1}^{n} {}^{n-1}\mathbf{C}_{m-1}\omega_m l^{\mu}_m. \tag{5.5}$$

The combinatorial factor is introduced for convenience. It also ensures we do not surpass the limits of m. The fluid variables receives following corrections,

$$\vartheta_\Lambda = \vartheta_{\Lambda o} + \tilde{\Delta}^{(n-1)}\vartheta_\Lambda, \qquad v^i = v^i_o + \tilde{\Delta}^{(n-1)}v^i. \tag{5.6}$$

Further, there is no parity-odd gauge invariant scalar at equilibrium on $\mathcal{M}_{2n-1}$, implying that $\tilde{\Delta}^{(n-1)}W^{\text{eqb}}_{(C)} = 0$. Now comparing eq. (2.8) with eq. (3.14) we will find the constraints at parity-odd leading derivative order:

$$\omega_m = -\frac{\vartheta^2 n}{\epsilon + P}\left[sC_{m-1} + qC_m + (n+1)\left(\frac{s}{m} + \frac{q\nu}{m+1}\right)C^{(2n)}\nu^m\right]. \tag{5.7}$$

And the corrections to fluid variables,

$$\tilde{\Delta}^{(n-1)}\vartheta = \tilde{\Delta}^{(n-1)}\nu = 0, \qquad \tilde{\Delta}^{(n-1)}v^i = \sum_{m=1}^{n} {}^{n-1}\mathbf{C}_{m-1}\alpha_{o(m)}l^i_{o(m)}, \tag{5.8}$$

where,

$$\alpha_m = -\frac{\vartheta^2 n}{\epsilon + P}\left[C_{m-1}\nu - C_m + \frac{(n+1)}{m(m+1)}C^{(2n)}\nu^{m+1}\right]. \tag{5.9}$$

Here we present these relations for completion as well as to set up our notations and conventions. We would also like to make some interesting observations about these functions. One can verify that

$$s\alpha_m + q\alpha_{m+1} = \nu\omega_m - \omega_{m+1} \qquad \forall\ m \in \{1, n-1\}, \tag{5.10}$$

$$P^{(1,0)}\omega_m = s\left(P^{(1,0)}\alpha_m\right)^{(0,1)} + q\left(P^{(1,0)}\alpha_{m+1}\right)^{(0,1)} \qquad \forall\ m \in \{1, n-1\}. \tag{5.11}$$

Here pressure $P(\vartheta, \nu)$ is function of temperature ϑ and redefined chemical potential ν. For any function $Q(\vartheta, \nu)$ we define $Q^{(m,n)} = \frac{\partial^{m+n}}{\partial^m\vartheta\partial^n\nu}Q$. These will come handy in subleading order calculation.

JHEP05(2015)010

Note that the entire leading order parity-odd sector is determined in terms of unknown coefficients C_m's. It turns out that some of them are forced to be zero by requiring CPT invariance, while the others can be related to the gravitational and mixed anomalies by keeping the fluid on a cone [11]. These relations to gravitational and mixed anomalies however, do not show up in derivative counting.

5.2.2 Parity-even

The most generic current corrections at parity-even leading derivative order are (see table 1):

$$\Upsilon^{\mu}_{(1)} = \sum_{\Lambda=1}^{3} \lambda_{\Lambda} V^{\mu}_{\Lambda}, \qquad \Pi^{\mu\nu}_{(1)} = -2\eta\sigma^{\mu\nu} - \zeta P^{\mu\nu}\Theta, \tag{5.12}$$

while at equilibrium the only surviving contributions are:

$$\Upsilon^{\mu}_{o(1)} = \sum_{\Lambda=1}^{2} \lambda_{o\Lambda} V^{\mu}_{o\Lambda}, \qquad \Pi^{\mu\nu}_{o(1)} = 0. \tag{5.13}$$

There are no gauge-invariant parity-even scalars at equilibrium that appear at this order. Therefore, $\Delta^{(n-1)} W^{\rm eqb}_{(C)} = 0$. Now comparing eq. (2.8) with eq. (3.14) we will find at parity-even leading derivative order that all corrections vanish

$$\pi^{ij}_{o(1)} = \varsigma^{i}_{o(1)} = \Delta^{(1)}\vartheta = \Delta^{(1)}\nu = \Delta^{(1)} v^{i} = 0. \tag{5.14}$$

We hence get the constraints:

$$\lambda_1 = \lambda_2 = 0. \tag{5.15}$$

So finally the form of currents is

$$\Upsilon^{\mu}_{(1)} = \lambda_3 V^{\mu}_{3}, \qquad \Pi^{\mu\nu}_{(1)} = -2\eta\sigma^{\mu\nu} - \zeta P^{\mu\nu}\Theta. \tag{5.16}$$

We also get to know that no fluid quantities (ϑ, ν, v^i) get order one parity-even correction.

5.3 Subleading order fluid

In this section, we shall describe the constraints on charged fluid in arbitrary even dimensions at subleading derivative order (i.e. n order), in presence of U(1) anomaly. Where as, the subleading correction to parity-even sector comes at second order in derivative expansion. Some aspects of four dimensional fluids at sub-leading order have already been performed in [7, 16].

5.3.1 Parity-odd

Sub-leading order parity-odd fluid dynamics in four spacetime dimensions has already been discussed in [7]. Here, we generalize the results in arbitrary even dimensions and find the constraints on the transport coefficients. We see that, much like in [7], the higher dimensional transport coefficients depend on first order transport coefficients η, ζ.

JHEP05(2015)010

From counting we can see that the n order parity-odd corrections (at eqb.) are given by (see table 5)

$$\tilde{\Upsilon}^{\mu}_{o(n)} = \sum_{m=1}^{n-1} {}^{n-2}\mathbf{C}_{m-1}\left(\tilde{\nu}_{o1,m}\tilde{\mathbf{V}}^{\mu}_{o1,m} + \tilde{\nu}_{o2,m}\tilde{\mathbf{V}}^{\mu}_{o2,m}\right), \tag{5.17}$$

$$\begin{aligned}\tilde{\Pi}^{\mu\nu}_{o(n)} = &\sum_{m=1}^{n-1} {}^{n-2}\mathbf{C}_{m-1}\left(\tilde{\tau}_{o1,\Lambda m}\tilde{\mathbf{T}}^{\mu\nu}_{o1,\Lambda m} + \tilde{\tau}_{o2,\Lambda\Gamma m}\tilde{\mathbf{T}}^{\mu\nu}_{o2,\Lambda\Gamma m}\right) + \sum_{m=1}^{n-1} {}^{n-1}\mathbf{C}_{m-1}\tilde{\tau}_{o3,\Lambda m}\tilde{\mathbf{T}}^{\mu\nu}_{o3,\Lambda m}\\ &+ P^{\mu\nu}_{o}\sum_{m=1}^{n} {}^{n-1}\mathbf{C}_{m-1}\tilde{\sigma}_{o\Lambda m}\tilde{\mathbf{S}}_{o\Lambda m}.\end{aligned} \tag{5.18}$$

Sum over the relevant 'Λ, Γ' indices is understood. We explicitly write the m index contraction to emphasize that the sum runs over different values for different terms. We do not state non-equilibrium contributions as they won't be required in this computation.

From eq. (2.8) and eq. (3.14) we get,

$$\Delta^{(n)}\bar{T} = \mathrm{e}^{2\sigma}\tilde{\Delta}^{(n)}\epsilon = \mathrm{e}^{2\sigma}\vartheta_o^2\frac{\delta W^{\mathrm{eqb}}_{(C)}}{\delta\vartheta_o}, \tag{5.19}$$

$$\Delta^{(n)}\bar{J} = -\mathrm{e}^{\sigma}\tilde{\Delta}^{(n)}q = -\mathrm{e}^{\sigma}\frac{\delta W^{\mathrm{eqb}}_{(C)}}{\delta\nu_o}. \tag{5.20}$$

Now,

$$\tilde{\Delta}^{(n)}\epsilon = \left(\frac{\partial\epsilon}{\partial\vartheta}\right)_o\tilde{\Delta}^{(n)}\vartheta + \left(\frac{\partial\epsilon}{\partial\nu}\right)_o\tilde{\Delta}^{(n)}\nu, \qquad \tilde{\Delta}^{(n)}q = \left(\frac{\partial q}{\partial\vartheta}\right)_o\tilde{\Delta}^{(n)}\vartheta + \left(\frac{\partial q}{\partial\nu}\right)_o\tilde{\Delta}^{(n)}\nu. \tag{5.21}$$

Therefore from eq. (5.19) and eq. (5.20) we can write,

$$\tilde{\Delta}^{(n)}\vartheta_\Lambda = \vartheta_o\mathbf{E}_{o\Lambda\Gamma}\frac{\delta\tilde{\Delta}^{(n)}W^{\mathrm{eqb}}_{(C)}}{\delta\vartheta_{o\Gamma}}, \tag{5.22}$$

where,

$$\mathbf{E}_{\Lambda\Gamma} = \begin{pmatrix}\vartheta\frac{\partial\vartheta}{\partial\epsilon}\big|_q, & \frac{1}{\vartheta}\frac{\partial\vartheta}{\partial q}\big|_\epsilon\\ \vartheta\frac{\partial\nu}{\partial\epsilon}\big|_q, & \frac{1}{\vartheta}\frac{\partial\nu}{\partial q}\big|_\epsilon\end{pmatrix}, \qquad \vartheta_\Lambda = \{\vartheta, \nu\}. \tag{5.23}$$

Similarly comparing the $\bar{T}^i$ equations in (2.8) and (3.14) we get,

$$\frac{\epsilon_o + P_o}{\vartheta_o}\tilde{\Delta}^{(n)}v^i = \vartheta_o\nu_o\frac{\partial W^{\mathrm{eqb}}_{(C)}}{\partial A_i} - \mathrm{e}^{-\sigma}\frac{\partial W^{\mathrm{eqb}}_{(C)}}{\partial a_i}, \tag{5.24}$$

which can be written as,

$$\tilde{\Delta}^{(n)}v^i = (-)^{\Lambda}\frac{\mu_{o\Lambda}}{P_o^{(1,0)}}\frac{\delta\tilde{\Delta}^{(n)}W^{\mathrm{eqb}}_{(C)}}{\delta a_{\Lambda i}}, \tag{5.25}$$

where,

$$\mathbf{A}_\Lambda = \partial_\Gamma P\mathbf{E}_{\Gamma\Lambda} = \left(\vartheta\frac{\partial P}{\partial\epsilon}\big|_q, \frac{1}{\vartheta}\frac{\partial P}{\partial q}\big|_\epsilon\right), \qquad \mu_\Lambda = \{\vartheta, \nu\vartheta\}, \qquad a^i_\Lambda = \left\{\tilde{\vartheta}a^i, A^i\right\}. \tag{5.26}$$

JHEP05(2015)010

One can check that $\mathbf{E}_{\Lambda\Gamma}$ is symmetric matrix and $\partial_\Lambda = \frac{\partial}{\partial\vartheta_\Lambda}$. We would like to emphasize that these are purely notations, to make the calculations tractable and easy to digest. There is a summation on repeated Λ, Γ indices. Now comparing $\bar{T}^{ij}$ and $\bar{J}^i$ in eq. (2.8) with eq. (3.14) at parity-odd subleading derivative order, we have corrections to constitutive relations

$$\begin{aligned}
\frac{1}{\vartheta_o}\tilde{\pi}^{ij}_{o(n)} &= 2\frac{\delta\tilde{\Delta}^{(n)}W^{\rm eqb}_{(C)}}{\delta g_{ij}} - g^{ij}\mathbf{A}_{o\Lambda}\frac{\delta\tilde{\Delta}^{(n)}W^{\rm eqb}_{(C)}}{\delta\vartheta_{o\Lambda}} - \frac{1}{\vartheta_o}\tilde{\Delta}^{(n-1)}\pi^{ij}_{(1)},\\
P^{(1,0)}_o\tilde{\varsigma}^i_{o(n)} &= \vartheta_o\mathbf{S}_{o\Lambda}\frac{\delta\tilde{\Delta}^{(n)}W^{\rm eqb}_{(C)}}{\delta a_{\Lambda i}} - P^{(1,0)}_o\tilde{\Delta}^{(n-1)}\varsigma^i_{(1)},
\end{aligned} \tag{5.27}$$

where,

$$\mathbf{S}_\Lambda = \frac{\partial P}{\partial\mu_{\bar{\Lambda}}} = \{q, s\}\,. \tag{5.28}$$

$\bar{\Lambda}$ swaps the value of $\Lambda : 1 \leftrightarrow 2$. The generating functional $\tilde{\Delta}^{(n)}W^{\rm eqb}_{(C)}$ contain all scalars $\tilde{\mathbf{S}}_{o\Lambda m}$. But one can check that $\tilde{\mathbf{S}}_{o1m}$ can be connected to $\tilde{\mathbf{S}}_{o2m}$ by a total derivative. So we take the partition function

$$\tilde{\Delta}^{(n)}W^{\rm eqb}_{(C)} = \int \mathrm{d}^{2n-1}x\sqrt{g}\sum_{m=1}^{n}{}^{n-1}\mathbf{C}_{m-1}\tilde{\mathcal{S}}_m\tilde{\mathbf{S}}_{o2,m}. \tag{5.29}$$

