Spectrum of the quantum integrable $D_2^{(2)}$ spin chain with generic boundary fields

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ABSTRACT: Exact solution of the quantum integrable $D_2^{(2)}$ spin chain with generic integrable boundary fields is constructed. It is found that the transfer matrix of this model can be factorized as the product of those of two open staggered anisotropic XXZ spin chains. Based on this identity, the eigenvalues and Bethe ansatz equations of the $D_2^{(2)}$ model are derived via off-diagonal Bethe ansatz.

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Contents

1 Introduction

The $D_2^{(2)}$ spin chain model is one of the most representative integrable system associated with quantum algebra beyond A-series. The exact solution of the $D_2^{(2)}$ spin chain is also the foundation to solve the high rank $D_n^{(2)}$ models with nested analytical methods. Particularly, the $D_2^{(2)}$ spin chain has many applications in the string theory and black hole. For an example, Robertson, Jacobsen and Saleur found [1] that an open $D_2^{(2)}$ spin chain with some integrable boundary condition possesses the lattice regularisation of a non-compact boundary conformal field theory and is closely related to the SL(2, $\mathbb{R}$)/U(1) Euclidean black hole [2–6].

The eigenvalues of the transfer matrix of the periodic $D_n^{(2)}$ model firstly was obtained by the analytical Bethe ansatz [7] and then by the algebraic Bethe ansatz [8]. For open boundary conditions, besides the R-matrix, the reflection matrices should also be used to construct the transfer matrix which generates the conserved quantities including the model Hamiltonian [9–11]. The Hamiltonian with diagonal boundary fields was exactly solved via both the coordinate Bethe ansatz [12] and the analytical Bethe ansatz [13, 14]. Recently, Robertson, Pawelkiewicz, Jacobsen and Saleur [15] reported that the R-matrix of $D_2^{(2)}$ model [16–18] is related to the antiferromagnetic Potts model and the staggered XXZ spin chain [19–24]. Based on this idea, Nepomechie and Retore [25] obtained the exact solutions of transfer matrices of both the closed $D_2^{(2)}$ spin chain and the open one with a special boundary condition by using the factorization identities and algebraic Bethe ansatz.

In this paper, we study the exact solution of the $D_2^{(2)}$ spin chain with generic non-diagonal boundary fields. Because the reflection matrix and the dual one can not be diagonalized simultaneously, the U(1) symmetry of the system is broken. The structure of the present paper is as follows. In section 2, we give a brief description of the $D_2^{(2)}$ model with open boundary condition. The R-matrix, reflection matrices and generating functional of conserved quantities are introduced. In section 3, we show that the transfer matrix can be

factorized as the product of two open staggered XXZ spin chains. In section 4, by using the fusion techniques, we obtain the exact solution of the system via off-diagonal Bethe ansatz. The inhomogeneous $T - Q$ relations and related Bethe ansatz equations are given. The summary of main results and some concluding remarks are presented in section 5. Appendix A provides the results for another inequivalent generic non-diagonal boundary fields.

2 $D_2^{(2)}$-model

The conserved quantities including the model Hamiltonian of the $D_2^{(2)}$ spin chain are generated by the transfer matrix $t(u)$

$$t(u) = tr_0\{K_0^+(u)T_0(u)K_0^-(u)\hat{T}_0(u)\}. \tag{2.1}$$

Here u is the spectral parameter, the subscript 0 means the four-dimensional auxiliary space V_0, tr_0 means taking trace only in the auxiliary space V_0, $K_0^+(u)$ is the boundary reflection matrix defined in the auxiliary space at one end, $K_0^-(u)$ is the reflection matrix at the other end, $T_0(u)$ and $\hat{T}_0(u)$ are the monodromy matrices constructed by the 16×16 R-matrix as

$$\begin{aligned} T_0(u) &= R_{01}(u)R_{02}(u)\cdots R_{0N}(u), \\ \hat{T}_0(u) &= R_{N0}(u)R_{N-10}(u)\cdots R_{10}(u). \end{aligned} \tag{2.2}$$

Here the subscript $j = 1, \cdots, N$ denotes the four-dimensional quantum space V_j of j-th site, which means that the spin of the $D_2^{(2)}$ chain at j-th site has four components, and N is the number of sites. Thus $T_0(u)$ and $\hat{T}_0(u)$ are defined in the tensor space $V_0 \otimes V_1 \otimes \cdots \otimes V_N$ and $\otimes_{j=1}^N V_j$ is the quantum or physical space.

The integrability of the system requires that the transfer matrices (2.1) with different spectral parameters commute with each other

$$[t(u), t(v)] = 0. \tag{2.3}$$

Thus all the expansion coefficients of $t(u)$ with respect to u are commutative. The coefficients or their combinations are the conserved quantities. The commutation relation (2.3) is achieved by that the R-matrices in eq. (2.2) satisfy the Yang-Baxter equation

$$R_{12}(u - v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u - v), \tag{2.4}$$

and the reflection matrices in eq. (2.1) for the given R-matrix satisfy the reflection equations [9–11]

$$\begin{aligned} R_{12}(-u + v)&K_1^+(u)M_1^{-1}R_{21}(-u - v + 4\eta)M_1K_2^+(v) \\ &= K_2^+(v)M_1R_{12}(-u - v + 4\eta)M_1^{-1}K_1^+(u)R_{21}(-u + v), \end{aligned} \tag{2.5}$$

$$R_{12}(u - v)K_1^-(u)R_{21}(u + v)K_2^-(v) = K_2^-(v)R_{12}(u + v)K_1^-(u)R_{21}(u - v). \tag{2.6}$$

The solution of Yang-Baxter equation (2.4) associated with the twisted $D_2^{(2)}$ quantum algebra, gives the 16×16 R-matrix defined in the tensor space $V_1 \otimes V_2$ as [12–14]

$$
\begin{aligned}
R_{12}(u) = e^{-2(u+2\eta)} \Bigg\{ &(e^{2u}-e^{4\eta})(e^{2u}-e^{4\eta}) \sum_{\alpha \neq 2,3} [e_1]_\alpha^\alpha \otimes [e_2]_\alpha^\alpha + e^{2\eta}(e^{2u}-1)(e^{2u}-e^{4\eta}) \\
&\times \sum_{\substack{\alpha \neq \beta,\beta' \\ \alpha\,\text{or}\,\beta \neq 2,3}} [e_1]_\alpha^\alpha \otimes [e_2]_\beta^\beta - \frac{1}{2}(e^{4\eta}-1)(e^{2u}-e^{4\eta})\Bigg[(e^u+1)\Bigg(\sum_{\alpha=1,\beta=2,3} + e^u \sum_{\alpha=4,\beta=2,3}\Bigg) \\
&\times \Big([e_1]_\beta^\alpha \otimes [e_2]_\alpha^\beta + [e_1]_{\alpha'}^{\beta'} \otimes [e_2]_{\beta'}^{\alpha'}\Big) + (e^u-1)\Bigg(-\sum_{\alpha=1,\beta=2,3} + e^u \sum_{\alpha=4,\beta=2,3}\Bigg) \\
&\times \Big([e_1]_\beta^\alpha \otimes [e_2]_\alpha^{\beta'} + [e_1]_{\alpha'}^{\beta'} \otimes [e_2]_\beta^{\alpha'}\Big)\Bigg] + \sum_{\alpha,\beta\neq2,3} a_{\alpha\beta}(u)[e_1]_\beta^\alpha \otimes [e_2]_{\beta'}^{\alpha'} + \frac{1}{2}\sum_{\alpha\neq2,3,\beta=2,3} \\
&\times \Big[b_\alpha^+(u)\Big([e_1]_\beta^\alpha \otimes [e_2]_{\beta'}^{\alpha'} + [e_1]_{\alpha'}^{\beta'} \otimes [e_2]_\alpha^\beta\Big) + b_\alpha^-(u)\Big([e_1]_\beta^\alpha \otimes [e_2]_\beta^{\alpha'} + [e_1]_{\alpha'}^{\beta'} \otimes [e_2]_\alpha^\beta\Big)\Big] \\
&+ \sum_{\alpha=2,3}\Big(c^+(u)[e_1]_\alpha^\alpha \otimes [e_1]_\alpha^{\alpha'} + c^-(u)[e_1]_\alpha^\alpha \otimes [e_2]_\alpha^\alpha + d^+(u)[e_1]_{\alpha'}^\alpha \otimes [e_2]_\alpha^{\alpha'} \\
&+ d^-(u)[e_1]_{\alpha'}^\alpha \otimes [e_2]_{\alpha'}^\alpha\Big)\Bigg\}.
\end{aligned}
\tag{2.7}
$$

