

Topological orders, braiding statistics, and mixture of two types of twisted BF theories in five dimensions

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ABSTRACT: Topological orders are a prominent paradigm for describing quantum many-body systems without symmetry-breaking orders. We present a topological quantum field theoretical (TQFT) study on topological orders in five-dimensional spacetime (5D) in which *topological excitations* include not only point-like *particles*, but also two types of spatially extended objects: closed string-like *loops* and two-dimensional closed *membranes*. Especially, membranes have been rarely explored in the literature of topological orders. By introducing higher-form gauge fields, we construct exotic TQFT actions that include mixture of two distinct types of BF topological terms and many twisted topological terms. The gauge transformations are properly defined and utilized to compute level quantization and classification of TQFTs. Among all TQFTs, some are not in Dijkgraaf-Witten cohomological classification. To characterize topological orders, we concretely construct all braiding processes among topological excitations, which leads to very exotic links formed by closed spacetime trajectories of particles, loops, and membranes. For each braiding process, we construct gauge-invariant Wilson operators and calculate the associated braiding statistical phases. As a result, we obtain expressions of link invariants all of which have manifest geometric interpretation. Following Wen's definition, the boundary theory of a topological order exhibits gravitational anomaly. We expect that the characterization and classification of 5D topological orders in this paper encode information of 4D gravitational anomaly. Further consideration, e.g., putting TQFTs on 5D manifolds with boundaries, is left to future work.

KEYWORDS: Anyons, Topological States of Matter, Gauge Symmetry

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1 Introduction

As the most notable examples of beyond-symmetry-breaking orders, topological orders have attracted a lot of attentions for decades [1–4]. One of most prominent features of the research in topological orders is joint efforts from inter-disciplinary fields. Triggered by the condensed matter side, e.g., the fractional quantum Hall effect, topological orders have also shed light on frontiers of high-energy physics, mathematical physics, and quantum information science [5–9]. In condensed matter physics, topological orders are gapped phases of matter and lack of any local order parameters that characterize symmetry-breaking

patterns, which calls for a new revolution on the traditional solid-state physics and statistical physics of phases and phase transitions. The low-energy effective field theories of topologically ordered phases of matter are usually topological quantum field theories (TQFTs) [10, 11]. The 2D boundary of 3D topological orders is governed by conformal field theory (CFT) [12] and gravitational anomaly in a more general sense [13, 14].¹ The algebraic theory of topological orders in 3D belongs to a subclass of tensor category [15], which play a very fundamental role similar to group theory in symmetry-breaking orders. As a generalization of fermionic and bosonic statistics, anyonic statistics in 3D turns out to be described by the mathematics of braid group, which was previously discovered through the path-integral of indistinguishable particles [16]. The field of topological orders has also borrowed many exciting ideas from quantum information science, which leads to rapid developments we have been witnesses to, such as the long-range entanglement nature of topological orders and stabilizer codes as exactly solvable lattice models that admit topological orders [17–20].

Topological excitations are a central concept of topological orders. Above the topologically ordered ground states, topological excitations look very odd and behave like fractionalized degrees of freedom. For example, anyons, a kind of exotic particles in 3D world, carry fractionalized electron charge and fractionalized statistics, which can be regarded as a consequence of electron fractionalization in the fractional quantum Hall systems. In addition to point-like excitations, *spatially extended excitations*, e.g., string-like loop excitations and two-dimensional closed membrane excitations have been constructed in topological orders of 4D and higher dimensions. Their exotic entanglement, symmetry enrichment, braiding statistics, TQFTs, and higher-category theory have been studied intensively [21–46]. Recently, the so-called “fracton physics” of spatially extended excitations is also discussed, where both mobility and deformability of spatially extended excitations are restricted either partially or completely [47–49].

Braiding statistics among topological excitations is served as a topological order parameter to classify and characterize topological orders [2]. In 3D, braiding statistics among anyons is known to form S and T matrices. In 4D, particles cannot be anyonic but the presence of loop excitations makes braiding statistics even more bizarre. First, we can consider a discrete gauge group $G = \prod_i \mathbb{Z}_{N_i}$. All particles carry and thus are labeled by periodic gauge charges. Likewise, all loops carry and thus are labeled by periodic gauge fluxes. Then, braiding statistics can be particle-loop braiding [50–55], multi-loop braiding [25], and particle-loop-loop braiding (i.e., Borromean-Rings braiding) [23]. Recently, ref. [46] exhausted all possible combinations of braiding processes in order to obtain a complete list of topological orders in 4D. The basic guiding principle there is to find TQFTs that have well-defined gauge transformations. All TQFTs for describing these braiding processes have structures of a multi-component BF term [56] in the presence of some twists. The BF term in 4D has a form of BdA where B and A are respectively 2- and 1-form gauge fields. Twists consist of $AAdA$, $AAAA$ [42], and AAB [23].

¹*Important convention:* in this paper, when we mention 2D, 3D, etc., we refer to the dimension of spacetime. If the 3-dimensional space, instead of spacetime, is considered, we would emphasize it as 3D space, etc.

Although topological orders in 4D and 3D are already interesting and directly relevant to experimentally realizable systems of condensed matter physics, in this paper, we move forward to investigate topological orders and TQFTs in 5D, which turns out to exhibit highly unexplored features of both physics and mathematics. *Firstly*, one of the most attractive features in 5D topological orders is the existence of membrane excitations, which are geometrically two-dimensional compact manifolds and form three-dimensional world-volumes. Membrane excitations can participate in nontrivial braiding processes that are expected to go beyond braidings in 3D and 4D. *Secondly*, understanding 5D topological orders is also useful for understanding gravitational anomalies in 4D according to Wen's definition [13, 14]. In general, the boundary of a topologically ordered state on an open manifold has gravitational anomaly, which strictly forbids the consistent existence of the boundary theory alone by removing the bulk topological ordered state. This fact is true and robust even though all global symmetries are broken. Therefore, we are allowed to investigate gravitational anomalies in 4D by means of topological orders in 5D. *Thirdly*, when gauge group is still $G = \prod_{i=1}^n \mathbb{Z}_{N_i}$, TQFTs in 5D may contain two types of BF terms, i.e., CdA and $\tilde{B}dB$, where C is 3-form, B and \tilde{B} are two different 2-form, A is 1-form. Therefore, for each \mathbb{Z}_{N_i} subgroup, there are two choices for assignment of gauge charges and corresponding BF terms. In such a mixed BF theory, if we further add twisted terms (i.e., twists), e.g., $AAAAA$, $AAAdA$, $AdAdA$, AAC , $AAAB$, ABB , $BAdA$, $AAdB$, the resulting gauge theories are expected to be very complex and host exotic topological orders. Given BF terms, the additional twists are not always compatible with each other, as some of combinations of twists unavoidably violate gauge invariance of either *usual* type or *large* type. This phenomenon is similar to but much more intricate than that in 4D studied in ref. [46] due to the presence of two types of BF terms.

In this paper, for understanding low energy physics of 5D topological orders, we construct gauge-invariant TQFT actions and define gauge transformations for all gauge fields. More specifically, TQFT actions consist of two distinct types of BF terms with twisted topological terms, dubbed BF theories. By means of gauge transformations, we obtain the quantization and periods of the coefficients of all topological terms, which leads to TQFT classification. Some of TQFTs are beyond Dijkgraaf-Witten cohomological classification $H^5(G, \mathbb{R}/\mathbb{Z})$. Furthermore, we construct gauge-invariant observables (generalized Wilson operators) of TQFT actions. In 5D topological orders, a braiding process results in a nontrivial link formed by the closed world-lines of particles, world-sheets of loops, and/or world-volumes of membranes. It is natural to expect a linking number or link invariant in 5D to characterize such a braiding process. For our 5D BF theories, we relate the expectation value of a Wilson operator to counting intersections of sub-manifolds in 5D. It should be noted that the link invariants are obtained from our physical theory via the principle of gauge invariance. In this sense, our physical theory provides an alternative route to understand link theory of higher-dimensional compact manifolds. The latter manifolds are physically realized as closed spacetime trajectories of topological excitations in our 5D condensed matter systems.

This paper is organized as follows. In section 2, we define topological excitations in the hydrodynamical approach, and then introduce two types of BF terms. We also add

twisted terms to BF terms and study the resulting TQFT. Typically, we calculate the expectation values of Wilson operators and reveal their geometric interpretation. Next, in section 3, we exhaust all possible TQFTs that are gauge invariant such that all braiding processes encoded in a given TQFT are mutually compatible. Classification of these TQFTs is discussed for different gauge groups, collected in table 1 (\mathbb{Z}_{N_1} , $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$, and $\prod_{i=1}^3 \mathbb{Z}_{N_i}$). For $G = \prod_{i=1}^n \mathbb{Z}_{N_i}$ with $n \geq 4$, details are presented in section 3.4. Section 4 is devoted to a conclusion and outlook. Some technical details are collected in appendices.

2 Topological excitations and braiding statistics in 5-dimensional topological orders

Besides the robust ground state degeneracy, topological excitations and their braiding statistics also serve as part of the definition of topological orders. In section 2.1, we intuitively explain what topological excitations in 5D look like and how to understand the braiding processes among them after reviewing the cases in 3D and 4D. Then, in section 2.2, we introduce two types of BF terms that describe braiding processes of two topological excitations. Last, as shown in section 2.3 and 2.4, by adding a twisted term to BF terms, we construct TQFT actions that describe braiding processes involving multiple topological excitations. Especially, for each braiding process, we find a gauge-invariant observable (generalized Wilson operator) whose vacuum expectation value is expressed by the intersection of some sub-manifolds in 5D. We find that such gauge-invariant observables are closely related with linking numbers of spatially extended topological excitations in 5D topological orders.

2.1 Topological excitations and their braiding processes in 3D, 4D, and 5D spacetime

It is beneficial for us to review topological excitations and their braiding processes in 3D and 4D before we move into the 5-dimensional spacetime. Anyon in the fractional quantum Hall effect (FQHE), a typical 3D topological order, may be the most well-known example of topological excitations in 3D. The self-statistics of anyons is captured by the phase difference after exchanging two anyons in the 2D space. If viewed in 3D, the world-lines of two anyons form a braid during the exchange. Geometrically, anyons are particle excitations, the only possible excitation in 3D topological orders.² In 4D topological orders, topological excitation spectrum is composed by particle excitations and loop excitations [21, 22]. Braiding processes in 4D topological orders can be divided into three classes [46]: particle-loop braiding [11, 50, 56–59], multi-loop (three or four) braiding [25, 42], and particle-loop-loop braiding [23]. In particle-loop braiding, a particle excitation moves around a loop excitation such that its spatial trajectory and the loop excitation form a Hopf link. This Hopf link can also be identified from the intersection of the world-line of the particle and the

²In 3D topological orders, loop excitations are regarded equivalent to particle excitations. More concretely, because a loop excitation is impenetrable in 2D space, it actually behaves like a particle excitation. Therefore, we do not consider loop excitations in 3D topological orders.

world-sheet of the loop.³ In multi-loop braiding, three or four loops are linked in a special manner and their intersecting world-sheets indicate the topological invariant characterizing such braiding. In particle-loop-loop braiding, one particle moves around two unlinked loops such that its spatial trajectory and these two loops form Borromean rings [23, 60], or in general a Brunnian link.

In 5D, besides particles and loops, there is a kind of exotic topological excitations, dubbed as *membranes*, which look like closed 2-dimensional surfaces.⁴ The membrane excitations in 3D space have not been considered because they are impenetrable. However, in 4D space, the interior of a membrane excitation becomes accessible due to the extra dimension. Therefore, nontrivial braiding processes involving particles, loops and membranes are possible in 5D.

In a braiding process of 5D topological order, excitations move in 5D spacetime in a particular manner, resulting in their 1D world-line, 2D world-sheet, and/or 3D world-volume intersecting in a corresponding fashion. The 5D TQFT can tell us about how these manifolds intersect in 5D though which is not easy to imagine for us living in 3D. To visualize braiding processes in 5D topological order, we need to present them in lower dimensions. For this purpose, we can perform the following procedures. First, noticed that only the relative motion of excitations matters in a braiding process, we can assume one of the excitations to be static in 4D space, then we project the world-lines/world-sheets/world-volumes from 5D spacetime to 4D space. For the static excitation, it is just fixed in 4D space; for other excitations doing relative motion, their spatial trajectories are left in 4D space after projection. An intuitive example is the anyon braiding in 3D, shown in figure 1, in which the world-lines of anyons can be projected to spatial trajectories in 2D plane. Yet, the static excitation and spatial trajectories in 4D space are still strange to observer in 3D space. Our strategy is to furthermore project these spatial trajectories to slices of 4D space, i.e., 3D space. As an example, figure 2 shows the projection of spatial trajectory of particle-loop braiding from 3D space to 2D plane. This example shows how an observer in 2D space can understand braiding process in 4D spacetime. In summary, by projecting the general world-lines in 5D spacetime to spatial trajectories in 4D space, then viewing them in 3D slices, we can observe the braiding processes in 5D topological order. This method will be explained by several pictures when discussing braiding and 5D TQFT in the following main text.

2.2 Two types of BF terms

As the effective theory of topological order, TQFT describes braiding processes by revealing how the world-lines (in general meaning) of excitations intersect in spacetime. A braiding process is nontrivial only if the world-lines of excitations form a link that is homotopic invariant. By constructing gauge-invariant observables, we know how to count the intersections of world-lines to obtain a result that is invariant under homotopic mapping

³For simplicity, when we mention particle, loop, or membrane, we refer to topological excitations in topological orders.

⁴We only consider membrane excitation that has a shape of a 2-sphere in this paper. Membrane excitations with other shapes, e.g., 2-torus, may have more fascinating properties.

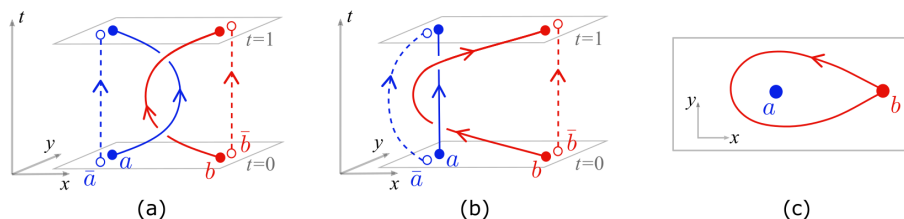


Figure 1. Braiding of two particles (e.g., anyons) in 3D spacetime, world-lines of particles and their projection in 2D space. (a) At $t = 0$, two pairs of particles (solid circles) and antiparticle (empty circles) are created. Two particles (a and b) move in 3D spacetime such that their world-lines (solid lines) cross each other. At $t = 1$, two particles meet their own antiparticles (\bar{a} and \bar{b}). The $a\bar{a}$ and $b\bar{b}$ pairs are annihilated. Since the world-line of an antiparticle (dash line) can be understood as that of a particle with a reverse direction, the particle-antiparticle pair's world-lines can be viewed as a single closed one of the particle. For a braiding process of two particles in 3D, the two closed world-lines are linked. (b) One of the two particles (e.g., a) can be assumed static in the xy -plane. By slightly modifying the world-line of b and keeping the two closed world-lines linked, the braiding process contributes the same phase shift. (c) The world-lines of a and b illustrated in (b) are projected to the xy -plane. After that, b 's world-line is appeared as a closed spatial trajectory encircling the static particle a . In FQHE, an anyon moving around another one is exactly the braiding of them.

of world-lines. In this and following sections, we use TQFT actions and gauge-invariant observables to study braiding processes in 5D topological order.

The study of BF term has a long history [56]. Discovered from quantum gravity and high energy physics, now BF theories have been introduced into frontiers of condensed matter physics. In an n -dimensional spacetime manifold M , the BF term is the wedge product of a p -form \mathcal{B} and an $(n-p)$ -form $\mathcal{F} = d\mathcal{A}$ where \mathcal{A} is an $(n-p-1)$ -form: $\mathcal{B} \wedge d\mathcal{A}$. BF term in 3D and 4D reads $\tilde{A}dA$ and BdA respectively, where \tilde{A} and A are 1-form, B is 2-form. Theories with BF terms have been applied to many different physical systems [26, 31, 32, 43, 50, 59, 61, 62]. It is easy to verify that $\tilde{A}dA$ (BdA) is the only BF term in 3D (4D). However, in 5D, by taking $p = 2$ or $p = 3$, there are two types of BF terms, i.e., CdA and $\tilde{B}dB$ which are respectively called type-I and type-II in this paper. Here, C is a 3-form, \tilde{B} and B are two different 2-form's. These two types of BF terms in 5D are the cornerstone of this paper. With the action $S \sim \int_M \mathcal{B}d\mathcal{A}$, one can calculate the linking number of a p and $(n-p-1)$ -dimensional sub-manifolds [42, 56]. In this section, we study the action with type-I and type-II BF term in details, revealing the connection between braiding processes in 5D topological orders and BF terms.

We first look at the action with type-I BF term:

$$S = \int \frac{N_1}{2\pi} C^1 dA^1, \quad (2.1)$$

which is invariant up to boundary term under gauge transformations

$$A^1 \rightarrow A^1 + d\chi^1, \quad C^1 \rightarrow C^1 + dT^1, \quad (2.2)$$

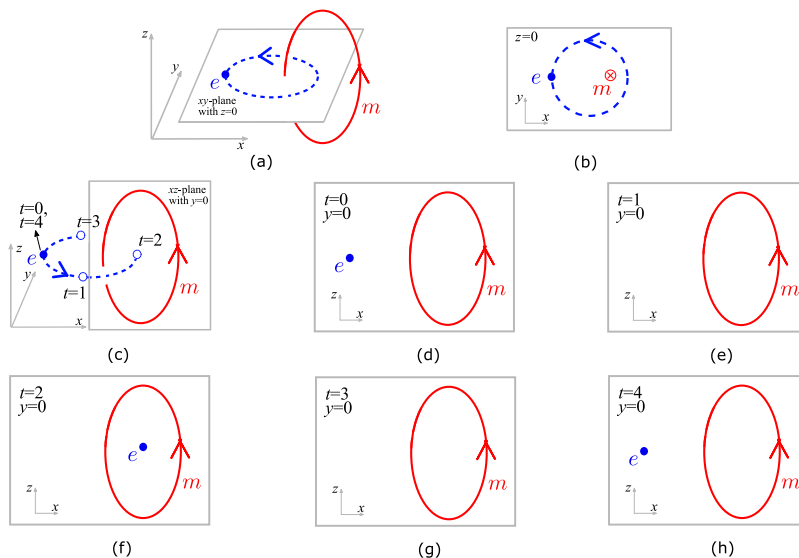


Figure 2. A particle-loop braiding in 3D space and its projection on 2D planes. (a) Particle-loop braiding is one important braiding process in 4D topological order. In this braiding process, if we assume the loop m to be static, what we see is that the particle e encircles the loop such that its spatial trajectory and loop m form a Hopf link. (b) For an observer living on the xy -plane with $z = 0$, the particle-loop braiding in 3D appears as a particle e encircling a flux that penetrates this plane. (c) The same particle-loop braiding in 3D space as that in (a). The particle's positions at different moments are labeled. (d)-(e) Snapshots of particle-loop braiding observed in xz -plane with $y = 0$ at different moments. In this plane, the loop excitation appears as its complete form while the spatial trajectory of the particle is not continuous any longer. An observer living on this plane can only see the particle at $t = 0, 2, 4$. At $t = 1$ or $t = 3$, this particle is located at xz -planes with different y -coordinates. To this planar observer's knowledge, a particle cannot cross a loop excitation to reach its interior area. What illustrated in (d)-(h) happens due to an extra dimension (the particle is able to move in y -direction) that cannot be seen by this observer. On the other hand, an observer in 2D plane can detect a 3rd dimension by such an “anomalous” phenomenon. This inspire us that we can learn about braiding process in 4D space (5D spacetime) by observing it in different 3D spaces with a fixed 4th coordinate.

where 0-form χ^1 and 2-form T^1 are $\mathbb{U}(1)$ gauge parameters with $\oint d\chi^1 \in 2\pi\mathbb{Z}$ and $\oint dT^1 \in 2\pi\mathbb{Z}$. We consider the following gauge invariant observable (Wilson operator)

$$\begin{aligned} \mathcal{W}(\omega_1, \gamma_1) &= \exp\left(im_1 \int_{\omega_1} C^1\right) \exp\left(ie_1 \int_{\gamma_1} A^1\right) \\ &= \exp\left[im_1 \int C^1 \wedge \delta^\perp(\omega_1) + ie_1 \int A^1 \wedge \delta^\perp(\gamma_1)\right], \end{aligned} \quad (2.3)$$

where ω_1 is a closed 3D volume and γ_1 is a closed 1D curve; charges $m_1, e_1 \in \mathbb{Z}$.⁵ The expectation value is given by $\langle \mathcal{W}(\omega_1, \gamma_1) \rangle = \frac{1}{Z} \int \mathcal{D}A \mathcal{D}C \mathcal{W}(\omega_1, \gamma_1) \exp(iS)$ where the

⁵Subscripts (1, 2, \dots , or general i, j) are used to distinguish manifolds that labeled by the same Greek letter. In this paper, γ_i labels different 1D closed curves; σ_i and $\tilde{\sigma}_i$ label different 2D closed surfaces; ω_i stands for different 3D closed volumes. Σ_i , Ω_i , and Ξ_i are the Seifert (hyper)surfaces: $\partial\Sigma_i = \gamma_i$, $\partial\Omega_i = \sigma_i$, and $\partial\Xi_i = \omega_i$. $\mu_i|_X$, $\nu_i|_X$, and $\lambda_i|_X$ respectively stand for general 1D open curve, 2D open surface, and 3D open volume on manifold X .

partition function is defined as $\mathcal{Z} = \int \mathcal{D}C \mathcal{D}A \exp(iS)$. By integrating out C^1 we get $dA^1 = -\frac{2\pi m_1}{N_1} \delta^\perp(\omega_1)$ which can be solved by $A^1 = -\frac{2\pi m_1}{N_1} \delta^\perp(\Xi_1)$ with $\partial \Xi_1 = \omega_1$, i.e., Ξ_1 is a Seifert hypersurface bounded by ω_1 . $\delta^\perp(\omega_1)$ is the 2-form valued delta function distribution supported on ω_1 and $\delta^\perp(\Xi_1)$ is similarly defined. Plugging the solution back to $\langle \mathcal{W}(\omega_1, \gamma_1) \rangle$, we get

$$\begin{aligned} \langle \mathcal{W}(\omega_1, \gamma_1) \rangle &= \exp \left[-\frac{i2\pi e_1 m_1}{N_1} \int \delta^\perp(\Xi_1) \wedge \delta^\perp(\gamma_1) \right] \\ &= \exp \left[-\frac{i2\pi e_1 m_1}{N_1} \#(\Xi_1 \cap \gamma_1) \right], \end{aligned} \tag{2.4}$$

i.e., $\langle \mathcal{W}(\omega_1, \gamma_1) \rangle$ is determined by counting the intersection of Ξ_1 and γ_1 . In 3D, the linking number of two closed curves γ_i and γ_j is defined by the intersection number $\#(\Sigma_i \cap \gamma_j)$ where Σ_i is a Seifert surface of γ_i . Analogous to this, $\#(\Xi_1 \cap \gamma_1)$ defines the linking number of γ_1 and ω_1 (whose Seifert hypersurface is Ξ_1) in 5D. In section 2.1, we mention that 5D TQFT describes braiding process via intersection pattern of world-lines of excitations in 5D. The action (2.1) and $\langle \mathcal{W}(\omega_1, \gamma_1) \rangle$ describe the braiding process of a particle and a membrane once we interpret γ_1 (ω_1) as the closed world-line (world-volume) of particle (membrane). Figure 3 illustrates this particle-membrane braiding by projecting it in different 3D space.

Next we focus on the action that consists of a type-II BF term:

$$S = \int \frac{N_1}{2\pi} \tilde{B}^1 dB^1 \tag{2.5}$$

which is invariant up to boundary terms under

$$B^1 \rightarrow B^1 + dV^1, \quad \tilde{B}^1 \rightarrow \tilde{B}^1 + d\tilde{V}^1. \tag{2.6}$$

Here, 1-form V^1 and 1-form \tilde{V}^1 are $\mathbb{U}(1)$ gauge parameters with $\oint dV^1 \in 2\pi\mathbb{Z}$ and $\oint d\tilde{V}^1 \in 2\pi\mathbb{Z}$. We point out that \tilde{B}^1 and B^1 are two independent 2-form gauge field variables. The corresponding gauge-invariant observable is

$$\mathcal{W}(\tilde{\sigma}_1, \sigma_1) = \exp \left(im_1 \int_{\tilde{\sigma}_1} \tilde{B}^1 \right) \exp \left(ie_1 \int_{\sigma_1} B^1 \right), \tag{2.7}$$

where $\tilde{\sigma}_1$ and σ_1 are two closed 2D surfaces; charges $m_1, e_1 \in \mathbb{Z}$. Similar to the calculation of $\langle \mathcal{W}(\omega_1, \gamma_1) \rangle$ above, we get the solution $B^1 = -\frac{2\pi m_1}{N_1} \delta^\perp(\tilde{\Omega}_1)$ with $\partial \tilde{\Omega}_1 = \tilde{\sigma}_1$, i.e., $\tilde{\Omega}_1$ is a Seifert hypersurface bounded by $\tilde{\sigma}_1$. Plugging this solution back, we obtain

$$\begin{aligned} \langle \mathcal{W}(\tilde{\sigma}_1, \sigma_1) \rangle &= \exp \left[-\frac{i2\pi e_1 m_1}{N_1} \int \delta^\perp(\tilde{\Omega}_1) \wedge \delta^\perp(\sigma_1) \right] \\ &= \exp \left[-\frac{i2\pi e_1 m_1}{N_1} \#(\tilde{\Omega}_1 \cap \sigma_1) \right], \end{aligned} \tag{2.8}$$

which indicates that $\langle \mathcal{W}(\tilde{\sigma}_1, \sigma_1) \rangle$ is determined by counting the intersection of $\tilde{\Omega}_1$ and σ_1 . Similarly, $\tilde{\sigma}_1$ and σ_1 can be viewed as closed world-sheets of two loop excitations whose linking number is given by $\#(\tilde{\Omega}_1 \cap \sigma_1)$. Therefore, the type-II BF term $\tilde{B}^1 dB^1$ describes to the braiding process of two loops and the braiding statistical phase can be extracted from $\langle \mathcal{W}(\tilde{\sigma}_1, \sigma_1) \rangle$. Figure 4 provides a diagrammatic representation of this loop-loop braiding by projecting the world-sheets to spatial trajectories of loops from 5D to 4D space.

