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Bilocal holography and locality in the bulk

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Abstract: Bilocal holography provides a constructive approach to the vector model/higher spin gravity duality. It has two ingredients: a change of field variables and a change of space time coordinates. The change of field variables ensures that the loop expansion parameter becomes $\frac{1}{N}$. The change of coordinates solves the Clebsch-Gordan problem of moving from the tensor product basis (in which the collective bilocal field is written) to the direct sum basis (appropriate for the description of the gravity fields). We argue that the change of space time coordinates can be deduced by requiring that operators constructed in the bilocal collective field theory are dual to local operators in the AdS bulk.

Keywords: 1/N Expansion, AdS-CFT Correspondence, Gauge-Gravity Correspondence, Higher Spin Gravity

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Contents

1 Introduction

Bilocal holography [1] is a constructive approach to the duality [2, 3] between $O(N)$ vector models and higher spin gravity[4, 5]. It is a concrete example of the general framework of collective field theory [6, 7], which provides a constructive approach to the AdS/CFT duality [8–10] and other gauge theory/gravity dualities [11]. Concretely, bilocal holography gives an explicit formula for the higher spin gravity fields in terms of the operators of the conformal field theory. This realizes a construction of 'precursors' i.e. bulk fields in terms of boundary operators [12], as collective fields. The mapping between the boundary and bulk theories is achieved by matching the independent degrees of freedom in the conformal field theory to the independent degrees of freedom in the completely gauge fixed higher spin gravity. The fact that this can actually be carried out in complete detail is thanks to impressive work of Metsaev [13–18] which achieves a complete gauge fixing (to light cone gauge) of the higher spin gravity and the reduction to independent degrees of freedom on both sides of the duality.

Bilocal holography has two ingredients: there is a change of field variables and a change of space time coordinates. The change of field variables ensures that while the loop expansion parameter before transformation is $\hbar$, it becomes $\frac{1}{N}$ after the transformation. The change of coordinates solves the Clebsch-Gordan problem of moving from the tensor product basis (in which the collective bilocal field is written) to the direct sum basis (appropriate for the gravity fields). This change of coordinates is highly non-trivial and the correctness of this step was verified by showing that the complete set of generators of the conformal field theory are mapped into those of the higher spin gravity [19]. Bilocal holopgraphy has been developed in a number of interesting directions [20–33].

Focus for now on the duality between CFT_3 and higher spin gravity on AdS_4. Before gauge fixing and reducing to independent degrees of freedom on both sides, the higher spin gravity includes a bulk scalar as well as a gauge field of every even integer spin, while the single trace primaries of the conformal field theory are a scalar primary of dimension $\Delta = 1$ and a family of conserved spinning currents, of dimension $s+1$ and spin s for every even integer s. After gauge fixing and reduction the complete set of conformal field theory degrees of freedom are packaged in a single $\text{O}(N)$ invariant equal time bilocal field, while on the gravity side one remains with two polarizations ($A^{XX\cdots X}$ and $A^{X\cdots XZ}$) of the spinning gauge with spin $2s$ for $s > 0$, as well as the bulk scalar. Given that we consider a light cone gauge fixing, it is natural to pursue a light front quantization of both theories. In terms of the original scalar field ϕ^a of the $\mathrm{O}(N)$ model, the bilocal field is written as

$$\sigma(x^+, x_1^-, x_1, x_2^-, x_2) = \phi^a(x^+, x_1^-, x_1)\phi^a(x^+, x_2^-, x_2) \tag{1.1}$$

where $x^\pm$ are the light cone coordinates of CFT_3 and x is the single coordinate transverse to the light cone. The bilocal field develops a large N expectation value, which we denote as $\sigma_0(x^+, x^-, x_1, x_2^-, x_2)$. Expanding about this background defines the fluctuation $\eta(x^+, x_1^-, x_1, x_2^-, x_2)$ as follows

$$\sigma(x^+, x_1^-, x_1, x_2^-, x_2) = \sigma_0(x^+, x_1^-, x_1, x_2^-, x_2) + \eta(x^+, x_1^-, x_1, x_2^-, x_2) \tag{1.2}$$

It is the fluctuation $\eta(x^+, x_1^-, x_1, x_2^-, x_2)$ that is identified with the dynamical fields of the dual higher spin gravity. This is a single field that depends on 5 coordinates. On the higher spin gravity side, it is useful to package the complete collection of higher spin gauge fields and the bulk scalar into a single field, with the help of a book keeping coordinate θ, as follows

$$\Phi(X^+, X^-, X, Z, \theta) = \sum_{s=0}^{\infty} \left(\frac{A^{XX\cdots XX}(X^+, X^-, X, Z)}{Z} \cos(2s\theta) + \frac{A^{XX\cdots XZ}(X^+, X^-, X, Z)}{Z} \sin(2s\theta) \right) \tag{1.3}$$

The fields on the right hand side are all tangent space fields. See [13] for a useful discussion. Following [27], the mapping of the fields is most easily written in a mixed position/momentum space representation. In both the conformal field theory and in the higher spin gravity there is a symmetry under translations of x^- and X^- respectively. This allows us to perform a Fourier transform which trades x_1^- and x_2^- for p_1^+ and p_2^+ in the conformal field theory, and X^- for P^+ in the higher spin gravity. The relation between the field can then be written as

$$\Phi(X^+, P^+, X, Z, \theta) = \mu(p_1^+, p_2^+)\eta(x^+, p_1^+, x_1, p_2^+, x_2) \tag{1.4}$$

where

$$\mu(p_1^+, p_2^+) = \sqrt{p_1^+ p_2^+} \tag{1.5}$$

The relation between the coordinates of the conformal field theory and the bulk AdS_4 space time is given by $x^+ = X^+$ as well as

$$P^+ = p_1^+ + p_2^+ \qquad X = \frac{p_1^+ x_1 + p_2^+ x_2}{p_1^+ + p_2^+}$$

$$Z = \frac{\sqrt{p_1^+ p_2^+}}{p_1^+ + p_2^+}|x_1 - x_2| \qquad \theta = 2\arctan\sqrt{\frac{p_2^+}{p_1^+}} \tag{1.6}$$

There are many tests that can be carried out once the bilocal holography mapping, as stated above, is given. For example, bilocal holography provides an explicit bulk reconstruction for complete set bulk gauge fields as well as the scalar field. This entails proving that the field $\Phi(X^+, P^+, X, Z, \theta)$ obeys the correct equation of motion [19] as well as the correct GKPW boundary conditions as $Z \to 0$ [32]. It is also interesting to explore whether or not information localizes in the bilocal holography construction as is expected in a theory quantum gravity. By restricting to a single subregion of the conformal field theory, one finds [27] that the subregion of the bulk that can be reconstructed is in perfect accord with entanglement wedge reconstruction [34–39]. Further by using the mapping it is possible to translate the statement of the holography of information [40–43][1] in the gravity theory into a statement in the conformal field theory. Using the usual operator product expansion formula the holography of information can be demonstrated directly in the conformal field theory [28].