We compute the variation of generating functional with respect to different fields and find that

$$\begin{aligned}
\frac{\delta\tilde{\Delta}^{(n)}W^{\rm eqb}_{(C)}}{\delta g_{ij}} &= 0,\\
\frac{\delta\tilde{\Delta}^{(n)}W^{\rm eqb}_{(C)}}{\delta\vartheta_{o\Lambda}} &= -(-)^\Lambda\sum_{m=1}^{n}{}^{n-1}\mathbf{C}_{m-1}\tilde{\mathcal{S}}^{(1,0)}_m\tilde{\mathbf{S}}_{o\bar{\Lambda},m},\\
\frac{\delta\tilde{\Delta}^{(n)}W^{\rm eqb}_{(C)}}{\delta a_{\Lambda i}} &= (n-1)\sum_{m=1}^{n-1}{}^{n-2}\mathbf{C}_{m-1}\tilde{\mathcal{S}}^{(1,0)}_{m+2-\Lambda}\tilde{\mathbf{V}}^i_{o2,m}.
\end{aligned} \tag{5.30}$$

Using the form of lower order currents corrections from eq. (5.16) we can write,

$$\begin{aligned}
\tilde{\Delta}^{(n-1)}\pi^{ij}_{(1)} &= -2\eta_o\tilde{\Delta}^{(n-1)}\sigma^{ij} - \zeta_o g^{ij}\tilde{\Delta}^{(n-1)}\Theta\\
&= -2\eta_o\vartheta_o\,{}^{n-1}\mathbf{C}_{m-1}\partial_\Lambda\left(\frac{\alpha_{o(m)}}{\vartheta_o}\right)\tilde{\mathbf{T}}^{ij}_{o1,\Lambda m}\\
&\quad - \eta_o\,{}^{n-2}\mathbf{C}_{m-1}(n-1)\alpha_{o(m+2-\Lambda)}\tilde{\mathbf{T}}^{ij}_{o2,\Lambda m}\\
&\quad - g^{ij}\zeta_o\vartheta_o\,{}^{n-1}\mathbf{C}_{m-1}\partial_\Lambda\left(\frac{\alpha_{o(m)}}{\vartheta_o}\right)\tilde{\mathbf{S}}_{o\Lambda m}
\end{aligned} \tag{5.31}$$

$$\begin{aligned}
\tilde{\Delta}^{(n-1)}\varsigma^i_{(1)} &= \lambda_{o3}\tilde{\Delta}^{(n-1)}\mathcal{E}^i\\
&= \lambda_{o3}\,{}^{n-2}\mathbf{C}_{m-1}(n-1)\left[\alpha_{o(m+1)} + \nu_o\alpha_{o(m)}\right]\tilde{\mathbf{V}}^i_{o1m}.
\end{aligned} \tag{5.32}$$

JHEP05(2015)010

One can now use the results, obtained in eqs. (5.30) and (5.32) in eq. (5.27) and comparing these expressions with eq. (5.18) to get the constraints,

$$\tilde{\tau}_{1,\Lambda m} = 2\eta\vartheta\partial_\Lambda\left(\frac{\alpha_m}{\vartheta}\right), \qquad \tilde{\tau}_{2,\Lambda m} = \eta(n-1)\alpha_{(m+2-\Lambda)}, \qquad \tilde{\tau}_{3,\Lambda\Gamma m} = 0,$$
$$-\frac{1}{\mathbf{A}_2}\left(\tilde{\sigma}_{1m} - \frac{\zeta}{2\eta}\tilde{\tau}_{1,1m}\right) = \frac{1}{\mathbf{A}_1}\left(\tilde{\sigma}_{2m} - \frac{\zeta}{2\eta}\tilde{\tau}_{1,2m}\right) = P^{(1,0)}\tilde{\chi}_m,$$
$$\tilde{\nu}_{1m} = -\lambda_3(n-1)\left(\alpha_{m+1} + \nu\alpha_m\right), \qquad \tilde{\nu}_{2m} = -(n-1)\left(q\tilde{\chi}_{m+1} + s\tilde{\chi}_m\right). \tag{5.33}$$

Hence everything is determined in terms of a known function α_m and a new coefficient $\tilde{\chi}_m$. Note that if we had used the $2n$-assymetrization condition eq. (4.17) to get rid of one traceless symmetric tensor to start with; a consistent choice would have been to remove $\tilde{\mathbf{T}}^{\mu\nu}_{3,\Lambda 1m}$ entirely and $\tilde{\mathbf{T}}^{\mu\nu}_{3,\Lambda 2m}$ for $m = 1$ (see table 5). The coefficients of these terms are set to zero already by our constraints, which means the other leftover constraints are independent.

Finally we get the corrections to fluid variables using eq. (5.25) as

$$\tilde{\Delta}^{(n)}\vartheta_\Lambda = {}^{n-1}\mathbf{C}_{m-1}(-)^\Gamma\mathbf{E}_{o\Lambda\Gamma}\chi_{o(m)}\tilde{\mathbf{S}}_{o\bar{\Gamma},m},$$
$$\tilde{\Delta}^{(n)}v^i = (n-1)\sum_{m=1}^{n-1}{}^{n-2}\mathbf{C}_{m-1}\left(\chi_{o(m+1)} - \nu_o\chi_{o(m)}\right)\tilde{\mathbf{V}}^i_{o2,m}. \tag{5.34}$$

5.3.2 Parity-even

Next, we present the results for sub-leading order (two-derivative) parity even sector for the fluid. From counting we can verify that at the second order, parity-even corrections (at eqb.) are given by (see table 2):

$$\Upsilon^\mu_{o(2)} = \sum_{\#}\nu_{o\#}\mathbf{V}^\mu_{o\#}, \qquad \frac{1}{\vartheta_o}\Pi^{\mu\nu}_{o(2)} = \sum_{\#}\tau_{o\#}\mathbf{T}^{\mu\nu}_{o\#} + P^{\mu\nu}_o\sum_{\#}\sigma_{o\#}\mathbf{S}_{o\#}. \tag{5.35}$$

refers to sum over all relevant indices. Now comparing eq. (2.8) with eq. (3.14) at parity-even subleading derivative order, and performing a similar manipulation as last section, we have corrections to constitutive relations:

$$\frac{1}{\vartheta_o}\pi^{ij}_{o(2)} = 2\frac{\delta\Delta^{(2)}W^{\rm eqb}_{(C)}}{\delta g_{ij}} - g^{ij}\mathbf{A}_{o\Lambda}\frac{\delta\tilde{\Delta}^{(2)}W^{\rm eqb}_{(C)}}{\delta\vartheta_{o\Lambda}} - \boxed{P^{(1,0)}_o\tilde{\Delta}^{(1)}v^{\langle i}\tilde{\Delta}^{(1)}v^{j\rangle}}$$
$$+\boxed{g^{ij}\tilde{\Delta}^{(1)}v_k\left\{\mathbf{A}_{o2}\tilde{\varsigma}^k_{o(1)} + \left(\frac{1}{\vartheta_o}\mathbf{A}_{o1}P^{(1,0)}_o + \frac{1}{2\vartheta_o}\mathbf{A}_{o2}P^{(0,1)}_o - \frac{1}{3}P^{(1,0)}_o\right)\tilde{\Delta}^{(1)}v^k\right\}},$$
$$P^{(1,0)}_o\varsigma^i_{o(2)} = \vartheta_o\mathbf{S}_{o\Lambda}\frac{\delta\Delta^{(2)}W^{\rm eqb}_{(C)}}{\delta a_{\Lambda i}} - \boxed{P^{(1,0)}_o\tilde{\Delta}^{(1)}\tilde{\varsigma}^i_{(1)}}, \tag{5.36}$$

while the fluid variables get the corrections:

$$\tilde{\Delta}^{(2)}\vartheta_\Lambda = \vartheta_o\mathbf{E}_{o\Lambda\Gamma}\frac{\delta\tilde{\Delta}^{(2)}W^{\rm eqb}_{(C)}}{\delta\vartheta_{o\Gamma}} - \boxed{\left(\mathbf{A}_{o\Lambda} - \frac{1}{2}P^{(0,1)}_o\mathbf{E}_{o\Lambda 2}\right)\tilde{\Delta}^{(1)}v_i\tilde{\Delta}^{(1)}v^i} - \boxed{\vartheta_o\mathbf{E}_{o\Lambda 2}\tilde{\Delta}^{(1)}v_i\tilde{\varsigma}^i_{o(1)}},$$
$$\Delta^{(2)}v^i = (-)^\Lambda\frac{\mu_{o\Lambda}}{P^{(1,0)}_o}\frac{\delta\Delta^{(2)}W^{\rm eqb}_{(C)}}{\delta a_{\Lambda i}}. \tag{5.37}$$

JHEP05(2015)010

Notice that the boxed terms only contribute for four dimensional fluids ($n = 2$). Out of the scalars enlisted in table 2, $\mathbf{S}_{o1\Lambda}$ can be related to others by a total derivative. Hence $\Delta^{(2)}W^{\rm eqb}_{(C)}$ is given by:

$$\Delta^{(2)}W^{\rm eqb}_{(C)} = -\frac{1}{2}\int \left\{\mathrm{d}x^i\right\}\sqrt{g}\left\{\mathcal{S}_R\mathbf{S}_{o4} + \mathcal{S}_{f\Lambda\Gamma}\mathbf{S}_{o3(\Lambda\Gamma)} + \mathcal{S}_{\vartheta\Lambda\Gamma}\mathbf{S}_{o2(\Lambda\Gamma)}\right\}. \tag{5.38}$$

Now we can find the variations of $\Delta^{(2)}W^{\rm eqb}_{(C)}$,

$$\begin{aligned}
2\frac{\delta\Delta^{(2)}W^{\rm eqb}_{(C)}}{\delta g_{ij}} &= -\partial_\Lambda\mathcal{S}_R\mathbf{T}^{ij}_{o1,\Lambda} - \left(\partial_\Lambda\partial_\Gamma\mathcal{S}_R - \mathcal{S}_{\vartheta\Lambda\Gamma}\right)\mathbf{T}^{ij}_{o2,\Lambda\Gamma} + 2\mathcal{S}_{f\Lambda\Gamma}\mathbf{T}^{ij}_{o3,\Lambda\Gamma} + \mathcal{S}_R\mathbf{T}^{ij}_{o4} \\
&\quad + g^{ij}\Bigg[\left(1-\frac{1}{d-1}\right)\partial_\Lambda\mathcal{S}_R\mathbf{S}_{o1,\Lambda} \\
&\qquad + \left(\partial_\Lambda\partial_\Gamma\mathcal{S}_R - \frac{1}{d-1}\partial_\Lambda\partial_\Gamma\mathcal{S}_R - \frac{1}{2}\mathcal{S}_{\vartheta\Lambda\Gamma} + \frac{1}{d-1}\mathcal{S}_{\vartheta\Lambda\Gamma}\right)\mathbf{S}_{o2,\Lambda\Gamma} \\
&\qquad - \frac{1}{2}\left(1-\frac{4}{d-1}\right)\mathcal{S}_{f\Lambda\Gamma}\mathbf{S}_{o3,\Lambda\Gamma} - \frac{1}{2}\left(1-\frac{2}{d-1}\right)\mathcal{S}_R\mathbf{S}_{o4}\Bigg],
\end{aligned} \tag{5.39}$$

$$\begin{aligned}
\frac{\delta\Delta^{(2)}W^{\rm eqb}_{(C)}}{\delta\vartheta_{o\Sigma}} &= \mathcal{S}_{\vartheta\Sigma\Lambda}\mathbf{S}_{o1,\Lambda} + \left(\partial_{(\Gamma}\mathcal{S}_{\vartheta\Lambda)\Sigma} - \frac{1}{2}\partial_\Sigma\mathcal{S}_{\vartheta\Lambda\Gamma}\right)\mathbf{S}_{o2,\Lambda\Gamma} \\
&\quad - \frac{1}{2}\partial_\Sigma\mathcal{S}_{f\Lambda\Gamma}\mathbf{S}_{o3,\Lambda\Gamma} - \frac{1}{2}\partial_\Sigma\mathcal{S}_R\mathbf{S}_{o4},
\end{aligned} \tag{5.40}$$

$$\frac{\delta\Delta^{(2)}W^{\rm eqb}_{(C)}}{\delta a_{\Lambda i}} = 2\mathcal{S}_{f\Lambda\Gamma}\mathbf{V}^i_{o1,\Gamma} - 2\partial_\Sigma\mathcal{S}_{f\Lambda\Gamma}\mathbf{V}^i_{o2,\Gamma\Sigma}. \tag{5.41}$$