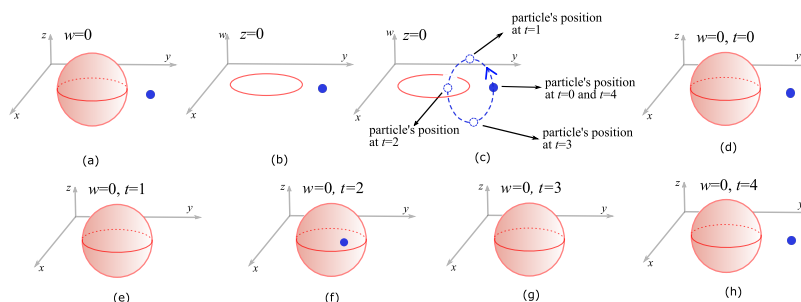


Figure 3. Observing the particle-membrane braiding described by CdA term in 3D space. (a) A membrane (red) with the shape of a 2-sphere in xyz -space with $w = 0$ and a particle (blue) are illustrated. This membrane is assumed to be static in space. (b) If observed in xyw -space with $z = 0$, this membrane appears as a “loop”. (c) In xyw -space with $z = 0$, the particle-membrane braiding in 4D space looks like a particle-loop braiding. Starting at $t = 0$, the particle moves in the xyw -space with $z = 0$ and returns to its initial position at $t = 4$. The particle’s spatial trajectory forms a closed 1D curve that is linked with the “loop” (projection of membrane in this 3D space). The w -coordinate of particle, w_p , varies in the following way: at $t = 0$, $w_p = 0$; for $0 < t < 2$, $w_p > 0$; at $t = 2$, $w_p = 0$ again; for $2 < t < 4$, $w_p < 0$; finally, the particle returns to its initial position thus $w_p = 0$ at $t = 4$. (d)–(h) Snapshots of particle-membrane braiding at different moments in xyz -space with $w = 0$. At $t = 0$, the particle is located outside the membrane. At $t = 2$, the particle appears inside the membrane. At $t = 4$, this particle returns to its starting position; in xyz -space with $w = 0$, it appears outside the membrane again. For other moments, the particle is located in xyz -space with $w \neq 0$. In 4D space, Ξ_1 in eq. (2.4) is projected to a 3D Seifert hypersurface and γ_1 in that appears as a closed 1D spatial trajectory. In xyw -space with $w = 0$ shown in (c), the aforementioned 3D Seifert hypersurface appears as a 2D one whose boundary is exactly the “loop”. The spatial trajectory and the 2D Seifert surface intersect at one point at $t = 2$ implying that $|\#(\Xi_1 \cap \gamma_1)| = 1$ with a sign determined by orientation. In xyz -space with $w = 0$ shown in (d)–(h), the spatial trajectory is the union of particle’s positions at different moments which intersects with the 3D Seifert hypersurface at $t = 2$. This also indicates $|\#(\Xi_1 \cap \gamma_1)| = 1$.

2.3 Braidings in type-I BF theory with a twist

So far we have investigated TQFT actions that consist of single BF term. The coefficient $\frac{N_1}{2\pi}$ of the BF term encodes the \mathbb{Z}_{N_1} gauge group. If a $\prod_{i=1}^n \mathbb{Z}_{N_i}$ topological order is considered, i.e., the gauge group is $G = \prod_{i=1}^n \mathbb{Z}_{N_i}$, there are twisted terms allowed in TQFT actions besides BF terms. As examples, we present some TQFT actions that consist of BF terms and *one* twisted term in 4D $\prod_{i=1}^n \mathbb{Z}_{N_i}$ topological orders [23, 32, 42, 44, 63, 64]. With twisted term $AAdA$ or $AAAA$, TQFT actions $S = \int \sum_i \frac{N_i}{2\pi} B^i dA^i + qA^i A^j dA^k$ and $S = \int \sum_i \frac{N_i}{2\pi} B^i dA^i + qA^i A^j A^k A^l$ describe multi-loop braidings (three-loop and four-loop, respectively; q is the proper coefficient). With twisted term AAB , the action $S = \int \sum_i \frac{N_i}{2\pi} B^i dA^i + qA^i A^j B^k$ describes the particle-loop-loop braiding. It is natural to expect that TQFT actions of BF terms and twisted terms in 5D are related to braiding processes of topological excitations. We divide the TQFT actions into 3 classes according to the type of their BF terms, namely type-I BF theory, type-II BF theory, and mixed BF theory. In the remaining part of this section, we consider BF theory with only *one* twisted term. Consistent combination of all twisted terms will be explored in section 3.

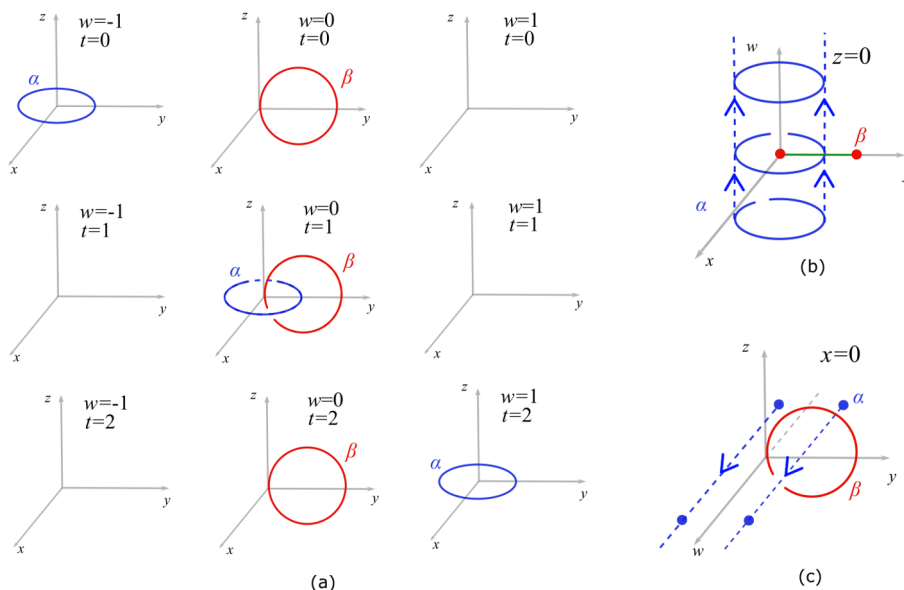


Figure 4. Diagrammatic representation of loop-loop braiding described by $\tilde{B}dB$ term. (a) Loop-loop braiding process viewed in xyz -space with different w - and t -coordinates. Loop α (blue) lives on xy -plane with $z_\alpha = 0$ and $w_\alpha = -1$ at $t = 0$. Loop β (red) lives on yz -plane with $x_\beta = 0$ and $w_\beta = 0$. Loop β is assumed to be static in space. When $t = 0$, loop α can only be observed in xyz -space with $w = -1$ and so does loop β in xyz -space with $w = 0$; as for xyz -space with $w = 1$, neither of two loops could be observed. Starting at $t = 0$, loop α moves in w -direction such that at $t = 1$ $w_\alpha = 0$ and loop α and β are linked in xyz -space with $w = 0$. Next, loop α continues to move such that $w_\alpha = 1$ at $t = 2$. At $t = 2$, loop α and β are not linked obviously. Then loop α returns to its initial position in such a way that it would be linked with loop β again. During this process, these two loops never intersect with each other since no points of them share a same 4D spatial coordinate. (b) Part of the spatial trajectory of loop α (indicated by blue dash lines) observed in xyw -space with $z = 0$. In this 3D space, loop β appears as two points (red solid circles). In 4D space, $\tilde{\Omega}_1$ in eq. (2.8) is projected to a 2D Seifert surface and σ_1 becomes a 2D closed spatial trajectory. A further projection to xyw -space with $z = 0$ makes this 2D Seifert surface appear as a 1D one (the green segment) whose boundary is two points (red solid circles, exactly the projection of loop β in this 3D space). This 1D Seifert surface intersect with loop α 's spatial trajectory at one point, which implies $|\#(\tilde{\Omega}_1 \cap \sigma_1)| = 1$ with a sign determined by orientation. (c) Viewed in wyz -space with $x = 0$, loop α appears as two points (blue solid circles) and part of its spatial trajectory is shown as the blue dash lines. This spatial trajectory intersect with loop β 's 2D Seifert surface at one point, indicating that $|\#(\tilde{\Omega}_1 \cap \sigma_1)| = 1$ as well.

We start with type-I BF terms with a twisted term formed by A and C , i.e., AAC , $AdAdA$, $AAAdA$, and $AAAAA$.

AAC twisted topological term. By considering three $C^i dA^i$ terms, we can introduce the twisted term $A^i A^j C^k$. The corresponding TQFT action is

$$S = S_{BF} + S_{AAC} = \int \sum_{i=1}^3 \frac{N_i}{2\pi} C^i dA^i + qA^1 A^2 C^3, \tag{2.9}$$

where q is a quantized and periodic coefficient, $q = \frac{pN_1N_2N_3}{(2\pi)^2N_{123}}$, $p \in \mathbb{Z}_{N_{123}}$, N_{123} is the greatest common divisor of N_1 , N_2 and N_3 . The property of coefficient q results from two requirements on the TQFT action: large gauge invariance and flux identification. In appendix A we derive the quantization and periodicity of AAC and other twisted terms. It should be pointed out that the indices i , j , and k in $A^iA^jC^k$ have to be mutually different. For example, the $A^1A^2C^2$ term is prohibited, because either A^2 or C^2 has to be the Lagrange multiplier, which means that they cannot simultaneously appear in the twisted term. A more rigorous explanation is that the an action with $A^1A^2C^2$ term would no more be gauge-invariant, which is similar to the 4D case $S \sim \int \sum_{i=1}^2 B^i dA^i + A^1A^2B^2$ studied in the section IV A of ref. [46].

This action (2.9) is invariant under gauge transformations:

$$\begin{aligned}
 A^1 &\rightarrow A^1 + d\chi^1, \quad C^1 \rightarrow C^1 + dT^1 - \frac{2\pi q}{N_1} \left(\chi^2 C^3 - A^2 T^3 + \chi^2 dT^3 \right), \\
 A^2 &\rightarrow A^2 + d\chi^2, \quad C^2 \rightarrow C^2 + dT^2 + \frac{2\pi q}{N_2} \left(\chi^1 C^3 - A^1 T^3 + \chi^1 dT^3 \right), \\
 C^3 &\rightarrow C^3 + dT^3, \quad A^3 \rightarrow A^3 + d\chi^3 - \frac{2\pi q}{N_3} \left[\left(\chi^1 A^2 + \frac{1}{2} \chi^1 d\chi^2 \right) - \left(\chi^2 A^1 + \frac{1}{2} \chi^2 d\chi^1 \right) \right],
 \end{aligned} \tag{2.10}$$

where χ^i and T^i are 0-form and 2-form gauge parameters, respectively. The gauge-invariant observable is

$$\begin{aligned}
 \mathcal{W} = \exp &\left\{ i \int_{\omega_1} e_1 \left[C^1 + \frac{1}{2} \frac{2\pi q}{N_1} \left(d^{-1} A^2 C^3 - d^{-1} C^3 A^2 \right) \right] \right. \\
 &+ i \int_{\omega_2} e_2 \left[C^2 + \frac{1}{2} \frac{2\pi q}{N_2} \left(d^{-1} C^3 A^1 - d^{-1} A^1 C^3 \right) \right] \\
 &\left. + i \int_{\gamma_3} e_3 \left[A^3 + \frac{1}{2} \frac{2\pi q}{N_3} \left(d^{-1} A^1 A^2 - d^{-1} A^2 A^1 \right) \right] \right\}
 \end{aligned} \tag{2.11}$$

with e_1 , e_2 , and e_3 being integers. The derivation of gauge transformation (2.10) and verification of gauge-invariance of (2.11) is detailed in appendix B. The operation d^{-1} is defined as $d^{-1}C^3|_{\omega_i} \equiv \int_{\lambda_3|_{\omega_i}} C^3$ where $\lambda_3|_{\omega_i}$ is an open 3D volume on ω_i with $i = 1, 2$. As a 2-form, $d^{-1}C^3$ is well-defined on ω_i if and only if C^3 is exact on ω_i , i.e., $\int_{\omega_i} C^3 = 0$. The action of d^{-1} on A^i is defined as $d^{-1}A^i|_{\omega_j} \equiv \int_{\mu_i|_{\omega_j}} A^i$ and $d^{-1}A^i|_{\gamma_3} \equiv \int_{\mu_i|_{\gamma_3}} A^i$, where μ_i is an open curve on ω_j or γ_3 . As a 0-form, $d^{-1}A^i$ is well-defined on ω_j or γ_3 if and only if the integral of A^i over any 1-dimensional closed sub-manifolds of ω_j or γ_3 is zero. Since A^1 , A^2 , and C^3 are required to be exact on specific manifolds, it is straightforward that dA^1 , dA^2 , and dC^3 are zero on corresponding manifolds. This in fact guarantees the gauge invariance of (2.11). From a geometric perspective, the exactness condition implies restrictions that some sub-manifolds are not linked. In other words, the gauge invariance of observable is associated with the geometric interpretation of braiding process. The gauge fields and their fluxes are defined on specific sub-manifolds. For example, consider a gauge field \mathcal{A} whose flux $d\mathcal{A}$ is defined on a closed 1-manifold γ_1 : $d\mathcal{A} = \delta^\perp(\gamma_1)$. Say, \mathcal{A} is exact on another closed 1-manifold γ_2 , this means that $\int_\Sigma d\mathcal{A} = \int_{\gamma_2} \mathcal{A} = 0$ with $\partial\Sigma = \gamma_2$. Since $\int_\Sigma d\mathcal{A} = \int \delta^\perp(\Sigma) \wedge \delta^\perp(\gamma_1) = \#(\Sigma \cap \gamma_1)$, the linking number of γ_1 and γ_2 is 0 due to \mathcal{A} is

exact on γ_2 . On the other hand, the gauge invariance of \mathcal{W} ensures that $\langle \mathcal{W} \rangle$ is invariant under homotopic mapping of world-lines of excitations. However, sometimes $\langle \mathcal{W} \rangle$ does not clearly show how world-lines are linked. The exactness condition, required to meet gauge invariance, will give some hints since they reveal which sub-manifolds are not linked.

The expectation value is given by $\langle \mathcal{W} \rangle = \frac{1}{Z} \int \mathcal{D}A \mathcal{D}C \mathcal{W} \exp(iS)$. Integrating out C^1 , C^2 , and A^3 implies $A^{1,2} = -\frac{2\pi e_{1,2}}{N_{1,2}} \delta^\perp(\Xi_{1,2})$ and $C^3 = -\frac{2\pi e_3}{N_3} \delta^\perp(\Sigma_3)$ with $\partial\Xi_{1,2} = \omega_{1,2}$ and $\partial\Sigma_3 = \gamma_3$. Here, ω stands for 3D closed volume, γ stands for 1D closed curve, etc, as explained in footnote 5. Putting these solutions back in $\langle \mathcal{W} \rangle$, we have

$$\begin{aligned} \int q A^1 A^2 C^3 &= -\frac{(2\pi)^3 q e_1 e_2 e_3}{N_1 N_2 N_3} \int \delta^\perp(\Xi_1) \wedge \delta^\perp(\Xi_2) \delta^\perp(\Sigma_3), \\ &= -\frac{p N_1 N_2 N_3}{(2\pi)^2 N_{123}} \frac{(2\pi)^3 e_1 e_2 e_3}{N_1 N_2 N_3} \#(\Xi_1 \cap \Xi_2 \cap \Sigma_3) \\ &= -\frac{2\pi p e_1 e_2 e_3}{N_{123}} \#(\Xi_1 \cap \Xi_2 \cap \Sigma_3); \end{aligned} \quad (2.12)$$

$$\begin{aligned} \int_{\omega_1} e_1 \left[\frac{q}{2} (d^{-1} A^2 C^3 - d^{-1} C^3 A^2) \right] &= \frac{\pi p e_1 e_2 e_3}{N_{123}} \left[\int \delta^\perp(\omega_1) \wedge \delta^\perp(\Sigma_3) \wedge d^{-1} \delta^\perp(\Xi_2)|_{\omega_1} \right. \\ &\quad \left. - \int \delta^\perp(\omega_1) \wedge \delta^\perp(\Xi_2) \wedge d^{-1} \delta^\perp(\Sigma_3)|_{\omega_1} \right]. \end{aligned} \quad (2.13)$$

By definition, $d^{-1} \delta^\perp(\Xi_2)|_{\omega_1} = \int_{\mu_2|\omega_1} \delta^\perp(\Xi_2) = \int \delta^\perp(\mu_2|\omega_1) \wedge \delta^\perp(\Xi_2)$, etc. Thus

$$\begin{aligned} \int_{\omega_1} e_1 \left[\frac{q}{2} (d^{-1} A^2 C^3 - d^{-1} C^3 A^2) \right] &= \frac{\pi p e_1 e_2 e_3}{N_{123}} \left[\int \delta^\perp(\omega_1) \wedge \delta^\perp(\Sigma_3) \wedge \int \delta^\perp(\mu_2|\omega_1) \wedge \delta^\perp(\Xi_2) \right. \\ &\quad \left. - \int \delta^\perp(\omega_1) \wedge \delta^\perp(\Xi_2) \wedge \int \delta^\perp(\lambda_3|\omega_1) \wedge \delta^\perp(\Sigma_3) \right] \\ &= \frac{\pi p e_1 e_2 e_3}{N_{123}} [\#(\omega_1 \cap \Sigma_3 \cap \mu_2|\omega_1 \cap \Xi_2) \\ &\quad - \#(\omega_1 \cap \Xi_2 \cap \lambda_3|\omega_1 \cap \Sigma_3)]. \end{aligned} \quad (2.14)$$

In a similar manner, we can calculate the remaining parts of $\langle \mathcal{W} \rangle$ and end up with

$$\begin{aligned} \langle \mathcal{W} \rangle &= \exp \left\{ -\frac{i 2\pi p e_1 e_2 e_3}{N_{123}} \#(\Xi_1 \cap \Xi_2 \cap \Sigma_3) \right. \\ &\quad + \frac{i \pi p e_1 e_2 e_3}{N_{123}} [\#(\omega_1 \cap \Sigma_3 \cap \mu_2|\omega_1 \cap \Xi_2) - \#(\omega_1 \cap \Xi_2 \cap \lambda_3|\omega_1 \cap \Sigma_3)] \\ &\quad + \frac{i \pi p e_1 e_2 e_3}{N_{123}} [\#(\omega_2 \cap \Xi_1 \cap \lambda_3|\omega_2 \cap \Sigma_3) - \#(\omega_2 \cap \Sigma_3 \cap \mu_1|\omega_2 \cap \Xi_1)] \\ &\quad \left. + \frac{i \pi p e_1 e_2 e_3}{N_{123}} [\#(\gamma_3 \cap \Xi_2 \cap \mu_1|\gamma_3 \cap \Xi_1) - \#(\gamma_3 \cap \Xi_1 \cap \mu_2|\gamma_3 \cap \Xi_2)] \right\}. \end{aligned} \quad (2.15)$$

Geometrically, $\langle \mathcal{W} \rangle$ is determined by counting the intersections of several sub-manifolds in 5D. According to their dimension, we can interpret their relations with excitations. The 1D γ_i can be understood as the closed world-line of particle. The 3D ω_i can be regarded as the closed world-volume of membrane. Σ_i and Ξ_i are Seifert (hyper)surfaces: $\partial\Sigma_i = \gamma_i$ and $\partial\Xi_i = \omega_i$. We can see that all these sub-manifolds are related to one

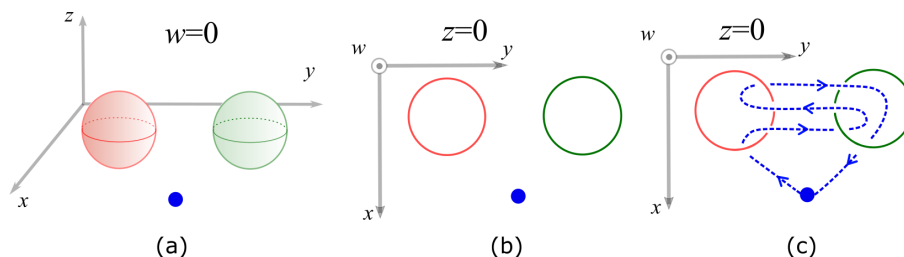


Figure 5. Particle-membrane-membrane braiding described by AAC term. (a) Two membranes and one particle viewed in xyz -space with $w = 0$. These two membranes are assumed to be static in space. (b) These two membrane excitations appears as two loops in xyw -space with $z = 0$. (c) If viewed in xyw -space with $z = 0$, the spatial trajectory of particle and two loops (projections of two membranes) form Borromean rings. In Borromean rings, any two of the two loops and particle's trajectory are not linked. This means that the particle's spatial trajectory is not linked with any of two membranes and neither the two membranes, which matches the exactness conditions that imply specific sub-manifolds are not linked. This particle-membrane-membrane braiding can be considered as the 5D analog of particle-loop-loop braiding described by AAB term in 4D [23].

particle and two membranes. In this sense, we believe that the AAC term along with its gauge-invariant observable (2.11) describe the braiding process of one particle and two membranes. Exploited with the exactness conditions, figure 5 illustrates this particle-membrane-membrane braiding in 3D space.

In addition, the expectation value (2.15) is expressed as the sum of several terms: $\langle \mathcal{W} \rangle = \exp\left(-\frac{i2\pi p e_1 e_2 e_3}{N_{123}} \mathfrak{L}\right)$ with

$$\begin{aligned}
 \mathfrak{L} = & \#(\Xi_1 \cap \Xi_2 \cap \Sigma_3) \\
 & - \frac{1}{2} [\#(\omega_1 \cap \Sigma_3 \cap \mu_2|_{\omega_1 \cap \Xi_2}) - \#(\omega_1 \cap \Xi_2 \cap \lambda_3|_{\omega_1 \cap \Sigma_3})] \\
 & - \frac{1}{2} [\#(\omega_2 \cap \Xi_1 \cap \lambda_3|_{\omega_2 \cap \Sigma_3}) - \#(\omega_2 \cap \Sigma_3 \cap \mu_1|_{\omega_2 \cap \Xi_1})] \\
 & - \frac{1}{2} [\#(\gamma_3 \cap \Xi_2 \cap \mu_1|_{\gamma_3 \cap \Xi_1}) - \#(\gamma_3 \cap \Xi_1 \cap \mu_2|_{\gamma_3 \cap \Xi_2})]. \tag{2.16}
 \end{aligned}$$

$\#(\Xi_1 \cap \Xi_2 \cap \Sigma_3)$ counts the signed intersections of Ξ_1 , Ξ_2 and Σ_3 and other terms have similar geometric meanings. Each term in \mathfrak{L} (2.16) is not homotopic invariant because it depends on the choices of Seifert (hyper)surfaces. However, \mathfrak{L} as the sum of these terms, is homotopic invariant due to the gauge invariance of $\langle \mathcal{W} \rangle$. We see that a mathematical invariant is obtained via physical gauge-invariant field theory. The formula of \mathfrak{L} is similar to the equation (9) of ref. [23] which computes the Milnor's triple linking number from a 4D TQFT. In fact, the Milnor's triple linking number of 3 closed curves is also expressed as the sum of several homotopic-variant terms [65, 66]. Analogously, eq. (2.16) can be regarded as the ‘‘triple linking number’’ of closed world-lines (in general meaning) of two membranes and one particle in 5D. In the following main text, we will see more observables whose expectation values are expressed in a similar manner. It is interesting to reveal the mathematical structure behind them.