Our goal in this paper is explore another feature of the holographic duality, namely bulk locality. We will see that constructing fields that are local in the bulk AdS spacetime naturally leads us to the coordinate map (1.6) appearing in bilocal holography. This result which is the central result of this paper, is a useful observation since it outlines a deductive approach to determine the change of coordinates appearing in the bilocal holography map.

In what follows we consider the general case of the duality between CFT_d and AdS_{d+1}.[2] In this case the bilocal $\eta(x^+, x_1^-, \vec{x}_1, x_2^-, \vec{x}_2)$ is a function of $2d-1$ coordinates and the vector $\vec{x}$ is a $d-2$ dimensional vector transverse to the light cone. The bulk higher spin fields are again collected into a single field[3]

$$\Phi(X^+, X^-, \vec{X}, Z, \alpha^i) = \sum_{s=0}^{\infty} \alpha_{a_1}\alpha_{a_2}\cdots\alpha_{a_{2s}} \frac{A^{a_1 a_2 \cdots a_{2s}}(X^+, X^-, \vec{X}, Z)}{Z^{\frac{d-1}{2}}}|0\rangle \tag{1.7}$$

The index a_i on the oscillators runs over Z and the $d-2$ directions $\vec{X}$ transverse to the lightcone. Only $d-2$ of these oscillators are independent since we must impose that $\Phi(X^+, X^-, \vec{X}, Z, \alpha^i)$ is traceless [32]. There are $d+1$ coordinates $X^\pm, \vec{X}, Z$ needed to specify an event in the AdS bulk and $d-2$ independent oscillators so that $\Phi(X^+, X^-, \vec{X}, Z, \alpha^i)$ is a function of $2d-1$ coordinates. The holographic mapping between the fields in this case is

$$\Phi(X^+, P^+, \vec{X}, Z, \alpha) = \mu(p_1^+, p_2^+, Z)\eta(x^+, p_1^+, \vec{x}_1, p_2^+, \vec{x}_2) \tag{1.8}$$

where

$$\mu(p_1^+, p_2^+, Z) = (p_1^+ p_2^+)^{\frac{4-d}{2}} Z^{\frac{3-d}{2}} \tag{1.9}$$

In the rest of the article we will determine the relation between the coordinates of the d dimensional bilocal field theory $(x^+, p_1^+, \vec{x}_1, p_2^+, \vec{x}_2)$ and those of the bulk AdS_{d+1} space time $(X^+, P^+, \vec{X}, Z, \alpha^i)$, as well as the form for $\mu(p_1^+, p_2^+, Z)$ quoted above. This coordinate mapping will be determined by requiring that the bilocal field of the conformal field theory is dual to a local operator in the bulk.

[1]For a beautiful set of lectures, incredibly helpful when learning this material, go to ref. [44].

[2]We limit ourselves to $d \geq 3$. The $d = 2$ case is more subtle because in this case the free scalar fields are not primary operators.

[3]We are using the very useful oscillator representation introduced in [13].

2 Bulk locality

A point in the bulk AdS_{d+1} space time is specified by $d+1$ coordinates[4] $X^M = (X^\mu, Z)$, where X^μ is a set of d coordinates and Z is the radial (holographic) coordinate. The bulk metric is

$$ds^2 = \frac{dX^\mu dX^\nu \eta_{\mu\nu} + dZ^2}{Z^2} \tag{2.1}$$

where $\eta = \text{diag}(-1, 1, \cdots, 1)$. The isometries of this metric are generated by

$$\begin{aligned} P^\mu &= \partial^\mu & J^{\mu\nu} &= X^\mu \partial^\nu - X^\nu \partial^\mu \\ D &= X^\mu \partial_\mu + Z\partial_Z & K^\mu &= -\frac{1}{2}(X^\mu X_\mu + Z^2)\partial^\mu + X^\mu D \end{aligned} \tag{2.2}$$

These generators generate the group SO(2,d). In what follows we employ lightcone coordinates, obtained by setting

$$X^\pm = \frac{1}{\sqrt{2}}\left(X^{d-1} \pm X^0\right) \tag{2.3}$$

The vector $\vec{X}$ has components X^i for $i = 1, 2, \cdots, d-2$ which are coordinates transverse to the lightcone. Z is the coordinate for the extra holographic dimension.

AdS spacetime is an example of a symmetric space[45]. Starting from any particular point in the AdS spacetime, we can reach any other point with the action of some element of SO(2,d). Not every element of SO(2,d) will move a particular point: each point has an isotropy group under whose action the given point is fixed. For the case we are discussing here, the isotropy group is SO(1,d) so that we can identify AdS spacetime as the coset SO(2,d)/SO(1,d). As a particularly simple example, consider the bulk point $X^\mu = 0$ and $Z = Z_p$ where Z_p is a definite value. Fixing these values of the coordinates the generators become

$$P^\mu = \partial^\mu \qquad J^{\mu\nu} = 0 \qquad D = Z_p\partial_Z \qquad K^\mu = -\frac{1}{2}Z_p^2\partial^\mu \tag{2.4}$$

Thus, the generators $J^{\mu\nu}$ and $K^\mu + \frac{1}{2}Z_p^2 P^\mu$ both vanish when acting on our bulk point. They are non-zero acting on any other point so we can define this bulk location as the point that is annihilated by the group generated by $\{J^{\mu\nu}, K^\mu + \frac{1}{2}Z_p^2 P^\mu\}$. This group is SO(1,$d$) - it is the isotropy group of the bulk point we are considering. Note that the coset SO(2,d)/SO(1,d) is a $d+1$ dimensional space and each class in this coset corresponds to a point in the bulk of the AdS_{d+1} spacetime. Choosing a different bulk point leads to a different isotropy group, which is still isomorphic to SO(1,d). The data of the isotropy group and the bulk point are equivalent pieces of data. The isotropy group is giving us information about bulk locality - it tells us what it means to be localized at a specific bulk point.