Using the form of lower order corrections from eq. (5.5) for $n = 2$ we can write,

$$\begin{aligned}
\tilde{\Delta}^{(1)}\tilde{\varsigma}^i_{(1)} &= \omega_{o\Sigma}l^i_{o\Sigma} \\
&= (-)^\Sigma\omega_{o\Sigma}\left\{\mu_{o\bar{\Sigma}}\alpha_{\bar{\Lambda}}\mathbf{V}^i_{o1,\Lambda} - \partial_\Gamma\left(\mu_{o\bar{\Sigma}}\alpha_{\bar{\Lambda}}\right)\mathbf{V}^i_{o2,\Lambda\Gamma}\right\}.
\end{aligned} \tag{5.42}$$

We can now put the variations of generating functional along with lower order corrections worked out above in eq. (5.36). Using eqs. (5.10), (5.11) and eliminating partition function coefficients $\mathcal{S}$'s we will find following 7 constraints,

$$\tau_{1,\Lambda} + \partial_\Lambda\tau_4 = 0, \tag{5.43}$$

$$\sigma_{o1,\Lambda} = \frac{d-2}{d-1}\partial_\Lambda\tau_4 - \mathbf{A}_\Sigma\partial_\Sigma\partial_\Lambda\tau_4 - \mathbf{A}_\Sigma\tau_{2,\Sigma\Lambda}, \tag{5.44}$$

$$\begin{aligned}
2\sigma_{2,\Lambda\Gamma} &= \partial_\Lambda\partial_\Gamma\tau_4 - \mathbf{A}_\Sigma\partial_\Sigma\partial_\Lambda\partial_\Gamma\tau_4 - \frac{d-3}{d-1}\tau_{2,\Lambda\Gamma} \\
&\quad -2\mathbf{A}_\Sigma\partial_{(\Lambda}\tau_{2,\Gamma)\Sigma} + \mathbf{A}_\Sigma\partial_\Sigma\tau_{2,\Lambda\Gamma},
\end{aligned} \tag{5.45}$$

$$4\sigma_{3,\Lambda\Gamma} = -\frac{d-5}{d-1}\tau_{3,\Lambda\Gamma} + \mathbf{A}_\Sigma\partial_\Sigma\tau_{3,\Lambda\Gamma}, \tag{5.46}$$

$$2\sigma_4 = -\frac{d-3}{d-1}\tau_4 + \mathbf{A}_\Sigma\partial_\Sigma\tau_4, \tag{5.47}$$

$$\nu_{1,\Lambda} = \frac{\vartheta}{P^{(1,0)}}\mathbf{S}_\Gamma\tau_{3,\Lambda\Gamma}, \tag{5.48}$$

$$\nu_{2,\Gamma\Lambda} = -\frac{\vartheta}{P^{(1,0)}}\mathbf{S}_\Sigma\partial_\Lambda\tau_{3,\Gamma\Sigma}. \tag{5.49}$$

JHEP05(2015)010

Coincidently none of the constraints depend on $n = 2$ special contributions. On the other hand fluid variables corrections are given by eq. (5.37):

$$\begin{aligned}\tilde{\Delta}^{(2)}\vartheta_\Omega = \vartheta_o \mathbf{E}_{o\Omega\Sigma}\Bigg[& \left(\tau_{o2,\Sigma\Lambda} + \partial_\Lambda\partial_\Sigma\tau_{o4}\right)\mathbf{S}_{o1,\Lambda} \\ & + \left(\partial_{(\Gamma}\tau_{o2,\Lambda)\Sigma} - \frac{1}{2}\partial_\Sigma\tau_{o2,\Lambda\Gamma} + \frac{1}{2}\partial_\Sigma\partial_\Lambda\partial_\Gamma\tau_{o4}\right)\mathbf{S}_{o2,\Lambda\Gamma} \\ & - \frac{1}{4}\partial_\Sigma\tau_{o3,\Lambda\Gamma}\mathbf{S}_{o3(\Lambda\Gamma)} - \frac{1}{2}\partial_\Sigma\tau_{o4}\mathbf{S}_{o4}\Bigg] \\ & + \boxed{\frac{1}{2}\Bigg[\frac{1}{2}\vartheta_o\mathbf{E}_{o\Omega\Sigma}\partial_\Sigma\left(P_o^{(1,0)}\alpha_{o\bar{\Lambda}}\alpha_{o\bar{\Gamma}}\right) - \mathbf{A}_{o\Omega}\alpha_{o\bar{\Lambda}}\alpha_{o\bar{\Gamma}} \\ & \qquad + \vartheta_o\mathbf{E}_{o\Omega 2}\alpha_{o\bar{\Lambda}}\left(\frac{1}{2}q_o\alpha_{o\bar{\Gamma}} - \omega_{o\bar{\Gamma}}\right)\Bigg]\mathbf{S}_{o3(\Lambda\Gamma)}}\end{aligned} \tag{5.50}$$

$$\begin{aligned}\Delta^{(2)}v^i = (-)^\Lambda\frac{\mu_{o\Lambda}}{P_o^{(1,0)}}\Bigg[& \left(\tau_{o3,\Lambda\Gamma} - \boxed{P_o^{(1,0)}\alpha_{o\bar{\Lambda}}\alpha_{o\bar{\Gamma}}}\right)\mathbf{V}^i_{o1,\Gamma} \\ & - \partial_\Sigma\left(\tau_{o3,\Lambda\Gamma} - \boxed{P_o^{(1,0)}\alpha_{o\bar{\Lambda}}\alpha_{o\bar{\Gamma}}}\right)\mathbf{V}^i_{o2,\Gamma\Sigma}\Bigg]\end{aligned} \tag{5.51}$$

This completes our calculation of subsubleading derivative order fluid.

6 Conclusions

In this paper we computed the energy momentum tensor and charge current for a fluid system in $2n$ dimensions with U(1) anomaly up to subleading order in derivative expansion (for both parity odd and parity even sectors) from the equilibrium partition function of the fluid. We described a novel counting prescription to construct the fluid data. However, an important issue we encountered here is that it is non-trivial to find independent vectors and tensors at arbitrary derivative order. But we were still able to find the independent transport coefficients and distinct constitutive relations. We showed that the knowledge of independent scalars at the required derivative order is sufficient for this purpose. This is a powerful observation and it enables us to carry on the computation at $(n+1)$ derivative order, where, we could find the independent scalars. We observe that the parity odd transport coefficients which appear at n derivative order in constitutive relations are constrained and some of them depend on the first order transport coefficients like η, ζ etc. It would be interesting to find the similar dependence in a holographic set up [17]. We plan to explore the holographic computation in future.

It is also interesting to find the fluid constitutive relations in presence of both U(1) and gravitational anomaly in arbitrary $2n$ dimensions. But, since the gravitational anomaly appears at two higher derivative level compared to the U(1) anomaly in derivative expansion[1], it requires to carry on our analysis to one higher derivative (sub-sub-leading) order, i.e. to $(n+1)$ derivative order. Fortunately, as mentioned earlier, even at this order, we could determine the independent scalars and hence, in principle, the computation is possible. We have carried a large part of it in appendix B.

JHEP05(2015)010

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A Kaluza-Klein decomposition

If a $(d+1)$-dim spacetime $\mathcal{M}_{(d+1)}$ has a preferred time-like direction ω^μ, it can be decomposed into $S^1 \times \mathcal{M}_{(d)}$, where S^1 is the euclidean time circle. A k-rank tensor decompose in 2^k parts in this scheme:

1. $\mathcal{S}$ does not decompose.
2. $\mathcal{V}^\mu$ decompose in $\omega^\mu V_\mu$ and $P^{\mu\nu}V_\nu$.
3. $\mathcal{T}^{\mu\nu}$ decompose in $\omega^\mu\omega^\nu T_{\mu\nu}$, $\omega^\mu P_o^{\alpha\nu}T_{\mu\nu}$, $P_o^{\alpha\mu}\omega^\nu T_{\mu\nu}$ and $P_o^{\alpha\mu}P_o^{\beta\nu}T_{\mu\nu}$,

and so on. Where $P_o^{\mu\nu} = G^{\mu\nu} - \frac{\omega^\mu\omega^\nu}{G_{\alpha\beta}\omega^\alpha\omega^\beta}$ is the projection operator. If we are studying theory at equilibrium, we already have a preferred direction along the Killing vector of the theory $\omega^\mu = \partial_0$. In this case we know that a $(d+1)$-vector $\mathcal{V}^\mu$ will yield a scalar:

$$\omega^\mu \mathcal{V}_\mu \Rightarrow \mathcal{V}_0 := V, \tag{A.1}$$

and a (d)-vector:

$$P_o^{\mu\nu}\mathcal{V}_\nu \Rightarrow \mathcal{V}^i := V^i. \tag{A.2}$$

Hence we see that a U(1) gauge field $\mathcal{A}^\mu$ will be decomposed in $\{\mathcal{A}_0(\vec{x}), \mathcal{A}^i(\vec{x})\}$. Similarly a tensor $\mathcal{T}^{\mu\nu}$ decomposes in $\mathcal{T}_{00}$, $\mathcal{T}^i_{\ 0}$, $\mathcal{T}_0^{\ i}$, $\mathcal{T}^{ij}$. It is the similar way the metric $G^{\mu\nu}$ on $\mathcal{M}_{(d+1)}$ decomposes, hence we define:

$$G_{00} = -\mathrm{e}^{2\sigma}, \quad G^i_{\ 0} = 0, \quad G^{ij} = g^{ij}, \tag{A.3}$$

where we define g^{ij} as metric on $\mathcal{M}_{(d)}$. Now using the diffeomorphic invariance one can work out the full form of $G^{\mu\nu}$

$$\mathrm{d}s^2 = G_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -\mathrm{e}^{2\sigma(\vec{x})}\left(\mathrm{d}t + a_i(\vec{x})\mathrm{d}x^i\right)^2 + g_{ij}(\vec{x})\mathrm{d}x^i\mathrm{d}x^j, \tag{A.4}$$

$$G_{\mu\nu} = \begin{bmatrix} -\mathrm{e}^{2\sigma} & -\mathrm{e}^{2\sigma}a_j \\ -\mathrm{e}^{2\sigma}a_i & \left(g_{ij} - a_i a_j \mathrm{e}^{2\sigma}\right) \end{bmatrix}, \qquad G^{\mu\nu} = \begin{bmatrix} (-\mathrm{e}^{-2\sigma} + a^2) & -a^j \\ -a^i & g^{ij} \end{bmatrix}, \tag{A.5}$$

where time redefinition invariance requires that a^i is an independent gauge field, named as Kaluza-Klein gauge field. Using the euclidean time period $\tilde{\beta}$ we can define the local equilibrium temperature of the theory as: $\vartheta_o = 1/\beta_o = \mathrm{e}^{-\sigma}/\tilde{\beta}$. Our higher dimensional metric is hence disintegrated in a scalar (Temperature), a gauge field and a lower dimensional metric.

JHEP05(2015)010

We can now use the metric $G^{\mu\nu}$ to raise/lower the components of vectors:

$$\mathcal{V}_i = g_{ij}\mathcal{V}^j + a_i\mathcal{V}_0, \qquad \mathcal{V}^0 = -\mathrm{e}^{-2\sigma}\mathcal{V}_0 - a_j\mathcal{V}^j. \tag{A.6}$$

which are not Kaluza-Klein gauge invariant. From here we read out the (d)-covectors:

$$V_i = (\mathcal{V}_i - a_i\mathcal{V}_0)\,. \tag{A.7}$$

Determinant of metric in two spaces can be related as:

$$G = -\det G_{\mu\nu} = \mathrm{e}^{2\sigma}\det g_{ij} = \mathrm{e}^{2\sigma}g. \tag{A.8}$$

We have the Levi-Civita symbol in lower spatial dimensions:

$$\epsilon^{ijk\ldots} = \mathrm{e}^{\sigma}\epsilon^{0ijk\ldots} = -\mathrm{e}^{-\sigma}\epsilon_0{}^{ijk\ldots}, \tag{A.9}$$

where $\epsilon^{0123\ldots} = 1/\sqrt{G}$ and $\epsilon^{123\ldots} = 1/\sqrt{g}$.

