

RECEIVED: December 30, 2021

REVISED: March 21, 2022

ACCEPTED: March 22, 2022

PUBLISHED: April 7, 2022

Quantum nonequilibrium dynamics from Knizhnik-Zamolodchikov equations

Tigran A. Sedrakyan^a and Hrachya M. Babujian^b

^a*Department of Physics, University of Massachusetts, Amherst, MA 01003, U.S.A.*

^b*A. Alikhanyan National Scientific Laboratory, Yerevan Physics Institute, Yerevan, Armenia*

E-mail: tsedrakyan@umass.edu, babujian@yerphi.am

ABSTRACT: We consider a set of non-stationary quantum models. We show that their dynamics can be studied using links to Knizhnik-Zamolodchikov (KZ) equations for correlation functions in conformal field theories. We specifically consider the boundary Wess-Zumino-Novikov-Witten model, where equations for correlators of primary fields are defined by an extension of KZ equations and explore the links to dynamical systems. As an example of the workability of the proposed method, we provide an exact solution to a dynamical system that is a specific multi-level generalization of the two-level Landau-Zenner system known in the literature as the Demkov-Osherov model. The method can be used to study the nonequilibrium dynamics in various multi-level systems from the solution of the corresponding KZ equations.

KEYWORDS: Bethe Ansatz, Conformal Field Theory, Integrable Field Theories, Quantum Dissipative Systems

ARXIV EPRINT: [2112.12866](https://arxiv.org/abs/2112.12866)

Contents

1	Introduction	1
2	KZ equations for boundary WZNW model	4
2.1	Boundary WZNW term	5
2.2	KZ equations in the presence of the boundary term	5
3	Multi-level Landau-Zenner problem and its descendants	7
3.1	The solution of Demkov-Osherov and bow-tie models	8
3.2	Demkov-Osherov model and its alternations: the link to KZ equations	10
3.3	Solution of KZ equations	12
4	Conclusions	15
A	First term with derivative in the null-vector condition (2.11)	16
B	Second term in the null-vector condition (2.11)	17

1 Introduction

There has been considerable interest in nonequilibrium dynamics of quantum systems (see e.g., refs. [1, 2]) over the last couple of decades that surged again recently in connection with quantum information problems [3–19]. These include the dynamical discrete-state Bardeen-Cooper-Schrieffer (BCS) pairing models [5, 20–33], where for example the interaction strength can be made time dependent, and various multi-level Landau-Zenner tunneling models and their many body generalizations [34–46].

The cornerstone in the integrable properties of the discrete-state BCS hamiltonian is Richardson’s exact solution [20, 21] and its extensions to a class of models with far-reaching applications in the theory of strongly correlated quantum matter. The exact solution is achieved by employing the connections to an exactly solvable Gaudin magnet [47]. The connection is established upon rewriting the bilinears of spinful fermion operators in $s = 1/2$ operators using Schwinger fermion construction. As a result, the Richardson Hamiltonian is mapped to an exactly solvable spin-1/2 magnet with mutually commuting integrals of motion. Notably, the number of the integrals of motion is equal to the number of degrees of freedom in the system, a property that guarantees integrability. Moreover, these integrals of motion of the Richardson model are a linear combination of the corresponding integral of motion of the Gaudin magnet and S_l^3 component of the corresponding $s = 1/2$ operator.

Another important connection is between the Gaudin magnet and a two-dimensional conformal theory [48] (CFT) associated with the $SU(2)$ group, namely the Wess-Zumino-Novikov-Witten (WZNW) theory. Generally, it is well established that theories, where

current algebra is a fundamental symmetry, are governed by WZNW action, which has topological properties. The connection with magnet is through the correlation functions of the primary fields in CFT that satisfy a series of Knizhnik-Zamolodchikov (KZ) [49] equations involving constants of motion of the Gaudin model. The correlation functions in $SU(2)$ WZNW have been obtained from the solution of KZ equations with constants of motion of the Gaudin model by one of us and a collaborator in refs. [50–52]. This approach was further developed in ref. [30], where a boundary $SU(2)$ WZNW model was identified and it was shown that the correlation functions of the primary fields there satisfy the series of modified KZ equations, where the Gaudin operators are replaced by constants of motion of the Richardson model. The approach was also used to obtain exact solutions to a variety of dynamical systems in ref. [30]. Notably, in ref. [53] the method of refs. [50–52] was applied to a broad class Gaudin magnets with spin operators, S_n , being the generators of most simple complex Lie algebras.¹ A different method for diagonalization of Gaudin Hamiltonians was developed in ref. [54]. Here one employs a bosonization approach (using the bosonic representation of the corresponding affine algebra) and treats the diagonalization problem and KZ equations on the same footing. As revealed in refs. [54, 55], certain ordinary differential operators called opers could encode the spectrum of the quantum Gaudin Hamiltonians corresponding to a simple Lie algebra. Amazingly, the opers that carry all the information about the spectra of the Gaudin model are in direct correspondence with the Langlands dual/twisted Lie algebra (rather than the simple Lie algebra itself). This can be regarded as a particular case of the geometric Langlands correspondence [56–58]

Other profound implications of KZ equations in gauge theories and quantum spin chains have recently been reported in refs. [59, 60]. Bethe states and KZ equations were thoroughly investigated for generalized $sl(2)$ Gaudin models in [61, 62]. Moreover, a relation between integrable Kondo problems in products of chiral $SU(2)$ WZNW models and affine $SU(2)$ Gaudin models was recently reported in [63]. Another relationship between the quantum Gaudin models with boundary and the classical many-body integrable systems of Calogero-Moser type was established in [64]. More interesting correspondences between Gaudin models, conformal field theories, and BCS -Richardson pairing models are reported in refs. [65–70].

In the last 30 years, interesting, purely mathematical approaches have been developed to construct integrable models linked to loop groups and current algebras. In refs. [71–73], a momentum map, namely a Hamiltonian (Lie) group action on a symplectic manifold, was used to construct conserved quantities for the action in the context of KZ equations. It gave a dual formulation of integrable Hamiltonian systems as isospectral flows in the two-loop algebras. The momentum map of Poisson manifold of rank- k perturbations of a fixed $n \times n$ matrices to dual of pairs of loop algebras $\tilde{gl}(k)$ and $\tilde{gl}(n)$ presented a base for generating integrable Gaudin type of models linked to current algebra.

In refs. [74, 75], a holonomic system of differential equations with rational coefficients was found, which are compatible with an extension of the rational KZ equations by an element of the Cartan subalgebra. These equations were termed rational dynamical differ-

¹The statement applies to all simple complex Lie algebras except those of E_8 , G_2 , and F_4 .

ential (DD) equations. Note that the first such extension of KZ equations by σ^z in SU(2) case was formulated in ref. [76] for the Maxwell-Bloch system. Further, in ref. [75] it was shown, that (gl_k, gl_n) duality, which plays an important role in the representation theory and the classical invariant theory, [77, 78], gives rise a duality between KZ equations (for the Lie algebra gl_n) and DD equations for gl_k .

A comprehensive study of dualities between various quantum and classical many-body systems, including dualities of KZ equations, with application to the calculation of integrable probabilities in the stochastic process are presented in ref. [79]. The probabilities in the stochastic process are treated within the conventional machinery of integrable models, including transfer matrices and Bethe ansatz equations. An interesting KZ interpretation of duality in stochastic processes is discussed in [80].

Therefore, strikingly it turns out that the pairing Hamiltonian and a boundary WZNW CFT are related to one another in a nontrivial manner via the modification of KZ equations for primary fields in CFT that include integrals of motion of the pairing Hamiltonian. Consequently, it is expected that analytic properties of the correlation functions of conformal field theories and the spectral and dynamical properties of the pairing Hamiltonian can be explored on the same footing. One of the aims of the present work is to manifest this relation explicitly. To this end, we discuss the precise modification of the KZ equations and that these are satisfied by the correlation functions of primary fields in the boundary WZNW model. The main goal of the present work, however, is to study several dynamical systems that can be solved employing the connection through modified KZ equations, where time-dependence is explicit, but can be treated exactly upon solving the set of KZ equations.

As we see, through nontrivial correspondences, seemingly disconnected domains of theoretical physics are linked one with another. Following the same prescription above, several dynamical systems discussed below can be mapped on a variant of the pairing Hamiltonian, with the interaction parameter playing the role of a function of the time variable. Then, it can be shown that the Schrödinger equations for wave functions in these models lead to the specific extension of KZ equations for correlation functions of primary fields in SU(2) current algebra. Using the relation to the KZ equations, one can find the solutions of such dynamical models leading to various incarnations of the modified KZ equations in different situations. To this end, in the second part of the paper, we consider first the multi-level generalization of the two-level Landau-Zenner system, namely the Demkov-Osherov model, and its alternations. We show that the Demkov-Osherov model and its partner generalized bow-tie models are linked to an extension of KZ equations, which can be solved exactly. We present this analytical solution and discuss its implications.

The remainder of the paper is organized as follows. In section 2, we construct the boundary WZNW model, which leads to an extended version of KZ equations. In section 3, we consider the dynamical multi-level Landau-Zenner problem and its simplest realizations: the Demkov-Osherov model and its modification. A class of exact solutions to these dynamical systems is presented. In the appendix, we present the details of the construction of the boundary term in the WZNW model.