Since the isotropy group leaves the bulk point fixed, it will only shuffle the different polarizations of a spinning field amongst each other. Thus, for example, to construct

[4] In what follows we always use capital letters for coordinates of the bulk AdS spacetime and little letters for the coordinates of the space time of the CFT.

an operator O_Ψ in the bilocal field theory that is dual to a bulk operator localized at $X^M = (0, Z_p)$ in the AdS bulk, we solve the equations[5]

$$[\, J^{\mu\nu}\, ,\, O_\Psi] = 0 \;=\; \left[K^\mu + \frac{1}{2} Z_p^2 P^\mu\, ,\, O_\Psi\right] \tag{2.5}$$

This is the minimal requirement for operators in the bilocal field theory to be dual to operators localized in the AdS bulk. See [46–49] for related discussion. The second equation above is telling: an operator localized at a boundary point is primary, whilst an operator located in the bulk corresponds to a non-trivial combination of the primary and its descendents. So AdS/CFT geometrizes the space of CFT operators, introducing an extra holographic dimension with coordinate Z and placing the primaries on the boundary of the AdS bulk. Since the single trace primaries generate the complete set of gauge invariant operators, all the information sits on the boundary, consistent with the holography of information.

Our goal is now to use the requirement of bulk locality to derive the coordinate transformation of bilocal holography. Towards that end, consider the set of bulk points given by $X^+ = 0$, $\vec{X} = \vec{X}_p$, $Z = Z_p$ where $\vec{X}_p$ and Z_p are definite values. Notice that X^- is left arbitrary i.e. we are considering a light like line of points in the AdS$_{d+1}$ bulk. For this line of points the momentum P^+ and special conformal transformation K^+ become

$$P^+ = \partial^+ \qquad\qquad K^+ \;=\; -\frac{1}{2}(\vec{X}\cdot\vec{X} + Z_p^2)\partial^+ \tag{2.6}$$

The second of (2.5) provides a non trivial differential equation that a bulk field localized on the light like line must obey

$$\left(K^+ + \frac{1}{2}(\vec{X}\cdot\vec{X} + Z_p^2)P^+\right)O_\Psi = 0 \tag{2.7}$$

This equation is completely general, holding for fields of any spin, thanks to the form of K^+ given in (3.71) of [13]. The coordinate transformation (1.6) was determined by matching the generators of conformal transformations of the bilocal conformal field theory with those of the higher spin gravity. To obtain a relation between the coordinates of the field theory and those of the gravity, we will insert the bilocal generators into the above equation. Further, if we now take a Fourier transform on X^- in gravity, the differential operator ∂^+ is replaced by the variable P^+. Since we have localized only to the light like line parameterized by X^-, the Fourier transformed field obeys the same bulk locality condition. Similarly, if we take a Fourier transform in the bilocal theory, ∂_1^+ and ∂_2^+ are replaced by p_1^+ and p_2^+. Bilocal holography matches the Hilbert space of the bilocal theory with that of the higher spin gravity. Since these are each defined at fixed light cone time we should identify $X^+ = x^+$. Inserting the bilocal expression for the generators into (2.7), we obtain

$$\left(-\frac{1}{2}(\vec{x}_1\cdot\vec{x}_1 p_1^+ + \vec{x}_2\cdot\vec{x}_2 p_2^+) + \frac{1}{2}(\vec{X}\cdot\vec{X} + Z_p^2)(p_1^+ + p_2^+)\right)O_\Psi = 0 \tag{2.8}$$

[5]Here we are studying a scalar state for simplicity. If we have a spinning state we would again use oscillators and add the spin contribution $M^{\mu\nu} = \alpha^\mu\bar{\alpha}^\nu - \alpha^\nu\bar{\alpha}^\mu$ to $J^{\mu\nu}$ and $M^{\mu\nu}x_\nu$ to K^μ. In addition $J^{\mu\nu}$ would no longer annihilate the state, but would mix the different spin states amongst each other.

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Notice that this is a polynomial multiplied by the field. Since the field does not vanish the polynomial does and in the end we obtain the equation

$$-\frac{1}{2}(\vec{x}_1 \cdot \vec{x}_1 p_1^+ + \vec{x}_2 \cdot \vec{x}_2 p_2^+) + \frac{1}{2}(\vec{X}_p \cdot \vec{X}_p + Z_p^2)(p_1^+ + p_2^+) = 0 \tag{2.9}$$

which relates the bulk AdS_{d+1} coordinates $\vec{X}_p$ and Z_p to the coordinates $\vec{x}_1$, $\vec{x}_2$, p_1^+ and p_2^+ of the conformal field theory.

3 Holographic mapping

In this section we will solve (2.9) to explicitely demonstrate the link between bulk locality and the bilocal holography mapping. One of the key pieces of evidence motivating the discovery of AdS/CFT was a matching between the global symmetries of the conformal field theory and the isometries of the dual AdS gravity. Our solution of (2.9) will also make use of this fact.

The symmetry $X^- \to X^- + a$ in the bulk corresponds to $x_1^- \to x_1^- + a$ and $x_2^- \to x_2^- + a$ in the bilocal collective field theory. Thus, the generator producing an infinitesimal transformation of X^- in the bulk must generate an infinitesimal transformation of both x_1^- and x_2^- which implies that

$$P^+ = p_1^+ + p_2^+ \tag{3.1}$$

An identical argument for the directions transverse to the light cone allows us to conclude that

$$P^i = p_1^i + p_2^i \tag{3.2}$$

Now consider an SO(d-2) transformation $R^i{}_j$ which acts as

$$X^i \to R^i{}_j X^j \tag{3.3}$$

in the bulk, and as

$$x_1^i \to R^i{}_j x_1^j \qquad x_2^i \to R^i{}_j x_2^j \tag{3.4}$$

in the bilocal field theory. Thus, $\vec{X}$, $\vec{x}_1$ and $\vec{x}_2$ are all in the $d-2$ dimensional vector representation of SO(d-2). Higher powers of these coordinates are not generally even in an ireducible representation, and in particular, they are not in the vector representation. A simple way to ensure that all three are in the vector representation is to take a linear relation between them[6]

$$\vec{X} = \alpha\vec{x}_1 + \beta\vec{x}_2 \tag{3.5}$$

Next, the translation $\vec{X} \to \vec{X} + \vec{a}$ in the bulk corresponds to $\vec{x}_1 \to \vec{x}_1 + \vec{a}$ and $\vec{x}_2 \to \vec{x}_2 + \vec{a}$ in the bilocal field theory. This forces $\beta + \alpha = 1$ so that we have

$$\vec{X} = \alpha\vec{x}_1 + (1-\alpha)\vec{x}_2 \tag{3.6}$$

[6]In principle one could add a constant $\vec{X}_0$ on the right hand side. This can always be removed with a judicious choice of origin.