It is useful to see how higher dimensional contractions behave in lower dimensions:

$$\mathcal{A}^\mu\mathcal{B}_\mu = -\mathrm{e}^{-2\sigma}AB + A^iB_i \tag{A.10}$$

$$\epsilon^{i\mu_1\mu_2\ldots\mu_{n-1}}\mathcal{A}_{\mu_1\mu_2\ldots\mu_{n-1}}, = \mathrm{e}^{-\sigma}\epsilon^{ij_1j_2\ldots j_{n-2}}\sum_a(-1)^a A_{j_1\ldots j_{a-1}0j_a\ldots j_{n-2}}. \tag{A.11}$$

A.1 Derivatives of metric

Once the metric is known we can reduce the derivatives of metric, i.e. the Christoffel Symbol and the Riemann Tensor. The Christoffel Symbol is defined by:

$$\hat{\Gamma}^\lambda_{\mu\nu} = \frac{1}{2}G^{\lambda\alpha}\left(\partial_\mu G_{\alpha\nu} + \partial_\nu G_{\alpha\mu} - \partial_\alpha G_{\mu\nu}\right). \tag{A.12}$$

Pretending it to be a tensor at the moment, if we define its indices to be raised and lowered with the metric $G^{\mu\nu}$. We can reduce it for Kaluza-Klein form of the metric:

$$\begin{aligned}
&\hat{\Gamma}_{000} = 0, && \hat{\Gamma}^i{}_{00} = -\mathrm{e}^{2\sigma}\frac{\partial^i\vartheta_o}{\vartheta_o}, && \hat{\Gamma}_0{}^i{}_0 = \hat{\Gamma}_{00}{}^i = \mathrm{e}^{2\sigma}\frac{\partial^i\vartheta_o}{\vartheta_o},\\
&\hat{\Gamma}^{ij}{}_0 = \hat{\Gamma}^i{}_0{}^j = \frac{1}{2}\mathrm{e}^{2\sigma}f^{ij}, && \hat{\Gamma}_0{}^{ij} = -\frac{1}{2}\mathrm{e}^{2\sigma}g^{ia}g^{jb}\left(\partial_a a_b + \partial_b a_a\right), && \hat{\Gamma}^{kij} = g^{il}g^{jm}\Gamma^k{}_{lm},
\end{aligned} \tag{A.13}$$

where $\Gamma^k{}_{ij}$ is Christoffel Symbol on $\mathcal{M}_{(d)}$, which is raised and lowered by g_{ij}. Also we define KK field tensor:

$$f^{ij} = \nabla^i a^j - \nabla^j a^i. \tag{A.14}$$

$\hat{\Gamma}_0{}^{ij}$ is not KK gauge invariant, even though it has time index down and spatial index up, which is the manifestation of $\hat{\Gamma}$ not being a tensor.

Lets define the higher dimensional covariant derivative as $\hat{\nabla}$ and lower dimensional as ∇, whereas the usual derivative is given by ∂. We can check that:

$$\begin{aligned}
\tilde{\nabla}^i\mathcal{V}^j &= \nabla^i V^j + \frac{1}{2}f^{ij}V,\\
\hat{\nabla}^i\mathcal{V}_0 &= \nabla^i V + \beta_o V\nabla^i\vartheta_o + \frac{1}{2}\mathrm{e}^{2\sigma}f^{ij}V_j,\\
\hat{\nabla}_0\mathcal{V}^i &= V\frac{\nabla^i\vartheta_o}{\vartheta_o} + \frac{1}{2}\mathrm{e}^{2\sigma}f^{ij}V_j,\\
\hat{\nabla}_0\mathcal{V}_0 &= \mathrm{e}^{2\sigma}V^i\frac{\nabla_i\vartheta_o}{\vartheta_o},
\end{aligned} \tag{A.15}$$

JHEP05(2015)010

similarly,

$$\begin{aligned}
\hat{\nabla}^i \mathcal{V}^{jk} &= \nabla^i \mathcal{V}^{jk} + \frac{1}{2} f^{ij} \mathcal{V}_0{}^k + \frac{1}{2} f^{ik} \mathcal{V}^j{}_0, \\
\hat{\nabla}^i \mathcal{V}^j{}_0 &= \nabla^i \mathcal{V}^j{}_0 + \frac{1}{2} f^{ij} \mathcal{V}_{00} + = \mathcal{V}^j{}_0 \frac{\nabla^i \vartheta_o}{\vartheta_o} + \frac{1}{2} \mathrm{e}^{2\sigma} f^i{}_k \mathcal{V}^{jk}, \\
\hat{\nabla}^i \mathcal{V}_{00} &= \nabla^i \mathcal{V}_{00} + 2\mathcal{V}_{00} \frac{\nabla^i \vartheta_o}{\vartheta_o} + \frac{1}{2} \mathrm{e}^{2\sigma} f^i{}_k \left(\mathcal{V}_0{}^k + \mathcal{V}^k{}_0 \right), \\
\hat{\nabla}_0 \mathcal{V}^{ij} &= \mathcal{V}^i{}_0 \frac{\nabla^j \vartheta_o}{\vartheta_o} + \mathcal{V}_0{}^j \frac{\nabla^i \vartheta_o}{\vartheta_o} + \frac{1}{2} \mathrm{e}^{2\sigma} f^j{}_k \mathcal{V}^{ik} + \frac{1}{2} \mathrm{e}^{2\sigma} f^i{}_k \mathcal{V}^{kj}, \\
\hat{\nabla}_0 \mathcal{V}^i{}_0 &= \mathcal{V}_{00} \frac{\nabla^i \vartheta_o}{\vartheta_o} + \frac{1}{2} \mathrm{e}^{2\sigma} f^i{}_k \mathcal{V}^k{}_0 + \mathrm{e}^{2\sigma} \mathcal{V}^{ij} \frac{\nabla_j \vartheta_o}{\vartheta_o}, \\
\hat{\nabla}_0 \mathcal{V}_{00} &= \mathrm{e}^{2\sigma} \left(\mathcal{V}^i{}_0 + \mathcal{V}_0{}^i \right) \frac{\nabla_i \vartheta_o}{\vartheta_o}.
\end{aligned} \tag{A.16}$$

Finally the Riemann Curvature Tensor is defined using an arbitrary vector X^μ as:

$$\mathcal{R}_{\mu\nu\rho\sigma} X^\sigma \ = \frac{1}{2} \left(\hat{\nabla}_\mu \hat{\nabla}_\nu - \hat{\nabla}_\nu \hat{\nabla}_\mu \right) X_\rho, \tag{A.17}$$

using which we can define:

$$\mathcal{R}_{\mu\nu} = \mathcal{R}_{\mu\alpha\nu}{}^\alpha, \qquad \mathcal{R} = \mathcal{R}^\alpha_\alpha. \tag{A.18}$$

Now a straight away computation will yield:

$$\begin{aligned}
\mathcal{R} &= R - 4 \frac{1}{\vartheta_o^2} \nabla_i \vartheta_o \nabla^i \vartheta_o + 2 \frac{1}{\vartheta_o} \nabla^i \nabla_i \vartheta_o + \frac{1}{4} \mathrm{e}^{2\sigma} f^{ij} f_{ij}, \\
u^\mu u^\nu \mathcal{R}_{\mu\nu} = \mathrm{e}^{-2\sigma} \mathcal{R}_{00} &= 2 \frac{1}{\vartheta_o^2} \nabla_i \vartheta_o \nabla^i \vartheta_o - \frac{1}{\vartheta_o} \nabla_i \nabla^i \vartheta_o + \frac{1}{4} \mathrm{e}^{2\sigma} f^{ij} f_{ij}, \\
u_\mu \mathcal{R}^{i\mu} = \mathrm{e}^{-\sigma} \mathcal{R}^i{}_0 &= \mathrm{e}^{\sigma} \frac{1}{2} \left(\nabla_k f^{ki} + \frac{3}{\vartheta_o} f^{ik} \nabla_k \vartheta_o \right), \\
\mathcal{R}^{ij} &= R^{ij} - 2 \frac{1}{\vartheta_o^2} \nabla^i \vartheta_o \nabla^j \vartheta_o + \frac{1}{\vartheta_o} \nabla^i \nabla^j \vartheta_o + \frac{1}{2} \mathrm{e}^{2\sigma} f^i{}_a f^{ja}, \\
u^\alpha u^\beta \mathcal{R}^{i\ j}_{\ \alpha\ \beta} = \mathrm{e}^{-2\sigma} \mathcal{R}^{i\ j}_{\ 0\ 0} &= 2 \frac{1}{\vartheta_o^2} \nabla^i \vartheta_o \nabla^j \vartheta_o - \frac{1}{\vartheta_o} \nabla^i \nabla^j \vartheta_o + \frac{1}{4} \mathrm{e}^{2\sigma} f^i{}_a f^{ja}, \\
\mathcal{R}^{ijk\alpha} u_\alpha = \mathrm{e}^{-\sigma} \mathcal{R}^{ijk}{}_0 &= \frac{1}{2\vartheta_o} \nabla^k f_1^{ij} - \frac{1}{\vartheta_o^2} \left(f_1^{ij} \nabla^k \vartheta_o + \frac{1}{2} f_1^{ik} \nabla^j \vartheta_o - \frac{1}{2} f_1^{jk} \nabla^i \vartheta_o \right).
\end{aligned} \tag{A.19}$$

Here R^{ijkl} is defined to be lower dimensional Riemann tensor, and $R^{ij} = R^{ikj}{}_k$, $R = R^i{}_i$.

A.2 Derivatives of gauge field

Now let us have a look at derivatives of gauge field $\mathcal{A}^\mu$. Being a vector it decomposes as:

$$A = \mathcal{A}_0 = -\mathrm{e}^{2\sigma} \left(\mathcal{A}^0 + a_j A^j \right), \qquad A^i = \mathcal{A}^i, \qquad A_i = \left(\mathcal{A}_j - a_j A \right). \tag{A.20}$$

The gauge transformation $\mathcal{A}_\mu \to \mathcal{A}_\mu + \partial_\mu \Lambda$ translates to:

$$A \to A, \qquad A_i \to A_i + \partial_i \Lambda. \tag{A.21}$$

Hence A^i is a gauge field on $\mathcal{M}_{(d)}$, while A is a scalar. Using $\tilde{\beta}$ (euclidean temperature) we define the local equilibrium potential $\nu_o = \tilde{\beta} A$. Higher dimensional field tensor however decomposes as:

$$\mathcal{F}^{\mu\nu} = \hat{\nabla}^\mu \mathcal{A}^\nu - \hat{\nabla}^\nu \mathcal{A}^\mu \quad \Rightarrow \quad \mathcal{F}^{ij} = F^{ij} + \mathrm{e}^\sigma \vartheta_o \nu_o f^{ij}, \qquad \mathcal{F}^i{}_0 = \mathrm{e}^\sigma \vartheta_o \nabla^i \nu_o, \tag{A.22}$$

where,

$$F^{ij} = \nabla^i A^j - \nabla^j A^i. \tag{A.23}$$

Now we define the four vector electric field:

$$\mathcal{E}^\mu = \mathcal{F}^{\mu\nu} u_\nu \quad \Rightarrow \quad \mathcal{E}_0 = -\mathrm{e}^\sigma \vartheta_o v_i \nabla^i \nu_o, \quad \mathcal{E}^i = -\mathrm{e}^{-2\sigma} v \mathrm{e}^\sigma \vartheta_o \nabla^i \nu_o + v_j \left(F^{ij} + \mathrm{e}^\sigma \vartheta_o \nu_o f^{ij} \right). \tag{A.24}$$

B Subsubleading order fluid

In this appendix we extend the counting discussed in section 4 to subsubleading order fluid. We form a complete set of data at this order and classify the respective scalars, vectors and symmetric tensors. Later using the independent scalars at this order we construct an equilibrium partition function and compute its variation. We were however unable to process the constraints explicitly, as the calculations are analytically intractable.

B.1 Counting at equilibrium

At subsubleading order ($D = n+1$, $s = 3$), index families required are: $2D = 2n+2$ ($\mathbf{V}_\epsilon^{CC}$), $2D-1 = 2n+1$ ($\mathbf{T}_\epsilon^C$), $2D-2 = 2n$ ($\mathbf{V}_\epsilon^C$), $2D-3 = 2n-1$ ($\mathbf{T}_\epsilon$) and $2D-4 = 2n-2$ ($\mathbf{V}_\epsilon$). We only compute terms surviving at equilibrium, as non-equilibrium pieces are not required till subsubsubleading order parity-even or subsubsubsubleading order parity-odd calculation.