2 KZ equations for boundary WZNW model

Another connection is between the gapless generalization of the Heisenberg model to spin- s [83–85]. Specifically, it was shown by Affleck and Haldane, [86] that the connection reveals itself in identifying the quantum field theory corresponding to the critical point with the WZNW model with action [87, 88]

$$S_{WZNW}(g) = \frac{k}{16\pi} \int_{S^2} dz d\bar{z} \operatorname{tr} \left[\partial_a g^\dagger \partial^a g \right] - \frac{ik}{24\pi} \int_{\Sigma^3} d^3x \varepsilon^{\mu\nu\rho} \operatorname{tr} \left[g^\dagger \partial_\mu g g^\dagger \partial_\nu g g^\dagger \partial_\rho g \right]. \quad (2.1)$$

Here integration in the first term is over a two-dimensional manifold, S^2 , corresponding to a compactified complex plane parameterized by (w, \bar{w}) and group elements $g(w, \bar{w}) \in \text{SU}(2)$. Integration in the second topological WZNW term is over a three-dimensional manifold Σ^3 , with $x = (w, \bar{w}, \xi) \in \Sigma^3$, whose boundary at $\xi = 0$ is the aforementioned sphere, $S^2 = \partial\Sigma^3$ and the function $g(w, \bar{w}; \xi = 0)$ is extended into the interior of the ball $\xi \in [0, 1]$ in a non-unique way. The parameter k in eq. (2.1) is an integer number, which is the level of the corresponding CFT and linked to spin s of the chain model as $k = 2s$. The WZNW action is invariant under conformal and non-Abelian current algebras.

In the seminal work ref. [49], Knizhnik and Zamolodchikov have shown that N -point correlation functions, $G(w_1 \dots w_N \mid \bar{w}_1 \dots \bar{w}_N) = \left\langle \phi_{s_1}^{\mu_1}(w_1, \bar{w}_1) \dots \phi_{s_N}^{\mu_N}(w_N, \bar{w}_N) \right\rangle_{S_{WZNW}}$ of primary fields with spins $0 \leq s_i \leq k/2$, $i = 1, \dots, N$ of the WZNW model eq. (2.1), satisfy a system of first-order differential KZ equations. In the holomorphic sector these equations can be written as

$$\left[(k+2)\partial_{w_l} - \hat{H}_l^G \right] G(\{w_l\}) = 0, \quad (2.2)$$

where

$$\hat{H}_l^G = \sum_{l' \neq l} \frac{\hat{\mathbf{S}}_l \cdot \hat{\mathbf{S}}_{l'}}{w_l - w_{l'}} \quad (2.3)$$

are the integrals of motion of the Gaudin magnet model [47] with $\hat{\mathbf{S}}_l$ being a set of generators of $\text{SU}(2)$ with $l = \overline{1N}$. Gaudin model Hamiltonian \hat{H}^G is integrable and is linear combination of integrals of motion $\hat{H}^G = 2 \sum_{l=1}^N w_l \hat{H}_l^G$. As we pointed in the introduction, a class of models including the celebrated Richardson's pairing Hamiltonian [5, 20–33], whose conserved integrals of motion can be regarded as a generalization of those of the Gaudin magnet [89–91]

$$\hat{H}_l^R = \lambda S_l^3 + \hat{H}_l^G, \quad [\hat{H}_l^R, \hat{H}_{l'}^R] = 0, \quad (2.4)$$

leading to extended by λS_l^3 term KZ equations (EKZ). Namely,

$$\left[(k+2)\partial_{w_l} - \hat{H}_l^R \right] G(\{w_l\}) = 0. \quad (2.5)$$

Below we will follow the idea put forward in ref. [30] and show that EKZ equations (2.5), with $l = 1 \dots N$, are satisfied by the primary fields of the boundary WZNW CFT.

2.1 Boundary WZNW term

Consider left and right boundary terms of the WZNW model based on left-flowing and right-flowing currents $J^a(w) = \text{tr}[\hat{\mathbf{S}}^a g^\dagger(w) \partial_w g(w)]$, $\bar{J}^a(\bar{w}) = \text{tr}[\hat{\mathbf{S}}^a g^\dagger(\bar{w}) \partial_{\bar{w}} g(\bar{w})]$ $a = 1, 2, 3$, with spin- s generators of SU(2) algebra, $\hat{\mathbf{S}}^a$, and

$$\begin{aligned} S_{\text{bound}}^L(\mathcal{C}) &= \alpha \oint_{\mathcal{C}} dw w J^3(w), \\ S_{\text{bound}}^R(\bar{\mathcal{C}}) &= \alpha \oint_{\bar{\mathcal{C}}} d\bar{w} \bar{w} \bar{J}^3(\bar{w}). \end{aligned} \tag{2.6}$$

Here the coefficient $\alpha \in \mathcal{R}$ is real. Then, upon adding these currents to the action of the WZNW model, $S_{WZNW}(g)$, we obtain the model dubbed boundary WZNW (BWZNW):

$$S_{BWZNW}(g) = S_{WZNW}(g) + S_{\text{bound}}^L(\mathcal{C}) + S_{\text{bound}}^R(\bar{\mathcal{C}}). \tag{2.7}$$

We show below that the primary fields in this boundary theory satisfy EKZ equations eq. (2.5). In eq. (2.6) contours \mathcal{C} and $\bar{\mathcal{C}}$ are largest contours encompassing complex numbers w_l and approaching \mathcal{C}_∞ . Contour $\bar{\mathcal{C}}$ have clockwise rotation. Both contours should contain all points w_l of the correlators, which are under consideration (see figure 1). Note also that due to conformal invariance, the left currents $J^a(w)$, $a = 1, 2, 3$, do not depend on \bar{w} and likewise the right currents, $\bar{J}^a(\bar{w})$, do not depend on w .

Generally, we are interested in calculating a correlation function of arbitrary spin $0 \leq s_i \leq (k/2)$ primary fields [48] at points $w_i, i = 1 \dots N$ in the SU(2) boundary WZNW model (see figure 1). Namely $G(w_1, \dots w_N) = \langle \phi_{s_1}(w_1) \dots \phi_{s_N}(w_N) \rangle_{S_{BWZNW}}$, where functional averaging is defined in a standard way

$$\langle \phi_{s_1}(w_1) \dots \phi_{s_N}(w_N) \rangle_S = \int [\mathcal{D}\phi] \phi_{s_1}(w_1) \dots \phi_{s_N}(w_N) e^{-S_{WZNW} - S_{\text{bound}}(\mathcal{C})}. \tag{2.8}$$

Due to this relation, one can evaluate any correlator in the boundary model via its relation to the a correlation function of the same primary fields multiplied with $\Phi[\mathcal{C}] = e^{-S_{\text{bound}}(\mathcal{C})}$ but with average in the bulk WZNW model. Namely,

$$G(z_1, \dots z_N) = \langle \Phi[\mathcal{C}] \phi_{s_1}(z_1) \dots \phi_{s_N}(z_N) \rangle_{S_{WZNW}}. \tag{2.9}$$

With this rearrangement and conceptually new averaging procedure, one can show that the primary fields satisfy the EKZ equation following the standard procedure of deriving the KZ equations in the ordinary WZNW model. We present a detailed discussion of the outlined procedure in the next subsection.

2.2 KZ equations in the presence of the boundary term

The emergence of ordinary KZ equations in CFT is the consequence of the fact that by definition, primary fields $\phi_s(w)$ are null vectors in current algebra SU(2). Namely the primary fields fulfill constraints $[L_0 - \frac{1}{k+2} J_0^a J_0^a] \phi_s(w) = 0$, where operators J_0^a and J_{-1}^a are 0 and -1 coefficients of the Laurent series of the current $J^a(w)$, while $L_0 = \partial_w$ is the

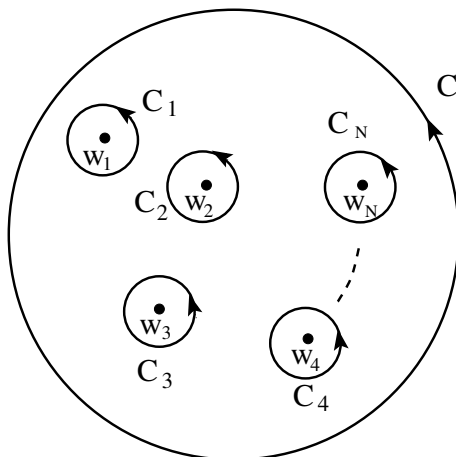


Figure 1. Boundary contour C corresponds to the limit of large distances. It shrinks to the sum of contours C_i , $i = 1, \dots, N$ encircling points w_i : $C = \bigsqcup_{i=1}^N C_i$.

zero component of the Virasoro algebra. When they are acting on any field at $w \in C_1$, then one has

$$J_0^a = \oint_{C_1} J^a(u) du, \quad J_{-1}^a = \oint_{C_1} \frac{J^a(u)}{u-w} du. \tag{2.10}$$

The null vector condition is the same also in the boundary model. Therefore it gives the following equation for correlation functions

$$\begin{aligned} & \langle \left[\partial_{w_1} - \frac{1}{k+2} J_{-1}^a J_0^a \right] \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N) \rangle_{\text{BWZNW}} = \\ & = \langle e^{\Phi(C)} \left[\partial_{w_1} - \frac{1}{k+2} J_{-1}^a J_0^a \right] \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N) \rangle_{\text{WZNW}} = 0. \end{aligned} \tag{2.11}$$

Here we consider equation for the first primary field with argument w_1 , therefore operator $J_{-1}^a J_0^a$ is acting on $\phi_{s_1}(w_1)$. This means that in the integral representation of these operators, eq. (2.10), contours should circle w_1 . As one can see, in eq. (2.11), there are two terms: the first term is the derivative, ∂_{w_1} . The second term is $J_{-1}^a J_0^a$. By definition, the primary fields are those that are eigenstates of the current operator, $J^a(u)$. Namely, they fulfill the following operator algebra relation

$$J^a(u) \phi_{s_i}(w_i) = \frac{S_i^a}{u-w_i} \phi_{s_i}(w_i) + \dots \tag{2.12}$$

