

An integrable road to a perturbative plateau

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ABSTRACT: As has been known since the 90s, there is an integrable structure underlying two-dimensional gravity theories. Recently, two-dimensional gravity theories have regained an enormous amount of attention, but now in relation with quantum chaos — superficially nothing like integrability. In this paper, we return to the roots and exploit the integrable structure underlying dilaton gravity theories to study a late time, large $e^{S_{\text{BH}}}$ double scaled limit of the spectral form factor. In this limit, a novel cancellation due to the integrable structure ensures that at each genus g the spectral form factor grows like T^{2g+1} , and that the sum over genera converges, realising a perturbative approach to the late-time plateau. Along the way, we clarify various aspects of this integrable structure. In particular, we explain the central role played by ribbon graphs, we discuss intersection theory, and we explain what the relations with dilaton gravity and matrix models are from a more modern holographic perspective.

KEYWORDS: 2D Gravity, AdS-CFT Correspondence, Black Holes

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1 Introduction

Without dynamical gravity, correlation functions measured in black hole backgrounds decay all the way to zero for late times. However, as pointed out initially by Maldacena [1], in any full theory of quantum gravity this should not be possible. This is because black holes themselves are finite entropy quantum systems.

This predicted behavior for late time correlators was later sharpened by [2]. They considered a toy model of the thermal two point function, known as the spectral form factor (H is the Hamiltonian of the holographically dual quantum system)

$$Z(\beta + iT, \beta - iT) = \text{Tr}\left(e^{-(\beta+iT)H}\right) \text{Tr}\left(e^{-(\beta-iT)H}\right) = \sum_{i=1}^{\dim(H)} \sum_{j=1}^{\dim(H)} e^{-\beta(E_i+E_j)} e^{-iT(E_i-E_j)}, \tag{1.1}$$

and argued that its late-time behavior is universally given by a ramp-and plateau structure (on time scales $\sim e^S$ with S the black hole entropy)¹

$$Z(\beta + iT, \beta - iT) = \int_0^\infty dE e^{-2\beta E} \min(\rho(E), T/2\pi), \quad \rho(E) = e^{S(E)}. \tag{1.2}$$

In particular for $T \rightarrow \infty$ this goes to a *non-zero* constant $Z(2\beta)$. That this does not decay all the way to zero indeed follows from the fact that black holes are discrete (or finite entropy) quantum systems: the value $Z(2\beta)$ arises from the terms $i = j$ in the sum (1.1). If we instead had a continuous spectrum those terms would measure zero, so the correlator would instead indeed decay to zero.

It is interesting to ask how the bulk gravitational path integral reproduces this ramp-and plateau structure; one, because they are universal; and two, because the plateau is a signature of microstructure (and hence unitarity) in gravity. In gravity, one computes the spectral form factor by path integrating over all geometries with two asymptotically AdS boundaries (with appropriate boundary conditions implementing the $\beta \pm iT$). It was found [3] that the linear ramp (the $T/2\pi$ piece in (1.2)) is explained by wormhole geometries connecting both boundaries²

$$Z(\beta + iT, \beta - iT)_{\text{conn}} \supset \text{1 wormhole} \tag{1.3}$$

¹Actually, this simple ramp-and plateau structure is only visible after some time-averaging, or other types of averaging [2]. We will exclusively be concerned with the gravitational interpretation of such smeared quantities in this work.

²Similarly wormholes were found to be important for understanding late-time correlators [4–7], the Page curve [8, 9], the fate of late-time infalling observers [10] and more.

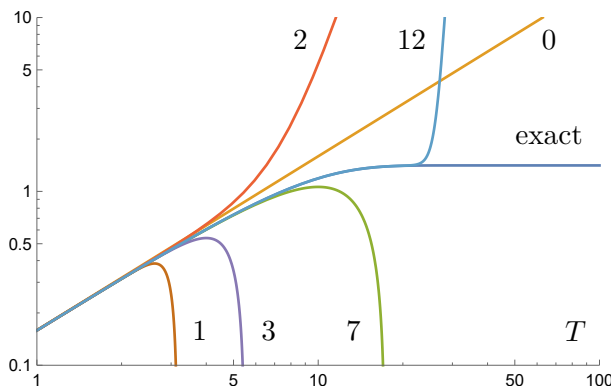


Figure 1. The double scaling limit of the sum over genus g wormholes in the Airy model, up to $g = g_{\max}$ (numbers shown) with $e^{S_0} = 10$ and $\beta = 1/2$.

The factual case-by-case limitation of the maximal power of T to T^{2g+1} seems miraculous. It depends on very nontrivial cancellations in large sums involving the expansion coefficients of these polynomial volumes $V_{g,2}(b_1, b_2)$. To derive the universality of T^{2g+1} (1.6) using the gravitational path integral, we thus need to prove that these cancellations in the volumes happen at any genus g , and for any dilaton gravity theory (1.4).

In section 3 and section 4 we derive these cancellations using the relations between our gravity models (1.4) and an infinite set of differential equations, known as the KdV hierarchy. In particular, one can use these differential equations to prove [26, 27] some set of cancellations in sums of intersection numbers $\langle \tau_{d_1} \dots \tau_{d_n} \rangle$, which are essentially integrals of products of specific two forms over the moduli space of Riemann surfaces. We then prove that those cancellations in intersection numbers imply the cancellations that we need in the volumes, for all theories (1.4). One key ingredient is a duality between exponentials of operators τ_k and changes in the dilaton gravity potential, which we find in section 5

$$\exp\left(\sum_{k=2}^{\infty} t_k \tau_k\right) \Leftrightarrow \exp\left(\int d^2x \sqrt{g} \sum_{k=2}^{\infty} \frac{(-1)^k}{(2k-1)!!} t_k \Phi^{2k} e^{-2\pi\Phi}\right). \quad (1.9)$$

This can be viewed as an application of the fact that intersection numbers can be viewed as correlators of cusp defects in JT gravity, which we also derive. This also leads to a new understanding of the KdV equations directly in terms of dilaton gravity variables: they express how observables change when one alters certain parameters in the gravitational action.

In section 3 we give a gentle introduction to intersection numbers and their relation with gravity, with more intuitive comments gathered in appendix C, as we did not want to assume that the readers were familiar with these more mathematical constructions.

Relation with other work. Part of section 2 is based upon discussions with Saad, Shenker, Stanford and Yang. In particular, the observation that for Airy and JT gravity equation (2.2) and (2.5) gives the exact spectral form factor in the limit $T \rightarrow \infty$ and $e^{S_0} \rightarrow \infty$ with $T e^{-S_0}$ fixed, and that this leads to a series with a non-zero radius of convergence (matched by topological recursion) which gives the plateau (A.1) and (A.5), is due to

Saad, Stanford, Yang and Yao [22]. The fact that equation (2.2) implies cancellations in Weil-Petersson volumes for JT gravity was independently observed and investigated by [28].

2 The plateau from the perturbative sum over wormhole geometries

In this section we derive a simple integral representation of the spectral form factor in the limit T and e^{S_0} go to infinity keeping their ratio fixed (the τ -scaling limit [21, 29]), as advertised in the introduction. This integral admits a power series in T^{2g+1} , which is reproduced by computing the genus g wormhole amplitudes in gravity. In this τ -scaling limit, the sum over genus converges to the plateau, without the need for non-perturbative (in e^{S_0}) corrections.

2.1 Powers of time

The connected spectral form factor is related with the spectral correlation via a Laplace transform

$$Z(\beta + iT, \beta - iT)_{\text{conn}} = \int_{-\infty}^{+\infty} dE_1 \int_{-\infty}^{+\infty} dE_2 e^{-\beta(E_1+E_2)} e^{iT(E_1-E_2)} \rho(E_1, E_2)_{\text{conn}}. \quad (2.1)$$

We will consider 2d dilaton gravities with an exact dual description as random matrix theories [11, 18, 19, 30]. In the full matrix integral, $\rho(E_1, E_2)_{\text{conn}}$ is a smooth function of E_1 and E_2 , except for a contact term. In random matrix theory that in the limit $T \rightarrow \infty$ and $e^{S_0} \rightarrow \infty$ with $\tau = Te^{-S_0}$ fixed [15, 21, 22, 29] (τ -scaling limit), this integral simplifies to

$$Z(\beta + iT, \beta - iT)_{\text{conn}} = \int_{-\infty}^{+\infty} dE_1 \int_{-\infty}^{+\infty} dE_2 e^{-\beta(E_1+E_2)} e^{iT(E_1-E_2)} \rho(E_1, E_2)_{\text{conn eff}}, \quad (2.2)$$

where the effective spectral correlation features the sine kernel [31]⁴

$$\rho(E_1, E_2)_{\text{conn eff}} = \delta(E_1 - E_2)\rho_0(E) - \frac{\sin^2(\pi\rho_0(E)(E_1 - E_2))}{\pi^2(E_1 - E_2)^2}. \quad (2.3)$$

Changing variables to $E_1 - E_2 = \omega$ and $E_1 + E_2 = 2E$ this becomes

$$Z(\beta+iT, \beta-iT)_{\text{conn}} = \int_{-\infty}^{+\infty} dE e^{-2\beta E} \rho_0(E) - \int_{-\infty}^{+\infty} dE e^{-2\beta E} \int_{-\infty}^{+\infty} d\omega e^{iT\omega} \frac{\sin^2(\pi\rho_0(E)\omega)}{\pi^2\omega^2}. \quad (2.4)$$

The ω -integral is a standard Fourier-transform, which gives the familiar ramp-and plateau [2]

$$\begin{aligned} Z(\beta + iT, \beta - iT)_{\text{conn}} &= \int_{-\infty}^{+\infty} dE e^{-2\beta E} \min(\rho_0(E), T/2\pi) \\ &= \int_{-\infty}^{+\infty} dE e^{-2\beta E} \rho_0(E) - \int_{E(T)}^{+\infty} dE e^{-2\beta E} (\rho_0(E) - T/2\pi), \\ \rho_0(E(T)) &= T/2\pi, \end{aligned} \quad (2.5)$$

⁴The generic proof of this fact goes through Efetov's non-linear sigma model description of random matrix theory [15, 32], see also recently [13, 14, 33, 34]. We can alternatively use D-brane calculus as explained in [11, 12].

with $E(T)$ determined by solving $\rho_0(E(T)) = T/2\pi$.⁵ Choosing the energy axis such that $\rho_0(0) = 0$,⁶ and using integration by parts, we arrive at

$$\begin{aligned} Z(\beta + iT, \beta - iT)_{\text{conn}} &= \int_0^{E(T)} dE e^{-2\beta E} \rho_0(E) + \frac{1}{2\beta} \rho_0(E(T)) e^{-2\beta E(T)} \\ &= \frac{1}{2\beta} \int_0^{T/2\pi} d\rho_0 e^{-2\beta E(\rho_0)}. \end{aligned} \tag{2.6}$$

This is the final formula for the spectral form factor in the τ -scaling limit. One might say that we have put non-perturbative information into the calculation at this point, but here we just use the matrix model answers. The objective is to reproduce the final formula from gravity and explain its convergence properties, solely using perturbation theory.

As already alluded to, we consider 2d dilaton gravity (1.4) with a matrix integral dual and for these theories the spectrum has an expansion in powers of $E^{k+1/2}$ [35–37],

$$\rho_0(E) = \frac{e^{S_0}}{2\pi} \sum_{k=0}^{\infty} f_k E^{1/2+k} = \frac{e^{S_0}}{2\pi} \left(w + \sum_{k=1}^{\infty} f_k w^{2k+1} \right), \tag{2.7}$$

where we've introduced the notation $E = w^2$, and without loss of generality we have set $f_0 = 1$ (by changing S_0). The function in round brackets will be referred to as $f(w)$ and its inverse (which exists by assumption) can be obtained using the Lagrange inversion theorem. Notably, this has a powers series expansion in odd powers of f too

$$w(f) = f + \sum_{k=1}^{\infty} w_k f^{2k+1}, \tag{2.8}$$

and as a result the Taylor series

$$e^{-2\beta w(f)^2} = 1 - 2\beta f^2 + 4\pi\beta \sum_{n=2}^{\infty} (2n+1) P_{n-1}(\beta) f^{2n}, \tag{2.9}$$

is even in f . Here $P_n(\beta)$ is a degree n polynomial in β , which can easily be computed explicitly for any fixed n . The tau-scaled spectral form factor (2.6) then expands as

$$\begin{aligned} Z(\beta + iT, \beta - iT)_{\text{conn}} &= \frac{e^{S_0}}{4\pi\beta} \int_0^{Te^{-S_0}} df e^{-2\beta E(f)} \\ &= \sum_{g=0}^{\infty} P_{g-1}(\beta) T^{2g+1} e^{-2gS_0} = \frac{T}{4\pi\beta} - \frac{1}{6\pi} T^3 e^{-2S_0} + \dots \end{aligned} \tag{2.10}$$

This is the main result of this section. Computing the tau-scaled limit of the (connected) spectral form factor using the non-perturbatively exact matrix integral formulation of 2d dilaton gravities results in a *universal* series expansion in $T^{2g+1} e^{-2gS_0}$. It is tempting to interpret the term at order e^{-2gS_0} as the τ -scaling limit of the perturbative genus g wormhole amplitude in gravity. We will confirm below in section 2.2 that this is indeed the case. Let us point out some noteworthy features of this expansion.

⁵Here we assume $\rho_0(E)$ grows monotonically, in order to have a unique solution. The conclusion (2.10) remains true for non-monotonic spectra though, see appendix A.

⁶Some dilaton gravities have a non-zero threshold energy, but such modifications are straightforward to incorporate.

1. In the limit $T \rightarrow \infty$ this series approaches the (leading order) plateau

$$\frac{e^{S_0}}{4\pi\beta} \int_0^\infty df e^{-2\beta w(f)^2} = \int_0^\infty dE \rho_0(E) e^{-2\beta E} = Z_0(2\beta). \quad (2.11)$$

Depending on the case, the series expansion can have a finite or an infinite radius of convergence (as function of T). However in all cases (with invertible spectrum) the series converges to the exact answer for small enough T , and that answer has a unique analytic continuation to all positive T . The examples in appendix A should clarify this.

So, in the τ -scaling limit, the plateau is *perturbatively* accessible in the genus expansion. Notice also that this is different from Borel resummation, since we are taking a limit where some badly growing terms in the genus g go away.

2. The first two terms in the expansion (2.10) are theory-independent, meaning they do not depend on the f_k . The polynomial $P_n(\beta)$ depends only on $f_2 \dots f_n$, so the higher the genus the more we probe the UV part of the spectrum. The sign of each term depends on these f_k (and β) and are not-universal.
3. Theories with a nonzero Hagedorn temperature never reach a plateau when $2\beta < \beta_H$, because for these cases the would-be plateau $Z_0(2\beta)$ is divergent. We can appreciate this also using

$$\partial_T Z(\beta + iT, \beta - iT)_{\text{conn}} = \frac{1}{4\pi\beta} e^{-2\beta E(Te^{-S_0})} \quad (2.12)$$

For Hagedorn spectra $f(E) \sim e^{\beta_H E}$ one finds $E(f) = \log(f)/\beta_H + \text{constant}$, and hence indeed the τ -scaled spectral form factor grows without bounds for late times. This Hagedorn growth at high energies only occurs for non-local theories, such as string theories.

4. The sine kernel (2.3) or the associated level repulsion is generally considered to be the hallmark feature of chaotic quantum systems, it is essentially synonymous with random matrix universality. The scaling T^{2g+1} contains the same information as this sine kernel, and we consider it therefore to be the real-time version of random matrix universality. Therefore these should be an argument why in any gravity model (beyond 2d dilaton gravity) there are contributions growing like T^{2g+1} in the gravitational path integral. This explanation we think should be intrinsically Lorentzian. A proposal based on topology changing processes for how these powers of time can be explained, analogous to the double-cone [3], will be presented elsewhere [38].

We further discuss this integral and its series expansion (2.10) for several examples, as well as the modifications for non-monotonic spectra, in appendix A, in order not to disrupt the flow of the paper.

2.2 Euclidean wormholes and cancellations in volumes

We now switch gears, and move to the gravitational computation. For this we do an expansion of the connected two-boundary observable in genus

$$Z(\beta + iT, \beta - iT)_{\text{conn}} = \sum_{g=0}^{\infty} e^{-2gS_0} Z_g(\beta + iT, \beta - iT)_{\text{conn}} + \text{non-perturbative}, \quad (2.13)$$

where the genus g gravitational wormhole amplitude is computed as

$$Z_g(\beta_1, \beta_2)_{\text{conn}} = \int_0^\infty db_1 b_1 \frac{1}{2\pi^{1/2} \beta_1^{1/2}} e^{-\frac{b_1^2}{4\beta_1}} \int_0^\infty db_2 b_2 \frac{1}{2\pi^{1/2} \beta_2^{1/2}} e^{-\frac{b_2^2}{4\beta_2}} V_{g,2}(b_1, b_2), \quad (2.14)$$

with $V_{g,2}(b_1, b_2)$ the (deformed) Weil-Petersson volume⁷ for a genus g wormhole with (geodesic) boundaries with length b_1 and b_2 . They are symmetric polynomial in b_1^2 and b_2^2 with a maximum total degree of $3g - 1$. The way to compute these polynomials $V_{g,2}(b_1, b_2)$ in practice is using the Eynard-Orantin topological recursion [11, 23, 25, 39], with a spectral curve $y(z)$ (originating from (2.7)) given by⁸

$$y(z) = \frac{1}{2} \sum_{k=0}^{\infty} (-1)^k f_k z^{1+2k}. \quad (2.15)$$

We stress that these polynomials are the result of doing the gravitational path integral over all metrics which are topologically a genus g connected wormhole in the dilaton gravity with genus zero spectrum (2.7). In order to compare (2.14) to (2.10), we compute (2.14) explicitly by writing the WP volume as

$$V_{g,2}(b_1, b_2) = \sum_{d_1, d_2=0}^{d_1+d_2=3g-1} V_{g,2}^{d_1, d_2} \frac{b_1^{2d_1}}{4^{d_1} d_1!} \frac{b_2^{2d_2}}{4^{d_2} d_2!}, \quad (2.16)$$

with some symmetric constants $V_{g,2}^{d_1, d_2}$. We obtain

$$Z_g(\beta_1, \beta_2)_{\text{conn}} = \frac{1}{\pi} \sum_{d_1, d_2=0}^{d_1+d_2=3g-1} V_{g,2}^{d_1, d_2} \beta_1^{1/2+d_1} \beta_2^{1/2+d_2}. \quad (2.17)$$

Continuing to Lorentzian signature and putting $\beta = 0$ (see below for finite β) this becomes

$$Z_g(iT, -iT)_{\text{conn}} = \frac{1}{\pi} \sum_{q=0}^{(3g-1)/2} (-1)^q T^{2q+1} \sum_{d=0}^{2q} (-1)^d V_{g,2}^{d, 2q-d}. \quad (2.18)$$

Comparing to (2.10), we claim that in the τ -scaling limit for generic dilaton gravities we should have

$$Z_g(iT, -iT)_{\text{conn}} = P_{g-1}(0) T^{2g+1}. \quad (2.19)$$

⁷Here we mean a slight generalisation of the Weil-Petersson volumes, which includes cases where we have summed over defects [18, 19]. One might call these deformed Weil-Petersson volumes.

⁸Only for the spectral curve $y(z) = \sin(2\pi z)/(4\pi)$ do we obtain the true Weil-Petersson volumes.

This is rather surprising because in (2.18) we have powers of T that are larger than $2g + 1$, and hence should dominate the τ -scaling limit. This indicates a novel cancellation between the various coefficients of the volumes $V_{g,2}(b_1, b_2)$! More precisely we are claiming that for *any* theory

$$\sum_{d=0}^{2q} (-1)^d V_{g,2}^{d,2q-d} = 0, \quad q > g, \quad \sum_{d=0}^{2g} (-1)^d V_{g,2}^{d,2g-d} = \pi(-1)^g P_{g-1}(0). \quad (2.20)$$

These cancellations for $q > g$ are quite surprising from the point of view of these polynomials, but they are nevertheless true, as one can check case by case. The simplest example is the genus $g = 3$ wormhole with $q = 4$. Based on dimensional analysis, one expects a term proportional to T^9 . However, this term is absent because of the cancellation⁹

$$\frac{1}{2} \sum_{d=0}^8 (-1)^d V_{3,2}^{d,8-d} = \frac{1}{324} - \frac{5}{324} + \frac{77}{1620} - \frac{503}{5670} + \frac{607}{11340} = 0. \quad (2.21)$$

This example is theory independent since it only depends on $f_1 = 1$, but we stress that more in general the $V_{g,2}^{d_1,d_2}$ depend on all f_k such that (2.10) predicts theory-dependent cancellations. One checks easily case by case that cancellations indeed occur, more examples are provided in appendix A.

To compute the τ -scaled spectral form factor for finite β , we can introduce symmetric polynomials $e_2 = \beta_1\beta_2$ and $e_1 = \beta_1 + \beta_2$, and reorganize (2.17) into

$$\begin{aligned} Z_g(\beta + iT, \beta - iT)_{\text{comm}} &= \frac{1}{\pi} \sum_{q=0}^{(3g-1)/2} (-1)^q e_2^{q+1/2} \sum_{m=2q}^{3g-1} e_1^{m-2q} \\ &\times 2 \sum_{d=0}^q (-1)^d V_{g,2}^{d,m-d} \frac{m/2 - d}{m - q - d} \frac{(m - q - d)!}{(q - d)!(m - 2q)!}, \end{aligned} \quad (2.22)$$

where now $e_2 = \beta^2 + T^2$ and $e_1 = 2\beta$.¹⁰ The constraint that this amplitude grows no faster than T^{2g+1} imposes that the coefficient of $e_2^{q+1/2}$ vanishes for $q > g$

$$\sum_{d=0}^q (-1)^d V_{g,2}^{d,m-d} \frac{m/2 - d}{m - q - d} \frac{(m - q - d)!}{(q - d)!(m - 2q)!} = 0, \quad q > g, m \geq 2q, \quad (2.23)$$

and furthermore there is a precise theory-*dependent* prediction for $q = g$

$$2 \sum_{m=2g}^{3g-1} (2\beta)^{m-2g} \sum_{d=0}^g (-1)^d V_{g,2}^{d,m-d} \frac{m/2 - d}{m - g - d} \frac{(m - g - d)!}{(g - d)!(m - 2g)!} = \pi(-1)^g P_{g-1}(\beta). \quad (2.24)$$

The case $m = 2q$ reduces to the $\beta = 0$ constraints (2.20) (taking into account the factor $1/2$ mentioned above). The constraints with $m > 2q$ represent additional cancellations which we claim are satisfied by the genus g wormhole amplitudes of *all* dilaton gravities (with matrix integral duals).

⁹For a list of volumes see for instance [40].

¹⁰For the term $m = 2q$ the term with $d = q$ receives an extra $1/2$, which we left implicit.

A happy consequence of the τ -scaling limit is that all terms with $q < g$ are subleading. These terms contain lower powers of T , but their coefficients are growing more and more rapidly the lower the power of T , ultimately growing like $(2g)!$. This was conjectured in [41] and proven in [42], see also (228) in [11] as Taylor series in b^2 . We make some further comments on this in section 6.1. Including such terms makes the sum over genus very complicated and one needs to resort to a Borel resummation in order to study it. The τ -scaling limit thus gets rid of these intricacies, by selecting only the $q = g$ term at each genus. This is why the series is convergent, in part, and non-perturbative corrections are *not* present.