AdAdA twisted topological term. Next, we consider the *AdAdA* term, i.e., the Chern-Simons term in 5D.⁶ For a gauge group $G = \prod_{i=1}^n \mathbb{Z}_{N_i}$, the type-I *BF* theory with *AdAdA* term has the form of

$$S = S_{BF} + S_{AdAdA} = \int \sum_{i=1}^n \frac{N_i}{2\pi} C^i dA^i + q A^i dA^j dA^k \quad (2.17)$$

with gauge transformations $A^i \rightarrow A^i + d\chi^i$ and $C^i \rightarrow C^i + dT^i$. The indices of $A^i dA^j dA^k$, same or different, take values from $\{1, 2, \dots, n\}$. The simplest type-I *BF* theory with *AdAdA* term is

$$S = \int \frac{N_1}{2\pi} C^1 dA^1 + q A^1 dA^1 dA^1 \quad (2.18)$$

with $q = \frac{p}{(2\pi)^2}$, $p \in \mathbb{Z}_{N_1}$. This action (2.18) describes a \mathbb{Z}_{N_1} 5D topological order. The gauge-invariant observable for (2.18) is

$$\mathcal{W} = \exp \left(i e_1 \int_{\omega_1} C^1 \right) = \exp \left[i e_1 \int C^1 \wedge \delta^\perp(\omega_1) \right] \quad (2.19)$$

with the expectation value

$$\langle \mathcal{W} \rangle = \exp \left[-\frac{i 2\pi p e_1 e_1 e_1}{N_1 N_1 N_1} \# (\Xi_1 \cap \omega_1 \cap \omega_1) \right]. \quad (2.20)$$

If the gauge group is $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$, the *AdAdA* twisted term can also be $A^1 dA^2 dA^1$ or $A^2 dA^1 dA^2$. They are the only two linearly independent terms because $d(A^1 A^2 dA^1) = dA^1 A^2 dA^1 - A^1 dA^2 dA^1$ and $d(A^1 A^2 dA^2) = dA^1 A^2 dA^2 - A^1 dA^2 dA^2$. The corresponding type-I *BF* theory and gauge-invariant observable is

$$S = \int \sum_{i=1}^2 \frac{N_i}{2\pi} C^i dA^i + q A^1 dA^2 dA^1, \quad (2.21)$$

$$\mathcal{W} = \exp \left(i \sum_{i=1}^2 e_i \int_{\omega_i} C^i \right) = \exp \left[i \sum_{i=1}^2 e_i \int C^i \wedge \delta^\perp(\omega_i) \right], \quad (2.22)$$

$$\langle \mathcal{W} \rangle = \exp \left[-\frac{i 2\pi p e_1 e_2 e_1}{N_1 N_2 N_1} \# (\Xi_1 \cap \omega_2 \cap \omega_1) \right]; \quad (2.23)$$

or

$$S' = \int \sum_{i=1}^2 \frac{N_i}{2\pi} C^i dA^i + q' A^2 dA^1 dA^2, \quad (2.24)$$

$$\mathcal{W}' = \exp \left(i \sum_{i=1}^2 e_i \int_{\omega_i} C^i \right) = \exp \left[i \sum_{i=1}^2 e_i \int C^i \wedge \delta^\perp(\omega_i) \right], \quad (2.25)$$

$$\langle \mathcal{W}' \rangle = \exp \left[-\frac{i 2\pi p e_2 e_1 e_2}{N_2 N_1 N_2} \# (\Xi_2 \cap \omega_1 \cap \omega_2) \right]. \quad (2.26)$$

The coefficients of twisted terms are $q = \frac{p}{(2\pi)^2}$, $p \in \mathbb{Z}_{N_{12}}$ and $q' = \frac{p'}{(2\pi)^2}$, $p' \in \mathbb{Z}_{N_{12}}$; $\mathbb{Z}_{N_{12}}$ is the greatest common divisor of N_1 and N_2 . The general case is that $G = \prod_{i=1}^3 \mathbb{Z}_{N_i}$

⁶When *BF* term is not included, the theory is studied in ref. [67].

in which the twisted term is $A^1 dA^2 dA^3$. In fact, $A^1 dA^2 dA^3$, $A^2 dA^3 dA^1$, and $A^3 dA^1 dA^2$ are identical up to a boundary term because $d(A^1 A^2 dA^3) = A^2 dA^3 dA^1 - A^1 dA^2 dA^3$ and $d(A^1 dA^2 A^3) = A^3 dA^1 dA^2 - A^1 dA^2 dA^3$. The type-I BF theory with $A^1 dA^2 dA^3$ is

$$S = \int \sum_{i=1}^3 \frac{N_i}{2\pi} C^i dA^i + q A^1 dA^2 dA^3 \quad (2.27)$$

with $q = \frac{p}{(2\pi)^2}$, $p \in \mathbb{Z}_{N_{123}}$. For this type-I BF theory (2.27), the gauge-invariant observable is

$$\mathcal{W} = \exp \left(i \sum_{i=1}^3 e_i \int_{\omega_i} C^i \right) = \exp \left[i \sum_{i=1}^3 e_i \int C^i \wedge \delta^\perp(\omega_i) \right], \quad (2.28)$$

and its expectation value is

$$\langle \mathcal{W} \rangle = \exp \left[-\frac{i 2\pi p e_1 e_2 e_3}{N_1 N_2 N_3} \#(\Xi_1 \cap \omega_2 \cap \omega_3) \right]. \quad (2.29)$$

We conclude that the $A^i dA^j dA^k$ twisted term describes the braiding process of three membranes, in which membranes move in 5D leaving their intersecting world-volumes described by $\#(\Xi_i \cap \omega_j \cap \omega_k)$. For the action $S \sim \int \sum B dA + A^i dA^j dA^k$, the expectation value of gauge-invariant observable takes the form of $\langle \mathcal{W} \rangle \sim \exp[\#(\Xi_i \cap \omega_j \cap \omega_k)]$. All these three sub-manifolds are related with membranes: ω_j and ω_k can be regarded as the closed world-volumes of membrane i and j ; Ξ_i is the Seifert hypersurface of ω_i , the closed world-volume of membrane i . $(\Xi_i \cap \omega_j \cap \omega_k)$ can be seen as the intersection of three membranes' world-volumes. Therefore, we believe that the $AdAdA$ term describes the braiding of three membrane whose phase shift is related to $\#(\Xi_i \cap \omega_j \cap \omega_k)$.

AAAdA twisted topological term. Then we consider the $AAAdA$ twisted term. Since A is 1-form, the three A 's in $AAAdA$ have to be different, thus $G = \prod_{i=1}^n \mathbb{Z}_{N_i}$ with $n \geq 3$ is required. If $G = \prod_{i=1}^3 \mathbb{Z}_{N_i}$, the twisted term can be $A^1 A^2 A^3 dA^1$, $A^1 A^2 A^3 dA^2$, or $A^1 A^2 A^3 dA^3$. Take $A^1 A^2 A^3 dA^1$ as an example, the type-I BF theory with $A^1 A^2 A^3 dA^1$ is

$$S = S_{BF} + S_{AAAdA} = \int \sum_{i=1}^3 \frac{N_i}{2\pi} C^i dA^i + q A^1 A^2 A^3 dA^1 \quad (2.30)$$

with $q = \frac{p N_1 N_2 N_3}{(2\pi)^3 N_{123}}$, $p \in \mathbb{Z}_{N_{123}}$. The corresponding gauge transformations are

$$\begin{aligned} A^1 &\rightarrow A^1 + d\chi^1, \quad C^1 \rightarrow C^1 + dT^1 - \frac{2\pi q}{N_1} (A^1 d\chi^2 A^3 + A^1 A^2 d\chi^3 + A^1 d\chi^2 d\chi^3), \\ A^2 &\rightarrow A^2 + d\chi^2, \quad C^2 \rightarrow C^2 + dT^2 + \frac{2\pi q}{N_2} (d\chi^1 A^3 A^1 + d\chi^1 d\chi^3 A^1), \\ A^3 &\rightarrow A^3 + d\chi^3, \quad C^3 \rightarrow C^3 + dT^3 + \frac{2\pi q}{N_3} (-d\chi^1 A^2 A^1 - d\chi^1 d\chi^2 A^1). \end{aligned} \quad (2.31)$$

If $G = \prod_{i=1}^4 \mathbb{Z}_{N_i}$, possible $AAAdA$ terms are $A^1 A^2 A^3 dA^4$, $A^2 A^3 A^4 dA^1$, $A^3 A^4 A^1 dA^2$, and $A^4 A^1 A^2 dA^3$. Since $d(A^1 A^2 A^3 A^4) = dA^1 A^2 A^3 A^4 - A^1 dA^2 A^3 A^4 + A^1 A^2 dA^3 A^4 - A^1 A^2 A^3 dA^4$, only three of them are linearly independent. We consider $A^1 A^2 A^3 dA^4$ as an example and the corresponding type-I BF theory is

$$S = S_{BF} + S_{AAAdA} = \int \sum_{i=1}^4 \frac{N_i}{2\pi} C^i dA^i + q A^1 A^2 A^3 dA^4 \quad (2.32)$$

with $q = \frac{pN_1N_2N_3}{(2\pi)^3N_{123}}$, $p \in \mathbb{Z}_{N_{1234}}$, N_{1234} is the greatest common divisor of N_1 , N_2 , N_3 , and N_4 . The gauge transformations are

$$A^i \rightarrow A^i + d\chi^i, \quad C^i \rightarrow C^i + dT^i + \frac{2\pi q}{N_i} \sum_{j,k} \epsilon^{ijk4} \left(A^j \chi^k dA^4 - \frac{1}{2} \chi^j d\chi^k dA^4 \right). \quad (2.33)$$

The gauge-invariant observable is

$$\mathcal{W} = \exp \left\{ i \sum_{i=1}^4 \int_{\omega_i} e_i \left[C^i + \frac{1}{2} \frac{2\pi q}{N_i} \sum_{j,k} \epsilon^{ijk4} (d^{-1} A^j) A^k dA^4 \right] \right\} \quad (2.34)$$

and its expectation value is

$$\begin{aligned} \langle \mathcal{W} \rangle = \exp & \left\{ \frac{i2\pi p e_1 e_2 e_3 e_4}{N_4 N_{123}} \# (\Xi_1 \cap \Xi_2 \cap \Xi_3 \cap \omega_4) \right. \\ & \left. + \frac{i\pi p e_1 e_2 e_3 e_4}{N_4 N_{123}} \sum_{i=1}^4 \sum_{j,k} \epsilon^{ijk4} \# (\omega_i \cap \Xi_k \cap \omega_4 \cap \mu_j |_{\omega_i \cap \Xi_j}) \right\}. \quad (2.35) \end{aligned}$$

Similar to the case of $AdAdA$ twisted term, ω_i and Ξ_i can be interpreted as membrane's closed world-volume and its Seifert hypersurface. Eq. (2.35) calculates the intersection of world-volumes of four membranes to give the phase shift. It makes sense to consider $AAAdA$ term describing the braiding of four membranes.

AAAAA twisted topological term. The last possible twisted term in type-I BF theory is $AAAAA$. The corresponding action is

$$S = S_{BF} + S_{AAAAA} = \int \sum_{i=1}^5 \frac{N_i}{2\pi} C^i dA^i + q A^1 A^2 A^3 A^4 A^5, \quad (2.36)$$

where $q = \frac{pN_1N_2N_3N_4N_5}{(2\pi)^4N_{12345}}$, $p \in \mathbb{Z}_{N_{12345}}$, $\mathbb{Z}_{N_{12345}}$ is the greatest common divisor of N_1 , N_2 , N_3 , N_4 , and N_5 . The gauge transformations are

$$\begin{aligned} A^i & \rightarrow A^i + d\chi^i, \\ C^i & \rightarrow C^i + dT^i - \frac{2\pi q}{N_i} \sum_{k<l<m} \epsilon^{ijklm} \chi^j A^k A^l A^m - \frac{2\pi q}{N_i} \sum_{j<k,l<m} \epsilon^{ijklm} \chi^j d\chi^k A^l A^m \\ & - \frac{2\pi q}{N_i} \sum_{j<k<l} \sum_{m=1}^5 \epsilon^{ijklm} \chi^j d\chi^k d\chi^l A^m - \frac{2\pi q}{N_i} \sum_{j<k<l<m} \epsilon^{ijklm} \chi^j d\chi^k d\chi^l d\chi^m. \quad (2.37) \end{aligned}$$

The gauge-invariant observable is

$$\mathcal{W} = \exp \left[i \sum_{i=1}^5 \int_{\omega_i} e_i \left(C^i + \frac{1}{4} \frac{2\pi q}{N_i} \sum_{k<l<m} \sum_{j=1}^5 \epsilon^{ijklm} d^{-1} A^j A^k A^l A^m \right) \right] \quad (2.38)$$

and its expectation value is

$$\begin{aligned} \langle \mathcal{W} \rangle = \exp & \left\{ \frac{i2\pi p \prod_{i=1}^5 e_i}{N_{12345}} \# (\Xi_1 \cap \Xi_2 \cap \Xi_3 \cap \Xi_4 \cap \Xi_5) \right. \\ & \left. + \sum_{i=1}^5 \frac{i2\pi p \prod_{i=1}^5 e_i}{4N_{12345}} \sum_{k<l<m} \sum_{j=1}^5 \epsilon^{ijklm} \# (\omega_i \cap \Xi_k \cap \Xi_l \cap \Xi_m \cap \mu_j |_{\omega_i \cap \Xi_j}) \right\}. \quad (2.39) \end{aligned}$$

As cases discussed above, ω_i and Ξ_i can be treated as membrane's closed world-volume and its Seifert hypersurface. Consider a braiding of five membranes, eq. (2.39) exactly gives the phase shift by counting the intersection of five membranes' world-volumes.

2.4 Braidings in mixed BF theory with a twist

In a type-II BF theory whose only BF term is $\tilde{B}dB$, there is no twisted term since 2-form \tilde{B} and B cannot make up a 5-form twisted term. Therefore the only possible type-II BF theory is $S = \int \sum_{i=1}^n \frac{N_i}{2\pi} \tilde{B}^i dB^i$ with gauge group $G = \prod_{i=1}^n \mathbb{Z}_{N_i}$. We will come back to this action in section 3.

Besides type-I or type-II BF theory, we study *mixed* BF theory with a twist which consists of *two types* of BF terms and one twisted term. Possible twisted terms in mixed BF theory include BBA , $BAdA$, $AAdB$, and $AAAB$.

BAdA twisted topological term. One example of mixed BF theory with $BAdA$ term is

$$S = S_{BF} + S_{BAdA} = \int \frac{N_1}{2\pi} C^1 dA^1 + \frac{N_2}{2\pi} \tilde{B}^2 dB^2 + qB^2 A^1 dA^1, \quad (2.40)$$

where $q = \frac{pN_2N_1}{(2\pi)^2 N_{12}}$, $p \in \mathbb{Z}_{N_{12}}$. This mixed BF action is invariant up to boundary terms under

$$\begin{aligned} A^1 &\rightarrow A^1 + d\chi^1, & C^1 &\rightarrow C^1 + dT^1 - \frac{2\pi q}{N_1} dV^2 A^1, \\ B^2 &\rightarrow B^2 + dV^2, & \tilde{B}^2 &\rightarrow \tilde{B}^2 + d\tilde{V}^2 - \frac{2\pi q}{N_2} d\chi^1 A^1. \end{aligned} \quad (2.41)$$

The gauge-invariant observable is

$$\mathcal{W} = \exp \left\{ i \left[e_1 \left(\int_{\omega_1} C^1 + \frac{2\pi q}{N_1} \int_{\Xi_1} B^2 dA^1 \right) + e_2 \left(\int_{\sigma_2} \tilde{B}^2 - \frac{2\pi q}{N_2} \int_{\Omega_2} A^1 dA^1 \right) \right] \right\}, \quad (2.42)$$

where $\partial\Omega_2 = \sigma_2$ and $\partial\Xi_1 = \omega_1$. The expectation value is

$$\langle \mathcal{W} \rangle = \exp \left[-\frac{i2\pi p e_2 e_1 e_1}{N_1 N_{12}} \# (\Omega_2 \cap \Xi_1 \cap \omega_1) \right]. \quad (2.43)$$

Other possible $BAdA$ terms include $B^3 A^1 dA^2$, $B^3 A^2 dA^1$, etc. Notice that $d(B^3 A^1 A^2) = dB^3 A^1 A^2 + B^3 dA^1 A^2 - B^3 A^1 dA^2$, only two of them are linearly independent. Thus the $AAdB$ twisted term can always be expressed by two proper $BAdA$ twisted terms, up to a boundary term. For now, we consider an example of mixed BF theory with $B^3 A^1 dA^2$:

$$S = \int \sum_{i=1}^2 \frac{N_i}{2\pi} C^i dA^i + \frac{N_3}{2\pi} \tilde{B}^3 dB^3 + qB^3 A^1 dA^2 \quad (2.44)$$

which is gauge-invariant under

$$\begin{aligned} A^1 &\rightarrow A^1 + d\chi^1, & C^1 &\rightarrow C^1 + dT^1 - \frac{2\pi q}{N_1} dV^3 A^2, \\ A^2 &\rightarrow A^2 + d\chi^2, & C^2 &\rightarrow C^2 + dT^2, \\ B^3 &\rightarrow B^3 + dV^3, & \tilde{B}^3 &\rightarrow \tilde{B}^3 + d\tilde{V}^3 - \frac{2\pi q}{N_3} d\chi^1 A^2. \end{aligned} \quad (2.45)$$

The coefficient q is quantized and periodic: $q = \frac{pN_3N_1}{(2\pi)^2N_{13}}$, $p \in \mathbb{Z}_{N_{123}}$. For this action, the gauge-invariant observable is

$$\mathcal{W} = \exp \left\{ i \left[e_1 \left(\int_{\omega_1} C^1 + \frac{2\pi q}{N_1} \int_{\Xi_1} B^3 dA^2 \right) + e_2 \int_{\omega_2} C^2 + e_3 \left(\int_{\sigma_3} B^3 - \frac{2\pi q}{N_3} \int_{\Omega_3} A^1 dA^2 \right) \right] \right\}. \quad (2.46)$$

with its expectation value being

$$\langle \mathcal{W} \rangle = \exp \left[-\frac{i2\pi p e_1 e_2 e_3}{N_2 N_{13}} \#(\Omega_3 \cap \Xi_1 \cap \omega_2) \right], \quad (2.47)$$

where $\omega_{1,2}$ are closed 3D volumes, σ_3 is closed 2D surface; Ξ_1 and Ω_3 are Seifert hyper-surfaces: $\partial\Xi_1 = \omega_1$ and $\partial\Omega_3 = \sigma_3$. If we consider the $A^1 A^2 dB^3$ or $A^2 B^3 dA^1$ term, the corresponding TQFT action, gauge transformation, and observable along with its expectation value, would be slightly different but share a similar pattern.

We notice that the $BAdA$ term shares a similar form of the $AAdA$ term in 4D TQFT which describes the 3-loop braiding in 4D topological order. Since $BAdA$ and $AAdA$ term both have an AdA part, the Chern-Simons term in 3D, their corresponding braiding processes may share some similarities.

We first briefly review the 3-loop braiding in 4D spacetime described by the $AAdA$ twisted term. Figure 6 illustrates a typical 3-loop braiding which can be described by the TQFT action

$$S = \int \sum_{i=1}^3 \frac{N_i}{2\pi} B^i dA^i + \frac{pN_1N_2}{(2\pi)^2 N_{12}} A^1 A^2 dA^3 \quad (2.48)$$

with $p \in \mathbb{Z}_{N_{123}}$. The gauge-invariant observable and its expectation value are [42]

$$\mathcal{W} = \exp \left\{ i \left[\sum_{i=1}^3 e_i \left(\int_{\sigma_i} B^i + \sum_j \frac{pe^{ij}}{2\pi N_j} \int_{\Omega_i} A^j dA^3 \right) \right] \right\}, \quad (2.49)$$

$$\langle \mathcal{W} \rangle = \exp \left[\frac{2\pi i p e_1 e_2 e_3}{N_{123}} \#(\Omega_1 \cap \Omega_2 \cap \sigma_3) \right]. \quad (2.50)$$

In eq. (2.50), σ_i is the closed 2D world-sheet of loop i and Ω_i is the 3D Seifert hypersurface with $\partial\Omega_i = \sigma_i$. Derived from a 4D field theory, $\Omega_1 \cap \Omega_2 \cap \sigma_3$, intersection of world-sheet and Seifert hypersurfaces, is a geometric object in 4D which is not so intuitive for an observer living in 3D. However, we can project it from 4D spacetime to 3D space so that we can have a better understanding. More concretely, being projected on 3D space, the closed world-sheet of loop appears as the closed spatial trajectory of loop, and the Seifert hypersurfaces are observed as 2D Seifert surfaces. Figure 6 illustrates how to calculate $\#(\Omega_1 \cap \Omega_2 \cap \sigma_3)$ from a 3-loop braiding.

Now we move back to the $BAdA$ twisted term and the braiding of one loop and two membranes it describes. For a $B^3 A^1 dA^2$ twisted term, the phase shift from the braiding process is related to $\#(\Omega_3 \cap \Xi_1 \cap \omega_3)$, see eq. (2.47), where Ω_3 is a 3D Seifert hypersurface, Ξ_3 is a 4D Seifert hypersurface, and ω_3 is a closed 3D world-volume. Similarly, we can project $\Omega_3 \cap \Xi_1 \cap \omega_3$ from 5D spacetime to 4-dimensional space: (1) Ω_3 is projected to be a 2D Seifert surface that can be viewed as the Seifert surface of a static loop; (2) Ξ_1

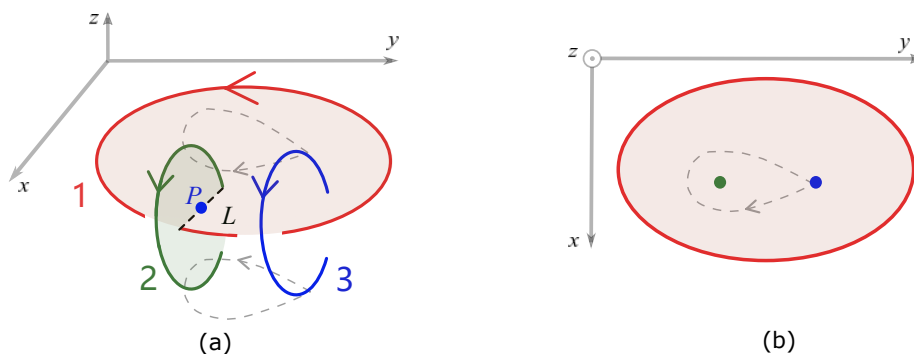


Figure 6. 3-loop braiding described by $AAdA$ term in 3D space. (a) The configuration of 3-loop braiding described by $A^1A^2dA^3$ term. For this 3-loop braiding, its phase shift is related to $\#(\Omega_1 \cap \Omega_2 \cap \sigma_3)$ as indicated in eq. (2.50). This linking number can be calculated via projecting Ω_1 , Ω_2 , and σ_3 from 4D spacetime to 3D space. After projection, Ω_1 appears as the Seifert surface (red shaded area) of the static base loop (loop 1); Ω_2 appears as the Seifert surface (green shaded area) of loop 2; σ_3 becomes the closed spatial trajectory of loop 3. We can see that the projection of $(\Omega_1 \cap \Omega_2)$ to 3D space is the segment L (black dash line). $(\Omega_1 \cap \Omega_2)$ is the world-sheet generated by L . Now we consider the spatial trajectory of loop 3: shrinking itself, passing through loop 2, expanding itself, finally back to its initial position, loop 3 swaps a closed two-dimensional surface. This surface intersects with segment L at a point P (blue solid circle). In other words, the intersection points, $(\Omega_1 \cap \Omega_2 \cap \sigma_3)$, share the same spatial coordinates as that of point P . Since the spatial trajectory of loop 3 only intersects with L once, $(\Omega_1 \cap \Omega_2)$ and σ_3 intersect only at one specific moment, i.e., $(\Omega_1 \cap \Omega_2 \cap \sigma_3)$'s temporal coordinate has only one value. At last, we can conclude that there is only 1 intersection point of Ω_1 , Ω_2 , and σ_3 , i.e., $|\#(\Omega_1 \cap \Omega_2 \cap \sigma_3)| = 1$ with a sign determined by orientation. (b) The 3-loop braiding can be viewed as an anyon braiding on the Seifert surface of the base loop.

appears as a 3D Seifert hypersurface that bounds a static membrane; (3) world-volume ω_3 is projected to be a closed spatial trajectory of a moving membrane. In 4D space, we see this braiding process as follows. The initial configuration is that a loop is linked with two membranes. If viewed in 3D space, see figure 7, a loop is linked with another two loops (two membranes are saw as two loops in 3D space). Then one membrane moves, passing through another membrane, such that in 3D space a 3-loop braiding is observed as shown in figure 7.

The 3-loop braiding described by $AAdA$ term can be understood as a braiding of two anyons on the Seifert surface of the base loop, see figure 6. The latter is captured by the AdA term, i.e., Chern-Simons term in 3D. Since $BAdA$ and $AAdA$ terms share a same AdA part, we argue that the braiding process described by $BAdA$ can be reduced to a braiding of two anyons on a Seifert surface of the loop excitation, as shown in figure 7.