Here η is the crossing parameter, α and β take the values from 1 to 4, $\alpha' = 5 - \alpha$, $\beta' = 5 - \beta$, $\bar{\alpha} = 2$ if $\alpha = 1$, $\bar{\alpha} = \frac{5}{2}$ if $\alpha = 2$ or $\alpha = 3$, and $\bar{\alpha} = 3$ if $\alpha = 4$. $[e_k]_\beta^\alpha$ $(k = 1, 2)$ is the 4×4 representation matrix of Weyl basis of k-th space. The coefficients $a_{\alpha\beta}(u)$ are defined as

$$
a_{\alpha\beta}(u) = \begin{cases} (e^{4\eta}e^{2u} - e^{4\eta})(e^{2u} - 1), & \alpha = \beta, \\ (e^{4\eta} - 1)[e^{4\eta}e^{2\eta(\bar{\alpha}-\bar{\beta})}(e^{2u} - 1) - \delta_{\alpha\beta'}(e^{2u} - e^{4\eta})], & \alpha < \beta, \\ (e^{4\eta} - 1)e^{2u}[e^{2\eta(\bar{\alpha}-\bar{\beta})}(e^{2u} - 1) - \delta_{\alpha\beta'}(e^{2u} - e^{4\eta})], & \alpha > \beta, \end{cases}
\tag{2.8}
$$

where $\alpha, \beta \neq 2, 3$. The functions $b_\alpha^\pm(u)$, $c^\pm(u)$ and $d^\pm(u)$ are given by

$$
b_\alpha^\pm(u) = \begin{cases} \pm e^{2\eta(\alpha-1/2)}(e^{4\eta} - 1)(e^{2u} - 1)(e^u \pm e^{2\eta}), & \alpha = 1, \\ e^{2\eta(\alpha-7/2)}(e^{4\eta} - 1)(e^{2u} - 1)e^u(e^u \pm e^{2\eta}), & \alpha = 4, \end{cases}
$$

$$
c^\pm(u) = \pm\frac{1}{2}(e^{4\eta} - 1)(e^{2\eta} + 1)e^u(e^u \mp 1)(e^u \pm e^{2\eta}) + e^{2\eta}(e^{2u} - 1)(e^{2u} - e^{4\eta}),
$$

$$
d^\pm(u) = \pm\frac{1}{2}(e^{4\eta} - 1)(e^{2\eta} - 1)e^u(e^u \pm 1)(e^u \pm e^{2\eta}).
\tag{2.9}
$$

The R-matrix (2.7) has following properties

$$
\begin{aligned}
&\text{Unitarity}: && R_{12}(u)\,R_{21}(-u) = \rho(u) = 16\sinh^2(u - 2\eta)\sinh^2(u + 2\eta), \\
&\text{Initial condition}: && R_{12}(0) = \rho(0)^{\frac{1}{2}}\mathcal{P}_{12}, \\
&\text{Crossing unitarity}: && R_{12}(u)^{t_1} M_1 R_{21}(-u + 4\eta)^{t_1} M_1^{-1} = \rho(u - 2\eta), \\
& && R_{12}(u)^{t_2} M_2^{-1} R_{21}(-u + 4\eta)^{t_2} M_2 = \rho(u - 2\eta),
\end{aligned}
\tag{2.10, 2.11}
$$

where $\mathcal{P}_{12}$ is the permutation operator with the matrix elements $[\mathcal{P}_{12}]^{\alpha\gamma}_{\beta\delta} = \delta_{\alpha\delta}\delta_{\beta\gamma}$, $R_{21}(u) = \mathcal{P}_{12}R_{12}(u)\mathcal{P}_{12}$, t_k denotes the transposition in the k-th space, M_k is the 4×4 diagonal constant matrix

$$M_k = \text{diag}(e^{2\eta}, 1, 1, e^{-2\eta}). \tag{2.12}$$

The solutions of reflection equations (2.5)–(2.6) with fixed R-matrix (2.7) give the reflection matrices $K_k^{\pm}(u)$ defined in the four-dimensional space V_k as [26–29]

$$K_k^+(u) = M_k K_k^-(-u + 2\eta)|_{\{s,s_1,s_2\}\to\{s',s'_1,s'_2\}}, \tag{2.13}$$

$$K_k^-(u) = \begin{pmatrix} k_{11}(u) & k_{12}(u) & k_{13}(u) & k_{14}(u) \\ k_{21}(u) & k_{22}(u) & k_{23}(u) & k_{24}(u) \\ k_{31}(u) & k_{32}(u) & k_{33}(u) & k_{34}(u) \\ k_{41}(u) & k_{42}(u) & k_{43}(u) & k_{44}(u) \end{pmatrix}, \tag{2.14}$$

where $\{s, s_1, s_2\}$ are the free boundary parameters at one end and $\{s', s'_1, s'_2\}$ are the ones at the other end. Here we should note that the reflection equation (2.6) has two inequivalent classes of generic non-diagonal solutions. Without losing generality, we consider one of the generic solutions, whose matrix elements are[1]

$$k_{11}(u) = \frac{1}{2}e^{-u}[\cosh(u-\eta)\sinh(u-2s) - 2s_1s_2\sinh\eta\sinh^2(u)],$$

$$k_{12}(u) = \frac{1}{2}s_1e^{-\frac{u}{2}}\sqrt{\cosh\eta}\sinh(2u)\cosh\frac{1}{2}(u-\eta-2s),$$

$$k_{13}(u) = -\frac{1}{2}s_1e^{-\frac{u}{2}}\sqrt{\cosh\eta}\sinh(2u)\sinh\frac{1}{2}(u-\eta-2s),$$

$$k_{14}(u) = \frac{1}{2}s_1^2\sinh u\sinh(2u),$$

$$k_{21}(u) = \frac{1}{2}s_2e^{-\frac{u}{2}}\sqrt{\cosh\eta}\sinh(2u)\cosh\frac{1}{2}(u-\eta-2s),$$