We can also understand this from Efetov's non-linear sigma model [13–15, 32] (which described the τ -scaling limit) formulation of these quantities. The resolvent is represented by a double integral over variables s_{11} and s_{22} . The energy dependence in the action is $e^{S_0} E(s_{11} - s_{22})$, and the double integral is dominated by two saddle-points. The expansion around the first saddle grows like $(2g)!$ in the energy domain, whereas the expansion around the second saddle gives a non-perturbative (in e^{S_0}) correction. We can now inverse Laplace transform this to get a non-linear sigma model computation of $Z(\beta)$. The energy integral gives a delta function $\delta(\beta - e^{S_0}(s_{11} - s_{22}))$, which collapses the double integral to a single integral. This single integral, notably, has just one saddle point, expanding around which produces the sum over wormholes. Because there is only one saddle, there are no non-perturbative corrections in the thermal ensemble. This calculation extends to the connected spectral form factor, it becomes a bit messier but the conclusion remains the same.

In summary, there are two reasons why this genus expansion is convergent: the τ -scaling limit gets rid of terms that grow too fast, and the genus expansion in the thermal ensemble is more convergent than that in the microcanonical ensemble.¹¹

3 Cancellations in topological gravity

In the previous section we found that there need to be non-trivial cancellations between the coefficients of (deformed) Weil-Petersson volumes. In this section and the next we explain how these relations come about. First we will consider topological gravity, i.e. the Airy model, where many of these cancellations are known [26, 27] in the context of intersection theory. In section 4 we will use open-closed duality to show that these very cancellations of [26, 27] actually imply the growth $T^{2g+1}e^{-2gS_0}$ for all double-scaled matrix integrals, including all models of dilaton gravity [18, 19, 30, 34, 43, 44].

We remind the reader that this $T^{2g+1}e^{-2gS_0}$ is the type of series that we found earlier will converge to the plateau. So, in part, these cancellations explain why the sum over genus g wormholes converges (in the τ -scaling limit) to the plateau.

¹¹One other way to see that perturbation theory in the microcanonical ensemble is more complicated is because the integrals over b with the density of states of the trumpet are not convergent and need to be defined using analytic continuation. This is true in the Airy case, in JT an additional complication is that the volumes themselves contain pieces that grow like $(2g)!$. The τ -scaling limit cures the second, but not the first complication, which is cured by going to the canonical ensemble.

3.1 Ribbon graphs and intersection numbers

In 1991, Witten [45] showed that topological gravity is related to a matrix model with spectral density

$$\rho_0(E) = e^{S_0} \frac{E^{1/2}}{2\pi}. \tag{3.1}$$

This spectral density occurs for any dilaton gravity at small enough energy, and is thus an important first case to consider.

Small energy means that we are at small temperature and so huge asymptotic boundaries. Since the partition functions $Z(\beta_1, \dots, \beta_n)$ are given by integrals over the (deformed) Weil-Petersson volumes (2.14), one might wonder what large β_i implies for those volumes. It is straightforward to see from the trumpet partition function [11]

$$Z_{\text{trumpet}}(\beta, b) = \frac{1}{2\pi^{1/2}\beta^{1/2}} e^{-\frac{b^2}{4\beta}}, \tag{3.2}$$

that we now get contributions mostly from the region where $b^2 \sim \beta$, hence b^2 is large. In other words, we can consider the leading large b_i terms in the (deformed) WP volumes. These are theory-independent terms which have a clear geometric interpretation, as we now point out, see also appendix C.3.

Since they are theory independent, we can start by considering the undeformed WP volumes, which compute the volumes of moduli spaces of hyperbolic Riemann surfaces with $R = -2$. As the boundaries of these Riemann surfaces are $K = 0$ geodesics we can use Gauss-Bonnet to show that the area of such Riemann surfaces of genus g and n geodesic boundary is given by

$$A = 2\pi(2g + n - 2), \tag{3.3}$$

and hence is a *constant*. This means that when we take the b_i very large, the Riemann surface needs to become a collection of very thin strips glued along trivalent vertices (trivalent because we can consider the pair of pants decomposition of the Riemann surface). In other words, the (deformed or undeformed) WP volumes reduce to the volume of moduli space of trivalent ribbon graphs, see also figure 2.

These volumes are simple to compute, one can parameterize the moduli space of ribbon graphs for a certain graph Γ by the lengths ℓ_j of the edges of the graph, those lengths are constrained only by the fact that the sum of the lengths ℓ_{j_i} of the edges forming a certain boundary i add up to b_i . Accounting for the standard symmetry factor in Feynman diagrams one thus obtains

$$V_{\Gamma, g, n}(b_1 \dots b_n) = \frac{1}{|\text{Aut}(\Gamma_{g, n})|} \prod_{j=1}^{6g-6+3n} \int_0^\infty d\ell_j \prod_{i=1}^n \delta\left(b_i - \sum_{j_i} \ell_{j_i}\right). \tag{3.4}$$

Summing over all $\Gamma_{g, n}$ with a certain genus g can then indeed be checked to reproduce the Airy volumes. But how is this related to intersection numbers? The crux of Kontsevich's seminal work [46] was the realization that these integrals can be represented equivalently as (see also [47, 48])

$$V_{g, n}(b_1 \dots b_n) = \sum_{\Gamma_{g, n}} V_{\Gamma_{g, n}}(b_1 \dots b_n) = \int_{\overline{\mathcal{M}}_{g, n}} \exp\left(\frac{1}{2} \sum_{i=1}^n b_i^2 \psi_i\right). \tag{3.5}$$

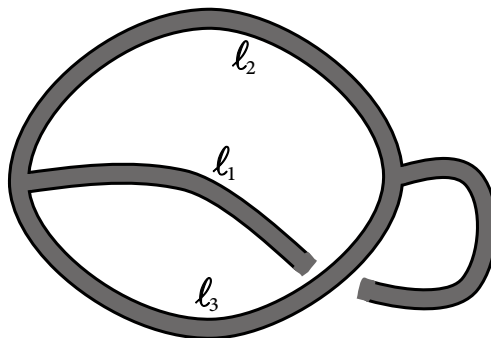


Figure 2. The torus with one puncture $V_{1,1}(b)$ becomes a ribbon graph, when the hole b becomes large. In this limit the moduli space is spanned by the lengths of the ribbons with the constraint $b = 2\ell_1 + 2\ell_2 + 2\ell_3$, and we integrate with the flat measure to recover $V_{1,1}(b) = b^2/48$.

The right hand side is an integral over the moduli space of Riemann surfaces. Just like for the ordinary (deformed) WP volumes we include degenerate Riemann surfaces, which results in the so-called Deligne-Mumford compactification of the moduli space of Riemann surfaces $\overline{\mathcal{M}}_{g,n}$. Here the ψ_i denote the first Chern classes $c_1(\mathcal{L}_i)$ of some bundles of one forms \mathcal{L}_i that are constructed as follows, see also appendix C. Take a punctured Riemann surface and consider the cotangent space at each puncture $x = x_i$. For each x_i these cotangent spaces depend on the moduli of the Riemann surface under consideration. The collection of all those spaces is a bundle \mathcal{L}_i over $\overline{\mathcal{M}}_{g,n}$.

The key to proving the relation (3.5) is to realize that one can choose a specific one form α in \mathcal{L}_i for which the curvature two-form $d\alpha = c_1(\mathcal{L}_i)$ takes on a simple form [46]. The first step is to use the fact that there is an equivalence between the moduli space of Riemann surfaces and the moduli spaces of ribbon graphs [49–51]. Concretely, one can associate to every Riemann surface with n boundaries of lengths b_i a unique ribbon graph with lengths ℓ_j with again the sum of ℓ_{j_i} constrained to b_i . This map is provided by the Jenkins-Strebel quadratic differential [51]. So we can exchange the fundamental domain in terms of the Teichmüller coordinates b_i and τ_i for a sum over ribbon graphs with the simple constraint that the sum of lengths ℓ_{j_i} is constrained to b_i .¹² In these coordinates, Kontsevich found a local expression for a one form α whose curvature is constant¹³

$$\psi_i = d\alpha = c_1(\mathcal{L}_i) = \frac{2}{b_i^2} \sum_{j < k} d\ell_{j_i} \wedge d\ell_{k_i} . \tag{3.6}$$

With this equation it is not hard to imagine that writing out the exponential in (3.5) generates simply the flat measure in (3.4). The actual proof still involves some combinatorics [46], but the point should be obvious. We want to emphasize that all this aside, the ψ_i are

¹²It is quite remarkable that the moduli space of Riemann surfaces is so simple in these coordinates ℓ_j . One can think of ℓ_j as the propagation times of open strings, and this simplicity of moduli space is roughly why open string field theory is simpler than closed string field theory.

¹³We are suppressing some factors of two and minus signs in the sum, associated for instance with cases where one edge contributes twice to b_i , see theorem 3.20 in [40]. These are not important for this intuitive argument.

simply two forms on $\overline{\mathcal{M}}_{g,n}$. In terms of Teichmüller coordinates b_i and τ_i they have rather complicated expressions, but in the ℓ_j coordinates they are constants.¹⁴

Introducing common notation for the $2k$ -forms (the k 'th power of the Chern classes)

$$\tau_k = \psi_i^k, \tag{3.7}$$

and writing out the exponentials in (3.5) we arrive at the relation between (deformed) WP volumes for large b_i and so-called intersection numbers $\langle \tau_{d_1} \dots \tau_{d_n} \rangle$ (for more about this name see appendix C)

$$V_{g,n}(b_1 \dots b_n) = \sum_{d_1=0}^{\infty} \frac{b_1^{2d_1}}{2^{d_1} d_1!} \dots \sum_{d_n=0}^{\infty} \frac{b_n^{2d_n}}{2^{d_n} d_n!} \langle \tau_{d_1} \dots \tau_{d_n} \rangle_g, \quad \langle \tau_{d_1} \dots \tau_{d_n} \rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n}. \tag{3.8}$$

Because first Chern classes ψ_i are two-forms, and the dimension of the space $\overline{\mathcal{M}}_{g,n}$ is $6g - 6 + 2n$, there is an obvious selection rule

$$\sum_{i=1}^n d_i = 3g - 3 + n, \tag{3.9}$$

and from the definition, the correlators $\langle \tau_{d_1} \dots \tau_{d_n} \rangle$ are symmetric under exchanging any two labels. In terms of the n -boundary finite temperature partition function we then obtain

$$\begin{aligned} Z(\beta_1 \dots \beta_n)_{\text{conn}} &= \frac{\beta_1^{1/2}}{\pi^{1/2}} \dots \frac{\beta_n^{1/2}}{\pi^{1/2}} \sum_{g=0}^{\infty} e^{-2gS_0} \sum_{d_1=0}^{\infty} (2\beta_1)^{d_1} \dots \sum_{d_2=0}^{\infty} (2\beta_n)^{d_n} \langle \tau_{d_1} \dots \tau_{d_n} \rangle_g \\ &= \frac{x_1^{1/2}}{(2\pi)^{1/2}} \dots \frac{x_n^{1/2}}{(2\pi)^{1/2}} \mathcal{F}(x_1 \dots x_n), \quad x_i = 2\beta_i e^{-2S_0/3}, \end{aligned} \tag{3.10}$$

with the generating functional

$$\mathcal{F}(x_1 \dots x_n) = \sum_{d_1=0}^{\infty} x_1^{d_1} \dots \sum_{d_n=0}^{\infty} x_n^{d_n} \langle \tau_{d_1} \dots \tau_{d_n} \rangle. \tag{3.11}$$

By taking the large b_i limit in Mirzakhani's recursion relations [25, 52] (either in the answer or in the derivation) one obtains a simple version of topological recursion for ribbon graphs, see appendix C.4. This can be translated to simple recursion relations for intersection numbers $\langle \tau_{d_1} \dots \tau_{d_n} \rangle$, which are the expansion coefficients of ribbon graph volumes (3.8).

¹⁴To avoid confusion, the Weil-Petersson measure $db_i \wedge d\tau_i$ becomes only flat in the ℓ_j coordinates for large b_i [40]. So in general one can view Weil-Petersson volumes as integrating over the moduli space of ribbon graphs, but with a non-flat measure. The integration domain is the same [49–51], but the integrand is different.

One finds eventually equation (25) in [53]^{15,16}

$$\begin{aligned}
 (2k+1) \left\langle \tau_0^2 \tau_k \prod_{j=0}^{\infty} \tau_j^{d_j} \right\rangle_g &= \prod_{j=0}^{\infty} \sum_{a_j=0}^{d_j} \frac{d_j!}{a_j!(d_j-a_j)!} \left\langle \tau_0 \tau_{k-1} \prod_{j=0}^{\infty} \tau_j^{d_j-a_j} \right\rangle_{g_1} \left\langle \tau_0^3 \prod_{j=0}^{\infty} \tau_j^{a_j} \right\rangle_{g_2} \\
 + 2 \prod_{j=0}^{\infty} \sum_{b_j=0}^{d_j} \frac{d_j!}{b_j!(d_j-b_j)!} &\left\langle \tau_0^2 \tau_{k-1} \prod_{j=0}^{\infty} \tau_j^{d_j-b_j} \right\rangle_{g_1} \left\langle \tau_0^2 \prod_{j=0}^{\infty} \tau_j^{b_j} \right\rangle_{g_2} + \frac{1}{4} \left\langle \tau_0^4 \tau_{k-1} \prod_{j=0}^{\infty} \tau_j^{d_j} \right\rangle_{g-1} .
 \end{aligned} \tag{3.12}$$

These recursion relations encode the famous Korteweg-de Vries (KdV) hierarchy [59], which is an infinite set of differential equations. The connection to the KdV hierarchy will prove useful in our work, so we summarize it now.

3.2 KdV equations and intersection numbers

The KdV hierarchy consists of an infinite set of differential equations [45]

$$(2n+1)\partial_n \partial_0^2 F = \partial_{n-1} \partial_0 F \partial_0^3 F + 2\partial_{n-1} \partial_0^2 F \partial_0^2 F + \frac{1}{4} \partial_{n-1} \partial_0^4 F, \quad n \geq 0. \tag{3.13}$$

Here F is a function of an infinite set of parameters $t_0, t_1, t_2 \dots$ called KdV times and $\partial_i = \partial/\partial t_i$. Often, these equations are presented in a slightly different way, by introducing functions R_n and u of the KdV times

$$R_n = \partial_{n-1} \partial_0 F, \quad R_1 = \partial_0^2 F = u. \tag{3.14}$$

By definition one then has

$$\partial_n u = \partial_0 R_{n+1}. \tag{3.15}$$

The functions R_n for $n \geq 0$ are then determined recursively by the KdV hierarchy (3.13) as functions of u

$$(2n+1)\partial_0 R_{n+1} = (\partial_0 u)R_n + 2u\partial_0 R_n + \frac{1}{4}\partial_0^3 R_n, \tag{3.16}$$

¹⁵This is quite tedious. We can rewrite the recursion relations of volumes in terms of recursion relations for intersection numbers as in [54], but now using the volumes of ribbon graphs and without the κ two-forms. What one obtains are the Dijkgraaf-Verlinde² [55] version of the recursion relations, equation (7.27) in [56] but with appropriate normalization. These recursion relations are not manifestly the same as the ones we wrote above, they correspond with the Virasoro constraint equations in the KdV formalism, whereas our equations above literally follow from the KdV equations themselves. That these two infinite sets of differential equations (and correspondingly, two infinite sets of recursion equations) are equivalent was proven again in [56], see also [57, 58]. We explain these steps more carefully in section 3.2.

¹⁶The labels d_j fix the genus g via the selection rule (3.9) and should be chosen such that g is a positive integer (otherwise the correlator trivially vanishes). Applied to this case the selection rule reads

$$\sum_{j=0}^{\infty} (j-1)d_j + (k-1) = 3g-1.$$

Similarly the correlators on the right side only get contributions from values a_j and b_j such that g_1 and g_2 are non-negative integers. We automatically have $g_1 + g_2 = g$, as one checks using the selection rules. There are no e^{S_0} in these equations, these are just statements about integrals of $2j$ forms over some symplectic manifold.

the $n = 0$ version of this equation

$$\partial_0 u = R_0 \partial_0 u + 2u \partial_0 R_0 + \frac{1}{4} \partial_0^3 R_0 \tag{3.17}$$

has the solution $R_0 = 1$. Using that $R_1 = u$, one can then solve the $n = 1$ version of (3.16) and obtain the solution¹⁷

$$R_2 = \frac{1}{2} u^2 + \frac{1}{12} \partial_0^2 u. \tag{3.18}$$

This can be continued recursively and n and so in principle one can find solutions to the KdV equations in terms of one unknown function u . These functions R_n that one obtains this way are (up to potential normalization factors appearing in various papers) known as Gelfand-Dikii polynomials.

In fact the KdV hierarchy has one more equation, known as the string equation [45]

$$\partial_0^2 F = t_0 + \sum_{i=0}^{\infty} t_{i+1} \partial_0 \partial_i F, \tag{3.19}$$

which, via the definition of the Gelfand-Dikii polynomials becomes

$$0 = \sum_{n=0}^{\infty} (t_n - \delta_{n,1}) R_n. \tag{3.20}$$

Inserting the expressions for the Gelfand-Dikii polynomials in terms of powers of u (and t_0 derivatives thereof) this becomes a highly complicated equation for u , which is all that remains to be solved of the KdV hierarchy of equations. But what does this have to do with the intersection numbers we discussed previously?

The relation with intersection numbers is that this unknown function F is a generating functional of intersection numbers [45, 46] (see appendix C.2 for more details about this)

$$F = \left\langle \exp \left(\sum_{k=0}^{\infty} t_k \tau_k \right) \right\rangle = \sum_{\{d_i\}} \prod_{i=0}^{\infty} \frac{t_i^{d_i}}{d_i!} \left\langle \prod_{j=0}^{\infty} \tau_j^{d_j} \right\rangle. \tag{3.21}$$

Bearing in mind that in the gravitational description one interprets $\langle \tau_{d_1} \dots \tau_{d_n} \rangle$ as connected correlators (3.10), it is also natural to consider the generating function of full (not per-se connected) correlators

$$Z = \exp(F). \tag{3.22}$$

There are approximately a gazillion proofs for (3.21), but in the spirit of the discussion we have had thus far we only mention one approach, due to Dijkgraaf-Verlinde² [56], and explained nicely by Witten [58]. The gist is that using standard manipulations one can show that the solution F of the KdV hierarchy also satisfies linear differential equations [58]

$$L_n Z = 0, \quad n \geq -1, \tag{3.23}$$

with L_n satisfying the Virasoro algebra¹⁸

$$[L_m, L_n] = (m - n) L_{m+n} + \delta_{n+m,0} \frac{m(m^2 - 1)}{12}. \tag{3.24}$$

¹⁷This is the standard Korteweg-de Vries equation $\partial_1 u = u \partial_0 u + \partial_0^3 u / 12$.

¹⁸As a result, you actually only need to consider the equations for $-1 \leq n \leq 2$ [58].

They are explicitly given by

$$L_n = \frac{1}{2} \sum_{m=-\infty}^{+\infty} \alpha_m \alpha_{n-m+1} + \frac{1}{16} \delta_{n,0}, \quad (3.25)$$

with creation and annihilation operators

$$\alpha_m = \begin{cases} \frac{1}{2^{1/2}} (2m-1)!! \partial_{m-1} & m \geq 1 \\ \frac{1}{2^{1/2}} \frac{1}{(-2m-1)!!} (t_{-m} - \delta_{m,-1}) & m \leq 0. \end{cases} \quad (3.26)$$

Then one can show that these differential equations are equivalent to the topological recursion relations that we found in appendix C.4 for the ribbon graph volumes $V_{g,n}(b_1 \dots b_n)$. In particular one can write the recursion relations of those volumes in terms of recursion relations for the intersection numbers as in [54], the result is the Dijkgraaf-Verlinde² recursion relation (equation (7.27) in [56] or (4.1) in [59])

$$\begin{aligned} & (2n+3)!! \left\langle \tau_{n+1} \prod_{i \in X} \tau_{k_i} \right\rangle_g \quad (3.27) \\ &= \sum_{j \in X} \frac{(2n+2k_j+1)!!}{(2k_j-1)!!} \left\langle \tau_{k_j+n} \prod_{i \in X/j} \tau_{k_i} \right\rangle_g \\ &+ \frac{1}{2} \sum_{m=1}^n (2m-1)!! (2n-2m-1)!! \left\langle \tau_{m-1} \tau_{n-m} \prod_{i \in X} \tau_{k_i} \right\rangle_{g-1} \\ &+ \frac{1}{2} \sum_{g'=0}^g \sum_{X_1 \cup X_2 = X} \sum_{m=1}^n (2m-1)!! (2n-2m+1)!! \left\langle \tau_{m-1} \prod_{i \in X_1} \tau_{k_{i_1}} \right\rangle_{g'} \left\langle \tau_{n-m} \prod_{i_2 \in X_2} \tau_{k_{i_2}} \right\rangle_{g-g'} \end{aligned}$$

We remind the reader that these correlators have an implicit genus, because of the selection rule (3.9). If we assign genus g to the correlators on the first line, one sees that the correlators on the second line have genus $g-1$, and the ones on the final line have genera g' and $g-g'$. It is obvious then where each term comes from in the Mirzakhani recursion relations for $V_{g,n}(b_1 \dots b_n)$.

This is identical to the equations that one finds from writing out (3.23). To see this, one writes Z in terms of F , and expands F as in (3.21) in powers of t_k . In such an expansion, a derivative ∂_k is the creator of an extra τ_k

$$\partial_k F = \sum_{\{d_i\}} \prod_{i=0}^{\infty} \frac{t_i^{d_i}}{d_i!} \left\langle \tau_k \prod_{j=0}^{\infty} \tau_j^{d_j} \right\rangle, \quad (3.28)$$

and multiplication with t_k is like removing or annihilating a τ_k

$$t_k F = \sum_{d_i} \prod_{i=0}^{\infty} \frac{t_i^{d_i}}{d_i!} \left\langle d_k \tau_k^{d_k-1} \prod_{j \neq k} \tau_j^{d_j} \right\rangle. \quad (3.29)$$

Applying this several times one indeed recovers (3.27) (by comparing terms with identical powers $t_i^{d_i}$). Before proceeding we mention two special cases of (3.27). The case $n = -1$ is

called the string equation, and is equivalent to (3.19)¹⁹

$$\left\langle \tau_0 \prod_{i=1}^m \tau_{k_i} \right\rangle_g = \sum_{j=1}^m \left\langle \tau_{k_j-1} \prod_{i \neq j}^m \tau_{k_i} \right\rangle_g, \quad (3.30)$$

and the case $n = 0$ is called the dilaton equation (use the selection rule (3.9) to simplify the prefactor)

$$\left\langle \tau_1 \prod_{i=1}^m \tau_{k_i} \right\rangle_g = -\chi \left\langle \prod_{i=1}^m \tau_{k_i} \right\rangle_g, \quad \chi = 2 - 2g - m \quad (3.31)$$

The first term in

$$L_{-1} = \frac{1}{4}t_0^2 - \frac{1}{2}\partial_0 + \sum_{m=1} \frac{1}{2}t_m \partial_{m-1}, \quad (3.32)$$

fixes the fact that $\langle \tau_0^n \rangle = \delta_{n,3}$. Similarly, we can understand the term $1/16$ in (3.25) as fixing $\langle \tau_1 \rangle = 1/24$. Note that using equation (3.4) one finds $V_{1,1}(b) = b^2/48$ (see also [10] and appendix C), from which we deduce $\langle \tau_1 \rangle = 1/24$. Furthermore from $V_{0,3}(b_1, b_2, b_3) = 1$ (the moduli space $\mathcal{M}_{0,3}$ is a point) one finds $\langle \tau_0^3 \rangle = 1$. These are the initial conditions for the recursion relations.