$2D$ family: $2D$ family was already discussed in section 4.3.1, but this time since four indices are free from ϵ, two $(2,4,2)$ can appear with two antisymmetric indices of R^{ijkl} contracted. We will find 3 combinations — $(19n-20)$ vectors of type $\mathbf{V}_\epsilon^{CC}$:

1. $2(2,4,2) \oplus (n-3)(2,2,1)$: 1 possibility — $(n-2)$ vectors

$$\left\langle {}^{m-1}_{n-m-2} \right\rangle^{ijklm} \mathcal{K}_{jk}{}^{ab} \mathcal{K}_{lmab} \Big|_{m=1}^{n-2}.$$

2. $(2,4,2) \oplus (n-1)(2,2,1)$ — 4 possibilities — $(8n-10)$ vectors

$$\left\langle {}^{m-1}_{n-m} \right\rangle^{i} \mathcal{K} \Big|_{m=1}^{n}, \quad \left\langle {}^{m-1}_{n-m-1} \right\rangle^{ijk} \mathcal{X}_{\Lambda ja} \mathcal{K}^a{}_k \Big|_{m=1}^{n-1}, \quad \left\langle {}^{m-1}_{n-m-1} \right\rangle^{ijk} \mathcal{K}_{jk}{}^{ab} \mathcal{X}_{\Lambda ab} \Big|_{m=1}^{n-1},$$
$$\left\langle {}^{m-1}_{n-m-2} \right\rangle^{ijklm} \mathcal{K}_{jk}{}^{ab} \mathcal{X}_{\Lambda al} \mathcal{X}_{\Gamma bm} \Big|_{m=1}^{n-2}.$$

3. $(n+1)(2,2,1)$ — 3 possibilities — $(10n-8)$ vectors

$$\left\langle {}^{m-1}_{n-m} \right\rangle^{i} \mathcal{X}_\Lambda^{jk} \mathcal{X}_{\Gamma jk} \Big|_{m=1}^{n}, \quad \left\langle {}^{m-1}_{n-m-1} \right\rangle^{ijk} \mathcal{X}_{\Lambda ja} \mathcal{X}_{\Gamma kb} \mathcal{X}_\Sigma^{ab} \Big|_{m=1}^{n-1},$$
$$\left\langle {}^{m-1}_{n-m-2} \right\rangle^{ijklm} \mathcal{X}_{1aj} \mathcal{X}_{2k}^a \mathcal{X}_{1bl} \mathcal{X}_{2m}^b \Big|_{m=1}^{n-2}.$$

JHEP05(2015)010

$2D-1$ family: $2D-1$ family was already discussed in section 4.3.2, but this time three indices are free from ϵ. So only one among $(2,4,2)$ and $(\frac{5}{3},5,3)$ can appear, and not more that once. We will find 5 combinations of type $\mathbf{T}^C_\epsilon$:

1. $(2,4,2)\oplus(n-2)(2,2,1)\oplus(1,1,1)$: 3 possibilities — $(8n-12)$ symmetric traceless tensors

$$\left\langle {m-1 \atop n-m-1} \right\rangle^{\langle ijk}\partial_j\vartheta_\Lambda\mathcal{K}_k^{\ a\rangle}\Big|_{m=1}^{n-1},\quad \left\langle {m-1 \atop n-m-1} \right\rangle^{\langle ijk}\mathcal{K}_{jk}^{\ \ a\rangle b}\partial_b\vartheta_\Lambda\Big|_{m=1}^{n-1},$$
$$\left\langle {m-1 \atop n-m-2} \right\rangle^{\langle ijklm}\partial_j\vartheta_\Lambda\mathcal{X}_{\Gamma kb}\mathcal{K}_{lm}^{\ \ a\rangle b}\Big|_{m=1}^{n-2}.$$

2. $(n)(2,2,1)\oplus(1,1,1)$: 5 possibilities

   (a) Contraction between $(2,2,1)$ and $(1,1,1)$ — $(12n-8)$ symmetric traceless tensors and $(4n)$ scalars

$$\left\langle {m-1 \atop n-m} \right\rangle^{\langle i}\mathcal{X}_\Lambda^{j\rangle k}\partial_k\vartheta_\Gamma\Big|_{m=1}^{n},\quad \left\langle {m-1 \atop n-m-1} \right\rangle^{\langle ijk}\mathcal{X}_{\Lambda j}^{l\rangle}\mathcal{X}_{\Gamma ka}\partial^a\vartheta_\Sigma\Big|_{m=1}^{n-1}.$$

   Scalars: we can take trace and get $4n$ scalars:

$$\left\langle {m-1 \atop n-m} \right\rangle_i\mathcal{X}_\Lambda^{ik}\partial_k\vartheta_\Gamma\Big|_{m=1}^{n}$$

   (b) Contraction between $(2,2,1)$ and $(2,2,1)$ — $(14n-18)$ traceless symmetric tensors and $(2n-2)$ scalars

$$\left\langle {m-1 \atop n-m-1} \right\rangle^{\langle ijk}\mathcal{X}_{1j}^a\mathcal{X}_{2ak}\partial^{l\rangle}\vartheta_\Lambda\Big|_{m=1}^{n-1}.\quad \left\langle {m-1 \atop n-m-1} \right\rangle^{\langle ijk}\mathcal{X}_{\Lambda ja}\mathcal{X}_\Gamma^{l\rangle a}\partial_k\vartheta_\Sigma\Big|_{m=1}^{n-1},$$
$$\left\langle {m-1 \atop n-m-2} \right\rangle^{\langle ijklm}\mathcal{X}_{1j}^a\mathcal{X}_{2ak}\mathcal{X}_{\Lambda l}^{n\rangle}\partial_m\vartheta_\Gamma\Big|_{m=1}^{n-2}.$$

   Scalars: taking trace we get $2n-2$ scalars:

$$\left\langle {m-1 \atop n-m-1} \right\rangle^{ijk}\mathcal{X}_{1j}^a\mathcal{X}_{2ak}\partial_i\vartheta_\Lambda\Big|_{m=1}^{n-1}.$$

3. $(2,4,2)\oplus(n-3)(2,2,1)\oplus(\frac{3}{2},3,2)$: 1 possibility — $(2n-4)$ traceless symmetric tensors

$$\left\langle {m-1 \atop n-m-2} \right\rangle^{\langle ijklm}\mathcal{K}_{jk}^{\ \ a\rangle b}\partial_b\mathcal{X}_{\Lambda lm}\Big|_{m=1}^{n-2}.$$

4. $(n-1)(2,2,1)\oplus(\frac{3}{2},3,2)$: 7 possibilities

   (a) Contraction within $(\frac{3}{2},3,2)$ — $(6n-4)$ traceless symmetric tensors and $(2n)$ scalars

$$\left\langle {m-1 \atop n-m} \right\rangle^{\langle i}\partial_k\mathcal{X}_\Lambda^{kj\rangle}\Big|_{m=1}^{n},\quad \left\langle {m-1 \atop n-m-1} \right\rangle^{\langle ijk}\mathcal{X}_{\Lambda j}^{a\rangle}\partial^b\mathcal{X}_{\Gamma bk}\Big|_{m=1}^{n-1}.$$

   Scalars: taking trace we get $2n$ scalars:

$$\left\langle {m-1 \atop n-m} \right\rangle^{i}\partial^k\mathcal{X}_{\Lambda ki}\Big|_{m=1}^{n}.$$

JHEP05(2015)010

(b) Contraction between $(2,2,1)$ and $(\frac{3}{2},3,2)$ — $(20n-28)$ traceless symmetric tensors and $(4n-4)$ scalars.

$$\left\langle {}^{m-1}_{n-m-1}\right\rangle^{\langle ijk}\mathcal{X}_{\Lambda jb}\partial^b\mathcal{X}^{a\rangle}_{\Gamma k}\Big|_{m=1}^{n-1},\quad \left\langle {}^{m-1}_{n-m-1}\right\rangle^{\langle ijk}\mathcal{X}_{\Lambda jb}\partial^{a\rangle}\mathcal{X}^{b}_{\Gamma k}\Big|_{m=1}^{n-1},$$

$$\left\langle {}^{m-1}_{n-m-1}\right\rangle^{\langle ijk}\mathcal{X}^{a\rangle b}_{\Lambda}\partial_b\mathcal{X}_{\Gamma jk}\Big|_{m=1}^{n-1},\quad \left\langle {}^{m-1}_{n-m-2}\right\rangle^{\langle ijklm}\mathcal{X}^{a\rangle}_{\Lambda j}\mathcal{X}_{\Gamma kb}\partial^b\mathcal{X}_{\Sigma lm}\Big|_{m=1}^{n-2}.$$

Scalars: taking trace we get $4n-4$ scalars:

$$\left\langle {}^{m-1}_{n-m-1}\right\rangle^{ijk}\mathcal{X}_{\Lambda ib}\partial^b\mathcal{X}_{\Gamma jk}\Big|_{m=1}^{n-1}.$$

(c) Contraction between $(2,2,1)$ and $(2,2,1)$ — $(2n-4)$ traceless symmetric tensors

$$\left\langle {}^{m-1}_{n-m-2}\right\rangle^{\langle ijklm}\mathcal{X}^{b}_{1j}\mathcal{X}_{2bk}\partial^{a\rangle}\mathcal{X}_{\Lambda lm}\Big|_{m=1}^{n-2}.$$

5. $(n-2)(2,2,1)\oplus(\frac{5}{3},5,3)$: 3 possibilities — $(4n-6)$ traceless symmetric tensors

$$\left\langle {}^{m-1}_{n-m-1}\right\rangle^{\langle ijk}\partial_a\mathcal{K}^{al\rangle}{}_{jk}\Big|_{m=1}^{n-1},\quad \left\langle {}^{m-1}_{n-m-1}\right\rangle^{\langle ijk}\partial_j\mathcal{K}_k{}^{a\rangle}\Big|_{m=1}^{n-1},$$

$$\left\langle {}^{m-1}_{n-m-2}\right\rangle^{\langle ijklm}\partial_j\mathcal{K}_{kl}{}^{a\rangle b}\mathcal{X}_{\Lambda bm}\Big|_{m=1}^{n-2}.$$

$2D-2$ family: $2D-2$ family was already discussed in section 4.3.2. Here again, one among $(2,4,2)$ and $(\frac{5}{3},5,3)$ can appear, and not more that once. We will find 7 combinations — $(39n-46)$ vectors of type $\mathbf{V}^C_\epsilon$:

1. $(2,4,2)\oplus(D-4)(2,2,1)\oplus 2(1,1,1)$: no combinations possible
2. $(n-1)(2,2,1)\oplus 2(1,1,1)$: 3 possibilities — $(12n-10)$ vectors

$$\left\langle {}^{m-1}_{n-m}\right\rangle^{i}\partial_k\vartheta_\Lambda\partial^k\vartheta_\Gamma\Big|_{m=1}^{n},\quad \left\langle {}^{m-1}_{n-m-1}\right\rangle^{ijk}\mathcal{X}_{\Lambda ja}\partial^a\vartheta_\Gamma\partial_k\vartheta_\Sigma\Big|_{m=1}^{n-1},$$

$$\left\langle {}^{m-1}_{n-m-2}\right\rangle^{ijklm}\mathcal{X}^{a}_{1j}\mathcal{X}_{2ak}\partial_l\vartheta_1\partial_m\vartheta_2\Big|_{m=1}^{n-2}.$$

3. $(n-1)(2,2,1)\oplus(1,2,2)$: 2 possibilities — $(6n-4)$ vectors

$$\left\langle {}^{m-1}_{n-m}\right\rangle^{i}\partial_k\partial^k\vartheta_\Lambda\Big|_{m=1}^{n},\quad \left\langle {}^{m-1}_{n-1-m}\right\rangle^{ijk}\mathcal{X}_{\Lambda ja}\partial^a\partial_k\vartheta_\Gamma\Big|_{m=1}^{n-1}.$$

4. $(n-2)(2,2,1)\oplus(\frac{3}{2},3,2)\oplus(1,1,1)$: 3 possibilities — $(16n-24)$ vectors

$$\left\langle {}^{m-1}_{n-m-1}\right\rangle^{ijk}\partial_j\vartheta_\Lambda\partial^a\mathcal{X}_{\Gamma ak}\Big|_{m=1}^{n-1},\quad \left\langle {}^{m-1}_{n-m-1}\right\rangle^{ijk}\partial_a\mathcal{X}_{\Lambda jk}\partial^a\vartheta_\Gamma\Big|_{m=1}^{n-1},$$

$$\left\langle {}^{m-1}_{n-m-2}\right\rangle^{ijklm}\mathcal{X}_{\Lambda ja}\partial^a\mathcal{X}_{\Gamma kl}\partial_m\vartheta_\Sigma\Big|_{m=1}^{n-2}.$$

5. $(n-2)(2,2,1)\oplus(\frac{4}{3},4,3)$: 1 possibility — $(2n-2)$ vectors

$$\left\langle {}^{m-1}_{n-m-1}\right\rangle^{ijk}\partial_a\partial^a\mathcal{X}_{\Gamma jk}\Big|_{m=1}^{n-1}.$$

JHEP05(2015)010

6. $(n-3)(2,2,1)\oplus(\frac{5}{3},5,3)\oplus(1,1,1)$: no possiblities

7. $(n-3)(2,2,1)\oplus 2(\frac{3}{2},3,2)$: 1 possibility — $(3n-6)$ vectors

$$\left\langle {}^{m-1}_{n-m-2}\right\rangle^{ijklm}\partial_a\mathcal{X}_{\Lambda jk}\partial^a\mathcal{X}_{\Gamma lm}\Big|_{m=1}^{n-2}.$$

$2D-3$ family: we are interested in combinations in $(2D-3)$ family which survive at equilibrium. We generated them through a Mathematica code and found 22 of them. We won't list all of them here, because it won't be required. Due to properties of $\mathbf{T}_\epsilon$, most of them will not contribute. We will be only left with 3 combinations — $(7n-9)$ symmetric traceless tensors:

1. $(n-2)(2,2,1)\oplus 3(1,1,1)$: 1 possibility — $(2n-2)$ symmetric traceless tensors

$$\left\langle {}^{m-1}_{n-m-1}\right\rangle^{\langle ijk}\partial_j\vartheta_1\partial_k\vartheta_2\partial^{l\rangle}\vartheta_\Lambda\Big|_{m=1}^{n-1}.$$

2. $(n-2)(2,2,1)\oplus(1,1,1)\oplus(1,2,2)$: 1 possibility — $(4n-4)$ symmetric traceless tensors

$$\left\langle {}^{m-1}_{n-m-1}\right\rangle^{\langle ijk}\partial_j\vartheta_\Lambda\partial_k\partial^{l\rangle}\vartheta_\Gamma\Big|_{m=1}^{n-1}.$$

3. $(n-3)(2,2,1)\oplus(\frac{3}{2},3,2)\oplus 2(1,1,1)$: 1 possibility — $(2n-4)$ symmetric traceless tensors

$$\left\langle {}^{m-1}_{n-m-2}\right\rangle^{\langle ijklm}\partial_j\vartheta_1\partial_k\vartheta_2\partial^{a\rangle}\mathcal{X}_{\Lambda lm}\Big|_{m=1}^{n-2}.$$

$2D-4$ family: there are 51 combinations in $(2D-4)$ family which survive at equilibrium. However none of them will contribute due to properties of $\mathbf{V}_\epsilon$.