Upon using this relation, the direct calculation of two terms in the null vector condition (2.11) for the first primary field at w_1 gives (see details in appendix A)

$$\langle e^{\alpha \oint_C dw w J^3(w)} \partial_{w_1} \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N) \rangle_{\text{WZNW}} = \partial_{w_1} \langle e^{\alpha \sum_{i=1}^N w_i S_i^3} \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N) \rangle_{\text{WZNW}} \tag{2.13}$$

and

$$\begin{aligned}
 & \langle e^{\alpha \oint_C dw w J^3(w)} J_{-1}^a J_0^a \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N) \rangle_{\text{WZNW}} \\
 &= \sum_{j=2}^N \frac{S_j^a S_1^a}{w_1 - w_j} e^{\alpha \sum_{i=2}^N w_i S_i^3} \langle \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N) \rangle_{\text{WZNW}} \\
 &+ c_1 \alpha S_1^3 e^{\alpha \sum_{i=2}^N w_i S_i^3} \langle \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N) \rangle_{\text{WZNW}},
 \end{aligned} \tag{2.14}$$

where $c_1 = S_1^a S_1^a$ and the summation over repeating indices, $a = 1, 2, 3$, is assumed (see appendix A for all pertinent calculations of the present subsection). Combining these two equations into the null-vector condition, eq. (2.11), for the correlation function

$$G(w_1, \cdots w_N) = \langle \phi_{s_1} \cdots \phi_{s_N} \rangle_{\text{BWZNW}} = \langle e^{\alpha \sum_{i=2}^N w_i S_i^3} \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N) \rangle_{\text{WZNW}}, \tag{2.15}$$

where we have used operator algebra (2.12) for primary field, we arrive at following EKZ equations

$$\left[(k+2) \partial_{w_1} - \lambda S_1^3 - \sum_{j=2}^N \frac{S_j^a S_1^a}{w_1 - w_j} \right] G(w_1, \cdots w_N) = 0. \tag{2.16}$$

Here the coefficient $\lambda = \alpha c_1$. Such an equation was established in ref. [76] for the Maxwell-Bloch system. Below we consider a class of nonequilibrium systems whose dynamics can be studied from the exact solution of the corresponding EKZ equations.

3 Multi-level Landau-Zenner problem and its descendants

In a related context, certain integrable aspects of models with time-dependent Hamiltonians have recently been studied in refs. [30, 42, 44, 46]. Here we present the solutions of the Demkov-Osherov [34], bow-tie (BT) [35] and generalized version of the bow-tie (GBT) models [36, 37]. This is achieved from the exact solutions of the associated EKZ equations. We show that GBT contains a set of integrals of motion, which form EKZ for correlation functions, where $n - 2$ spins should be considered classical or, equivalently, large-spins $s \gg 1$. Using the off-shell Algebraic Bethe Ansatz (OSABA) technique developed in refs. [51, 52, 76], we find the solution of this EKZ for correlation functions in GBT.

In its general formulation, the multilevel LZ problem is defined by a time-dependent Hamiltonian of the form

$$H_{LZ} = \hat{A} + \hat{B}t, \tag{3.1}$$

where \hat{A} and \hat{B} are $(n+1) \times (n+1)$ Hermitian matrices. In general, for arbitrary \hat{A} and \hat{B} matrices, corresponding time-dependent Schrödinger equation,

$$-i \partial_t \Psi(t) = H_{LZ} \Psi(t), \tag{3.2}$$

can be solved formally via utilizing the T -exponent representation of the wave function. The latter however is practically not very useful for computation of observables and other applications. Though, in some simple cases, a solution can be found. The most simple case is when matrix \hat{B} has n coinciding eigenvalues, b_2 , and only one is different, $b_1 \neq b_2$.

Then the unitary matrix U , which diagonalizes \hat{B} , is degenerate on a subgroup of $n \times n$ unitary matrices. The latter can be used to diagonalize $n \times n$ minor of the matrix \hat{A} . To simplify the described t -dependent Hamiltonian, one can subtract from the Hamiltonian eq. (3.3) the diagonal matrix, $b_2 t \times \text{id}$, and rescale the time by the factor $b_1 - b_2$. After this procedure, the Hamiltonian H_{LZ} acquires the form

$$H_{DO} = \begin{pmatrix} t + a_{00} & v_{01} & \cdot & \cdot & \cdot & v_{0n} \\ v_{01} & a_{01} & 0 & \cdot & \cdot & 0 \\ v_{02} & 0 & a_{02} & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ v_{0n-1} & 0 & \cdot & \cdot & a_{0n-1} & 0 \\ v_{0n} & 0 & \cdot & \cdot & 0 & a_{0n} \end{pmatrix}, \quad (3.3)$$

which was first defined in ref. [34] and subsequently dubbed Demkov-Osherov (DO) model. BT model (parametrized by a set of constants $\{r_i\}$) is equivalent to the DO model and their Hamiltonians are linked by the following linear transformation:

$$a_{0i} = r_i t + t. \quad (3.4)$$

These models and their generalizations were discussed in refs. [41–43].

3.1 The solution of Demkov-Osherov and bow-tie models

We look for the eigenvalues and wave functions of the Hamiltonian, $H_{DO} |x_m(t)\rangle = E_m^0(t) |x_m(t)\rangle$, in the form

$$|x_m(t)\rangle = \sum_{k=0}^n \frac{\gamma_k}{x_m(t) - \epsilon_k} |k\rangle, \quad (3.5)$$

where

$$|k\rangle = (0, \dots, 1, \dots, 0)^T, \text{ and } k = 0, 1, \dots, n. \quad (3.6)$$

Here in ket $|k\rangle$ the unity entry stands at the k 'th place where the leftmost entry has number zero and we have introduced a set of yet unknown parameters, $(\gamma_0; \{\gamma_i\})$, $i = 1, \dots, N$. Now, we assume that the energy eigenvalues corresponding to wave function, $|x_m(t)\rangle$, are of the form

$$E_m^0(t) = \frac{\gamma_0^2}{x_m(t) - \epsilon_0}, \quad (3.7)$$

where the functions $x_m(t)$ and the connection between γ_k and ϵ_k and parameters entering the DO Hamiltonian eq. (3.3) are yet to be determined from self-consistency relations. This so far is an ansatz, which will simplify the final result considerably. In this basis, the DO Hamiltonian (3.3) acquires a simple form

$$H_{DO} = (t + a_{00})|0\rangle\langle 0| + \sum_{i=1}^n a_{0i}|i\rangle\langle i| + \sum_{i=1}^n v_{0i}(|0\rangle\langle i| + |i\rangle\langle 0|). \quad (3.8)$$

We substitute now the expressions in eq. (3.5) for eigenstate and eq. (3.7) for energy into the Schrödinger equation. This yields

$$H_{DO} |x_m(t)\rangle = E_m^0(t) |x_m(t)\rangle. \quad (3.9)$$

Consider the equation corresponding to basis element $|k = 0\rangle$. The equation thus will acquire the following algebraic form:

$$(a_{00} + t) \frac{\gamma_0}{x_m(t) - \epsilon_0} + \sum_i \frac{v_{0i}\gamma_i}{x_m(t) - \epsilon_i} = \frac{\gamma_0^2}{x_m(t) - \epsilon_0} \cdot \frac{\gamma_0}{x_m(t) - \epsilon_0}. \quad (3.10)$$

For basis elements $|i\rangle$, with $i = 1, \dots, n$, the Schrödinger equation eq. (3.9) yields

$$\frac{a_{0i}\gamma_i}{x_m(t) - \epsilon_i} + \frac{v_{0i}\gamma_0}{x_m(t) - \epsilon_0} = \frac{\gamma_0^2}{x_m(t) - \epsilon_0} \frac{\gamma_i}{x_m(t) - \epsilon_i}. \quad (3.11)$$

It is straightforward to check that the eigenvalue equations eqs. (3.10) and (3.11) will be fulfilled if the original parameters v_{01}, \dots, v_{0n} and a_{01}, \dots, a_{0n} are self-consistently connected with γ_i and $\epsilon_i - \epsilon_0$ for $i = 1, \dots, n$ as

$$\begin{aligned} v_{0i} &= \frac{\gamma_0\gamma_i}{\epsilon_0 - \epsilon_i}, \\ a_{0i} &= \frac{\gamma_0^2}{\epsilon_i - \epsilon_0}, \quad i = 1, \dots, n. \end{aligned} \quad (3.12)$$

From eq. (3.10), the parameters a_{00} and the time variable, t , are bound with relations

$$a_{00} = \sum_{i=1}^n \frac{\gamma_i^2}{\epsilon_i - \epsilon_0}, \quad (3.13)$$

$$t = \sum_{i=0}^n \frac{\gamma_i^2}{x_m(t) - \epsilon_i}. \quad (3.14)$$

Eq. (3.13) yields a condition under which the eigenvalue equation has a solution, while eq. (3.14) for time gives polynomial equation of $n + 1$ order for $x_m(t)$. That polynomial t -dependent equation defines $m = 0, \dots, n$ solutions in the form of functions for $x_m(t)$. Importantly, the relation eq. (3.14) appears when we equate the time dependent terms in the left and right hand sides of eq. (3.10).

From eqs. (3.12) and (3.13), one can find

$$\frac{v_{0i}}{a_{0i}} = -\frac{\gamma_i}{\gamma_0}, \quad i = 1, \dots, n, \quad (3.15)$$

$$a_{00} = \sum_i^n \frac{v_{0i}^2}{a_{0i}}. \quad (3.16)$$

These equations directly relate parameters in the DO Hamiltonian eq. (3.3), with the set of new parameters $(\gamma_0; \{\gamma_i\})$. Note that the relation (3.16) does not essentially restrict the DO Hamiltonian (3.3). Upon starting from the Hamiltonian with arbitrary diagonal entries, a'_{0i} , $i = 0, 1, \dots, n$, and making a constant shift $a_{0i} = a'_{0i} + a$, one will always

fulfill the condition eq. (3.16) by solving the polynomial equation of $n + 1$ order for a . This procedure is equivalent to shifting the Hamiltonian by a constant, which does not change anything. So for the general class of polynomials that support real solutions for a , eq. (3.16) does not limit the general DO model and its solution.