As mentioned in footnote 18, beyond the $n = -1$ and $n = 0$, we only need the $n = 1$ and $n = 2$ Virasoro conditions. Imposing these conditions results in non-linearities. Whereas before acting with L_{-1} and L_0 is first order in the derivatives w.r.t. t_m , L_1 and L_2 contain second order derivatives acting on $Z = e^F$, resulting, for instance, in terms of the form $(\partial_0 F)^2 + \partial_0^2 F$ when converting to the free energy and giving rise to the final term in (3.27).

Instead of expanding the Virasoro constraints (3.23) in intersection numbers, we can also just expand the KdV hierarchy (3.13) (and string equation). Since both sets of equations carry identical information, so do the resulting recursion relations. One matches in a straightforward manner each consecutive term in (3.13) with those in the recursion relations (3.12). We proceed with (3.12) now.

3.3 Simplest cancellations

One can use this recursive version of the KdV equations (3.12) to get differential equations for $\mathcal{F}(x_1 \dots x_n)$ (3.11) [53] which can then be solved exactly [53, 60]. For $\mathcal{F}(x)$ we start from (3.12) with $d_j = 0$

$$(2n + 1) \langle \tau_0^2 \tau_n \rangle = \langle \tau_0 \tau_{n-1} \rangle \langle \tau_0^3 \rangle + 2 \langle \tau_0^2 \tau_{n-1} \rangle \langle \tau_0^2 \rangle + \frac{1}{4} \langle \tau_0^4 \tau_{n-1} \rangle, \quad (3.33)$$

which, using the string equation (3.30) and $\langle \tau_0^n \rangle = \delta_{n,3}$ simplifies to

$$2n \langle \tau_{n-2} \rangle - \frac{1}{4} \langle \tau_{n-5} \rangle = 0. \quad (3.34)$$

¹⁹Here and below g is determined using the selection rule (3.9), the correlator vanishes whenever g is not a positive integer

$$\sum_{i=0}^m k_i = 3g - 2 + m.$$

We can rewrite this recursion relation into a differential equation for $\mathcal{F}(x)$ by basically doing the inverse of what took us from the KdV equations to (3.12).²⁰ Multiplying with x^{n-1} and summing over n we find

$$2\partial_x \sum_n x^n \langle \tau_{n-2} \rangle - \frac{1}{4} \sum_n x^{n-1} \langle \tau_{n-5} \rangle = 0 \quad \Rightarrow \quad \left(2\partial_x x^2 - \frac{1}{4}x^4 \right) \mathcal{F}(x) = 0. \quad (3.35)$$

This equation has a unique solution once one impose that the linear term $\mathcal{F}(x) \supset x \langle \tau_1 \rangle = x/24$

$$\mathcal{F}(x) = \frac{1}{x^2} \exp\left(\frac{x^3}{24}\right). \quad (3.36)$$

With the relation (3.10) this computes the exact one-boundary partition function for topological gravity with spectrum (3.1), which reproduces the answer one obtains from double scaling the exactly solvable Gaussian matrix integral [11]

$$Z(\beta) = \frac{e^{S_0}}{4\pi^{1/2}\beta^{3/2}} \exp\left(\frac{1}{3}\beta^3 e^{-2S_0}\right). \quad (3.37)$$

We can obtain the two-boundary partition function in similar manner. For $\mathcal{F}(x_1, x_2)$ one start from (3.12) with $d_j = \delta_{j,q}$

$$(2n+1) \langle \tau_0^2 \tau_n \tau_q \rangle = \langle \tau_0 \tau_{n-1} \rangle \langle \tau_0^3 \tau_q \rangle + \langle \tau_0 \tau_{n-1} \tau_q \rangle + 2 \langle \tau_0^2 \tau_{n-1} \rangle \langle \tau_0^2 \tau_q \rangle + \frac{1}{4} \langle \tau_0^4 \tau_{n-1} \tau_q \rangle. \quad (3.38)$$

Using the string equation we find the relation

$$\mathcal{F}(x_1, x_2, 0) = \sum_{n=0}^{\infty} \sum_{q=0}^{\infty} x_1^n x_2^q \langle \tau_0 \tau_n \tau_q \rangle = (x_1 + x_2) \sum_{n=0}^{\infty} \sum_{q=0}^{\infty} x_1^n x_2^q \langle \tau_n \tau_q \rangle = (x_1 + x_2) \mathcal{F}(x_1, x_2). \quad (3.39)$$

Using tricks like this we find that multiplying (3.38) with $x_1^n x_2^q$ and summing over n and q results term by term in

$$\begin{aligned} 2\partial_1 x_1(x_1 + x_2) \mathcal{F}(x_1, x_2, 0) - (x_1 + x_2) \mathcal{F}(x_1, x_2, 0) \\ = x_2 \mathcal{F}(0, x_2, 0) \mathcal{F}(x_1, 0, 0) + x_1 \mathcal{F}(x_1, x_2, 0) \\ + 2x_1 \mathcal{F}(0, x_2, 0) \mathcal{F}(x_1, 0, 0) + \frac{1}{4}x_2(x_1 + x_2)^3 \mathcal{F}(x_1, x_2, 0). \end{aligned} \quad (3.40)$$

Using again the string equation $\mathcal{F}(x_1, 0, 0) = x_1^2 \mathcal{F}(x_1) = \exp(x_1^3/24)$ one can rearrange this into

$$\left(1 - \frac{1}{4}x_1 x_2 (x_1 + x_2) + 2 \frac{x_1(x_1 + x_2)}{x_2 + 2x_1} \partial_1 \right) \mathcal{F}(x_1, x_2, 0) \exp\left(-\frac{x_1^3 + x_2^3}{24}\right) = 1. \quad (3.41)$$

This simplifies tremendously by introducing the coordinate $a = x_1 x_2 (x_1 + x_2)$ and naming the function outside the braces $f(a)$, then Mathematica spits out the unique solution (the boundary condition comes from taking $x_1 = 0$ and using $\mathcal{F}(0, x_2, 0) = \exp(x_2^3/24)$ as above)

$$\left(1 - \frac{1}{4}a + 2a\partial_a \right) f(a) = 1, \quad f(0) = 1 \quad \Rightarrow \quad f(a) = \frac{(2\pi)^{1/2}}{a^{1/2}} \exp\left(\frac{a}{8}\right) \text{Erf}\left(\frac{a^{1/2}}{2^{3/2}}\right). \quad (3.42)$$

²⁰The difference is that here we consider all n , but only one insertion from the exponential, whereas in the KdV hierarchy we consider the whole exponential but fixed n .

From this one then obtains finally the exact answer for $\mathcal{F}(x_1, x_2)$

$$\mathcal{F}(x_1, x_2) = \frac{(2\pi)^{1/2}}{(x_1 x_2)^{1/2} (x_1 + x_2)^{3/2}} \exp\left(\frac{1}{24}(x_1 + x_2)^3\right) \operatorname{Erf}\left(\frac{1}{2^{3/2}}(x_1 x_2 (x_1 + x_2))^{1/2}\right), \quad (3.43)$$

and using (3.10) one finds the exact connected two-boundary partition function [21]

$$Z(\beta_1, \beta_2)_{\text{conn}} = \frac{e^{S_0}}{4\pi^{1/2}(\beta_1 + \beta_2)^{3/2}} \exp\left(\frac{1}{3}(\beta_1 + \beta_2)^3 e^{-2S_0}\right) \operatorname{Erf}\left(\left((\beta_1 \beta_2 (\beta_1 + \beta_2))^{1/2} e^{-S_0}\right)\right). \quad (3.44)$$

Note that this expressions indeed reduces to (A.1) in the τ -scaling limit.

A comment on unstable surfaces. We pause to make one comment about (3.36) and (3.43). Notice that they contain the terms $1/x^2$ and $1/(x_1 + x_2)$, which, from our conversion from the recursion relation to the differential equation should not be there. Nevertheless, these are correct and should be included, they represent the contributions at $(g, n) = (0, 1)$ and $(0, 2)$, which are degenerate surfaces, i.e they have zero area. In the mathematics literature they go under the name unstable surfaces. The matrix integral calculations naturally include these cases and also in the gravity calculations they should be there. From the recursion relation we can also see that they are there if we continue to negative n and q . For instance from (3.34) we see that the boundary condition $\langle \tau_1 \rangle = 1/(24)$ determines $\langle \tau_{-2} \rangle = 1$, but $\langle \tau_n \rangle$ with $n < -2$ (along side with $n = -1, 0$) still vanish. This gives the $1/x^2$ indeed. Furthermore, this also implies that $\langle \tau_0 \tau_{-1} \rangle = 1$.

Cancellations for two boundaries. As explained in the recent paper by Eynard, Lewanski and Ooms [27], something interesting happens when we write $\mathcal{F}(x_1, x_2)$ as a series expansion (using the known expansions of Erf and exponentials, see for instance equation (2.13) in [60]) in elementary symmetric polynomials $e_1 = x_1 + x_2$ and $e_2 = x_1 x_2$:

$$\mathcal{F}(x_1, x_2) = \sum_{g=0}^{\infty} \sum_{m=0}^g e_2^m e_1^{3g-1-2m} \frac{3^m}{24g m! (g-m)!} \frac{1}{2m+1} (-1)^m. \quad (3.45)$$

The label g refers to genus, because of the weight e^{-2gS_0} that each term acquires in terms of β_i variables (3.10). Let us compare this with the generic expansion of $\mathcal{F}(x_1, x_2)$ in elementary symmetric polynomials and intersection numbers, following from (3.11) (this is elementary, but slightly tedious to find)

$$\begin{aligned} \mathcal{F}(x_1, x_2) &= \sum_g \sum_{m=0}^{(3g-1)/2} e_2^m e_1^{3g-1-2m} \\ &\times \sum_{p=0}^m \langle \tau_p \tau_{3g-1-p} \rangle \frac{3g-1-2p}{3g-1-p-m} \frac{(3g-1-p-m)!}{(m-p)!(3g-1-2m)!} (-1)^{p+m}. \end{aligned} \quad (3.46)$$

Notice that this is identical to the term of top degree $m = 3g - 1$ in (2.22), which is no accident because topological gravity is the case $t_2, t_3 \dots = 0$ in section 2 in which case only the top degree survives. This should become more clear in section 4. We left implicit the factor $1/2$ for the case $p = m = (3g - 1)/2$.

The noteworthy observation is that the exact expression (3.45) at fixed genus g has a maximal power e_2^g , where the selection rule (3.9) of intersection theory allows also for powers e_2^m with $g < m < (3g - 1)/2$. This implies highly non-trivial cancellations in intersection numbers

$$\sum_{p=0}^m \langle \tau_p \tau_{3g-1-p} \rangle \frac{3g-1-2p}{3g-1-p-m} \frac{(3g-1-p-m)!}{(m-p)!(3g-1-2m)!} (-1)^p = 0, \quad g < m < (3g-1)/2. \tag{3.47}$$

The simplest case $m = (3g - 1)/2$ has appeared in the literature before, for instance [26] (theorem 8)

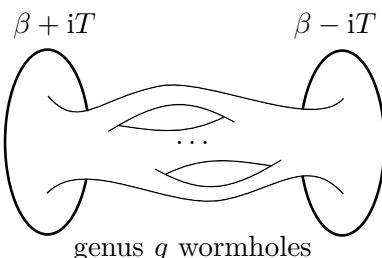
$$\sum_{d_1+d_2=3g-1} \langle \tau_{d_1} \tau_{d_2} \rangle (-1)^{d_1} = 0. \tag{3.48}$$

Now, these cancellations (3.47) are exactly the same as the highest degree cancellations $m = 3g - 1$ that we predicted for volumes (2.23) associated with generic spectra. This is obvious because the expansion coefficients in (2.16) for the volumes are exactly identical to the intersection number in (3.8)

$$V_{g,d_1,d_2} = 2^{d_1+d_2} \langle \tau_{d_1} \tau_{d_2} \rangle_g, \tag{3.49}$$

which for topological gravity are nonzero only if $d_1 + d_2 = 3g - 1$ (3.9). So, for topological gravity the only non-trivial cancellations in (2.23) are $m = 3g - 1$, the other cases are trivially satisfied because of the selection rule (3.9).

The fact that (3.45) has a maximal power e_2^g directly implies that the genus g wormhole in topological gravity grows for late times no faster than

$$Z_g(\beta + iT, \beta - iT)_{\text{conn}} = \text{genus } g \text{ wormholes} \sim T^{2g+1} e^{-2gS_0}, \tag{3.50}$$


because for late enough times $e_1 \sim \beta$ and $e_2 \sim T^2$. We have shown that the cancellations in the volumes required to make this happen can be derived directly from the KdV equations. The question then arises whether this is also true for *generic* dilaton gravity theories, or generic spectral curves $\rho_0(E)$. We will demonstrate in section 4 that this *is* the case, the universal scaling (3.50) can be derived using the KdV hierarchy.

To set up that argument, we first introduce a generalization of the cancellations (3.45) in $\mathcal{F}(x_1, x_2)$ to the n -boundary correlators in topological gravity $\mathcal{F}(x_1 \dots x_n)$.

3.4 Multi-boundary cancellations

Just like for the cases $\mathcal{F}(x_1)$ and $\mathcal{F}(x_1, x_2)$ that we presented in section 3.3, one can get an exact formula for the n -boundary correlators in topological gravity $\mathcal{F}(x_1 \dots x_n)$ via the KdV recursion relations. This was shown by Liu and Xu [61], who found a recursive formula to

compute $\mathcal{F}(x_1 \dots x_n)$ from $\mathcal{F}(x_1 \dots x_m)$ with $m < n$. The proof, nor the particular equation, is particularly insightful, the point is that precise formulas exist and follow purely from KdV.

These formulas can then be expanded in elementary symmetric polynomials [27], like we did for the two point function in (3.45). From the intersection number formula (3.11) and simple dimensional analysis we get an expansion of the type

$$\mathcal{F}(x_1 \dots x_n) = \sum_{g=0}^{\infty} \sum_{m_1=0}^{\infty} \dots \sum_{m_n=0}^{\infty} e_1^{m_1} e_2^{m_2} \dots e_n^{m_n} C_g(m_2 \dots m_n), \quad \sum_{j=1}^n j m_j = 3g - 3 + n, \tag{3.51}$$

where the constraint on the powers comes from the selection rule (3.9). For clarity

$$e_1 = \sum_{j=1}^n x_j, \quad e_2 = \sum_{j_1 < j_2 = 1}^n x_{j_1} x_{j_2}, \quad e_3 = \sum_{j_1 < j_2 < j_3 = 1}^n x_{j_1} x_{j_2} x_{j_3}, \tag{3.52}$$

etcetera. Crucially, Eynard, Lewanski and Ooms [27] found that many of the naively allowed expansion coefficients identically vanish

$$C_g(m_2 \dots m_n) = 0, \quad \sum_{j=2}^n m_j > g. \tag{3.53}$$

This generalizes the fact that the highest power in (3.45) is e_2^g . They impressively checked this vanishing for all n and $g \leq 7$, and for all g and $n \leq 3$, using the explicit formulas of Liu and Xu [61] (see also [60]) and the KdV equations (in various forms).

Taken at face value, this new constraint (3.53) does not seem too constraining on the individual powers of $e_3, e_4 \dots$, because the original selection rule does not allow m_3 to grow faster than g , or m_4 faster than $3g/4$ etcetera. So one may wonder if, beyond $m = 2$, these cancellations (3.53) have any intuitive physical interpretation, we will now demonstrate that they do.

4 Universal cancellations in dilaton gravity via open-closed duality

In this section we show that the constraint (3.53) on the total power of $e_2, e_3 \dots$ implies (and in fact is identical to) the universal maximal growth (3.50) for all double scaled matrix models (with square root edges (2.7)), not just topological gravity. This late-time wormhole universality, in turn, is key for the emergence of the plateau, which, in turn, is gravity's way of saying it is a discrete quantum system [2].

Readers familiar with how those models are described using the KdV hierarchy may skip to section 4.2, for didactic purposes we will go slower.

4.1 KdV equations around different backgrounds

We have learned that because of the relation of the volumes with ψ_i classes for topological gravity (3.5)

$$V_{g,n}(b_1 \dots b_n) = \int_{\overline{\mathcal{M}}_{g,n}} \exp\left(\frac{1}{2} \sum_{i=1}^n b_i^2 \psi_i\right) = \sum_{d_1=0}^{\infty} \frac{b_1^{2d_1}}{2^{d_1} d_1!} \dots \sum_{d_n=0}^{\infty} \frac{b_n^{2d_n}}{2^{d_n} d_n!} \langle \tau_{d_1} \dots \tau_{d_n} \rangle_g, \tag{4.1}$$

that n -boundary partition functions can be expressed in terms of intersection numbers, as in (3.10).

More in general, there is such a relation for all double scaled matrix integrals with a leading order spectrum of the type (2.7)

$$\rho_0(E) = \frac{e^{S_0}}{2\pi} \sum_{k=0}^{\infty} f_k E^{k+1/2}, \quad f_0 = 1. \tag{4.2}$$

In this more general case the volumes are computed in terms of intersection numbers as

$$V_{g,n}(b_1 \dots b_n) = \sum_{d_1=0}^{\infty} \frac{b_1^{2d_1}}{2^{d_1} d_1!} \dots \sum_{d_n=0}^{\infty} \frac{b_n^{2d_n}}{2^{d_n} d_n!} \left\langle \tau_{d_1} \dots \tau_{d_n} \exp \left(\sum_{k=2}^{\infty} \gamma_k \tau_k \right) \right\rangle_g, \tag{4.3}$$

with a relation $f_k(\gamma_i)$ that we have yet to determine. Notice that we can get the correlators on the right from the same generating functional F that satisfies the KdV hierarchy (3.21), by expanding around a different set of KdV times $t_{k \text{ total}} = \gamma_k + t_k$

$$F = \left\langle \exp \left(\sum_{k=0}^{\infty} t_k \text{total} \tau_k \right) \right\rangle = \prod_{i=0}^{\infty} \sum_{d_i} \frac{t_i^{d_i}}{d_i!} \left\langle \prod_{j=0}^{\infty} \tau_j^{d_j} \exp \left(\sum_{k=2}^{\infty} \gamma_k \tau_k \right) \right\rangle = \prod_{i=0}^{\infty} \sum_{d_i} \frac{t_i^{d_i}}{d_i!} \left\langle \prod_{j=0}^{\infty} \tau_j^{d_j} \right\rangle_{\gamma_k}, \tag{4.4}$$

where in the last equality we have introduced shorthand notation $\langle \dots \rangle_{\gamma_k}$ for correlators in the presence of an extra exponential of τ_k operators.

There are several ways to appreciate that (4.3) calculates the correlators of a double scaled matrix integral [11] with a spectral curve of the type (4.2) (or some model of two dimensional (dilaton) gravity). One way is to write out the Virasoro constraints (3.23); but with the more generally correct form for the creation and annihilation operators (we can always set $\gamma_0 = \gamma_1 = 0$)

$$\alpha_m = \begin{cases} \frac{1}{2^{1/2}} (2m-1)!! \partial_{m-1} & m \geq 1 \\ \frac{1}{2^{1/2}} \frac{1}{(-2m-1)!!} (t_{-m} - \delta_{m,-1} + \gamma_{-m}) & m \leq 0. \end{cases} \tag{4.5}$$

We can use the definition of F (4.4) to see that we still have the property that a derivative with respect to t_k is the creator of an τ_k

$$\partial_k F = \sum_{\{d_i\}} \prod_{i=0}^{\infty} \frac{t_i^{d_i}}{d_i!} \left\langle \tau_k \prod_{j=0}^{\infty} \tau_j^{d_j} \right\rangle_{\gamma_k}, \tag{4.6}$$

and furthermore multiplication with t_k is still like removing or annihilating a τ_k

$$t_k F = \sum_{\{d_i\}} \prod_{i=0}^{\infty} \frac{t_i^{d_i}}{d_i!} \left\langle d_k \tau_k^{d_k-1} \prod_{j \neq k}^{\infty} \tau_j^{d_j} \right\rangle_{\gamma_k}. \tag{4.7}$$

Then we can write out the generators L_n (3.25) and use $Z = \exp(F)$ to obtain recursion relations that are analogous to the Dijkgraaf-Verlinde² ones (3.27) but which depend explicitly on the γ_k , because γ_k appears in (4.5). These recursive equations to compute

$\langle \tau_{d_1} \dots \tau_{d_n} \rangle_{\gamma_k}$ can then be recovered alternatively from the topological recursion [62] relations between volumes with spectral curve (4.2), demonstrating the equivalence.

We emphasize again that we are always dealing with the same function F , the same KdV hierarchy (3.13) and the same string equation (which features the invariant $t_{k \text{ total}} = t_k + \gamma_k$)

$$0 = \sum_{k=0}^{\infty} (t_k + \gamma_k - \delta_{k,1}) R_k, \quad (4.8)$$

which in a correlation function reduces to the generalization of (3.30)

$$\left\langle \tau_0 \prod_{i=1}^n \tau_{k_i} \right\rangle_{\gamma_k} - \sum_{j=1}^{\infty} \gamma_{j+1} \left\langle \tau_j \prod_{i=1}^n \tau_{k_i} \right\rangle_{\gamma_k} = \sum_{m=1}^n \left\langle \tau_{k_m-1} \prod_{i \neq m}^n \tau_{k_i} \right\rangle_{\gamma_k}. \quad (4.9)$$

For multi-critical points with $E^{k+1/2}$ edge only γ_{k+1} is nonzero and we recover the $n = 0$ case of formula (4.1) in [59] (the formulas are more symmetric if we view the $-\delta_{k,1}$ as $\gamma_1 = -1$ instead, which we will adopt below).