All the subsubleading parity-odd data surviving at equilibrium has been summarized in tables 6, 7, 8.

Independent scalars: as we discussed in section 4.4, we only need to construct independent scalars which enter in equilibrium partition function. At subsubleading order one can find antisymmetrizations which will determine $\mathbb{S}_{3\Lambda m}$ and $\mathbb{S}_{4\Lambda\Gamma m}$ in terms of $\mathbb{S}_{1\Lambda m}$ and $\mathbb{S}_{2\Lambda\Gamma m}$ respectively:

$$\mathcal{X}_1^{[i_1j_1}\dots\mathcal{X}_1^{i_{m-1}j_{m-1}}\mathcal{X}_2^{i_mj_m}\dots\mathcal{X}_2^{i_{n-1}j_{n-1}}\mathcal{X}^a_{\Lambda b}\hat{\nabla}^{b]}\vartheta_\Gamma\Big|_{m=1}^{n}=0, \tag{B.1}$$

$$\mathcal{X}_1^{[i_1j_1}\dots\mathcal{X}_1^{i_{m-1}j_{m-1}}\mathcal{X}_2^{i_mj_m}\dots\mathcal{X}_2^{i_{n-1}j_{n-1}}\hat{\nabla}_b\mathcal{X}_\Lambda^{ba]}\Big|_{m=1}^{n}=0. \tag{B.2}$$

Each of $\mathbb{S}_{1\Lambda m}$ and $\mathbb{S}_{2\Lambda\Gamma m}$, on the other hand is a unique scalar per choice of the parity-even tensor used to construct it by contracting with ϵ. Since antisymmetrization conditions cannot alter the tensor structure, these scalars are independent.

JHEP05(2015)010

Name	Term	Equilibrium
$\tilde{\mathbb{V}}^{\mu}_{1\Lambda m}\big\vert^{n}_{m=1}$	$l^{\mu}\mathbf{S}_{1\Lambda}$	$\left\langle {}^{m-1}_{n-m}\right\rangle^{i}\nabla^{k}\nabla_{k}\vartheta_{\Lambda o}$
$\tilde{\mathbb{V}}^{\mu}_{2(\Lambda\Gamma)m}\big\vert^{n}_{m=1}$	$l^{\mu}\mathbf{S}_{2(\Lambda\Gamma)}$	$\left\langle {}^{m-1}_{n-m}\right\rangle^{i}\nabla^{k}\vartheta_{\Lambda o}\nabla_{k}\vartheta_{\Gamma o}$
$\tilde{\mathbb{V}}^{\mu}_{3(\Lambda\Gamma)m}\big\vert^{n}_{m=1}$	$l^{\mu}\mathbf{S}_{3(\Lambda\Gamma)}$	$\left\langle {}^{m-1}_{n-m}\right\rangle^{i}f^{ab}_{\Lambda}f_{\Gamma ab}$
$\tilde{\mathbb{V}}^{\mu}_{4m}\big\vert^{n}_{m=1}$	$l^{\mu}\mathbf{S}_{4}$	$\left\langle {}^{m-1}_{n-m}\right\rangle^{i}R$
$\tilde{\mathbb{V}}^{\mu}_{5\Lambda\Gamma m}\big\vert^{n-1}_{m=1}$	$\left\langle {}^{m-1}_{n-m-1}\right\rangle^{\mu\nu\rho\sigma}u_{\nu}V_{\Lambda\rho}\mathbf{V}_{1\Gamma\sigma}$	$\left\langle {}^{m-1}_{n-m-1}\right\rangle^{ijk}\nabla_{j}\vartheta_{\Lambda o}\nabla^{a}f_{\Gamma ak}$
$\tilde{\mathbb{V}}^{\mu}_{6\Lambda\Gamma\Sigma m}\big\vert^{n-1}_{m=1}$	$\left\langle {}^{m-1}_{n-m-1}\right\rangle^{\mu\nu\rho\sigma}u_{\nu}V_{\Lambda\rho}\mathbf{V}_{2\Gamma\Sigma\sigma}$	$\left\langle {}^{m-1}_{n-m-1}\right\rangle^{ijk}\nabla_{j}\vartheta_{\Lambda o}f_{\Gamma ka}\nabla^{a}\vartheta_{\Sigma o}$
$\tilde{\mathbb{V}}^{\mu}_{7\Lambda\Gamma m}\big\vert^{n-1}_{m=1}$	$\left\langle {}^{m-1}_{n-1-m}\right\rangle^{\mu\nu\rho\sigma}u_{\nu}\mathcal{X}_{\Lambda\rho\alpha}\hat{\nabla}^{\alpha}\hat{\nabla}_{\sigma}\vartheta_{\Gamma}$	$\left\langle {}^{m-1}_{n-1-m}\right\rangle^{ijk}f_{\Lambda ja}\nabla^{a}\nabla_{k}\vartheta_{\Gamma o}$
$\tilde{\mathbb{V}}^{\mu}_{8\Lambda m}\big\vert^{n-1}_{m=1}$	$\left\langle {}^{m-1}_{n-1-m}\right\rangle^{\mu\nu\rho\sigma}u_{\nu}\mathcal{X}_{\Lambda\rho\alpha}\mathcal{K}^{\alpha}_{\sigma}$	$\left\langle {}^{m-1}_{n-1-m}\right\rangle^{ijk}f_{\Lambda ja}R^{a}_{\ k}$
$\tilde{\mathbb{V}}^{\mu}_{9(\Lambda\Gamma)\Sigma m}\big\vert^{n-1}_{m=1}$	$\left\langle {}^{m-1}_{n-m-1}\right\rangle^{\mu\nu\rho\sigma}u_{\nu}\mathcal{X}_{\Lambda\rho\alpha}\mathcal{X}_{\Gamma\sigma\beta}\mathcal{X}^{\alpha\beta}_{\Sigma}$	$\left\langle {}^{m-1}_{n-m-1}\right\rangle^{ijk}f_{\Lambda ja}f_{\Gamma kb}f^{ab}_{\Sigma}$
$\tilde{\mathbb{V}}^{\mu}_{10\Lambda\Gamma m}\big\vert^{n-1}_{m=1}$	$\left\langle {}^{m-1}_{n-m-1}\right\rangle^{\mu\nu\rho\sigma}u_{\nu}V^{\alpha}_{\Lambda}\hat{\nabla}_{\alpha}\mathcal{X}_{\Gamma\rho\sigma}$	$\left\langle {}^{m-1}_{n-m-1}\right\rangle^{ijk}\nabla^{a}\vartheta_{\Lambda o}\nabla_{a}f_{\Gamma jk}$
$\tilde{\mathbb{V}}^{\mu}_{11\Lambda m}\big\vert^{n-1}_{m=1}$	$\left\langle {}^{m-1}_{n-m-1}\right\rangle^{\mu\nu\rho\sigma}u_{\nu}P^{\alpha\beta}\hat{\nabla}_{\alpha}\hat{\nabla}_{\beta}\mathcal{X}_{\Lambda\rho\sigma}$	$\left\langle {}^{m-1}_{n-m-1}\right\rangle^{ijk}\nabla^{a}\nabla_{a}f_{\Lambda jk}$
$\tilde{\mathbb{V}}^{\mu}_{12\Lambda m}\big\vert^{n-1}_{m=1}$	$\left\langle {}^{m-1}_{n-m-1}\right\rangle^{\mu\nu\rho\sigma}u_{\nu}\mathcal{K}_{\rho\sigma}{}^{\alpha\beta}\mathcal{X}_{\Lambda\alpha\beta}$	$\left\langle {}^{m-1}_{n-m-1}\right\rangle^{ijk}R_{jk}{}^{ab}f_{\Lambda ab}$
$\tilde{\mathbb{V}}^{\mu}_{13m}\big\vert^{n-2}_{m=1}$	$\left\langle {}^{m-1}_{n-m-2}\right\rangle^{\mu\nu\rho\sigma\alpha\beta}u_{\nu}\mathcal{X}_{1\kappa\rho}\mathcal{X}^{\kappa}_{2\sigma}\hat{\nabla}_{\alpha}\vartheta_{1}\hat{\nabla}_{\beta}\vartheta_{2}$	$\left\langle {}^{m-1}_{n-m-2}\right\rangle^{ijklm}f_{1aj}f^{a}_{2k}\nabla_{l}\vartheta_{1o}\nabla_{m}\vartheta_{2o}$
$\tilde{\mathbb{V}}^{\mu}_{14m}\big\vert^{n-2}_{m=1}$	$\left\langle {}^{m-1}_{n-m-2}\right\rangle^{\mu\nu\rho\sigma\alpha\beta}u_{\nu}\mathcal{X}_{1\kappa\rho}\mathcal{X}^{\kappa}_{2\sigma}\mathcal{X}_{1\delta\alpha}\mathcal{X}^{\delta}_{2\beta}$	$\left\langle {}^{m-1}_{n-m-2}\right\rangle^{ijklm}f_{1aj}f^{a}_{2k}f_{1bl}f^{b}_{2m}$
$\tilde{\mathbb{V}}^{\mu}_{15\Lambda\Gamma\Sigma m}\big\vert^{n-2}_{m=1}$	$\left\langle {}^{m-1}_{n-m-2}\right\rangle^{\mu\nu\rho\sigma\alpha\beta}u_{\nu}\mathcal{X}_{\Lambda\kappa\rho}\hat{\nabla}^{\kappa}\mathcal{X}_{\Gamma\sigma\alpha}V_{\Sigma\beta}$	$\left\langle {}^{m-1}_{n-m-2}\right\rangle^{ijklm}f_{\Lambda aj}\nabla^{a}f_{\Gamma kl}\nabla_{m}\vartheta_{\Sigma o}$
$\tilde{\mathbb{V}}^{\mu}_{16(\Lambda\Gamma)m}\big\vert^{n-2}_{m=1}$	$\left\langle {}^{m-1}_{n-m-2}\right\rangle^{\mu\nu\rho\sigma\alpha\beta}u_{\nu}P^{\delta\kappa}\hat{\nabla}_{\delta}\mathcal{X}_{\Lambda\rho\sigma}\hat{\nabla}_{\kappa}\mathcal{X}_{\Gamma\alpha\beta}$	$\left\langle {}^{m-1}_{n-m-2}\right\rangle^{ijklm}\nabla_{a}f_{\Lambda jk}\nabla^{a}f_{\Gamma lm}$
$\tilde{\mathbb{V}}^{\mu}_{17(\Lambda\Gamma)m}\big\vert^{n-2}_{m=1}$	$\left\langle {}^{m-1}_{n-m-2}\right\rangle^{\mu\nu\rho\sigma\alpha\beta}u_{\nu}\mathcal{K}_{\rho\sigma}{}^{\kappa\delta}\mathcal{X}_{\Lambda\kappa\alpha}\mathcal{X}_{\Gamma\delta\beta}$	$\left\langle {}^{m-1}_{n-m-2}\right\rangle^{ijklm}R_{jk}{}^{ab}f_{\Lambda al}f_{\Gamma bm}$
$\tilde{\mathbb{V}}^{\mu}_{18m}\big\vert^{n-2}_{m=1}$	$\left\langle {}^{m-1}_{n-m-2}\right\rangle^{\mu\nu\rho\sigma\alpha\beta}u_{\nu}\mathcal{K}_{\rho\sigma}{}^{\delta\kappa}\mathcal{K}_{\alpha\beta\delta\kappa}$	$\left\langle {}^{m-1}_{n-m-2}\right\rangle^{ijklm}R_{jk}{}^{ab}R_{lmab}$

Table 6. Subsubleading order parity-odd vectors at equilibrium.