$$k_{22}(u) = -\frac{1}{2}\cosh u[\sinh u + \cosh\eta\sinh(2s)],$$

$$k_{23}(u) = -\frac{1}{2}\sinh u[\sinh\eta\cosh(2s) + 2s_1s_2\sinh u\cosh(u-\eta)],$$

$$k_{24}(u) = -\frac{1}{2}s_1e^{\frac{u}{2}}\sqrt{\cosh\eta}\sinh(2u)\sinh\frac{1}{2}(u-\eta+2s),$$

$$k_{31}(u) = -\frac{1}{2}s_2e^{-\frac{u}{2}}\sqrt{\cosh\eta}\sinh(2u)\sinh\frac{1}{2}(u-\eta-2s),$$

$$k_{32}(u) = -\frac{1}{2}\sinh u[\sinh\eta\cosh(2s) + 2s_1s_2\sinh(u)\cosh(u-\eta)],$$

$$k_{33}(u) = \frac{1}{2}\cosh u[\sinh u - \cosh\eta\sinh(2s)],$$

$$k_{34}(u) = -\frac{1}{2}s_1e^{\frac{u}{2}}\sqrt{\cosh\eta}\sinh(2u)\cosh\frac{1}{2}(u-\eta+2s),$$

$$k_{41}(u) = \frac{1}{2}s_2^2\sinh u\sinh(2u),$$

$$k_{42}(u) = -\frac{1}{2}s_2e^{\frac{u}{2}}\sqrt{\cosh\eta}\sinh(2u)\sinh\frac{1}{2}(u-\eta+2s),$$

---

[1]The solution is different from that given in [29] even it has the same number of free boundary parameters.

$$k_{43}(u) = -\frac{1}{2}s_2 e^{\frac{u}{2}}\sqrt{\cosh\eta}\sinh(2u)\cosh\frac{1}{2}(u-\eta+2s),$$

$$k_{44}(u) = -\frac{1}{2}e^u[\cosh(u-\eta)\sinh(u+2s)-2s_1 s_2 \sinh\eta\sinh^2(u)]. \qquad (2.15)$$

For the K-matrices $K_k^{\pm}(u)$ given by (2.15) and (2.13) satisfy $tr_0 K_0^+(0) \neq 0$. The Hamiltonian can be given in terms of the transfer matrix by the standard way[2] [10]

$$H = \frac{\partial \ln t(u)}{2\partial u}\Big|_{u=0,\{\theta_j\}=0} - \frac{tr_0 K_0^+(0)'}{2tr_0 K_0^+(0)} = \sum_{k=1}^{N-1}H_{k\,k+1} + \frac{K_N^-(0)'}{2K_N^-(0)} + \frac{tr_0\{K_0^+(0)H_{10}\}}{tr_0 K_0^+(0)}, \qquad (2.16)$$

where $H_{k\,k+1} = \rho(0)^{-1}R_{k\,k+1}(0)\frac{\partial}{\partial u}R_{k\,k+1}(u)\mid_{u=0}$.

Another solution with 3 free boundary parameters is given by (A.1) below, which agrees with that obtained in [29]. It is easy to check that $K^+(u)$ and $K^-(u)$ can not be diagonalized simultaneously for generic choices of 6 boundary parameters. Then the traditional algebraic Bethe ansatz can not be applied to solve the eigenvalues of transfer matrix (2.1) because of the absence of an obvious reference state [30].

3 Factorization of the reflection matrices

To obtain the eigenvalues of the transfer matrix (2.1), we first consider the decomposition of space. The four-dimensional space can be regarded as the tensor of two two-dimensional spaces. For example, $V_1 = V_{1'} \otimes V_{2'}$ and $V_2 = V_{3'} \otimes V_{4'}$. Then the R-matrix (2.7) can be factorized as the product of R-matrices of the anisotropic XXZ spin chain with suitable global transformation [1, 15, 25, 31]

$$R_{12}(u) = 2^4[S\otimes S]\tilde{R}_{1'4'}(u+i\pi)\tilde{R}_{1'3'}(u)\tilde{R}_{2'4'}(u)\tilde{R}_{2'3'}(u-i\pi)[S\otimes S]^{-1}, \qquad (3.1)$$

$$R_{21}(u) = 2^4[S\otimes S]\tilde{R}_{3'2'}(u+i\pi)\tilde{R}_{4'2'}(u)\tilde{R}_{3'1'}(u)\tilde{R}_{4'1'}(u-i\pi)[S\otimes S]^{-1}, \qquad (3.2)$$

where the transformation matrix S is

$$S = S^{-1} = \begin{pmatrix} 1 & & & \\ & \frac{\cosh\frac{\eta}{2}}{\sqrt{\cosh\eta}} & -\frac{\sinh\frac{\eta}{2}}{\sqrt{\cosh\eta}} & \\ & -\frac{\sinh\frac{\eta}{2}}{\sqrt{\cosh\eta}} & -\frac{\cosh\frac{\eta}{2}}{\sqrt{\cosh\eta}} & \\ & & & 1 \end{pmatrix}, \qquad (3.3)$$

and the R-matrix reads

$$\tilde{R}_{1'2'}(u) = \begin{pmatrix} \sinh(-\frac{u}{2}+\eta) & & & \\ & \sinh\frac{u}{2} & e^{-\frac{u}{2}}\sinh\eta & \\ & e^{\frac{u}{2}}\sinh\eta & \sinh\frac{u}{2} & \\ & & & \sinh(-\frac{u}{2}+\eta) \end{pmatrix}. \qquad (3.4)$$

---

[2]It is remarked that one can define the Hamiltonian by (2.16) for the case of $t(0) \neq 0$ (i.e., $tr_0 K_0^+(0) \neq 0$), however for the case of $t(0) = 0$ one needs to adopt other way [1] instead to construct a meaningful Hamiltonian.

The R-matrix (3.4) has following properties

$$
\begin{aligned}
\text{Quasi}-\text{period}: \quad & \tilde{R}_{1'2'}(u+2i\pi) = -\tilde{R}_{1'2'}(u), \\
\text{PT}-\text{symmetry}: \quad & \tilde{R}_{1'2'}^{t_{1'}t_{2'}}(u) = \tilde{R}_{1'2'}(u), \\
\text{Unitarity}: \quad & \tilde{R}_{1'2'}(u)\,\tilde{R}_{2'1'}(-u) = \rho_s(u) = \sinh\left(-\frac{u}{2}+\eta\right)\sinh\left(\frac{u}{2}+\eta\right), \\
\text{Initial condition}: \quad & \tilde{R}_{1'2'}(0) = \rho_s(0)^{\frac{1}{2}}\tilde{\mathcal{P}}_{1'2'}, \\
\text{Crossing unitarity}: \quad & \tilde{R}_{1'2'}(u)^{t_{1'}}\tilde{M}_{1'}\tilde{R}_{2'1'}(-u+4\eta)^{t_{1'}}\tilde{M}_{1'}^{-1} = \rho_s(u-2\eta), \quad (3.5)
\end{aligned}
$$

where $\tilde{\mathcal{P}}_{1'2'}$ is the permutation operator defined in the tensor space $V_{1'}\otimes V_{2'}$, $t_{k'}$ ($k'=1',2'$) denotes the transposition in the k'-th subspace, and $\tilde{M}_{k'}$ is the diagonal constant matrix with the form of $\tilde{M}_{k'} = \text{diag}(e^\eta, e^{-\eta})$. Besides, the R-matrix (3.4) also satisfies the Yang-Baxter equation

$$
\tilde{R}_{12}(u-v)\tilde{R}_{13}(u)\tilde{R}_{23}(v) = \tilde{R}_{23}(v)\tilde{R}_{13}(u)\tilde{R}_{12}(u-v). \quad (3.6)
$$