Let us now compute the spectrum $\rho_0(E)$ that follows from (4.4). For this we need to be a bit more clever with the genus counting parameter and define [48]

$$F(e^{S_0}, \gamma_k) = \sum_{g=0}^{\infty} e^{-2gS_0} \left\langle \exp \left(\sum_{k=0}^{\infty} \gamma_k \tau_k \right) \right\rangle_g, \quad R_{k+1}(e^{S_0}, \gamma_k) = \partial_0 \partial_k F(e^{S_0}, \gamma_k), \quad (4.10)$$

with again $u(e^{S_0}, \gamma_k) = R_1(e^{S_0}, \gamma_k)$. Using the selection rule (3.9) we observe that this is related to F via $F(e^{S_0}, \gamma_k) = e^{-2S_0} F(\gamma_k e^{S_0 2(1-k)/3})$. Using these relations one rewrites the KdV recursion relations for R_n (3.16) as (all equations below implicitly have the functions of $u(e^{S_0}, \gamma_k)$ that we just defined)

$$(2n+1)\partial_0 R_{n+1} = \partial_0 u R_n + 2u\partial_0 R_n + e^{-2S_0} \frac{1}{4} \partial_0^3 R_n, \quad 0 = \sum_{k=0}^{\infty} \gamma_k R_k. \quad (4.11)$$

This redefinition may seem a bit like wasted energy, but the benefit is that we can now very easily solve the recursion relation to leading order in e^{S_0} , because we can neglect the final term. With seed $R_0 = 1$, one finds $R_k = u^k/k!$ and the leading order string equation becomes quite simple

$$-\gamma_0 = \sum_{k=1}^{\infty} \gamma_k \frac{u^k}{k!} = \mathcal{G}(u). \quad (4.12)$$

One can then use this simple structure to find $Z(\beta)$ to leading order as follows [48, 63–65]. In all the expressions we have written down so far t_0 was set to zero, but this makes using the KdV hierarchy impossible since there are ∂_0 s everywhere. Thus we need to consider a partition function $Z(\beta, t_0)$ with non-zero t_0 . For that we need to use the contributions from unstable surfaces discussed below (3.43). In particular, consider

$$\partial_0 Z(\beta, t_0) = e^{S_0} \frac{\beta^{1/2}}{\pi^{1/2}} \sum_d (2\beta)^d \langle \tau_0 \tau_d \rangle_{\gamma_k} = e^{S_0} \frac{1}{2\pi^{1/2} \beta^{1/2}} \sum_{d=0}^{\infty} (2\beta)^d R_d = e^{S_0} \frac{e^{2\beta u}}{2\pi^{1/2} \beta^{1/2}}, \quad (4.13)$$

where we used that a derivative w.r.t. to t_0 inserts a factor of τ_0 ,²¹ the definition $R_{k+1} = \partial_0 \partial_k F$ and that $R_d = u^d/d!$. Integrating (4.13), using that $\gamma_0 = -\mathcal{G}(u)$ such that $d\gamma_0 = -\mathcal{G}'(u)du$, and putting $t_0 = 0$ again one obtains the leading order solution [48]

$$Z(\beta) = \frac{e^{S_0}}{2\pi^{1/2}\beta^{1/2}} \int_{-2u_0}^{\infty} du \mathcal{G}'(-u/2) e^{-\beta u}, \quad (4.14)$$

with u_0 the smallest positive solution to $\mathcal{G}(u_0) = 0$. The spectral density is then found to be

$$\rho_0(E) = \frac{e^{S_0}}{\pi} \int_{E_0}^E du \frac{\mathcal{G}'(-u/2)}{\sqrt{E-u}}, \quad (4.15)$$

with $E_0 = 2u_0$. In the case of JT we have

$$\mathcal{G}(-u/2) = \frac{u^{1/2}}{2\pi} I_1(2\pi u^{1/2}), \quad (4.16)$$

which gives $E_0 = 0$ and the density to be the sinh. More in general, one finds an expansion of $\rho_0(E)$ in terms of $E^{k+1/2}$ as in (2.7) but with expansion coefficients that depend on γ_k non-linearly because E_0 depends non-linearly on these couplings. To sum things up: (4.3) computes volumes of two dimensional (dilaton) gravity theories, and the leading order spectrum can be found more or less directly from the KdV equations.²²

Before we proceed let us make two comments that will be important later.

1. JT gravity corresponds to the case

$$\gamma_{n+1} = (-1)^{n+1} \frac{(2\pi^2)^n}{n!}, \quad (4.17)$$

which indeed reproduces the series expansion of $\rho_0(E) = e^{S_0} \sinh(2\pi E^{1/2})/4\pi^2$. Of course we also have the known expression for the Weil-Petersson symplectic form [47]

$$V_{g,n}(b_1 \dots b_n) = \int_{\overline{\mathcal{M}}_{g,n}} \exp\left(2\pi^2 \kappa + \frac{1}{2} \sum_{i=1}^n b_i^2 \psi_i\right), \quad (4.18)$$

where the κ class represents the Weil-Petersson two form on punctured Riemann surfaces. If we compare this with our current construction we recover the known duality [47, 66]

$$e^{2\pi^2 \kappa} \Leftrightarrow \exp\left(\sum_{k=2}^{\infty} (-1)^k \frac{(2\pi^2)^{k-1}}{(k-1)!} \tau_k\right), \quad (4.19)$$

which is to be understood as holding within expectation values.

2. This relation between an exponential of τ_k operators and matrix integrals with different spectral densities should be considered an example of open-closed (string)

²¹To be very precise here: we work to first order in perturbation theory in t_0 , which will be enough as we will later set t_0 to zero again.

²²Some extra overall $e^{E_0\beta}$ in the integral (2.10) does not change any of the statements about the power series.

duality. For instance in (4.3) the left-hand side is the “closed” picture, we compute n -boundary amplitudes in a specific gravity theory. But, the right-hand side is a sum over $m \geq n$ -boundary amplitudes in topological gravity

$$\begin{aligned} \langle \tau_{d_1} \dots \tau_{d_n} \rangle_{\gamma_k} &= \langle \tau_{d_1} \dots \tau_{d_n} \rangle + \sum_{k_1=2}^{\infty} \gamma_{k_1} \langle \tau_{d_1} \dots \tau_{d_n} \tau_{k_1} \rangle \\ &+ \frac{1}{2} \sum_{k_1=2}^{\infty} \sum_{k_2=2}^{\infty} \gamma_{k_1} \gamma_{k_2} \langle \tau_{d_1} \dots \tau_{d_n} \tau_{k_1} \tau_{k_2} \rangle + \dots \end{aligned} \quad (4.20)$$

This should be viewed as the “open” description [67]. The identity (4.19) should be interpreted in the same way. The κ two forms are integrated over $\overline{\mathcal{M}}_{g,n}$, whereas the τ_k correspond with Chern classes $c_1(L_a)^{\wedge k}$ associated with $m - n$ extra marked points x_a integrated over $\overline{\mathcal{M}}_{g,m}$ with $m \geq n$. In string language, the exponential of τ_k 's is like inserting D-branes (exponentials of boundaries), see also figure 3.

3. Another side of the coin is the relation with ribbon graphs. We demonstrated in section 3.1 that topological gravity can be thought of in the same way as JT gravity, but where the higher genus Riemann surfaces one glues to are ribbon graphs with large lengths b_i . In turn, the full JT theory also has a ribbon graph interpretation; actually two, related by open-closed duality.

The open side of the interpretation is essentially rewriting (4.20) in terms of ribbon graphs using Kontsevich’s insights, which we explained in section 3.1. The crucial thing there is then that the JT ribbon graphs in the open description are sums of the usual (Airy) ribbon graphs, but with additional faces that are weighted by the γ_i instead of the $b_i^2/2$.

The closed perspective is that one takes the usual (Airy) ribbon graphs, but when computing the volume of moduli space one uses not the flat measure, rather the κ class implies a non-trivial measure

$$\mu(\{\hat{\ell}_i\}, \{b_i\})_{\text{JT}} = \exp\left(2\pi^2\kappa + \frac{1}{2} \sum_{i=1}^n b_i^2 \psi_i\right) = \exp\left(- \sum_{i<j}^{6g-6+2n} W_{ij}^{-1} d\hat{\ell}_i \wedge d\hat{\ell}_j\right), \quad (4.21)$$

with W_{ij} the matrix

$$W_{ij} = \sum_{p \in \gamma_i \cap \gamma_j} \cos \theta_p. \quad (4.22)$$

Here γ_i are $6g - 6 + 2n$ distinct simple closed geodesics with length $\hat{\ell}_i$, and θ_p is the intersection angle between two of them. These $\hat{\ell}_i$ are crucially not the same as the lengths ℓ_j we encountered around equation (3.4). For $\hat{\ell}_i$ we picked a set of simple closed geodesics γ_i on the surface, whereas in (3.4) we looked at the edges of a ribbon graph associated to the surface. How to construct the γ_i from the ribbon graph is non-trivial. From the ribbon graph, as explained in [40] there is an algorithmic way of constructing the simple closed geodesics (these can be disjoint unions of curves also) but since the ribbon graph is trivalent²³ one will get $6g - 6 + 3n$ geodesics, which are

²³The moduli space of ribbon graphs of other valency have dimension strictly less than $6g - 6 + 2n$.

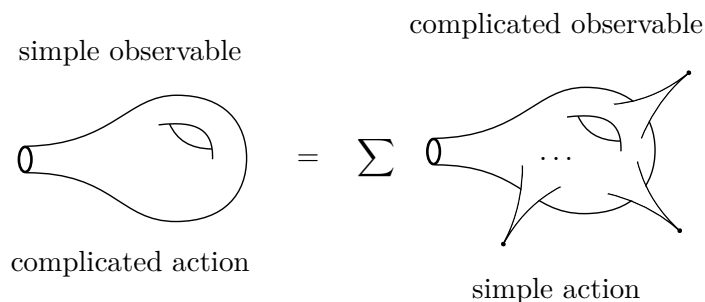


Figure 3. Open-closed duality in a nutshell. Here we have in mind computing $V_{1,1}(b)$ in the theory with an exponential of τ_k inserted in the action, see (5.9) below. Either we use the complicated action (5.9) (left); or we expand out the deformations (right), in which case we have a sum over cusp-defects but we just use the simple JT gravity action at the cost of computing a more complicated observable with many boundaries (or operators). Alternatively, in the context of this section, we could think of the left picture as JT gravity and the right picture as Airy with an exponential of (4.19) inserted. The idea is always the same.

n too many. One can eliminate them by noticing that the analogue of the matrix W_{ij} will have rank $6g - 6 + 2n$ and so there is a set of rows and columns one can select to get the right geodesics. This is thus rather cumbersome, but luckily at large b_i the ℓ_i and $\hat{\ell}_i$ have a simple relation [40], resulting in (3.6).

At any rate, the closed picture is thus more complicated only in the sense that we would have to write (4.21) in terms of the edge lengths ℓ_i of the usual ribbon graphs.

Another point to note is that the relation with the perhaps more familiar measure in terms of the Fenchel-Nielsen length and twist coordinates (b_j, τ_j) is also not easy. One would have to write the twist and length variables in terms of the edge variables ℓ_j of ribbon graphs. The relation with the $\hat{\ell}_i$ is a bit easier, but then one still needs to convert $\hat{\ell}_i$ to the ℓ_i variables, which as explained above, is hard.

The point is that you can compute complicated things in a simple theory (many-boundary observables in topological gravity) and learn about simple things in a complicated theory (few-boundary observables in generic two-dimensional gravity models). Eynard, Lewanski and Ooms have such a complicated claim (3.53) in a simple theory, we will recast this into a simple claim (2.10) in complicated theories.

4.2 Universal cancellations in all theories

Consider now the two-boundary correlator in a generic background γ_k , or for any double scaled matrix integral with square root spectral edge. According to (3.10) and (4.3) we have

$$Z(\beta_1, \beta_2)_{\text{conn}} = \frac{\beta_1^{1/2}}{\pi^{1/2}} \frac{\beta_2^{1/2}}{\pi^{1/2}} \mathcal{F}(2\beta_1, 2\beta_2)_{\gamma_k} = \frac{\beta_1^{1/2}}{\pi^{1/2}} \frac{\beta_2^{1/2}}{\pi^{1/2}} \sum_{g=0}^{\infty} e^{-2gS_0} \mathcal{F}_g(2\beta_1, 2\beta_2)_{\gamma_k}, \quad (4.23)$$

with the generating functional

$$\mathcal{F}_g(x_1, x_2)_{\gamma_k} = \sum_{d_1=0}^{\infty} \sum_{d_2=0}^{\infty} x_1^{d_1} x_2^{d_2} \left\langle \tau_{d_1} \tau_{d_2} \exp \left(\sum_{k=2}^{\infty} \gamma_k \tau_k \right) \right\rangle_g. \quad (4.24)$$

Now let us introduce a new infinite set of variables $y_i(t_k)$ such that

$$\gamma_k = \sum_{i=0}^{\infty} y_i^k. \quad (4.25)$$

Then expanding out the generating functional gives

$$\mathcal{F}_g(x_1, x_2)_{\gamma_k} = \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{i_1=0}^{\infty} \cdots \sum_{i_m=0}^{\infty} \mathcal{F}_g(x_1, x_2, y_{i_1} \cdots y_{i_m}), \quad (4.26)$$

which features the $n \geq 2$ -boundary correlators of topological gravity

$$\begin{aligned} \mathcal{F}_g(x_1 \dots x_n) &= \sum_{d_1=0}^{\infty} \cdots \sum_{d_n=0}^{\infty} x_1^{d_1} \dots x_n^{d_n} \langle \tau_{d_1} \dots \tau_{d_n} \rangle_g \\ &= \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} e_1^{m_1} e_2^{m_2} \dots e_n^{m_n} C_g(m_2 \dots m_n), \end{aligned} \quad (4.27)$$

which as we discussed around (3.53) has a constrained expansion in elementary symmetric polynomials

$$C_g(m_2 \dots m_n) = 0, \quad \sum_{j=2}^n m_j > g. \quad (4.28)$$

For late times in the spectral form factor computation $x_1 \sim iT$ and $x_2 \sim -iT$, but none of the y_i scale in any way with T , these are just parameters inherent to the theory. This means that $e_2 \sim e_3 \sim \dots \sim T^2$ and therefore

$$e_1^{m_1} e_2^{m_2} \dots e_n^{m_n} \sim T^{2 \sum_{j=2}^n m_j}. \quad (4.29)$$

The power of T that appears here is precisely the expression that was constrained by the KdV equations to be upper bound by $2g$. Hence we have derived that $\mathcal{F}_g(x_1, x_2)_{\gamma_k}$ grows no faster than T^{2g} for generic γ_k

$$C_g(m_2 \dots m_n) = 0, \quad \sum_{j=2}^n m_j > g \quad \Rightarrow \quad Z_g(\beta + iT, \beta - iT)_{\text{conn}} \sim T^{2g+1} e^{-2gS_0}. \quad (4.30)$$

In summary, we have used the KdV equations to prove a universal late-time scaling behavior for genus g wormhole amplitudes in generic double scaled matrix integrals (or two dimensional gravity models). We explained in section 2 why this behavior was key for the emergence of the plateau.

In fact we believe that the arrow works both ways

$$C_g(m_2 \dots m_n) = 0, \quad \sum_{j=2}^n m_j > g \quad \Leftrightarrow \quad Z_g(\beta + iT, \beta - iT)_{\text{conn}} \sim T^{2g+1} e^{-2gS_0}, \quad (4.31)$$

in other words this new type of wormhole universality at late times also implies all of the cancellations that Eynard, Lewanski and Ooms [27] found. We demonstrate this in appendix B.

5 KdV equations in dilaton gravity and the matrix integral

In the previous sections we primarily focused on the intersection theory and its integrable structure by itself, but there is actually a natural interpretation in dilaton gravity and the matrix integral of all this stuff as well. The general message that we want to convey here is that the KdV equations seem to have a lot of applications in gravity, it seems like we have just scratched the surface.

In section 5.1 we prove that the τ_k can be interpreted as local operators in dilaton gravity which create cusps (sharp defects [18, 19, 68]). KdV backgrounds with different γ_k correspond with inserting a gas of such cusps in JT gravity, this changes the dilaton potential, and in that sense the different KdV backgrounds are different dilaton gravity models.

The KdV equations mostly describe how observables in dilaton gravity change under changes in the action, the answer is basically that we compute things with a new spectral curve. Likewise, the string-and dilaton equations have an interpretation in dilaton gravity as transformation rules for observables under changes in the action. They reduce to analogues of the Dijkgraaf-Verlinde² loop equations [56].

In section 5.2 we present a clean matrix integral dual for the τ_k operators, improving on equations that appeared in [36, 37, 43, 69, 70].

5.1 Dilaton gravity

We start with remembering the relation (4.3) between Weil-Petersson volumes and τ_k correlators in the JT gravity background (4.19)

$$V_{g,n}(b_1 \dots b_n) = \sum_{d_1=0}^{\infty} \frac{b_1^{2d_1}}{2^{d_1} d_1!} \dots \sum_{d_n=0}^{\infty} \frac{b_n^{2d_n}}{2^{d_n} d_n!} \langle \tau_{d_1} \dots \tau_{d_n} \rangle_{\kappa}. \quad (5.1)$$

Furthermore [19, 71–74] we also know that correlators of defects with opening angle $\alpha_i < \pi$ in JT gravity are analytic continuations of Weil-Petersson volumes to $b_i = i\alpha_i$

$$\begin{aligned} \langle \mathcal{O}_D(\alpha_1) \dots \mathcal{O}_D(\alpha_n) \rangle_{g_{\text{conn}}} &= V_{g,n}(i\alpha_1 \dots i\alpha_n) \\ &= \sum_{d_1=0}^{\infty} \frac{(-1)^{d_1}}{2^{d_1} d_1!} \alpha_1^{2d_1} \dots \sum_{d_n=0}^{\infty} \frac{(-1)^{d_n}}{2^{d_n} d_n!} \alpha_n^{2d_n} \langle \tau_{d_1} \dots \tau_{d_n} \rangle_{\kappa}, \end{aligned} \quad (5.2)$$

with $\mathcal{O}_D(\alpha)$ the dilaton gravity operator that creates a defect with opening angle α ,²⁴ [68] (using Gauss-Bonnet one can check that this indeed creates the expected angular defect [68, 76])

$$\mathcal{O}_D(\alpha) \Leftrightarrow \int d^2x \sqrt{g} e^{-(2\pi-\alpha)\Phi} \Leftrightarrow \dots \begin{array}{c} \curvearrowright \\ \alpha \\ \curvearrowleft \end{array} \quad (5.3)$$

conical defect

²⁴One can use several quantization schemes for defects in JT gravity [18, 75], leading to slightly different expressions for the defect operator. We are using the most intuitive conventions of [18]. The first equality in (5.4) is scheme-independent, the second equality can be directly modified to a different scheme. The scheme of [75] may be more appropriate for doing semiclassics with the action in (5.9).

Equation (5.2) basically means that this defect operator is a generating functional of τ_k operators

$$\tau_k \Leftrightarrow (-1)^k \frac{2^k k!}{(2k)!} \partial_\alpha^{2k} \mathcal{O}_D(\alpha)|_{\alpha=0} \Leftrightarrow \frac{(-1)^k}{(2k-1)!!} \int dx^2 \sqrt{g} \Phi^{2k} e^{-2\pi\Phi}, \quad (5.4)$$

where in the second equality we inserted the dilaton gravity meaning of $\mathcal{O}_D(\alpha)$. Another way of stating this is that we have the obvious identity

$$\int d^2x \sqrt{g} e^{-(2\pi-\alpha)\Phi} \Leftrightarrow \exp\left(\frac{1}{2}(\mathrm{i}\alpha)^2 \psi_{\text{extra}}\right), \quad \alpha < \pi. \quad (5.5)$$

Here ψ_{extra} is the first Chern class attached to the extra marked point, constructed in a similar way as (3.5). The same equation (5.4) is found by working directly with geodesic boundaries and expanding around $b = 0$ whilst using the dilaton gravity description of a geodesic boundary [34]²⁵

$$\mathcal{O}_G(b) \Leftrightarrow e^{-S_0} \int d^2x \sqrt{g} e^{-2\pi\Phi} \cos(b\Phi) \Leftrightarrow \dots \text{ (diagram) } b \quad (5.6)$$

geodesic boundary

We learn from (5.4) that the τ_k operators should be interpreted in JT gravity as cusps (infinitely sharp defects $\alpha = 0$) with an additional insertion of Φ^{2k} at the cusps. This fits nicely with the stringy intuition that local vertex operators (here τ_k) can be viewed as closed string states coming in from ∞ , the infinity is the fact that the cusp is infinitely sharp and the state is specified by the Φ^{2k}

$$\tau_k \Leftrightarrow \frac{(-1)^k}{(2k-1)!!} \int dx^2 \sqrt{g} \Phi^{2k} e^{-2\pi\Phi} \Leftrightarrow \dots \text{ (diagram) } \alpha = 0 \quad (5.7)$$

(nearly) cusp defect

Suppose now that we take the KdV hierarchy and we consider the following total background γ_k

$$\exp\left(2\pi^2\kappa + \sum_{k=2}^{\infty} t_k \tau_k\right) = \exp\left(\sum_{k=2}^{\infty} \gamma_k \tau_k\right), \quad (5.8)$$

where the relation between γ_k and t_k follows from the expansion of the κ class in τ_k operators (4.19). Because the $2\pi^2\kappa$ is specifying JT gravity, we can view the remainder of the exponential as an operator insertion in JT gravity. Using the dictionary (5.4) we find that, because the $t_k \tau_k$ are in the exponential, we obtain a deformation of the JT gravity dilaton potential

$$\exp\left(2\pi^2\kappa + \sum_{k=2}^{\infty} t_k \tau_k\right) \Leftrightarrow \exp\left(S_0\chi + \frac{1}{2} \int d^2x \sqrt{g} \Phi(R+2) + \int d^2x \sqrt{g} \sum_{k=2}^{\infty} \frac{(-1)^k}{(2k-1)!!} t_k \Phi^{2k} e^{-2\pi\Phi}\right). \quad (5.9)$$

²⁵The reader should not dwell on the e^{-S_0} prefactor here, which just signifies that we choose to count this as a hole rather than a local operator topologically. For defects and τ_k operator insertions there is no such prefactor.

These deformations span the class of dilaton gravities that were discussed in [18, 19, 77]. This formula is exactly true as long as the deformation decays for large Φ no slower than $e^{-\pi\Phi}$.

Consider now the string equation (4.9) which for JT gravity $\gamma_{j+1} = -(-2\pi^2)^j/j!$ becomes

$$\sum_{a=0}^{\infty} \frac{(-2\pi^2)^a}{a!} \left\langle \tau_a \prod_{i=1}^n \tau_{k_i} \right\rangle_{\kappa} = \sum_{m=1}^n \left\langle \tau_{k_m-1} \prod_{i \neq m}^n \tau_{k_i} \right\rangle_{\kappa}. \quad (5.10)$$

Note that the operator that is being “inserted” on the left (replacing the role of the so-called puncture operator τ_0 in topological gravity) is related through (5.2) with a Weil-Petersson volume evaluated at $\alpha = 2\pi$. We can then rewrite the string equation by summing over the other τ_k in the correlator with appropriate prefactors, such that they correspond with inserting geodesic boundaries too as in (5.1). One finds that the string equation becomes

$$V_{g,n+1}(2\pi i, b_1 \dots b_n) = \int_0^{b_1} da_1 a_1 V_{g,n}(a_1, b_2 \dots b_n) + \dots + \int_0^{b_n} da_n a_n V_{g,n}(b_1 \dots b_{n-1}, a_n). \quad (5.11)$$

One can indeed check this explicitly, for instance

$$V_{1,2}(2\pi i, b) = \frac{b^4}{192} + \pi^2 \frac{b^2}{24} = \int_0^b daa V_{1,1}(a), \quad V_{1,1}(a) = \frac{a^2}{48} + \frac{\pi^2}{12}. \quad (5.12)$$

This relation (5.11) should be viewed as the first term in the Taylor expansion of Mirzakhani’s recursion around $b = 2\pi i$. Indeed, Mirzakhani’s recursion relations are [25, 52]

$$\begin{aligned} & b V_{g,n+1}(b, B) \\ &= \frac{1}{2} \int_0^{\infty} db' b' db'' b'' D(b, b', b'') \left(V_{g-1, n+2}(b', b'', B) + \sum_{\text{stable}} V_{h_1, n_1}(b', B_1) V_{h_2, n_2}(b'', B_2) \right) \\ &+ \sum_{i=1}^n \int_0^{\infty} db' b' (b - T(b, b', b_i)) V_{g, n-1}(b', B/b_i), \end{aligned} \quad (5.13)$$

with

$$D(b, b', b'') = 2 \log \left(\frac{e^{b/2} + e^{(b'+b'')/2}}{e^{-b/2} + e^{(b'+b'')/2}} \right), \quad T(b, b', b'') = \log \left(\frac{\cosh \frac{b''}{2} + \cosh \frac{b-b'}{2}}{\cosh \frac{b''}{2} + \cosh \frac{b+b'}{2}} \right). \quad (5.14)$$

Inserting $b = 2\pi i$ (and being careful with branchcuts upon doing the analytic continuation from real b) one obtains

$$D(2\pi i, b', b'') = 0, \quad T(2\pi i, b', b_k) = 2\pi i \theta(b' - b_k), \quad (5.15)$$

which reduces the recursion relation (5.13) directly to (5.11). The other Virasoro constraints (3.23) in the KdV hierarchy compute subleading terms in the Taylor expansion of the volumes around $b = 2\pi i$.