B.2 Attempt for fluid constraints

In the equilibrium partition function $\tilde{\Delta}^{(n+1)}W^{\text{eqb}}_{(C)}$ we can include the scalars: $\tilde{\mathbb{S}}_{o1\Lambda m}$, $\tilde{\mathbb{S}}_{o2\Lambda\Gamma m}$. But it can be checked that antisymmetric part of $\tilde{\mathbb{S}}_{o2[\Lambda\Gamma]m}$ can be related through a total derivative to $\tilde{\mathbb{S}}_{o1\Lambda m}$. So we have:

$$\tilde{\Delta}^{(n+1)}W^{\text{eqb}}_{(C)}=\int\left\{x^{i}\right\}\sqrt{g}\ {}^{n-2}\mathbf{C}_{m-1}\left\{\mathcal{Q}_{1,\Lambda m}\tilde{\mathbb{S}}_{o1,\Lambda m}+\mathcal{Q}_{2,\Lambda\Gamma m}\tilde{\mathbb{S}}_{o2,(\Lambda\Gamma)m}\right\}. \tag{B.3}$$

Sum over relevant indices is understood. Varying the partition function we will get:

$$\begin{aligned}\frac{\delta\tilde{\Delta}^{(n+1)}W^{\text{eqb}}_{(C)}}{\delta g_{ij}}&={}^{n-2}\mathbf{C}_{m-1}\left[\mathcal{Q}_{1,\Lambda m}\left(\tilde{\mathbb{T}}^{ij}_{o9,12\Lambda m}+\tilde{\mathbb{T}}^{ij}_{o8,21\Lambda m}\right)+2\mathcal{Q}_{2,\Lambda\Gamma m}\tilde{\mathbb{T}}^{ij}_{o7,\Lambda\Gamma m}\right]\\&\quad-{}^{n-3}\mathbf{C}_{m-1}(n-2)\left[\mathcal{Q}_{1,\Lambda m}\tilde{\mathbb{T}}^{ij}_{o18,2\Lambda m}-\mathcal{Q}_{2,\Lambda\Gamma(m+2-\Sigma)}\tilde{\mathbb{T}}^{ij}_{o19,\Lambda\Sigma\Gamma m}\right],\end{aligned} \tag{B.4}$$

$$\frac{\delta\tilde{\Delta}^{(n+1)}W^{\text{eqb}}_{(C)}}{\delta\vartheta_{o\Lambda}}={}^{n-2}\mathbf{C}_{m-1}\Big(2\partial_{[\Lambda}\mathcal{Q}_{1,\Gamma]m}\tilde{\mathbb{S}}_{o1,\Gamma m}-\mathcal{Q}_{1,\Lambda m}\tilde{\mathbb{S}}_{o2,[12]m}+\partial_{\Lambda}\mathcal{Q}_{2,\Gamma\Sigma m}\tilde{\mathbb{S}}_{o2,(\Gamma\Sigma)m}\Big), \tag{B.5}$$

JHEP05(2015)010

Name	Term	Equilibrium
$\tilde{\mathbb{T}}^{\mu\nu}_{1\Lambda m}\big\vert_{m=1}^{n}$	$l_m^{\langle\mu}\mathbf{V}_{1\Lambda}^{\nu\rangle}$	$\left\langle{}^{m-1}_{n-m}\right\rangle^{\langle i}\nabla_k f_\Lambda^{kj\rangle}$
$\tilde{\mathbb{T}}^{\mu\nu}_{2\Lambda m}\big\vert_{m=1}^{n}$	$l_m^{\langle\mu}\mathbf{V}_{2\Lambda\Gamma}^{\nu\rangle}$	$\left\langle{}^{m-1}_{n-m}\right\rangle^{\langle i} f_\Lambda^{j\rangle k}\nabla_k\vartheta_\Gamma$
$\tilde{\mathbb{T}}^{\mu\nu}_{3\Lambda m}\big\vert_{m=1}^{n-1}$	$\tilde{\mathbf{V}}_{1m}^{\langle\mu} V_\Lambda^{\nu\rangle}$	$\left\langle{}^{m-1}_{n-m-1}\right\rangle^{\langle ijk} f_{1j}^a f_{2ak}\nabla^{l\rangle}\vartheta_\Lambda$
$\tilde{\mathbb{T}}^{\mu\nu}_{4\Lambda m}\big\vert_{m=1}^{n-1}$	$\tilde{\mathbf{V}}_{2m}^{\langle\mu} V_\Lambda^{\nu\rangle}$	$\left\langle{}^{m-1}_{n-m-1}\right\rangle^{\langle ijk}\nabla_j\vartheta_{1o}\nabla_k\vartheta_{2o}\nabla^{a\rangle}\vartheta_{\Lambda o}$
$\tilde{\mathbb{T}}^{\mu\nu}_{5\Lambda\Gamma m}\big\vert_{m=1}^{n-1}$	$\left\langle{}^{m-1}_{n-m-1}\right\rangle^{\langle\mu\nu\rho\sigma} u_\nu V_{\Lambda\rho}\hat{\nabla}_\sigma\hat{\nabla}^{\alpha\rangle}\vartheta_\Gamma$	$\left\langle{}^{m-1}_{n-m-1}\right\rangle^{\langle ijk}\nabla_j\vartheta_{\Lambda o}\nabla_k\nabla^{a\rangle}\vartheta_{\Gamma o}$
$\tilde{\mathbb{T}}^{\mu\nu}_{6\Lambda m}\big\vert_{m=1}^{n-1}$	$\left\langle{}^{m-1}_{n-m-1}\right\rangle^{\langle\mu\nu\rho\sigma} u_\nu V_{\Lambda\rho}\mathcal{K}_\sigma^{\alpha\rangle}$	$\left\langle{}^{m-1}_{n-m-1}\right\rangle^{\langle ijk}\nabla_j\vartheta_{\Lambda o} R_k^{a\rangle}$
$\tilde{\mathbb{T}}^{\mu\nu}_{7\Lambda\Gamma m}\big\vert_{m=1}^{n-1}$	$\left\langle{}^{m-1}_{n-m-1}\right\rangle^{\langle\mu\nu\rho\sigma} u_\nu \mathcal{X}_{\Lambda\rho}^{\alpha\rangle}\mathbf{V}_{1\Gamma\sigma}$	$\left\langle{}^{m-1}_{n-m-1}\right\rangle^{\langle ijk} f_{\Lambda j}^{a\rangle}\nabla^b f_{\Gamma bk}$
$\tilde{\mathbb{T}}^{\mu\nu}_{8\Lambda\Gamma\Sigma m}\big\vert_{m=1}^{n-1}$	$\left\langle{}^{m-1}_{n-m-1}\right\rangle^{\langle\mu\nu\rho\sigma} u_\nu \mathcal{X}_{\Lambda\rho}^{\alpha\rangle}\mathbf{V}_{2\Gamma\Sigma\sigma}$	$\left\langle{}^{m-1}_{n-m-1}\right\rangle^{\langle ijk} f_{\Lambda j}^{a\rangle} f_{\Gamma kb}\nabla^b\vartheta_{\Sigma o}$
$\tilde{\mathbb{T}}^{\mu\nu}_{9\Lambda\Gamma\Sigma m}\big\vert_{m=1}^{n-1}$	$\left\langle{}^{m-1}_{n-m-1}\right\rangle^{\langle\mu\nu\rho\sigma} u_\nu \mathcal{X}_{\Lambda\rho\alpha}\mathcal{X}_\Gamma^{\delta\rangle\alpha} V_{\Sigma\sigma}$	$\left\langle{}^{m-1}_{n-m-1}\right\rangle^{\langle ijk} f_{\Lambda jb} f_\Gamma^{a\rangle b}\nabla_k\vartheta_{\Sigma o}$
$\tilde{\mathbb{T}}^{\mu\nu}_{10\Lambda m}\big\vert_{m=1}^{n-1}$	$\left\langle{}^{m-1}_{n-m-1}\right\rangle^{\langle\mu\nu\rho\sigma} u_\nu \mathcal{X}_{\Lambda\rho\delta}\hat{\nabla}^{\alpha\rangle}\mathcal{X}_{\Gamma\sigma}^{\delta}$	$\left\langle{}^{m-1}_{n-m-1}\right\rangle^{\langle ijk} f_{\Lambda jb}\nabla^{a\rangle} f_{\Gamma k}^b$
$\tilde{\mathbb{T}}^{\mu\nu}_{11\Lambda\Gamma m}\big\vert_{m=1}^{n-1}$	$\left\langle{}^{m-1}_{n-m-1}\right\rangle^{\langle\mu\nu\rho\sigma} u_\nu \mathcal{X}_{\Lambda\rho\delta}\hat{\nabla}^\delta\mathcal{X}_{\Gamma\sigma}^{\alpha\rangle}$	$\left\langle{}^{m-1}_{n-m-1}\right\rangle^{\langle ijk} f_{\Lambda jb}\nabla^b f_{\Gamma k}^{a\rangle}$
$\tilde{\mathbb{T}}^{\mu\nu}_{12\Lambda\Gamma m}\big\vert_{m=1}^{n-1}$	$\left\langle{}^{m-1}_{n-m-1}\right\rangle^{\langle\mu\nu\rho\sigma} u_\nu \mathcal{X}_\Lambda^{\alpha\rangle\delta}\hat{\nabla}_\delta\mathcal{X}_{\Gamma\rho\sigma}$	$\left\langle{}^{m-1}_{n-m-1}\right\rangle^{\langle ijk} f_\Lambda^{a\rangle b}\nabla_b f_{\Gamma jk}$
$\tilde{\mathbb{T}}^{\mu\nu}_{13\Lambda m}\big\vert_{m=1}^{n-1}$	$\left\langle{}^{m-1}_{n-m-1}\right\rangle^{\langle\mu\nu\rho\sigma} u_\nu V_{\Lambda\alpha}\mathcal{K}_{\rho\sigma}^{\kappa\rangle\alpha}$	$\left\langle{}^{m-1}_{n-m-1}\right\rangle^{\langle ijk}\nabla_b\vartheta_{\Lambda o} R_{jk}^{a\rangle b}$
$\tilde{\mathbb{T}}^{\mu\nu}_{14m}\big\vert_{m=1}^{n-1}$	$\left\langle{}^{m-1}_{n-m-1}\right\rangle^{\langle\mu\nu\rho\sigma} u_\nu\hat{\nabla}_\alpha\mathcal{K}_{\rho\sigma}^{\kappa\rangle\alpha}$	$\left\langle{}^{m-1}_{n-m-1}\right\rangle^{\langle ijk}\nabla_b R_{jk}^{a\rangle b}$
$\tilde{\mathbb{T}}^{\mu\nu}_{15m}\big\vert_{m=1}^{n-1}$	$\left\langle{}^{m-1}_{n-m-1}\right\rangle^{\langle\mu\nu\rho\sigma} u_\nu\hat{\nabla}_\rho\mathcal{K}_\sigma^{\alpha\rangle}$	$\left\langle{}^{m-1}_{n-m-1}\right\rangle^{\langle ijk}\nabla_j R_k^{a\rangle}$
$\tilde{\mathbb{T}}^{\mu\nu}_{16\Lambda m}\big\vert_{m=1}^{n-2}$	$\left\langle{}^{m-1}_{n-m-2}\right\rangle^{\langle\mu\nu\rho\sigma\alpha\beta} u_\nu\hat{\nabla}_\rho\vartheta_1\hat{\nabla}_\sigma\vartheta_2\hat{\nabla}^{\kappa\rangle}\mathcal{X}_{\Lambda\alpha\beta}$	$\left\langle{}^{m-1}_{n-m-2}\right\rangle^{\langle ijklm}\nabla_j\vartheta_{1o}\nabla_k\vartheta_{2o}\nabla^{a\rangle} f_{\Lambda lm}$
$\tilde{\mathbb{T}}^{\mu\nu}_{17\Lambda m}\big\vert_{m=1}^{n-2}$	$\left\langle{}^{m-1}_{n-m-2}\right\rangle^{\langle\mu\nu\rho\sigma\alpha\beta} u_\nu\mathcal{X}_{1\rho}^{\delta}\mathcal{X}_{2\delta\sigma}\hat{\nabla}^{\kappa\rangle}\mathcal{X}_{\Lambda\alpha\beta}$	$\left\langle{}^{m-1}_{n-m-2}\right\rangle^{\langle ijklm} f_{1j}^b f_{2bk}\nabla^{a\rangle} f_{\Lambda lm}$
$\tilde{\mathbb{T}}^{\mu\nu}_{18\Lambda\Gamma m}\big\vert_{m=1}^{n-2}$	$\left\langle{}^{m-1}_{n-m-2}\right\rangle^{\langle\mu\nu\rho\sigma\alpha\beta} u_\nu\mathcal{X}_{1\rho}^{\delta}\mathcal{X}_{2\delta\sigma}\mathcal{X}_{\Lambda\alpha}^{\kappa\rangle} V_{\Gamma\beta}$	$\left\langle{}^{m-1}_{n-m-2}\right\rangle^{\langle ijklm} f_{1j}^b f_{2bk} f_{\Lambda l}^{a\rangle}\nabla_m\vartheta_{\Gamma o}$
$\tilde{\mathbb{T}}^{\mu\nu}_{19\Lambda\Gamma\Sigma m}\big\vert_{m=1}^{n-2}$	$\left\langle{}^{m-1}_{n-m-2}\right\rangle^{\langle\mu\nu\rho\sigma\alpha\beta} u_\nu\mathcal{X}_{\Lambda\rho}^{\kappa\rangle}\mathcal{X}_{\Gamma\sigma}^{\delta}\hat{\nabla}_\delta\mathcal{X}_{\Sigma\alpha\beta}$	$\left\langle{}^{m-1}_{n-m-2}\right\rangle^{\langle ijklm} f_{\Lambda j}^{a\rangle} f_{\Gamma k}^b\nabla_b f_{\Sigma lm}$
$\tilde{\mathbb{T}}^{\mu\nu}_{20\Lambda\Gamma m}\big\vert_{m=1}^{n-2}$	$\left\langle{}^{m-1}_{n-m-2}\right\rangle^{\langle\mu\nu\rho\sigma\alpha\beta} u_\nu V_{\Lambda\rho}\mathcal{X}_{\Gamma\sigma\delta}\mathcal{K}_{\alpha\beta}^{\kappa\rangle\delta}$	$\left\langle{}^{m-1}_{n-m-2}\right\rangle^{\langle ijklm}\nabla_j\vartheta_{\Lambda o} f_{\Gamma kb} R_{lm}^{a\rangle b}$
$\tilde{\mathbb{T}}^{\mu\nu}_{21\Lambda m}\big\vert_{m=1}^{n-2}$	$\left\langle{}^{m-1}_{n-m-2}\right\rangle^{\langle\mu\nu\rho\sigma\alpha\beta} u_\nu\hat{\nabla}_\delta\mathcal{X}_{\Lambda\rho\sigma}\mathcal{K}_{\alpha\beta}^{\kappa\rangle\delta}$	$\left\langle{}^{m-1}_{n-m-2}\right\rangle^{\langle ijklm}\nabla_b f_{\Lambda jk} R_{lm}^{a\rangle b}$
$\tilde{\mathbb{T}}^{\mu\nu}_{22\Lambda m}\big\vert_{m=1}^{n-2}$	$\left\langle{}^{m-1}_{n-m-2}\right\rangle^{\langle\mu\nu\rho\sigma\alpha\beta} u_\nu\mathcal{X}_{\Lambda\delta\rho}\hat{\nabla}_\sigma\mathcal{K}_{\alpha\beta}^{\kappa\rangle\delta}$	$\left\langle{}^{m-1}_{n-m-2}\right\rangle^{\langle ijklm} f_{\Lambda bj}\nabla_k R_{lm}^{a\rangle b}$

