

$\mathcal{N} = 4$ SYM, Argyres-Douglas theories, and an exact graded vector space isomorphism

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ABSTRACT: In this first of two papers, we explain in detail the simplest example of a broader set of relations between apparently very different theories. Our example relates $\mathfrak{su}(2)$ $\mathcal{N} = 4$ super Yang-Mills (SYM) to a theory we call “(3, 2)”. This latter theory is an exactly marginal diagonal $SU(2)$ gauging of three $D_3(SU(2))$ Argyres-Douglas (AD) theories. We begin by observing that the Schur indices of these two theories are related by an algebraic transformation that is surprisingly reminiscent of index transformations describing spontaneous symmetry breaking on the Higgs branch. However, this transformation breaks half the supersymmetry of the SYM theory as well as its full $\mathcal{N} = 2$ $SU(2)_F$ flavor symmetry. Moreover, it does so in an interesting way when viewed through the lens of the corresponding 2D vertex operator algebras (VOAs): affine currents of the small $\mathcal{N} = 4$ super-Virasoro algebra at $c = -9$ get mapped to the $\mathcal{A}(6)$ stress tensor and some of its conformal descendants, while the extra supersymmetry currents on the $\mathcal{N} = 4$ side get mapped to higher-dimensional fermionic currents and their descendants on the $\mathcal{A}(6)$ side. We prove these relations are facets of an exact graded vector space isomorphism (GVSI) between these two VOAs. This GVSI respects the $U(1)_r$ charge of the parent 4D theories. We briefly sketch how more general $\mathfrak{su}(N)$ $\mathcal{N} = 4$ SYM theories are related to an infinite class of AD theories via generalizations of our example. We conclude by showing that, in this class of theories, the $\mathcal{A}(6)$ VOA saturates a new inequality on the number of strong generators.

KEYWORDS: Conformal and W Symmetry, Conformal Field Theory, Extended Supersymmetry, Supersymmetric Gauge Theory

ARXIV EPRINT: [2012.13209](https://arxiv.org/abs/2012.13209)

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1 Introduction

In this note, we revisit a 4D $\mathcal{N} = 2$ superconformal field theory (SCFT) we first studied in [1] and find some remarkable relations it has to $\mathfrak{su}(2)$ $\mathcal{N} = 4$ super Yang-Mills (SYM). We will refer to the $\mathcal{N} = 2$ theory in question as the $(3, 2)$ SCFT.¹ This theory consists of three copies of the so-called $D_3(\text{SU}(2)) \simeq (A_1, A_3)$ Argyres-Douglas (AD) theory² with a gauged diagonal $\text{SU}(2)$ symmetry having vanishing beta function (see figure 1). In what follows, we will refer to these $D_3(\text{SU}(2))$ SCFTs as “ D_3 ” theories.

Modulo having the same $\mathfrak{su}(2)$ gauge algebra, $\mathfrak{su}(2)$ $\mathcal{N} = 4$ SYM and the $(3, 2)$ theory seem to be very different beasts. For example, in the former case, the $\mathcal{N} = 2$ matter sector (prior to gauging) consists of a free adjoint-valued hypermultiplet while, in the latter case, it consists of three strongly interacting D_3 SCFTs.

However, this picture begins to change when one thinks of the so-called “Schur sector” of operators in both SCFT matter sectors (prior to gauging). In particular, the resemblance becomes starker when one thinks in terms of the related 2D vertex operator algebras (VOAs) that the general correspondence in [10] assigns to the matter sectors in question. On the SYM side, we have a VOA (strongly) generated by adjoint-valued dimension 1/2 symplectic bosons while, on the $(3, 2)$ side, we have a VOA (strongly) generated by adjoint-valued affine currents for each D_3 matter sector [11–14].

This realization motivates a cursory glance at some of the basic observables of the 4D theories we are discussing and reveals the following: in both cases $a = c$, and, as we will explain in section 2, the Schur indices of the SYM theory and $(3, 2)$ are related in a simple way

$$\mathcal{I}_{(3,2)}(q) = \mathcal{I}_{\mathfrak{su}(2)}\left(q^3; q^{\frac{1}{2}}\right), \tag{1.1}$$

¹In [1], we referred to this SCFT somewhat unimaginatively as $\hat{\mathcal{T}}$. It has also been studied in [2], where it was called the $(E_6^{1,1}, \text{SU}(2))$ SCFT. This theory has been further studied in [3–5].

²These theories were originally discovered in [6]. The nomenclature is borrowed from [7–9].

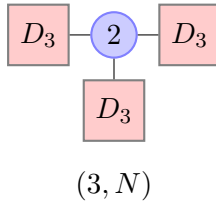


Figure 1. Our AD theory of interest, $(3, 2)$, consists of an exactly marginal diagonal $SU(2)$ gauging of three $D_3(SU(2))$ SCFTs.

where the label “ $\mathfrak{su}(2)$ ” refers to the corresponding $\mathcal{N} = 4$ theory. In writing (1.1), we have set $x = q^{\frac{1}{2}}$, where x is the flavor $SU(2)_F$ fugacity that arises when we think of the $\mathcal{N} = 4$ theory as an $\mathcal{N} = 2$ theory.

As we will explain in section 3, these features have a deeper explanation in terms of a mathematically precise exact graded vector space isomorphism (GVSI) between the 2D vertex operator algebras (VOAs) that correspond to $(3, 2)$ [1] and $\mathfrak{su}(2)$ $\mathcal{N} = 4$ SYM [10]. In other words, we find a GVSI between the $\mathcal{A}(6)$ algebra of [15, 16] and the small $\mathcal{N} = 4$ super-Virasoro algebra at $c = -9$.

We suggest that the GVSI and its consequences can be thought of physically as comprising a distant cousin of spontaneous symmetry breaking on the Higgs branch. At the level of moduli spaces there is a superficial similarity to moving onto the Higgs branch and removing a decoupled Nambu-Goldstone (NG) multiplet: $\mathfrak{su}(2)$ $\mathcal{N} = 4$ SYM (thought of as an $\mathcal{N} = 2$ theory) has a 2-complex dimensional Higgs branch, while the $(3, 2)$ theory has a trivial Higgs branch. Moreover, just as in the case of two theories related by Higgs branch renormalization group flows, the $(3, 2)$ and $\mathfrak{su}(2)$ $\mathcal{N} = 4$ SYM theories have the same value of the $U(1)_r \subset SU(2)_R \times U(1)_r$ anomaly, namely

$$\mathrm{Tr}|_{(3,2)} U(1)_r = \mathrm{Tr}|_{\mathfrak{su}(2)} U(1)_r = 0. \quad (1.2)$$

This result follows from the fact that $a_{(3,2)} - c_{(3,2)} = a_{\mathfrak{su}(2)} - c_{\mathfrak{su}(2)}$.

Yet another similarity arises when one thinks of moving onto the Higgs branch in terms of the index [17]. More precisely, to describe the Higgsing of the $\mathcal{N} = 4$ theory in terms of the index, we can, just as in (1.1), set $x = q^{\frac{1}{2}}$. Performing this substitution corresponds to leaving unsuppressed certain contributions to the Schur index from the $SU(2)_F$ lowest-weight component of the holomorphic moment map, μ_- . The resulting divergence of the index is interpreted as setting $\langle \mu_- \rangle \neq 0$ and moving onto the moduli space. The corresponding residue of the index describes the IR theory that remains (after removing the NG multiplet).

