The mass of the adjoint pion in $\mathcal{N} = 1$ supersymmetric Yang-Mills theory

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ABSTRACT: In Monte Carlo simulations of $\mathcal{N} = 1$ supersymmetric Yang-Mills theory the mass of the unphysical adjoint pion, which is easily obtained numerically, is being used for the tuning to the limit of vanishing gluino mass. In this article we show how to define the adjoint pion in the framework of partially quenched chiral perturbation theory and we derive a relation between its mass and the mass of the gluino analogous to the Gell-Mann-Oakes-Renner relation of QCD.

The $\mathcal{N} = 1$ supersymmetric Yang-Mills theory (SYM) is the supersymmetric extension of non-Abelian gauge theory. It describes gluons, belonging to gauge group $SU(N_c)$, interacting with their superpartners, the gluinos. The gluons are represented by non-Abelian gauge fields $A_\mu(x) = A_\mu^a(x)T^a$, $a = 1, \ldots, N_c^2 - 1$, where T^a are the generators of the gauge group. The gluinos $\lambda(x) = \lambda^a(x)T^a$ are spin $1/2$ Majorana fermions. They are in the adjoint representation of the gauge group and their gauge covariant derivative is given by $\mathcal{D}_\mu \lambda^a = \partial_\mu \lambda^a + g\, f_{abc} A_\mu^b \lambda^c$. The (on-shell) Lagrangian of SYM is

$$\mathcal{L} = \mathrm{tr}\left[-\frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \mathrm{i}\, \bar{\lambda}\gamma^\mu \mathcal{D}_\mu \lambda - m_g \bar{\lambda}\lambda \right], \tag{1}$$

where $F_{\mu\nu}$ is the non-Abelian field strength. A gluino mass term has been added, that breaks supersymmetry softly. In the limit $m_g = 0$ the action is invariant under a supersymmetry transformation.

In recent years Monte Carlo simulations of SYM have been performed in order to study its non-perturbative properties, in particular to determine the spectrum of low-lying bound states; see [1, 2] and references therein. In these calculations the lattice regularisation of SYM proposed by Curci and Veneziano [3] has been employed. Here the gluinos are represented by Wilson fermions. Both supersymmetry and chiral symmetry are broken by the lattice discretisation. In order to approach these symmetries in the continuum limit, a fine-tuning of the bare gluino mass parameter is necessary [3, 4]. As in the Curci-Veneziano formulation the gluino mass term is not protected against additive renormalisation, the point of vanishing gluino mass is not given a priori, but has to be determined on the basis of suitable observables.

One possibility to determine the gluino mass is to employ the lattice supersymmetric Ward identities as discussed in [5]. Another, numerically much easier way is to monitor the mass of the adjoint pion (a–π), which is the pion in the corresponding theory with two Majorana fermions in the adjoint representation. The a–π is not a physical particle in SYM, which only contains one Majorana fermion. Its mass can, however, obtained unambiguously from the corresponding correlation function, which is obtained as follows. One of the mesonic bound states described by SYM is the so-called adjoint η' (a–η'), which is a colourless pseudoscalar particle with interpolating field $\bar{\lambda}(x)\gamma_5\lambda(x)$. Its correlation function contains connected and disconnected fermionic contributions. The connected part

$$C(x,y) = \langle \mathrm{tr}_{sc}[\gamma_5(\gamma^\mu \mathcal{D}_\mu)^{-1}(x,y)\gamma_5(\gamma^\mu \mathcal{D}_\mu)^{-1}(y,x)]\rangle, \tag{2}$$

where tr_{sc} denotes a trace over Dirac and colour indices, yields the a–π-correlation function.

The adjoint pion mass is expected to vanish in the limit of a massless gluino according to

$$m_{\mathrm{a}-\pi}^2 \propto m_g, \tag{3}$$

analogous to the Gell-Mann-Oakes-Renner (GOR) relation of QCD [6], as has been argued on the basis of the OZI-approximation of SYM [7]. Indeed, numerical investigations of both the gluino mass from supersymmetric Ward identities and the adjoint pion mass [8] have shown that the points of their vanishing are consistent with each other, and that $m_{\mathrm{a}-\pi}^2$

is proportional to m_g. In practice the a–π is being used for tuning since it yields a more precise signal than the supersymmetric Ward identities.

It is the purpose of this article to demonstrate that the adjoint pion can be defined in a partially quenched setup, in which the model is supplemented by a second species of gluinos and the corresponding bosonic ghost gluinos, in the same way as for one-flavour QCD [9], and to show that the behaviour indicated in eq. (3) is indeed found in partially quenched chiral perturbation theory.