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The arguments we have considered so far allow the parameter α to be an arbitrary function of p_1^+ and p_2^+.

Under a translation $X^i \to X^i + a^i$ the bulk coordinate Z is unchanged. This translation takes $x_1^i \to x_1^i + a^i$ and $x_2^i \to x_2^i + a^i$ in the bilocal field theory so that Z is a function only of $x_1^i - x_2^i$. Next, under the SO(d-2) transformation $R^i{}_j$ we know that Z is unchanged. This implies that Z is a function only of $|\vec{x}_1 - \vec{x}_2|$. If we now assume that Z is a linear function of $|\vec{x}_1 - \vec{x}_2|$ we can write

$$Z = \delta|\vec{x}_1 - \vec{x}_2| \tag{3.7}$$

where again, our arguments allow δ to be an arbitrary function of p_1^+ and p_2^+. The assumption of linearity will be motivated below.

Recall that the equation we wish to solve, (2.9), is given by

$$-(\vec{x}_1 \cdot \vec{x}_1 p_1^+ + \vec{x}_2 \cdot \vec{x}_2 p_2^+) + (\vec{X} \cdot \vec{X} + Z^2)(p_1^+ + p_2^+) = 0 \tag{3.8}$$

A simple computation shows that

$$\begin{aligned} \vec{X} \cdot \vec{X} + Z^2 &= \alpha^2 \vec{x}_1 \cdot \vec{x}_1 + (1 - 2\alpha + \alpha^2)\vec{x}_2 \cdot \vec{x}_2 + 2\alpha(1-\alpha)\vec{x}_1 \cdot \vec{x}_2 \\ &\quad + \delta^2(\vec{x}_1 \cdot \vec{x}_1 + \vec{x}_2 \cdot \vec{x}_2 - 2\vec{x}_1 \cdot \vec{x}_2) \end{aligned} \tag{3.9}$$

Inserting (3.9) into (3.8) and equating the coefficients of $\vec{x}_1 \cdot \vec{x}_1$, $\vec{x}_2 \cdot \vec{x}_2$ and $\vec{x}_1 \cdot \vec{x}_2$ to zero, we obtain the following three equations

$$\begin{aligned} -p_1^+ + (\alpha^2 + \delta^2)(p_1^+ + p_2^+) &= 0 \\ -p_2^+ + (1 - 2\alpha + \alpha^2 + \delta^2)(p_1^+ + p_2^+) &= 0 \\ (2\alpha(1-\alpha) - 2\delta^2)(p_1^+ + p_2^+) &= 0 \end{aligned} \tag{3.10}$$

The last equation above implies that $\delta^2 = \alpha(1-\alpha)$. Inserting this into the first equation above gives a linear equation for α so that α and δ are determined. This unique solution ensures that all three equations above are satisfied. The solution is

$$\alpha = \frac{p_1^+}{p_1^+ + p_2^+} \qquad 1 - \alpha = \frac{p_2^+}{p_1^+ + p_2^+} \qquad \delta = \frac{\sqrt{p_1^+ p_2^+}}{p_1^+ + p_2^+} \tag{3.11}$$

This argument therefore implies that

$$\vec{X} = \frac{p_1^+ \vec{x}_1 + p_2^+ \vec{x}_2}{p_1^+ + p_2^+} \qquad Z = \frac{\sqrt{p_1^+ p_2^+}}{p_1^+ + p_2^+}|\vec{x}_1 - \vec{x}_2| \tag{3.12}$$

which is precisely the coordinate transformation for the bulk AdS coordinates given by the bilocal holography mapping (1.6).

Let us now return to the assumption that Z is linear in $|\vec{x}_1 - \vec{x}_2|$. To get some insight into why this must be the case, note that we can rewrite (2.9) as

$$-(\vec{x}_1 \cdot \vec{x}_1 p_1^+ + \vec{x}_2 \cdot \vec{x}_2 p_2^+) + \vec{X} \cdot \vec{X}(p_1^+ + p_2^+) = -Z^2(p_1^+ + p_2^+) \tag{3.13}$$

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Recalling the formula (3.6), it is clear that all terms on the left hand side of this equation are quadratic in $\vec{x}_i$. Thus, the right hand side must be too and this forces Z to be a linear function of $|\vec{x}_1 - \vec{x}_2|$.

An alternative way to approach (2.9) is to take its commutator with P^i, which gives

$$\begin{aligned} 0 &= -\frac{1}{2}\left([P^i, \vec{x}_1 \cdot \vec{x}_1]p_1^+ + [P^i, \vec{x}_2 \cdot \vec{x}_2]p_2^+\right) + \frac{1}{2}\left([P^i, \vec{X}_p \cdot \vec{X}_p] + [P^i, Z_p^2]\right)(p_1^+ + p_2^+) \\ &= -x_1^i p_1^+ - x_2^i p_2^+ + X_p^i(p_1^+ + p_2^+) \end{aligned} \tag{3.14}$$

which immediately implies that

$$X_p^i = \frac{x_1^i p_1^+ + x_2^i p_2^+}{p_1^+ + p_2^+} \tag{3.15}$$

Inserting this into (2.9) we find

$$\begin{aligned} Z_p^2 &= \frac{(\vec{x}_1 \cdot \vec{x}_1 p_1^+ + \vec{x}_2 \cdot \vec{x}_2 p_2^+)}{p_1^+ + p_2^+} - \vec{X}_p \cdot \vec{X}_p \\ &= \frac{p_1^+ p_2^+}{(p_1^+ + p_2^+)^2}|\vec{x}_1 - \vec{x}_2|^2 \end{aligned} \tag{3.16}$$

which is the result we obtained above.