From the perspective of the index, only the fact that we simultaneously rescale $q \rightarrow q^3$ in (1.1) keeps $\mathcal{I}_{\mathfrak{su}(2)}$ finite and tells the index to describe physics different from Higgsing. As we will see, this rescaling leads to interesting phenomena like the fact that, at the level of the corresponding VOAs, some of the (strong) generators of the $c = -9$ small $\mathcal{N} = 4$ super-Virasoro algebra are mapped to generators of the $\mathcal{A}(6)$ chiral algebra while others are mapped to descendants. More generally, the number of derivatives acting on an operator

in one theory is not preserved under the mapping to its cousin in the other theory (see section 3 for details).

However, even this peculiar mapping of operators is somewhat reminiscent of spontaneous symmetry breaking. Indeed, consider a flavor symmetry current for some spontaneously broken symmetry in a general quantum field theory (QFT) in $d > 2$. In the deep IR, this current is mapped to a descendant of the NG field

$$j_\mu \rightarrow f_\pi \partial_\mu \pi. \tag{1.3}$$

More generally, we may expect the following mapping between operators in the UV and IR

$$\partial^n \mathcal{O}_{UV} \rightarrow \partial^m \mathcal{O}_{IR}, \tag{1.4}$$

where it may happen that $n \neq m$.³

In the context of $\mathcal{N} = 2$ theories, it is useful to think of the above discussion in terms of Higgs branch operators and their Hall-Littlewood (HL) generalizations [10, 18]. Thinking along these lines, we easily find examples of (1.3) and (1.4). Such situations often arise due to the fact that certain Higgs branch operators of the UV theory get mapped to operators that, in the deep IR, have support only in the low energy effective QFT describing the decoupled NG multiplets and vanish in the remaining IR QFT (e.g., the $\mathcal{N} = 2$ flavor symmetry current multiplets for spontaneously broken flavor symmetries). In terms of symbols, we have

$$\mathcal{O}_{UV}^{\text{HL}} \rightarrow \mathcal{O}_{\text{IR}}|_{\text{rem.}} = 0, \tag{1.5}$$

where $\mathcal{O}_{UV}^{\text{HL}}$ is an HL operator in the UV theory, $\mathcal{O}_{\text{IR}} \neq 0$ is its IR avatar supported in the NG effective QFT, and “ $|_{\text{rem.}}$ ” denotes the restriction of this operator to the part of the IR theory decoupled from the NG multiplets.

We will see that many of these phenomena have cousins in our case. For example, all HL operators of the $\mathfrak{su}(2)$ $\mathcal{N} = 4$ SYM theory will be mapped to non-HL operators in the $(3, 2)$ theory (whose Higgs branch and HL sector vanishes). In other words, instead of (1.5), we will have

$$\mathcal{O}_{\mathfrak{su}(2)}^{\text{HL}} \rightarrow \mathcal{O}_{(3,2)}|_{\text{HL}} = 0, \tag{1.6}$$

where $\mathcal{O}_{(3,2)} \neq 0$ is an operator of the $(3, 2)$ SCFT, and “ $|_{\text{HL}}$ ” denotes the restriction to the (trivial) HL ring.

The plan of this paper is as follows. In the next section we describe the ingredients that lead to the index relation (1.1). With this groundwork out of the way, in section 3 we proceed to describe the GVSIs between the VOAs discussed above. In section 4, we tease a generalization of the relation between $(3, 2)$ and $\mathfrak{su}(2)$ $\mathcal{N} = 4$ SYM that will be discussed in much greater detail in [26]. In particular, we show that the $\mathcal{A}(6)$ VOA saturates an inequality on the number of strong generators for VOAs in the class of theories discussed in [26]. We conclude with some open questions and future directions.

³We have been schematic about contraction of indices above, but both sides of the above relation must transform in the same way under the Lorentz group.

2 The index relation

In this section we derive the index relation (1.1) and explain some of its consequences before setting the stage for a discussion of the GVSI in section 3.

Let us begin by briefly reminding the reader that the Schur index [18] is a specialization of the $\mathcal{N} = 2$ superconformal index counting certain, at worst 1/4-BPS, local operators. In particular, the Schur index is a refined signed trace over the Hilbert space of local operators, \mathcal{H}

$$\mathcal{I}(q; \vec{x}) := \text{Tr}_{|\mathcal{H}} (-1)^F e^{-\beta\Delta} q^{E-R} \prod_i (x_i)^{f_i}, \quad (2.1)$$

where $|q| < 1$, \vec{x} is a vector of flavor fugacities corresponding to weights \vec{f} , F is fermion number, E is the conformal dimension, R is the $SU(2)_R$ weight, and $\Delta := \left\{ \mathcal{Q}_{2^-}, (\mathcal{Q}^{2^-})^\dagger \right\}$.

Given this definition, we would like to construct $\mathcal{I}_{(3,2)}$, $\mathcal{I}_{\mathfrak{su}(2)}$, and check (1.1). For ease of reference in what follows, we reproduce this relation below

$$\mathcal{I}_{(3,2)}(q) = \mathcal{I}_{\mathfrak{su}(2)}(q^3; q^{\frac{1}{2}}), \quad (2.2)$$

where we again remind the reader that “ $\mathfrak{su}(2)$ ” stands for $\mathfrak{su}(2)$ $\mathcal{N} = 4$ SYM.

The simple relation in (2.2) can, at some level, be anticipated from the simple form the Schur index of D_3 takes:⁴ it has simple “single letter” contributions when written as in [19–21]. In particular, we have

$$\mathcal{I}_{D_3}(q; y) = \text{P.E.} \left[\left(\frac{q(1+q)}{1-q^3} \right) \chi_{\text{adj}}(y) \right], \quad (2.3)$$

where y is a fugacity for the D_3 ’s $SU(2)$ flavor symmetry that we gauge to produce the (3, 2) SCFT, and we define the “plethystic exponential” to be $\text{P.E.}[g(q; x_1, \dots, x_p)] := \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} g(q^n; x_1^n, \dots, x_p^n) \right)$. Notice that the expression in (2.3) is not too different from the index of the hypermultiplets in the SYM theory

$$\mathcal{I}_{\text{hyp}}(q; y, x) = \text{P.E.} \left[\frac{q^{\frac{1}{2}}}{1-q} (x + x^{-1}) \chi_{\text{adj}}(y) \right], \quad (2.4)$$

where y is the fugacity for the $\mathfrak{su}(2)$ flavor symmetry we gauge to produce the SYM theory, and x is a fugacity for the remaining $SU(2)_F$ $\mathcal{N} = 2$ flavor symmetry we have discussed at length above.

To generate the relation in (2.2) given the building blocks in (2.3) and (2.4), we need only integrate the vector multiplet contribution to the index

$$\mathcal{I}_{\text{vec}}(q; y) = \text{P.E.} \left[-\frac{2q}{1-q} \chi_{\text{adj}}(y) \right], \quad (2.5)$$

over the $\mathfrak{su}(2)$ Haar measure and include the appropriate matter contributions. Doing so, we find

$$\mathcal{I}_{\mathfrak{su}(2)}(q; x) = \int d\mu_{\mathfrak{su}(2)}(y) \cdot \mathcal{I}_{\text{vec}}(q; y) \cdot \mathcal{I}_{\text{hyp}}(q; y, x), \quad (2.6)$$

⁴This phenomenon is familiar from the general $D_p(SU(N))$ theories: they are very much the closest AD cousins of free theories [22–25].

and

$$\mathcal{I}_{(3,2)}(q) = \int d\mu_{\text{SU}(2)}(y) \cdot \mathcal{I}_{\text{vec}}(q; y) \cdot (\mathcal{I}_{D_3}(q; y))^3, \quad (2.7)$$

from which (2.2) easily follows upon taking $q \rightarrow q^3$ and $x \rightarrow q^{\frac{1}{2}}$ in (2.6).