Relations (3.12)–(3.14) show that we can always absorb the parameter ϵ_0 into ϵ_i , $i = 1, \dots, n$ and $x_m(t)$ as $\epsilon_i - \epsilon_0 \rightarrow \epsilon_i$ and $x_m(t) - \epsilon_0 \rightarrow x_m(t)$. Nevertheless, we keep it for completeness. After these remarks, it is clear that transformation of parameters $(a_{0i}, v_{0i}, t) \rightarrow (\epsilon_i, \gamma_i, \gamma_0)$ is essentially exact and retains the correct number of degrees of freedom.

3.2 Demkov-Osherov model and its alternations: the link to KZ equations

DO, and BT models can be altered by redefining the properties of eigenvalues of the matrix \hat{B} . Let us assume that the matrix \hat{B} has $n - 1$ identical eigenvalues, b_2 , and two other eigenvalues, b_1 . Then the diagonalizing unitary matrix, U , will have degeneracy over subgroup of $SU(n - 1)$ matrices, which will allow diagonalizing $(n - 1) \times (n - 1)$ diagonal minor of \hat{A} . Then, subtracting from the Hamiltonian eq. (3.1) the diagonal matrix, $b_2 t \times \text{id}$, and after appropriate rescaling of the time by the factor $b_1 - b_2$, one arrives at the following alternation of Demkov-Osherov model (ADO) given in the form of a simple matrix Hamiltonian:

$$H_{\text{ADO}} = \begin{pmatrix} t + v_{00} & v_{01} & v_{02} & \cdot & v_{0j} & \cdot & \cdot & v_{0n} \\ v_{01} & t + v_{11} & v_{12} & \cdot & v_{1j} & \cdot & \cdot & v_{1n} \\ v_{02} & v_{12} & a_2 & 0 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ v_{0j} & v_{1j} & 0 & \cdot & a_j & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{n-1} & 0 \\ v_{0n} & v_{1n} & 0 & \cdot & 0 & \cdot & 0 & a_n \end{pmatrix} \quad (3.17)$$

Since the Hamiltonian is Hermitian, all parameters, v_{ij} and a_j , $j = 2, \dots, n$, that are present in it, are real. In the bra-ket notations, the Hamiltonian eq. (3.17) reads

$$H_{\text{ADO}} = \sum_{k=0,1} (t + v_{kk}) |k\rangle \langle k| + \sum_{i=2}^n a_i |i\rangle \langle i| + v_{01} (|0\rangle \langle 1| + |1\rangle \langle 0|) + \sum_{k=0,1;i=2}^n v_{ki} (|k\rangle \langle i| + |i\rangle \langle k|). \quad (3.18)$$

In ref. [43], the model was studied semiclassically for $n = 3$, within the independent level crossing approximation.

Below we are going to link the time-dependent Schrödinger equation $-i\partial_t \Psi = H_{\text{GBT}} \Psi$, with $\Psi = (\psi_0, \psi_1, \dots, \psi_n)^T$, with EKZ equations and show that the solution of the latter satisfies the former. To this end, after Fourier transformation, the Schrödinger equation

acquires the form of $\omega\Psi = H_\omega\Psi$. The latter in detailed form reads

$$\begin{aligned}\omega\psi_0 &= (-i\partial_\omega + v_{00})\psi_0 + v_{01}\psi_1 + \sum_{k=2}^n v_{0k}\psi_k \\ \omega\psi_1 &= (-i\partial_\omega + v_{11})\psi_1 + v_{01}\psi_0 + \sum_{k=2}^n v_{1k}\psi_k \\ \omega\psi_k &= a_k\psi_k + v_{0k}\psi_0 + v_{1k}\psi_1, \quad k = 2, \dots, n.\end{aligned}\tag{3.19}$$

The solution of the last set of equations for ψ_k with $k = 2, \dots, n$ is straightforward, and yields

$$\psi_k = \frac{v_{0k}\psi_0 + v_{1k}\psi_1}{\omega - a_k}, \quad k = 2, \dots, n.\tag{3.20}$$

Substituting these expressions for ψ_k into the first two equations in (3.19), we obtain two equations for $\psi_{0,1}$. The latter can be simplified further by the following transformation of the wave functions $\psi_{0,1} = e^{i\omega^2/2}\bar{\psi}_{0,1}$. After this transformation, the linear in ω term can be eliminated from the equations. Finally, we see that these two equations reduce to the following 2×2 matrix equation

$$i\partial_\omega\Phi = \sum_{\mu=0}^3 \left[b_1^\mu S_1^\mu + \sum_{k=2}^n \frac{S_1^\mu b_k^\mu}{\omega - a_k} \right] \Phi, \quad \Phi = (\bar{\psi}_0, \bar{\psi}_1)^T\tag{3.21}$$

where the matrix $S_1^\mu = (1, \sigma_1, \sigma_2, \sigma_3)$, $\mu = 0, 1, 2, 3$ is given by the unity and the set of three Pauli matrices (the presence of the unity operator in the set of spin operators implied that we have algebra $u(2)$ instead of $su(2)$). From now on, we will assume summation over repeating Greek indices μ, ν to run from $0, \dots, 3$. In (3.21) vectors b_k are found to be

$$\begin{aligned}b_1^0 &= \frac{(v_{00} + v_{11})}{2}, & b_1^3 &= \frac{(v_{00} - v_{11})}{2}, & b_1^+ &= b_1^- = v_{01}, \\ b_k^0 &= \frac{(v_{0k}^2 + v_{1k}^2)}{2}, & b_k^3 &= \frac{(v_{0k}^2 - v_{1k}^2)}{2}, & b_k^+ &= b_k^- = v_{0k}v_{1k}.\end{aligned}\tag{3.22}$$

One can see that eq. (3.22) is surprisingly similar to one of the EKZ equations, eq. (2.16), but instead of the full set of n quantum spin generators, we have only one spin generator S_1^μ associated with energy ω as a spectral parameter. Other $n - 1$ spins, associated with spectral parameters $a_k, k = 2, 3, \dots, n$, are classical vectors, b_k^μ . They can be treated quasi-classically as large quantum spin limits. The $b_1^\mu S_1^\mu$ term corresponds to λS^3 modification in EKZ equations (2.16).

Now it is natural to ask whether there is a sufficient number of integrals of motion for the Hamiltonian in the Schrödinger equation (3.21), namely

$$H_1 = b_1^\mu S_1^\mu + \sum_{k=2}^n \frac{S_1^\mu b_k^\mu}{\omega - a_k},\tag{3.23}$$

to be integrable? One can expect that the set of operators

$$H_k = \sum_{k'>k}^n \frac{b_k^\mu b_{k'}^\mu}{a_k - a_{k'}} + \frac{b_k^\mu S_1^\mu}{a_k - \omega}, \quad k = 2, \dots, n\tag{3.24}$$

are candidates of integrals of motion, since they are large spin quasiclassical limits of the operators H_j in the set of EKZ equations (2.4)–(2.5). Indeed, it appears that their commutators are equal to

$$\begin{aligned}
 [H_i, H_j] &= \frac{1}{(a_i - \omega)(a_j - \omega)} b_i^\mu b_j^\nu [S_1^\mu, S_1^\nu] \\
 &= \sum_{a,b,c=1}^3 \frac{b_i^a b_j^b \epsilon_{abc} S_1^c}{(a_i - \omega)(a_j - \omega)}, \quad i, j = 1, 2, \dots, n.
 \end{aligned}
 \tag{3.25}$$

These operators will commute if all n classical limits of spins, b_k^μ , $k = 1, 2, \dots, n$, are parallel: $b_k \parallel b_{k'}$. A straightforward analysis of expressions in eq. (3.22) for b_k^μ shows that the parallel property will be fulfilled if

$$v_{ij} = \gamma_i \gamma_j, \quad i, j = 0, 1, \dots, n. \tag{3.26}$$

Then we have a complete set of EKZ equations, where ω plays the role of the first spectral parameter while others are a_k , $k = 2, \dots, n$. Moreover, EKZ equations, eq. (2.5), with this set of Hamiltonians have a solution because zero curvature condition is fulfilled

$$\partial_{a_i} H_j - \partial_{a_j} H_i - [H_i, H_j] = 0, \quad i, j = 0, 2, \dots, n, \tag{3.27}$$

where $a_0 = \omega$. Hence, the solution of the ADO model, with “parallel” conditions eq. (3.26) on parameters, is defined by the solution of the EKZ equations with one quantum and $n - 1$ classical spins. Wave function of ADO model, $\Phi(\omega, a_2, \dots, a_n)$, thus can be treated as correlation function $G(\omega, a_i)$ in EKZ equation.

Interestingly, parallelism of b_k^μ vectors indicates that the 2×2 matrix $b^\mu S_1^\mu = b_k^0 \mathbf{1} + \mathbf{b}_k \mathbf{S}_1$ has rank one and the interaction term in eqs. (3.23)–(3.24) coincides with the momentum map in refs. [72, 73]. The latter gives a dual characterization of integrable Hamiltonian systems as isospectral flow in the two loop algebras.

3.3 Solution of KZ equations

The solutions of KZ and EKZ equations based on OSABA were formulated in refs. [51, 52, 76]. The situation in the ADO model is simpler. Since all vectors, b_i are parallel it follows that all matrices $b_k^\mu S_1^\mu$ are commuting. Therefore, one can work with them as with ordinary commuting numbers.