To complete the holographic mapping we still have to determine the angles that are used to package the higher spin fields into a single field. Following [20] we will show how the spin components of the angular momentum generators are determined. We start by considering the rotations transverse to the light cone[7]

$$J^{ij} = X^i P^j - P^i X^j + m^{ij} \tag{3.17}$$

The first two terms are the orbital contribution while the third term in the spin part. It is the spin part that we would like to determine. It is therefore convenient to perform the analysis at the bulk point $X^i = 0$, which sets the orbital terms to zero. In this case we have

$$J^{ij} = m^{ij} \tag{3.18}$$

This should be matched to the generator of the bilocal field theory, which reads

$$J^{ij} = x_1^i \frac{\partial}{\partial x_1^j} - x_1^j \frac{\partial}{\partial x_1^i} + x_2^i \frac{\partial}{\partial x_2^j} - x_2^j \frac{\partial}{\partial x_2^i} \tag{3.19}$$

To perform the comparison, note that since the angles are translation invariant, they can only be a function only of $x_1^i - x_2^i$. Setting $X^i = 0$ implies that $\vec{x}_1$ and $\vec{x}_2$ are not independent variables, but rather they obey the relation

$$p_1^+ \vec{x}_1 + p_2^+ \vec{x}_2 = 0 \tag{3.20}$$

Using this relation, it is simple to argue that

$$\vec{x}_1 = \frac{p_2^+}{p_1^+ + p_2^+}(\vec{x}_1 - \vec{x}_2) \qquad\qquad \vec{x}_2 = -\frac{p_1^+}{p_1^+ + p_2^+}(\vec{x}_1 - \vec{x}_2) \tag{3.21}$$

[7]This formula assumes that $d > 3$.

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After using these identities in (3.19) and equating the result to (3.18) we learn that

$$m^{ij} = \frac{(x_1-x_2)^i}{p_1^+ + p_2^+}(p_2^+ p_1^j - p_1^+ p_2^j) - \frac{(x_1-x_2)^j}{p_1^+ + p_2^+}(p_2^+ p_1^i - p_1^+ p_2^i) \tag{3.22}$$

which is in complete agreement with the known result [20].

Next consider the m^{iz} contribution to the spin angular momentum. It proves useful to study the special conformal generator K^i which is given by [13]

$$K^i = -\frac{1}{2}\left(2X^+X^- + \vec{X}\cdot\vec{X} + Z^2\right)P^i + X^i D + m^{ij}X^j + m^{iZ}Z + m^{i-}X^+ \tag{3.23}$$

This can be simplified dramatically if we situate ourselves at the light like line of bulk points specified by $X^+ = 0 = X^i$ and any X^-. In this case K^i becomes

$$K^i = -\frac{1}{2}Z^2 P^i + m^{iZ} Z \tag{3.24}$$

The only unknown in this expression is m^{iZ}. We know that the spin generator m^{iZ} is translation invariant which implies that it is again only a function only of $x_1^i - x_2^i$. Equating this to the corresponding expression in the bilocal field theory, we obtain

$$\begin{aligned}
K^i &= \mu(p_1^+, p_2^+, Z)\left(-\frac{1}{2}(\vec{x}_1\cdot\vec{x}_1 p_1^i + \vec{x}_2\cdot\vec{x}_2 p_2^i) + x_1^i D_1 + x_2^i D_2\right)\frac{1}{\mu(p_1^+, p_2^+, Z)} \\
&= \mu(p_1^+, p_2^+, Z)\Bigg(-\frac{1}{2}\left(\frac{(p_2^+)^2}{(p_1^+ + p_2^+)^2}|\vec{x}_1 - \vec{x}_2|^2 p_1^i + \frac{(p_1^+)^2}{(p_1^+ + p_2^+)^2}|\vec{x}_1 - \vec{x}_2|^2 p_2^i\right) \\
&\qquad + \frac{p_2^+}{p_1^+ + p_2^+}(x_1^i - x_2^i)D_1 - \frac{p_1^+}{p_1^+ + p_2^+}(x_1^i - x_2^i)D_2\Bigg)\frac{1}{\mu(p_1^+, p_2^+, Z)}
\end{aligned} \tag{3.25}$$

where when acting on $\phi^a(x^+, x_1^-, x_1)$ we have

$$\begin{aligned}
D_1 &= x^+ p_1^- + x_1^- p_1^+ + x_1^j p_1^j + \frac{d-2}{2} \\
&= x_1^- p_1^+ + \frac{p_2^+}{p_1^+ + p_2^+}(x_1^j - x_2^j)p_1^j + \frac{d-2}{2}
\end{aligned} \tag{3.26}$$

and when acting on $\phi^a(x^+, x_2^-, x_2)$ we have

$$D_2 = x_2^- p_2^+ - \frac{p_1^+}{p_1^+ + p_2^+}(x_1^j - x_2^j)p_2^j + \frac{d-2}{2} \tag{3.27}$$

Equating (3.24) and (3.25) we easily find

$$\begin{aligned}
m^{iZ} &= \frac{x_1^i - x_2^i}{|\vec{x}_1 - \vec{x}_2|}\left[\sqrt{p_1^+ p_2^+}(x_1^- - x_2^-) + \frac{((p_1^+)^2 p_2^j + (p_2^+)^2 p_1^j)(x_1^j - x_2^j)}{(p_1^+ + p_2^+)\sqrt{p_1^+ p_2^+}}\right] \\
&\quad + \frac{1}{2}\frac{p_1^+ - p_2^+}{p_1^+ + p_2^+}|\vec{x}_1 - \vec{x}_2|\left(p_1^i\sqrt{\frac{p_2^+}{p_1^+}} - p_2^i\sqrt{\frac{p_1^+}{p_2^+}}\right)
\end{aligned} \tag{3.28}$$

which is in complete agreement with the known result [20, 32]. This computation also fixes

$$\mu(p_1^+, p_2^+, Z) = f(Z)(p_1^+ p_2^+)^{\frac{4-d}{2}} \tag{3.29}$$

Finally, by matching the action of the CFT dilatation operator on $\mu(p_1^+, p_2^+, Z)\eta(x^+, p_1^+, \vec{x}_1, p_2^+, \vec{x}_2)$ with the action of the bulk dilatation operator on $\Phi(X^+, P^+, \vec{X}, Z, \alpha)$, we fix

$$\mu(p_1^+, p_2^+, Z) = (p_1^+ p_2^+)^{\frac{4-d}{2}} Z^{\frac{3-d}{2}} \tag{3.30}$$

This completes the derivation of the coordinate transformation, by making use of bulk locality.

4 A concrete example

In the previous section we have written down the generators of the angular momentum. In this section, for the specific case that $d = 4$, we will translate these angular momenta into a collection of angles. Denote the coordinates of the conformal field theory by $x^\mu = (t, w, x, y)$. Light cone coordinates are defined as $x^\pm = t \pm w$ and the bilocal field depends on 7 coordinates $\sigma(x^+, x_1^-, x_1, y_1, x_2^-, x_2, y_2)$. Bilocal holography relates these 7 coordinates to the AdS_5 coordinates X^+, X^-, X, Y, Z and two angles θ, φ. The two angles will be extracted from the angular momenta m^{XY}, m^{XZ} and m^{YZ}.