The very factorization (3.1)–(3.2) of the R-matrices allows us, after a tedious calculation, to have that the reflection matrices (2.13)–(2.14) with the elements (2.15) can be expressed in terms of the factorization form as

$$
\begin{aligned}
K_1^+(u) &= [\rho_s(i\pi)]^{-\frac{1}{2}}S\tilde{R}_{2'1'}(i\pi)\tilde{K}_{2'}^+(u)\tilde{M}_{2'}^{-1}\tilde{R}_{1'2'}(-2u+4\eta-i\pi)\tilde{M}_{2'}\tilde{K}_{1'}^+(u)S^{-1}, \\
K_1^-(u) &= [\rho_s(i\pi)]^{-\frac{1}{2}}S\tilde{K}_{1'}^-(u+i\pi)\tilde{R}_{2'1'}(2u+i\pi)\tilde{K}_{2'}^-(u)\tilde{R}_{1'2'}(-i\pi)S^{-1}, \quad (3.7)
\end{aligned}
$$

where $\tilde{K}_{k'}^{\pm}(u)$ are the 2×2 generic non-diagonal reflection matrices of the XXZ spin chain

$$
\tilde{K}_{k'}^+(u) = \tilde{M}_{k'}\,\tilde{K}_{k'}^-(-u+2\eta)|_{\{s,s_1,s_2\}\to\{s',s_1',s_2'\}}, \quad (3.8)
$$

$$
\tilde{K}_{k'}^-(u) = \begin{pmatrix} -e^{-\frac{u}{2}}\sinh(\frac{u}{2}-s) & s_1\sinh u \\ s_2\sinh u & e^{\frac{u}{2}}\sinh(\frac{u}{2}+s) \end{pmatrix}, \quad (3.9)
$$

which satisfy the reflection equations

$$
\begin{aligned}
\tilde{R}_{1'2'}(-u+v)\tilde{K}_{1'}^+(u)&\tilde{M}_{1'}^{-1}\tilde{R}_{2'1'}(-u-v+4\eta)\tilde{M}_{1'}\tilde{K}_{2'}^+(v) \\
&= \tilde{K}_{2'}^+(v)\tilde{M}_{1'}\tilde{R}_{1'2'}(-u-v+4\eta)\tilde{M}_{1'}^{-1}\tilde{K}_{1'}^+(u)\tilde{R}_{2'1'}(-u+v), \quad (3.10) \\
\tilde{R}_{1'2'}(u-v)\tilde{K}_{1'}^-(u)\tilde{R}_{2'1'}(u+v)&\tilde{K}_{2'}^-(v) = \tilde{K}_{2'}^-(v)\tilde{R}_{1'2'}(u+v)\tilde{K}_{1'}^-(u)\tilde{R}_{2'1'}(u-v). \quad (3.11)
\end{aligned}
$$

Some remarks are in order. The boundary parameters s, $s_1\,s_2$ are the same as those of (2.13)–(2.14). Here it should also be addressed that when $\tilde{K}^-(u)=1$ in (3.9), the resulting $K^-(u)$ given by (3.7) is just that discussed in refernce [25] with $\epsilon=0$. When $s_1=s_2=0$ in (3.9), the resulting $K^-(u)$ given by (3.7) is the second case discussed in [12]. Due to the fact that $\tilde{K}^\pm(u)$ are all diagonal ones,[3] it is only special cases that one can adopt coordinate/algebraic Bethe ansatz to solve the corresponding $D_2^{(2)}$ model [15, 25].

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[3]However, the resulting $K^\pm(u)$ obtained by the relation (3.7) even have non-diagonal matrix elements [12, 25].

Based on the R-matrix (3.4) and reflection matrices (3.8)–(3.9), we construct the transfer matrix $\tilde{t}(u)$ of the inhomogeneous XXZ spin chain as

$$\tilde{t}(u) = tr_{0'}\{\tilde{K}_{0'}^+(u)\tilde{T}_{0'}(u)\tilde{K}_{0'}^-(u)\hat{\tilde{T}}_{0'}(u)\}, \tag{3.12}$$

where $0'$ means the auxiliary space, $\tilde{T}_{0'}(u)$ and $\hat{\tilde{T}}_{0'}(u)$ are the monodromy matrices

$$\tilde{T}_{0'}(u) = \tilde{R}_{0'1'}(u-\theta_1)\tilde{R}_{0'2'}(u-\theta_2)\cdots\tilde{R}_{0'(2N)'}(u-\theta_{2N}),$$
$$\hat{\tilde{T}}_{0'}(u) = \tilde{R}_{0'(2N)'}(u+\theta_{2N})\tilde{R}_{0'(2N-1)'}(u+\theta_{2N-1})\cdots\tilde{R}_{0'1'}(u+\theta_1), \tag{3.13}$$

and $\{\theta_j | j = 1, \cdots, 2N\}$ are the inhomogeneous parameters. We should note that the quantum space of transfer matrix $\tilde{t}(u)$ for the XXZ spin chain and that of $t(u)$ for the $D_2^{(2)}$ model are the same. Thus the number of sites in eq. (3.13) is extended to $2N$ to ensure the dimension of Hilbert space is 4^N. The monodromy matrices (3.13) satisfy the Yang-Baxter relations

$$\tilde{R}_{0'0''}(u-v)\tilde{T}_{0'}(u)\tilde{T}_{0''}(v) = \tilde{T}_{0''}(v)\tilde{T}_{0'}(u)\tilde{R}_{0'0''}(u-v),$$
$$\tilde{R}_{0''0'}(u-v)\hat{\tilde{T}}_{0'}(u)\hat{\tilde{T}}_{0''}(v) = \hat{\tilde{T}}_{0''}(v)\hat{\tilde{T}}_{0''}(u)\tilde{R}_{0''0'}(u-v). \tag{3.14}$$

From the reflection equations (3.10)–(3.11) and Yang-Baxter relation (3.14), we can prove that the transfer matrices $\tilde{t}(u)$ with different spectral parameters commutate with each other

$$[\tilde{t}(u), \tilde{t}(v)] = 0. \tag{3.15}$$

Interestingly, we find that if the inhomogeneous parameters are staggered, i.e., $\theta_j = 0$ for the odd j and $\theta_j = i\pi$ for the even j, the transfer matrix (2.1) of the $D_2^{(2)}$ spin chain can be factorized as the product of transfer matrices of two staggered XXZ spin chains with fixed spectral difference

$$t(u) = 2^{8N}\rho_s(2u+i\pi-2\eta)\,\tilde{t}_s(u+i\pi)\,\tilde{t}_s(u), \tag{3.16}$$

where $\tilde{t}_s(u) = \tilde{t}(u)|_{\{\theta_j\}=\{0,i\pi\}}$. The proof is as follows. For simplicity, we denote

$$\tilde{T}_{0'}^s(u) = \tilde{T}_{0'}(u)|_{\{\theta_j\}=\{0,i\pi\}}, \quad \hat{\tilde{T}}_{0'}^s(u) = \hat{\tilde{T}}_{0'}(u)|_{\{\theta_j\}=\{0,i\pi\}}. \tag{3.17}$$