BBA twisted topological term. An example of mixed BF theory with BBA term is

$$S = S_{BF} + S_{BBA} = \int \frac{N_1}{2\pi} \tilde{B}^1 dB^1 + \frac{N_2}{2\pi} \tilde{B}^2 dB^2 + \frac{N_3}{2\pi} C^3 dA^3 + qB^1 B^2 A^3 \quad (2.51)$$

where q is a quantized and periodic coefficient: $q = \frac{pN_1N_2N_3}{(2\pi)^2N_{123}}$, $p \in \mathbb{Z}_{N_{123}}$. The gauge

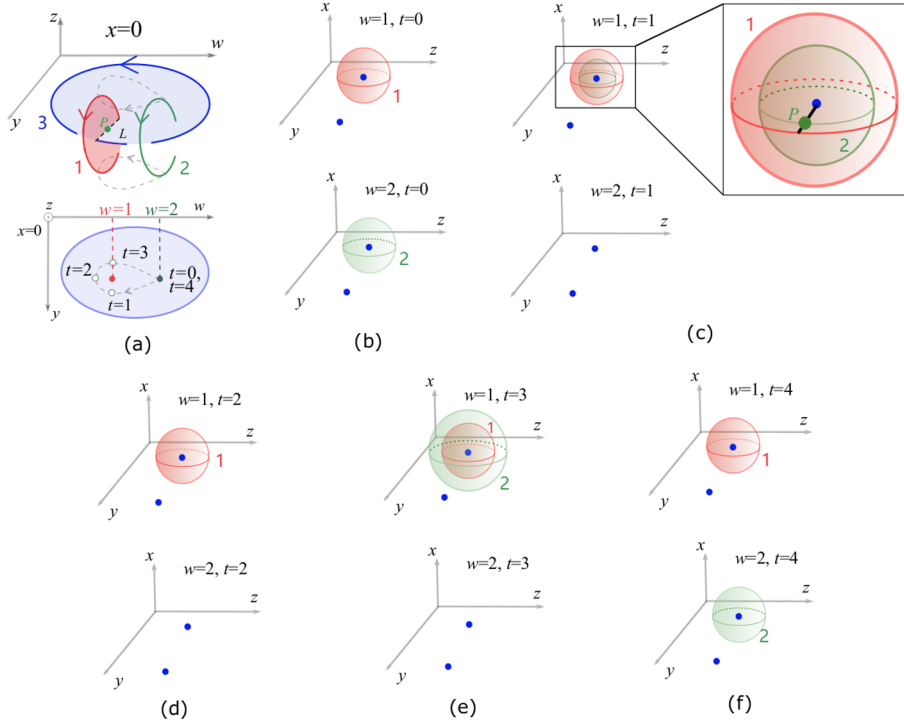


Figure 7. Loop-membrane-membrane braiding described by $BAdA$ term viewed in 3D space. (a) Viewed in ywz -space with $x = 0$, the two membranes (1 and 2) appears as two loops that are linked with the loop excitation (loop 3, blue) respectively. Loop 3 and membrane 1 are assumed to be static in space. This loop-membrane-membrane braiding looks like a 3-loop braiding in ywz -space with $x = 0$ while this 3-loop braiding can be further viewed as a braiding of two anyons on the Seifert surface of loop 3. The Ω_3 in eq. (2.47) is projected to the 2D Seifert surface of loop 3. Ξ_1 is projected to the 3D Seifert surface of static membrane 1. Since in this ywz -space with $x = 0$, static membrane 1 appears as a static loop, Ξ_1 's projection shows as the this loop's 2D Seifert surface. ω_2 is projected to the closed spatial trajectory of membrane 2 (loop 2 in this 3D space). We can see that these manifolds intersect at point P (green solid circle, also refer to figure. 6). This means that $|\#(\Omega_3 \cap \Xi_1 \cap \omega_2)| = 1$ with a sign determined by orientation. (b)-(f) Two membranes viewed in yzx -space with $w = 1$ and $w = 2$ respectively at different moments. The position of membrane 2 at different t 's are also labeled in (a). In these 3D spaces, loop 3 appears as two points (blue solid circles). In yzx -space with $w = 1$, Ω_3 appears as the 1D Seifert surface of two points (loop 3); Ξ_1 appears the 3D Seifert surface of static membrane 3; ω_2 is the spatial trajectory of membrane 2 that is the union of membrane 2's locations at $t = 1$ and $t = 3$ thus being discrete. The zoomed-in picture in (c) shows that these manifolds resulted from projection intersect at the point P (green solid circle), corresponding to that illustrated in (a). This also means that $|\#(\Omega_3 \cap \Xi_1 \cap \omega_2)| = 1$ with a sign determined by orientation.

transformations are

$$\begin{aligned}
 B^1 &\rightarrow B^1 + dV^1, \quad \tilde{B}^1 \rightarrow \tilde{B}^1 + d\tilde{V}^1 + \frac{2\pi q}{N_1} \left(V^2 A^3 + B^2 \chi^3 + V^2 d\chi^3 \right), \\
 B^2 &\rightarrow B^2 + dV^2, \quad \tilde{B}^2 \rightarrow \tilde{B}^2 + d\tilde{V}^2 + \frac{2\pi q}{N_2} \left(V^1 A^3 + B^1 \chi^3 + V^1 d\chi^3 \right), \\
 A^3 &\rightarrow A^3 + d\chi^3, \quad C^3 \rightarrow C^3 + dT^3 + \frac{2\pi q}{N_3} \left(-V^1 B^2 - B^1 V^2 - \frac{1}{2} V^1 dV^2 - \frac{1}{2} dV^1 V^2 \right),
 \end{aligned} \tag{2.52}$$

where V^i and \tilde{V}^i are different 1-form gauge parameters. The gauge-invariant observable is

$$\begin{aligned} \mathcal{W} = \exp \left\{ i \int_{\sigma_1} e_1 \left[\tilde{B}^1 - \frac{1}{2} \frac{2\pi q}{N_1} \left(d^{-1} B^2 A^3 + d^{-1} A^3 B^2 \right) \right] \right. \\ \left. + i \int_{\sigma_2} e_2 \left[\tilde{B}^2 - \frac{1}{2} \frac{2\pi q}{N_2} \left(d^{-1} B^1 A^3 + d^{-1} A^3 B^1 \right) \right] \right. \\ \left. + i \int_{\omega_3} e_3 \left[C^3 - \frac{1}{2} \frac{2\pi q}{N_3} \left(d^{-1} B^1 B^2 + d^{-1} B^2 B^1 \right) \right] \right\} \end{aligned} \quad (2.53)$$

and its expectation value is

$$\begin{aligned} \langle \mathcal{W} \rangle = \exp \left\{ -\frac{i2\pi p e_1 e_2 e_3}{N_{123}} \#(\Omega_1 \cap \Omega_2 \cap \Xi_3) \right. \\ \left. + \frac{i\pi p e_1 e_2 e_3}{N_{123}} [\#(\sigma_1 \cap \Xi_3 \cap \nu_2|_{\sigma_1} \cap \Omega_2) + \#(\sigma_1 \cap \Omega_2 \cap \mu_3|_{\sigma_1} \cap \Xi_3)] \right. \\ \left. + \frac{i\pi p e_1 e_2 e_3}{N_{123}} [\#(\sigma_2 \cap \Xi_3 \cap \nu_1|_{\sigma_2} \cap \Omega_1) + \#(\sigma_2 \cap \Omega_1 \cap \mu_3|_{\sigma_2} \cap \Xi_3)] \right. \\ \left. + \frac{i\pi p e_1 e_2 e_3}{N_{123}} [\#(\omega_3 \cap \Omega_1 \cap \nu_1|_{\omega_3} \cap \Omega_1) + \#(\omega_3 \cap \Omega_2 \cap \nu_2|_{\omega_3} \cap \Omega_2)] \right\}. \end{aligned} \quad (2.54)$$

For a braiding process of two loops and one membrane, the phase shift should be obtained via counting the intersections of loops' world-sheets and membrane's world-volume. Eq. (2.54) just produces this phase shift once we see σ_i (ω_i) as closed world-sheet (world-volume) of loop (membrane) and notice that $\partial\Omega_i = \sigma_i$ and $\partial\Xi_i = \omega_i$. In this sense, we can say that the BBA term corresponds to the braiding of two loops and one membrane.

AAAB twisted topological term. Finally we consider the *AAAB* twisted term. An example of mixed *BF* theory with *AAAB* is

$$S = S_{BF} + S_{AAAB} = \int \sum_{i=1}^3 \frac{N_i}{2\pi} C^i dA^i + \frac{N_4}{2\pi} \tilde{B}^4 dB^4 + q A^1 A^2 A^3 B^4, \quad (2.55)$$

with $q = \frac{p N_1 N_2 N_3 N_4}{(2\pi)^3 N_{1234}}$, $p \in \mathbb{Z}_{N_{1234}}$. The gauge transformations are

$$\begin{aligned} C^i &\rightarrow C^i + dT^i \\ &+ \frac{2\pi q}{N_i} \sum_{j,k} \epsilon^{ijk} \left(-\chi^j A^k B^4 - \frac{1}{2} A^j A^k V^4 - \frac{1}{2} \chi^j d\chi^k B^4 - \chi^j A^k dV^4 + \frac{1}{2} d\chi^j \chi^k dV^4 \right), \\ \tilde{B}^4 &\rightarrow \tilde{B}^4 + d\tilde{V}^4 \\ &+ \frac{2\pi q}{N_4} \sum_{i,j,k} \epsilon^{4ijk} \left(-\frac{1}{2} \chi^i A^j A^k + \frac{1}{2} A^i \chi^j d\chi^k - \frac{1}{6} \chi^i d\chi^j d\chi^k \right). \end{aligned} \quad (2.56)$$

The gauge-invariant observable is

$$\begin{aligned} \mathcal{W} = \exp \left\{ i \sum_{i=1}^3 \int_{\omega_i} e_i \left[C^i - \frac{1}{3} \frac{2\pi q}{N_i} \sum_{j,k} \epsilon^{ijk} \left(B^4 A^j d^{-1} A^k - \frac{1}{2} A^j A^k d^{-1} B^4 \right) \right] \right. \\ \left. + i \int_{\sigma_4} e_4 \left(\tilde{B}^4 - \frac{1}{6} \frac{2\pi q}{N_4} \sum_{i,j,k} \epsilon^{ijk} A^i A^j d^{-1} A^k \right) \right\}. \end{aligned} \quad (2.57)$$

Its expectation value is

$$\begin{aligned}
 \langle \mathcal{W} \rangle = \exp & \left\{ \frac{i2\pi p \prod_{i=1}^4 e_i}{N_{1234}} \# (\Xi_1 \cap \Xi_2 \cap \Xi_3 \cap \Omega_4) \right. \\
 & - \frac{i2\pi p \prod_{i=1}^4 e_i}{3N_{1234}} \sum_{i,j,k} \epsilon^{ijk} \left[\# (\omega_i \cap \Omega_4 \cap \Xi_j \cap \mu_k |_{\omega_i \cap \Xi_k}) - \frac{1}{2} \# (\omega_i \cap \Xi_j \cap \Xi_k \cap \nu_4 |_{\omega_i \cap \Omega_4}) \right] \\
 & \left. - \frac{i2\pi p \prod_{i=1}^4 e_i}{6N_{1234}} \sum_{i,j,k} \epsilon^{ijk} \# (\sigma_4 \cap \Xi_i \cap \Xi_j \cap \mu_k |_{\omega_i \cap \Xi_k}) \right\}. \tag{2.58}
 \end{aligned}$$

In eq. (2.58), σ_i stands for 2D closed world-sheet of loop, ω_i stands for 3D closed world-volume of membrane; $\partial\Xi_i = \omega_i$, $\partial\Omega_i = \sigma_i$. If we consider a braiding of three membranes and one loop, eq. (2.58) just counts the intersections of these excitations' world-sheets and world-volume in a gauge-invariant way and gives the phase shift. Thus we believe that this *AAAB* term along with its gauge-invariant observable describes the braiding of three membranes and one loop.

Before closing this section, we point out that the *BC* term is also a possible 5-form twisted term but it is not taken into consideration in the present paper. The *BC* term in 5D *BF* theory looks like the *BB* term in 4D. Both of which are quadratic terms just like *BF* terms. So, inclusion of *BC* term may complicate the analysis. In 4D, *BB* term can drastically change the gauge group, which means that the coefficients of *BF* terms cannot uniquely determine the gauge group G [63, 64]. We regard that *BC* term may have similar effect. In addition, *BB* term in 4D can change the self-statistics (bosonic or fermionic) of particles [26, 45, 63, 64, 68, 69]. *BC* term may also play important role in transmuting self-statistics of particles. Since we are only interested in 5D topological orders with all bosons in this paper, we do not take *BC* term into account. But it will be definitely exciting to incorporate *BC* term in the future and study canonical quantization and equation of motion in the presence of source terms [64].

3 TQFTs in 5 dimensions: within and beyond Dijkgraaf-Witten cohomological classification

In section 2, we have studied type-I and mixed *BF* theories with *single* twisted term as well as the corresponding braiding processes in 5D topological orders. It is natural to ask what if more twisted terms are considered in these *BF* theories. Similar to the case in 4D [46], the compatibility of twisted terms in 5D should also be considered. Using the technique developed in ref. [46], we can exhaust 5D TQFT actions with all allowed twisted terms once the gauge group is given. Then, we study the classification of TQFT actions, which is the main purpose of this section. The complexity of classification of 5D TQFT actions mostly comes from the fact that there exist two types of *BF* terms in 5D. For a given gauge group $G = \prod_{i=1}^n \mathbb{Z}_{N_i}$, since each \mathbb{Z}_{N_i} cyclic gauge subgroup can be encoded in one of two types of *BF* terms, different distributions of \mathbb{Z}_{N_i} to type-I and type-II *BF* terms lead to different twisted terms. Once the *BF* terms are determined, all possible twisted terms can be figured out. An action with all allowed twisted terms which are

detailed in section 2.3 and section 2.4, can be written and its classification can be obtained by counting the coefficients of twisted terms. In the remaining part of this section, we will discuss classification of some BF theories with multiple twisted terms for $G = \mathbb{Z}_{N_1}$ (section 3.1), $G = \prod_{i=1}^2 \mathbb{Z}_{N_i}$ (section 3.2), $G = \prod_{i=1}^3 \mathbb{Z}_{N_i}$ (section 3.3), and $G = \prod_{i=1}^n \mathbb{Z}_{N_i}$ with $n \geq 4$ (section 3.4). We find that only a part of type-I BF theories with twisted terms are consistent with Dijkgraaf-Witten cohomological classification. The other part of type-I BF theories and all mixed BF theories are totally beyond the group cohomology classification.

Some notations used in this section need to be explained here before we move forward. We use a set $\{N_i\}_{i=1}^n = \{N_1, \dots, N_n\}$ to label the gauge group $G = \prod_{i=1}^n \mathbb{Z}_{N_i}$. Then, a subset $\alpha \subset \{N_i\}_{i=1}^n$ is introduced to denote which \mathbb{Z}_{N_i} gauge subgroup is encoded in type-I BF term. For example, consider $\{N_i\}_{i=1}^5$ and let $\alpha = \{N_1, N_2, N_3\}$, in this case the BF terms are $\sum_{i=1}^3 \frac{N_i}{2\pi} C^i dA^i + \sum_{i=4}^5 \frac{N_i}{2\pi} \tilde{B}^i dB^i$; if we let $\alpha' = \{N_2, N_4, N_5\}$, the BF terms are $\sum_{i=2,4,5} \frac{N_i}{2\pi} C^i dA^i + \sum_{i=1,3} \frac{N_i}{2\pi} \tilde{B}^i dB^i$ instead. Naturally, α and α' lead to different twisted terms, hence different TQFT actions and classifications. However, α and α' have the same cardinality, the two different TQFT actions and classifications can be connected by rearranging the indices, as can be seen in the examples below. Therefore, for simplicity, we only present results for one configuration of α when the cardinality of the set α is fixed.

3.1 $G = \mathbb{Z}_{N_1}$

If $\alpha = \emptyset$, the action is

$$S = \int \frac{N_1}{2\pi} \tilde{B}^1 dB^1. \tag{3.1}$$

Obviously, the coefficient N_i is fixed, resulting in only one TQFT. The classification is denoted as “ \mathbb{Z}_1 ”.

When $G = \mathbb{Z}_{N_1}$, the only nontrivial choice of α is $\alpha = \{N_1\}$. The action is

$$S = \int \frac{N_1}{2\pi} C^1 dA^1 + \left\langle A^1 dA^1 dA^1 \right\rangle, \tag{3.2}$$

whose classification is \mathbb{Z}_{N_1} .⁷ We point out that the action (3.2) is classified by the 5th cohomology group $H^5(\mathbb{Z}_{N_1}, \mathbb{R}/\mathbb{Z})$. Classification of actions when $G = \mathbb{Z}_{N_1}$ is summarized in table 1. We conclude that there are in total $1 + N_1$ different BF theories when $G = \mathbb{Z}_{N_1}$.

3.2 $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$

If $\alpha = \emptyset$, the action is

$$S = \int \sum_{i=1}^2 \frac{N_i}{2\pi} \tilde{B}^i dB^i \tag{3.3}$$

whose classification is \mathbb{Z}_1 . In fact, when $\alpha = \emptyset$, the action can only be the type-II BF theory without twisted terms, whose classification is automatically equal to \mathbb{Z}_1 . For $\alpha \neq \emptyset$, the TQFT actions are mixed BF theories or type-I BF theories with twisted terms.

⁷We use $\langle \text{twisted term 1, twisted term 2, } \dots \rangle$ to simply denote a summation of twisted terms that appear in actions. In this notation, all coefficients of twisted terms, which are properly quantized as mentioned in section 2.3 and 2.4, are omitted for the notational convenience.

G	α	Classification		
\mathbb{Z}_{N_1}	\emptyset	\mathbb{Z}_1		
	$\{N_1\}$	\mathbb{Z}_{N_1}	★	
$\prod_{i=1}^2 \mathbb{Z}_{N_i}$	\emptyset	\mathbb{Z}_1		
	$\{N_1\}$	$\mathbb{Z}_{N_1} \times (\mathbb{Z}_{N_{12}})^2$		
	$\{N_2\}$	$\mathbb{Z}_{N_2} \times (\mathbb{Z}_{N_{12}})^2$		
	$\{N_1, N_2\}$	$\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times (\mathbb{Z}_{N_{12}})^2$	★	
$\prod_{i=1}^3 \mathbb{Z}_{N_i}$	\emptyset	\mathbb{Z}_1		
	$\{N_1\}$	$\mathbb{Z}_{N_1} \times (\mathbb{Z}_{N_{12}})^2 \times (\mathbb{Z}_{N_{13}})^2 \times \mathbb{Z}_{N_{123}}$		
	$\{N_2\}$	$\mathbb{Z}_{N_2} \times (\mathbb{Z}_{N_{12}})^2 \times (\mathbb{Z}_{N_{23}})^2 \times \mathbb{Z}_{N_{123}}$		
	$\{N_3\}$	$\mathbb{Z}_{N_3} \times (\mathbb{Z}_{N_{13}})^2 \times (\mathbb{Z}_{N_{23}})^2 \times \mathbb{Z}_{N_{123}}$		
	$\{N_1, N_2\}$	$\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times (\mathbb{Z}_{N_{12}})^2 \times (\mathbb{Z}_{N_{13}})^2 \times (\mathbb{Z}_{N_{23}})^2 \times (\mathbb{Z}_{N_{123}})^2$		
	$\{N_1, N_3\}$	$\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_3} \times (\mathbb{Z}_{N_{13}})^2 \times (\mathbb{Z}_{N_{12}})^2 \times (\mathbb{Z}_{N_{23}})^2 \times (\mathbb{Z}_{N_{123}})^2$		
	$\{N_2, N_3\}$	$\mathbb{Z}_{N_2} \times \mathbb{Z}_{N_3} \times (\mathbb{Z}_{N_{23}})^2 \times (\mathbb{Z}_{N_{12}})^2 \times (\mathbb{Z}_{N_{13}})^2 \times (\mathbb{Z}_{N_{123}})^2$		
	$\{N_1, N_2, N_3\}$	no AAC: $\prod_{i=1}^3 \mathbb{Z}_{N_i} \times \prod_{1 \leq i < j \leq 3} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{123}})^4$		★
		with $A^1 A^2 C^3$: $(\mathbb{Z}_{N_{123}} \setminus \{0\}) \times \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times (\mathbb{Z}_{N_{12}})^2$		
		with $A^2 A^3 C^1$: $(\mathbb{Z}_{N_{123}} \setminus \{0\}) \times \mathbb{Z}_{N_2} \times \mathbb{Z}_{N_3} \times (\mathbb{Z}_{N_{23}})^2$		
with $A^3 A^1 C^2$: $(\mathbb{Z}_{N_{123}} \setminus \{0\}) \times \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_3} \times (\mathbb{Z}_{N_{13}})^2$				

Table 1. Classification of BF theories with twisted terms for different gauge groups G , which are detailed in sections 3.1, 3.2, and 3.3. α is a set, which denotes \mathbb{Z}_{N_i} gauge subgroups that are encoded in the type-I BF term, e.g., $\alpha = \{N_1, N_2\}$ means $\frac{N_1}{2\pi} C^1 dA^1 + \frac{N_2}{2\pi} C^2 dA^2$. Only a part of 5D BF theories, denoted by a ★ symbol, are consistent with Dijkgraaf-Witten cohomological classification, i.e., $H^5(G, \mathbb{R}/\mathbb{Z})$. The remaining are beyond the group cohomology classification. When $G = \prod_{i=1}^3 \mathbb{Z}_{N_i}$ and $\alpha = \{N_1, N_2, N_3\}$, depending on different choices of AAC term, the actions and their classification are different, as discussed in section 3.3. The reason is that AAC term may be incompatible with other twisted terms, hence cannot be included in the action.

If $\alpha = \{N_1\}$, the action is

$$S = \int \frac{N_1}{2\pi} C^1 dA^1 + \frac{N_2}{2\pi} \tilde{B}^2 dB^2 + \left\langle A^1 dA^1 dA^1, B^2 B^2 A^1, B^2 A^1 dA^1 \right\rangle, \quad (3.4)$$

whose classification is $\mathbb{Z}_{N_1} \times (\mathbb{Z}_{N_{12}})^2$. For the case of $\alpha = \{N_2\}$, one just needs to switch the indices 1 and 2 to obtain the corresponding classification: $\mathbb{Z}_{N_2} \times (\mathbb{Z}_{N_{12}})^2$, as shown in table 1.

If $\alpha = \{N_1, N_2\}$, the action is

$$S = \int \sum_{i=1}^2 \frac{N_i}{2\pi} C^i dA^i + \left\langle A^1 dA^1 dA^1, A^2 dA^2 dA^2, A^1 dA^2 dA^1, A^2 dA^1 dA^1 \right\rangle, \quad (3.5)$$

which is classified by the 5th cohomology group of $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$: $H^5(\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times (\mathbb{Z}_{N_{12}})^2$.

The number of different BF theories for $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$ can be obtained as follows. For $\alpha = \emptyset$, there is one BF theory. For $\alpha = \{N_1\}$, there are $|\mathbb{Z}_{N_1} \times (\mathbb{Z}_{N_{12}})^2| = N_1 (N_{12})^2$ different BF theories. For $\alpha = \{N_2\}$ and $\alpha = \{N_1, N_2\}$, this number is $N_2 (N_{12})^2$ and $N_1 N_2 (N_{12})^2$, respectively. Therefore, the total number of different BF theories for $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$ is $1 + N_1 (N_{12})^2 + N_2 (N_{12})^2 + N_1 N_2 (N_{12})^2$.

3.3 $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \mathbb{Z}_{N_3}$

Follow the same line of thinking, we can obtain the classification of BF theories for $G = \prod_{i=1}^3 \mathbb{Z}_{N_i}$, which is collected in table 1.

If $\alpha = \emptyset$, the action is type-II BF theory without twisted terms, whose classification is simply \mathbb{Z}_1 .

If $\alpha = \{N_1\}$, the action is

$$S = \int \frac{N_1}{2\pi} C^1 dA^1 + \sum_{i=2,3} \frac{N_i}{2\pi} \tilde{B}^i dB^i + \left\langle A^1 dA^1 dA^1, \sum_{i=2,3} B^i B^i A^1 + B^i A^1 dA^1, B^2 B^3 A^1 \right\rangle, \quad (3.6)$$

whose classification is $\mathbb{Z}_{N_1} \times (\mathbb{Z}_{N_{12}})^2 \times (\mathbb{Z}_{N_{13}})^2 \times \mathbb{Z}_{N_{123}}$. For the case of $\alpha = \{N_2\}$ or $\alpha = \{N_3\}$, the results are similar, as shown in table 1.

If $\alpha = \{N_1, N_2\}$, the action is

$$S = \int \sum_{i=1}^2 \frac{N_i}{2\pi} C^i dA^i + \frac{N_3}{2\pi} \tilde{B}^3 dB^3 + \left\langle \sum_{i=1,2} A^i dA^i dA^i, A^1 dA^2 dA^1, A^2 dA^1 dA^2, \sum_{i=1,2} (B^3 B^3 A^i + B^3 A^i dA^i), A^1 A^2 dB^3, B^3 A^2 dA^1 \right\rangle, \quad (3.7)$$

whose classification is $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \prod_{1 \leq i < j \leq 3} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{123}})^2$. For other choices of α , the corresponding classifications are listed in table 1.

If $\alpha = \{N_1, N_2, N_3\}$, we get a type-I BF theory with twists. Possible twisted terms include AAC , $AdAdA$, and $AAAdA$. Due to the possible incompatibility between AAC and $AdAdA$ or $AAAdA$ terms [46], we need to treat these twisted terms carefully to obtain correct classifications. First we do not consider AAC terms in this type-I BF theory. The action is

$$S = \int \sum_{i=1}^3 \frac{N_i}{2\pi} C^i dA^i + \left\langle \sum_{i=1}^3 A^i dA^i dA^i, \sum_{1 \leq i < j \leq 3} (A^i dA^j dA^i + A^j dA^i dA^j), A^1 dA^2 dA^3, \sum_{i=1}^3 A^1 A^2 A^3 dA^i \right\rangle, \quad (3.8)$$

being classified by the 5th cohomology group of $\prod_{i=1}^3 \mathbb{Z}_{N_i}$:

$$H^5 \left(\prod_{i=1}^3 \mathbb{Z}_{N_i}, \mathbb{R}/\mathbb{Z} \right) = \prod_{i=1}^3 \mathbb{Z}_{N_i} \times \prod_{1 \leq i < j \leq 3} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{123}})^4. \quad (3.9)$$

Then we consider an AAC term in the action. Without loss of generality, we add the $A^1A^2C^3$ term in twisted terms. We point out that $A^2A^3C^1$ and $A^3A^1C^2$ are also possible AAC terms, yet neither of them is compatible with $A^1A^2C^3$ [46]. If $A^2A^3C^1$ or $A^3A^1C^2$, instead of $A^1A^2C^3$, is included in twisted terms, the discussion and result are similar, as shown in table 1. Since $AdAdA$ and $AAAdA$ with A^3 or dA^3 are incompatible with $A^1A^2C^3$ [46], the action is

$$S = \int \sum_{i=1}^3 \frac{N_i}{2\pi} C^i dA^i + \frac{pN_1N_2N_3}{(2\pi)^2 N_{123}} A^1A^2C^3 + \left\langle \sum_{i=1,2} A^i dA^i dA^i, A^1dA^2dA^1, A^2dA^1dA^2 \right\rangle \quad (3.10)$$

with $p \in \mathbb{Z}_{N_{123}} \setminus \{0\}$. The reason for this incompatibility is that either A^3 or C^3 has to be the Lagrange multiplier, meaning that they cannot appear in twisted terms in the same time. More concretely, if the action consists of $A^1A^2C^3$ and other twisted terms with A^3 or dA^3 , it would unavoidably break the gauge-invariance, just like the illegitimate $A^1A^2C^2$ term discussed in “ AAC twisted topological term” in section 2.3. The classification of action (3.10) is

$$(\mathbb{Z}_{N_{123}} \setminus \{0\}) \times \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times (\mathbb{Z}_{N_{12}})^2, \quad (3.11)$$

where $\mathbb{Z}_{N_{123}} \setminus \{0\}$ is the set obtained by removing the identity element $\{0\}$ from the cyclic group $\mathbb{Z}_{N_{123}}$, corresponding to the existence of $\frac{pN_1N_2N_3}{(2\pi)^2 N_{123}} A^1A^2C^3$, i.e., $p \neq 0$. Therefore (3.11) is not a group any more but it is still a set:

$$\begin{aligned} & (\mathbb{Z}_{N_{123}} \setminus \{0\}) \times \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times (\mathbb{Z}_{N_{12}})^2 \\ & = \{(x_1, x_2, x_3, x_4, x_5) \mid x_1 \in \mathbb{Z}_{N_{123}} \setminus \{0\}, x_2 \in \mathbb{Z}_{N_1}, x_3 \in \mathbb{Z}_{N_2}, x_4 \in \mathbb{Z}_{N_{12}}, x_5 \in \mathbb{Z}_{N_{12}}\}. \end{aligned} \quad (3.12)$$

Each element is given by a group of 5 numbers, i.e., $(x_1, x_2, x_3, x_4, x_5)$, denoting a choice of periodic coefficients of twisted terms in action (3.10): x_1 for $A^1A^2C^3$, x_2 for $A^1dA^1dA^1, \dots$, and x_5 for $A^2dA^1dA^2$. There are in total $(N_{123} - 1) N_1 N_2 (N_{12})^2$ elements in (3.11), which means $(N_{123} - 1) N_1 N_2 (N_{12})^2$ different actions when $G = \prod_{i=1}^3 \mathbb{Z}_{N_i}$, $\alpha = \{N_1, N_2, N_3\}$, and the $A^1A^2C^3$ term is considered. Compare to the classification of action (3.8), we see that the existence of $A^1A^2C^3$ term indeed excludes some twisted terms.