Using the results of the previous section, we have

$$m^{XY} = (x_1 - x_2)\frac{p_2^+ p_1^y - p_1^+ p_2^y}{p_1^+ + p_2^+} - (y_1 - y_2)\frac{p_2^+ p_1^x - p_1^+ p_2^x}{p_1^+ + p_2^+} \tag{4.1}$$

Now, to interpret this formula note that

$$\left[\frac{p_2^+ p_1^i - p_1^+ p_2^i}{p_1^+ + p_2^+}, \frac{p_1^+ x_1^j + p_2^+ x_2^j}{p_1^+ + p_2^+}\right] = 0 \qquad \left[\frac{p_2^+ p_1^i - p_1^+ p_2^i}{p_1^+ + p_2^+}, x_1^j - x_2^j\right] = \delta^{ij} \tag{4.2}$$

so that we can interpret the momentum $\mathcal{P}^i = \frac{p_2^+ p_1^i - p_1^+ p_2^i}{p_1^+ + p_2^+}$ as the momentum conjugate to the relative coordinate. The total momentum, which must commute with the relative coordinate and is conjugate to the $\vec{X}$ coordinate, also behaves as expected

$$\left[p_1^i + p_2^i, x_1^j - x_2^j\right] = 0 \qquad \left[p_1^i + p_2^i, \frac{p_1^+ x_1^j + p_2^+ x_2^j}{p_1^+ + p_2^+}\right] = \delta^{ij} \tag{4.3}$$

Consequently, if we set

$$x_1 - x_2 = |\vec{x}_1 - \vec{x}_2| \cos(\varphi) \qquad y_1 - y_2 = |\vec{x}_1 - \vec{x}_2| \sin(\varphi) \tag{4.4}$$

where

$$|\vec{x}_1 - \vec{x}_2| \equiv \sqrt{(x_1 - x_2)^2 - (y_1 - y_2)^2} \tag{4.5}$$

then we have

$$m^{XY} = \frac{\partial}{\partial \varphi} \tag{4.6}$$

Next consider the pair of generators

$$
\begin{aligned}
m^{XZ} &= \frac{x_1-x_2}{|\vec{x}_1-\vec{x}_2|} \\
&\quad\times\left[\sqrt{p_1^+p_2^+}(x_1^- - x_2^-)+\frac{((p_1^+)^2p_2^x+(p_2^+)^2p_1^x)(x_1-x_2)+((p_1^+)^2p_2^y+(p_2^+)^2p_1^y)(y_1-y_2)}{(p_1^++p_2^+)\sqrt{p_1^+p_2^+}}\right] \\
&\quad+\frac{1}{2}\frac{p_1^+-p_2^+}{p_1^++p_2^+}|\vec{x}_1-\vec{x}_2|\left(p_1^x\sqrt{\frac{p_2^+}{p_1^+}}-p_2^x\sqrt{\frac{p_1^+}{p_2^+}}\right) \\
&= \cos\varphi\frac{\partial}{\partial\theta}-\cot\theta\sin\varphi\frac{\partial}{\partial\varphi} \\
&= \cos\varphi P_\theta-\cot\theta\sin\varphi P_\varphi \qquad (4.7)\\
m^{YZ} &= \frac{y_1-y_2}{|\vec{x}_1-\vec{x}_2|} \\
&\quad\times\left[\sqrt{p_1^+p_2^+}(x_1^- - x_2^-)+\frac{((p_1^+)^2p_2^x+(p_2^+)^2p_1^x)(x_1-x_2)+((p_1^+)^2p_2^y+(p_2^+)^2p_1^y)(y_1-y_2)}{(p_1^++p_2^+)\sqrt{p_1^+p_2^+}}\right] \\
&\quad+\frac{1}{2}\frac{p_1^+-p_2^+}{p_1^++p_2^+}|\vec{x}_1-\vec{x}_2|\left(p_1^y\sqrt{\frac{p_2^+}{p_1^+}}-p_2^y\sqrt{\frac{p_1^+}{p_2^+}}\right) \\
&= \sin\varphi\frac{\partial}{\partial\theta}+\cot\theta\cos\varphi\frac{\partial}{\partial\varphi} \\
&= \sin\varphi P_\theta+\cot\theta\cos\varphi P_\varphi \qquad (4.8)
\end{aligned}
$$

where the angles θ, φ are given by

$$
\theta = 2\arctan\sqrt{\frac{p_2^+}{p_1^+}} \qquad\qquad \varphi = \arctan\frac{y_1-y_2}{x_1-x_2} \tag{4.9}
$$

which agrees with the angle θ appearing in the $d = 3$ bilocal holography map. Thus, we are indeed able to identify two extra angles. Finally, the momenta conjugate to these angles are

$$
\begin{aligned}
P_\varphi &= m^{XY} \qquad (4.10)\\
P_\theta &= \sqrt{p_1^+p_2^+}(x_1^- - x_2^-)+\frac{x_1-x_2}{2}\left(\sqrt{\frac{p_2^+}{p_1^+}}p_1^x+\sqrt{\frac{p_1^+}{p_2^+}}p_2^x\right)+\frac{y_1-y_2}{2}\left(\sqrt{\frac{p_2^+}{p_1^+}}p_1^y+\sqrt{\frac{p_1^+}{p_2^+}}p_2^y\right)
\end{aligned}
$$

5 Discussion and conclusions

Collective field theory provides a constructive approach to the AdS/CFT duality and other gauge theory/gravity dualities. It formulates the dynamics of quantum field theory in terms of gauge invariant collective fields. This reorganization of the degrees of freedom has an important consequence: while the loop expansion parameter of the original theory is $\hbar$, after the change of field variables the loop expansion parameter becomes $\frac{1}{N}$, matching that of the dual gravity description. For the specific case of the duality between $O(N)$ vector models and higher spin gravity, the collective fields are bilocal and the resulting collective construction is called bilocal holography.

Bilocal holography has two ingredients: the change of field variables we have just discussed, as well as a change of space time coordinates. The change of space time coordinates is needed to develop the physical interpretation of the theory. The scalar field of the O(N) model transforms in the spin zero and dimension $\frac{d-2}{2}$ representation of SO(2,d). The bilocal transforms in a tensor product of two copies of this representation. This representation is reducible and each irreducible component corresponds to a different bulk field in the gravitational description. Consequently, to develop the physical interpretation of the bilocal collective field theory we must solve the Clebsch-Gordan problem of moving from the tensor product basis (in which the collective bilocal field is written) to the direct sum basis (appropriate for the gravity fields). This is accomplished by a change of space time coordinates.