From the direct calculation, we have

$$\tilde{t}_s(u+i\pi)\tilde{t}_s(u)$$
$$= [\rho_s(2u+i\pi-2\eta)]^{-1}tr_{0'0''}\Big\{\tilde{K}_{0''}^+(u)\tilde{M}_{0''}^{-1}\tilde{R}_{0'0''}(-2u+4\eta-i\pi)\tilde{M}_{0''}$$
$$\times \tilde{K}_{0'}^+(u)\tilde{T}_{0'}^s(u+i\pi)\tilde{T}_{0''}^s(u)\tilde{K}_{0'}^-(u+i\pi)\tilde{R}_{0''0'}(2u+i\pi)\tilde{K}_{0''}^-(u)\hat{\tilde{T}}_{0'}^s(u+i\pi)\hat{\tilde{T}}_{0''}^s(u)\Big\}. \tag{3.18}$$

By using the Yang-Baxter equation (3.6), we obtain

$$\tilde{R}_{l'0'}(u+2i\pi)\tilde{R}_{l'0''}(u+i\pi)\tilde{R}_{0'0''}(-i\pi)\tilde{R}_{0''0'}(i\pi)\tilde{R}_{j'0'}(u+i\pi)\tilde{R}_{j'0''}(u)$$
$$= \tilde{R}_{0'0''}(-i\pi)\tilde{R}_{l'0''}(u+i\pi)\tilde{R}_{l'0'}(u+2i\pi)\tilde{R}_{j'0''}(u)\tilde{R}_{j'0'}(u+i\pi)\tilde{R}_{0''0'}(i\pi), \tag{3.19}$$

which gives the identity

$$\hat{\tilde{T}}_{0'}^{s}(u+i\pi)\hat{\tilde{T}}_{0''}^{s}(u) = [\rho_s(i\pi)]^{-1}\tilde{R}_{0'0''}(-i\pi)\hat{\tilde{T}}_{0''}^{s}(u)\hat{\tilde{T}}_{0'}^{s}(u+i\pi)\tilde{R}_{0''0'}(i\pi). \tag{3.20}$$

Substituting eq. (3.20) into (3.18), we have

$$\begin{aligned}
\tilde{t}_s(u+i\pi)\,\tilde{t}_s(u) &= [\rho_s(2u+i\pi-2\eta)\rho_s(i\pi)]^{-1}tr_{0'0''}\Big\{\tilde{R}_{0''0'}(i\pi)\tilde{K}_{0''}^{+}(u)\tilde{M}_{0''}^{-1} \\
&\quad\times \tilde{R}_{0'0''}(-2u+4\eta-i\pi)\tilde{M}_{0''}K_{0'}^{+}(u)\tilde{T}_{0'}^{s}(u+i\pi)\tilde{T}_{0''}^{s}(u) \\
&\quad\times \tilde{K}_{0'}^{-}(u+i\pi)\tilde{R}_{0''0'}(2u+i\pi)\tilde{K}_{0''}^{-}(u)\tilde{R}_{0'0''}(-i\pi)\hat{\tilde{T}}_{0''}^{s}(u)\hat{\tilde{T}}_{0'}^{s}(u+i\pi)\Big\} \\
&= 2^{-8N}[\rho_s(2u+i\pi-2\eta)]^{-1}\mathcal{S}^{-1}t(u)\mathcal{S},
\end{aligned}$$

where $\mathcal{S} = S\otimes S\otimes\ldots\otimes S$. Then we arrive at the conclusion (3.16).

4 Exact solution

Now, we derive the eigenvalue of transfer matrix $t(u)$ of the $D_2^{(2)}$ spin chain based on the factorization identity (3.16). According to eq. (3.15), we know that $\tilde{t}_s(u+i\pi)$ and $\tilde{t}_s(u)$ have common eigenstates. Acting eq. (3.16) on a common eigenstate, we obtain

$$\Lambda(u) = 2^{8N}\rho_s(2u+i\pi-2\eta)\,\tilde{\Lambda}_s(u+i\pi)\,\tilde{\Lambda}_s(u), \tag{4.1}$$

where $\Lambda(u)$, $\tilde{\Lambda}_s(u+i\pi)$ and $\tilde{\Lambda}_s(u)$ are the eigenvalues of the transfer matrices $t(u)$, $\tilde{t}_s(u+i\pi)$ and $\tilde{t}_s(u)$, respectively.

In order to obtain the eigenvalue of transfer matrix $\tilde{t}_s(u)$ of the staggered XXZ spin chain, we should diagonalize the transfer matrix $\tilde{t}(u)$ of the inhomogeneous XXZ spin chain first. The method is fusion [32–37]. The main idea of fusion is that the R-matrix at the some special points can degenerate into the projector operators. For the present case, at the point of $u = 2\eta$, the R-matrix (3.4) degenerates into

$$\tilde{R}_{1'2'}(2\eta) = P_{1'2'}^{(1)}S_{1'2'}^{(1)}, \tag{4.2}$$

where $S_{1'2'}^{(1)}$ is an irrelevant constant matrix omitted here, $P_{1'2'}^{(1)}$ is the one-dimensional projector operator

$$P_{1'2'}^{(1)} = |\psi_0\rangle\langle\psi_0|, \quad |\psi_0\rangle = \frac{1}{\sqrt{2\cosh\eta}}\left(e^{-\frac{\eta}{2}}|12\rangle + e^{\frac{\eta}{2}}|21\rangle\right), \tag{4.3}$$

and $\{|1\rangle,|2\rangle\}$ are the orthogonal bases of the 2-dimensional linear space $V_{1'}$ (or $V_{2'}$). From the Yang-Baxter equation (3.6) and using the properties of projector, we obtain

$$\begin{aligned}
P_{2'1'}^{(1)}\tilde{R}_{1'3'}(u)\tilde{R}_{2'3'}(u+2\eta)P_{2'1'}^{(1)} &= -\sinh\left(\frac{u}{2}+\eta\right)\sinh\left(\frac{u}{2}-\eta\right), \\
P_{1'2'}^{(1)}\tilde{R}_{3'1'}(u)\tilde{R}_{3'2'}(u+2\eta)P_{1'2'}^{(1)} &= -\sinh\left(\frac{u}{2}+\eta\right)\sinh\left(\frac{u}{2}-\eta\right).
\end{aligned} \tag{4.4}$$