Due to this commutativity property, the inspection yields that

$$\partial_\omega (\omega - a_i)^{b^\mu S_1^\mu} = \frac{b^\nu S_1^\nu}{\omega - a_i} (\omega - a_i)^{b^\mu S_1^\mu}. \tag{3.28}$$

Therefore, one can derive the general solution of EKZ equations based on the Hamiltonian H_1 given in eq. (3.23) and integrals of motion H_i , $i = 2, \dots, n$ given by eq. (3.24). Namely, the solution of

$$\partial_\omega \Phi(\omega, \{a_i\}) = -i H_1 \Phi(\omega, \{a_i\}), \quad \partial_{a_i} \Phi(\omega, \{a_i\}) = -i H_i \Phi(\omega, \{a_i\}), \tag{3.29}$$

we construct in several steps. As the first step, one can look for $\Phi(\omega, \{a_i\})$ in the form

$$\Phi(\omega, \{a_i\}) = \prod_{j>i} (a_i - a_j)^{-ib_i^\mu b_j^\mu} \Phi^0(\omega, \{a_i\}). \quad (3.30)$$

Here vectors b_k^μ are defined by expressions provided in eqs. (3.22) and (3.26). For the new “wavefunction” Φ_0 we obtain a new equation that is similar to eq. (3.29), wherein the r.h.s. of the second equation in the operator H_i one retains only the second term. As the second step, in the same second equation one can separate the $\mu = 0$ component of vectors b_k^μ, b_1^μ from the rest in terms that contain them. This gives

$$\begin{aligned} \partial_\omega \Phi^0(\omega, \{a_i\}) &= -i \left(b_1^0 + \mathbf{b}_1 \mathbf{S} + \sum_{k=2}^n \frac{b_k^0}{\omega - a_k} + \sum_{k=2}^n \frac{\mathbf{b}_k \mathbf{S}}{\omega - a_k} \right) \Phi^0(\omega, \{a_i\}), \\ \partial_{a_i} \Phi^0(\omega, \{a_i\}) &= -i \left(\frac{b_i^0}{a_i - \omega} + \frac{\mathbf{b}_i \mathbf{S}}{a_i - \omega} \right) \Phi^0(\omega, \{a_i\}). \end{aligned} \quad (3.31)$$

Here we define the $\mathbf{S} = (\sigma^1, \sigma^2, \sigma^3)$, where $\sigma^j, j = 1, 2, 3$, are Pauli matrices. For Φ^0 one will then obtain

$$\Phi_0(\omega, \{a_i\}) = e^{-i\omega b_1^0} \prod_{k=2}^n (\omega - a_k)^{-ib_k^0} \Phi_s(\omega, \{a_i\}), \quad (3.32)$$

where $\Phi_s(\omega, \{a_i\})$ is still unknown. After this one arrives at the equations for the function $\Phi_s(\omega, a_i)$ of the form

$$\begin{aligned} \partial_\omega \Phi_s(\omega, \{a_i\}) &= -i \left(\mathbf{b}_1 \mathbf{S} + \sum_{k=2}^n \frac{\mathbf{b}_k \mathbf{S}}{\omega - a_k} \right) \Phi_s(\omega, \{a_i\}) \\ \partial_{a_i} \Phi_s &= -i \frac{\mathbf{b}_i \mathbf{S}}{a_i - \omega} \Phi_s(\omega, \{a_i\}). \end{aligned} \quad (3.33)$$

Earlier we have observed that commutativity of the Hamiltonians H_i for different i leads to the condition of all vectors \mathbf{b}_i being parallel to each other, i.e.

$$\mathbf{b}_i = \beta_i \mathbf{n}, \quad (3.34)$$

where \mathbf{n} is an arbitrary unit vector and β_i are the norms of vectors \mathbf{b}_i . Below all these norms will be linked to the parameters of the ADO Hamiltonian.

Let us now rewrite eqs. (3.31) using the newly introduced notations:

$$\begin{aligned} \partial_\omega \Phi_s(\omega, \{a_i\}) &= -i \left(\beta_1 \mathbf{nS} + \sum_{k=2}^n \frac{\beta_k \mathbf{nS}}{\omega - a_k} \right) \Phi_s(\omega, \{a_i\}), \\ \partial_{a_i} \Phi_s(\omega, \{a_i\}) &= -i \left(\frac{\beta_i \mathbf{nS}}{a_i - \omega} \right) \Phi_s(\omega, \{a_i\}). \end{aligned} \quad (3.35)$$

The eigenvectors and eigenvalues of \mathbf{nS} here are defined as

$$\mathbf{nS} \xi_m = m \xi_m, \quad (3.36)$$

where ξ_m represents a two component spinor eigenvector with eigenvalue $m = \pm 1$. One can directly compute ξ_m by solving eigenvalue equations directly in components. Using this fact we are now able to finally write the solution Φ_s^m in the form

$$\Phi_s^m = \exp(-i\omega\beta_1 m) \prod_{k=2}^n (\omega - a_k)^{-i\beta_k m} \xi_m. \quad (3.37)$$

As the last step, we collect all parts of the solution and putting them together obtain

$$\begin{aligned} \Phi^m = & \exp[-i\omega(b_1^0 + \beta_1 m)] \prod_{j>i} (a_i - a_j)^{-i(b_i^0 b_j^0 + \beta_i \beta_j)} \\ & \times \prod_{k=2}^n (\omega - a_k)^{-i(b_k^0 + \beta_k m)} \xi_m. \end{aligned} \quad (3.38)$$

As our equations are linear, the general solution will be a linear combination of the two different solutions corresponding to two eigenvalues with $m = \pm 1$.

One can express final answer in terms of parameters $\{\gamma_i\}$, $i = 1, \dots, n$ by use of the expressions in eqs. (3.22) and (3.26) ensuring parallelism of b vectors. In terms of parameters γ , the b vectors have the following form

$$\begin{aligned} b_1^0 &= \frac{\gamma_0^2 + \gamma_1^2}{2}, & b_1^3 &= \frac{\gamma_0^2 - \gamma_1^2}{2}, & b_1^\pm &= \gamma_0 \gamma_1, \\ b_k^0 &= \frac{\gamma_0^2 + \gamma_1^2}{2} \gamma_k^2, & b_k^3 &= \frac{\gamma_0^2 - \gamma_1^2}{2} \gamma_k^2, & b_k^\pm &= \gamma_0 \gamma_1 \gamma_k^2, \end{aligned} \quad (3.39)$$

from where it follows that

$$\begin{aligned} \beta_1 &= b_1^0 = \frac{\gamma_0^2 + \gamma_1^2}{2}, \\ \beta_k &= b_k^0 = \frac{\gamma_0^2 + \gamma_1^2}{2} \gamma_k^2, \quad k = 2, 3, \dots, N. \end{aligned} \quad (3.40)$$

Finally, by use of eqs. (3.39) and (3.40), one can simplify the expression in eq. (3.38). Taking into account the factor $e^{i\omega^2/2}$ [we remind the reader about the transformation we introduced in the text below eq. (3.20)], the exact solution of ADO model acquires the form

$$\begin{aligned} \Phi(\omega, \{a_i\}) &= e^{i\omega^2/2} \prod_{j>i} (a_i - a_j)^{-2i\beta_i \beta_j} \times \prod_{j=2}^n (\omega - a_j)^{-i\beta_j(1+m)} e^{-i\omega\beta_1(1+m)} \xi_m, \\ &\text{with } m = \pm 1. \end{aligned} \quad (3.41)$$

This is one of the key results of the present work. Fourier transform of eq. (3.41) for $m = \pm 1$ defines the real-time evolution of the wave function

$$\Phi(t, a_2, \dots, a_n) = \int_{-\infty}^{\infty} d\omega \Phi(\omega, a_2, \dots, a_n) e^{i\omega t}. \quad (3.42)$$

Appropriately normalized wave function gives the system's evolution starting from the initial state at $t = -\infty$. Namely

$$\Phi(t, a_2, \dots, a_n) = G(t, a_2, \dots, a_n) \Phi(-\infty, a_2, \dots, a_n), \quad (3.43)$$

where $G(t, a_2, \dots a_n) = T \exp\{i \int_{-\infty}^t dt' H_{ADO}\}$ is the time evolution operator where T stands for time ordering. It defines the amplitudes of transmission probabilities between diabatic states at $t = -\infty$ and $t = \infty$. We can see from here that the state with $m = -1$ has trivial ω dependence, $\Phi(\omega) \sim e^{i\omega^2/2}$, which translates into a similar oscillatory time-dependence of the wave-function $\Phi(t) \sim e^{-it^2/2}$, that gives zero transmission probability between diabatic states.

The analytical form of Fourier transformation of the wave function for arbitrary n is cumbersome. However, for the simple case of $n = 2$, it is straightforward, yielding the exact transmission probability P of the Landau-Zener problem

$$P = e^{-2\pi(\gamma_0^2 + \gamma_1^2)\gamma_2^2}. \tag{3.44}$$

This is in full accordance with Landau-Zener result of two band crossing [92], for parallel levels.

4 Conclusions

We have presented a set of quantum dynamical systems, which are linked to modified Knizhnik-Zamolodchikov equations and can be solved exactly upon employing this connection. Those are the BCS pairing model (based on Richardson's exact solution), multi-level Landau-Zener tunneling models, and their realizations as a generalization of Demkov-Osherov and bow-tie models. In these systems, interaction couplings can be considered to be time-dependent linearly, leading to the extension of KZ equations. This link is surprising and based on the fact that all these models contain integrals of motion of Gaudin magnets. This inherent property of integrability leads to EKZ equations. Using the link and integrable properties, we solve ADO and GBT models exactly. We believe this link of dynamical systems to EKZ equations is not limited and can be extended to other systems with time-dependent Hamiltonians.

Moreover, it is known that KZ equations are written for correlation functions of WZWN models. Here we have revisited the question posed in ref. [30], namely what is the CFT for which EKZ defines correlation functions? Using a different method, we have shown that the corresponding model is the WZWN model with a boundary term discussed first in ref. [30].