It is interesting to note that one consequence of the index relation we have derived in (2.2) is that the $U(1)_r$ anomalies discussed in the introduction must match. Indeed, since the $SU(2)_F$ flavor symmetry has vanishing linear 't Hooft anomaly,⁵ we see that $\mathcal{I}_{(3,2)}$ and $\mathcal{I}_{\mathfrak{su}(2)}$ have the same “high-temperature” behavior and hence the same value of $a - c$ (see the discussion in [11]). Indeed,

$$\begin{aligned} a_{(3,2)} = c_{(3,2)} = 2 &\Rightarrow \text{Tr}|_{(3,2)} U(1)_r = 0, \\ a_{\mathfrak{su}(2)} = c_{\mathfrak{su}(2)} = \frac{3}{4} &\Rightarrow \text{Tr}|_{\mathfrak{su}(2)} U(1)_r = 0. \end{aligned} \quad (2.8)$$

While the above discussion strongly suggests that the Schur sectors of $\mathfrak{su}(2)$ $\mathcal{N} = 4$ SYM and the (3, 2) SCFT are related, it is not at all obvious from the facts presented thus far that there is a particularly simple map between these two sectors. Indeed, it turns out that both the (3, 2) and $\mathfrak{su}(2)$ $\mathcal{N} = 4$ theories have fermionic and bosonic Schur operators.⁶ Therefore, even though the Schur indices are closely related, the $(-1)^F$ in the definition (2.1) can sweep various differences in operator content under the rug. For example, at the level of the index, there is no general way to distinguish relations amongst bosons from fermionic Schur operators with the same quantum numbers (and vice versa). The main result of the next section will be to show that, in spite of this possibility, there is in fact a GVSIs between the two Schur sectors when analyzed at the levels of the corresponding VOAs.

3 The exact graded vector space isomorphism

To gain further insight into the mechanism that explains the index relation in (2.2), it is useful to consider the associated 2D VOAs in the sense of [10]. Indeed, for any 4D $\mathcal{N} = 2$ SCFT, \mathcal{T} , [10] showed there is a corresponding 2D chiral algebra, $\chi[\mathcal{T}]$, living on a plane $\mathcal{P} \simeq \mathbb{R}^2 \subset \mathbb{R}^4$. Via an $SU(2)_R$ twisting on \mathcal{P} , one can show that each Schur operator resides in a cohomology class (with respect to a nilpotent supercharge) that is in one-to-one correspondence with a state of the VOA living on \mathcal{P} .

We can make contact with the discussion of the previous section by noting that the torus partition function for $\chi[\mathcal{T}]$ takes the form

$$Z_{T^2; \chi[\mathcal{T}]}(q; z; \vec{x}) = \text{Tr} z^{M^\perp} q^{L_0 - \frac{c_{2d}}{24}} \prod_i (x_i)^{f_i}, \quad (3.1)$$

where $M^\perp = j_1 - j_2 = -r$ is the spin transverse to \mathcal{P} , $c_{2d} = -12c_{4d}$ is central charge of the VOA in terms of the 4D central charge, L_0 is the holomorphic weight, and x_i is a

⁵This statement is more generally true of any $\mathcal{N} = 2$ flavor symmetry [27].

⁶This statement can be seen by carefully combining the vector multiplet gaugino Schur operators into composites with the bosonic matter sector Schur operators. This procedure was carried out for the (3, 2) theory in [1].

flavor fugacity (sometimes referred to in the VOA literature as a Jacobi variable) with corresponding weight f_i . Note here that r is just the $U(1)_r$ charge and that it is a conserved (though non-local) charge of the VOA.

Since they count essentially the same states, it should come as no surprise that [10]

$$Z_{T^2; \chi[\mathcal{T}]}(q; (-1); \vec{x}) = q^{-\frac{c_2 d}{24}} \mathcal{I}_{\mathcal{T}}(q; \vec{x}). \tag{3.2}$$

In this relation, the holomorphic dimension maps as follows: $h = E - R$. Therefore, we can translate (2.2) into the 2D statement that

$$Z_{T^2, \chi[(3,2)]}(q; (-1)) = q^{\frac{1}{8}} Z_{T^2, \chi[\mathfrak{su}(2)]}(q^3; (-1); q^{\frac{1}{2}}). \tag{3.3}$$

However, our goal is to go beyond this relation and to understand if there is a non-trivial mapping of states in the two VOAs. On the $\mathcal{N} = 4$ side we have the following:

1. From [10, 28], we know that $\chi[\mathfrak{su}(2)] = \text{sVir}_{\text{sm } \mathcal{N}=4}^{c=-9}$, i.e., the VOA associated with the $\mathfrak{su}(2)$ $\mathcal{N} = 4$ theory is the small $\mathcal{N} = 4$ super-Virasoro algebra at $c = -9$.
2. The bosonic strong generators of this VOA are the three affine currents of $\widehat{\mathfrak{su}(2)}_{-\frac{3}{2}}$

$$\text{Bos. strong gens. : } J^{0, \pm} \in \widehat{\mathfrak{su}(2)}_{-\frac{3}{2}} \subset \text{sVir}_{\text{sm } \mathcal{N}=4}^{c=-9}. \tag{3.4}$$

These currents have $h = 1$, $r = 0$, and are related to 4D Higgs branch operators (i.e., moment map primaries of $\widehat{\mathcal{B}}_1$ type with $R = 1$ in the language of [29] that correspond to $SU(2)_F$; see also [30]). Therefore $J^{0, \pm}$ are also the bosonic generators of the HL chiral ring. Here, T is not an independent generator (it is the Sugawara stress tensor).

3. The fermionic strong generators are the four $\mathcal{N} = 4$ supercurrents of $h = \frac{3}{2}$

$$\text{Ferm. strong gens. : } G^{\pm}, \tilde{G}^{\pm}, \tag{3.5}$$

where G^{\pm} have $r = 1/2$ and \tilde{G}^{\pm} have $r = -1/2$. These latter currents are also HL generators (they reside in multiplets of type $\widehat{\mathcal{D}}_{\frac{1}{2}(0,0)}$) and the former are not (they reside in multiplets of type $\mathcal{D}_{\frac{1}{2}(0,0)}$).

On the (3, 2) side we have the following:

1. From [1], we know that $\chi[(3,2)] = \mathcal{A}(6)$, i.e., the VOA associated with the (3, 2) theory is the $c = -24$ $\mathcal{A}(6)$ chiral algebra of Feigin, Feigin, and Tipunin [15, 16].
2. The only bosonic strong generator of this VOA is the energy momentum tensor

$$\text{Bos. strong gens. : } T \in \text{Vir}_{c=-24} \subset \mathcal{A}(6). \tag{3.6}$$

This operator has $h = 2$, $r = 0$, and is related to the 4D $SU(2)_R$ current (it is a superconformal descendant in the $\widehat{\mathcal{C}}_{0(0,0)}$ stress tensor multiplet). This is not a Higgs branch or HL operator.