From the example above we see that even if α is fixed, the classification and number of different BF theories may still depend on the choice of AAC terms. Generally speaking, though the BF terms, i.e., α , are determined, there are still different combinations of compatible twisted terms [46] which lead to different BF theories and classification. This feature is more obvious and important when the G contains more cyclic subgroups, e.g., cases discussed in section 3.4. The complete classification of BF theories for $G = \prod_{i=1}^3 \mathbb{Z}_{N_i}$ is given in table 1. In order to find out the total number of BF theories, one just need to sum up the corresponding cardinality of each classification in table 1.

3.4 $G = \prod_{i=1}^4 \mathbb{Z}_{N_i}$ and generalization to $G = \prod_{i=1}^n \mathbb{Z}_{N_i}$ with $n \geq 5$

In this section, we investigate BF theories and their classifications when the gauge group is $G = \prod_{i=1}^4 \mathbb{Z}_{N_i}$. In this case, there are much more twisted terms and complicated compatibility issues, which makes a long list of BF theories as well as their classification. For

sake of simplicity, we aim to provide some typical examples of BF theories and classifications for $G = \prod_{i=1}^4 \mathbb{Z}_{N_i}$ which indicate regularities for generalization to all BF theories and classifications.

If $\alpha = \emptyset$, the BF theories can only be

$$S = \int \sum_{i=1}^4 \frac{N_i}{2\pi} \tilde{B}^i dB^i \quad (3.13)$$

with the classification of \mathbb{Z}_1 .

If $\alpha = \{N_1\}$, the action is

$$S = \int \frac{N_1}{2\pi} C^1 dA^1 + \sum_{i=2}^4 \frac{N_i}{2\pi} \tilde{B}^i dB^i + \left\langle A^1 dA^1 dA^1, \sum_{i=2}^4 \left(B^i B^i A^1 + B^i A^1 dA^1 \right), \sum_{2 \leq i < j \leq 4} B^i B^j A^1 \right\rangle, \quad (3.14)$$

whose classification is $\mathbb{Z}_{N_1} \times \prod_{i=2}^4 (\mathbb{Z}_{N_{1i}})^2 \times \mathbb{Z}_{N_{123}} \times \mathbb{Z}_{N_{124}} \times \mathbb{Z}_{N_{134}}$.

If $\alpha = \{N_1, N_2\}$, the action is

$$S = \int \sum_{i=1}^2 \frac{N_i}{2\pi} C^i dA^i + \sum_{i=3}^4 \frac{N_i}{2\pi} \tilde{B}^i dB^i + \left\langle \sum_{i=1}^2 A^i dA^i dA^i, A^1 dA^2 dA^1, A^2 dA^1 dA^2, B^3 B^4 A^1, B^3 B^4 A^2, \sum_{i=3,4} \left(B^i B^i A^1 + B^i A^1 dA^1 + B^i B^i A^2 + B^i A^2 dA^2 + A^1 A^2 dB^i + B^i A^2 dA^1 \right) \right\rangle, \quad (3.15)$$

whose classification is

$$\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times (\mathbb{Z}_{N_{12}})^2 \times \mathbb{Z}_{N_{134}} \times \mathbb{Z}_{N_{234}} \times \prod_{i=3}^4 \left[(\mathbb{Z}_{N_{1i}})^2 \times (\mathbb{Z}_{N_{2i}})^2 \times (\mathbb{Z}_{N_{12i}})^2 \right].$$

If $\alpha = \{N_1, N_2, N_3\}$, BF terms are $\sum_{i=1}^3 \frac{N_i}{2\pi} C^i dA^i + \frac{N_4}{2\pi} \tilde{B}^4 dB^4$. Possible twisted terms include AAC , $AdAdA$, $AAAdA$, BBA , $BAdA$, and $AAdB$. Once again, we need to take care of the incompatibility of AAC and other twisted terms, as discussed in section 3.3. There are two situations: the action includes AAC term or not.

1. If there is no AAC term in twisted terms, the action is

$$S = \int \sum_{i=1}^3 \frac{N_i}{2\pi} C^i dA^i + \frac{N_4}{2\pi} \tilde{B}^4 dB^4 + \left\langle \sum_{i=1}^3 A^i dA^i dA^i, \sum_{1 \leq i < j \leq 3} \left(A^i dA^j dA^i + A^j dA^i dA^j \right), A^1 dA^2 dA^3, \sum_{i=1}^3 A^1 A^2 A^3 dA^i, \sum_{i=1}^3 \left(B^4 B^4 A^i + B^4 A^i dA^i \right), \sum_{1 \leq i < j \leq 3} \left(A^i A^j dB^4 + B^4 A^j dA^i \right), A^1 A^2 A^3 B^4 \right\rangle, \quad (3.16)$$

whose classification is $\prod_{i=1}^3 \mathbb{Z}_{N_i} \times \prod_{1 \leq i < j \leq 3} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{123}})^4 \times \prod_{i=1}^3 (\mathbb{Z}_{N_{i4}})^2 \times \prod_{1 \leq i < j \leq 3} (\mathbb{Z}_{N_{ij4}})^2 \times \mathbb{Z}_{N_{1234}}$.

2. If one AAC term is added to the action, without loss of generality, let it be $A^1A^2C^3$, the action is

$$S = \int \sum_{i=1}^3 \frac{N_i}{2\pi} C^i dA^i + \frac{N_4}{2\pi} \tilde{B}^4 dB^4 + \frac{pN_1N_2N_3}{(2\pi)^2 N_{123}} A^1A^2C^3 + \left\langle \sum_{i=1,2} A^i dA^i dA^i, \right. \\ \left. A^1 dA^2 dA^1, A^2 dA^1 dA^2, \sum_{i=1,2} \left(B^4 B^4 A^i + B^4 A^i dA^i \right), A^1 A^2 dB^4, B^4 A^2 dA^1 \right\rangle \quad (3.17)$$

with $p \in \mathbb{Z}_{N_{123}} \setminus \{0\}$. As mentioned in section 3.3, when there are only 3 elements in α , different AAC terms are incompatible with each other. Therefore, only *one* AAC term can be added to the action when $|\alpha| = 3$, e.g., $\alpha = \{N_1, N_2, N_3\}$. The classification of action (3.17) is

$$(\mathbb{Z}_{N_{123}} \setminus \{0\}) \times \prod_{i=1}^2 \mathbb{Z}_{N_i} \times (\mathbb{Z}_{N_{12}})^2 \times (\mathbb{Z}_{N_{14}})^2 \times (\mathbb{Z}_{N_{24}})^2 \times (\mathbb{Z}_{N_{124}})^2. \quad (3.18)$$

Compared to action (3.10), the difference between their classifications is due to extra twisted terms resulted from $\tilde{B}^4 dB^4$ and type-I BF terms. Appendix C presents the corresponding action and classification if other AAC term is considered.

If $\alpha = \{N_1, N_2, N_3, N_4\}$, the action is a type-I BF theory with twisted terms. Similarly, we need to study two situations in which the action consists of AAC terms or not. First, we consider the action without any AAC terms,

$$S = \int \sum_{i=1}^4 \frac{N_i}{2\pi} C^i dA^1 + \left\langle \sum_{i=1}^4 A^i dA^i dA^i, \sum_{1 \leq i < j \leq 4} \left(A^i dA^j dA^i + A^j dA^i dA^j \right), \sum_{1 \leq i < j < k \leq 4} A^i dA^j dA^k, \right. \\ \left. \sum_{1 \leq i < j < k \leq 4} \left(\sum_{l=i,j,k} A^i A^j A^k dA^l \right), A^1 A^2 A^3 dA^4, A^3 A^2 A^4 dA^1, A^1 A^3 A^4 dA^2 \right\rangle, \quad (3.19)$$

whose classification is

$$\prod_{i=1}^4 \mathbb{Z}_{N_i} \times \prod_{1 \leq i < j \leq 4} \left(\mathbb{Z}_{N_{ij}} \right)^2 \times \prod_{1 \leq i < j < k \leq 4} \left(\mathbb{Z}_{N_{ijk}} \right)^4 \times (\mathbb{Z}_{N_{1234}})^3 = H^5 \left(\prod_{i=1}^4 \mathbb{Z}_{N_i}, \mathbb{R}/\mathbb{Z} \right), \quad (3.20)$$

same as the result obtained from Dijkgraaf-Witten model. Then we study the case in which AAC terms are added to the action. Since there are 4 type-I BF terms, some AAC terms may be compatible, unlike the above case of $\alpha = \{N_1, N_2, N_3\}$. We first discuss the action and its classification when only one AAC term is considered. Then, following the compatibility principle [46], we add more allowed AAC terms to the action and figure out the corresponding classification.

1. Consider the action with only one AAC term, without loss of generality, $A^1A^2C^4$,

$$S = \int \sum_{i=1}^4 \frac{N_i}{2\pi} C^i dA^1 + \frac{pN_1N_2N_4}{(2\pi)^2 N_{124}} A^1A^2C^4 + \left\langle \sum_{i=1}^3 A^i dA^i dA^i, \right. \\ \left. \sum_{1 \leq i < j \leq 3} \left(A^i dA^j dA^i + A^j dA^i dA^j \right), A^1 dA^2 dA^3, \sum_{i=1}^3 A^1 A^2 A^3 dA^i \right\rangle, \quad (3.21)$$

where $p \in \mathbb{Z}_{N_{124}} \setminus \{0\}$. Its classification is $(\mathbb{Z}_{N_{124}} \setminus \{0\}) \times \prod_{i=1}^3 \mathbb{Z}_{N_i} \times \prod_{1 \leq i < j \leq 3} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{123}})^4$. There are other AAC terms compatible with $A^1 A^2 C^4$, e.g., $A^1 A^3 C^4$ and $A^2 A^3 C^4$ can also be added to action (3.21).

2. If $A^1 A^2 C^4$ and $A^1 A^3 C^4$ are included in twisted terms, the action is

$$S = \int \sum_{i=1}^4 \frac{N_i}{2\pi} C^i dA^1 + \frac{p N_1 N_2 N_4}{(2\pi)^2 N_{124}} A^1 A^2 C^4 + \frac{p' N_1 N_3 N_4}{(2\pi)^2 N_{134}} A^1 A^3 C^4 + \left\langle \sum_{i=1}^3 A^i dA^i dA^i, \sum_{1 \leq i < j \leq 3} (A^i dA^j dA^i + A^j dA^i dA^j), A^1 dA^2 dA^3, \sum_{i=1}^3 A^1 A^2 A^3 dA^i \right\rangle \quad (3.22)$$

with $p \in \mathbb{Z}_{N_{124}} \setminus \{0\}$ and $p' \in \mathbb{Z}_{N_{134}} \setminus \{0\}$. Its classification is $(\mathbb{Z}_{N_{124}} \setminus \{0\}) \times (\mathbb{Z}_{N_{134}} \setminus \{0\}) \times \prod_{i=1}^3 \mathbb{Z}_{N_i} \times \prod_{1 \leq i < j \leq 3} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{123}})^4$.

3. If twisted terms include $A^1 A^2 C^4$, $A^1 A^3 C^4$, and $A^2 A^3 C^4$, the classification of corresponding action is $\prod_{1 \leq i < j \leq 3} (\mathbb{Z}_{N_{ij4}} \setminus \{0\}) \times \prod_{i=1}^3 \mathbb{Z}_{N_i} \times \prod_{1 \leq i < j \leq 3} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{123}})^4$.

4. On the other hand, $A^1 A^2 C^3$ is also compatible with $A^1 A^2 C^4$, but not compatible with $A^1 A^3 C^4$ or $A^2 A^3 C^4$. The action with $A^1 A^2 C^4$ and $A^1 A^2 C^3$ is

$$S = \int \sum_{i=1}^4 \frac{N_i}{2\pi} C^i dA^1 + \frac{p N_1 N_2 N_4}{(2\pi)^2 N_{124}} A^1 A^2 C^4 + \frac{p' N_1 N_2 N_3}{(2\pi)^2 N_{123}} A^1 A^2 C^3 + \left\langle \sum_{i=1}^2 A^i dA^i dA^i, A^1 dA^2 dA^1, A^2 dA^1 dA^2 \right\rangle \quad (3.23)$$

with $p \in \mathbb{Z}_{N_{124}} \setminus \{0\}$ and $p' \in \mathbb{Z}_{N_{123}} \setminus \{0\}$. The corresponding classification is $(\mathbb{Z}_{N_{124}} \setminus \{0\}) \times (\mathbb{Z}_{N_{123}} \setminus \{0\}) \times \prod_{i=1}^2 \mathbb{Z}_{N_i} \times (\mathbb{Z}_{N_{12}})^2$.

At last, we point out that if other AAC term, e.g., $A^2 A^3 C^4$, and its compatible AAC terms are considered in the action, the action and classification can be obtained by a similar manner. In appendix C, we list actions and their classifications for all possible combinations of AAC terms.

In section 3.1, 3.2, 3.3, and 3.4, we have studied the classification of BF theories for gauge group consisting of up 4 cyclic groups. Such discussion can be generalized to the cases of $G = \prod_{i=1}^n \mathbb{Z}_{N_i}$ with $n \geq 5$. Some typical BF theories and their classifications are given in appendix D, which is helpful for considering general cases of $G = \prod_{i=1}^5 \mathbb{Z}_{N_i}$. In short, one needs to first determine BF terms. Then, all possible twisted terms can be found out according to BF terms. Next, one should check the compatibility between twisted terms, which can be done with the guidance provided in ref. [46]. Finally, an action with all allowed twisted terms can be written and its classification can be obtained by counting the coefficients of twisted terms.

We point out that when $G = \prod_{i=1}^5 \mathbb{Z}_{N_i}$, $\alpha = \{N_1, N_2, N_3, N_4, N_5\}$, and no *AAC* terms are taken in account, the action is

$$\begin{aligned}
 S = \int \sum_{i=1}^5 \frac{N_i}{2\pi} C^i dA^i + \left\langle \sum_{i=1}^5 A^i dA^i dA^i, \sum_{1 \leq i < j \leq 5} \left(A^i dA^j dA^i + A^j dA^i dA^j \right), \right. \\
 \sum_{1 \leq i < j < k \leq 5} \left(A^i dA^j dA^k + \sum_{l=i,j,k} A^i A^j A^k dA^l \right), \\
 \left. \sum_{1 \leq i < j < k < l \leq 5} \left(A^i A^j A^k dA^l + A^k A^j A^l dA^i + A^i A^k A^l dA^j \right), A^1 A^2 A^3 A^4 A^5 \right\rangle, \quad (3.24)
 \end{aligned}$$

which is classified by

$$\begin{aligned}
 H^5 \left(\prod_{i=1}^5 \mathbb{Z}_{N_i}, \mathbb{R}/\mathbb{Z} \right) = \prod_{i=1}^5 \mathbb{Z}_{N_i} \times \prod_{1 \leq i < j \leq 5} \left(\mathbb{Z}_{N_{ij}} \right)^2 \times \prod_{1 \leq i < j < k \leq 5} \left(\mathbb{Z}_{N_{ijk}} \right)^4 \\
 \times \prod_{1 \leq i < j < k < l \leq 5} \left(\mathbb{Z}_{N_{ijkl}} \right)^3 \times \mathbb{Z}_{N_{12345}}. \quad (3.25)
 \end{aligned}$$

Once *AAC* terms are considered or for other choices of α , the action's classification is totally beyond that obtained from group cohomology, which is an important feature of TQFT in 5D.

4 Conclusions and outlook

In summary, we study 5D topological orders from the field-theoretical aspect. In 5D topological orders, topological excitations include particles, loops, and membranes, whose braiding processes are complicated and not been fully understood yet. With the help of TQFT, we write down topological terms, including *BF* terms and twisted terms, in 5D. More concretely, there are *two* types of *BF* terms in 5D, unlike the case in 3D or 4D. Such two types of *BF* terms are studied in details. By combining *BF* terms and twisted terms, we write down TQFT actions that are invariant under gauge transformations. For each TQFT action, we construct the gauge-invariant Wilson operator whose expectation value can be expressed as intersection patterns of geometric objects in 5D, as detailed in section 2.2, 2.3, and 2.4. These results are obtained from field theory in a gauge-invariant fashion, and correspond to link invariants of links formed by closed spacetime trajectories of topological excitations in 5D. In addition, the observable phase of braiding process is given by the Wilson operator. In section 3, we study classifications of TQFT action consisting of *BF* terms and twisted terms. Depending on the type of *BF* term, these TQFT actions are dubbed type-I or mixed *BF* theories. We find that, only some of type-I *BF* theories is classified by group cohomology, i.e., consistent with Dijkgraaf-Witten model. For other type-I *BF* theories and all mixed *BF* theories, their classifications are beyond Dijkgraaf-Witten cohomological classification. Table 1 summarizes classification of *BF* theories for $G = \mathbb{Z}_{N_1}$, $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$, and $G = \prod_{i=1}^3 \mathbb{Z}_{N_i}$. Some interesting questions still remain open, which are left for future study:

1. Gauge transformations for TQFT actions with twisted terms usually contain shift terms. How to understand these shift terms from mathematical perspective (e.g., fibre bundle)? We hope future work could give more insight to this problem.
2. It should be noticed that the classification of topological orders is not identical to that of TQFTs. For example, consider a 3 + 1D all-boson topological order (all particle excitations are bosons) with gauge group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, we can write down a TQFT action for it: $S \sim \int \sum_{i=1}^2 \frac{N_i}{2\pi} B^i dA^i + q_1 A^1 A^2 dA^1 + q_2 A^1 A^2 dA^2$. This gauge theory is classified by $H^4(G, \mathbb{R}/\mathbb{Z}) = \mathbb{Z}_2 \times \mathbb{Z}_2$. However, this topological order is actually classified by $H_{\text{Aut}}^4(G) = H^4(G, \mathbb{R}/\mathbb{Z}) / \text{Aut} = \mathbb{Z}_2$, where ‘‘Aut’’ stands for the automorphism of G . This can be understood by the redundancy of relabelling the two \mathbb{Z}_2 fluxes. For all TQFTs within cohomology classification [21], we expect that group automorphism can be used to find the classification of topological orders. For other TQFTs, more careful considerations are needed for the classification of corresponding topological orders. Physically, in order to find out classification of 5D topological orders, one should consider the inequivalent data sets formed by physical observables. In the future, it would be important to thoroughly study observables in 5D topological orders and find out the complete classification.
3. The world we live in is a 4D spacetime. Nevertheless, the 4D anomalous theory cannot exist alone unless it appears as the boundary theory of a 5D topologically ordered state, which is the phenomenon of gravitational anomaly [13, 14, 37]. Since 5D $\prod_i^n \mathbb{Z}_{N_i}$ gauge theories are investigated in this paper, it would be interesting to study the relation between 5D theories and 4D anomalous theories. For the purpose, we need to consider TQFTs on a manifold with boundary.
4. In this paper, linking number in 5D is obtained via a field-theoretical approach, whose topological invariance is guaranteed by gauge invariance of Wilson operator. Our work may shed light on the study of links and knots in higher dimensions, which still call for joint efforts from physicists and mathematicians.
5. By adding global symmetry in 5D topological orders, we can study symmetry fractionalization on membranes, based on the field-theoretical framework in ref. [36] where symmetry fractionalization on loops is characterized and classified. Putting the theory on a manifold with boundary may lead to anomalous symmetry fractionalization patterns, which may answer the question asked at the end of ref. [37].
6. By noting that K -matrix Chern-Simons theory (i.e., $\frac{K^{IJ}}{4\pi} \int A^I dA^J$ with an symmetric integer matrix K) can be used to describe all Abelian topological orders in 3D spacetime, It is interesting to consider K -matrix BdB theory with the action $S = \frac{K^{IJ}}{4\pi} B^I dB^J$ where K is an antisymmetric integer matrix. The above BF term $Bd\tilde{B}$ considered in this paper is just the off-diagonal term in this theory. Then, the boundary theory and canonical quantization of this action can be studied systematically.

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A Quantization and periodicity of twisted terms

In this appendix, we will derive the quantized and periodic coefficients for specific twisted terms as examples. The basic guiding principle is that the TQFT action should satisfy *large gauge invariance* and *flux identification* [39]. These derivations can be easily generalized to other twisted terms.

A.1 Twisted terms from type-I BF terms

Example: AAC term studied in section 2.3.

Consider $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \mathbb{Z}_{N_3}$. The TQFT action is

$$S = S_{BF} + S_{AAC} = \int \sum_{i=1}^3 \frac{N_i}{2\pi} C^i dA^i + q A^1 A^2 C^3. \quad (\text{A.1})$$

After integrating out the Lagrange multipliers C^1, C^2 and A^3 , the action S reduces to $S_{AAC} = \int q A^1 A^2 C^3$, where the fields A^1, A^2 and C^3 are set to be closed with $\oint A^1 \in \frac{2\pi}{N_1} \mathbb{Z}_{N_1}$, $\oint A^2 \in \frac{2\pi}{N_2} \mathbb{Z}_{N_2}$ and $\oint C^3 \in \frac{2\pi}{N_3} \mathbb{Z}_{N_3}$. Consider putting this action on the spacetime manifold $\mathcal{M} = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^3$. Under the gauge transformations $A^{1,2} \rightarrow A^{1,2} + d\chi^{1,2}$ and $C^3 \rightarrow C^3 + dT^3$, S_{AAC} changes as

$$\begin{aligned} S_{AAC} &\rightarrow S'_{AAC} = \int_{\mathcal{M}} q \left(A^1 + d\chi^1 \right) \left(A^2 + d\chi^2 \right) \left(C^3 + dT^3 \right) \\ &= S_{AAC} + \int_{\mathcal{M}} q \left(A^1 d\chi^2 C^3 + d\chi^1 A^2 C^3 + A^1 A^2 dT^3 \right) \\ &\quad + \int_{\mathcal{M}} q \left(A^1 d\chi^1 dT^3 + d\chi^1 A^2 dT^3 \right) + \int_{\mathcal{M}} q d\chi^1 d\chi^2 dT^3 \\ &\equiv S_{AAC} + \Delta S_{AAC}^{(1)} + \Delta S_{AAC}^{(2)} + \Delta S_{AAC}^{(3)}. \end{aligned} \quad (\text{A.2})$$

The *large gauge invariance* requires that $\Delta S_{AAC} = \Delta S_{AAC}^{(1)} + \Delta S_{AAC}^{(2)} + \Delta S_{AAC}^{(3)}$ should be 0 mod 2π . Suppose that

$$\oint_{\mathbb{S}^1} A^1 = \frac{2\pi n_1}{N_1}, \quad \oint_{\mathbb{S}^1} d\chi^1 = 2\pi m_1, \quad (\text{A.3})$$

$$\oint_{\mathbb{S}^1} A^2 = \frac{2\pi n_2}{N_2}, \quad \oint_{\mathbb{S}^1} d\chi^2 = 2\pi m_2, \quad (\text{A.4})$$

$$\oint_{\mathbb{S}^3} C^3 = \frac{2\pi n_3}{N_3}, \quad \oint_{\mathbb{S}^3} dT^3 = 2\pi m_3, \quad (\text{A.5})$$

where n_i and m_i ($i = 1, 2, 3$) are integers. Then

$$\int_{\mathcal{M}} A^1 d\chi^2 C^3 = \oint_{S^1} A^1 \oint_{S^1} d\chi^2 \oint_{S^3} C^3 = \frac{(2\pi)^3 n_1 m_2 n_3}{N_1 N_3}, \quad (\text{A.6})$$

$$\int_{\mathcal{M}} d\chi^1 A^2 C^3 = \oint_{S^1} d\chi^1 \oint_{S^1} A^2 \oint_{S^3} C^3 = \frac{(2\pi)^3 m_1 n_2 n_3}{N_2 N_3}, \quad (\text{A.7})$$

$$\int_{\mathcal{M}} A^1 A^2 dT^3 = \oint_{S^1} A^1 \oint_{S^1} A^2 \oint_{S^3} dT^3 = \frac{(2\pi)^3 n_1 n_2 m_3}{N_1 N_2}, \quad (\text{A.8})$$

$$\int_{\mathcal{M}} A^1 d\chi^1 dT^3 = \oint_{S^1} A^1 \oint_{S^1} d\chi^1 \oint_{S^3} dT^3 = \frac{(2\pi)^3 n_1 m_2 m_3}{N_1}, \quad (\text{A.9})$$

$$\int_{\mathcal{M}} d\chi^1 A^2 dT^3 = \oint_{S^1} d\chi^1 \oint_{S^2} A^2 \oint_{S^3} dT^3 = \frac{(2\pi)^3 m_1 n_2 m_3}{N_2}, \quad (\text{A.10})$$

$$\int_{\mathcal{M}} d\chi^1 d\chi^2 dT^3 = \oint_{S^1} d\chi^1 \oint_{S^1} d\chi^2 \oint_{S^3} dT^3 = (2\pi)^3 m_1 m_2 m_3. \quad (\text{A.11})$$

The large gauge invariance of S_{AAC} is guaranteed by the following constraints

$$\Delta S_{AAC}^{(1)} = (2\pi)^3 \cdot \left(\frac{qn_1 m_2 n_3}{N_1 N_3} + \frac{qm_1 n_2 n_3}{N_2 N_3} + \frac{qn_1 n_2 m_3}{N_1 N_2} \right) = 0 \pmod{2\pi}, \quad (\text{A.12})$$

$$\Delta S_{AAC}^{(2)} = (2\pi)^3 \cdot \left(\frac{qn_1 m_2 m_3}{N_1} + \frac{qm_1 n_2 m_3}{N_2} \right) = 0 \pmod{2\pi}, \quad (\text{A.13})$$

$$\Delta S_{AAC}^{(3)} = (2\pi)^3 \cdot qm_1 m_2 m_3 = 0 \pmod{2\pi}. \quad (\text{A.14})$$

For arbitrary value of integers n_i and m_i , the above constraints are satisfied by quantizing the coefficient q :

$$q = \frac{pN_1 N_2 N_3}{(2\pi)^2 N_{123}}, p \in \mathbb{Z}, \quad (\text{A.15})$$

where N_{123} is the greatest common divisor of N_1 , N_2 and N_3 .