Acknowledgments

The authors are grateful to A. G. Sedrakyan and R. H. Poghossian for many useful discussions and to N. A. Sinitsyn for valuable comments. The research was supported by startup funds from the University of Massachusetts, Amherst (T.A.S.), and H.M.B. was supported by the Armenian State Committee of Science in the framework of the research projects 191T-008, 20RF-142, and 21AG-1C024.

A First term with derivative in the null-vector condition (2.11)

From the operator algebra eq. (2.12), where the current $J^a(u)$ is acting on primary field at the point w_j , we have

$$J_0^a \phi_{s_i}(w_i) = \oint_{C_i} du J^a(u) \phi_{s_i}(w_i) = \oint_{C_i} \frac{S_i^a}{u - w_i} \phi_{s_i}(w_i) = S_i^a \phi_{s_i}(w_i). \quad (\text{A.1})$$

Here we have used the fact that since $J^a(u)$ is acting on the primary field at w_i , the contour C_i is circling the position w_i , after which Cauchy integration is applied.

The first term in eq. (2.11) reads

$$\langle e^\alpha \oint_{\mathcal{C}} dww J^3(w) \partial_{w_1} \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N) \rangle = \langle \sum_k \frac{\alpha^k}{k!} \left(\oint_{\mathcal{C}} dww J^3(w) \right)^k \partial_{w_1} \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N) \rangle \quad (\text{A.2})$$

Using eq. (2.12), it is easy to calculate the linear in the current term in these series. Since, according to CFT, singularities in this operator product expansion may appear only at the positions w_i of the primary field, we can shrink boundary contour \mathcal{C} into the sum of circles C_i around those points: $\mathcal{C} = \sqcup_i C_i$ (see figure 1). Then each term defined by the contour C_i gives the action of the current on primary field $\phi_{s_i}(w_i)$. Using eq. (2.12) for the action of current on the primary field, we obtain

$$\begin{aligned} & \langle \sum_{i=1}^N \oint_{C_i} dww J^3(w) \partial_{w_1} \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N) \rangle \\ &= \sum_{i=1}^N \oint_{C_i} dww \partial_{w_1} \frac{S_i^3}{w - w_i} \langle \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N) \rangle \\ &= \sum_{i=1}^N \partial_{w_1} [w_i S_i^3 \langle \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N) \rangle] \end{aligned} \quad (\text{A.3})$$

Higher-order terms of current $J^3(w)$ will produce series, where $\sum_i \oint_{C_i} dww J^3(w)$ is replaced by $\sum_i w_i S_i^3$ and, therefore, one will get

$$\langle e^\alpha \oint_{\mathcal{C}} dww J^3(w) \partial_{w_1} \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N) \rangle = \partial_{w_1} [e^\alpha \sum_{i=1}^N w_i S_i^3 \langle \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N) \rangle]. \quad (\text{A.4})$$

It is important to notice that the anomalous contractions $J^3(w)J^3(w') \sim k/(w - w')^2$ between different currents in higher-order terms in the expansion of the exponent does not contribute into eq. (A.4). This happens because after regularization of multiple integrals $(\oint_{\mathcal{C}} dww J^3(w))^p$ by introducing infinitesimal shifts of p contours, which is equivalent to the normal-ordered form of operators, one will not have poles within the smallest contour and thus the Cauchy integral will produce zero.

Eq. (A.4) reproduces eq. (2.13) of the main text.

B Second term in the null-vector condition (2.11)

After expanding the exponent and using expressions (2.10) for the currents, the second term in (2.11) reads

$$\begin{aligned}
 I &= \langle \Phi(\mathcal{C})(J_{-1}^a J_0^a) \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N) \rangle \\
 &= \sum_{k=1}^{\infty} \frac{\alpha^k}{k!} \left\langle \left(\oint_{\mathcal{C}} dw w J^3(w) \right)^k \oint_{C_1} du \frac{J^a(u)}{u-w_1} S_1^a \right. \\
 &\quad \left. \times \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N) \right\rangle
 \end{aligned} \tag{B.1}$$

Here we have used the relation eq. (A.1) for the action of J_0^a on primary fields and expression eq. (2.10) for J_{-1}^a . Eq. (B.1) shows that besides primary fields at w_i , there are also currents $J^a(u)$ with which $J^3(w)$ in the exponent will have contractions according to SU(2) current algebra relations

$$J^3(w) J^\pm(u) = \pm \frac{J^\pm(u)}{w-u}. \tag{B.2}$$

Therefore contour \mathcal{C} will include singularities not only at w_i , $i = 1 \cdots N$, but also in u (see figure 2). Hence boundary contour again will shrink into the sum $\mathcal{C} = \bigsqcup_{j=1}^N C_j$. As in eq. (A.4), the contribution of the contours C_j , $2 = 1, \cdots, N$ is equivalent to the replacement of $\sum_{i=2}^N \oint_{C_i} dw w J^3(w)$ by $\sum_{i=2}^N w_i S_i^3$. As a result, from eq. (B.1), one obtains

$$\begin{aligned}
 I &= \left\langle \sum_{k=1}^{\infty} \frac{\alpha^k}{k!} \left(\oint_{C_1} dw w J^3(w) \right)^k \oint_{\bar{C}_1} du \frac{J^a(u)}{u-w_1} S_1^a \right. \\
 &\quad \left. \times e^{\alpha \sum_{i=2}^N w_i S_i^3} \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N) \right\rangle
 \end{aligned} \tag{B.3}$$

The situation with C_1 is different. According to the CFT rules, the order of currents $J^3(w)$ and $J^a(u)$ defines the size of the contours in their integrals, namely, $C_1 \supset \bar{C}_1$, see figure 2.

Now, let us first analyze the linear term in the series of exponential in eq. (B.3). After using the current algebra relation eq. (B.1) and Cauchy integration over w , one obtains

$$\begin{aligned}
 \oint_{C_1} dw w \oint_{\bar{C}_1} du J^3(w) \frac{J^\pm(u)}{u-w_1} &= \pm \oint_{C_1} dw w \oint_{\bar{C}_1} du \frac{J^\pm(u)}{(u-w_1)(w-u)} \\
 &= \pm \oint_{\bar{C}_1} du u \frac{J^\pm(u)}{(u-w_1)}.
 \end{aligned} \tag{B.4}$$

Hence, for the exponent in eq. (B.1), one will have

$$e^{\alpha \oint_{C_1} dw w J^3(w)} \oint_{\bar{C}_1} du \frac{J^\pm(u)}{(u-w_1)} = \oint_{\bar{C}_1} du \frac{e^{\pm u} J^\pm(u)}{(u-w_1)} \tag{B.5}$$

and the entire second term eq. (B.1) becomes

$$I = \left\langle \oint_{\bar{C}_1} du \frac{e^{\alpha u} J^+(u) S_1^- + e^{-\alpha u} J^-(u) S_1^+}{u-w_1} \times e^{\alpha \sum_{i=2}^N w_i S_i^3} \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N) \right\rangle \tag{B.6}$$

Using the extension of the relation eq. (2.12) for primary fields to their product

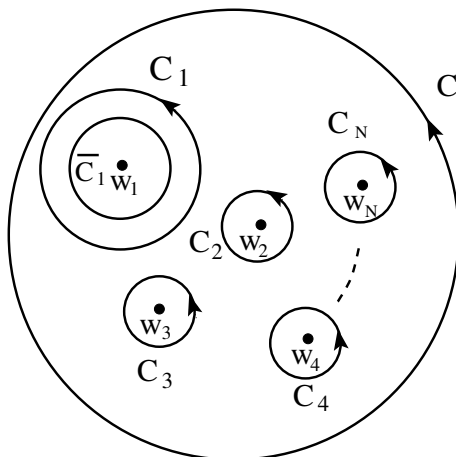


Figure 2. Boundary contour C shrinks to the sum of contours C_i , $i = 1, \dots, N$. Here \bar{C}_1 and C_i , $i = 2, \dots, N$ are the contours in the integral expression of the operators J_{-1}^a , which they act on the primary fields, $\phi_{s_i}(w_i)$, at points w_i , $i = 1, \dots, N$. We have that $C_1 \supset \bar{C}_1$.

$$J^a(u)\phi_{s_1}(w_1) \cdots \phi_{s_1}(w_N) = \sum_{i=1}^N \frac{S_i^a}{u - w_i} \phi_{s_i}(w_i) \cdots \phi_{s_1}(w_N). \tag{B.7}$$

and after Cauchy integration over u , one obtains

$$I = \left[\sum_{i=2}^N \frac{e^{\alpha w_1} S_i^+ S_1^- + e^{-\alpha w_1} S_i^- S_1^+}{w_1 - w_i} + \oint_{C_1} du \frac{e^{\alpha u} S_1^+ S_1^- + e^{-\alpha u} S_1^- S_1^+}{(u - w_1)^2} \right] \times e^\alpha \sum_{i=2}^N w_i S_i^3 \langle \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N) \rangle \tag{B.8}$$

Now, after using the identity

$$S_1^\pm e^{\mp \alpha w_1} = S_1^\pm e^{\alpha w_1 S_1^3} \tag{B.9}$$

and performing Cauchy integration, one arrives at

$$I = \left[\sum_{i=2}^N \frac{S_i^a S_1^a}{w_1 - w_i} + \alpha S_1^a S_1^a S_1^3 \right] \times e^\alpha \sum_{i=2}^N w_i S_i^3 \langle \phi_{s_1}(w_1) \cdots \phi_{s_N}(w_N) \rangle. \tag{B.10}$$

After defining $c_1 = \sum_{a=1}^3 S_1^a S_1^a$, we see that eq. (B.10) coincides with the relation eq. (2.14) of the main text.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] A. Kamenev, *Field theory of non-equilibrium systems*, Cambridge University Press, Cambridge, U.K. (2011).
- [2] A. Smith, M.S. Kim, F. Pollmann and J. Knolle, *Simulating quantum many-body dynamics on a current digital quantum computer*, *npj Quant. Inf.* **5** (2019) 106.