3. The fermionic currents are two $h = 4$ currents

$$\text{Ferm. strong gens. : } \Psi, \tilde{\Psi}, \tag{3.7}$$

where Ψ has $r = 1/2$ and $\tilde{\Psi}$ has $r = -1/2$. Using the D_3 Macdonald index [14] or conformal perturbation theory, one can argue that these operators have to be of $\widehat{\mathcal{C}}_{\frac{3}{2}(\frac{1}{2},0)} \oplus \widehat{\mathcal{C}}_{\frac{3}{2}(0,\frac{1}{2})}$ type [1]. Hence, as discussed in the introduction, the (3,2) theory has no Higgs branch or HL operators.⁷

Clearly the above VOAs are not isomorphic: the central charges and number of strong generators are different. Moreover the HL rings and Higgs branches do not match. Still, given the result in (3.3) we can hope for a non-trivial isomorphism of the VOAs when thought of as $(-1)^F$ (or equivalently $U(1)_r$) graded vector spaces. In particular, we would like to see if we can map operators to operators and null states to null states while preserving statistics and $U(1)_r$. In other words, we would like to see if we can construct the GVSI promised in the introduction. Such a GVSI implies (2.2) and (3.3), but, as discussed in section 2, it is a much stronger result.

To motivate such a GVSI, let us work out what such a map

$$\varphi : \chi[\mathfrak{su}(2)] \rightarrow \chi[(3,2)], \tag{3.8}$$

would look like for small values of h . Crucially, in addition to preserving $U(1)_r$, we are forced by (3.3) to set

$$h_{(3,2)} = 3h_{\mathfrak{su}(2)} + \frac{1}{2}f, \tag{3.9}$$

where $h_{(3,2)}$ is the holomorphic dimension in $\mathcal{A}(6)$, $h_{\mathfrak{su}(2)}$ is the corresponding quantity in $\text{sVir}_{\mathcal{N}=4}^{c=-9}$, and f is the weight under $SU(2)_F$.

The lowest-dimensional non-trivial operator in $\mathcal{A}(6)$ is the stress tensor at $h_{(3,2)} = 2$. The constraint (3.9) fixes

$$\varphi(J^-) = T. \tag{3.10}$$

The next non-trivial state is ∂T at $h_{(3,2)} = 3$, and again (3.9) fixes a unique choice

$$\varphi(J^0) = \partial T. \tag{3.11}$$

At $h_{(3,2)} = 4$ we have two bosonic states: $\partial^2 T$ and T^2 . At the level of a GVSI, we can set $\varphi(J^+)$ and $\varphi((J^-)^2)$ to any two independent linear combinations of these states. However, it is natural to also demand that the normal-ordered product is respected so that

$$\varphi(J^+) = \partial^2 T, \quad \varphi((J^-)^2) = T^2. \tag{3.12}$$

⁷Consistency with the bosonic generator in (3.6) suggests that these operators cannot come from 4D superconformal descendants of $\mathcal{D}_{R(0,j_2)} \oplus \overline{\mathcal{D}}_{R(j_1,0)}$ multiplets. Indeed, otherwise we would find the HL ring only contains fermionic generators whereas all examples of HL rings we are aware of (for interacting theories) contain bosonic generators as well. It would be interesting to see if one can prove a theorem forbidding HL rings with purely fermionic generators for interacting theories.

We also have two $h_{(3,2)} = 4$ fermionic operators: Ψ and $\tilde{\Psi}$. These are uniquely identified via the requirement that φ respect $U(1)_r$ as

$$\varphi(G^-) = \Psi, \quad \varphi(\tilde{G}^-) = \tilde{\Psi}. \tag{3.13}$$

The relations in (3.10), (3.11), (3.12), and (3.13) succinctly express the idea behind our generalization of Higgsing: the VOA generators that map to HL states on the SYM Higgs branch (i.e., the $\mathcal{N} = 2$ NG multiplet for the $SU(2)_F$ symmetry breaking and its $\mathcal{N} = 4$ vector multiplet partner) are set to zero in the (trivial) $(3, 2)$ HL ring (as in (1.6)). However, unlike the case of motion onto the Higgs branch, the HL operators are mapped to non-trivial but non-HL states in the $(3, 2)$ theory.

Before moving on to a proof, it is also worth considering operators with $h_{(3,2)} = 5$ in order to understand how derivatives get mapped by φ and to understand the mapping of the remaining fermionic generators of $\text{sVir}_{\text{sm}}^{c=-9, \mathcal{N}=4}$. To that end, the bosonic operators at $h_{(3,2)} = 5$ are $\partial^3 T$ and $T\partial T$. We can again choose φ to preserve the normal ordered product by taking

$$\varphi(\partial J^-) = \partial^3 J^-, \quad \varphi(J^0 J^-) = (\partial T)T. \tag{3.14}$$

The fermionic operators at $h_{(3,2)} = 5$ are G^+ and \tilde{G}^+ . The fact that φ respects $U(1)_r$ means that

$$\varphi(G^+) = \partial\Psi, \quad \varphi(\tilde{G}^+) = \partial\tilde{\Psi}. \tag{3.15}$$

Finally, it is worth further motivating our proof by considering the fermionic operators with $h_{(3,2)} = 6$, since this is the first level with null vectors in $\mathcal{A}(6)$ [15, 16]

$$\kappa\partial^2\Psi + T\Psi = \kappa\partial^2\tilde{\Psi} + T\tilde{\Psi} = 0, \quad \kappa \neq 0. \tag{3.16}$$

These two null vectors are important for our candidate GVSI, φ , to work. Indeed, (3.16) means that there are now just two fermionic states at $h_{(3,2)} = 6$: $T\Psi \sim \partial^2\Psi$ and $T\tilde{\Psi} \sim \partial^2\tilde{\Psi}$. This number is just right because there are only two candidate states in $\text{sVir}_{\text{sm}}^{c=-9, \mathcal{N}=4}$ that can map onto these $\mathcal{A}(6)$ fermions, and $U(1)_r$ fixes this mapping uniquely

$$\varphi(J^- G^-) = T\Psi, \quad \varphi(J^- \tilde{G}^-) = T\tilde{\Psi}. \tag{3.17}$$

In addition to these tests, we also explicitly checked the remaining states at $h_{(3,2)} = 6$ as well as the states with $h_{(3,2)} = 7, 8$ and found that they are all consistent with the existence of the above GVSI.⁸

This discussion makes plausible the following theorem:

Theorem 1. The map φ in (3.8) is a GVSI respecting (3.9) with the following additional properties:

1. φ respects $U(1)_r$ charge: $r(\mathcal{O}) = r(\varphi(\mathcal{O}))$ for all $\mathcal{O} \in \chi[\mathfrak{su}(2)]$. By the taxonomy of Schur operators, this means φ respects the Bose/Fermi statistics of the operators it maps.

⁸We thank J. Shafiq for rechecking some of these results.

2. There exists a basis $\{\mathcal{O}_i\}$ of $\chi[\mathfrak{su}(2)]$ such that every \mathcal{O}_i is a normal ordered product of strong generators and/or their derivatives, and $\varphi(\mathcal{O}_i) = \varphi(\mathcal{O}_{i,1}) \cdots \varphi(\mathcal{O}_{i,k_i})$ when $\mathcal{O}_i = \mathcal{O}_{i,1} \cdots \mathcal{O}_{i,k_i}$.⁹
3. $\varphi(\partial^k \mathcal{O}) = \partial^{3k} \varphi(\mathcal{O})$ for all strong generators \mathcal{O} of $\chi[\mathfrak{su}(2)]$ in the basis mentioned in property 2.
4. The non-local $\mathfrak{su}(2)$ under which (G^\pm, \tilde{G}^\pm) transform as doublets is mapped to a non-local $\mathfrak{su}(2)$ under which $(\Psi, \tilde{\Psi})$ transform as a doublet.