For instance, working out the L_0 constraint (or dilaton equation) for JT gravity gives

$$\sum_{a=0}^{\infty} (2a+3) \frac{(-2\pi^2)^a}{a!} \left\langle \tau_{a+1} \prod_{i=1}^n \tau_{k_i} \right\rangle_{\kappa} = \sum_{m=1}^n (2k_m+1) \left\langle \prod_{i=1}^n \tau_{k_i} \right\rangle_{\kappa}, \quad (5.16)$$

which in terms of volumes becomes

$$\left(\frac{2}{b}\partial_b + \partial_b^2\right)V_{g,n+1}(b = 2\pi i, b_1 \dots b_n) = \left(n + \sum_{i=1}^n b_i \partial_{b_i}\right)V_{g,n}(b_1 \dots b_n), \quad (5.17)$$

this relation one can again check explicitly, for instance for $V_{1,2}(b, b_1)$ and $V_{1,1}(b_1)$. In terms of partition functions (3.10) the string equation (5.11) becomes

$$Z(b = 2\pi i, \beta_1 \dots \beta_n)_{\text{conn}} = 2 \sum_{i=1}^n \beta_i Z(\beta_1 \dots \beta_n)_{\text{conn}}, \quad (5.18)$$

and the dilaton equation (5.17) becomes

$$\left(\frac{2}{b}\partial_b + \partial_b^2\right)Z(b = 2\pi i, \beta_1 \dots \beta_n)_{\text{conn}} = 2 \sum_{i=1}^n \beta_i \partial_{\beta_i} Z(\beta_1 \dots \beta_n)_{\text{conn}}. \quad (5.19)$$

These identities are the equivalent of what Dijkgraaf-Verlinde² [56] called loop equations for topological gravity. In the latter case the right hand side of the dilaton equation is also equal to $-3\chi Z(\beta_1 \dots \beta_n)_{\text{conn}}$, as in (3.31). This is not true for JT gravity, because the selection rule (3.9) is violated by the κ classes in the exponential. However, it turns out [40] we also have an equation

$$\frac{1}{b}\partial_b Z(b = 2\pi i, \beta_1 \dots \beta_n)_{\text{conn}} = -\chi Z(\beta_1 \dots \beta_n)_{\text{conn}}, \quad (5.20)$$

such that the dilaton equation simplifies to

$$\partial_b^2 Z(b = 2\pi i, \beta_1 \dots \beta_n)_{\text{conn}} = \left(2 \sum_{i=1}^n \beta_i \partial_{\beta_i} - 2\chi\right) Z(\beta_1 \dots \beta_n)_{\text{conn}}. \quad (5.21)$$

We are interested in understanding these equations (and if possible the other L_n constraints) directly from the dilaton gravity path integral, in the spirit of having a better gravity understanding of the KdV hierarchy. Based on equation (5.5) and

$$\sum_{a=0}^{\infty} \frac{(-2\pi^2)^a}{a!} \tau_a = \exp\left(\frac{1}{2}(2\pi i)^2 \psi_{\text{extra}}\right), \quad (5.22)$$

one might think that this corresponds with a blunt defect $\alpha = 2\pi$, which is essentially an area operator (or a marked point which is integrated over spacetime, without creating any source of curvature in that spacetime as backreaction)

$$\sum_{a=0}^{\infty} \frac{(-2\pi^2)^a}{a!} \tau_a \stackrel{?}{\Leftrightarrow} \int d^2x \sqrt{g} = A \Leftrightarrow \text{marked point without curvature source}. \quad (5.23)$$

This turns out to be *almost* correct, but not quite. Correlators of blunt defects $\alpha = 2\pi$ were discussed around equation (4.23) and (4.18) in [75]. The JT gravity connected n -boundary

path integral with m such defects inserted results in²⁶

$$\int_{\beta_1 \dots \beta_n \text{ conn}} \mathcal{D}g \mathcal{D}\Phi \left(\int d^2x \sqrt{g} \right)^m e^{-I_{\text{JT}}[g, \Phi]} = \left(2 \sum_{i=1}^n \beta_i \right)^m Z(\beta_1 \dots \beta_n)_{\text{conn}}, \quad (5.24)$$

which exponentiates indeed to equation (4.23) in [75]

$$\int_{\beta_1 \dots \beta_n \text{ conn}} \mathcal{D}g \mathcal{D}\Phi \exp \left(\lambda \int d^2x \sqrt{g} \right) e^{-I_{\text{JT}}[g, \Phi]} = \exp \left(2\lambda \sum_{i=1}^n \beta_i \right) Z(\beta_1 \dots \beta_n)_{\text{conn}}, \quad (5.25)$$

meaning we have shifted the overall energy scale with $E_0 = 2\lambda$. Our operator reproduces this behavior, up to contact terms. Indeed we have for instance (using again the string equation)

$$\sum_{a_1=0}^{\infty} \frac{(-2\pi^2)^{a_1}}{a_1!} \left\langle \tau_{a_1} \sum_{a_2=0}^{\infty} \frac{(-2\pi^2)^{a_2}}{a_2!} \tau_{a_2} \right\rangle_{\kappa} = \left\langle \sum_{a_2=0}^{\infty} \frac{(-2\pi^2)^{a_2+1}}{(a_2+1)!} \tau_{a_2} \right\rangle_{\kappa}, \quad (5.26)$$

which translated to Weil-Petersson volumes means

$$V_{g,2}(2\pi i, 2\pi i) = \int_0^{2\pi i} da a V_{g,1}(a) \neq 0. \quad (5.27)$$

These contributes from two operators coming into contact give corrections such as the second term in

$$Z(b=2\pi i, b=2\pi i, \beta_1 \dots \beta_n)_{\text{conn}} = \left(2 \sum_{i=1}^n \beta_i \right)^2 Z(\beta_1 \dots \beta_n)_{\text{conn}} + \int_0^{2\pi i} da a Z(b=a, \beta_1 \dots \beta_n)_{\text{conn}}, \quad (5.28)$$

which are absent for the blunt defects $\alpha = 2\pi$. In contrast, for topological gravity [55] the operator on the left-hand side of the string equation is τ_0 , and there are then no contact terms because there is no τ_{-1} . This is why in topological gravity τ_0 is exactly the area operator, also called puncture operator P .

The blunt defects $\alpha = 2\pi$ in JT gravity likewise have no contact terms [75] (in hyperbolic geometry this is because there is no geodesic that only surrounds a number of these blunt defects, but no handles). Our operators do, because they are built up out of an infinite number of sharp cusps. Their amplitudes are by construction analytic continuations of Weil-Petersson volumes, the blunt defects differ from this by the subtraction of contact terms. Therefore for $\alpha > \pi$ we rather have a relation like

$$\exp \left(\frac{1}{2} (i\alpha)^2 \psi_{\text{extra}} \right) \text{ minus contact terms} \Leftrightarrow \int d^2x \sqrt{g} \frac{2\pi}{\gamma(1-\alpha/2\pi)} e^{-(2\pi-\alpha)\Phi}, \quad \alpha > \pi. \quad (5.29)$$

So a description of blunt defects in terms of cohomology is a but more subtle, we understand that such a description is currently under construction [78].

²⁶If we are more careful with prefactors in [75] we see that actually the relevant operator is ϵA , then the Weil-Petersson volumes contribute $\epsilon 2\pi\chi = 0$ and the trumpets contribute $\epsilon \int du \sqrt{h} = \beta_i$ indeed. We use the quantization scheme of [75] here because it is more suitable for making contact with semiclassics, in that scheme for $\alpha \sim 2\pi$ there is roughly an extra prefactor $2\pi - \alpha$ in (5.3), this is the $2\pi/\gamma(1 - \alpha/2\pi)$ in (5.29) when α is close to 2π , and $\epsilon = 2\pi - \alpha$.

Note also that on the path integral level, the fact that we get a simple shift in the energy can also be understood from the fact that adding the area integral εA to the JT action can be removed by shifting the dilaton, which in turn causes a boundary term to appear that induces the shift in energy.

In summary, we can trust the dilaton gravity interpretation of the τ_k in (5.4) completely (also when inserting these τ_k in the action), as long as we do not insert infinite sums that conspire to defects with $\alpha < \pi$, at which point contact terms (dis)appear.

We end this section with some further comments and open questions.

1. It would be nice to have an exact description of our infinite sum (5.23) in dilaton gravity, then one could write up the dilaton gravity action for all minimal models (by turning off the KdV times γ_k (4.19) that took us from topological gravity to JT in the first place).²⁷ There is a description of topological gravity as pure Einstein-Hilbert gravity [56] (section 7.2), where the other minimal models have corrections in the action of the type (notice that equation (3.31) is then obvious via Gauss-Bonnet)

$$\tau_n \sim \int d^2x \sqrt{g} R^n. \tag{5.30}$$

It would be interesting to see how that worldsheet metric g and the JT gravity metric-and dilaton are related, particularly because there are no AdS₂ asymptotics in the first description.

2. It would be interesting to have some dilaton gravity understanding of the operator on the left-hand side of the dilaton equation(s) (5.20) and (5.21), analogous to how the string equation involves an area operator (5.23), in the spirit of having a better gravity understanding of the KdV hierarchy. In the quantization scheme of [19] we can immediately understand (5.20)²⁸

$$\sum_{a=0}^{\infty} \frac{(-2\pi^2)^a}{a!} \tau_{a+1} \quad \text{minus contact terms} \quad \Leftrightarrow \quad \int d^2x \sqrt{g} \frac{\epsilon\Phi - 1}{2\pi} = \frac{L - A}{2\pi} = -\chi, \tag{5.31}$$

where we used the boundary conditions on Φ to show that the boundary contribution (the length L) to the area A cancels in this observable, and the bulk part of the area evaluates indeed to $2\pi\chi$. To be clear, in the \Leftrightarrow we used our dictionary (5.7), and then we used the JT gravity description to find that the operator on the left computes the Euler character. This precisely reproduces the equation (5.20), which can be viewed as a subset of the KdV equations.

The operator in the dilaton equation (5.21) generates a (infinitesimal) metric rescaling (hence the name dilaton equation), which boils down to rescaling the boundary lengths

$$\sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \left(2 \sum_{i=1}^n \beta_i \partial_{\beta_i} - 2\chi \right)^m Z(\beta_1 \dots \beta_n)_{\text{conn}} = e^{-2\lambda\chi} Z(\beta_1 e^{2\lambda} \dots \beta_n e^{2\lambda})_{\text{conn}}. \tag{5.32}$$

²⁷See [30, 79, 80] for some (not obviously related) progress in that direction.

²⁸The $-A/2\pi$ comes from the $\alpha^{-1}\partial_\alpha$ working on the $2\pi - \alpha$ prefactor in the defect operator $\mathcal{O}_D(\alpha)$ in the quantization scheme of [19].

We can appreciate this by translating the operator in the left hand side of (5.21) to dilaton gravity variables, which indeed gives the generator of metric rescalings in the JT gravity action (for the χ part we remind ourselves of the relation between S_0 and Φ_0 , see for instance [81])

$$\sum_{a=0}^{\infty} (2a+1) \frac{(-2\pi^2)^a}{a!} \tau_{a+1} \text{ minus contact terms} \Leftrightarrow \int d^2x \sqrt{g} 2\Phi + \int_{\partial} du \sqrt{h} \Phi. \tag{5.33}$$

So the simplest L_{-1} and L_0 of the KdV equations have JT gravity interpretations as describing the covariant transformation rules of observables under certain changes of parameters in the action. This makes sense, as the KdV equations essentially state how F changes with the KdV times t_k . The other KdV equations $L_n Z = 0$ with $n > 0$ are essentially equivalent to (5.9) in combination with the statement that these deformed theories are matrix integrals with different spectral curves $\rho_0(E)$ [19]. These constraints state how observables change under changes in the dilaton gravity action with t_k and $k > 1$, namely by changing the spectral curve.

The string-and dilaton equation are special cases describing the flow under changes of t_0 and t_1 . This means that given the dictionary (5.7) one can derive the KdV hierarchy from the JT gravity path integral formulation, essentially.

3. Let us emphasize that without summing over the τ_k , that the KdV equations should be interpreted as recursion relations between cusps in JT gravity on closed manifolds. Without knowledge of explicit Weil-Petersson volumes these relations seem miraculous. It would be interesting to find an independent dilaton gravity calculation of even the simplest examples, for instance the correlator of one cusp on the torus

$$\int_{\text{torus}} \mathcal{D}g \mathcal{D}\Phi e^{-I_{\text{JT}}} \int d^2x \sqrt{g} e^{-2\pi\Phi} \Phi^{2k} = \left(\text{Diagram of a torus with a cusp} \right) k = 0, \quad k \neq 1. \tag{5.34}$$

Intersection theory predicts in particular that this vanishes for all $k \neq 1$ and it would be interesting to have a direct dilaton gravity argument for this.

Remember furthermore the relation (5.5)

$$\int d^2x \sqrt{g} e^{-(2\pi-\alpha)\Phi} \Leftrightarrow \exp\left(\frac{1}{2}(i\alpha)^2 \psi_{\text{extra}}\right), \quad \alpha < \pi, \tag{5.35}$$

and notice that the right-hand side is even in α . This implies that correlators of cusps multiplied with odd powers Φ^{2k+1} vanish identically in JT gravity (on any surface, and regardless of other operators)

$$\int d^2x \sqrt{g} e^{-2\pi\Phi} \Phi^{2k+1} = 0. \tag{5.36}$$

This too is mysterious without explicit knowledge of the volumes, it boils down to understanding why the Weil-Petersson volumes are even polynomials of b_i^2 . One

can view this as a redundancy, or a null deformation of the type discussed in [44]. Indeed, one can add terms like (5.36) to the JT action as in (5.9) without affecting any observables. So theories whose actions differ by terms of the type (5.36) are completely equivalent, and having them or not having them is a gauge choice. As a simple check on this, notice that a potential $U(\Phi)$ as in (5.36) is in the kernel of equation (1.4) in [18], the spectral curve (and thus all observables) remain unaffected. This redundancy is also why (5.6) is an analytic continuation of (5.5), one can replace the cosine by either one of the exponentials without changing any observables.

It is natural to wonder if there is some contour-integral argument for (5.36), directly in the dilaton gravity formulation (without going through the volumes).

5.2 Matrix integral

As the final element in the web of dualities between intersection numbers, matrix integrals and dilaton gravities we present the matrix integral interpretation of the ψ -classes. We claim that every insertion of τ_k corresponds with the following counterclockwise contour integral around the real axis (a contour above and below the real axis, the latter is denoted by R)

$$\tau_k \Leftrightarrow \frac{(-1)^{k+1}}{i} \frac{1}{(2k+1)!!} \oint_R dE E^{k+1/2} \text{Tr} \delta(H-E) = \frac{(-1)^{k+1}}{i} \frac{1}{(2k+1)!!} \oint_R dE E^{k+1/2} \rho(E). \tag{5.37}$$

We will derive this equation momentarily, but let us first give a few basic checks that it is correct, and show how this equation is useful in practice. (To our knowledge, this equation and its practical use are new, whereas versions of (5.56) were known.)

As a first check notice that $\rho_0(E)$ has no poles on the real axis, for instance for JT gravity $\rho_0(E) = e^{S_0} \sinh(2\pi E^{1/2})/4\pi^2$ for positive energies and zero otherwise. So we recover the statement that $\langle \tau_k \rangle_0 = 0$, which is true for generic backgrounds γ_p . In intersection theory this follows from the selection rule (3.9), which for $g = 0$ we can rewrite as

$$\sum_{i=1}^n (k_i - 1) + k = -2, \quad k_i \geq 2, \tag{5.38}$$

where the k_i come from expanding out the background and $k_i \geq 2$ because $\gamma_0 = \gamma_1 = 0$ (in particular, $\gamma_0 = 0$ is important). The left-hand side is non-negative so this is never satisfied, so $\langle \tau_k \rangle_0 = 0$ indeed.²⁹

At non-zero genus we obtain finite answer, because $E^{k+1/2} \rho(E)_g$ generically has poles at $E = 0$. To appreciate how this arises we can consider the inverse Laplace transform of (2.14)

$$Z_g(\beta) = \int_0^\infty dE e^{-\beta E} \rho_g(E) \quad \Rightarrow \quad \rho_g(E) = \int_0^\infty db b \frac{1}{2\pi E^{1/2}} \cos(bE^{1/2}) V_{g,1}(b), \tag{5.39}$$

where the volume is an even polynomial (2.16)

$$V_{g,1}(b) = \sum_{d=0}^\infty V_{g,d} \frac{b^{2d}}{4^d d!} \quad \Rightarrow \quad \rho_g(E) = \frac{1}{2\pi} \sum_{d=0}^\infty V_{g,d} (-1)^{d+1} \frac{(2d+1)!!}{2^d} E^{-d-3/2}. \tag{5.40}$$

²⁹Notice that the cases $k \leq -2$ can and do give nonzero answers at genus zero, as we found around equation (3.36).

Using this expressions and picking up the pole at the origin we obtain

$$\langle \tau_k \rangle_g = \frac{(-1)^{k+1}}{i} \frac{1}{(2k+1)!!} \oint_R dE E^{k+1/2} \rho(E)_g \quad (5.41)$$

$$= \sum_{d=0}^{\infty} V_{g,d} (-1)^{k+d} \frac{1}{2^d} \frac{1}{2\pi i} \oint_0 dE E^{-1+k-d} = \frac{1}{2^k} V_{g,k}, \quad (5.42)$$

which means that the operator (5.37) computes the expansion coefficients of the volumes. In other words our matrix integral definition (5.37) is recovering the expansion of volumes in intersection numbers (4.3) (in a generic background γ_k)

$$V_{g,n}(b_1 \dots b_n) = \sum_{d_1=0}^{\infty} \frac{b_1^{2d_1}}{2^{d_1} d_1!} \dots \sum_{d_n=0}^{\infty} \frac{b_n^{2d_n}}{2^{d_n} d_n!} \langle \tau_{d_1} \dots \tau_{d_n} \rangle_g. \quad (5.43)$$

The equation above was for $n = 1$ but it is obvious that this extends to generic n .

As a further check on this we can compute the disk with a puncture. This example involves picking up the pole at $E_1 = E_2$ of the genus zero wormhole, equation (139) in [11],

$$\langle \rho(M) \tau_k \rangle_{0 \text{ conn}} = \frac{(-1)^{k+1}}{i} \frac{1}{(2k+1)!!} \oint_R dE E^{k+1/2} \langle \rho(M) \rho(E) \rangle_{0 \text{ conn}} \quad (5.44)$$

$$= \frac{(-1)^k}{i} \frac{1}{(2k+1)!!} \frac{1}{4\pi^2 M^{1/2}} \oint_M dE E^k (E+M) \frac{1}{(E-M)^2}$$

$$= \frac{1}{2\pi} \frac{(-1)^k}{(2k-1)!!} M^{k-1/2}, \quad (5.45)$$

such that

$$\langle Z(\beta) \tau_k \rangle_{0 \text{ conn}} = \frac{1}{2\pi} \frac{(-1)^k}{(2k-1)!!} \int_0^\infty dM e^{-\beta M} M^{k-1/2} = \frac{(-1)^k}{2^{k+1} \pi^{1/2}} \beta^{-1/2-k}. \quad (5.46)$$

We can compare this with the prediction that one gets by viewing the trumpet as a generating function for these one-point functions, along the lines of (5.6)

$$Z_{\text{trumpet}}(\beta, b) = \frac{1}{2\pi^{1/2} \beta^{1/2}} e^{-\frac{b^2}{4\beta}} = \sum_{k=0}^{\infty} \frac{b^{2k}}{2^k k!} \frac{(-1)^k}{2^{k+1} \pi^{1/2}} \beta^{-1/2-k} = \sum_{k=0}^{\infty} \frac{b^{2k}}{2^k k!} \langle Z(\beta) \tau_k \rangle_{0 \text{ conn}}, \quad (5.47)$$

which indeed gives the same answer on the nose. Finally we also recover the statement that the cylinder amplitude vanishes in intersection theory (or the genus zero two point function)³⁰

$$\langle \tau_n \tau_k \rangle_0 = \frac{(-1)^{n+1}}{i} \frac{1}{(2n+1)!!} \oint_R dE E^{n+1/2} \langle \rho(E) \tau_k \rangle_{0 \text{ conn}}$$

$$= \frac{(-1)^{n+1+k}}{2\pi i} \frac{1}{(2n+1)!! (2k-1)!!} \oint_R dE E^{n+k} = 0. \quad (5.48)$$

³⁰Actually, if we include the contribution from unstable surfaces, we should continue to all integers n and k , in which case we get $\langle \tau_{-1-k} \tau_k \rangle_0 = (-1)^k$. This is consistent with the $1/(x_1 + x_2)$ in (3.43).

For generic backgrounds γ_k this follows from the same logic as around (5.38) (but with -1 on the right).

Now we explain how to derive this equation (5.37) for τ_k . The first step is proving an identity that appears in [37, 69], which holds miraculously for each arbitrary (but fixed) number E_0

$$\begin{aligned} \text{Tr} \left(\frac{1}{y-H} \right) - \frac{\text{Tr}(1)}{y} &= \sum_{k=1}^{\infty} (-1)^{k+1} (-y-E_0)^{-k-1/2} (-y+E_0)^{-1/2} \text{Tr} \left((H+E_0)^{k-1/2} (H-E_0)^{1/2} \right)_+ . \end{aligned} \quad (5.49)$$

The branchcuts of the roots are chosen in the usual way, we have $(-y-E_0)^{-k-1/2} > 0$ when $y < -E_0$ and $(-y+E_0)^{-1/2} > 0$ when $y < E_0$. Let us prove this equation for $y > E_0$ and real, after collecting all the signs it simplifies slightly

$$\begin{aligned} \text{Tr} \left(\frac{1}{y-H} \right) - \frac{\text{Tr}(1)}{y} &= \sum_{k=1}^{\infty} (y+E_0)^{-k-1/2} (y-E_0)^{-1/2} \text{Tr} \left((H+E_0)^{k-1/2} (H-E_0)^{1/2} \right)_+ \\ &= \sum_{q=1}^{\infty} \frac{1}{y^{q+1}} \text{Tr}(H^q) . \end{aligned} \quad (5.50)$$

The $+$ means we should expand in powers of $1/H$ and keep only the terms with positive powers of H in the resulting series. To check this, one explicitly expands the binomials in $1/H$, and then rearranges the resulting triple sum to collect the terms that multiply $\text{Tr}(H^q)$, for fixed q . The double sum at fixed q surprisingly spits out $1/y^{q+1}$, and we recover the large y expansion of the left-hand side [43].