Apart from the classical $U(1)_A$ axial symmetry, which is anomalous in the quantum theory, SYM does not have a continuous chiral symmetry. Therefore it also does not show spontaneous chiral symmetry breaking and does not have (pseudo-) Goldstone bosons like pions, whose masses would vanish in the chiral limit. The symmetry can, however, be enhanced artificially by adding additional flavours of gluinos $\lambda_i(x)$, $i = 2, \ldots, N$. If these additional gluinos were dynamical, the resulting theory would be different from SYM and would not be supersymmetric. On the other hand, if the additional gluinos are *quenched*, which means that they are not taken into account in the fermionic functional integral, the dynamical content of the model is identical to SYM and the correlation functions of the original fields are unchanged. This situation can be called *partially quenched*. It can be described theoretically by the introduction of bosonic ghost fermions [10], in our case ghost gluinos. The contribution of the ghost gluinos exactly cancels the contribution of the additional gluinos, and only the contribution of the original single gluino remains.

In the partially quenched setup adjoint pions can be formed out of the gluinos $\lambda_1 \equiv \lambda$ and λ_2 by means of $\bar{\lambda}_i \gamma_5 (\tau_\alpha)_{ij} \lambda_j$, where τ_α are the Pauli matrices.

Let us begin by considering the introduction of $N-1$ additional gluino fields. Of central importance for chiral perturbation theory and its partially quenched variant is the flavour symmetry group and its spontaneous breakdown. The fermionic kinetic term $\bar{\lambda}_i \gamma^\mu \mathcal{D}_\mu \lambda_i$ in the Lagrangian has the same form as the corresponding quark term in QCD. Due to the Majorana condition $\lambda = C \bar{\lambda}^T$ the left and right handed parts of the gluino fields are not independent of each other and consequently the chiral symmetry group is not equal to $SU(N)_L \otimes SU(N)_R$ but to some subgroup of it. If the hermitian generators of $SU(N)$ flavour transformations are denoted T_α, a short calculation reveals that the generators of the subgroup of $SU(N)_L \otimes SU(N)_R$ consistent with the Majorana condition are given by

$$\text{those} \quad T_\alpha, \quad \text{for which} \quad T_\alpha = -T_\alpha^*, \tag{4}$$

$$\text{and those} \quad T_\alpha \gamma_5, \quad \text{for which} \quad T_\alpha = T_\alpha^*. \tag{5}$$

They generate a subgroup isomorphic to $SU(N)$, which is the chiral symmetry group of N gluinos. Another way to view this group is to write the gluinos in terms of two-component Weyl fermions χ,

$$\lambda = \begin{pmatrix} \chi \\ -\epsilon \chi^* \end{pmatrix}, \tag{6}$$

where ϵ is the two-dimensional antisymmetric spinor-metric, and to represent the kinetic term as

$$\mathcal{L}_g = \chi_i^\dagger \bar{\sigma}^\mu \mathcal{D}_\mu \chi_i. \tag{7}$$

From this expression one directly sees that $\mathrm{SU}(N)$ transformations of the Weyl fields χ_i leave the kinetic term invariant.

The gluino mass term proportional to $\bar{\lambda}_i \lambda_i$ is invariant under the subgroup H of $\mathrm{SU}(N)$, which is generated by the $N(N-1)/2$ imaginary T_α, i.e. $T_\alpha = -T_\alpha^*$. The corresponding group elements $h = \exp(\mathrm{i} h_\alpha T_\alpha)$ are real orthogonal matrices, and we see that $H = \mathrm{SO}(N)$.

Assuming that the chiral symmetry group $\mathrm{SU}(N)$ of gluinos is spontaneously broken, accompanied by a non-vanishing gluino condensate $\langle \bar{\lambda}_i \lambda_j \rangle \propto \delta_{ij}$, the breakdown from $G = \mathrm{SU}(N)$ to $H = \mathrm{SO}(N)$ is precisely one of the three scenarios for spontaneous symmetry breakdown discussed by Peskin [11], adapted to Majorana fermions [12]. The Goldstone boson manifold is the coset space G/H. Chiral perturbation theory is based on an effective field theory for Goldstone bosons. For its formulation a suitable parameterisation of the Goldstone boson manifold is needed. The general procedure for formulating effective theories and finding the associated effective Lagrangians has been developed in ref. [13] and leads to nonlinear representations of the chiral symmetry group.

As for the discussion of the adjoint pion in SYM it is sufficient to consider only one additional gluino, we shall consider the case $N = 2$ for definiteness in the following. So we have $G = \mathrm{SU}(2)$ and $H = \mathrm{SO}(2) = \mathrm{U}(1)$. The subgroup H is generated by $T_2 = \sigma_2/2$. The homogeneous space $\mathrm{SU}(2)/\mathrm{U}(1)$ is isomorphic to the sphere S^2. Therefore it would be possible to represent the Goldstone boson field by a real unit vector field $\vec{n}(x)$ and formulate the effective Lagrangian as a non-linear σ-model for $\vec{n}(x)$. In our case there is, however, another way, which is more convenient for explicit calculations.