The key idea that leads to a proof of the theorem is to compare a decomposition of $\text{sVir}_{\text{sm}}^{c=-9} \mathcal{N}=4$ in terms of Weyl modules of the universal affine vertex algebra, $V_{-\frac{3}{2}}(\mathfrak{su}(2))$, given in [31]

$$\text{sVir}_{\text{sm}}^{c=-9} \mathcal{N}=4 \simeq \bigoplus_{m=0}^{\infty} \left(\pi_{m+1} \otimes V_{-\frac{3}{2}}(m\omega) \right), \quad (3.18)$$

with a corresponding decomposition of the $\mathcal{A}(6)$ VOA in terms of Virasoro modules given in [15]

$$\mathcal{A}(6) \simeq \bigoplus_{m=0}^{\infty} \left(\pi_{m+1} \otimes M_{m+1,1;6} \right). \quad (3.19)$$

In (3.18) and (3.19), π_{m+1} are $m + 1$ -dimensional irreducible representations of the $\mathfrak{su}(2)$ mentioned in bullet 4 of the above theorem.

Proof. We wish to first show that, under the identification in (3.9)

$$M_{m+1,1;6} \simeq V_{-\frac{3}{2}}(m\omega), \quad (3.20)$$

as a linear equivalence of graded vector spaces. Here $M_{m+1,1;6}$ is the Virasoro module with Kac labels $(m + 1, 1)$ at $c = -24$, and $V_{-\frac{3}{2}}(m\omega)$ is the Weyl module of $V_{-\frac{3}{2}}(\mathfrak{su}(2))$ associated with $m\omega$ (where ω is the fundamental weight of $\mathfrak{su}(2)$).

To that end, as discussed in [31], $V_{-\frac{3}{2}}(m\omega)$ has no singular vectors and is therefore spanned by all vectors of the form

$$|s, \{a_k\}, \{b_k\}, \{c_k\}\rangle_{\mathfrak{su}(2)} := \left(\prod_{k=1}^{\infty} (J_{-k}^+)^{a_k} (J_{-k}^0)^{b_k} (J_{-k}^-)^{c_k} \right) |s\rangle_{\mathfrak{su}(2)}, \quad (3.21)$$

where the product is taken so that J_{-k}^A is on the right of $J_{-\ell}^B$ if $k < \ell$, a_k, b_k , and c_k are non-negative integers, and $s = 0, \dots, m$ labels eigenstates of J_0^0 with eigenvalue $s - \frac{m}{2}$.¹⁰ Therefore, $V_{-\frac{3}{2}}(m\omega)$ is linearly isomorphic to the Virasoro Verma module spanned by

$$|s, \{a_k\}, \{b_k\}, \{c_k\}\rangle_{\text{Vir}} := \left(\prod_{k=1}^{\infty} (L_{-3k-1})^{a_k} (L_{-3k})^{b_k} (L_{-3k+1})^{c_k} \right) L_{-1}^s |h\rangle, \quad (3.22)$$

⁹By definition of strong generators, every element of $\chi[\mathfrak{su}(2)]$ is written as a linear combination of normal ordered products of strong generators and/or their derivatives. However, non-trivial null operator relations imply that some of these normal ordered products are linearly dependent. Our statement here is that there exists a set $\{\mathcal{O}_i\}$ of linearly independent normal ordered products for which φ preserves the structure of the normal ordering.

¹⁰Note that f in (3.9) is twice the eigenvalue of J_0^0 .

where we set

$$h = 3 \frac{(m\omega, m\omega + 2\rho)}{2(k_{\mathfrak{su}(2)} + h_{\mathfrak{su}(2)}^{\vee})} - \frac{m}{2} = \frac{m(3m + 5)}{2}, \quad (3.23)$$

in order to guarantee that $|s\rangle_{\mathfrak{su}(2)} \rightarrow (L_{-1}^s)|h\rangle$ under our identification (3.9). We also have the following mapping of modes that is manifestly compatible with (3.9)

$$J_{-k}^{\pm} \rightarrow L_{-(3k\pm 1)}, \quad J_{-k}^0 \rightarrow L_{-3k}. \quad (3.24)$$

We can write the space spanned by our states in (3.22) more succinctly as the quotient of the Verma module, $\text{Verma}(h)$, by the subspace generated by $(L_{-1})^{m+1}|h\rangle$, and so we find

$$V_{-\frac{3}{2}}(m\omega) \simeq \text{Verma}(h)/\text{Vir}_-(L_{-1})^{m+1}|h\rangle, \quad (3.25)$$

where Vir_- is the subalgebra spanned by L_{-k} with $k > 0$.

Let us now analyze $M_{m+1,1;6}$. As discussed in [15], this is the quotient of $\text{Verma}(h_{m+1,1})$ by a subspace generated by a singular vector $|\psi\rangle$ at level $m + 1$

$$M_{m+1,1;6} \simeq \text{Verma}(h_{m+1,1})/\text{Vir}_-|\psi\rangle, \quad (3.26)$$

where

$$h_{m+1,1} = \frac{(6(m+1) - 1)^2 - 25}{24} = \frac{m(3m+5)}{2}. \quad (3.27)$$

Note that this holomorphic dimension coincides with the one in (3.23). Therefore, we have proven (3.20).

In the rest of this proof, we show that the above graded linear isomorphism can be equipped with the additional properties listed in theorem 1. As shown in the proof of theorem 2.5 of [31], the highest weight state of $\pi_{m+1} \otimes V_{-\frac{3}{2}}(m\omega)$ in (3.18) corresponds to $G^+ \partial G^+ \partial^2 G^+ \dots \partial^{m-1} G^+$,¹¹ where the $\mathfrak{su}(2)$ action is such that (G^+, \tilde{G}^+) transforms as a doublet. Therefore the decomposition (3.18) implies that $\text{sVir}_{\text{sm}}^{c=-9}_{\mathcal{N}=4}$ is spanned by the normal ordered products

$$\prod_{k=1}^{\infty} (\partial^{k-1} J^+)^{a_k} (\partial^{k-1} J^0)^{b_k} (\partial^{k-1} J^-)^{c_k} G_{i_1}^{(s_1)} \partial G_{i_2}^{(s_2)} \dots \partial^{m-1} G_{i_m}^{(s_m)}, \quad (3.28)$$

where $G_1^0 := G^-$, $G_1^1 := G^+$, $G_2^0 := \tilde{G}^-$, $G_2^1 := \tilde{G}^+$, $s_k \in \{0, 1\}$, $i_k \in \{1, 2\}$, and we used the state operator map $J_{-k}^A|0\rangle \rightarrow \partial^{k-1} J^A(0)$ for $A = 0, \pm$ and $k \geq 1$. Note that both the sub-scripts and super-scripts of G_i^s are completely symmetric in (3.28).

We now turn to the $\mathcal{A}(6)$ side. We use the free field realization of $\mathcal{A}(6)$ discussed in [15]; $T = \frac{1}{2}(\partial\phi)^2 + \frac{5}{2\sqrt{3}}\partial^2\phi$, $\Psi = e^{-\sqrt{3}\phi}$ and $\tilde{\Psi} = [\frac{1}{2\pi i} \oint dz e^{\sqrt{3}\phi(z)}, e^{-\sqrt{3}\phi}]$ where ϕ is a free field such that $\phi(z)\phi(0) \sim \log z$. With this realization, the highest weight state of $\pi_{m+1} \otimes M_{m+1,1;6}$ in (3.19) is written as $e^{-m\sqrt{3}\phi} \propto \Psi \partial^3 \Psi \partial^6 \Psi \dots \partial^{3m-3} \Psi$,¹² where the

¹¹Here the ‘‘highest weight’’ is in the sense of $\mathfrak{su}(2) \otimes V_{-\frac{3}{2}}(\mathfrak{su}(2))$.