Now we can consider the following double scaling limit of (5.49) where we take $y = -E_0 - z^2$, shift also H by E_0 and send $E_0 \rightarrow \infty$

$$\text{Tr} \left(\frac{1}{z^2+H} \right) - \frac{\text{Tr}(1)}{E_0+z^2} = \sum_{k=1}^{\infty} (-1)^k z^{-2k-1} W_{k-1}(H) , \quad (5.51)$$

where we have introduced the so called scaling polynomials of [37]

$$W_{k-1}(H) = \lim_{E_0 \rightarrow \infty} (2E_0)^{-1/2} \text{Tr} \left(H^{k-1/2} (H-2E_0)^{1/2} \right)_+ . \quad (5.52)$$

The $+$ is confusing in this context, because it is unclear what the expansion in powers of H means upon double scaling. Below we will give a more correct formula that does not involve the subscript $+$ anymore, but is purely based on a contour integral, and therefore does make sense in the double scaling continuum limit.

Using the definition of an FZZT brane [11, 12, 30, 34, 69, 79, 80, 82–88]

$$\mathcal{O}_{\text{FZZT}}(z) = \text{Tr} \log(z^2+H) - \text{Tr}(1) \log(z^2) = \int_{\infty}^z dw 2w \left(\text{Tr} \left(\frac{1}{w^2+H} \right) - \frac{\text{Tr}(1)}{w^2} \right) \quad (5.53)$$

and its relation with a geodesic boundary [34] (we are not giving any topological weight to FZZT branes here, this is an irrelevant choice)

$$\mathcal{O}_{\text{G}}(b) = -\frac{e^{-S_0}}{2\pi i} \int_{-i\infty}^{+i\infty} dz e^{bz} \mathcal{O}_{\text{FZZT}}(z) , \quad (5.54)$$

we obtain an expansion of the geodesic boundary in scaling polynomials³¹

$$\begin{aligned}
 \mathcal{O}_G(b) &= \frac{e^{-S_0}}{b} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz e^{bz} 2z \left(\text{Tr} \left(\frac{1}{z^2 + H} \right) - \frac{\text{Tr}(1)}{E_0 + z^2} \right) \\
 &= 2 e^{-S_0} \sum_{k=1}^{\infty} (-1)^k W_{k-1}(H) \frac{1}{b} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz e^{bz} z^{-2k} \\
 &= e^{-S_0} \sum_{k=0}^{\infty} (-1)^{k+1} \frac{b^{2k}}{(2k)!} \frac{2}{2k+1} W_k(H). \tag{5.55}
 \end{aligned}$$

Comparing with the expansion of a geodesic boundary in cusp defects (5.6), we find the identification of the τ_k insertions with the scaling polynomials [37]

$$\mathcal{O}_G(b) = e^{-S_0} \sum_{k=0}^{\infty} \frac{b^{2k}}{2^k k!} \tau_k \quad \Rightarrow \quad \tau_k \Leftrightarrow (-1)^{k+1} \frac{2}{(2k+1)!} W_k(H). \tag{5.56}$$

Now the question is what the correct implementation of $W_k(H)$ in the double scaling limit is. In (5.52) the last square root $(H - 2E_0)^{1/2}$ is on the branchcut, so depending on whether H takes values slightly above or below the real axis we get respectively a factor $\pm i$. We thus have two possible contours for the eigenvalues of H in this observables. We claim that the correct combination is the average of both contours

$$W_k(H) = \frac{i}{2} \int_{R+i\epsilon} dE E^{k+1/2} \text{Tr} \delta(H - E) - \frac{i}{2} \int_{R-i\epsilon} dE E^{k+1/2} \text{Tr} \delta(H - E) \tag{5.57}$$

We can flip the orientation of the first contour, so that this becomes a counterclockwise contour integral around the real axis, which picks up any poles that might occur on the real axis in the integrand

$$W_k(H) = \frac{1}{2i} \oint_R dE E^{k+1/2} \text{Tr} \delta(H - E). \tag{5.58}$$

This reproduces our original claim (5.37), which we have independently proven to be correct above, by reproducing all intersection numbers from it.

To close off this section we remark how this relates to the Kontsevich matrix integral [46] and the equation (5.9). We have

$$\begin{aligned}
 \prod_{i=1}^{\infty} \det(1 + H/z_i^2) &= \exp \left(\sum_{i=1}^{\infty} \mathcal{O}_{\text{FZZT}}(z_i) \right) \Leftrightarrow \exp \left(\sum_{k=2}^{\infty} t_k(z_j) \tau_k \right), \\
 t_k(z_j) &= -(2k-1)!! \sum_{i=1}^{\infty} z_i^{-2k-1}, \tag{5.59}
 \end{aligned}$$

where in the first equality we used $\det(A) = \exp(\text{Tr}(\log(A)))$, and in the second equality we computed the w integral in (5.53) using the expansion in scaling polynomials (5.51), and identified the τ_k 's using (5.56). Kontsevich proved that this identity is correct, namely

³¹It is important that the FZZT branes actually have their poles at $z = -\epsilon \pm i\sqrt{H}$ [43], such that in reality the integrand contains $(z + \epsilon)^{-2k}$. Since $b > 0$ we can close the contour to $z = -\infty$, and there is no pole at infinity because of the second term on the first line.

inserting the left hand side in a matrix integral with a spectral curve corresponding to KdV times γ_k generates a matrix integral whose spectral curve corresponds with KdV times $\gamma_k + t_k(z_j)$. He did this by proving that the right-hand side equals F for those KdV times (with F the function that satisfies the KdV hierarchy (3.13)), see also appendix C.2.

In essence, this relation is the *raison d'être* of the scaling polynomials, we can start with a Gaussian matrix integral (which double scales to topological gravity [11]) and turn on some scaling polynomials in the action to get a double scaled theory whose spectral curve has $t_k(z_j)$ turned on.

6 Concluding remarks

We end this paper with three comments.

In section 6.1 we use the KdV equations (3.12) to demonstrate that the $(2g)!$ growth in volumes $V_{g,2}(b_1, b_2)$ for generic models comes from terms in the polynomials with order one powers of b_1^2 and b_2^2 . This is part of the reason why the genus expansion converges in the τ -scaling limit.

In section 6.2 we contemplate the multi-boundary generalization of our discussion, in particular we consider the multi-boundary generalization of the τ -scaled spectral form factor and discuss cancellations.

In section 6.3 we explain the logical need for a Lorentzian interpretation of the cancellations and the universal scaling behavior (3.50) of late-time genus g wormholes, and preview how this comes about by thinking about Lorentzian topology change in JT gravity [38].

6.1 Open-closed duality and factorial growth

As a final application of the KdV equations in gravity we want to give an intuitive argument explaining that the famous $(2g)!$ growth in volumes $V_{g,2}(b_1, b_2)$ for generic models of dilaton gravity comes from terms in the polynomials with order one powers of b_1^2 and b_2^2 (so not scaling with g). Therefore they will not survive in our $T \rightarrow \infty$ limit, which acts at each genus g individually because we also send $e^{S_0} \rightarrow \infty$. This is part of the explanation why the genus expansion is convergent in our setup, see section 2.2.

For concreteness we consider the constant term in the polynomials (4.3)

$$V_{g,2}(0, 0) = \left\langle \tau_0^2 \exp \left(\sum_{k=2}^{\infty} \gamma_k \tau_k \right) \right\rangle_g. \tag{6.1}$$

We would like to argue that this grows as $(2g)!$ for large genus g , and that the terms with large powers of b_1^2 and b_2^2 do not. The basic intuition is that for small powers of b_i^2 , following the selection rule (3.9), we can get many, many low-dimensional forms such as τ_2 coming out of the exponential, and it makes sense that such correlators with many, many operators would grow factorially. On the other hand, for large powers of b_i^2 the selection rule (3.9) does not allow for that many operators to come down from the exponential (for the maximal power no operators come from the exponential and we are effectively computing Airy volumes again). With relatively fewer operators, it is hard to imagine factorial growth.

We believe this intuition should universally hold, but to make our point concrete we will here focus on the (2, 3) minimal string with spectrum [11, 69]

$$2\pi e^{-S_0} \rho_0(E) = E^{1/2} + \frac{t_2}{3} E^{3/2}. \tag{6.2}$$

This has an eigenvalue instanton saddle near $E = -3/t_2$ in the matrix integral and therefore, following the techniques of section 5.6 in [11] one expects factorial growth of the type (in this section we ignore for simplicity of presentation most sub-exponential g -dependence)

$$V_{g,2}(0, 0) = \langle \tau_0^2 \exp(t_2 \tau_2) \rangle_g \sim (2g)! t_2^{3g}, \tag{6.3}$$

where we used (4.15) and the fact that $u_0 = 0$ for this deformation. This arises from intersection theory as follows. Using the selection rule (3.9) we immediately obtain the correct scaling with t_2

$$V_{g,2}(0, 0) = \frac{t_2^{3g-1}}{(3g-1)!} \langle \tau_0^2 \tau_2^{3g-1} \rangle_g. \tag{6.4}$$

Now we want to prove that the correlator $\langle \tau_0^2 \tau_2^{3g-1} \rangle_g$ for large genus grows as $(5g)!$, such that combined with the $1/(3g)!$ we get the predicted double factorial growth. For this we can use the happy fact that we can rewrite the KdV recursion (3.12) entirely as a recursion relation for

$$f(g) = \langle \tau_0 \tau_1 \tau_2^{3g-2} \rangle_g = \frac{1}{3g-1} \langle \tau_0^2 \tau_2^{3g-1} \rangle_g, \tag{6.5}$$

by considering $k = 2$ and $d_2 = 3g - 2$ with all other $d_j = 0$, and repeatedly using the string-and dilaton equations (3.30) and (3.31) to eliminate excess τ_0 and τ_1 's in all terms. For large genus, the dominant contributions to (3.12) in this setup come from the first, and the last term. One can check this by first assuming that this is true, then one finds the recursion relation

$$f(g) \sim g^5 f(g-1) \quad \Rightarrow \quad f(g) \sim (5g)!. \tag{6.6}$$

One then checks that with this $g!^5$ growth the other terms in the recursion relation become subleading indeed. Being more careful with all the prefactors and sub-factorial g -dependence one can recover the full prediction of the eigenvalue instanton from this recursion relation (3.12).

6.2 Multi-boundary generalization

Let us briefly comment on the multi-boundary generalization of the story that we have presented here.³² For instance, we could consider the three-boundary generalization of the spectral form factor

$$Z(\beta + iT_1, \beta + iT_2, \beta + iT_3)_{\text{conn}}, \quad T_1 + T_2 + T_3 = 0, \tag{6.7}$$

in the generalized τ -scaling limit where $\tau_1 = T_1 e^{-S_0}$ and $\tau_2 = T_2 e^{-S_0}$ remain finite (and their difference remains finite as well). The triple energy integral in (2.1) is then dominated

³²We thank Douglas Stanford for discussions on this.

by E_1 , E_2 and E_3 all close together, and we can use the three-boundary generalization of the sine-kernel (2.3), which can be found for instance in [12]. Doing the Fourier transforms over the small energy differences explicitly, one finds a generalization of the ramp-and-plateau (2.5)

$$Z(\beta+iT_1, \beta+iT_2, \beta-iT_1-iT_2)_{\text{conn}} = \int_0^\infty dE e^{-3\beta E} \begin{cases} 0 & \frac{T_1}{2\pi} + \frac{T_2}{2\pi} < \rho_0(E) \\ \frac{T_1}{2\pi} + \frac{T_2}{2\pi} - \rho_0(E) & \frac{T_1}{2\pi} < \rho_0(E) < \frac{T_1}{2\pi} + \frac{T_2}{2\pi} \\ \frac{T_2}{2\pi} & \frac{T_2}{2\pi} < \rho_0(E) < \frac{T_1}{2\pi} \\ \rho_0(E) & \rho_0(E) < \frac{T_2}{2\pi}, \end{cases} \quad (6.8)$$

where we have specialized to $T_2 < T_1$. Following the steps that led to (2.10), this can be massaged into

$$Z(\beta+iT_1, \beta+iT_2, \beta-iT_1-iT_2)_{\text{conn}} = \frac{e^{S_0}}{6\pi\beta} \int_0^{\tau_2} df e^{-3\beta E(f)} - \frac{e^{S_0}}{6\pi\beta} \int_{\tau_1}^{\tau_1+\tau_2} df e^{-3\beta E(f)}, \quad (6.9)$$

for instance for topological gravity $E(f) = f^2$ this is simply the sum of three Erf functions. For infinite times the first term produces the generalized plateau value $Z(3\beta)$, which comes from terms in the triple sum where all energies coincide. However, notice that if one Taylor expands this in e^{-S_0} that we again get a series in powers of e^{-2S_0} starting at e^{-2S_0} , whereas the three-boundary connected amplitudes scale as $e^{-(2g+1)S_0}$ with g the genus. So, unlike for the two-boundary spectral form factor we do not see any obvious way to reproduce this tau-scaled answer from the sum over wormhole geometries with three boundaries, at least for now.

In some sense this emphasizes how nice it is that this worked for two boundaries, but on the other hand it also shows that we might still need D-brane effects to understand more complicated phenomena.

One thing that does happen for any number of boundaries is that the cancellations in intersection theory (3.53) constrain the volumes in a non-trivial manner for all theories. Namely with the constraint $T_1 + \dots + T_n = 0$ we find the generalization of (4.29) for any of the times $T_i \rightarrow \infty$

$$e_1^{m_1} e_2^{m_2} \dots e_n^{m_n} \sim T_i^2 \sum_{j=2}^n m_j. \quad (6.10)$$

The power of T that appears here is precisely the expression that was constrained by the KdV equations to be upper bound by $2g$, therefore the maximal power of any of the times T_i is constrained

$$Z_g(\beta + iT_1 \dots \beta + iT_n)_{\text{conn}} \sim T_i^{2g+1} e^{-(2g+n-2)S_0}, \quad T_1 + \dots + T_n = 0. \quad (6.11)$$

On the other hand dimensional analysis of the volumes for generic dilaton gravities would again suggest that naively we could get powers up to T_i^{3g-2+n} . So there are also major cancellations in $V_{g,n}(b_1 \dots b_n)$ for generic dilaton gravity models. It remains to be seen whether or not those are related with explaining a generalization of the ramp-and-plateau such as (6.9). That would be interesting.

6.3 Universal powers of time via Lorentzian topology change

We want to stress that recovering the universal growth T^{2g+1} is highly non-trivial from the Euclidean gravitational path integral at genus g . In 2d dilaton gravity, we understand why this happens, namely because of known cancellations in intersection numbers. But the scaling of wormhole amplitudes with T^{2g+1} for late times should hold for essentially any gravity model in a regime dominated by black holes, because of random matrix universality [15].

It is likely that one could argue that late-time physics is effectively dominated by the near-horizon region of black holes, where one often recovers JT gravity. Nevertheless we do not think an explanation in terms of intersection numbers for real-life black holes is fully satisfactory.

Instead, we view the fact that the scaling T^{2g+1} is highly non-trivial in Euclidean signature (requiring quite miraculous, exact cancellations) as a sign that we should really be looking for a Lorentzian picture. After all, without Lorentzian large times there is nothing in the genus g amplitudes alerting us of any cancellations. Because of the universality, we think the Lorentzian picture should be sufficiently simple (unlike the Euclidean story), in the sense that one could imagine a generalization to higher dimensional black holes.

We believe that we have found such an interpretation by thinking about Lorentzian topology change. The idea is that we can build Lorentzian wormhole geometries using the crotch singularities of Louko-Sorkin [76, 89].³³ For late enough times, the locations of the crotches (these are the places where baby universes or wormholes are born, and die) should behave approximately as zero modes. These $2g$ zero modes give a volume factor T^{2g} , which along with the usual factor T from the rotational zero mode of the double cone [3] reproduces the universal T^{2g+1} .

We will present this picture elsewhere in more detail, with concrete calculations in JT gravity [38].

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A Examples

We invite interested readers to take their favorite spectral curve $y(z)$ of the form (2.15), compute the genus g wormhole using topological recursion, and compare the result with the expansion (2.10). It is satisfactory to see that they match, especially given how non-trivial the cancellations (2.23) are.

³³For other recent appearances of singular spacetimes in Lorentzian signature see for instance [90].

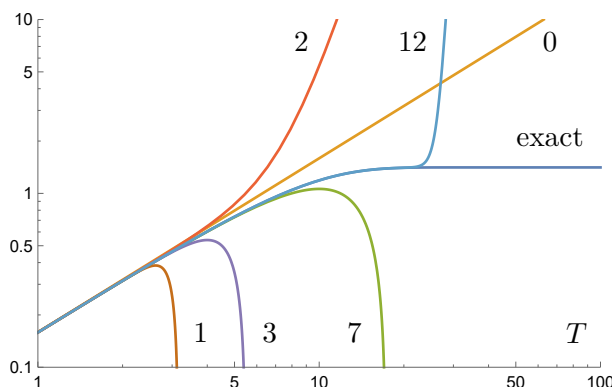


Figure 4. The τ -scaling limit of the sum over genus g wormholes in the Airy model (A.2) up to $g = g_{\max}$ (numbers shown) with $e^{S_0} = 10$ and $\beta = 1/2$. This series converges to the exact expression (A.1) for all T (infinite radius of convergence) and in particular it converges to the plateau.

It is quite elementary to prove this experimentally for fixed genus and generic spectral curve (2.15) by computing both sides for generic f_k , but we leave explicit expressions out to spare the reader’s eyes. Instead let us go through a few basic examples where we also have an analytic formula for the inverse of the spectrum $E(f)$.

Topological gravity. The Airy model has spectral curve $\rho_0(E) = e^{S_0} E^{1/2}/2\pi$, so the inverse spectrum is $E(f) = f^2$ and (2.6) evaluates to (we introduce $\tau = T e^{-S_0}$) [21, 22]

$$Z(\beta + iT, \beta - iT)_{\text{conn}} = \frac{e^{S_0}}{4\pi\beta} \int_0^\tau df e^{-2\beta E(f)} = \frac{e^{S_0}}{2^{7/2}\pi^{1/2}\beta^{3/2}} \text{Erf}\left(2^{1/2}\beta^{1/2}T e^{-S_0}\right) \quad (\text{A.1})$$

Using $\text{Erf}(\infty) = 1$ we see the plateau value $Z_0(2\beta)$ or the Airy model. The (convergent) Taylor expansion is

$$Z(\beta + iT, \beta - iT)_{\text{conn}} = \frac{1}{4\pi\beta} T - \frac{1}{6\pi} T^3 e^{-2S_0} + \frac{\beta}{10\pi} T^5 e^{-4S_0} - \frac{\beta^2}{21\pi} T^7 e^{-6S_0} + \frac{\beta^3}{54\pi} T^9 e^{-8S_0} \dots \quad (\text{A.2})$$

This expansion is recovered on the nose using topological recursion with the spectral curve $y(z) = z/2$, see figure 4. The volumes to be used in (2.14) in this case are

$$\begin{aligned} V_{1,2}(b_1, b_2) &= \frac{b_1^4 + 2b_2^2 b_1^2 + b_2^4}{192} \\ V_{2,2}(b_1, b_2) &= \frac{b_1^{10} + 15b_2^2 b_1^8 + 58b_2^4 b_1^6 + 58b_2^6 b_1^4 + 15b_2^8 b_1^2 + b_2^{10}}{4423680} \\ V_{3,2}(b_1, b_2) &= \frac{5b_1^{16} + 200b_2^2 b_1^{14} + 2156b_2^4 b_1^{12} + 8048b_2^6 b_1^{10} + 12140b_2^8 b_1^8 + \text{symmetric}}{4280706662400} \end{aligned} \quad (\text{A.3})$$

In the final term in (A.2), a possible $T^9 e^{-6S_0}$ does not appear because of the cancellation (2.21). Notice that the cancellations are exact, in line with the statement that (2.3) is effectively exact in the τ -scaling limit. At genus 4 there is another cancellation responsible for the lack of an $T^{11} e^{-8S_0}$ term, and so on.

JT gravity. For JT gravity the spectrum is $\rho_0(E) = e^{S_0} \sinh(2\pi E^{1/2})/4\pi^2$ [91–94] so the inverse spectrum is $E(f) = \operatorname{arcsinh}^2(2\pi f)/4\pi^2$. Using

$$\int_0^{E(\tau)} dE \partial_E f(E) e^{-2\beta E} = \int_0^{E(\tau)^{1/2}} dw \partial_w f(w) e^{-2\beta w^2} = \int_0^{E(\tau)^{1/2}} dw \cosh(2\pi w) e^{-2\beta w^2}, \tag{A.4}$$

one computes the exact tau-scaled connected spectral form factor (2.10) as [22]

$$\begin{aligned} Z(\beta+iT, \beta-iT)_{\text{conn}} &= \frac{e^{S_0}}{2^{9/2}\pi^{1/2}\beta^{3/2}} e^{\frac{\pi^2}{2\beta}} \operatorname{Erf}\left(2^{1/2}\beta^{1/2}\left(\operatorname{arcsinh}\left(2\pi T e^{-S_0}\right)/2\pi - \pi/2\beta\right)\right) \\ &\quad - \frac{e^{S_0}}{2^{9/2}\pi^{1/2}\beta^{3/2}} e^{\frac{\pi^2}{2\beta}} \operatorname{Erf}\left(2^{1/2}\beta^{1/2}\left(\operatorname{arcsinh}\left(-2\pi T e^{-S_0}\right)/2\pi - \pi/2\beta\right)\right). \end{aligned} \tag{A.5}$$

We recognize the JT disk for $T = \infty$ using $\operatorname{Erf}(\pm\infty) = \pm 1$. For large β this reduces to the Airy answer since at low energies the spectrum reduces to the Airy spectrum. The Taylor series in T^{2g+1} starts with

$$\begin{aligned} Z(\beta+iT, \beta-iT)_{\text{conn}} &= \frac{1}{4\pi\beta} T - \frac{1}{6\pi} T^3 e^{-2S_0} + \frac{3\beta + 4\pi^2}{30\pi} T^5 e^{-4S_0} \\ &\quad - \frac{15\beta^2 + 60\pi^2\beta + 64\pi^2}{315\pi} T^7 e^{-6S_0} + \dots \end{aligned} \tag{A.6}$$

These are the same coefficients found by taking the Weil-Peterson volumes, doing the Laplace transforms with the trumpet partition functions (2.14), and then taking the τ -scaling limit. The first such volume is

$$V_{1,2}(b_1, b_2) = \frac{1}{192} (4\pi^2 + b_1^2 + b_2^2)(12\pi^2 + b_1^2 + b_2^2) \tag{A.7}$$

The higher genus examples you should keep hidden away in your Mathematica code. One feature worth noting is that for $b_1 \gg 1$ and $b_2 \gg 1$ these reduce to the Airy volumes (A.3), see section 3.1. Notice again there is no $T^9 e^{-6S_0}$ because of the cancellation (2.21).