So far, we have determined the quantization of q . Next, we find out the periodicity of q by *flux identification*. When the \mathbb{Z}_{N_i} flux is inserted as n_i multiple units of $\frac{2\pi}{N_i}$, we have

$$\int q A^1 A^2 C^3 = q \oint_{S^1} A^1 \oint_{S^1} A^2 \oint_{S^3} C^3 = \frac{pN_1 N_2 N_3}{(2\pi)^2 N_{123}} \frac{2\pi n_1}{N_1} \frac{2\pi n_2}{N_2} \frac{2\pi n_3}{N_3} = \frac{2\pi p n_1 n_2 n_3}{N_{123}}. \quad (\text{A.16})$$

For arbitrary value of $n_1 n_2 n_3$, $\exp(i \int q A^1 A^2 C^3)$ is invariant when $\frac{2\pi p}{N_{123}}$ shifts by 2π , i.e., $p \rightarrow p + N_{123}$, which implies that p should be identified with $p + N_{123}$.

Combined with $p \in \mathbb{Z}$, we conclude that the quantization and periodic condition on the coefficient q is:

$$q = \frac{pN_1 N_2 N_3}{(2\pi)^2 N_{123}}, p \in \mathbb{Z}_{N_{123}}. \quad (\text{A.17})$$

In the above example, we have shown how to get quantization of the coefficient of the AAC twisted term based on large gauge invariance and flux identification. The similar derivation can be applied on other twisted terms.

Example: AdAdA term $(A^i dA^i dA^i, A^i dA^i dA^j, A^i dA^j dA^k)$ studied in section 2.3.

Consider $G = \mathbb{Z}_{N_1}$. The TQFT action is

$$S = S_{BF} + S_{AdAdA} = \int \frac{N_1}{2\pi} C^1 dA^1 + q A^1 dA^1 dA^1 \quad (\text{A.18})$$

and the gauge transformation is

$$\begin{aligned} A^1 &\rightarrow A^1 + d\chi^1, \\ C^1 &\rightarrow C^1 + dT^1. \end{aligned} \quad (\text{A.19})$$

After integrating out the Lagrange multipliers C^1 , the action S reduces to $S_{AdAdA} = \int q A^1 dA^1 dA^1$, where the field A^1 is set to be closed with $\oint A^1 \in \frac{2\pi}{N_1} \mathbb{Z}_{N_1}$. After the gauge transformations,

$$S_{AdAdA} \rightarrow S_{AdAdA} + \int q d\chi^1 dA^1 dA^1. \quad (\text{A.20})$$

The large gauge invariance requires that

$$\Delta S_{AdAdA} = \int q d\chi^1 dA^1 dA^1 = 0 \pmod{2\pi}. \quad (\text{A.21})$$

Consider putting this TQFT on the spacetime manifold $\mathbb{S}^1 \times \mathbb{S}^2 \times \mathbb{S}^2$ and assuming

$$\oint_{\mathbb{S}^1} d\chi^1 = 2\pi m_1, \quad \oint_{\mathbb{S}^2} dA^1 = 2\pi n_{a1}, \quad (\text{A.22})$$

we have

$$\Delta S_{AdAdA} = \int q d\chi^1 dA^1 dA^1 = q \oint_{\mathbb{S}^1} d\chi^1 \oint_{\mathbb{S}^2} dA^1 \oint_{\mathbb{S}^2} dA^1 = (2\pi)^3 q m_1 n_{a1} n_{a1} = 0 \pmod{2\pi}, \quad (\text{A.23})$$

which indicates

$$q = \frac{p}{(2\pi)^2}, p \in \mathbb{Z}. \quad (\text{A.24})$$

Now we consider the periodicity of q . Consider the \mathbb{Z}_{N_1} flux is inserted as n_1 multiple units of $\frac{2\pi}{N_1}$, we have

$$\int q A^1 dA^1 dA^1 = \frac{p}{(2\pi)^2} \oint_{\mathbb{S}^1} A^1 \oint_{\mathbb{S}^2} dA^1 \oint_{\mathbb{S}^2} dA^1 = \frac{2\pi p n_1 n_a n_a}{N_1}. \quad (\text{A.25})$$

For arbitrary values of n_1 and n_a , the partition function should be invariant under a shift by 2π , which means $\exp(i \int q A^1 dA^1 dA^1) \simeq \exp(i \int q A^1 dA^1 dA^1 + 2\pi i)$, thus we have

$$\frac{2\pi p n_1 n_a n_a}{N_1} \simeq \frac{2\pi p n_1 n_a n_a}{N_1} + 2\pi \Rightarrow p \simeq p + N_1. \quad (\text{A.26})$$

In conclusion, the coefficient q of $A^1 dA^1 dA^1$ twisted term is

$$q = \frac{p}{(2\pi)^2}, p \in \mathbb{Z}_{N_1}. \quad (\text{A.27})$$

Consider $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$. Possible twisted terms are $A^1 dA^2 dA^1$, $A^1 dA^1 dA^2$, $A^1 dA^2 dA^2$, $A^2 dA^1 dA^2$, $A^2 dA^2 dA^1$ and $A^2 dA^1 dA^1$. Notice that

$$A^1 dA^2 dA^1 = A^1 dA^1 dA^2, \quad (\text{A.28})$$

$$A^2 dA^1 dA^2 = A^2 dA^2 dA^1, \quad (\text{A.29})$$

and

$$d(A^1 A^2 dA^1) = dA^1 A^2 dA^1 - A^1 dA^2 dA^1, \quad (\text{A.30})$$

$$d(A^1 A^2 dA^2) = dA^1 A^2 dA^2 - A^1 dA^2 dA^2. \quad (\text{A.31})$$

The linearly independent twisted terms are $A^1 dA^2 dA^1$ and $A^2 dA^1 dA^2$. Take $A^1 dA^2 dA^1$ as an example:

$$S_{A^1 dA^2 dA^1} = \int \sum_{i=1}^2 \frac{N_i}{2\pi} C^i dA^i + q A^1 dA^2 dA^1. \quad (\text{A.32})$$

Similarly, integrating out Lagrange multipliers C^1 and C^2 implies constraints that A^1 and A^2 are closed with $\oint A^1 \in \frac{2\pi}{N_1} \mathbb{Z}_{N_1}$ and $\oint A^2 \in \frac{2\pi}{N_2} \mathbb{Z}_{N_2}$.

The large gauge invariance,

$$\Delta S_{A^1 dA^2 dA^1} = \int q d\chi^1 dA^2 dA^1 = q (2\pi)^3 m_1 n_{a2} n_{a1} = 0 \pmod{2\pi}, \quad (\text{A.33})$$

requires that

$$q = \frac{p}{(2\pi)^2}, p \in \mathbb{Z}. \quad (\text{A.34})$$

According to flux identification, $\int q A^1 dA^2 dA^1 = \frac{2\pi p n_1 n_{a2} n_{a1}}{N_1}$ indicates that

$$\frac{2\pi p}{N_1} \simeq \frac{2\pi p}{N_1} + 2\pi \Rightarrow p \simeq p + N_1 \Rightarrow p \in \mathbb{Z}_{N_1}. \quad (\text{A.35})$$

Notice that

$$d(A^1 A^2 dA^1) = dA^1 A^2 dA^1 - A^1 dA^2 dA^1, \quad (\text{A.36})$$

we can view $A^1 dA^2 dA^1$ and $A^2 dA^1 dA^1$ as the same topological term up to a boundary term. Thus

$$\begin{aligned} \exp\left(i \int q A^1 dA^2 dA^1\right) &= \exp\left[i \int q dA^1 A^2 dA^1 - i \int q d(A^1 A^2 dA^1)\right] \\ &= \exp\left(\frac{i 2\pi p n_{a1} n_2 n_{a1}}{N_2}\right), \end{aligned} \quad (\text{A.37})$$

which tells us that

$$p \simeq p + N_2 \Rightarrow p \in \mathbb{Z}_{N_2}. \quad (\text{A.38})$$

Together with $p \in \mathbb{Z}_{N_1}$, the coefficient q of $A^1 dA^2 dA^1$ twisted term is

$$q = \frac{p}{(2\pi)^2}, p \in \mathbb{Z}_{N_{12}}. \quad (\text{A.39})$$

Consider $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \mathbb{Z}_{N_3}$. The $AdAdA$ term can be $A^1 dA^2 dA^3$, $A^2 dA^3 dA^1$, or $A^3 dA^1 dA^2$. But we will see that these 3 twisted terms are not linearly independent. Consider the TQFT action

$$S = \int \sum_{i=1}^3 \frac{N_i}{2\pi} C^i dA^i + q A^1 dA^2 dA^3 \quad (\text{A.40})$$

with the gauge transformations

$$\begin{aligned} A^i &\rightarrow A^i + d\chi^i, \\ C^i &\rightarrow C^i + dT^i. \end{aligned} \tag{A.41}$$

Integrating out C^i reduces the action S to $\int qA^1dA^2dA^3$ where the fields A^i are enforced to be closed with $\oint A^i \in \frac{2\pi}{N_i}\mathbb{Z}_{N_i}$.

The quantization of coefficient q can be derived in the same manner:

$$q = \frac{p}{(2\pi)^2}, p \in \mathbb{Z}_{N_1}. \tag{A.42}$$

On the other hand, notice that

$$d(A^1A^2dA^3) = A^2dA^3dA^1 - A^1dA^2dA^3, \tag{A.43}$$

$$d(A^1dA^2A^3) = A^3dA^1dA^2 - A^1dA^2dA^3; \tag{A.44}$$

up to boundary terms we have

$$\begin{aligned} \exp\left(i \int qA^1dA^2dA^3\right) &= \exp\left[i \int qA^2dA^3dA^1 - i \int qd(A^1A^2dA^3)\right] \\ &= \exp\left(\frac{2\pi ipn_2n_{a3}n_{a1}}{N_2}\right) \end{aligned} \tag{A.45}$$

and

$$\begin{aligned} \exp\left(i \int qA^1dA^2dA^3\right) &= \exp\left[i \int qA^3dA^1dA^2 - i \int qd(A^1dA^2A^3)\right] \\ &= \exp\left(\frac{2\pi ipn_3n_{a1}n_{a2}}{N_3}\right). \end{aligned} \tag{A.46}$$

The flux identification of eq. (A.45) and eq. (A.46) respectively leads to

$$p \simeq p + N_2, \tag{A.47}$$

$$p \simeq p + N_3. \tag{A.48}$$

Together with $p \simeq p + N_1$, we can see that the period of p is N_{123} , i.e.,

$$p \simeq p + N_{123}. \tag{A.49}$$

Therefore, the coefficient q of topological term $A^1dA^2dA^3$ is

$$q = \frac{p}{(2\pi)^2}, p \in \mathbb{Z}_{N_{123}}. \tag{A.50}$$

Example: AAAAdA term ($A^iA^jA^kdA^i, A^iA^jA^kdA^i$) studied in section 2.3.

Consider $G = \prod_{i=1}^4 \mathbb{Z}_{N_i}$. Possible twisted terms: $A^2A^3A^4dA^1, A^3A^4A^1dA^2, A^4A^1A^2dA^3$ and $A^1A^2A^3dA^4$. We should notice that these 4 twisted terms are not linearly independent.⁸ Take $A^1A^2A^3dA^4$ as an example, the TQFT action is

$$S = S_{BF} + S_{AAAAdA} = \int \sum_{i=1}^4 \frac{N_i}{2\pi} C^i dA^i + qA^1A^2A^3dA^4 \tag{A.51}$$

⁸ $d(A^1A^2A^3A^4) = dA^1A^2A^3A^4 - A^1dA^2A^3A^4 + A^1A^2dA^3A^4 - A^1A^2A^3dA^4.$

and the gauge transformations are

$$\begin{aligned}
 A^i &\rightarrow A^i + d\chi^i, \\
 C^1 &\rightarrow C^1 + dT^1 + \frac{2\pi q}{N_1} \left(A^2 \chi^3 dA^4 - A^3 \chi^2 dA^4 - \frac{1}{2} \chi^2 d\chi^3 dA^4 + \frac{1}{2} \chi^3 d\chi^2 dA^4 \right), \\
 C^2 &\rightarrow C^2 + dT^2 + \frac{2\pi q}{N_2} \left(-A^1 \chi^3 dA^4 + A^3 \chi^1 dA^4 + \frac{1}{2} \chi^1 d\chi^3 dA^4 - \frac{1}{2} \chi^3 d\chi^1 dA^4 \right), \\
 C^3 &\rightarrow C^3 + dT^3 + \frac{2\pi q}{N_3} \left(A^1 \chi^2 dA^4 - A^2 \chi^1 dA^4 - \frac{1}{2} \chi^1 d\chi^2 dA^4 + \frac{1}{2} \chi^2 d\chi^1 dA^4 \right), \\
 C^4 &\rightarrow C^4 + dT^4.
 \end{aligned} \tag{A.52}$$

Once we integrate out the Lagrange multipliers C^i , we obtain constraints on A^i : $dA^i = 0$ and $\oint A^i \in \frac{2\pi}{N_i} \mathbb{Z}_{N_i}$. Now we determine the coefficient q . First we find out the quantization of q . Under the gauge transformations $A^i \rightarrow A^i + d\chi^i$ ($i = 1, 2, 3, 4$),

$$S_{AAAdA} \rightarrow S_{AAAdA} + \Delta S_{AAAdA}^{(1)} + \Delta S_{AAAdA}^{(2)} + \Delta S_{AAAdA}^{(3)}, \tag{A.53}$$

where

$$\Delta S_{AAAdA}^{(1)} = \int q \left(d\chi^1 A^2 A^3 dA^4 + A^1 d\chi^2 A^3 dA^4 + A^1 A^2 d\chi^3 dA^4 \right), \tag{A.54}$$

$$\Delta S_{AAAdA}^{(2)} = \int q \left(d\chi^1 d\chi^2 A^3 dA^4 + d\chi^1 A^2 d\chi^3 dA^4 + A^1 A^2 d\chi^3 dA^4 \right), \tag{A.55}$$

$$\Delta S_{AAAdA}^{(3)} = \int q d\chi^1 d\chi^2 d\chi^3 dA^4. \tag{A.56}$$

The large gauge invariance requires that $\Delta S_{AAAdA} = 0 \pmod{2\pi}$, i.e.,

$$\Delta S_{AAAdA}^{(1)} = \frac{(2\pi)^4 q m_1 n_2 n_3 n_{a4}}{N_2 N_3} + \frac{(2\pi)^4 q n_1 m_2 n_3 n_{a4}}{N_1 N_3} + \frac{(2\pi)^4 q n_1 n_2 m_3 n_{a4}}{N_1 N_2} = 0 \pmod{2\pi}, \tag{A.57}$$

$$\Delta S_{AAAdA}^{(2)} = \frac{(2\pi)^4 q m_1 m_2 n_3 n_{a4}}{N_3} + \frac{(2\pi)^4 q m_1 n_2 m_3 n_{a4}}{N_2} + \frac{(2\pi)^4 q n_1 m_2 m_3 n_{a4}}{N_1} = 0 \pmod{2\pi}, \tag{A.58}$$

$$\Delta S_{AAAdA}^{(3)} = (2\pi)^4 q m_1 m_2 m_3 n_{a4}. \tag{A.59}$$

These constraints lead to

$$q = \frac{p N_1 N_2 N_3}{(2\pi)^3 N_{123}}, p \in \mathbb{Z}. \tag{A.60}$$

Next we find out the period of q . We have

$$\int q A^1 A^2 A^3 dA^4 = \frac{p N_1 N_2 N_3}{(2\pi)^3 N_{123}} \cdot \frac{2\pi n_1}{N_1} \cdot \frac{2\pi n_2}{N_2} \cdot \frac{2\pi n_3}{N_3} \cdot 2\pi n_{a4} = \frac{2\pi p n_1 n_2 n_3 n_{a4}}{N_{123}}. \tag{A.61}$$

The flux identification tells us that no matter what values n_i and n_{a4} are, the partition function is invariant under a shift by 2π , which means

$$\exp \left(i \int q A^1 A^2 A^3 dA^4 \right) = \exp \left[\frac{i 2\pi p n_1 n_2 n_3 n_{a4}}{N_{123}} \right] \simeq \exp \left[\frac{i 2\pi (p+1) n_1 n_2 n_3 n_{a4}}{N_{123}} \right]. \tag{A.62}$$

leading to

$$p \simeq p + N_{123}. \tag{A.63}$$

However, there is another constraint on the period of p . Notice that

$$d(A^1 A^2 A^3 A^4) = dA^1 A^2 A^3 A^4 - A^1 dA^2 A^3 A^4 + A^1 A^2 dA^3 A^4 - A^1 A^2 A^3 dA^4, \tag{A.64}$$

up boundary terms we have

$$\begin{aligned} & \exp \left[i \int q \left(A^2 A^3 A^4 dA^1 - A^3 A^4 A^1 dA^2 + A^4 A^1 A^2 dA^3 \right) - i \int q d \left(A^1 A^2 A^3 A^4 \right) \right] \\ &= \exp \left[\frac{i2\pi p N_1 n_2 n_3 n_4 n_{a1}}{N_4 N_{123}} - \frac{i2\pi p N_2 n_3 n_4 n_1 n_{a2}}{N_4 N_{123}} + \frac{i2\pi p N_3 n_4 n_1 n_2 n_{a3}}{N_4 N_{123}} \right] \\ &= \exp \left[\frac{i2\pi p n_4}{N_4} \cdot \left(\frac{N_1 n_2 n_3 n_{a1} - N_2 n_3 n_1 n_{a2} + N_3 n_1 n_2 n_{a3}}{N_{123}} \right) \right] \\ &\simeq \exp \left[\frac{i2\pi (p+1) n_4}{N_4} \cdot \left(\frac{N_1 n_2 n_3 n_{a1} - N_2 n_3 n_1 n_{a2} + N_3 n_1 n_2 n_{a3}}{N_{123}} \right) \right], \end{aligned} \tag{A.65}$$

which means that

$$p \simeq p + N_4. \tag{A.66}$$

Together with $p \simeq p + N_{123}$, the period of p is given by $\text{gcd}(N_{123}, N_4) = N_{1234}$. In conclusion, the coefficient q of $A^1 A^2 A^3 dA^4$ is

$$q = \frac{p N_1 N_2 N_3}{(2\pi)^3 N_{123}}, p \in \mathbb{Z}_{N_{1234}}. \tag{A.67}$$

Example: AAAAA term studied in section 2.3.

This twisted term is possible when there are at least 5 \mathbb{Z}_{N_i} gauge subgroups, e.g., $\prod_{i=1}^5 \mathbb{Z}_{N_i}$, $A^1 A^2 A^3 A^4 A^5$. The action is

$$S = S_{BF} + S_{AAAAA} = \int \sum_{i=1}^5 \frac{N_i}{2\pi} C^i dA^i + q A^1 A^2 A^3 A^4 A^5. \tag{A.68}$$

The large gauge invariance and flux identification conditions indicate the quantization and periodicity of q in a similar manner:

$$q = \frac{p N_1 N_2 N_3 N_4 N_5}{(2\pi)^4 N_{12345}}, p \in \mathbb{Z}_{N_{12345}}. \tag{A.69}$$

A.2 Twisted terms from mixture of type-I and type-II BF terms

Examples: BAdA term and AAdB term studied in section 2.4.

Consider $G = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \mathbb{Z}_{N_3}$. Notice that $d(B^3 A^1 A^2) = dB^3 A^1 A^2 + B^3 dA^1 A^2 - B^3 A^1 dA^2$, thus only two of them are linearly independent. Take the $B^3 A^1 dA^2$ term as an example, the TQFT action is

$$S = \int \sum_{i=1}^2 \frac{N_i}{2\pi} C^i dA^i + \frac{N_3}{2\pi} \tilde{B}^3 dB^3 + q B^3 A^1 dA^2 \tag{A.70}$$

which is gauge-invariant under

$$\begin{aligned}
 A^1 &\rightarrow A^1 + d\chi^1, & C^1 &\rightarrow C^1 + dT^1 - \frac{2\pi q}{N_1} dV^3 A^2 \\
 A^2 &\rightarrow A^2 + d\chi^2, & C^2 &\rightarrow C^2 + dT^2 \\
 B^3 &\rightarrow B^3 + dV^3, & \tilde{B}^3 &\rightarrow \tilde{B}^3 + d\tilde{V}^3 - \frac{2\pi q}{N_3} d\chi^1 A^2.
 \end{aligned}
 \tag{A.71}$$

After integrating out C^1 , C^2 and \tilde{B}^3 , the action reduces to $\int q B^3 A^1 dA^2$ where the fields are set to be closed with $\oint A^1 \in \frac{2\pi}{N_1} \mathbb{Z}_{N_1}$, $\oint A^2 \in \frac{2\pi}{N_2} \mathbb{Z}_{N_2}$ and $\oint B^3 \in \frac{2\pi}{N_3} \mathbb{Z}_{N_3}$. In a similar manner, it can be derived that the coefficient q of $B^3 A^1 dA^2$ is

$$q = \frac{p N_3 N_1}{(2\pi)^2 N_{13}}, p \in \mathbb{Z}_{N_{13}}.
 \tag{A.72}$$

However, there is another constraint on the coefficient q . Noticed that

$$d(B^3 A^1 A^2) = dB^3 A^1 A^2 + B^3 dA^1 A^2 - B^3 A^1 dA^2,
 \tag{A.73}$$

we have

$$\begin{aligned}
 \exp\left(i \int q B^3 A^1 dA^2\right) &= \exp\left[i \int q A^1 A^2 dB^2 + i \int q B^3 A^2 dA^1 - i \int q d(B^3 A^1 A^2)\right] \\
 &= \exp\left[\frac{2\pi i p N_3 N_1 n_1 n_2}{N_{13} \cdot N_1 N_2} + \frac{2\pi i p N_3 N_1 n_3 n_2}{N_{13} \cdot N_3 N_2}\right] \\
 &= \exp\left[\frac{2\pi i p n_2}{N_2} \frac{n_1 N_3 + n_3 N_1}{N_{13}}\right].
 \end{aligned}
 \tag{A.74}$$

Since $\frac{n_1 N_3 + n_3 N_1}{N_{13}}$ is an integer, $n_2 \cdot \frac{n_1 N_3 + n_3 N_1}{N_{13}}$ is also an integer. In order to keep \mathcal{Z} invariant for arbitrary n_1 , n_2 and n_3 , $\frac{2\pi p}{N_2}$ is required to be identical to $\frac{2\pi p}{N_2} + 2\pi$, i.e.,

$$\frac{2\pi p}{N_2} \simeq \frac{2\pi p}{N_2} + 2\pi \Rightarrow p \simeq p + N_2.
 \tag{A.75}$$

Together with $p \in \mathbb{Z}_{N_{13}}$, we can conclude that the actual period of p is

$$\gcd(N_{13}, N_2) = N_{123}.
 \tag{A.76}$$

Thus the quantization and period of the coefficient q of $B^3 A^1 dA^2$ are actually given by

$$q = \frac{p N_3 N_1}{(2\pi)^2 N_{13}}, p \in \mathbb{Z}_{N_{123}}
 \tag{A.77}$$

If we consider the $B^3 A^2 dA^1$ term and a TQFT action

$$S_1 = \int \sum_{i=1}^2 \frac{N_i}{2\pi} C^i dA^i + \frac{N_3}{2\pi} \tilde{B}^3 dB^3 + q_1 B^3 A^2 dA^1,
 \tag{A.78}$$

the gauge transformations are

$$\begin{aligned}
 A^1 &\rightarrow A^1 + d\chi^1, \\
 A^2 &\rightarrow A^2 + d\chi^2, \\
 B^3 &\rightarrow B^3 + dV^3, \\
 \tilde{B}^3 &\rightarrow \tilde{B}^3 + d\tilde{V}^3 - \frac{2\pi q_1}{N_3} d\chi^2 A^1, \\
 C^1 &\rightarrow C^1 + dT^1, \\
 C^2 &\rightarrow C^2 + dT^2 - \frac{2\pi q_1}{N_2} dV^3 A^1;
 \end{aligned}
 \tag{A.79}$$

the coefficient q_1 is determined in a similar manner:

$$q_1 = \frac{p_1 N_3 N_2}{(2\pi)^2 N_{23}}, p_1 \in \mathbb{Z}_{N_{123}}.
 \tag{A.80}$$

If we consider the $A^1 A^2 dB^3$ term and a TQFT action

$$S_2 = \int \sum_{i=1}^2 \frac{N_i}{2\pi} C^i dA^i + \frac{N_3}{2\pi} \tilde{B}^3 dB^3 + q_2 A^1 A^2 dB^3,
 \tag{A.81}$$

the gauge transformations are

$$\begin{aligned}
 A^1 &\rightarrow A^1 + d\chi^1, \\
 A^2 &\rightarrow A^2 + d\chi^2, \\
 B^3 &\rightarrow B^3 + dV^3, \\
 \tilde{B}^3 &\rightarrow \tilde{B}^3 + d\tilde{V}^3, \\
 C^1 &\rightarrow C^1 + dT^1 + \frac{2\pi q_2}{N_1} d\chi^2 B^3, \\
 C^2 &\rightarrow C^2 + dT^2 - \frac{2\pi q_2}{N_2} d\chi^1 B^3;
 \end{aligned}
 \tag{A.82}$$

the coefficient q_2 is

$$q_2 = \frac{p_2 N_1 N_2}{(2\pi)^2 N_{12}}, p_2 \in \mathbb{Z}_{N_{123}}.
 \tag{A.83}$$

B Gauge-invariance of $S = S_{BF} + S_{AAC}$ and its Wilson operator

In this appendix, we derive the gauge transformations for the action (2.9), studied in section 2.3,

$$S = S_{BF} + S_{AAC} = \int \sum_{i=1}^3 \frac{N_i}{2\pi} C^i dA^i + q A^1 A^2 C^3
 \tag{B.1}$$

with $q = \frac{pN_1N_2N_3}{(2\pi)^2N_{123}}$ and $p \in \mathbb{Z}_{N_{123}}$. The basic idea can be applied on other actions studied in this paper. Then we verify that the corresponding Wilson operator

$$\begin{aligned} \mathcal{W} = \exp & \left\{ i \int_{\omega_1} e_1 \left[C^1 + \frac{1}{2} \frac{2\pi q}{N_1} \left(d^{-1} A^2 C^3 - d^{-1} C^3 A^2 \right) \right] \right. \\ & + i \int_{\omega_2} e_2 \left[C^2 + \frac{1}{2} \frac{2\pi q}{N_2} \left(d^{-1} C^3 A^1 - d^{-1} A^1 C^3 \right) \right] \\ & \left. + i \int_{\gamma_3} e_3 \left[A^3 + \frac{1}{2} \frac{2\pi q}{N_3} \left(d^{-1} A^1 A^2 - d^{-1} A^2 A^1 \right) \right] \right\} \end{aligned} \quad (\text{B.2})$$

is indeed gauge-invariant under the gauge transformations.