¹²One can show that $e^{-m\sqrt{3}\phi} \propto \Psi \partial^3 \Psi \partial^6 \Psi \dots \partial^{3m-3} \Psi$ as follows. First note that $e^{-\sqrt{3}\phi(z_1)} \dots e^{-\sqrt{3}\phi(z_m)} = e^{-\sqrt{3}(\phi(z_1) + \dots + \phi(z_m))} \prod_{1 \leq i < j \leq m} (z_i - z_j)^3$. Differentiating this identity and using $\Psi = e^{-\sqrt{3}\phi}$, we see that $\Psi(z_1) \partial^3 \Psi(z_2) \partial^6 \Psi(z_3) \dots \partial^{3m-3} \Psi(z_m) \propto e^{-m\sqrt{3}\phi(z_m)} + X$, where X vanishes in the limit of $z_k \rightarrow z_m$ for $k = 1, \dots, m-1$. This implies that the normal ordered product $\Psi \partial^3 \Psi \dots \partial^{3m-3} \Psi$ is proportional to $e^{-m\sqrt{3}\phi}$.

$\mathfrak{su}(2)$ -action is such that $(\Psi, \tilde{\Psi})$ transforms as a doublet. Then (3.19) and our proof of (3.20) imply that $\mathcal{A}(6)$ is spanned by

$$\prod_{k=1}^{\infty} (\partial^{3k-1}T)^{a_k} (\partial^{3k-2}T)^{b_k} (\partial^{3k-3}T)^{c_k} \partial^s \left(\Psi_{(i_1)} \partial^3 \Psi_{i_2} \cdots \partial^{3m-3} \Psi_{i_m} \right) \quad (3.29)$$

where $\Psi_1 \equiv \Psi$, $\Psi_2 \equiv \tilde{\Psi}$, $s \in \{0, \dots, m\}$, $i_k \in \{1, 2\}$, and we used the state operator map $L_{-k}|0\rangle \rightarrow \partial^{k-2}T(0)$ for $k \geq 2$.

Let us now consider the linear map $\varphi : \text{sVir}_{\text{sm}}^{c=-9, \mathcal{N}=4} \rightarrow \mathcal{A}(6)$ that satisfies the properties listed in theorem 1 for the basis (3.28) of $\text{sVir}_{\text{sm}}^{c=-9, \mathcal{N}=4}$. Here we set $\varphi(J^-) = T$, $\varphi(J^0) = \partial T$, $\varphi(J^+) = \partial^2 T$, $\varphi(G^-) = \Psi$, $\varphi(G^+) = \partial \Psi$, $\varphi(\tilde{G}^-) = \tilde{\Psi}$ and $\varphi(\tilde{G}^+) = \partial \tilde{\Psi}$. Then φ maps the basis (3.28) to

$$\prod_{k=1}^{\infty} (\partial^{3k-1}T)^{a_k} (\partial^{3k-2}T)^{b_k} (\partial^{3k-3}T)^{c_k} \frac{1}{m!} \sum_{\sigma \in S_m} \partial^{s_{\sigma(1)}} \Psi_{(i_1)} \partial^{3+s_{\sigma(2)}} \Psi_{i_2} \cdots \partial^{3m-3+s_{\sigma(m)}} \Psi_{i_m}, \quad (3.30)$$

where S_m is the symmetric group of degree m (note from comparing (3.28) and (3.30), properties 1–3 mentioned in the theorem follow). Below we show that this φ is a graded linear isomorphism. It is straightforward to show that φ is compatible with (3.9) and therefore is a graded linear map. Then all we need to show is that (3.29) and (3.30) span the same space so that φ is a linear isomorphism.

The fact that (3.29) and (3.30) span the same space can be seen as follows. We focus on the sub-space spanned by the $\mathfrak{su}(2)$ highest weight states, since its orthogonal complement is generated by the action of the $\mathfrak{su}(2)$ lowering generator. We start with the fact that $\partial^s(\Psi \partial^3 \Psi \cdots \partial^{3m-3} \Psi) \propto \partial^s e^{-m\sqrt{3}\phi} = P_s(\partial\phi) e^{-m\sqrt{3}\phi}$, where $P_s(\partial\phi)$ is a degree- s differential polynomial of $\partial\phi$ involving s derivatives. Using $T = \frac{1}{2}(\partial\phi)^2 + \frac{5}{2\sqrt{3}}\partial^2\phi$, we see that $P_s(\partial\phi) e^{-m\sqrt{3}\phi} \propto (\partial\phi)^s e^{-m\sqrt{3}\phi} + \sum_{k=1}^s Q_k(T) (\partial\phi)^{s-k} e^{-m\sqrt{3}\phi}$, where $Q_k(T)$ are differential polynomials of T . This implies that (3.29) for $a_k, b_k, c_k \geq 0$ and $0 \leq s \leq m$ span the same space as

$$\prod_{k=1}^{\infty} (\partial^{3k-1}T)^{a_k} (\partial^{3k-2}T)^{b_k} (\partial^{3k-3}T)^{c_k} (\partial\phi)^s e^{-m\sqrt{3}\phi}, \quad (3.31)$$

for the same ranges of a_k, b_k, c_k and s . Similarly, one can show that (3.30) for $a_k, b_k, c_k \geq 0$ and $s_k \in \{0, 1\}$ span the same space as (3.31) where we identify $s = s_1 + \cdots + s_m$.¹³ Hence, the spaces spanned by (3.29) and (3.30) are identical, and therefore φ is a graded linear isomorphism satisfying properties 1–4. \square

¹³Indeed, since $e^{-\sqrt{3}\phi(z_1)} \cdots e^{-\sqrt{3}\phi(z_m)} = e^{-\sqrt{3}(\phi(z_1)+\cdots+\phi(z_m))} \prod_{1 \leq i < j \leq m} (z_i - z_j)^3$, it is straight forward to show that $\partial^{s_1} \Psi \partial^{3+s_2} \Psi \cdots \partial^{3m-3+s_m} \Psi$ is written as $P_{s_1+\cdots+s_m}(\partial\phi) e^{-m\sqrt{3}\phi}$ for a differential polynomial $P_{s_1+\cdots+s_m}(\partial\phi)$. Since $s_k \in \{0, 1\}$, $s_1 + \cdots + s_m \in \{0, \dots, m\}$. Therefore, the same argument as above implies (3.30) for $a_k, b_k, c_k \geq 0$ and $s_k \in \{0, 1\}$ span the same space as (3.31).

4 $|\chi[(3, 2)]|$ versus $|\chi[\mathfrak{su}(2)]|$ and an inequality

In this section we will describe some additional relations between the $(3, 2)$ VOA and its $\mathcal{N} = 4$ SYM VOA cousin. Our starting point is to recall that the number of strong generators in $\mathcal{A}(6) = \chi[(3, 2)]$ saturates a universal bound on the number of strong generators in VOAs related to 4D $\mathcal{N} = 2$ theories with exactly marginal gauge couplings [1]

$$|\chi[\mathcal{T}]| \geq 3, \tag{4.1}$$

where \mathcal{T} is any 4D $\mathcal{N} = 2$ SCFT with an exactly marginal gauge coupling, and $|\chi[\mathcal{T}]|$ denotes the number of strong generators of $\chi(\mathcal{T})$ (here $|\mathcal{A}(6)| = 3$ since, as discussed in the previous section, it is strongly generated by T, Ψ , and $\tilde{\Psi}$).