Based on the reflections (3.10)–(3.11), the fusion of reflection matrices gives

$$P^{(1)}_{1'2'}\tilde{K}^+_{2'}(u+2\eta)\tilde{M}_{1'}\tilde{R}_{1'2'}(-2u+2\eta)\tilde{M}^{-1}_{1'}\tilde{K}^+_{1'}(u)P^{(1)}_{2'1'}$$
$$= 2\sinh(u-2\eta)\frac{1}{\alpha'}\cosh\frac{u+\alpha'_1}{2}\cosh\frac{u-\alpha'_1}{2}\cosh\frac{u+\alpha'_2}{2}\cosh\frac{u-\alpha'_2}{2}, \qquad (4.5)$$

$$P^{(1)}_{2'1'}\tilde{K}^-_{1'}(u)\tilde{R}_{2'1'}(2u+2\eta)\tilde{K}^-_{2'}(u+2\eta)P^{(1)}_{1'2'}$$
$$= -2\sinh(u+2\eta)\frac{1}{\alpha}\cosh\frac{u+\alpha_1}{2}\cosh\frac{u-\alpha_1}{2}\cosh\frac{u+\alpha_2}{2}\cosh\frac{u-\alpha_2}{2}, \qquad (4.6)$$

where the related constants are defined as

$$\alpha = \frac{1}{2s_1 s_2}, \qquad \beta = \sqrt{\frac{8s_1 s_2\cosh(2s)+16(s_1 s_2)^2+1}{16(s_1 s_2)^2}}, \quad \cosh\alpha_1 = \frac{\alpha}{2}+\beta,$$

$$\cosh\alpha_2 = \frac{\alpha}{2}-\beta, \qquad \alpha' = \frac{1}{2s'_1 s'_2}, \quad \beta' = \sqrt{\frac{8s'_1 s'_2\cosh(2s')+16(s'_1 s'_2)^2+1}{16(s'_1 s'_2)^2}},$$

$$\cosh\alpha'_1 = \frac{\alpha'}{2}+\beta', \quad \cosh\alpha'_2 = \frac{\alpha'}{2}-\beta'.$$

The Yang-Baxter relations (3.14) at certain points give

$$\tilde{T}_{0'}(\theta_j)\tilde{T}_{0''}(\theta_j+2\eta) = P^{(1)}_{0''0'}\tilde{T}_{0'}(\theta_j)\tilde{T}_{0''}(\theta_j+2\eta),$$
$$\hat{\tilde{T}}_{0'}(-\theta_j)\hat{\tilde{T}}_{0''}(-\theta_j+2\eta) = P^{(1)}_{0'0''}\hat{\tilde{T}}_{0'}(-\theta_j)\hat{\tilde{T}}_{0''}(-\theta_j+2\eta), \qquad (4.7)$$

which show two ways to generate the projector operator in the transfer matrix.

Considering the physical quantity $\tilde{t}(\pm\theta_j)\tilde{t}(\pm\theta_j+2\eta)$ and using the fusion relations (4.4)–(4.7), we obtain

$$\tilde{t}(\pm\theta_j)\tilde{t}(\pm\theta_j+2\eta) = \frac{4\sinh(\pm\theta_j-2\eta)\sinh(\pm\theta_j+2\eta)}{\alpha\alpha'\sinh(\pm\theta_j-\eta)\sinh(\pm\theta_j+\eta)}\cosh\frac{\pm\theta_j-\alpha_1}{2}$$
$$\times\cosh\frac{\pm\theta_j+\alpha_1}{2}\cosh\frac{\pm\theta_j-\alpha_2}{2}\cosh\frac{\pm\theta_j+\alpha_2}{2}\cosh\frac{\pm\theta_j-\alpha'_1}{2}\cosh\frac{\pm\theta_j+\alpha'_1}{2}$$
$$\times\cosh\frac{\pm\theta_j-\alpha'_2}{2}\cosh\frac{\pm\theta_j+\alpha'_2}{2}\prod_{i=1}^{M}\sinh\frac{\pm\theta_j-\theta_i-2\eta}{2}$$
$$\times\sinh\frac{\pm\theta_j-\theta_i+2\eta}{2}\sinh\frac{\pm\theta_j+\theta_i-2\eta}{2}\sinh\frac{\pm\theta_j+\theta_i+2\eta}{2}, \; j=1,\cdots,2N. \qquad (4.8)$$

We see that the product of two transfer matrices with fixed spectral parameters is a c-number equaling to the quantum determinant at the point of $u = \theta_j$. We shall note that the fusion identities (4.8) hold only at the discrete inhomogeneous points. Besides, from the direct calculation and using the properties (3.5), we also obtain the values of $\tilde{t}(u)$ at the points of $u = 0, 2\eta, i\pi$ as

$$\tilde{t}(0) = \tilde{t}(2\eta) = 2\cosh\eta\sinh s\sinh s'\prod_{j=1}^{2N}\rho_s(\theta_j),$$

$$\tilde{t}(i\pi) = 2\cosh\eta\cosh s\cosh s'\prod_{j=1}^{2N}\rho_s(\theta_j+i\pi). \qquad (4.9)$$

The asymptotic behavior of $\tilde{t}(u)$ when the spectral parameter tends to infinity reads

$$\tilde{t}(u)|_{u\to\pm\infty} = -2^{-4N-2}e^{\pm[(2N+2)(u-\eta)]}(e^{-\eta}s_1 s_2' + e^{\eta}s_2 s_1'). \qquad (4.10)$$

From the definition (3.12), we know that the transfer matrix $\tilde{t}(u)$ is an operator polynomial of e^u with the degree $4N + 4$, which can be completely determined by $4N + 5$ constraints. Thus the above $4N$ fusion identities (4.8) and 5 additional conditions (4.9)–(4.10) give us sufficient information to determine the eigenvalue $\tilde{\Lambda}(u)$ of $\tilde{t}(u)$. After some algebras, we express the eigenvalue $\tilde{\Lambda}(u)$ as the inhomogeneous $T - Q$ relation

$$\begin{aligned}
\tilde{\Lambda}(u) = {} & \frac{2\sinh(u-2\eta)}{\sinh(u-\eta)\sqrt{\alpha\alpha'}}\cosh\frac{u+\alpha_1}{2}\cosh\frac{u+\alpha_2}{2}\cosh\frac{u+\alpha_1'}{2}\cosh\frac{u+\alpha_2'}{2}a(u)\frac{Q(u+2\eta)}{Q(u)} \\
& + \frac{2\sinh u}{\sinh(u-\eta)\sqrt{\alpha\alpha'}}\cosh\frac{u-2\eta-\alpha_1}{2}\cosh\frac{u-2\eta-\alpha_2}{2}\cosh\frac{u-2\eta-\alpha_1'}{2} \\
& \times \cosh\frac{u-2\eta-\alpha_2'}{2}d(u)\frac{Q(u-2\eta)}{Q(u)} + x\sinh u\sinh(u-2\eta)\frac{a(u)d(u)}{Q(u)},
\end{aligned} \qquad (4.11)$$

where the functions $Q(u)$, $a(u)$, $d(u)$ and parameter x are

$$\begin{aligned}
Q(u) & = \prod_{l=1}^{2N}\sinh\frac{1}{2}(u-\mu_l)\sinh\frac{1}{2}(u+\mu_l-2\eta), \\
a(u) & = \prod_{j=1}^{2N}\sinh\frac{1}{2}(u-\theta_j-2\eta)\sinh\frac{1}{2}(u+\theta_j-2\eta) = d(u-2\eta), \\
x & = -2\sqrt{s_1 s_2 s_1' s_2'}\cosh\left[(2N+1)\eta+\frac{\alpha_1+\alpha_2+\alpha_1'+\alpha_2'}{2}\right] - (e^{-\eta}s_1 s_2' + e^{\eta}s_2 s_1'). \qquad (4.12)
\end{aligned}$$