In this action $S = S_{BF} + S_{AAC} + \int \sum_{i=1}^3 \frac{N_i}{2\pi} C^i dA^i + q A^1 A^2 C^3$, C^1 , C^2 and A^3 serve as Lagrange multipliers which respectively enforce the \mathbb{Z}_{N_1} , \mathbb{Z}_{N_2} and \mathbb{Z}_{N_3} cyclic group structures. These cyclic group structures imply that the gauge transformations of A^1 , A^2 and C^3 are

$$A^1 \rightarrow A^1 + d\chi^1, \quad (\text{B.3})$$

$$A^2 \rightarrow A^2 + d\chi^2, \quad (\text{B.4})$$

$$C^3 \rightarrow C^3 + dT^3, \quad (\text{B.5})$$

where χ^i and T^i are 0-form and 2-form gauge parameters. In order to compensate the variation of S_{AAC} after gauge transformation, the Lagrange multipliers C^1 , C^2 and A^3 transform with extra shift terms:

$$C^1 \rightarrow C^1 + dT^1 + X^1, \quad (\text{B.6})$$

$$C^2 \rightarrow C^2 + dT^2 + X^2, \quad (\text{B.7})$$

$$A^3 \rightarrow A^3 + d\chi^3 + Y^3. \quad (\text{B.8})$$

Our goal is to determine these shift terms X^1 , X^2 and Y^3 . After gauge transformation, the action changes as

$$\begin{aligned} & \int \sum_{i=1}^3 \frac{N_i}{2\pi} C^i dA^i + q A^1 A^2 C^3 \\ \rightarrow & \int \sum_{i=1}^2 \frac{N_i}{2\pi} \left(C^i + dT^i + X^i \right) dA^i + \frac{N_3}{2\pi} \left(A^3 + d\chi^3 + Y^3 \right) dC^3 \\ & + q \left(A^1 + d\chi^1 \right) \left(A^2 + d\chi^2 \right) \left(C^3 + dT^3 \right) \\ = & \int \sum_{i=1}^3 \frac{N_i}{2\pi} C^i dA^i + q A^1 A^2 C^3 + \frac{N_1}{2\pi} dT^1 dA^1 + \frac{N_2}{2\pi} dT^2 dA^2 + \frac{N_3}{2\pi} d\chi^3 dC^3 \\ & + \frac{N_1}{2\pi} X^1 dA^1 + \frac{N_2}{2\pi} X^2 dA^2 + \frac{N_3}{2\pi} Y^3 dC^3 \\ & + q \left(d\chi^1 A^2 C^3 + A^1 d\chi^2 C^3 + d\chi^1 d\chi^2 C^3 A^1 A^2 dT^3 + d\chi^1 A^2 dT^3 + A^1 d\chi^2 dT^3 \right) \\ & + q d\chi^1 d\chi^2 dT^3. \end{aligned} \quad (\text{B.9})$$

Because we only concern the definition of gauge transformations here, we can drop the boundary terms and obtain

$$\begin{aligned}\Delta S &= \int \frac{N_1}{2\pi} X^1 dA^1 + \frac{N_2}{2\pi} X^2 dA^2 + \frac{N_3}{2\pi} Y^3 dC^3 \\ &\quad + q \left(d\chi^1 A^2 C^3 + A^1 d\chi^2 C^3 + d\chi^1 d\chi^2 C^3 \right) \\ &\quad + q \left(A^1 A^2 dT^3 + d\chi^1 A^2 dT^3 + A^1 d\chi^2 dT^3 \right).\end{aligned}\tag{B.10}$$

Gauge-invariance requires $\Delta S = \int$ (total derivatives) thus ΔS can be neglected. Noticed that

$$d\chi^1 A^2 C^3 = d\left(\chi^1 A^2 C^3\right) - \chi^1 C^3 dA^2 + \chi^1 A^2 dC^3,\tag{B.11}$$

$$A^1 d\chi^2 C^3 = -d\left(A^1 \chi^2 C^3\right) + \chi^2 C^3 dA^1 - A^1 \chi^2 dC^3,\tag{B.12}$$

$$d\chi^1 d\chi^2 C^3 = \frac{1}{2} d\left(\chi^1 d\chi^2 C^3\right) - \frac{1}{2} d\left(d\chi^1 \chi^2 C^3\right) + \frac{1}{2} \chi^1 d\chi^2 dC^3 - \frac{1}{2} \chi^2 d\chi^1 dC^3,\tag{B.13}$$

$$A^1 A^2 dT^3 = d\left(A^1 A^2 T^3\right) - A^2 T^3 dA^1 + A^1 T^3 dA^2,\tag{B.14}$$

$$d\chi^1 A^2 dT^3 = d\left(\chi^1 A^2 dT^3\right) - \chi^1 dT^3 dA^2,\tag{B.15}$$

$$A^1 d\chi^2 dT^3 = -d\left(A^1 \chi^2 dT^3\right) + \chi^2 dT^3 dA^1,\tag{B.16}$$

we see that for

$$X^1 = -\frac{2\pi q}{N_1} \left(\chi^2 C^3 - A^2 T^3 + \chi^2 dT^3\right),\tag{B.17}$$

$$X^2 = \frac{2\pi q}{N_2} \left(\chi^1 C^3 - A^1 T^3 + \chi^1 dT^3\right),\tag{B.18}$$

$$Y^3 = -\frac{2\pi q}{N_3} \left[\left(\chi^1 A^2 + \frac{1}{2} \chi^1 d\chi^2\right) - \left(\chi^2 A^1 + \frac{1}{2} \chi^2 d\chi^1\right)\right],\tag{B.19}$$

ΔS can be written as

$$\begin{aligned}\Delta S &= \int q \left[d\left(\chi^1 A^2 C^3\right) - d\left(A^1 \chi^2 C^3\right) + \frac{1}{2} d\left(\chi^1 d\chi^2 C^3\right) - \frac{1}{2} d\left(d\chi^1 \chi^2 C^3\right) \right] \\ &\quad + q \left[d\left(A^1 A^2 T^3\right) + d\left(\chi^1 A^2 dT^3\right) - d\left(A^1 \chi^2 dT^3\right) \right] \\ &= \int \text{(total derivatives)},\end{aligned}\tag{B.20}$$

which ensure the action (B.1) is invariant after gauge transformation. Therefore, we conclude that the gauge transformations for $S = \int \sum_{i=1}^3 \frac{N_i}{2\pi} C^i dA^i + q A^1 A^2 C^3$ are

$$\begin{aligned}A^1 &\rightarrow A^1 + d\chi^1, \quad C^1 \rightarrow C^1 + dT^1 - \frac{2\pi q}{N_1} \left(\chi^2 C^3 - A^2 T^3 + \chi^2 dT^3\right), \\ A^2 &\rightarrow A^2 + d\chi^2, \quad C^2 \rightarrow C^2 + dT^2 + \frac{2\pi q}{N_2} \left(\chi^1 C^3 - A^1 T^3 + \chi^1 dT^3\right), \\ C^3 &\rightarrow C^3 + dT^3, \quad A^3 \rightarrow A^3 + d\chi^3 - \frac{2\pi q}{N_3} \left[\left(\chi^1 A^2 + \frac{1}{2} \chi^1 d\chi^2\right) - \left(\chi^2 A^1 + \frac{1}{2} \chi^2 d\chi^1\right)\right],\end{aligned}\tag{B.21}$$

same as eq. (2.10) in section 2.3.

Next, we verify that the Wilson operator for this action,

$$\begin{aligned} \mathcal{W} = \exp & \left\{ i \int_{\omega_1} e_1 \left[C^1 + \frac{1}{2} \frac{2\pi q}{N_1} \left(d^{-1} A^2 C^3 - d^{-1} C^3 A^2 \right) \right] \right. \\ & + i \int_{\omega_2} e_2 \left[C^2 + \frac{1}{2} \frac{2\pi q}{N_2} \left(d^{-1} C^3 A^1 - d^{-1} A^1 C^3 \right) \right] \\ & \left. + i \int_{\gamma_3} e_3 \left[A^3 + \frac{1}{2} \frac{2\pi q}{N_3} \left(d^{-1} A^1 A^2 - d^{-1} A^2 A^1 \right) \right] \right\}, \end{aligned} \quad (\text{B.22})$$

is invariant under the gauge transformations. This Wilson operator is composed of three similar terms. Below we show that $C^1 + \frac{1}{2} \frac{2\pi q}{N_1} (d^{-1} A^2 C^3 - d^{-1} C^3 A^2)$ is invariant under transformation (B.21); the other components of \mathcal{W} can be proven gauge-invariant in a similar way. After gauge transformation (B.21),

$$\begin{aligned} & \int_{\omega_1} C^1 + \frac{1}{2} \frac{2\pi q}{N_1} \left(d^{-1} A^2 C^3 - d^{-1} C^3 A^2 \right) \\ \rightarrow & \int_{\omega_1} C^1 + dT^1 - \frac{2\pi q}{N_1} \left(\chi^2 C^3 - A^2 T^3 + \chi^2 dT^3 \right) \\ & + \frac{1}{2} \frac{2\pi q}{N_1} \left(d^{-1} A^2 + \chi^2 \right) \left(C^3 + dT^3 \right) - \frac{1}{2} \frac{2\pi q}{N_1} \left(d^{-1} C^3 + T^3 \right) \left(A^2 + d\chi^2 \right) \\ = & \int_{\omega_1} C^1 + dT^1 - \frac{2\pi q}{N_1} \left(\chi^2 C^3 - A^2 T^3 + \chi^2 dT^3 \right) \\ & + \frac{1}{2} \frac{2\pi q}{N_1} \left(d^{-1} A^2 C^3 + \chi^2 C^3 + d^{-1} A^2 dT^3 + \chi^2 dT^3 \right) \\ & - \frac{1}{2} \frac{2\pi q}{N_1} \left(d^{-1} C^3 A^2 + T^3 A^2 + d^{-1} C^3 d\chi^2 + T^3 d\chi^2 \right) \\ = & \int_{\omega_1} C^1 + \frac{1}{2} \frac{2\pi q}{N_1} \left(d^{-1} A^2 C^3 - d^{-1} C^3 A^2 \right) + dT^1 - \frac{1}{2} \frac{2\pi q}{N_1} \chi^2 C^3 + \frac{1}{2} \frac{2\pi q}{N_1} A^2 T^3 \\ & - \frac{1}{2} \frac{2\pi q}{N_1} \chi^2 dT^3 - \frac{1}{2} \frac{2\pi q}{N_1} T^3 d\chi^2 + \frac{1}{2} \frac{2\pi q}{N_1} d^{-1} A^2 dT^3 - \frac{1}{2} \frac{2\pi q}{N_1} d^{-1} C^3 d\chi^2. \end{aligned} \quad (\text{B.23})$$

Because $d(\chi^2 T^3) = d\chi^2 T^3 + \chi^2 dT^3 = T^3 d\chi^2 + \chi^2 dT^3$,

$$\int_{\omega_1} \left(-\frac{1}{2} \frac{2\pi q}{N_1} \chi^2 dT^3 - \frac{1}{2} \frac{2\pi q}{N_1} T^3 d\chi^2 \right) = -\frac{1}{2} \frac{2\pi q}{N_1} \int_{\omega_1} \left(\chi^2 dT^3 + T^3 d\chi^2 \right) = -\frac{1}{2} \frac{2\pi q}{N_1} \int_{\omega_1} d(\chi^2 T^3). \quad (\text{B.24})$$

Meanwhile, using

$$\int_{\omega_1} d(d^{-1} A^2 T^3) = 0 = \int_{\omega_1} A^2 T^3 + \int_{\omega_1} d^{-1} A^2 dT^3, \quad (\text{B.25})$$

$$\int_{\omega_1} d(d^{-1} C^3 \chi^2) = 0 = \int_{\omega_1} C^3 \chi^2 + \int_{\omega_1} d^{-1} C^3 d\chi^2, \quad (\text{B.26})$$

we have

$$\int_{\omega_1} \frac{1}{2} \frac{2\pi q}{N_1} A^2 T^3 + \int_{\omega_1} \frac{1}{2} \frac{2\pi q}{N_1} d^{-1} A^2 dT^3 = \frac{1}{2} \frac{2\pi q}{N_1} \left(\int_{\omega_1} A^2 T^3 + \int_{\omega_1} d^{-1} A^2 dT^3 \right) = 0 \quad (\text{B.27})$$

$$\int_{\omega_1} \left[-\frac{1}{2} \frac{2\pi q}{N_1} \left(\chi^2 C^3 \right) - \frac{1}{2} \frac{2\pi q}{N_1} d^{-1} C^3 d\chi^2 \right] = -\frac{1}{2} \frac{2\pi q}{N_1} \left(\int_{\omega_1} \chi^2 C^3 + \int_{\omega_1} d^{-1} C^3 d\chi^2 \right) = 0. \quad (\text{B.28})$$

Therefore, we can see that after gauge transformation,

$$\int_{\omega_1} C^1 + \frac{1}{2} \frac{2\pi q}{N_1} (d^{-1} A^2 C^3 - d^{-1} C^3 A^2) \rightarrow \int_{\omega_1} C^1 + \frac{1}{2} \frac{2\pi q}{N_1} (d^{-1} A^2 C^3 - d^{-1} C^3 A^2) + \int_{\omega_1} \text{boundary terms}, \quad (\text{B.29})$$

i.e., $\exp \left\{ i \int_{\omega_1} e_1 \left[C^1 + \frac{1}{2} \frac{2\pi q}{N_1} (d^{-1} A^2 C^3 - d^{-1} C^3 A^2) \right] \right\}$ is invariant. Similarly, \mathcal{W} can be verified as gauge-invariant.

Last but not least, it should be emphasized that the idea of derivation of gauge transformation can be generalized to all actions discussed in this paper, as long as we carefully deal with the shift terms for the gauge fields serving as Lagrange multipliers.

C Classification for all combinations of AAC terms when $G = \prod_{i=1}^4 \mathbb{Z}_{N_i}$

When the gauge group is $G = \prod_{i=1}^4 \mathbb{Z}_{N_i}$, the action and corresponding classification depend on the which AAC terms are considered if $\alpha = \{N_1, N_2, N_3\}$ or $\alpha = \{N_1, N_2, N_3, N_4\}$. In this appendix, we list all classifications for each compatible combination of AAC terms and other twisted terms when $G = \prod_{i=1}^4 \mathbb{Z}_{N_i}$.

In main text, for $\alpha = \{N_1, N_2, N_3\}$, we have demonstrate the situation in which the $A^1 A^2 C^3$ term is included in the action:

$$S = \int \sum_{i=1}^3 \frac{N_i}{2\pi} C^i dA^i + \frac{N_4}{2\pi} \tilde{B}^4 dB^4 + \frac{p N_1 N_2 N_3}{(2\pi)^2 N_{123}} A^1 A^2 C^3 + \left\langle \sum_{i=1,2} A^i dA^i dA^i, A^1 dA^2 dA^1, A^2 dA^1 dA^2, \sum_{i=1,2} (B^4 B^4 A^i + B^4 A^i dA^i), A^1 A^2 dB^4, B^4 A^2 dA^1 \right\rangle \quad (\text{C.1})$$

with $p \in \mathbb{Z}_{N_{123}} \setminus \{0\}$, whose classification is

$$(\mathbb{Z}_{N_{123}} \setminus \{0\}) \times \prod_{i=1}^2 \mathbb{Z}_{N_i} \times (\mathbb{Z}_{N_{12}})^2 \times (\mathbb{Z}_{N_{14}})^2 \times (\mathbb{Z}_{N_{24}})^2 \times (\mathbb{Z}_{N_{124}})^2. \quad (\text{C.2})$$

In fact, besides $A^1 A^2 C^3$, possible linearly independent AAC terms are $A^2 A^3 C^1$ and $A^3 A^1 C^2$. If $A^2 A^3 C^1$, instead of $A^1 A^2 C^3$, is added to the action, the action is

$$S = \int \sum_{i=1}^3 \frac{N_i}{2\pi} C^i dA^i + \frac{N_4}{2\pi} \tilde{B}^4 dB^4 + \frac{p N_1 N_2 N_3}{(2\pi)^2 N_{123}} A^2 A^3 C^1 + \left\langle \sum_{i=2,3} A^i dA^i dA^i, A^2 dA^3 dA^2, A^3 dA^2 dA^3, \sum_{i=2,3} (B^4 B^4 A^i + B^4 A^i dA^i), A^2 A^3 dB^4, B^4 A^3 dA^2 \right\rangle \quad (\text{C.3})$$

with $p \in \mathbb{Z}_{N_{123}} \setminus \{0\}$ and its classification is

$$(\mathbb{Z}_{N_{123}} \setminus \{0\}) \times \prod_{i=2,3} \mathbb{Z}_{N_i} \times (\mathbb{Z}_{N_{23}})^2 \times (\mathbb{Z}_{N_{24}})^2 \times (\mathbb{Z}_{N_{34}})^2 \times (\mathbb{Z}_{N_{234}})^2. \quad (\text{C.4})$$

If $A^3A^1C^2$ appears in the action, the action is

$$S = \int \sum_{i=1}^3 \frac{N_i}{2\pi} C^i dA^i + \frac{N_4}{2\pi} \tilde{B}^4 dB^4 + \frac{pN_1N_2N_3}{(2\pi)^2 N_{123}} A^3A^1C^2 + \left\langle \sum_{i=1,3} A^i dA^i dA^i, \right. \\ \left. A^3 dA^1 dA^3, A^1 dA^3 dA^1, \sum_{i=1,3} (B^4 B^4 A^i + B^4 A^i dA^i), A^3 A^1 dB^4, B^4 A^1 dA^3 \right\rangle \quad (\text{C.5})$$

with $p \in \mathbb{Z}_{N_{123}} \setminus \{0\}$ and its classification is

$$(\mathbb{Z}_{N_{123}} \setminus \{0\}) \times \prod_{i=1,3} \mathbb{Z}_{N_i} \times (\mathbb{Z}_{N_{13}})^2 \times (\mathbb{Z}_{N_{14}})^2 \times (\mathbb{Z}_{N_{34}})^2 \times (\mathbb{Z}_{N_{134}})^2. \quad (\text{C.6})$$

For $\alpha = \{N_1, N_2, N_3, N_4\}$, there are more possible AAC terms and more of them are compatible thus can be added to the action simultaneously. In the main text, we have shown the classifications for cases in which $A^1A^2C^4$, $A^1A^2C^4 + A^1A^3C^4$, $A^1A^2C^4 + A^1A^3C^4 + A^2A^3C^4$, and $A^1A^2C^4 + A^1A^2C^3$ are included in actions respectively. Below we exhaust all possible combinations of AAC terms and give corresponding classifications.

$A^2A^3C^1$ The action is

$$S = \int \sum_{i=1}^4 \frac{N_i}{2\pi} C^i dA^1 + \frac{pN_2N_3N_1}{(2\pi)^2 N_{123}} A^2A^3C^1 + \left\langle \sum_{i=2,3,4} A^i dA^i dA^i, \right. \\ \left. \sum_{2 \leq i < j \leq 4} (A^i dA^j dA^i + A^j dA^i dA^j), A^2 dA^3 dA^4, \sum_{i=2,3,4} A^2 A^3 A^4 dA^i \right\rangle \quad (\text{C.7})$$

with $p \in \mathbb{Z}_{N_{123}} \setminus \{0\}$. Its classification is

$$(\mathbb{Z}_{N_{123}} \setminus \{0\}) \times \prod_{i=2}^4 \mathbb{Z}_{N_i} \times \prod_{2 \leq i < j \leq 4} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{234}})^4. \quad (\text{C.8})$$

$A^2A^4C^1$ The classification is $(\mathbb{Z}_{N_{124}} \setminus \{0\}) \times \prod_{i=2}^4 \mathbb{Z}_{N_i} \times \prod_{2 \leq i < j \leq 4} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{234}})^4$.

$A^3A^4C^1$ The classification is $(\mathbb{Z}_{N_{134}} \setminus \{0\}) \times \prod_{i=2}^4 \mathbb{Z}_{N_i} \times \prod_{2 \leq i < j \leq 4} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{234}})^4$.

$A^2A^3C^1 + A^2A^4C^1$ The action is

$$S = \int \sum_{i=1}^4 \frac{N_i}{2\pi} C^i dA^1 + \frac{pN_2N_3N_1}{(2\pi)^2 N_{123}} A^2A^3C^1 + \frac{p'N_2N_4N_1}{(2\pi)^2 N_{124}} A^2A^4C^1 + \left\langle \sum_{i=2,3,4} A^i dA^i dA^i, \right. \\ \left. \sum_{2 \leq i < j \leq 4} (A^i dA^j dA^i + A^j dA^i dA^j), A^2 dA^3 dA^4, \sum_{i=2,3,4} A^2 A^3 A^4 dA^i \right\rangle \quad (\text{C.9})$$

with $p \in \mathbb{Z}_{N_{123}} \setminus \{0\}$ and $p' \in \mathbb{Z}_{N_{124}} \setminus \{0\}$. Its classification is

$$(\mathbb{Z}_{N_{123}} \setminus \{0\}) \times (\mathbb{Z}_{N_{124}} \setminus \{0\}) \times \prod_{i=2}^4 \mathbb{Z}_{N_i} \times \prod_{2 \leq i < j \leq 4} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{234}})^4. \quad (\text{C.10})$$

$A^2A^3C^1 + A^3A^4C^1$ The classification is

$$(\mathbb{Z}_{N_{123}} \setminus \{0\}) \times (\mathbb{Z}_{N_{134}} \setminus \{0\}) \times \prod_{i=2}^4 \mathbb{Z}_{N_i} \times \prod_{2 \leq i < j \leq 4} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{234}})^4.$$

$A^2A^4C^1 + A^3A^4C^1$ The classification is

$$(\mathbb{Z}_{N_{124}} \setminus \{0\}) \times (\mathbb{Z}_{N_{134}} \setminus \{0\}) \times \prod_{i=2}^4 \mathbb{Z}_{N_i} \times \prod_{2 \leq i < j \leq 4} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{234}})^4.$$

$A^2A^3C^1 + A^2A^4C^1 + A^3A^4C^1$ The classification is

$$(\mathbb{Z}_{N_{123}} \setminus \{0\}) \times (\mathbb{Z}_{N_{124}} \setminus \{0\}) \times (\mathbb{Z}_{N_{134}} \setminus \{0\}) \times \prod_{i=2}^4 \mathbb{Z}_{N_i} \times \prod_{2 \leq i < j \leq 4} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{234}})^4.$$

$A^3A^1C^2$ The action is

$$S = \int \sum_{i=1}^4 \frac{N_i}{2\pi} C^i dA^1 + \frac{pN_3N_1N_2}{(2\pi)^2 N_{123}} A^3 A^1 C^2 + \left\langle \sum_{i=1,3,4} A^i dA^i dA^i, \right. \\ \left. \sum_{\substack{1 \leq i < j \leq 4 \\ i,j \neq 2}} (A^i dA^j dA^i + A^j dA^i dA^j), A^1 dA^3 dA^4, \sum_{i=1,3,4} A^1 A^3 A^4 dA^i \right\rangle \quad (\text{C.11})$$

with $p \in \mathbb{Z}_{N_{123}} \setminus \{0\}$. Its classification is

$$(\mathbb{Z}_{N_{123}} \setminus \{0\}) \times \prod_{i=1,3,4} \mathbb{Z}_{N_i} \times \prod_{\substack{1 \leq i < j \leq 4 \\ i,j \neq 2}} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{134}})^4. \quad (\text{C.12})$$

$A^4A^1C^2$ The classification is $(\mathbb{Z}_{N_{124}} \setminus \{0\}) \times \prod_{i=1,3,4} \mathbb{Z}_{N_i} \times \prod_{\substack{1 \leq i < j \leq 4 \\ i,j \neq 2}} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{134}})^4.$

$A^3A^4C^2$ The classification is $(\mathbb{Z}_{N_{234}} \setminus \{0\}) \times \prod_{i=1,3,4} \mathbb{Z}_{N_i} \times \prod_{\substack{1 \leq i < j \leq 4 \\ i,j \neq 2}} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{134}})^4.$

$A^3A^1C^2 + A^4A^1C^2$ The classification is

$$(\mathbb{Z}_{N_{123}} \setminus \{0\}) \times (\mathbb{Z}_{N_{124}} \setminus \{0\}) \times \prod_{i=1,3,4} \mathbb{Z}_{N_i} \times \prod_{\substack{1 \leq i < j \leq 4 \\ i,j \neq 2}} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{134}})^4.$$

$A^3A^1C^2 + A^3A^4C^2$ The classification is

$$(\mathbb{Z}_{N_{123}} \setminus \{0\}) \times (\mathbb{Z}_{N_{234}} \setminus \{0\}) \times \prod_{i=1,3,4} \mathbb{Z}_{N_i} \times \prod_{\substack{1 \leq i < j \leq 4 \\ i,j \neq 2}} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{134}})^4.$$

$A^3A^4C^2 + A^4A^1C^2$ The classification is

$$(\mathbb{Z}_{N_{234}} \setminus \{0\}) \times (\mathbb{Z}_{N_{124}} \setminus \{0\}) \times \prod_{i=1,3,4} \mathbb{Z}_{N_i} \times \prod_{\substack{1 \leq i < j \leq 4 \\ i,j \neq 2}} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{134}})^4.$$

$A^3A^1C^2 + A^4A^1C^2 + A^3A^4C^2$ The classification is

$$(\mathbb{Z}_{N_{123}} \setminus \{0\}) \times (\mathbb{Z}_{N_{124}} \setminus \{0\}) \times (\mathbb{Z}_{N_{234}} \setminus \{0\}) \times \prod_{i=1,3,4} \mathbb{Z}_{N_i} \times \prod_{\substack{1 \leq i < j \leq 4 \\ i,j \neq 2}} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{134}})^4.$$

$A^1A^2C^3$ The action is

$$S = \int \sum_{i=1}^4 \frac{N_i}{2\pi} C^i dA^1 + \frac{pN_1N_2N_3}{(2\pi)^2 N_{123}} A^1A^2C^3 + \left\langle \sum_{i=1,2,4} A^i dA^i dA^i, \right. \\ \left. \sum_{\substack{1 \leq i < j \leq 4 \\ i,j \neq 3}} (A^i dA^j dA^i + A^j dA^i dA^j), A^1 dA^2 dA^4, \sum_{i=1,2,4} A^1 A^2 A^4 dA^i \right\rangle \quad (C.13)$$

with $p \in \mathbb{Z}_{N_{123}} \setminus \{0\}$. Its classification is

$$(\mathbb{Z}_{N_{123}} \setminus \{0\}) \times \prod_{i=1,2,4} \mathbb{Z}_{N_i} \times \prod_{\substack{1 \leq i < j \leq 4 \\ i,j \neq 3}} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{124}})^4. \quad (C.14)$$

$A^1A^4C^3$ The classification is $(\mathbb{Z}_{N_{134}} \setminus \{0\}) \times \prod_{i=1,2,4} \mathbb{Z}_{N_i} \times \prod_{\substack{1 \leq i < j \leq 4 \\ i,j \neq 3}} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{124}})^4$.