Table 7. Subsubleading order parity-odd symmetric traceless tensors at equilibrium.

$\tilde{\mathbb{S}}_{1\Lambda m}\big\vert_{m=1}^{n-1}$	$\tilde{\mathbf{V}}_{1m}^{\mu} V_{\Lambda\mu}$	$\left\langle{}^{m-1}_{n-m-1}\right\rangle^{ijk} f_{1ai} f_{2j}^a\nabla_k\vartheta_{\Lambda o}$
$\tilde{\mathbb{S}}_{2\Lambda\Gamma m}\big\vert_{m=1}^{n-1}$	$\left\langle{}^{m-1}_{n-m-1}\right\rangle^{\mu\nu\rho\sigma} u_\nu\mathcal{X}_{\Lambda\mu\delta}\hat{\nabla}^\delta\mathcal{X}_{\Gamma\rho\sigma}$	$\left\langle{}^{m-1}_{n-m-1}\right\rangle^{ijk} f_{\Lambda ia}\nabla^a f_{\Gamma jk}$
$\tilde{\mathbb{S}}_{3\Lambda m}\big\vert_{m=1}^{n}$	$l_m^{\mu}\mathbf{V}_{1\Lambda\mu}$	$\left\langle{}^{m-1}_{n-m}\right\rangle^{i}\nabla^k f_{\Lambda ki}$
$\tilde{\mathbb{S}}_{4\Lambda\Gamma m}\big\vert_{m=1}^{n}$	$l_m^{\mu}\mathbf{V}_{2\Lambda\Gamma\mu}$	$\left\langle{}^{m-1}_{n-m}\right\rangle^{i} f_{\Lambda ij}\nabla^j\vartheta_{\Gamma o}$

Table 8. Subsubleading order parity-odd scalars at equilibrium.

JHEP05(2015)010

$$\frac{\delta\tilde{\Delta}^{(n+1)}W^{\rm eqb}_{(C)}}{\delta a_{\Omega i}} = {}^{n-2}\mathbf{C}_{m-1}\Big[(-)^{\Omega}\partial_\Gamma\mathcal{Q}_{1,\Lambda m}\tilde{\mathbb{V}}^i_{o6,\Gamma\bar{\Omega}\Lambda m}-(-)^{\Omega}\mathcal{Q}_{1,\Lambda m}\left(\tilde{\mathbb{V}}^i_{o7,\bar{\Omega}\Lambda m}+\frac{1}{2}\tilde{\mathbb{V}}^i_{o10,\Lambda\bar{\Omega}m}\right)$$
$$-2\partial_\Sigma\partial_\Gamma\mathcal{Q}_{2,\Omega\Lambda m}\tilde{\mathbb{V}}^i_{o6,\Sigma\Lambda\Gamma m}+\partial_\Gamma\mathcal{Q}_{2,\Omega\Lambda m}\left(4\tilde{\mathbb{V}}^i_{o5,\Gamma\Lambda m}+2\tilde{\mathbb{V}}^i_{o7,\Lambda\Gamma m}+\tilde{\mathbb{V}}^i_{o10,\Gamma\Lambda m}\right)$$
$$+2\mathcal{Q}_{2,\Omega\Lambda m}\left(2\tilde{\mathbb{V}}^i_{o8,\Lambda m}+\tilde{\mathbb{V}}^i_{o11,\Lambda m}+\tilde{\mathbb{V}}^i_{o12,\Lambda m}\right)\Big]$$
$$+(n-2)\ {}^{n-3}\mathbf{C}_{m-1}\Big[-2\Big((\partial_\Gamma\mathcal{Q}_{2,\Omega\Lambda(m+2-\Sigma)}$$
$$-\frac{1}{2}\partial_\Gamma\mathcal{Q}_{2,\Lambda\Sigma(m+2-\Omega)}\Big)\tilde{\mathbb{V}}^i_{o15,(\Sigma\Lambda)\Gamma m}$$
$$+\left(\mathcal{Q}_{2,\Omega\Lambda(m+2-\Sigma)}-\frac{1}{2}\mathcal{Q}_{2,\Lambda\Sigma(m+2-\Omega)}\right)\left(\tilde{\mathbb{V}}^i_{o16,(\Sigma\Lambda)m}-2\tilde{\mathbb{V}}^i_{o17,(\Sigma\Lambda)m}\right)\Big] \quad \text{(B.6)}$$

On the other hand from counting we can see that the third order parity-odd corrections (at eqb.) are given by (see tables 6, 7, 8):

$$\tilde{\Upsilon}^\mu_{o(n+1)}=\sum_{\#}\phi_{o\#}\tilde{\mathbb{V}}^\mu_{o\#},\qquad \tilde{\Pi}^{\mu\nu}_{o(n+1)}=\sum_{\#}\varpi_{o\#}\tilde{\mathbb{T}}^{\mu\nu}_{o\#}+P^{\mu\nu}_o\sum_{\#}\gamma_{o\#}\tilde{\mathbb{S}}_{o\#}. \quad \text{(B.7)}$$

corresponds to all the relevant indices. Similar to subleading order, here also we will have special contributions for $n=2$, as 3 leading order $(n-1)$ parity odd corrections can combine to give a $3n-3$ order parity-odd corrections, which will be equal to $n+1$ only at $n=2$ (Remember we are not considering $n=1$ case). Now comparing eq. (2.8) with eq. (3.14) at parity-odd subsubleading derivative order, we have corrections to constitutive relations:

$$\frac{1}{\vartheta_o}\tilde{\pi}^{ij}_{o(n+1)}=2\frac{\delta\tilde{\Delta}^{(n+1)}W^{\rm eqb}_{(C)}}{\delta g_{ij}}-g^{ij}\mathbf{A}_{o\Lambda}\frac{\delta\tilde{\Delta}^{(n+1)}W^{\rm eqb}_{(C)}}{\delta\vartheta_{o\Lambda}}+g^{ij}\frac{2}{\vartheta_o}\mathbf{A}_{o\Lambda}\partial_\Lambda P_o\tilde{\Delta}^{(n-1)}v^k\Delta^{(2)}v_k$$
$$+g^{ij}\mathbf{A}_{o2}\tilde{\Delta}^{(n-1)}v_i\left(\varsigma^i_{o(2)}+\boxed{\tilde{\Delta}^{(1)}\tilde{\varsigma}^i_{(1)}}\right)$$
$$-g^{ij}\mathbf{A}_{o2}\Delta^{(2)}v_k\left(q_o\tilde{\Delta}^{(n-1)}v^k-\tilde{\varsigma}^k_{o(n-1)}\right)$$
$$-2P^{(1,0)}_o\tilde{\Delta}^{(n-1)}v^{(i}\Delta^{(2)}v^{j)}-\frac{1}{\vartheta_o}\left(\tilde{\Delta}^{(n-1)}\pi^{ij}_{(2)}+\tilde{\Delta}^{(n)}\pi^{ij}_{(1)}\right),$$
$$P^{(1,0)}_o\tilde{\varsigma}^i_{o(n+1)}=\vartheta_o\mathbf{S}_{o\Lambda}\frac{\delta\tilde{\Delta}^{(n+1)}W^{\rm eqb}_{(C)}}{\delta a_{\Lambda i}}-\left[P^{(1,0)}_o\Delta^{(2)}q-\frac{1}{\vartheta_o^2}P^{(0,1)}_o\Delta^{(2)}(\epsilon+P)\right]\tilde{\Delta}^{(n-1)}v^i$$
$$+q_o\tilde{\Delta}^{(n-1)}v_j\left(\boxed{\frac{3}{2}P^{(1,0)}_o\tilde{\Delta}^{(1)}v^i\tilde{\Delta}^{(1)}v^j}+\frac{1}{\vartheta_o}\pi^{ij}_{o(2)}\right)$$
$$-P^{(1,0)}_o\left(\Delta^{(2)}\tilde{\varsigma}^i_{(n-1)}+\tilde{\Delta}^{(n-1)}\varsigma^i_{(2)}+\tilde{\Delta}^{(n)}\varsigma^i_{(1)}\right), \quad \text{(B.8)}$$

while the fluid variables get corrections:

$$\tilde{\Delta}^{(n+1)}v^i=(-)^\Lambda\frac{\mu_{o\Lambda}}{P^{(1,0)}_o}\frac{\delta\tilde{\Delta}^{(n+1)}W^{\rm eqb}_{(C)}}{\delta a_{\Lambda i}}-\frac{1}{P^{(1,0)}_o}\tilde{\Delta}^{(n-1)}v_j\left(\frac{1}{\vartheta_o}\Delta^{(2)}(\epsilon+P)g^{ij}+\frac{1}{\vartheta_o}\pi^{ij}_{o(2)}\right)$$
$$-\boxed{\frac{3}{2}\tilde{\Delta}^{(1)}v^i\tilde{\Delta}^{(1)}v^j\tilde{\Delta}^{(1)}v_j},$$

JHEP05(2015)010

$$\tilde{\Delta}^{(n+1)}\vartheta_\Lambda = \vartheta_o \mathbf{E}_{o\Lambda\Gamma}\frac{\delta\tilde{\Delta}^{(n+1)}W^{\rm eqb}_{(C)}}{\delta\vartheta_{o\Gamma}} - 2\mathbf{A}_{o\Lambda}\tilde{\Delta}^{(n-1)}v^k\Delta^{(2)}v_k$$
$$- \mathbf{E}_{o\Lambda 2}\vartheta_o\tilde{\Delta}^{(n-1)}v_i\left(\varsigma^i_{o(2)} + \boxed{\tilde{\Delta}^{(1)}\bar{\varsigma}^i_{(1)}}\right) + \mathbf{E}_{o\Lambda 2}\vartheta_o\Delta^{(2)}v_i\left(q_o\tilde{\Delta}^{(n-1)}v^i - \tilde{\varsigma}^i_{o(n-1)}\right). \tag{B.9}$$

From here onwards in principle the way would be to solve eq. (B.8) and find constraints for transport coefficients appearing in eq. (B.7). To solve we would need to plug in the fluid variable corrections to all lower orders, along with corrections to lower order constitutive relations due to fluid variable corrections. The terms which were zero at equilibrium at lower orders will also start to contribute by gaining the fluid variable corrections. Leaving aside terms specifically for $n = 2$, still we would have to deal with a large mess in eq. (B.8) which is analytically not quite tractable. So we leave these expressions at this point for reference.

Readers are advised that expressions eqs. (B.8), (B.9) does not contain contributions from gravitational and mixed anomaly, and conserved Chern Simons form. Recall that while we set up relations eq. (3.14), we only used the form of anomalous currents eq. (2.2) and conserved Chern-Simons form eq. (3.9) to subleading derivative order. At subsubleading order, they will receive further gravitational corrections.

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JHEP05(2015)010