Because $\tilde{\Lambda}(u)$ is a polynomial, the singularities of right hand side of eq. (4.11) should be cancelled with each other, which gives that the Bethe roots $\{\mu_l\}$ should satisfy the Bethe ansatz equations

$$\begin{aligned}
& \frac{2\sinh(\mu_l-2\eta)}{\sinh(\mu_l-\eta)\sqrt{\alpha\alpha'}}\cosh\frac{\mu_l+\alpha_1}{2}\cosh\frac{\mu_l+\alpha_2}{2}\cosh\frac{\mu_l+\alpha_1'}{2}\cosh\frac{\mu_l+\alpha_2'}{2}\frac{Q(\mu_l+2\eta)}{d(\mu_l)} \\
& + \frac{2\sinh\mu_l}{\sinh(\mu_l-\eta)\sqrt{\alpha\alpha'}}\cosh\frac{\mu_l-2\eta-\alpha_1}{2}\cosh\frac{\mu_l-2\eta-\alpha_2}{2}\cosh\frac{\mu_l-2\eta-\alpha_1'}{2} \\
& \times \cosh\frac{\mu_l-2\eta-\alpha_2'}{2}\frac{Q(\mu_l-2\eta)}{a(\mu_l)} = -x\sinh\mu_l\sinh(\mu_l-2\eta), \quad l=1,\cdots,2N. \qquad (4.13)
\end{aligned}$$

Some remarks are in order. By solving the algebraic equations (4.13), we obtain the values of Bethe roots $\{\mu_l\}$. Substituting these values into the inhomogeneous $T - Q$ relation (4.11), we obtain the eigenvalue $\tilde{\Lambda}(u)$. The different sets of Bethe roots would give different eigenvalues. As shown in [38, 39], based on the numerical calculation and analytical analysis with the help of Bézout theorem, the $T - Q$ relation (4.11) can generate all the eigenvalues of $\tilde{t}(u)$. The eigenvalue $\tilde{\Lambda}(u)$ has the well-defined quasi-inhomogeneous limit $\{\theta_j\} = \{0, i\pi\}$. Substituting

$$\tilde{\Lambda}_s(u) = \tilde{\Lambda}(u)|_{\{\theta_j\}=\{0, i\pi\}}, \quad \tilde{\Lambda}_s(u+i\pi) = \tilde{\Lambda}(u+i\pi)|_{\{\theta_j\}=\{0, i\pi\}}, \qquad (4.14)$$

into eq. (4.1), we then are able to obtain the eigenvalue $\Lambda(u)$ of the transfer matrix $t(u)$ of the $D_2^{(2)}$ spin chain associated with the most generic non-diagonal K-matrices $K^{\pm}(u)$ given by (2.13)–(2.15). Therefore, the expression (4.1) gives the complete spectrum of the system via the relation (2.16).

5 Discussion

In this paper, we have studied the exact solutions of one-dimensional quantum integrable system connected with the twisted $D_2^{(2)}$ quantum algebra in the generic open boundary conditions, where the reflection matrices have non-diagonal elements. We find that the generating functional of the model can be factorized as the product of transfer matrices of two XXZ spin chains with staggered inhomogeneous parameters. Based on these factorization identities and using the method of fusion, we obtain the eigenvalues and corresponding Bethe ansatz equations of the model.

Based on the obtained eigenvalues, the eigenstate of the $D_2^{(2)}$ model can be retrieved by using the separation of variables [40–43] or the off-diagonal Bethe ansatz [44]. Then the correlation functions, norm, form factors and other interesting scalar products can be calculated. Staring from the obtained Bethe ansatz equations and using the finite size scaling analysis of the contribution of inhomogeneous term in the $T - Q$ relation (4.11), the physical quantities such as ground state energy density, surface energy and elementary excitations in the thermodynamic limit could also be studied. The results given in this paper are the foundations to exactly solve the high rank $D_n^{(2)}$ model by using the analytical methods such as the nested off-diagonal Bethe ansatz [30].

Acknowledgments

A Another non-diagonal boundary reflection

The reflection equation (2.6) has another inequivalent generic non-diagonal solution where the matrix elements are

$$k_{11}(u) = \frac{1}{2}e^{-u}\left[\sinh(u-\eta)\cosh(u-2s) - 2s_1 s_2 \sinh\eta\cosh^2 u\right],$$

$$k_{12}(u) = -\frac{1}{2}s_1 e^{-\frac{u}{2}+\frac{i\pi}{4}}\sqrt{\cosh\eta}\sinh(2u)\sinh\frac{1}{2}\left(u-\eta-2s+\frac{i\pi}{2}\right),$$

$$k_{13}(u) = \frac{1}{2}s_1 e^{-\frac{u}{2}+\frac{i\pi}{4}}\sqrt{\cosh\eta}\sinh(2u)\cosh\frac{1}{2}\left(u-\eta-2s+\frac{i\pi}{2}\right),$$

$$k_{14}(u) = -\frac{1}{2}s_1^2 \cosh u \sinh(2u),$$

$$k_{21}(u) = \frac{1}{2}s_2 e^{-\frac{u}{2}+\frac{i\pi}{4}}\sqrt{\cosh\eta}\sinh(2u)\cosh\frac{1}{2}\left(u-\eta-2s+\frac{i\pi}{2}\right),$$

$$k_{22}(u) = -\frac{1}{2}\cosh u \left[\sinh\eta\cosh(2s) - 2s_1 s_2 \cosh u \sinh(u-\eta)\right],$$

$$k_{23}(u) = \frac{1}{2}\sinh u \left[\cosh\eta\sinh(2s) - \sinh\left(u+\frac{i\pi}{2}\right)\right],$$

$$k_{24}(u) = \frac{1}{2}s_1 e^{\frac{u}{2}+\frac{i\pi}{4}}\sqrt{\cosh\eta}\sinh(2u)\cosh\frac{1}{2}\left(u-\eta+2s-\frac{i\pi}{2}\right),$$

$$k_{31}(u) = -\frac{1}{2}s_2 e^{-\frac{u}{2}+\frac{i\pi}{4}}\sqrt{\cosh\eta}\sinh(2u)\sinh\frac{1}{2}\left(u-\eta-2s+\frac{i\pi}{2}\right),$$

$$k_{32}(u) = -\frac{1}{2}\sinh u \left[\cosh\eta\sinh(2s) - \sinh\left(u+\frac{i\pi}{2}\right)\right],$$

$$k_{33}(u) = -\frac{1}{2}\cosh u \left[\sinh\eta\cosh(2s) - 2s_1 s_2 \cosh u \sinh(u-\eta)\right],$$

$$k_{34}(u) = \frac{1}{2}s_1 e^{\frac{u}{2}+\frac{i\pi}{4}}\sqrt{\cosh\eta}\sinh(2u)\sinh\frac{1}{2}(u-\eta+2s-\frac{i\pi}{2}),$$

$$k_{41}(u) = -\frac{1}{2}s_2^2 \cosh u \sinh(2u),$$

$$k_{42}(u) = \frac{1}{2}s_2 e^{\frac{u}{2}+\frac{i\pi}{4}}\sqrt{\cosh\eta}\sinh(2u)\sinh\frac{1}{2}\left(u-\eta+2s-\frac{i\pi}{2}\right),$$

$$k_{43}(u) = \frac{1}{2}s_2 e^{\frac{u}{2}+\frac{i\pi}{4}}\sqrt{\cosh\eta}\sinh(2u)\cosh\frac{1}{2}\left(u-\eta+2s-\frac{i\pi}{2}\right),$$

$$k_{44}(u) = \frac{1}{2}e^{u}\left[\sinh(u-\eta)\cosh(u+2s) - 2s_1 s_2 \sinh\eta\cosh^2 u\right]. \tag{A.1}$$