In the original work [19] this change of coordinates was discovered by brute force and then verified by demonstrating that it reproduces the bulk AdS isometry generators starting from the conformal generators in the bilocal field theory. In this paper we have outlined a deductive approach to determining this change of coordinates.

The idea is to formulate a minimal requirement for operators in the bilocal field theory to be dual to an operator located at a given bulk point. More correctly, we have considered the conditions needed to localize operators to a light like line in space time. The first step is the choice of the light like line in the bulk AdS space time. The isotropy group of this line is then determined. The generators of the isotropy group act by shuffling polarizations of the spinning field. Using a Fourier transform on the light like line, we have constructed a mixed momentum/position description. Within this description an algebraic equation was determined, which fixed the form of the AdS bulk coordinates in terms of those of the bilocal conformal field theory. Finally, additional angles used to package the complete collection of spinning bulk fields into a single field, were determined (implicitly) by evaluating the spin contribution to the conformal generators.

There are two different approaches which can be followed to rewrite the vector model in terms of bi-local fields which are then mapped to the bulk. The approach taken in this article uses the Hamiltonian language. In this description the bi-local operators are constructed from fields at different points in space but at the same time. This breaks manifest Lorentz invariance. A second approach [26, 33, 52] uses bilocal fields constructed from fields that are separated both in space and time. This preserves Lorentz invariance as well as the full conformal group. Both descriptions are useful. The two time description is useful as it makes the underlying conformal symmetry manifest. On the other hand, studying the theory at finite temperature is straightforward in the Hamiltonian approach, but it is non-trivial in the two time approach. The relation between the single time and two time descriptions has been considered in [20, 53]. In the two-time description one can impose constraints and perform a gauge fixing to the single time description.

Our construction is likely to be useful in more general applications of collective field theory to gauge theory/gravity dualities. An immediate application would be to gauge theories. In this case, the theory includes matrix fields transforming in the adjoint representation. Consequently, there is a much richer set of invariant variables: one can take traces of fields

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at different locations, so that bilocal operators, trilocal operators and in general k-local operators appear in the set of invariants. The construction of the change of coordinates needed for this case is highly non-trivial and has been an obstacle to progress. The application of bulk locality can be used to overcome this difficulty [50]. Another interesting application of bulk locality is to consider fermionic vector models [51] which are potentially relevant for an understanding of the holography of de Sitter space. This application would involve using the isotropy group of a point in de Sitter spacetime.

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References

[1] S.R. Das and A. Jevicki, *Large N collective fields and holography*, *Phys. Rev. D* **68** (2003) 044011 [hep-th/0304093] [INSPIRE].

[2] I.R. Klebanov and A.M. Polyakov, *AdS dual of the critical O(N) vector model*, *Phys. Lett. B* **550** (2002) 213 [hep-th/0210114] [INSPIRE].

[3] E. Sezgin and P. Sundell, *Massless higher spins and holography*, *Nucl. Phys. B* **644** (2002) 303 [hep-th/0205131] [INSPIRE].

[4] M.A. Vasiliev, *Consistent equation for interacting gauge fields of all spins in* $(3+1)$*-dimensions*, *Phys. Lett. B* **243** (1990) 378 [INSPIRE].

[5] M.A. Vasiliev, *Nonlinear equations for symmetric massless higher spin fields in* $(A)dS_d$, *Phys. Lett. B* **567** (2003) 139 [hep-th/0304049] [INSPIRE].

[6] A. Jevicki and B. Sakita, *The quantum collective field method and its application to the planar limit*, *Nucl. Phys. B* **165** (1980) 511 [INSPIRE].

[7] A. Jevicki and B. Sakita, *Collective field approach to the large N limit: Euclidean field theories*, *Nucl. Phys. B* **185** (1981) 89 [INSPIRE].

[8] J.M. Maldacena, *The large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200] [INSPIRE].

[9] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett. B* **428** (1998) 105 [hep-th/9802109] [INSPIRE].

JHEP04(2024)079

[10] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150] [INSPIRE].

[11] A.M. Polyakov, *The wall of the cave*, *Int. J. Mod. Phys. A* **14** (1999) 645 [hep-th/9809057] [INSPIRE].

[12] J. Polchinski, L. Susskind and N. Toumbas, *Negative energy, superluminosity and holography*, *Phys. Rev. D* **60** (1999) 084006 [hep-th/9903228] [INSPIRE].

[13] R.R. Metsaev, *Light cone form of field dynamics in anti-de Sitter space-time and AdS/CFT correspondence*, *Nucl. Phys. B* **563** (1999) 295 [hep-th/9906217] [INSPIRE].

[14] R.R. Metsaev, *Shadows, currents and AdS*, *Phys. Rev. D* **78** (2008) 106010 [arXiv:0805.3472] [INSPIRE].

[15] R.R. Metsaev, *CFT adapted gauge invariant formulation of arbitrary spin fields in AdS and modified de Donder gauge*, *Phys. Lett. B* **671** (2009) 128 [arXiv:0808.3945] [INSPIRE].

[16] R.R. Metsaev, *CFT adapted gauge invariant formulation of massive arbitrary spin fields in AdS*, *Phys. Lett. B* **682** (2010) 455 [arXiv:0907.2207] [INSPIRE].

[17] R.R. Metsaev, *Anomalous conformal currents, shadow fields and massive AdS fields*, *Phys. Rev. D* **85** (2012) 126011 [arXiv:1110.3749] [INSPIRE].

[18] R.R. Metsaev, *CFT adapted approach to massless fermionic fields, AdS/CFT, and fermionic conformal fields*, arXiv:1311.7350 [INSPIRE].

[19] R. de Mello Koch, A. Jevicki, K. Jin and J.P. Rodrigues, *AdS_4/CFT_3 construction from collective fields*, *Phys. Rev. D* **83** (2011) 025006 [arXiv:1008.0633] [INSPIRE].

[20] A. Jevicki, K. Jin and Q. Ye, *Collective dipole model of AdS/CFT and higher spin gravity*, *J. Phys. A* **44** (2011) 465402 [arXiv:1106.3983] [INSPIRE].

[21] A. Jevicki, K. Jin and Q. Ye, *Bi-local model of AdS/CFT and higher spin gravity*, in the proceedings of the 11$^{\text{th}}$ *workshop on non-perturbative quantum chromodynamics*, (2011) [arXiv:1112.2656] [INSPIRE].