In what follows, we would like to argue that $|\mathcal{A}(6)|$ saturates another bound on the number of strong generators, but this time from below.

However, to discuss this bound, we will preview some results that will appear in our upcoming work [26]. In particular, in [26] we will argue that one can generalize some of the above results (and find various new ones) by considering an infinite set of $\mathcal{N} = 2$ SCFTs that are closely related to the $(3, 2)$ theory.

We call these generalized theories (n, N) SCFTs (they have also been studied in [2] under different names¹⁴). We consider four infinite cases:

1. The $(3, N)$ SCFT with $\gcd(3, N) = 1$ as in figure 2. It consists of an exactly marginal $SU(N)$ gauging of three $D_3(SU(N)) := D_{3,N}$ SCFTs. The $(3, 2)$ SCFT discussed in the previous sections has $N = 2$ (and $D_{3,2} := D_3$).
2. The $(2, N)$ SCFT with $\gcd(2, N) = 1$ as in figure 3. It consists of an exactly marginal $SU(N)$ gauging of four $D_2(SU(N)) := D_{2,N}$ SCFTs.
3. The $(4, N)$ SCFT with $\gcd(4, N) = 1$ as in figure 4. It consists of an exactly marginal $SU(N)$ gauging of two $D_4(SU(N)) := D_{4,N}$ SCFTs and one $D_{2,N}$ theory.
4. The $(6, N)$ SCFT with $\gcd(6, N) = 1$ as in figure 5. It consists of an exactly marginal $SU(N)$ gauging of one $D_{2,N}$ SCFT, one $D_{3,N}$ theory, and one $D_6(SU(N)) := D_{6,N}$ SCFT.

Among other results, we will argue in [26] for the following generalization of (1.1), where $\mathfrak{su}(2)$ $\mathcal{N} = 4$ SYM is generalized to $\mathfrak{su}(N)$ $\mathcal{N} = 4$ SYM

$$\mathcal{I}_{(n,N)}(q) = \mathcal{I}_{\mathfrak{su}(N)}(q^n, q^{n/2-1}), \tag{4.2}$$

and the following generalization of (3.9)

$$h_{(n,N)} = n \cdot h_{\mathfrak{su}(N)} + \left(\frac{n}{2} - 1\right) f. \tag{4.3}$$

Generalizing section 3, we will argue in [26] that, among other things, there is an underlying GVSI

$$\varphi_{n,N} : \chi[\mathfrak{su}(N)] \rightarrow \chi[(n, N)], \tag{4.4}$$

that respects $U(1)_r$ and (4.3).

¹⁴Various subsets of these theories have also been studied in [3, 4].

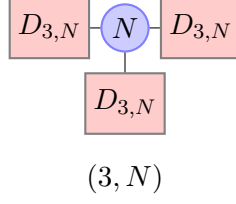


Figure 2. Our main theory of interest, $(3, 2)$, has $N = 2$ and is the simplest member of this larger class of theories (see figure 1). More generally, we may consider exactly marginal diagonal $SU(N)$ gaugings of three $D_3(SU(N)) := D_{3,N}$ SCFTs (where $D_{3,2} := D_3$). The only constraint on N is that $\gcd(3, N) = 1$.

Given this picture, we claim that $\mathcal{A}(6)$ saturates a new bound (this time from below) on the number of strong VOA generators in the infinite set of (n, N) SCFTs relative to the number of strong generators in the corresponding $\mathcal{N} = 4$ theory:

Claim 3. Assuming the conjecture in [10] for the VOA corresponding to $\mathfrak{su}(N)$ $\mathcal{N} = 4$ SYM, $\chi[\mathfrak{su}(N)]$,¹⁵ and assuming the existence of a GVSI, $\varphi_{n,N}$, described above, we have the following bound on the number of strong generators of $\chi[(n, N)]$ relative to the number of strong generators of $\chi[\mathfrak{su}(N)]$

$$|\chi[(n, N)]| \leq |\chi[\mathfrak{su}(N)]| - 4. \quad (4.5)$$

Moreover, $\mathcal{A}(6)$ is the unique theory saturating (4.5) in the class of (n, N) SCFTs.

Proof. Let us first consider the case of $n = 2$. We require that $\gcd(N, n) = 1$ and therefore that $N \geq 3$. From (4.3), we see that the $SU(2)_F$ generators of the SYM theory are mapped to dimension two operators: T , X_1 , and X_2 . Here T is the energy-momentum tensor and $X_{1,2}$ are other spin-two currents in $\chi[(2, N)]$. These must be strong generators since the $(2, N)$ theory has no flavor symmetries. By the conjecture for $\chi[\mathfrak{su}(N)]$ in [10], there are no other bosonic generators at $h_{(2,N)} = 2$.

At $h_{(2,N)} = 3$ we will have the bosonic operators ∂T , ∂X_1 , and ∂X_2 . There cannot be any relation involving just these operators (otherwise some linear combination of T , X_1 , and X_2 would be constant), and we also see from (4.3) that $\varphi_{2,N}(\partial J^-)$, $\varphi_{2,N}(\partial J^0)$, $\varphi_{2,N}(\partial J^+)$ can only contribute at $h_{(2,N)} = 4$. Therefore, we learn that three strong generators of $\chi[\mathfrak{su}(N)]$ must map to the $h_{(2,N)} = 3$ bosonic derivatives in $\chi[(2, N)]$. These are the 2D states arising from three of the four B_{ijk} operators of the $\mathfrak{su}(N)$ $\mathcal{N} = 4$ SYM VOA conjectured in [10] (note that here we are using the fact that $N \geq 3$).

At $h_{(2,N)} = 3$ we also have four fermionic states arising from mapping $\{G^\pm, \tilde{G}^\pm\}$ to $\{\Psi^\pm, \tilde{\Psi}^\pm\}$ under $\varphi_{2,N}$. By similar logic to the one used above in the bosonic case, we must have four independent derivatives at $h_{(2,N)} = 4$: $\partial\Psi^\pm$ and $\partial\tilde{\Psi}^\pm$. From (4.3) we see that these states cannot arise from images of $\varphi(\partial G^\pm)$ or $\varphi(\partial\tilde{G}^\pm)$. Therefore, these must be generated by four fermionic currents in $\chi[\mathfrak{su}(N)]$ (these are four of the 2D avatars of the

¹⁵Actually, we will only need to assume that the low-lying states of the conjectures in [10] are correct.

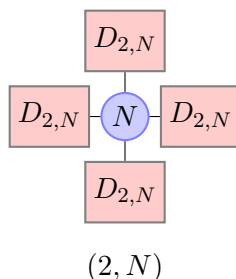


Figure 3. We may further generalize the (3, 2) theory by considering a diagonal exactly marginal $SU(N)$ gauging of four $D_2(SU(N)) := D_{2,N}$ theories. Here $\gcd(2, N) = 1$.

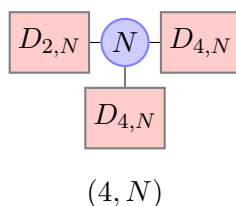


Figure 4. We may also generalize the (3, 2) theory by considering a diagonal exactly marginal $SU(N)$ gauging of two $D_4(SU(N)) := D_{4,N}$ theories and one $D_2(SU(N)) := D_{2,N}$ SCFT. Here $\gcd(4, N) = 1$.