In this case, the reflection matrices $K_1^{\pm}(u)$ of the $D_2^{(2)}$ spin chain can be also factorized as the product of reflection matrices $\tilde{K}_{1',2'}^{\pm}(u)$ of the XXZ spin chain by a different way from those of (3.7)

$$K_1^+(u) = S\tilde{\mathcal{P}}_{1'2'}\tilde{K}_{2'}^+\left(u+\frac{i\pi}{2}\right)\tilde{M}_{2'}^{-1}\tilde{R}_{1'2'}(-2u+4\eta-2i\pi)\tilde{M}_{2'}\tilde{K}_{1'}^+\left(u+\frac{3i\pi}{2}\right)S^{-1},$$

$$K_1^-(u) = S\tilde{K}_{1'}^-\left(u+\frac{3i\pi}{2}\right)\tilde{R}_{2'1'}(2u+2i\pi)\tilde{K}_{2'}^-\left(u+\frac{i\pi}{2}\right)\tilde{\mathcal{P}}_{1'2'}S^{-1}, \tag{A.2}$$

where the permutation operator $\tilde{\mathcal{P}}_{1'2'}$ is included. Here it should be addressed that when $\tilde{K}^-(u) = 1$ in (3.9), the resulting $K^-(u)$ given by (A.2) is that discussed in [25] with $\epsilon = 1$.

When $s_1 = s_2 = 0$ in (3.9), the resulting $K^-(u)$ given by (A.2) is the third case discussed in [12]. For $K^-(u)$ defined by (A.1), the corresponding $K^+(u)$ given by (2.13) indeed satisfies $tr_0 K_0^+(0) = 0$. For this case one has to, instead of (2.16), take the second order derivative of the transfer matrix to construct a meaningful Hamiltonian [1].

Motivated by the factorization (A.2), we construct the transfer matrix of the related XXZ spin chain as

$$\bar{t}(u) = tr_{0'} \left\{ \tilde{K}_{0'}^+ \left(u + \frac{i\pi}{2} \right) \tilde{T}_{0'}(u) \tilde{K}_{0'}^- \left(u + \frac{i\pi}{2} \right) \hat{\tilde{T}}_{0'}(u + i\pi) \right\}. \tag{A.3}$$

The transfer matrix $\bar{t}(u)$ can be obtained from $\tilde{t}(u)$ (3.12) by the mapping

$$\bar{t}(u) = \tilde{t}(u)|_{u \to u + \frac{i\pi}{2}, \{\theta_j \to \theta_j + \frac{i\pi}{2}\}}. \tag{A.4}$$

After some algebraic calculation, we find that if the inhomogeneous parameters in eq. (A.3) are staggered, i.e., $\theta_j = \frac{i\pi}{2}$ for odd j and $\theta_j = \frac{3i\pi}{2}$ for even j, the transfer matrix $t(u)$ of $D_2^{(2)}$ spin chain can be factorized as the product of transfer matrices of two staggered XXZ spin chains with fixed spectral difference

$$t(u) = 2^{8N} \rho_s(2u + 2i\pi - 2\eta) \, \bar{t}_s(u + i\pi) \, \bar{t}_s(u), \tag{A.5}$$

where

$$\bar{t}_s(u) = \bar{t}(u)|_{\{\theta_j\} = \{i\pi/2, 3i\pi/2\}}. \tag{A.6}$$

The proof is as follows. From the definition (A.6), we readily have

$$
\begin{aligned}
\bar{t}_s(u + i\pi) \, \bar{t}_s(u) = [\rho_s(2u + 2i\pi - 2\eta)]^{-1} tr_{0'0''} \bigg\{ & \tilde{K}_{0''}^+ \left(u + \frac{i\pi}{2} \right) \tilde{M}_{0''}^{-1} \\
& \times \tilde{R}_{0'0''}(-2u + 4\eta - 2i\pi) \tilde{M}_{0''} \tilde{K}_{0'} \left(u + \frac{3i\pi}{2} \right) \tilde{T}_{0'}^s(u + i\pi) \tilde{T}_{0''}^s(u) \tilde{K}_{0'}^- \left(u + \frac{3i\pi}{2} \right) \\
& \times \tilde{R}_{0''0'}(2u + 2i\pi) \tilde{K}_{0''}^- \left(u + \frac{i\pi}{2} \right) \hat{\tilde{T}}_{0'}^s(u + 2i\pi) \hat{\tilde{T}}_{0''}^s(u + i\pi) \bigg\}.
\end{aligned}
\tag{A.7}
$$

By using the property $\tilde{\mathcal{P}}_{0''0'}^2 = 1$, we obtain

$$
\begin{aligned}
& \tilde{R}_{j'0'}(u + 3i\pi) \tilde{R}_{l'0'}(u + 2i\pi) \tilde{R}_{j'0''}(u + 2i\pi) \tilde{R}_{l'0''}(u + i\pi) \\
& = \tilde{\mathcal{P}}_{0''0'} \tilde{R}_{j'0''}(u + 3i\pi) \tilde{R}_{l'0''}(u + 2i\pi) \tilde{R}_{j'0'}(u + 2i\pi) \tilde{R}_{l'0'}(u + i\pi) \tilde{\mathcal{P}}_{0''0'},
\end{aligned}
\tag{A.8}
$$

which gives

$$\hat{\tilde{T}}_{0'}^s(u + 2i\pi) \hat{\tilde{T}}_{0''}^s(u + i\pi) = \tilde{\mathcal{P}}_{0'0''} \hat{\tilde{T}}_{0''}^s(u) \hat{\tilde{T}}_{0'}^s(u + i\pi) \tilde{\mathcal{P}}_{0'0''}. \tag{A.9}$$

Substituting eq. (A.9) into (A.7), we obtain

$$\bar{t}_s(u + i\pi) \, \bar{t}_s(u) = 2^{-8N} [\rho_s(2u + 2i\pi - 2\eta)]^{-1} \mathcal{S}^{-1} t(u) \mathcal{S}, \tag{A.10}$$

which gives the conclusion (A.5).

Because the transfer matrices $\bar{t}(u)$ with different spectral parameters commutate with each other, they have the common eigenstates. Acting the factorization identity (A.5) on a common eigenstate, we obtain the eigenvalue $\Lambda(u)$ of the transfer matrix $t(u)$ of the $D_2^{(2)}$ spin chain as

$$\Lambda(u) = 2^{8N} \rho_s(2u + 2i\pi - 2\eta) \, \bar{\Lambda}_s(u + i\pi) \, \bar{\Lambda}_s(u), \tag{A.11}$$

where

$$\bar{\Lambda}_s(u) = \tilde{\Lambda}(u)|_{u \to u + \frac{i\pi}{2}, \{\theta_j\} = \{i\pi/2, 3i\pi/2\}, \{\mu_l \to \mu_l + \frac{i\pi}{2}\}}, \tag{A.12}$$

and $\tilde{\Lambda}(u)$ is given by eq. (4.11).

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