In this case plotting the first few terms of (A.6) does not immediately converge to the plateau. The reason is that the Taylor expansion of (A.5) has a finite radius of convergence $T e^{-S_0} < 1/2\pi$. This in turn happens because $E(f)$ has a branchcut for $2\pi i f < -1$, so the Taylor series is only guaranteed to converge in the disk with radius $1/2\pi$. This same argument applies to the exact answer (A.5) because of the $\operatorname{arcsinh}(2\pi T e^{-S_0})$. But for $T e^{-S_0} < 1/2\pi$ the series converges uniquely to (A.5), and its analytic continuation to all $T > 0$ is obviously real and smooth, and includes the plateau, see figure 5.

Other minimal strings. As final example for the fact that the cancellations (2.23) occur for all spectra, and that the resulting perturbative genus expansion (2.10) converges to the plateau, we consider

$$f(E) = 2\pi e^{-S_0} \rho_0(E) = E^{1/2} + \frac{t_2}{3} E^{3/2}. \tag{A.8}$$

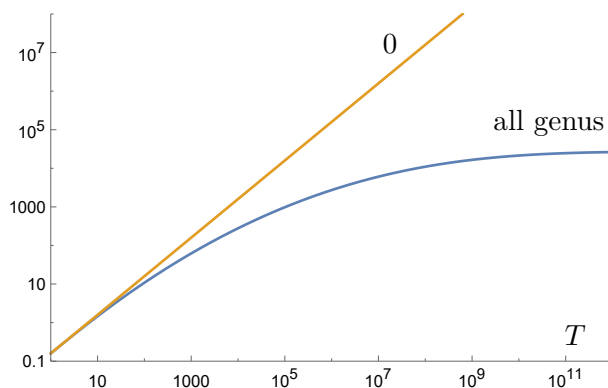


Figure 5. The τ -scaling limit of the sum over genus g wormholes in JT gravity (A.5) with $e^{S_0} = 10$ and $\beta = 1/2$. We have not shown individual terms in the series, because it converges only when $Te^{-S_0} < 1/2\pi$, which is a very short time on this plot. Importantly though, the sum over all genus uniquely gives the exact answer (A.5). The JT plateau is much higher and hence reached much later than the Airy plateau, because of the exponentially larger spectral density.

This is an example of the $(2, p)$ minimal string with odd p for which the spectrum is [11, 69]

$$2\pi e^{-S_0} \rho_0(E) = \frac{(2\kappa)^{1/2}}{p} \sinh\left(\frac{p}{2} \operatorname{arccosh}\left(1 + \frac{E}{\kappa}\right)\right) = \frac{1}{p} E^{1/2} U_{p-1}\left(\left(1 + \frac{E}{\kappa}\right)^{1/2}\right), \quad (\text{A.9})$$

where the even Chebychev polynomials give an expansion in positive integer powers of $1 + E/\kappa$. With $p = 3$ and $t_2 = 2/\kappa$ this becomes our example. These models have known dilaton gravity interpretations [30, 79, 95]. Using the first expression one finds (the same follows from Lagrange inversion)

$$E(f) = \frac{4}{t_2} \sinh^2\left(\frac{1}{3} \operatorname{arcsinh}\left(\frac{3t_2^{1/2}}{2} f\right)\right). \quad (\text{A.10})$$

The exact connected spectral form factor in the τ -scaling limit becomes

$$\begin{aligned} Z(\beta + iT, \beta - iT)_{\text{conn}} &= \frac{e^{S_0}}{4\pi\beta} \int_0^{E(\tau)^{1/2}} dw \partial_w f(w) e^{-2\beta w^2} \\ &= \frac{e^{S_0}}{8\pi\beta} \int_{-E(\tau)^{1/2}}^{+E(\tau)^{1/2}} dw (1 + t_2 w^2) e^{-2\beta w^2}. \end{aligned} \quad (\text{A.11})$$

A nice thing about the $(2, p)$ minimal models is that we know $E(f)$ as meromorphic functions, not just as Taylor series so we can plot the exact answer of this integral, see figure 6.

The resulting Taylor expansion

$$\begin{aligned} Z(\beta + iT, \beta - iT)_{\text{conn}} &= \frac{1}{4\pi\beta} T - \frac{1}{6\pi} T^3 e^{-2S_0} + \frac{3\beta + 2t_2}{30\pi} T^5 e^{-4S_0} \\ &\quad - \frac{6\beta^2 + 12t_2\beta + 7t_2^3}{126\pi} T^7 e^{-6S_0} + \dots \end{aligned} \quad (\text{A.12})$$

is again reproduced by the relevant volumes, which we emphasize requires t_2 -dependent cancellations. Much like in the JT example, the series expansion has some finite radius of

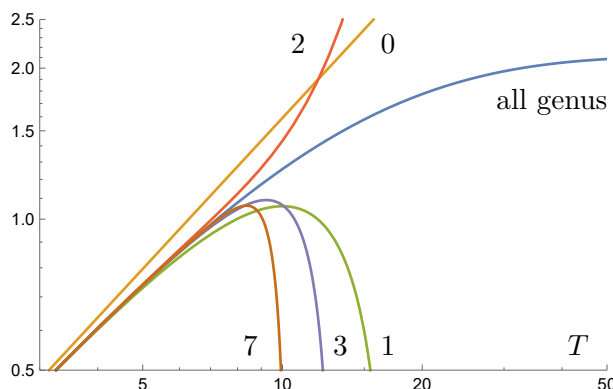


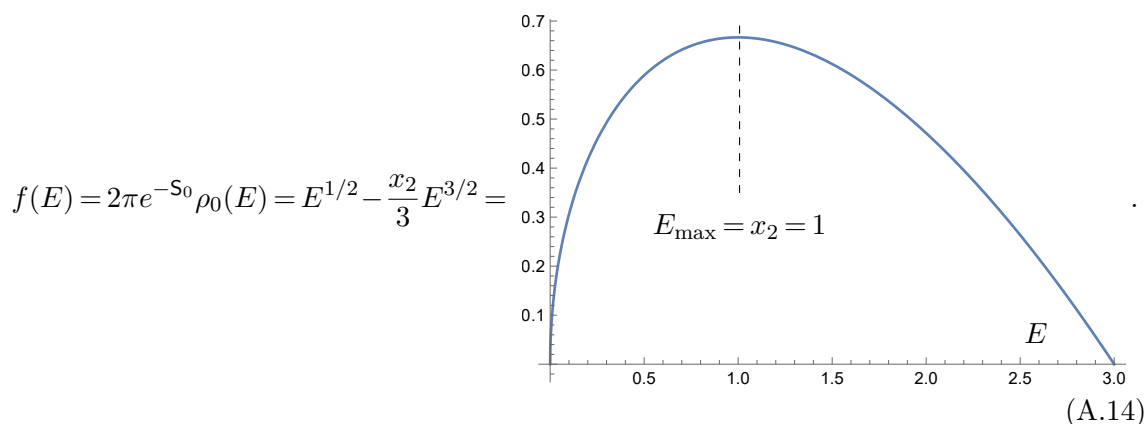
Figure 6. The τ -scaling limit of the sum over wormholes in the $(2, 3)$ minimal model (A.8) with $t_2 = 1$, $e^{S_0} = 10$ and $\beta = 1/2$. The series expansion converges to the exact answer for $T < 2e^{S_0}/3t_2^{1/2} = 20/3$ and diverges otherwise, but the resummed series has a unique analytic continuation which reproduces the exact answer for all $T > 0$.

convergence $Te^{-S_0} < 2/3t_2^{1/2}$ because $E(f)$ has a branchcut when $if < 2/3t_2^{1/2}$. For $t_2 > 0$ though $E(f)$ is smooth so the series will converge to the exact answer. This is the generic case for monotonic spectra, since there will be several point where $df/dE = 0$, indicating singular behavior in $E(f)$ and hence a finite radius of convergence. Nevertheless $E(\tau)$ for $\tau > 0$ is smooth for monotonic spectra, thus the series uniquely continues to the exact answer. As mentioned below (2.10), turning on t_3 only affects amplitudes for $g \geq 3$, for instance for generic t_k in (2.7) the expansion one obtains from either the integral (2.10) or topological recursion is

$$Z(\beta + iT, \beta - iT)_{\text{conn}} = \dots + \frac{3\beta + 2t_2}{30\pi} T^5 e^{-4S_0} - \frac{30\beta^2 + 60t_2\beta + 6t_3 + 35t_2^2}{630\pi} T^7 e^{-6S_0} + \dots \tag{A.13}$$

Non-monotonic spectra. For non-monotonic spectra in general there are several solution $E_1(f) \dots E_n(f)$ for the inverse spectrum and we get other corrections to for instance (2.6).

For concreteness we can first consider the case (A.8) but with $t_2 = -x_2 < 0$



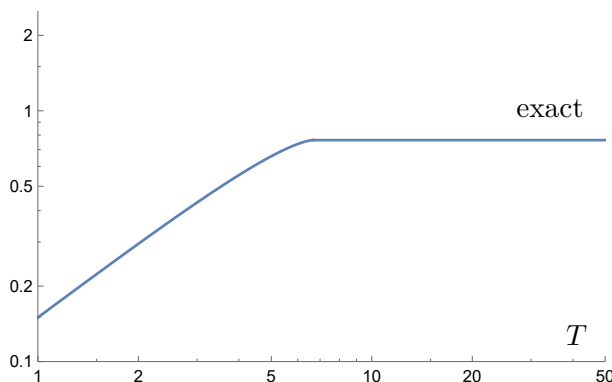


Figure 7. For non-monotonic spectra the tau-scaled connected spectral form factor has discontinuities at certain points, in the case (A.15) shown here one has a discontinuity in the first derivative. In the case (A.18) relevant for black holes the discontinuity is in the second derivative, we used $x_2 = 1$, $e^{S_0} = 10$ and $\beta = 1/2$.

This attains a maximum $2/3x_2^{1/2}$ at $E_{\max} = 1/x_2$ and vanishes beyond $E_{\text{edge}} = 3/x_2$. Starting from the first line in (2.5) one now obtains

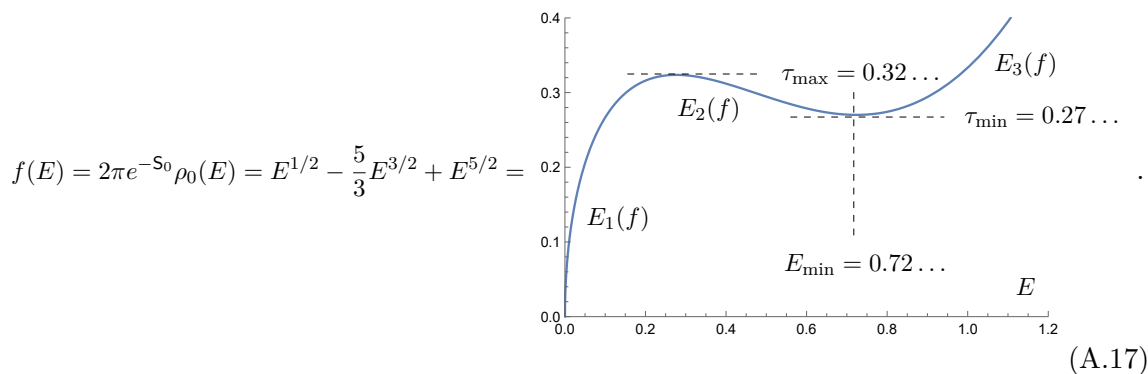
$$Z(\beta + iT, \beta - iT)_{\text{conn}} = \frac{e^{S_0}}{4\pi\beta} \int_0^{\min(\tau, \tau_{\max})} df e^{-2\beta E_1(f)} - \frac{e^{S_0}}{4\pi\beta} \int_0^{\min(\tau, \tau_{\max})} df e^{-2\beta E_2(f)}, \tag{A.15}$$

where the two solutions of the inverse spectrum are the two sheets of the analytic continuation of (A.10)

$$E_1(f) = \frac{4}{x_2} \sin^2\left(\frac{1}{3} \arcsin\left(\frac{3x_2^{1/2}}{2} f\right)\right), \quad E_2(f) = \frac{4}{x_2} \sin^2\left(\frac{\pi}{3} - \frac{1}{3} \arcsin\left(\frac{3x_2^{1/2}}{2} f\right)\right), \tag{A.16}$$

where we restricted the arcsin to the principle branch. The point now is that the gravitational expansion will only reproduce the first integral in (A.15). This is obvious from the discussion below (A.10), since the genus g Euclidean wormhole amplitudes have trivial analytic continuation to $t_2 < 0$. The radius of convergence of that expansion in this case is precisely $T = e^{S_0} \tau_{\max}$ for obvious reasons. One noteworthy new feature of is a discontinuity in the first derivative at τ_{\max} , an effect that the Taylor series does not capture, see figure 7

This example was not particularly relevant for black hole physics, where we want a spectrum with Cardy growth at large energies, not one that develops a second edge E_{edge} , as was the case above. The simplest non-monotonic case for which $\rho_0(E)$ grows for large E has a maximum and a minimum, as in the simple example with $t_2 = -5$ and $t_3 = -15$



Let us name the local maximum and minimum of f τ_{\max} and τ_{\min} . In these cases one obviously finds three real solutions $E_1(f) \dots E_3(f)$ for times $\tau_{\min} < \tau < \tau_{\max}$, and the exact answer for the tau-scaled connected spectral form factor that one obtains concordantly has three terms

$$Z(\beta+iT, \beta-iT)_{\text{conn}} = \frac{e^{S_0}}{4\pi\beta} \int_0^{\min(\tau, \tau_{\max})} df e^{-2\beta E_1(f)} - \frac{e^{S_0}}{4\pi\beta} \int_{\tau_{\min}}^{\min(\tau, \tau_{\max})} df e^{-2\beta E_2(f)} + \frac{e^{S_0}}{4\pi\beta} \int_{\tau_{\min}}^{\tau} df e^{-2\beta E_3(f)}. \quad (\text{A.18})$$

Again, one checks case by case via explicit computations that the genus g wormhole amplitude exactly reproduce the first of these integrals. For $T < e^{S_0} \tau_{\min}$ this is the full answer, so the sum over geometries reproduces the ramp-plateau exactly, for those early times. The corrections on the second line are non-perturbative corrections, and the salient feature is again that they come bearing a discontinuity in the second derivative of the tau-scaled spectral form factor at $T = e^{S_0} \tau_{\min}$. One can see this because near the minimum we have an expansion starting with (where $c > 0$ an order one number)

$$E_2(f) = E_{\min} - c(f - \tau_{\min})^{1/2} + \dots, \quad E_3(f) = E_{\min} + c(f - \tau_{\min})^{1/2} + \dots \quad (\text{A.19})$$

Thus the leading corrections from the second line in (A.18) look like (again $C > 0$ an order one number)

$$Z(\beta + iT, \beta - iT)_{\text{conn}} = \text{perturbative} - C(T - T_{\min})^{3/2} \theta(T - T_{\min}) e^{-S_0/2} + \dots \quad (\text{A.20})$$

This is non-perturbative, since the Heaviside makes these corrections not contribute in a Taylor series around $T = 0$. These corrections have no obvious geometric explanation, for instance one would require a fractional Euler character for a dual geometry.

B From powers of time to multi-boundary cancellations

In this appendix we argue that the arrow in (4.31) also works the other way, thus the universal scaling T^{2g+1} for all dilaton gravities implies also *all* of the cancellations of [27]. In a generic background we have

$$\mathcal{F}(x_1, x_2)_{\gamma_k} = \sum_{d_1, d_2} x_1^{d_1} x_2^{d_2} \langle \tau_{d_1} \tau_{d_2} \rangle_{\gamma_k}. \quad (\text{B.1})$$

The deformations we are interested in have $\gamma_0 = \gamma_1 = 0$. Suppose now we Taylor expand the exponential in (4.24) in γ_k

$$\begin{aligned} & \mathcal{F}_g(x_1, x_2)_{\gamma_k} \\ &= \sum_{d_1=0}^{3g-1} x_1^{d_1} x_2^{3g-1-d_1} \langle \tau_{d_1} \tau_{3g-1-d_1} \rangle + \gamma_2 \sum_{d_1=0}^{3g-2} x_1^{d_1} x_2^{3g-2-d_1} \langle \tau_{d_1} \tau_{3g-2-d_1} \tau_2 \rangle + \dots \\ &= \sum_{m=0}^{d/2} e_2^m e_1^{d-2m} \sum_{p=0}^m \langle \tau_p \tau_{d-p} \rangle \frac{d-2p}{d-p-m} \frac{(d-p-m)!}{(m-p)!(d-2m)!} (-1)^{p+m} \\ &+ \gamma_2 \sum_{m=0}^{d/2-1/2} e_2^m e_1^{d-1-2m} \sum_{p=0}^m \langle \tau_p \tau_{d-1-p} \tau_2 \rangle \frac{d-1-2p}{d-1-p-m} \frac{(d-1-p-m)!}{(m-p)!(d-1-2m)!} (-1)^{p+m} + \dots \end{aligned} \quad (\text{B.2})$$

In each term in the expansion one can then track the power of e_2 and e_1 to get more generic cancellation relations, namely the power of e_2 may never exceed g , because we should have T^{2g+1} for all γ_k . Hence

$$\sum_{p=0}^m \langle \tau_p \tau_{3g-2-p} \tau_2 \rangle \frac{3g-2-2p}{3g-2-p-m} \frac{(d-1-p-m)!}{(m-p)!(d-1-2m)!} (-1)^p = 0, \quad m > g. \quad (\text{B.3})$$

More generally the terms multiplying γ_k for $k \geq 2$ result in the new constraints

$$\begin{aligned} \sum_{p=0}^m \langle \tau_p \tau_{2m_{\max}-p} \tau_k \rangle \frac{2m_{\max}-2p}{2m_{\max}-p-m} \frac{(2m_{\max}-p-m)!}{(m-p)!(2m_{\max}-2m)!} (-1)^p &= 0, \\ 2m_{\max} &= 3g - k, \quad m > g. \end{aligned} \quad (\text{B.4})$$

The term where $m = m_{\max}$ is most symmetric and becomes

$$\sum_{p=0}^{3g-k} \langle \tau_p \tau_k \tau_{3g-p-k} \rangle (-1)^p = 0. \quad (\text{B.5})$$

We want to prove (4.31) by using open-closed duality for generic backgrounds γ_k

$$\begin{aligned} \mathcal{F}_g(x_1, x_2)_{\gamma_k} &= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{i_1 \dots i_m} \mathcal{F}_g(x_1, x_2, \alpha_{i_1} \dots \alpha_{i_m}) \\ &= \sum_{\substack{m_1+2m_2=3g-1 \\ m_1, m_2=0}} e_1(x)^{m_1} e_2(x)^{m_2} C_g(m_1, m_2)_{\gamma_k}, \end{aligned} \quad (\text{B.6})$$

combined with the fact that the maximal scaling T^{2g+1} implies

$$C_g(m_1, m_2)_{\gamma_k} = 0, \quad m_2 > g. \quad (\text{B.7})$$

The idea is to expand each term in the middle expression of (B.6) into powers of $e_1(x)$, $e_2(x)$ and $e_n(\alpha)$, and collecting the terms where all those powers are fixed numbers. Then we demand from (B.7) that when the power q of $e_2(x)$ exceeds g , that the function multiplying it must vanish. This function will depend on $e_1(x)$ and $e_n(\alpha)$, all of which are independent free parameters, indeed the infinite number of free parameters γ_k maps to ∞ free parameters α_i or equivalently to free parameters $e_n(\alpha)$. Therefore the vanishing of the functions multiplying $e_2(x)^q$ requires the vanishing of the coefficients of each term in the expansion of those functions in powers of $e_1(x)$ and $e_n(\alpha)$, giving a bunch of constraints. Because the terms for each m in (B.6) are homogeneous polynomials of inequivalent degree $3g-1+m$, terms with inequivalent m cannot contribute terms with the same powers of $e_1(x)$, $e_2(x)$ and $e_n(\alpha)$. Therefore the constraint that the coefficients multiplying $e_2(x)^q$ must vanish for $q > 2$ in the left of (B.6) implies the same property on each of the terms in the middle of (B.6), so

$$C_g(m_1, m_2)_{\gamma_k} = 0, \quad m_2 > g \quad \Rightarrow \quad \text{maximal power } e_2(x)^g \text{ in } \mathcal{F}_g(x_1, x_2, \alpha_1 \dots \alpha_m). \quad (\text{B.8})$$

We also used that each term in the sum over $i_1 \dots i_m$ is linearly independent, as there are ∞ independent free parameters α_i . We claim that for each fixed number of boundaries $2 + n$ we have

$$\text{maximal power } e_2(x)^g \text{ in } \mathcal{F}_g(x_1, x_2, \alpha_1 \dots \alpha_n) \Rightarrow C_g(m_2 \dots m_{n+2}) = 0, \quad \sum_{j=2}^{j+n} m_j > g. \quad (\text{B.9})$$

We now prove this for the three boundary case $n = 1$. In general, one can use the following property of elementary symmetric polynomials

$$e_n(x \oplus \alpha) = e_n(\alpha) + e_1(x)e_{n-1}(\alpha) + e_2(x)e_{n-2}(\alpha), \quad (\text{B.10})$$

to expand $\mathcal{F}_g(x_1, x_2, \alpha_1 \dots \alpha_n)$ into the building blocks discussed above. For $\mathcal{F}_g(x_1, x_2, \alpha)$ one requires

$$e_1(x \oplus \alpha) = e_1(x) + \alpha, \quad e_2(x \oplus \alpha) = e_1(x)\alpha + e_2(x), \quad e_3(x \oplus \alpha) = e_2(x)\alpha. \quad (\text{B.11})$$

Inserting this in the general equation, and absorbing some combinatorial prefactors in $C_g(m_2, m_3)$ for visual purposes, one finds

$$\mathcal{F}_g(x_1, x_2, \alpha) = \sum_{m_2, m_3=0}^{2m_2+3m_3=3g} \sum_{q=m_3}^{m_3+m_2} \sum_{p=m_2+2m_3-q}^{3g-m_2-m_3-q} \alpha^p e_1(x)^{3g-p-2q} e_2(x)^q C_g(m_2, m_3). \quad (\text{B.12})$$

The terms with $q > g$ can be rearranged as

$$\mathcal{F}_g(x_1, x_2, \alpha) \supset \sum_{q=g+1}^{3g/2} \sum_{p=0}^{3g-2q} \alpha^p e_1(x)^{3g-p-2q} e_2(x)^q \sum_{m_3=0}^p \sum_{k=0}^{\min(p-m_3, 3g/2-q-m_3/2)} C_g(q-m_3+k, m_3). \quad (\text{B.13})$$

The constraint that one should impose is that the coefficients of $\alpha^p e_2(x)^q$ vanish for all p and for $q > g$

$$\sum_{m_3=0}^p \sum_{k=0}^{\min(p-m_3, 3g/2-q-m_3/2)} C_g(q-m_3+k, m_3) = 0, \quad q > g. \quad (\text{B.14})$$

Clearly having the correct summation ranges is crucial. For $p = 0$ there is one term and the constraint becomes

$$C_g(q, 0) = 0, \quad q > g, \quad (\text{B.15})$$

which is just the old $n = 0$ two boundary constraint. To prove (4.31) it is key to impose the constraints in the correct order, working by induction in p . When we now consider $p = 1$, the term with $m_3 = 0$ already vanishes because of (B.15). When $m_3 = 1$ only $k = 0$ contributes and we find the new constraint

$$C_g(q-1, 1) = 0, \quad q > g. \quad (\text{B.16})$$

Since $q > g$ implies $q + k > g$ this means all terms with $m_3 = 1$ will vanish for arbitrary p . This process will continue, at fixed p we can prove that the terms with $m = p$ vanish such

that we always only need to deal with one term at $p + 1$, this is obvious using induction. Therefore we have obtained the set of constraints

$$C_g(q - r + k, r) = 0, \quad g < q \leq 3g/2, \quad 0 \leq r \leq 3g - 2q, \quad k \geq 0. \quad (\text{B.17})$$

Because $2m_2 + 3m_3 \leq 3g$ we have $m_3 \leq g$ and therefore one checks that (B.17) covers all cases in (4.31) for $n = 1$

$$C_g(m_2, m_3) = 0, \quad m_2 + m_3 > g, \quad 2m_2 + 3m_3 \leq 3g. \quad (\text{B.18})$$

It seems highly plausible that this proof extends to generic n , though it requires some combinatorics.