$A^2A^4C^3$ The classification is $(\mathbb{Z}_{N_{234}} \setminus \{0\}) \times \prod_{i=1,2,4} \mathbb{Z}_{N_i} \times \prod_{\substack{1 \leq i < j \leq 4 \\ i,j \neq 3}} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{124}})^4$.

$A^1A^2C^3 + A^1A^4C^3$ The classification is

$$(\mathbb{Z}_{N_{123}} \setminus \{0\}) \times (\mathbb{Z}_{N_{134}} \setminus \{0\}) \times \prod_{i=1,2,4} \mathbb{Z}_{N_i} \times \prod_{\substack{1 \leq i < j \leq 4 \\ i,j \neq 3}} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{124}})^4.$$

$A^1A^2C^3 + A^2A^4C^3$ The classification is

$$(\mathbb{Z}_{N_{123}} \setminus \{0\}) \times (\mathbb{Z}_{N_{234}} \setminus \{0\}) \times \prod_{i=1,2,4} \mathbb{Z}_{N_i} \times \prod_{\substack{1 \leq i < j \leq 4 \\ i,j \neq 3}} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{124}})^4.$$

$A^2A^4C^3 + A^1A^4C^3$ The classification is

$$(\mathbb{Z}_{N_{234}} \setminus \{0\}) \times (\mathbb{Z}_{N_{134}} \setminus \{0\}) \times \prod_{i=1,2,4} \mathbb{Z}_{N_i} \times \prod_{\substack{1 \leq i < j \leq 4 \\ i,j \neq 3}} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{124}})^4.$$

$A^1A^2C^3 + A^1A^4C^3 + A^2A^4C^3$ The classification is

$$(\mathbb{Z}_{N_{123}} \setminus \{0\}) \times (\mathbb{Z}_{N_{134}} \setminus \{0\}) \times (\mathbb{Z}_{N_{234}} \setminus \{0\}) \times \prod_{i=1,2,4} \mathbb{Z}_{N_i} \times \prod_{\substack{1 \leq i < j \leq 4 \\ i,j \neq 3}} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{124}})^4.$$

$A^1A^2C^4$ The action is

$$S = \int \sum_{i=1}^4 \frac{N_i}{2\pi} C^i dA^1 + \frac{pN_1N_2N_4}{(2\pi)^2 N_{124}} A^1A^2C^4 + \left\langle \sum_{i=1}^3 A^i dA^i dA^i, \right. \\ \left. \sum_{1 \leq i < j \leq 3} (A^i dA^j dA^i + A^j dA^i dA^j), A^1 dA^2 dA^3, \sum_{i=1}^3 A^1 A^2 A^3 dA^i \right\rangle \quad (C.15)$$

with $p \in \mathbb{Z}_{N_{124}} \setminus \{0\}$. Its classification is

$$(\mathbb{Z}_{N_{124}} \setminus \{0\}) \times \prod_{i=1}^3 \mathbb{Z}_{N_i} \times \prod_{1 \leq i < j \leq 3} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{123}})^4. \quad (C.16)$$

$A^1A^3C^4$ The classification is $(\mathbb{Z}_{N_{134}} \setminus \{0\}) \times \prod_{i=1}^3 \mathbb{Z}_{N_i} \times \prod_{1 \leq i < j \leq 3} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{123}})^4$.

$A^2A^3C^4$ The classification is $(\mathbb{Z}_{N_{234}} \setminus \{0\}) \times \prod_{i=1}^3 \mathbb{Z}_{N_i} \times \prod_{1 \leq i < j \leq 3} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{123}})^4$.

$A^1A^2C^4 + A^1A^3C^4$ The classification is $(\mathbb{Z}_{N_{124}} \setminus \{0\}) \times (\mathbb{Z}_{N_{134}} \setminus \{0\}) \times \prod_{i=1}^3 \mathbb{Z}_{N_i} \times \prod_{1 \leq i < j \leq 3} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{123}})^4$.

$A^1A^2C^4 + A^2A^3C^4$ The classification is $(\mathbb{Z}_{N_{124}} \setminus \{0\}) \times (\mathbb{Z}_{N_{234}} \setminus \{0\}) \times \prod_{i=1}^3 \mathbb{Z}_{N_i} \times \prod_{1 \leq i < j \leq 3} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{123}})^4$.

$A^2A^3C^4 + A^1A^3C^4$ The classification is $(\mathbb{Z}_{N_{234}} \setminus \{0\}) \times (\mathbb{Z}_{N_{134}} \setminus \{0\}) \times \prod_{i=1}^3 \mathbb{Z}_{N_i} \times \prod_{1 \leq i < j \leq 3} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{123}})^4$.

$A^1A^2C^4 + A^1A^3C^4 + A^2A^3C^4$ The classification is $(\mathbb{Z}_{N_{124}} \setminus \{0\}) \times (\mathbb{Z}_{N_{134}} \setminus \{0\}) \times (\mathbb{Z}_{N_{234}} \setminus \{0\}) \times \prod_{i=1}^3 \mathbb{Z}_{N_i} \times \prod_{1 \leq i < j \leq 3} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{123}})^4$.

$A^1A^2C^3 + A^1A^2C^4$ The action is

$$S = \int \sum_{i=1}^4 \frac{N_i}{2\pi} C^i dA^1 + \frac{pN_1N_2N_3}{(2\pi)^2 N_{123}} A^1 A^2 C^3 + \frac{pN_1N_2N_4}{(2\pi)^2 N_{124}} A^1 A^2 C^4 + \left\langle \sum_{i=1}^2 A^i dA^i dA^i, A^1 dA^2 dA^1, A^2 dA^1 dA^2 \right\rangle \quad (\text{C.17})$$

with $p \in \mathbb{Z}_{N_{123}} \setminus \{0\}$ and $p \in \mathbb{Z}_{N_{124}} \setminus \{0\}$. Its classification is

$$(\mathbb{Z}_{N_{123}} \setminus \{0\}) \times (\mathbb{Z}_{N_{124}} \setminus \{0\}) \times \prod_{i=1}^2 \mathbb{Z}_{N_i} \times (\mathbb{Z}_{N_{12}})^2. \quad (\text{C.18})$$

$A^1A^3C^4 + A^3A^1C^2$ The classification is $(\mathbb{Z}_{N_{134}} \setminus \{0\}) \times (\mathbb{Z}_{N_{123}} \setminus \{0\}) \times \prod_{i=1,3} \mathbb{Z}_{N_i} \times (\mathbb{Z}_{N_{13}})^2$.

$A^1A^4C^3 + A^4A^1C^2$ The classification is $(\mathbb{Z}_{N_{134}} \setminus \{0\}) \times (\mathbb{Z}_{N_{124}} \setminus \{0\}) \times \prod_{i=1,4} \mathbb{Z}_{N_i} \times (\mathbb{Z}_{N_{14}})^2$.

$A^2A^3C^4 + A^2A^3C^1$ The classification is $(\mathbb{Z}_{N_{234}} \setminus \{0\}) \times (\mathbb{Z}_{N_{123}} \setminus \{0\}) \times \prod_{i=2,3} \mathbb{Z}_{N_i} \times (\mathbb{Z}_{N_{23}})^2$.

$A^2A^4C^3 + A^2A^4C^1$ The classification is $(\mathbb{Z}_{N_{234}} \setminus \{0\}) \times (\mathbb{Z}_{N_{124}} \setminus \{0\}) \times \prod_{i=2,4} \mathbb{Z}_{N_i} \times (\mathbb{Z}_{N_{24}})^2$.

$A^3A^4C^1 + A^3A^4C^2$ The classification is $(\mathbb{Z}_{N_{134}} \setminus \{0\}) \times (\mathbb{Z}_{N_{234}} \setminus \{0\}) \times \prod_{i=3,4} \mathbb{Z}_{N_i} \times (\mathbb{Z}_{N_{34}})^2$.

D Typical examples of BF theories and classification when $G = \prod_{i=1}^5 \mathbb{Z}_{N_i}$

In this appendix, we give some representative examples of BF theories and their classifications. It is tedious to list all BF theories with twisted terms for $G = \prod_{i=1}^5 \mathbb{Z}_{N_i}$ due to lots of combinations of compatible twisted terms. Nevertheless, the discussion in section 3 and appendix C can be straightforward generalized to cases of gauge groups with arbitrary cyclic subgroups. Following the same line of thinking, we can figure out compatible twisted terms, write down the action, and find out the corresponding classification.

If $\alpha = \emptyset$, the action is

$$S = \int \sum_{i=1}^5 \frac{N_i}{2\pi} \tilde{B}^i dB^i, \quad (\text{D.1})$$

whose classification is \mathbb{Z}_1 .

If $\alpha = \{N_1\}$, the action is

$$S = \int \frac{N_1}{2\pi} C^1 dA^1 + \sum_{i=2}^5 \frac{N_i}{2\pi} \tilde{B}^i dB^i + \left\langle A^1 dA^1 dA^1, \sum_{i=2}^5 B^i B^i A^1, \sum_{i=2}^5 B^i A^1 dA^1, \sum_{2 \leq i < j \leq 5} B^i B^j A^1 \right\rangle, \quad (\text{D.2})$$

whose classification of topological gauge theories is $\mathbb{Z}_{N_1} \times \prod_{i=2}^5 (\mathbb{Z}_{N_{1i}})^2 \times \prod_{2 \leq i < j \leq 5} \mathbb{Z}_{N_{1ij}}$.

If $\alpha = \{N_1, N_2\}$, the action is

$$S = \int \sum_{i=1}^2 \frac{N_i}{2\pi} C^i dA^i + \sum_{i=3}^5 \frac{N_i}{2\pi} \tilde{B}^i dB^i + \left\langle \sum_{i=1}^2 A^i dA^i dA^i, A^1 dA^2 dA^1, A^2 dA^1 dA^2, \sum_{3 \leq i < j \leq 5} \sum_{k=1}^2 B^i B^j A^k, \sum_{i=3}^5 \sum_{j=1}^2 (B^i B^i A^j + B^i A^j dA^j), \sum_{i=3}^5 (A^1 A^2 dB^i + B^i A^2 dA^1) \right\rangle, \quad (\text{D.3})$$

whose classification is

$$\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times (\mathbb{Z}_{N_{12}})^2 \times \prod_{3 \leq i < j \leq 5} (\mathbb{Z}_{N_{1ij}} \times \mathbb{Z}_{N_{2ij}}) \times \prod_{i=3}^5 (\mathbb{Z}_{N_{1i}} \times \mathbb{Z}_{N_{2i}})^2 \times \prod_{i=3}^5 (\mathbb{Z}_{N_{12i}})^2.$$

If $\alpha = \{N_1, N_2, N_3\}$, we should be aware of keeping all twisted terms compatible with each other. Below we show two examples.

1. If we do not include AAC terms in twisted terms, the action is

$$S = \int \sum_{i=1}^3 \frac{N_i}{2\pi} C^i dA^i + \sum_{i=4}^5 \frac{N_i}{2\pi} \tilde{B}^i dB^i + \left\langle \sum_{i=1}^3 A^i dA^i dA^i, \sum_{1 \leq i < j \leq 3} (A^i dA^j dA^i + A^j dA^i dA^j), A^1 dA^2 dA^3, \sum_{i=1}^3 A^1 A^2 A^3 dA^i, \sum_{i=1}^3 B^4 B^5 A^i, \sum_{i=1}^3 \sum_{j=4}^5 (B^j B^j A^i + B^j A^i dA^j), \sum_{1 \leq i < j \leq 3} \sum_{k=4}^5 (A^i A^j dB^k + B^k A^j dA^i), \sum_{i=4}^5 A^1 A^2 A^3 B^i \right\rangle, \quad (\text{D.4})$$

whose classification is $\prod_{i=1}^3 \mathbb{Z}_{N_i} \times \prod_{1 \leq i < j \leq 3} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{123}})^4 \times \prod_{i=1}^3 \mathbb{Z}_{N_{i45}} \times \prod_{i=1}^3 [(\mathbb{Z}_{N_{i4}})^2 \times (\mathbb{Z}_{N_{i5}})^2] \times \prod_{1 \leq i < j \leq 3} [(\mathbb{Z}_{N_{ij4}})^2 \times (\mathbb{Z}_{N_{ij5}})^2] \times \prod_{i=4}^5 \mathbb{Z}_{N_{123i}}$.

2. If we add an AAC term, e.g., $A^1 A^2 C^3$, to twisted terms, the action is

$$S = \int \sum_{i=1}^3 \frac{N_i}{2\pi} C^i dA^i + \sum_{i=4}^5 \frac{N_i}{2\pi} \tilde{B}^i dB^i + \frac{pN_1 N_2 N_3}{(2\pi)^2 N_{123}} A^1 A^2 C^3 + \left\langle \sum_{i=1}^2 A^i dA^i dA^i, A^1 dA^2 dA^1, A^2 dA^1 dA^2, \sum_{i=1}^2 B^4 B^5 A^i, \sum_{i=1}^2 \sum_{j=4}^5 (B^j B^j A^i + B^j A^i dA^i), \sum_{i=4}^5 (A^1 A^2 dB^i + B^i A^1 dA^2) \right\rangle \quad (\text{D.5})$$

with $p \in \mathbb{Z}_{N_{123}} \setminus \{0\}$. Its classification is $(\mathbb{Z}_{N_{123}} \setminus \{0\}) \times \prod_{i=1}^2 \mathbb{Z}_{N_i} \times (\mathbb{Z}_{N_{12}})^2 \times \prod_{i=1}^2 \mathbb{Z}_{N_{i45}} \times \prod_{i=1}^2 (\mathbb{Z}_{N_{i4}} \times \mathbb{Z}_{N_{i5}})^2 \times \prod_{i=4}^5 (\mathbb{Z}_{N_{12i}})^2$.

If $\alpha = \{N_1, N_2, N_3, N_4\}$, we give an example for each situation.

1. If no AAC terms appear in this mixed BF theory, the action is

$$S = \int \sum_{i=1}^4 \frac{N_i}{2\pi} C^i dA^i + \frac{N_5}{2\pi} \tilde{B}^5 dB^5 + \left\langle \sum_{i=1}^4 A^i dA^i dA^i, \sum_{1 \leq i < j \leq 4} (A^i dA^j dA^i + A^j dA^i dA^j), \sum_{1 \leq i < j < k \leq 4} A^i dA^j dA^k, \sum_{1 \leq i < j < k \leq 4} \left(\sum_{s=i,j,k} A^i A^j A^k dA^s \right), A^1 A^2 A^3 dA^4, A^3 A^2 A^4 dA^1, A^1 A^3 A^4 dA^2, \sum_{i=1}^4 (B^5 B^5 A^i + B^5 A^i dA^i), \sum_{1 \leq i < j \leq 4} (A^i A^j dB^5 + B^5 A^j dA^i), \sum_{1 \leq i < j < k \leq 4} A^i A^j A^k B^5 \right\rangle, \quad (\text{D.6})$$

whose classification is $\prod_{i=1}^4 \mathbb{Z}_{N_i} \times \prod_{1 \leq i < j \leq 4} (\mathbb{Z}_{N_{ij}})^2 \times \prod_{1 \leq i < j < k \leq 4} (\mathbb{Z}_{N_{ijk}})^4 \times (\mathbb{Z}_{N_{1234}})^3 \times \prod_{i=1}^4 (\mathbb{Z}_{N_{i5}})^2 \times \prod_{1 \leq i < j \leq 4} (\mathbb{Z}_{N_{ij5}})^2 \times \prod_{1 \leq i < j < k \leq 4} \mathbb{Z}_{N_{ijk5}}$.

2. If twisted terms include $A^1 A^2 C^4 + A^1 A^3 C^4 + A^2 A^3 C^4$ terms, the action is

$$S = \int \sum_{i=1}^4 \frac{N_i}{2\pi} C^i dA^i + \frac{N_5}{2\pi} \tilde{B}^5 dB^5 + \frac{pN_1 N_2 N_4}{(2\pi)^2 N_{124}} A^1 A^2 C^4 + \frac{p'N_1 N_3 N_4}{(2\pi)^2 N_{134}} A^1 A^3 C^4 + \frac{p''N_2 N_3 N_4}{(2\pi)^2 N_{234}} A^2 A^3 C^4 + \left\langle \sum_{i=1}^3 A^i dA^i dA^i, \sum_{1 \leq i < j \leq 3} (A^i dA^j dA^i + A^j dA^i dA^j), A^1 dA^2 dA^3, \sum_{i=1}^3 A^1 A^2 A^3 dA^i, \sum_{i=1}^3 (B^5 B^5 A^i + B^5 A^i dA^i), \sum_{1 \leq i < j \leq 3} (A^i A^j dB^5 + B^5 A^j dA^i), A^1 A^2 A^3 B^5 \right\rangle \quad (\text{D.7})$$

with $p \in \mathbb{Z}_{N_{124}} \setminus \{0\}$, $p' \in \mathbb{Z}_{N_{134}} \setminus \{0\}$, and $p'' \in \mathbb{Z}_{N_{234}} \setminus \{0\}$. Its classification is $\prod_{1 \leq i < j \leq 3} (\mathbb{Z}_{N_{ij4}} \setminus \{0\}) \times \prod_{i=1}^3 \mathbb{Z}_{N_i} \times \prod_{1 \leq i < j \leq 3} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{123}})^4 \times \prod_{i=1}^3 (\mathbb{Z}_{N_{i5}})^2 \times \prod_{1 \leq i < j \leq 3} (\mathbb{Z}_{N_{ij5}})^2 \times \mathbb{Z}_{N_{1235}}$.

3. If $A^1 A^2 C^3 + A^1 A^2 C^4$ terms are added to twisted terms at the cost of excluding some other terms, the action is

$$\begin{aligned}
 S = & \int \sum_{i=1}^4 \frac{N_i}{2\pi} C^i dA^i + \frac{N_5}{2\pi} \tilde{B}^5 dB^5 + \frac{p N_1 N_2 N_3}{(2\pi)^2 N_{123}} A^1 A^2 C^3 + \frac{p' N_1 N_2 N_4}{(2\pi)^2 N_{124}} A^1 A^2 C^4 \\
 & + \left\langle \sum_{i=1}^2 A^i dA^i dA^i, A^1 dA^2 dA^1, A^2 dA^1 dA^2, \right. \\
 & \left. \sum_{i=1}^2 (B^5 B^5 A^i + B^5 A^i dA^i), A^1 A^2 dB^5, B^5 A^2 dA^1 \right\rangle \quad (D.8)
 \end{aligned}$$

with $p \in \mathbb{Z}_{N_{123}} \setminus \{0\}$ and $p' \in \mathbb{Z}_{N_{124}} \setminus \{0\}$. Its classification is $(\mathbb{Z}_{N_{123}} \setminus \{0\}) \times (\mathbb{Z}_{N_{124}} \setminus \{0\}) \times \prod_{i=1}^2 \mathbb{Z}_{N_i} \times (\mathbb{Z}_{N_{12}})^2 \times \prod_{i=1}^2 (\mathbb{Z}_{N_{i5}})^2 \times (\mathbb{Z}_{N_{125}})^2$.

If $\alpha = \{N_1, N_2, N_3, N_4, N_5\}$, i.e., type-I BF theory, similar to previous discussion, different combinations of twisted terms result in different actions and classifications. Some examples are listed as follow.

1. If there is no AAC terms in the action, i.e.,

$$\begin{aligned}
 S = & \int \sum_{i=1}^5 \frac{N_i}{2\pi} C^i dA^i + \left\langle \sum_{i=1}^5 A^i dA^i dA^i, \sum_{1 \leq i < j \leq 5} (A^i dA^j dA^i + A^j dA^i dA^j), \right. \\
 & \sum_{1 \leq i < j < k \leq 5} \left(A^i dA^j dA^k + \sum_{l=i,j,k} A^i A^j A^k dA^l \right), \\
 & \left. \sum_{1 \leq i < j < k < l \leq 5} (A^i A^j A^k dA^l + A^k A^j A^l dA^i + A^i A^k A^l dA^j), A^1 A^2 A^3 A^4 A^5 \right\rangle, \quad (D.9)
 \end{aligned}$$

whose classification is given by $H^5(\prod_{i=1}^5 \mathbb{Z}_{N_i}, \mathbb{R}/\mathbb{Z})$, consistent with the Dijkgraaf-Witten cohomology classification:

$$\prod_{i=1}^5 \mathbb{Z}_{N_i} \times \prod_{1 \leq i < j \leq 5} (\mathbb{Z}_{N_{ij}})^2 \times \prod_{1 \leq i < j < k \leq 5} (\mathbb{Z}_{N_{ijk}})^4 \times \prod_{1 \leq i < j < k < l \leq 5} (\mathbb{Z}_{N_{ijkl}})^3 \times \mathbb{Z}_{N_{12345}}. \quad (D.10)$$

2. If we put $\sum_{1 \leq i < j \leq 4} A^i A^j C^5$ terms in twisted terms, the action is

$$\begin{aligned}
 S = & \int \sum_{i=1}^5 \frac{N_i}{2\pi} C^i dA^i + \sum_{1 \leq i < j \leq 4} \frac{p_{ij5} N_i N_j N_5}{(2\pi)^2 N_{ij5}} A^i A^j C^5 + \left\langle \sum_{i=1}^4 A^i dA^i dA^i, \right. \\
 & \sum_{1 \leq i < j \leq 4} \left(A^i dA^j dA^i + A^j dA^i dA^j \right), \sum_{1 \leq i < j < k \leq 4} A^i dA^j dA^k, \\
 & \left. \sum_{1 \leq i < j < k \leq 4} \left(\sum_{l=i,j,k} A^i A^j A^k dA^l \right), A^1 A^2 A^3 dA^4, A^3 A^2 A^4 dA^1, A^1 A^3 A^4 dA^2 \right\rangle
 \end{aligned}
 \tag{D.11}$$

where $p_{ij5} \in \mathbb{Z}_{N_{ij5}} \setminus \{0\}$, $1 \leq i < j \leq 4$. Its classification is $\prod_{1 \leq i < j \leq 4} (\mathbb{Z}_{N_{ij5}} \setminus \{0\}) \times \prod_{i=1}^4 \mathbb{Z}_{N_i} \times \prod_{1 \leq i < j \leq 4} (\mathbb{Z}_{N_{ij}})^2 \times \prod_{1 \leq i < j < k \leq 4} (\mathbb{Z}_{N_{ijk}})^4 \times (\mathbb{Z}_{N_{1234}})^3$.

3. If $\sum_{1 \leq i < j \leq 3} (A^i A^j C^4 + A^i A^j C^5)$ terms are considered, the action is

$$\begin{aligned}
 S = & \int \sum_{i=1}^5 \frac{N_i}{2\pi} C^i dA^i + \sum_{1 \leq i < j \leq 3} \left(\frac{p_{ij4} N_i N_j N_4}{(2\pi)^2 N_{ij4}} A^i A^j C^4 + \frac{p_{ij5} N_i N_j N_5}{(2\pi)^2 N_{ij5}} A^i A^j C^5 \right) \\
 & + \left\langle \sum_{i=1}^3 A^i dA^i dA^i, \sum_{1 \leq i < j \leq 3} (A^i dA^j dA^i + A^j dA^i dA^j), A^1 dA^2 dA^3, \sum_{i=1,2,3} A^1 A^2 A^3 dA^i \right\rangle
 \end{aligned}
 \tag{D.12}$$

where $p_{ij4} \in \mathbb{Z}_{N_{ij4}} \setminus \{0\}$ and $p_{ij5} \in \mathbb{Z}_{N_{ij5}} \setminus \{0\}$, $1 \leq i < j \leq 3$. Its classification is $\prod_{1 \leq i < j \leq 3} (\mathbb{Z}_{N_{ij4}} \setminus \{0\}) \times \prod_{1 \leq i < j \leq 3} (\mathbb{Z}_{N_{ij5}} \setminus \{0\}) \times \prod_{i=1}^3 \mathbb{Z}_{N_i} \times \prod_{1 \leq i < j \leq 3} (\mathbb{Z}_{N_{ij}})^2 \times (\mathbb{Z}_{N_{123}})^4$.

4. If $A^1 A^2 C^3 + A^1 A^2 C^4 + A^1 A^2 C^5$ terms are collected in twisted terms, the action is

$$S = \int \sum_{i=1}^5 \frac{N_i}{2\pi} C^i dA^i + \sum_{i=3}^5 \frac{p_{12i} N_1 N_2 N_i}{(2\pi)^2 N_{12i}} A^1 A^2 C^i + \left\langle \sum_{i=1}^2 A^i dA^i dA^i, A^1 dA^2 dA^1, A^2 dA^1 dA^2 \right\rangle
 \tag{D.13}$$

where $p_{12i} \in \mathbb{Z}_{N_{12i}} \setminus \{0\}$, $3 \leq i \leq 5$. Its classification is $\prod_{i=1}^3 (\mathbb{Z}_{N_{12i}} \setminus \{0\}) \times \prod_{i=1}^2 \mathbb{Z}_{N_i} \times (\mathbb{Z}_{N_{12}})^2$.

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