Abstractly defined, the coset space G/H is equal to the set of cosets gH with $g \in G$. Every element of $\mathrm{SU}(2)$ (apart from exceptional points) can be parameterised uniquely as

$$g = \exp(\mathrm{i}\alpha_1 T_1 + \mathrm{i}\alpha_3 T_3)\,\exp(\mathrm{i}\alpha_2 T_2) \doteq uh \tag{8}$$

with real parameters α_k. Therefore the elements of the coset space G/H can be parameterised as

$$u = \exp(\mathrm{i}\alpha_1 T_1 + \mathrm{i}\alpha_3 T_3). \tag{9}$$

These matrices are unitary and symmetric, $u^T = u$. One could now set up chiral perturbation theory by introducing the field $\alpha(x) = \alpha_1(x) T_1 + \alpha_3(x) T_3$ via

$$u(x) = \exp\left(\mathrm{i}\frac{\alpha(x)}{F}\right), \tag{10}$$

where F is a dimensionful constant, analogous to the pion decay constant in chiral perturbation theory for QCD [14]. The transformation law of $u(x)$ under the chiral group $\mathrm{SU}(2)$ would, however, be complicated. Instead we introduce the field

$$U(x) = \exp\left(\mathrm{i}\frac{\phi(x)}{F}\right) \tag{11}$$

through

$$U(x) = u(x)^2 = u(x)u(x)^T. \tag{12}$$

This matrix valued field transforms in a simple way under SU(2), namely

$$U(x) \rightarrow U'(x) = VU(x)V^T, \qquad V \in \mathrm{SU}(2), \tag{13}$$

similar to the case of QCD. Two examples of invariant expressions are $\mathrm{tr}(AB^\dagger)$ and $\mathrm{tr}(AB^\dagger CD^\dagger)$ with A, B, C, D being derivatives of $U(x)$.

The effective Lagrangian can now be constructed in a standard way. Let

$$\chi = 2B_0 \mathcal{M} = 2B_0 m_g \mathbf{1} \tag{14}$$

be the mass term. The coefficient B_0 is the second low-energy constant, familiar from the case of QCD [6, 14]. It is equal to the ratio of the gluino condensate $\langle \bar{\lambda}\lambda \rangle$ to F^2. The leading order effective Lagrangian is given by

$$\mathcal{L}_2 = \frac{F^2}{4}\mathrm{tr}(\partial_\mu U \partial^\mu U^\dagger) + \frac{F^2}{4}\mathrm{tr}(\chi U^\dagger + U\chi^\dagger) \tag{15}$$

as for QCD. The next to leading order terms can be taken over from Gasser and Leutwyler [14], and are not reproduced here. They contain further low-energy constants L_i, the so-called Gasser-Leutwyler coefficients.

Let us now return to SYM, where in addition to the original sea gluino the additional valence gluino is introduced in a partially quenched manner. It is therefore accompanied by a ghost gluino $\rho(x)$, having the same Lorentz transformation properties, but being a boson. The chiral symmetry group is enhanced to the graded group SU(2—1) and the Goldstone boson field $\phi(x)$, appearing in the chiral field $U(x)$, is now a graded 3×3 matrix valued field,

$$\phi = \begin{pmatrix} \phi_{ss} & \phi_{sv} & \phi_{sg} \\ \phi_{vs} & \phi_{vv} & \phi_{vg} \\ \phi_{gs} & \phi_{gv} & \phi_{gg} \end{pmatrix} \tag{16}$$

where the labels s, v and g stand for sea, valence and ghost. For a graded matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{17}$$

where A is a 2×2 matrix and D a 1×1 matrix (number), the supertrace is defined by

$$\mathrm{str}(M) = \mathrm{tr}(A) - \mathrm{tr}(D). \tag{18}$$

The leading order effective Lagrangian reads

$$\mathcal{L}_2^{PQ} = \frac{F^2}{4}\mathrm{str}(\partial_\mu U \partial^\mu U^\dagger) + \frac{F^2}{4}\mathrm{str}(\chi U^\dagger + U\chi^\dagger). \tag{19}$$

Based on the effective Lagrangian the masses of the pseudo-Goldstone bosons can be calculated in partially quenched chiral perturbation theory [15, 16]. We have calculated the masses in next-to-leading order. Whereas the tree-level contributions are similar to the ones

in QCD, the loop contributions differ due to the different group structure. The adjoint pion is represented by ϕ_{sv}. For its mass $m_{\text{a}-\pi} = M_{sv}$ we find

$$M_{sv}^2 = 2B_0 m_g + \frac{(2B_0 m_g)^2}{F^2}(30L_8 - 2L_4 - 7L_5 + 8L_6), \tag{20}$$

with the low-energy coefficients L_i mentioned above. Interestingly the loop contribution vanishes so that no chiral logarithm appears in M_{sv}^2. For small m_g we recognise the desired GOR-relation

$$m_{\text{a}-\pi}^2 = 2B_0 m_g. \tag{21}$$

Acknowledgments

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