[22] R. de Mello Koch et al., *$S = 1$ in $O(N)$/HS duality*, *Class. Quant. Grav.* **30** (2013) 104005 [arXiv:1205.4117] [INSPIRE].

[23] R. de Mello Koch, A. Jevicki, J.P. Rodrigues and J. Yoon, *Holography as a gauge phenomenon in higher spin duality*, *JHEP* **01** (2015) 055 [arXiv:1408.1255] [INSPIRE].

[24] R. de Mello Koch, A. Jevicki, J.P. Rodrigues and J. Yoon, *Canonical formulation of $O(N)$ vector/higher spin correspondence*, *J. Phys. A* **48** (2015) 105403 [arXiv:1408.4800] [INSPIRE].

[25] M. Mulokwe and J.P. Rodrigues, *Large N bilocals at the infrared fixed point of the three dimensional $O(N)$ invariant vector theory with a quartic interaction*, *JHEP* **11** (2018) 047 [arXiv:1808.00042] [INSPIRE].

[26] R. de Mello Koch, A. Jevicki, K. Suzuki and J. Yoon, *AdS maps and diagrams of bi-local holography*, *JHEP* **03** (2019) 133 [arXiv:1810.02332] [INSPIRE].

[27] R. de Mello Koch, E. Gandote, N.H. Tahiridimbisoa and H.J.R. Van Zyl, *Quantum error correction and holographic information from bilocal holography*, *JHEP* **11** (2021) 192 [arXiv:2106.00349] [INSPIRE].

[28] R. de Mello Koch and G. Kemp, *Holography of information in AdS/CFT*, *JHEP* **12** (2022) 095 [arXiv:2210.11066] [INSPIRE].

JHEP04(2024)079

[29] C. Johnson, M. Mulokwe and J.P. Rodrigues, *Constructing the bulk at the critical point of three-dimensional large N vector theories*, Phys. Lett. B **829** (2022) 137056 [arXiv:2201.10214] [INSPIRE].

[30] R. de Mello Koch, *Microscopic entanglement wedges*, JHEP **08** (2023) 056 [arXiv:2307.05032] [INSPIRE].

[31] R. de Mello Koch, *Gravitational dynamics from collective field theory*, JHEP **10** (2023) 151 [arXiv:2309.11116] [INSPIRE].

[32] E. Mintun and J. Polchinski, *Higher spin holography, RG, and the light cone*, arXiv:1411.3151 [INSPIRE].

[33] O. Aharony, S.M. Chester and E.Y. Urbach, *A derivation of AdS/CFT for vector models*, JHEP **03** (2021) 208 [arXiv:2011.06328] [INSPIRE].

[34] B. Czech, J.L. Karczmarek, F. Nogueira and M. Van Raamsdonk, *The gravity dual of a density matrix*, Class. Quant. Grav. **29** (2012) 155009 [arXiv:1204.1330] [INSPIRE].

[35] M. Headrick, V.E. Hubeny, A. Lawrence and M. Rangamani, *Causality & holographic entanglement entropy*, JHEP **12** (2014) 162 [arXiv:1408.6300] [INSPIRE].

[36] A.C. Wall, *Maximin surfaces, and the strong subadditivity of the covariant holographic entanglement entropy*, Class. Quant. Grav. **31** (2014) 225007 [arXiv:1211.3494] [INSPIRE].

[37] D.L. Jafferis, A. Lewkowycz, J. Maldacena and S.J. Suh, *Relative entropy equals bulk relative entropy*, JHEP **06** (2016) 004 [arXiv:1512.06431] [INSPIRE].

[38] X. Dong, D. Harlow and A.C. Wall, *Reconstruction of bulk operators within the entanglement wedge in gauge-gravity duality*, Phys. Rev. Lett. **117** (2016) 021601 [arXiv:1601.05416] [INSPIRE].

[39] J. Cotler et al., *Entanglement wedge reconstruction via universal recovery channels*, Phys. Rev. X **9** (2019) 031011 [arXiv:1704.05839] [INSPIRE].

[40] A. Laddha, S.G. Prabhu, S. Raju and P. Shrivastava, *The holographic nature of null infinity*, SciPost Phys. **10** (2021) 041 [arXiv:2002.02448] [INSPIRE].

[41] C. Chowdhury, O. Papadoulaki and S. Raju, *A physical protocol for observers near the boundary to obtain bulk information in quantum gravity*, SciPost Phys. **10** (2021) 106 [arXiv:2008.01740] [INSPIRE].

[42] S. Raju, *Lessons from the information paradox*, Phys. Rept. **943** (2022) 1 [arXiv:2012.05770] [INSPIRE].

[43] S. Raju, *Failure of the split property in gravity and the information paradox*, Class. Quant. Grav. **39** (2022) 064002 [arXiv:2110.05470] [INSPIRE].

[44] *The Black Hole Information Paradox YouTube channel*, https://www.youtube.com/channel/UCJ-YA8uOwUlACfn49iD7TvA.

[45] *Symmetric space Wikipedia page*, .

[46] M. Miyaji et al., *Continuous multiscale entanglement renormalization ansatz as holographic surface-state correspondence*, Phys. Rev. Lett. **115** (2015) 171602 [arXiv:1506.01353] [INSPIRE].

[47] H. Verlinde, *Poking holes in AdS/CFT: bulk fields from boundary states*, arXiv:1505.05069 [INSPIRE].

[48] Y. Nakayama and H. Ooguri, *Bulk locality and boundary creating operators*, JHEP **10** (2015) 114 [arXiv:1507.04130] [INSPIRE].

JHEP04(2024)079

[49] D. Kabat and G. Lifschytz, *Local bulk physics from intersecting modular Hamiltonians*, JHEP **06** (2017) 120 [arXiv:1703.06523] [INSPIRE].

[50] R. de Mello Koch, P. Roy and H.J.R. Van Zyl, *Holography of a single free matrix*, work in progress.

[51] D. Das, S.R. Das, A. Jevicki and Q. Ye, *Bi-local construction of* Sp$(2N)/dS$ *higher spin correspondence*, JHEP **01** (2013) 107 [arXiv:1205.5776] [INSPIRE].

[52] R. de Mello Koch and J.P. Rodrigues, *Systematic* $1/N$ *corrections for bosonic and fermionic vector models without auxiliary fields*, Phys. Rev. D **54** (1996) 7794 [hep-th/9605079] [INSPIRE].

[53] K. Kamimura, *Elimination of relative time in bilocal model*, Prog. Theor. Phys. **58** (1977) 1947 [INSPIRE].