- [3] S. Ashhab, J.R. Johansson and F. Nori, *Decoherence in a scalable adiabatic quantum computer*, *Phys. Rev. A* **74** (2006) 052330.
- [4] J. Johansson et al., *Landau-Zener transitions in a superconducting flux qubit*, *Phys. Rev. B* **80** (2009) 012507.
- [5] T.A. Sedrakyan and V.M. Galitski, *Majorana path integral for nonequilibrium dynamics of two-level systems*, *Phys. Rev. B* **83** (2011) 134303.
- [6] F. Forster et al., *Characterization of qubit dephasing by Landau-Zener-Stückelberg-Majorana interferometry*, *Phys. Rev. Lett.* **112** (2014) 116803.
- [7] T. Albash, S. Boixo, D.A. Lidar and P. Zanardi, *Quantum adiabatic Markovian master equations*, *New J. Phys.* **14** (2012) 123016.
- [8] B. Bauer, T. Karzig, R. Mishmash, A. Antipov and J. Alicea, *Dynamics of Majorana-based qubits operated with an array of tunable gates*, *SciPost Phys.* **5** (2018) 004.
- [9] D. Guéry-Odelin, A. Ruschhaupt, A. Kiely, E. Torrontegui, S. Martínez-Garaot and J. Muga, *Shortcuts to adiabaticity: concepts, methods, and applications*, *Rev. Mod. Phys.* **91** (2019) 045001 [[arXiv:1904.08448](#)].
- [10] S. Matityahu, H. Schmidt, A. Bilmes, A. Shnirman, G. Weiss, A.V. Ustinov et al., *Dynamical decoupling of quantum two-level systems by coherent multiple Landau-Zener transitions*, *npj Quant. Inf.* **5** (2019) 114.
- [11] X. Xu, Z. Zhang and Z. Liang, *Nonequilibrium Landau-Zener tunneling in exciton-polariton condensates*, *Phys. Rev. A* **102** (2020) 033317.
- [12] A. Niranjana, W. Li and R. Nath, *Landau-Zener transitions and adiabatic impulse approximation in an array of two Rydberg atoms with time-dependent detuning*, *Phys. Rev. A* **101** (2020) 063415.
- [13] X. Yang, R. Liu, J. Li and X. Peng, *Optimizing adiabatic quantum pathways via a learning algorithm*, *Phys. Rev. A* **102** (2020) 012614.
- [14] D.L. Campbell et al., *Universal nonadiabatic control of small-gap superconducting qubits*, *Phys. Rev. X* **10** (2020) 041051 [[INSPIRE](#)].
- [15] Z.-T. Zhang and D.E. Liu, *Readout of Majorana bound states via Landau-Zener transition*, *Phys. Rev. B* **103** (2021) 195401.
- [16] A.D. Stehli, *Analog quantum simulator of the multistate Landau-Zener model*, Doctoral dissertation, [Karlsruhe Institute of Technology \(KIT\)](#), Karlsruhe, Germany, 20 May 2021.
- [17] A.R. Klots and L.B. Ioffe, *Set of holonomic and protected gates on topological qubits for a realistic quantum computer*, *Phys. Rev. B* **104** (2021) 144502.
- [18] J.R. McClean et al., *Low-depth mechanisms for quantum optimization*, *PRX Quantum* **2** (2021) 030312.
- [19] M. Rodriguez-Vega et al., *Real-time simulation of light-driven spin chains on quantum computers*, [arXiv:2108.05975](#).
- [20] R. Richardson, *A restricted class of exact eigenstates of the pairing-force hamiltonian*, *Phys. Lett.* **3** (1963) 277.
- [21] R.W. Richardson, *Pairing in the limit of a large number of particles*, *J. Math. Phys.* **18** (1977) 1802.

- [22] J. von Delft and D. Ralph, *Spectroscopy of discrete energy levels in ultrasmall metallic grains*, *Phys. Rept.* **345** (2001) 61.
- [23] J. von Delft and R. Poghossian, *Algebraic Bethe ansatz for a discrete-state BCS pairing model*, *Phys. Rev. B* **66** (2002) 134502 [[cond-mat/0106405](#)] [[INSPIRE](#)].
- [24] J. Dukelsky, S. Pittel and G. Sierra, *Colloquium: exactly solvable Richardson-Gaudin models for many-body quantum systems*, *Rev. Mod. Phys.* **76** (2004) 643 [[nucl-th/0405011](#)] [[INSPIRE](#)].
- [25] E.A. Yuzbashyan, A.A. Baytin and B.L. Altshuler, *Finite-size corrections for the pairing hamiltonian*, *Phys. Rev. B* **71** (2005) 094505.
- [26] E.A. Yuzbashyan, *Normal and anomalous solitons in the theory of dynamical Cooper pairing*, *Phys. Rev. B* **78** (2008) 184507.
- [27] R. Teodorescu, *Coherent oscillations in superconducting cold Fermi atoms and their applications*, in *Leading-edge superconductivity research developments*, T. Watanabe ed., Nova Science Publishers, (2008).
- [28] A.M. García-García, J.D. Urbina, E.A. Yuzbashyan, K. Richter and B.L. Altshuler, *Bardeen-Cooper-Schrieffer theory of finite-size superconducting metallic grains*, *Phys. Rev. Lett.* **100** (2008) 187001.
- [29] A. Faribault, P. Calabrese and J.-S. Caux, *Exact mesoscopic correlation functions of the pairing model*, *Phys. Rev. B* **77** (2008) 064503 [[arXiv:0710.4865](#)] [[INSPIRE](#)].
- [30] T.A. Sedrakyan and V. Galitski, *Boundary Wess-Zumino-Novikov-Witten model from the pairing hamiltonian*, *Phys. Rev. B* **82** (2010) 214502.
- [31] D. Fioretto, J.-S. Caux and V. Gritsev, *Exact out-of-equilibrium central spin dynamics from integrability*, *New J. Phys.* **16** (2014) 043024 [[arXiv:1211.5905](#)] [[INSPIRE](#)].
- [32] S.L. Muli, A. Poggialini and S. Bacca, *Muonic lithium atoms: nuclear structure corrections to the Lamb shift*, *SciPost Phys. Proc.* **3** (2020) 028 [[arXiv:1910.14370](#)] [[INSPIRE](#)].
- [33] Y. Shen, P. Isaac and J. Links, *Ground-state energy of a Richardson-Gaudin integrable BCS model*, *SciPost Phys. Core* **2** (2020) 001.
- [34] Y.N. Demkov and V.I. Osherov, *Stationary and non-stationary problems in quantum mechanics that can be solved by means of contour integration*, *Zh. Exp. Teor. Fiz.* **53** (1967) 1589 [*Sov. Phys. JETP* **26** (1968) 916].
- [35] V.N. Ostrovsky and H. Nakamura, *Exact analytical solution of the N-level Landau-Zener-type bow-tie model*, *J. Phys. A* **30** (1997) 6939.
- [36] Y.N. Demkov and V.N. Ostrovsky, *Multipath interference in a multistate Landau-Zener-type model*, *Phys. Rev. A* **61** (2000) 032705.
- [37] Y.N. Demkov and V.N. Ostrovsky, *The exact solution of the multistate Landau-Zener type model: the generalized bow-tie model*, *J. Phys. B* **34** (2001) 2419.
- [38] N.A. Sinitsyn, *Multiparticle Landau-Zener problem: application to quantum dots*, *Phys. Rev. B* **66** (2002) 205303.
- [39] A. Altland and V. Gurarie, *Many body generalization of the Landau-Zener problem*, *Phys. Rev. Lett.* **100** (2008) 063602.
- [40] A. Altland, V. Gurarie, T. Kriecherbauer and A. Polkovnikov, *Nonadiabaticity and large fluctuations in a many-particle Landau-Zener problem*, *Phys. Rev. A* **79** (2009) 042703.