C Intersection theory and volumes of moduli space

The heart of the paper is concerned with intersection theory and how various intersection numbers are associated with the volumes of moduli space of Riemann surfaces. The intersection theory we will need is a fascinating branch of mathematics that produced many beautiful and deep results. We now dive a bit into these mathematics, we will try to keep it as low brow as possible, combining various insights from [40, 46, 56, 96, 97].

First, let us start with a bit of motivation as to why we would want to study an intersection theory in the context of JT gravity. This has a rather extended history, but we can roughly summarize it as follows. The moduli space of Riemann surfaces of genus g and n punctured is a well motivated and studied object. To understand this space better, one can wonder what happens when one intersects the moduli space with various other subspaces with different co-dimensions. A particularly simple example of this is to consider an intersection such that generically the intersection is just a bunch of points. By counting these points one can learn some more about the geometry of the moduli space. In particular, since subspaces are dual to forms by Poincaré duality, such intersection numbers can even tell us the volume of the moduli space as we will explain below. These volumes are crucial for the understanding of higher-genus corrections to the JT path integral [11, 23, 47].

Another motivation as to why intersection numbers are interesting is that they have deep relation with a matrix integral, as conjectured by Witten [58] and proven by Kontsevich in [46]. This matrix integral describes the low energy sector of the matrix model dual to JT gravity and can also be extended to describe the full JT theory. In fact, as explained in section 3.1, these intersection numbers compute the volumes of moduli space of trivalent ribbon graphs. These ribbons are the propagators of a matrix integral a la t'Hooft [98] (the relevant matrix integral is the Kontsevich matrix integral, which is graph dual or color-flavor dual to the standard matrix integral that for instance [11] use).

C.1 What is being intersected?

Let us consider a Riemann surface $\Sigma_{g,n}$ with genus g and n punctures at fixed x_i . Such a surface can be obtained by considering a polygon in the hyperbolic disk and identifying the sides in the appropriate way. For instance by identifying the opposite sides of a hyperbolic

rectangle one gets a punctured torus. To understand what is being intersected, we will start off with defining various two-forms. These have a concrete definition, resulting in certain limiting cases in simple explicit expressions (3.6).

We start with our geometry $\Sigma_{g,n}$ and construct the cotangent space $T_x^* \Sigma_{g,n}$ of one-forms at $x = x_i$. This one-complex-dimensional space depends on the moduli of $\Sigma_{g,n}$ and so we can consider the collection of all cotangent spaces at $x = x_i$ as a function of the moduli of $\Sigma_{g,n}$. This collection is a complex line bundle, which we denote by \mathcal{L}_i . We can now choose an element of this bundle and compute its curvature two form, this two-form is usually referred to as the first Chern class of \mathcal{L}_i , and denoted $\psi_i = c_1(\mathcal{L}_i)$. Notice that by Poincare duality ψ_i corresponds to codimension two submanifolds of $\mathcal{M}_{g,n}$ and it is these submanifolds whose intersections one can count.

The intersection number of two co-dimension two submanifolds in some manifold can be represented by the wedge product of the two forms Poincare dual to the submanifolds. Likewise, wedge products of ψ_i integrated over $\mathcal{M}_{g,n}$ count the intersection of codimension two submanifolds in $\mathcal{M}_{g,n}$. This means that if we consider integrals of a top-dimensional form

$$\int_{\mathcal{M}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n}, \quad \sum_{i=1}^n d_i = 3g - 3 + n, \quad (C.1)$$

the integral describes the intersection of the n submanifolds (or rather cycles) dual to $\psi_i^{d_i}$. These are the intersection numbers discussed in the main text.³⁴

Before proceeding we make two comments.

1. Instead of $\mathcal{M}_{g,n}$ (which is non-compact) one considers the so-called Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ which includes degenerate Riemann surfaces at the boundary of moduli space. These surfaces have regions where locally the surface looks like two discs with their centers identified. See figure 8 (left) for an example that is counted in the moduli space of punctured tori $\overline{\mathcal{M}}_{1,1}$.
2. The spaces $\mathcal{M}_{g,n}$ are not manifolds, but orbifolds. We can understand this because we can view $\mathcal{M}_{g,n}$ as Teichmuller space $\mathcal{T}_{g,n} \sim \mathbb{H}^{3g-3+n}$ modulo identifications under the mapping class group, and these identifications can create \mathbb{Z}_m conical singularities in the resulting space $\mathcal{M}_{g,n}$, making them orbifolds. For instance, the keyhole $\mathcal{M}_{1,1} = \mathbb{H}/\text{SL}(2, \mathbb{Z})$ has one $\alpha = \pi$ and $\alpha = 2\pi/3$ conical singularity, see figure 8 (right) and [96].

For orbifolds Chern numbers need not be integers, in general they will be rational numbers. This makes the terminology intersection “number”, which one thinks of as an integer, odd. In general they are rational numbers for the compact orbifolds $\overline{\mathcal{M}}_{g,n}$, just like their Euler character.

³⁴For this to be a bona fide intersection number this should be a topological invariant and for that it is crucial to view the ψ_i as representatives of classes of the second cohomology group, and similarly the codimension two submanifolds should be viewed as codimension two cycles in the relevant homology group. For instance, see Bézout’s theorem for the simplest example of intersection numbers of curves in the real projective plane.

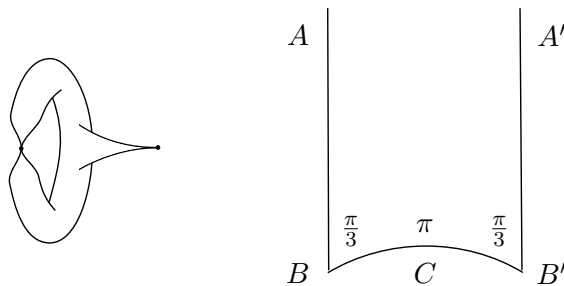


Figure 8. Degenerate once punctured torus that is included in the Deligne-Mumford compactification $\overline{\mathcal{M}}_{1,1}$ of $\mathcal{M}_{1,1}$ (left). The orbifold $\mathcal{M}_{1,1} = \mathbb{H}/\mathrm{SL}(2, \mathbb{Z})$ is the interior of the keyhole with extra identification of the half-lines AB and $A'B'$ and of the arcs CB and CB' [96] (right). There is an $\alpha = \pi$ total angle when one circles around C and an $\alpha = 2\pi/3$ total angle when one travels around $B = B'$. So, $\mathcal{M}_{1,1}$ is not a smooth manifold, but indeed an orbifold.

C.2 Kontsevich and Witten

The objective in intersection theory on the moduli space of Riemann surfaces is to calculate (C.1). This is a complicated endeavour, not only because at large genus and n there are many different integrals to consider, but also because the ψ_i itself are hard to find since $\mathcal{M}_{g,n}$ is complicated in general. However in the early nineties the work of Witten and Kontsevich [45, 46] showed that there is a combinatorial solution to the problem involving ribbon graphs. We have already explained how this works in section 3, here we provide an alternative (and historically more chronological) story.

One starts by considering the generating functional of the correlators (3.21)

$$F = \left\langle \exp \left(\sum_{k=0}^{\infty} t_k \tau_k \right) \right\rangle = \sum_{d_i} \prod_{i=0}^{\infty} \frac{t_i^{d_i}}{d_i!} \left\langle \prod_{j=0}^{\infty} \tau_j^{d_j} \right\rangle, \quad \langle \tau_{k_1} \cdots \tau_{k_n} \rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n}. \quad (\text{C.2})$$

Witten conjectured [45] that this F is the so-called τ -function of the KdV hierarchy, which is a fancy way to say that it satisfies the differential equations (3.13). As we explained in the main text, one can use this to get recursion relations (3.27) for the intersection numbers, which can be used to compute all intersection numbers given the seeds of the recursion

$$\langle \tau_0 \tau_0 \tau_0 \rangle_0 = 1, \quad \langle \tau_1 \rangle_1 = \frac{1}{24}. \quad (\text{C.3})$$

The first of these correlators is just a reflection of the fact that the moduli space of the three punctured sphere is a so the correlator can be normalized to one. The second correlator is a statement about the moduli space of the punctured torus and can be understood in various ways, for instance see [96].

The proof of this conjecture was given by Kontsevich [46], who found a way to simply compute the function F and checked that it was indeed a τ -function. His methods start with the story we presented in section 3.1. He first considered another generating function (3.8)

$$V_{g,n}(b_1 \cdots b_n) = \sum_{d_1=0}^{\infty} \frac{b_1^{2d_1}}{2^{d_1} d_1!} \cdots \sum_{d_n=0}^{\infty} \frac{b_n^{2d_n}}{2^{d_n} d_n!} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle_g = \int_{\overline{\mathcal{M}}_{g,n}} \exp \left(\frac{1}{2} \sum_{i=1}^n b_i^2 \psi_i \right), \quad (\text{C.4})$$

and realized that using Penner coordinates ℓ_j for $\overline{\mathcal{M}}_{g,n}$, the Chern classes ψ_i can be brought into the simple form (3.6), such that (after some combinatorics) this generating functional is seen to compute the volume of moduli space of ribbon graphs with fixed boundary lengths b_i (3.4)³⁵

$$V_{g,n}(b_1 \dots b_n) = \sum_{\Gamma_{g,n}} V_{\Gamma_{g,n}}(b_1 \dots b_n),$$

$$V_{\Gamma_{g,n}}(b_1 \dots b_n) = \frac{c_{\Gamma}}{|\text{Aut}(\Gamma)|} \prod_{j=1}^{6g-6+3n} \int_0^{\infty} d\ell_j \prod_{i=1}^n \delta\left(b_i - \sum_{j_i} \ell_{j_i}\right). \quad (\text{C.5})$$

These Penner coordinates ℓ_j (which are more open-string like whereas the Fenchel-Nielsen length-twist coordinates are more like a closed string parameterization) originate from constructing the Riemann surface by gluing together flat fixed-width strips of lengths ℓ_i at vertices where all the curvature is then localized. These strips naturally form a ribbon graph. The division by the mapping class group, which makes the moduli space so complicated in general, is avoided since there is no constraint on the lengths of the strips [46], except for the symmetry factor $1/|\text{Aut}(\Gamma)|$ from overcounting cases where say ℓ_1 and ℓ_2 label interchangeable edges, so one should only integrate over $\ell_1 > \ell_2$.

Equation (C.5) looks simple. For instance for $V_{1,1}(b)$ there is only one ribbon graph, see figure 2. There are three edges and so three lengths ℓ_i . The length b of the boundary is given by $2(\ell_1 + \ell_2 + \ell_3) = b$, since one traverses the edges twice. Since $\text{Aut}(\Gamma) = 6$ one finds the correct answer

$$V_{1,1}(b) = \frac{1}{|\text{Aut}(\Gamma)|} \int_0^{\infty} d\ell_1 \int_0^{\infty} d\ell_2 \int_0^{\infty} d\ell_3 \delta(\ell_1 + \ell_2 + \ell_3 - b/2) = \frac{1}{6} \int_0^{b/2} d\ell_1 \int_0^{b/2-\ell_1} d\ell_2 = \frac{b^2}{48}. \quad (\text{C.6})$$

Unfortunately life is not so simple. In practice it is hard to find all topologically distinct ribbon graphs for given g and n . Moreover, the linear constraints $b_i = A_{ij}\ell_j$ define a polytope, and it is not in general very simple to compute its volume because of these linear constraints. A simple way to get rid of those constraint is integrating over b_i . In particular, as Kontsevich noticed, matters simplify drastically when considering a Laplace transform. Doing the b_i integrals first, the volumes simplify tremendously

$$\prod_{i=1}^n \int_0^{\infty} db_i e^{-z_i b_i} \prod_{j=1}^{6g-6+3n} \int_0^{\infty} d\ell_j \prod_{i=1}^n \delta\left(b_i - \sum_{j_i} \ell_{j_i}\right) = \prod_{j=1}^{6g-6+3n} \frac{1}{z_{j_1} + z_{j_2}}, \quad (\text{C.7})$$

and applying the same Laplace transform to (C.4) one obtains Kontsevich's famous formula

$$\sum_{\Gamma} \frac{2^{2g-2+n}}{|\text{Aut}(\Gamma)|} \prod_{j=1}^{6g-6+3n} \frac{1}{z_{j_1} + z_{j_2}} = \sum_{k_i=0}^{\infty} \langle \tau_{k_1} \dots \tau_{k_n} \rangle_g \prod_{i=1}^n \frac{(2k_i)!}{2^{k_i} k_i!} \frac{1}{z_i^{2k_i+1}}. \quad (\text{C.8})$$

³⁵The constant c_{Γ} is the flat, but non-unit measure on the moduli space of ribbon graphs. Suppose the lengths constrains read $b_i = A_{ij}\ell_j$ and we form the square matrix B by using the first n columns of A , then $c_{\Gamma} = 2^{2g-2+n} / \det(B)$ [40].

At this point we have given the i 'th boundary some flavor z_i . Suppose that we have N flavors z_a at our disposal, then it makes sense to consider a sum of the above sum where we sum over all possible flavors for all boundaries, including a symmetry factor $1/(d_j)!$ in this sum when d_j physically indistinguishable boundaries τ_j appear. If we then also sum over n and g , the right hand side becomes precisely F , where the KdV times t_k in (C.2) are related with the flavors z_a as

$$t_k = \frac{(2k)!}{2^k k!} \sum_{a=0}^N \frac{1}{z_a^{2k+1}}. \tag{C.9}$$

The left hand side becomes a sum over all ribbon graphs Γ with any number of boundaries, any genus, and any flavor on any of the boundaries, weighed by a symmetry factor $1/|\text{Aut}(\Gamma)|$ that always appears in Feynman diagrams and with each edge contributing a weight $2/(z_{j_1} + z_{j_2})$.

Kontsevich recognized that these are nothing but the Feynman rules of an N by N matrix integral, but counting only fully connected diagrams. Taking into account the fact that fully connected diagrams exponentiate to the full theory, Kontsevich concluded [40]

$$e^F = \frac{\int dX \exp\left(\frac{i}{6} \text{Tr} X^3 - \frac{1}{2} \text{Tr} X^2 Z\right)}{\int dX \exp\left(-\frac{1}{2} \text{Tr} X^2 Z\right)}, \tag{C.10}$$

where the eigenvalues of Z are z_a , and X is integrated over all Hermitian matrices. The propagator is obviously correct, and the X^3 interaction generates trivalent ribbon graphs. Not only does this solve intersection theory, Kontsevich also checked explicitly that this expression for F is a τ -function, proving Witten's conjecture [45].

C.3 Relation between the moduli space of ribbon graphs and Riemann surfaces

We gather for clarity some discussion that already appeared in the main text about the relation between the volumes of ribbon graphs and the volumes of Riemann surfaces. Remember that using Gauss-Bonnet with $R = -2$ and $K = 0$

$$\chi = \frac{1}{4\pi} \int \sqrt{g} R - \frac{1}{2\pi} \int \sqrt{h} K = -\frac{1}{2\pi} \int \sqrt{g} = -\frac{A}{2\pi}, \tag{C.11}$$

that we deduced around (3.3) that Weil-Petersson volumes reduce to volumes of ribbon graphs for large b . Weil-Petersson volumes are calculated by invoking the symplectic structure and using the Weil-Petersson two-form $\Omega_{\text{WP}}(b_1, \dots, b_n)$ to get a volume form $e^{\Omega_{\text{WP}}}$

$$V_{g,n}(b_1 \cdots b_n) = \int_{\overline{\mathcal{M}}_{g,n}} e^{\Omega_{\text{WP}}} \tag{C.12}$$

Using (C.4) this implies

$$\Omega_{\text{WP}} \rightarrow \sum_{i=1}^n \frac{b_i^2}{2} \psi_i \tag{C.13}$$

Terms linear in b_i can be argued from the $SL(2, \mathbb{R})$ BF gauge theory to not exist as hyperbolic holonomies with positive and negative b_i are conjugate, so everything needs to be even in b_i . Therefore (see also [99])

$$\Omega_{\text{WP}} = \omega_{\text{WP}} + \sum_{i=1}^n \frac{b_i^2}{2} \psi_i, \tag{C.14}$$

with ω_{WP} the Weil-Petersson symplectic form on the moduli space of Riemann surfaces with $b_i = 0$.

This relation thus tells us that Weil-Petersson volumes are computed using intersection theory of not only the ψ classes, but together with the Weil-Petersson symplectic form ω_{WP} . It is this extra class that causes the WP volumes to obtain their intricate polynomial structure with coefficients that have various powers of π . In fact it is customary to write ω_{WP} as

$$\omega_{\text{WP}} = 2\pi^2 \kappa, \tag{C.15}$$

with κ a rather standard two-form one can also consider on the moduli space of Riemann surfaces, but whose definition is more complicated and will not be given here, but see [45].

To understand ω_{WP} more physically, note that when it is absent and we just discuss trivalent ribbon graphs, there is a matrix model (C.10) that also computes these volumes, with

$$F = \left\langle \exp \left(\sum_{k=0}^{\infty} t_k \tau_k \right) \right\rangle = \sum_{d_i} \prod_{i=0}^{\infty} \frac{t_i^{d_i}}{d_i!} \left\langle \prod_{j=0}^{\infty} \tau_j^{d_j} \right\rangle, \tag{C.16}$$

and

$$t_k = \frac{(2k)!}{2^k k!} \text{Tr} Z^{-(2k+1)} \tag{C.17}$$

The asymptotic expansion in Z means that when we want to compute the intersection theory relevant for the volumes of ribbon graphs, we need to take derivatives with respect to the t_k 's and then set all t_k 's to zero. The theory with all the $t_k = 0$ is the unperturbed matrix model and is precisely the double scaled Gaussian matrix model without any determinants inserted. To get more general double scaled models the only thing one then has to do is to simply not set all the t_k to zero afterwards, but set them to specific values γ_k . It turns out that one wants to set them to

$$\gamma_n = -\frac{(-2\pi^2)^{n-1}}{(n-1)!}. \tag{C.18}$$

To see how this connects to ω_{WP} , notice that the b dependent part will give us the τ correlators, which are obtained by taking derivatives of F and so the term $e^{\omega_{\text{WP}}}$ is equivalent to what we set the t_k 's to after we are done with taking derivatives. Thus the role of ω_{WP} is to deform the double scaled gaussian matrix model to one with a more general potential, specifically the one relevant for JT gravity. Thus we have the relation (4.19)

$$\langle (\dots) e^{\omega_{\text{WP}}} \rangle = \left\langle (\dots) e^{\sum_k \gamma_k \tau_k} \right\rangle. \tag{C.19}$$

To summarize what we have learned so far. We have introduced the notion of intersection numbers on the moduli space of Riemann surfaces. By invoking an open string parameterization of the moduli space we showed that the volume of moduli space of ribbon

graphs is related to these intersection numbers. Finally by noticing that for large b_i the bordered Riemann surfaces turn into ribbon graphs, we could make a clear relation between Weil-Petersson volumes and volumes of moduli space of ribbon graphs. In particular that deforming to the JT matrix model is equivalent to freezing some of the t_k to some non-trivial value given by (C.18).

C.4 Topological recursion for ribbon graphs

Here we discuss a simplification of Mirzakhani’s recursion relation for the Weil-Petersson volumes at large b_i and thus becoming a recursion relation for the volumes of moduli space of ribbon graphs. This was also shown in [40].

The derivation of Mirzakhani’s recursion [52] proceeds by considering an experiment where we take an external geodesic with length b and shoot in geodesics orthogonal to it into the surface. We track where those geodesics end up. The crux of the calculation comes down to doing this experiment within a single three holed sphere with boundaries b, b' and b'' from which one gets the identify

$$b = T(b, b', b'') + T(b, b'', b') + D(b, b', b'') \tag{C.20}$$

with symmetry

$$T(b, b', b'') = T(b', b, b''), \tag{C.21}$$

this measures the length along b of geodesics that end up on b' (or the other way around). The function $D(b, b', b'')$ is the length along b of geodesics that self-intersect before reaching a boundary, or that end up back on the b geodesic. The full recursion relation reads [25]

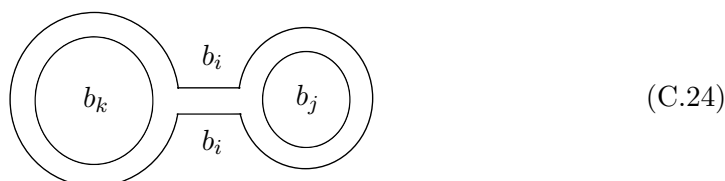
$$\begin{aligned} & b V_{g,n+1}(b, B) \\ &= \frac{1}{2} \int_0^\infty db' b, db'' b'' D(b, b', b'') \left(V_{g-1,n+2}(b', b'', B) + \sum_{\text{stable}} V_{h_1,n_1}(b', B_1) V_{h_2,n_2}(b'', B_2) \right) \\ &+ \sum_{i=1}^n \int_0^\infty db' b' (b - T(b, b', b_i)) V_{g,n-1}(b', B/b_i). \end{aligned} \tag{C.22}$$

One now just has to compute these functions. For ribbon graphs this is borderline trivial. One obtains a piecewise function

$$T(b_i, b_j, b_k) = \ell_{ij}, \quad D(t_i, t_j, t_k) = 2\ell_{ii}, \tag{C.23}$$

where ℓ_{ij} is the length of the ribbon graph shared between boundaries b_i and b_j . There are essentially two ribbon graphs that the three holed sphere can degenerate into, depending on the sizes b, b' and b'' .

1. When $b_i > b_j + b_k$ the Riemann surface becomes



and one reads off immediately for instance $T(b_i, b_j, b_k) = b_j$ etcetera, because a fraction b_j of the geodesics starting on b_i end up on b_j (or all geodesics starting on b_j end on b_i).

2. When neither of the lengths is bigger than the sum of the two others the Riemann surfaces looks like

$$b_k \left(\begin{array}{c} \text{---} b_i \text{---} \\ \text{---} b_j \text{---} \end{array} \right) b_k \tag{C.25}$$

and one can solve for the lengths of the edges, to find for instance $T(b_i, b_j, b_k) = (b_i + b_j - b_k)/2$.

These piecewise linear functions can also be obtained directly as limits of the formulas for finite lengths in [25]. One also checks obviously that this reproduces the correct Airy volumes. It is entirely consistent to think of Riemann surfaces with large boundaries as ribbon graphs, for all purposes.

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