- [41] A. Patra and E.A. Yuzbashyan, *Quantum integrability in the multistate Landau-Zener problem*, *J. Phys. A* **48** (2015) 245303 [[arXiv:1412.4926](#)] [[INSPIRE](#)].
- [42] E.A. Yuzbashyan, *Integrable time-dependent hamiltonians, solvable Landau-Zener models and Gaudin magnets*, *Ann. Phys.* **392** (2018) 323.
- [43] R.K. Malla and M.E. Raikh, *Loss of adiabaticity with increasing tunneling gap in nonintegrable multistate Landau-Zener models*, *Phys. Rev. B* **96** (2017) 115437.
- [44] N.A. Sinitsyn, E.A. Yuzbashyan, V.Y. Chernyak, A. Patra and C. Sun, *Integrable time-dependent quantum hamiltonians*, *Phys. Rev. Lett.* **120** (2018) 190402 [[INSPIRE](#)].
- [45] K. Nishimura and K. Takahashi, *Counterdiabatic hamiltonians for multistate Landau-Zener problem*, *SciPost Phys.* **5** (2018) 029.
- [46] V.Y. Chernyak and N.A. Sinitsyn, *Integrability in the multistate Landau-Zener model with time-quadratic commuting operators*, *J. Phys. A* **54** (2021) 115204.
- [47] M. Gaudin, *Diagonalisation d'une classe d'hamiltoniens de spin* (in French), *J. Phys. France* **37** (1976) 1087.
- [48] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, *Nucl. Phys. B* **241** (1984) 333 [[INSPIRE](#)].
- [49] V.G. Knizhnik and A.B. Zamolodchikov, *Current algebra and Wess-Zumino model in two-dimensions*, *Nucl. Phys. B* **247** (1984) 83 [[INSPIRE](#)].
- [50] H.M. Babujian, *Correlation function in WZNW model as a Bethe wave function for the Gaudin magnetics*, Tech. Rep. [YERPHI-1261-47-90-\[EREVAN\]](#), Akad. Nauk Armyanskoj. Inst. Fiz., Erevan, Armenia (1990).
- [51] H.M. Babujian, *Off-shell Bethe ansatz equation and N point correlators in $SU(2)$ WZNW theory*, *J. Phys. A* **26** (1993) 6981 [[hep-th/9307062](#)] [[INSPIRE](#)].
- [52] H.M. Babujian and R. Flume, *Off-shell Bethe ansatz equation for Gaudin magnets and solutions of Knizhnik-Zamolodchikov equations*, *Mod. Phys. Lett. A* **9** (1994) 2029 [[hep-th/9310110](#)] [[INSPIRE](#)].
- [53] V.V. Mkhitarian, *Gaudin magnet and off-mass-shell bethe wave functions*, *Theor. Math. Phys.* **113** (1997) 1217.
- [54] B. Feigin, E. Frenkel and N. Reshetikhin, *Gaudin model, Bethe ansatz and correlation functions at the critical level*, *Commun. Math. Phys.* **166** (1994) 27 [[hep-th/9402022](#)] [[INSPIRE](#)].
- [55] E. Frenkel, *Gaudin model andopers*, in *Workshop on infinite dimensional algebras and quantum integrable systems*, P. Kulish et al. eds., *Prog. Math.* **237** (2004) (2004) 1 [[math.QA/0407524](#)] [[INSPIRE](#)].
- [56] E. Frenkel, *Affine algebras, Langlands duality and Bethe ansatz*, in *Proceedings of the international congress of mathematical physics, Paris, France 1994*, D. Iagolnitzer ed., International Press, (1995), p. 606 [[q-alg/9506003](#)].
- [57] A. Beilinson and V. Drinfeld, *Quantization of Hitchin's integrable system and Hecke eigensheaves*, <http://math.uchicago.edu/~drinfeld/langlands/QuantizationHitchin.pdf>.
- [58] E. Frenkel, P. Koroteev, D.S. Sage and A.M. Zeitlin, *q -opers, QQ-systems, and Bethe ansatz*, [arXiv:2002.07344](#) [[INSPIRE](#)].

- [59] S. Jeong, N. Lee and N. Nekrasov, *Intersecting defects in gauge theory, quantum spin chains, and Knizhnik-Zamolodchikov equations*, *JHEP* **10** (2021) 120 [[arXiv:2103.17186](#)] [[INSPIRE](#)].
- [60] N. Nekrasov and A. Tsymbaliuk, *Surface defects in gauge theory and KZ equation*, [arXiv:2103.12611](#) [[INSPIRE](#)].
- [61] I. Salom, N. Manojlović and N. Cirilo António, *Generalized $sl(2)$ Gaudin algebra and corresponding Knizhnik-Zamolodchikov equation*, *Nucl. Phys. B* **939** (2019) 358 [[INSPIRE](#)].
- [62] I. Salom and N. Manojlović, *Bethe states and Knizhnik-Zamolodchikov equations of the trigonometric Gaudin model with triangular boundary*, *Nucl. Phys. B* **969** (2021) 115462 [[INSPIRE](#)].
- [63] D. Gaiotto, J.H. Lee, B. Vicedo and J. Wu, *Kondo line defects and affine Gaudin models*, *JHEP* **01** (2022) 175 [[arXiv:2010.07325](#)] [[INSPIRE](#)].
- [64] M. Vasilyev, A. Zabrodin and A. Zotov, *Quantum-classical duality for Gaudin magnets with boundary*, *Nucl. Phys. B* **952** (2020) 114931 [[arXiv:1911.11792](#)] [[INSPIRE](#)].
- [65] T. Skrypnyk and N. Manojlović, *Twisted rational r -matrices and algebraic Bethe ansatz: application to generalized gaudin and Richardson models*, *Nucl. Phys. B* **967** (2021) 115424.
- [66] T. Skrypnyk, *Anisotropic BCS-Richardson model and algebraic Bethe ansatz*, *Nucl. Phys. B* **975** (2022) 115679.
- [67] F. Delduc, S. Lacroix, M. Magro and B. Vicedo, *Assembling integrable σ -models as affine Gaudin models*, *JHEP* **06** (2019) 017 [[arXiv:1903.00368](#)] [[INSPIRE](#)].
- [68] I. Buric, S. Lacroix, J.A. Mann, L. Quintavalle and V. Schomerus, *Gaudin models and multipoint conformal blocks. Part II. Comb channel vertices in 3D and 4D*, *JHEP* **11** (2021) 182 [[arXiv:2108.00023](#)] [[INSPIRE](#)].
- [69] I. Buric, S. Lacroix, J.A. Mann, L. Quintavalle and V. Schomerus, *Gaudin models and multipoint conformal blocks: general theory*, *JHEP* **10** (2021) 139 [[arXiv:2105.00021](#)] [[INSPIRE](#)].
- [70] G.A. Kotousov and S.L. Lukyanov, *ODE/IQFT correspondence for the generalized affine $sl(2)$ Gaudin model*, *JHEP* **09** (2021) 201 [[arXiv:2106.01238](#)] [[INSPIRE](#)].
- [71] M.R. Adams, J.P. Harnad and E. Previato, *Isospectral Hamiltonian flows in finite and infinite dimensions. 1. Generalized Moser systems and moment maps into loop algebras*, *Commun. Math. Phys.* **117** (1988) 451 [[INSPIRE](#)].
- [72] M. Adams, J.P. Harnad and J. Hurtubise, *Dual moment maps into loop algebras*, *Lett. Math. Phys.* **20** (1990) 299 [[INSPIRE](#)].
- [73] J.P. Harnad, *Dual isomonodromic deformations and moment maps to loop algebras*, *Commun. Math. Phys.* **166** (1994) 337 [[hep-th/9301076](#)] [[INSPIRE](#)].
- [74] G. Felder, Y. Markov, V. Tarasov and A. Varchenko, *Differential equations compatible with KZ equations*, *Math. Phys. Anal. Geom.* **3** (2000) 139 [[math.QA/0001184](#)] [[INSPIRE](#)].
- [75] V. Tarasov and A. Varchenko, *Duality for Knizhnik-Zamolodchikov and dynamical equations*, [math.QA/0112005](#).
- [76] H.M. Babujian and A.V. Kitaev, *Generalized Knizhnik-Zamolodchikov equations and isomonodromy quantization of the equations integrable via the inverse scattering transform: Maxwell-Bloch system with pumping*, *J. Math. Phys.* **39** (1998) 2499 [[INSPIRE](#)].

- [77] D.P. Zhelobenko, *Compact Lie groups and their representations*, *Transl. Math. Mono.* **40**, Amer. Math. Soc., Providence, RI, U.S.A. (1983).
- [78] R. Howe, *Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond*, *Israel Math. Conf. Proc.* **8** (1995) 1.
- [79] A. Gorsky, M. Vasilyev and A. Zotov, *Dualities in quantum integrable many-body systems and integrable probabilities — I*, [arXiv:2109.05562](#) [INSPIRE].
- [80] Z. Chen, J. de Gier and M. Wheeler, *Integrable stochastic dualities and the deformed Knizhnik-Zamolodchikov equation*, *Int. Math. Res. Not.* **19** (2020) 5872 [[arXiv:1709.06227](#)] [INSPIRE].
- [81] F.D.M. Haldane, *Nonlinear field theory of large spin Heisenberg antiferromagnets. Semiclassically quantized solitons of the one-dimensional easy axis Néel state*, *Phys. Rev. Lett.* **50** (1983) 1153 [INSPIRE].
- [82] F.D.M. Haldane, *Continuum dynamics of the 1D Heisenberg antiferromagnetic identification with the $O(3)$ nonlinear sigma model*, *Phys. Lett. A* **93** (1983) 464 [INSPIRE].
- [83] L.A. Takhtajan, *The picture of low-lying excitations in the isotropic Heisenberg chain of arbitrary spins*, *Phys. Lett. A* **87** (1982) 479 [INSPIRE].
- [84] H.M. Babujian, *Exact solution of the one-dimensional isotropic Heisenberg chain with arbitrary spin S* , *Phys. Lett. A* **90** (1982) 479 [INSPIRE].
- [85] H.M. Babujian, *Exact solution of the isotropic Heisenberg chain with arbitrary spins: thermodynamics of the model*, *Nucl. Phys. B* **215** (1983) 317 [INSPIRE].
- [86] I. Affleck and F.D.M. Haldane, *Critical theory of quantum spin chains*, *Phys. Rev. B* **36** (1987) 5291 [INSPIRE].
- [87] A.M. Polyakov and P.B. Wiegmann, *Theory of non-Abelian Goldstone bosons*, *Phys. Lett. B* **131** (1983) 121 [INSPIRE].
- [88] A.M. Polyakov and P.B. Wiegmann, *Goldstone fields in two-dimensions with multivalued actions*, *Phys. Lett. B* **141** (1984) 223 [INSPIRE].
- [89] M.C. Cambiaggio, A.M.F. Rivas and M. Saraceno, *Integrability of the pairing Hamiltonian*, *Nucl. Phys. A* **624** (1997) 157 [[nucl-th/9708031](#)] [INSPIRE].
- [90] L. Amico, G. Falci and R. Fazio, *The BCS model and the off-shell Bethe ansatz for vertex models*, *J. Phys. A* **34** (2001) 6425 [[cond-mat/0010349](#)] [INSPIRE].
- [91] L. Amico and A. Osterloh, *Exact correlation functions of the BCS model in the canonical ensemble*, *Phys. Rev. Lett.* **88** (2002) 127003 [[cond-mat/0105141](#)] [INSPIRE].
- [92] R.K. Malla, V.Y. Chernyak and N.A. Sinitsyn, *Nonadiabatic transitions in Landau-Zener grids: integrability and semiclassical theory*, *Phys. Rev. B* **103** (2021) 144301.