4D $\text{Tr}Q_i Q_j \tilde{\lambda}_+$ and $\text{Tr}Q_i Q_j \lambda_+$ operators). As a result, we find the stronger result that

$$|\chi[(n, N)]| \leq |\chi[\mathfrak{su}(N)]| - 7 < \chi[\mathfrak{su}(N)] - 4. \tag{4.6}$$

Consider now the case of $n = 3$. Just as in the discussion of the (3, 2) theory in the previous sections, we see that (4.3) implies that the first non-trivial operator of $\chi[(3, N)]$ enters at $h_{(3,N)} = 2$: the stress tensor, T . We again have the unique identification, $\varphi(J^-) = T$. At $h_{(3,N)} = 3$ we have $\partial T \neq 0$. Since (4.3) implies $\varphi(\partial J^-)$ has $h_{(3,N)} = 5$, we see that a strong generator of $\chi[\mathfrak{su}(N)]$ must map to ∂T . By the conjecture in [10] we have that

$$\varphi(\alpha \cdot b_{111} + \beta \cdot J^0) = \partial T \in \chi[(2, N)], \tag{4.7}$$

for some constants α and β . Here $b_{111} = \chi(B_{111})$ where $B_{111} = \text{Tr}Q_1 Q_1 Q_1$ (“1” is a label denoting the lowest $SU(2)_F$ weight state in the doublet Q_i). An independent linear combination of $\varphi(b_{111})$ and $\varphi(J^0)$ map to a $\chi[(3, N)]$ strong generator, X , at $h_{(3,N)} = 3$ (this state was absent in the case $N = 2$, but it exists for $N \geq 4$).

Next consider the bosonic states at $h_{(3,N)} = 4$. We have T^2 , $\partial^2 T$, and ∂X . We can’t have a null state involving just $\partial^2 T$ and ∂X (since this would contradict ∂T and X being independent operators at $h_{(3,N)} = 3$).¹⁶ Therefore, we have that $\partial^2 T$ and ∂X are independent. Since $\varphi(\partial b_{111})$ has $h_{(3,N)} = 6$, we see that two strong generators of $\chi[\mathfrak{su}(N)]$ must map to the derivatives in question. This mapping occurs via two linear combinations of $\varphi(J^+)$, $\varphi(b_{(211)})$ (where the indices of $b_{(211)}$ are symmetrized), and $\varphi(b_{1111})$. We therefore see that there is an additional independent generator at this level, Y (again only for $N \geq 4$).

¹⁶Moreover, since $c \neq -\frac{22}{5}$, we also cannot have a null relation involving just $\partial^2 T$ and T^2 .

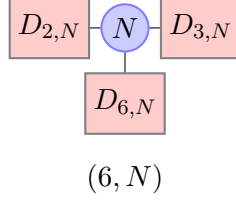


Figure 5. As a final generalization of the (3,2) theory we may consider a diagonal exactly marginal $SU(N)$ gauging of a $D_2(SU(N)) := D_{2,N}$ theory, a $D_3(SU(N)) := D_{3,N}$ SCFT, and a $D_6(SU(N)) := D_{6,N}$ theory. Here $\gcd(6, N) = 1$.

To finish off the case of $n = 3$, let us consider the fermionic states at $h_{(3,N)} = 4$. We have two states arising from the mapping of $\{G^-, \tilde{G}^-\}$ to $\{\Psi, \tilde{\Psi}\}$ under $\varphi_{3,N}$. Therefore, at $h_{(3,N)} = 5$, we have $\partial\Psi$ and $\partial\tilde{\Psi}$. By (3.9), these cannot arise from $\varphi(\partial G^-)$ or $\varphi(\partial\tilde{G}^-)$. As a result, two strong generators of $\chi[\mathfrak{su}(N)]$ must map onto them. We conclude that

$$|\chi[(3, N)]| \leq |\chi[\mathfrak{su}(N)]| - 4, \quad (4.8)$$

with the equality saturated if and only if $N = 2$.

Consider now the case of $n = 4$ (since $\gcd(4, N) = 1$, we see that $N \geq 3$). Again our first bosonic operator in $\chi[(4, N)]$ enters at $h_{(4,N)} = 2$ and has a unique mapping: $\varphi(J^-) = T$. As in the previous case, at $h_{(4,N)} = 3$ we have $\partial T \neq 0$. Since (4.3) implies that $\varphi(\partial J^-)$ has $h_{(4,N)} = 6$, we see that a strong generator of $\chi[\mathfrak{su}(N)]$ must map to ∂T . By the conjecture in [10], we have that $\varphi(b_{111}) = \partial T$. Next, at $h_{(4,N)} = 4$, we have T^2 and $\partial^2 T \neq 0$ (since $c \neq -\frac{22}{5}$ we also have $\partial^2 T$ is independent of T^2). Since $\varphi(\partial b_{111})$ has $h_{(4,N)} = 7$, we must have $\varphi(J^0)$, $\varphi(J^- J^-)$, and $\varphi(b_{1111})$ mapping into these stress tensor states and a new strong generator X (for $N \geq 5$; we see that for $N = 3$, X cannot exist using [10]). Since $\varphi(\partial J^0)$, $\varphi(\partial J^- J^-)$, $\varphi(\partial b_{1111})$ have $h_{(4,N)} = 8$, this means there is a strong generator that maps to $\partial^3 T$.

Now consider the lowest-dimensional fermions $\varphi(G^-) = \Psi$ and $\varphi(\tilde{G}^-) = \tilde{\Psi}$ at $h_{(4,N)} = 5$. These are strong generators of $\chi[(4, N)]$, and their derivatives $\partial\Psi$ and $\partial\tilde{\Psi}$ must correspond to other strong generators of the $\chi[\mathfrak{su}(N)]$ VOA by (4.3). Therefore, we have the stronger result that

$$|\chi[(4, N)]| \leq |\chi[\mathfrak{su}(N)]| - 5 < |\chi[\mathfrak{su}(N)]| - 4. \quad (4.9)$$

Finally, consider the case of $n = 6$ (since $\gcd(6, N) = 1$, we see that $N \geq 5$). As above, our first bosonic operator in $\chi[(6, N)]$ enters at $h_{(6,N)} = 2$ and has a unique mapping: $\varphi(J^-) = T$. Again at $h_{(6,N)} = 3$ we have $\partial T \neq 0$. Since (4.3) implies that $\varphi(\partial J^-)$ has $h_{(6,N)} = 8$, we see that a strong generator of $\chi[\mathfrak{su}(N)]$ must map to ∂T . By the conjecture in [10], we have that $\varphi(b_{111}) = \partial T$. Next, at $h_{(6,N)} = 4$ we have T^2 and $\partial^2 T \neq 0$ (again, since $c \neq -\frac{22}{5}$, we also have that $\partial^2 T$ is independent of T^2). Since $\varphi(\partial b_{111})$ has $h_{(6,N)} = 9$, we must have $\varphi(J^- J^-)$ and $\varphi(b_{1111})$ mapping into linear combinations of T^2 and $\partial^2 T$. Since $\varphi(\partial b_{1111})$ has $h_{(6,N)} = 10$, we must have another strong generator mapping into $\partial^3 T$.

Now consider the lowest-dimensional fermions $\varphi(G^-) = \Psi$ and $\varphi(\tilde{G}^-) = \tilde{\Psi}$ at $h_{(6,N)} = 7$. These are strong generators of $\chi[(6, N)]$, and their derivatives, $\partial\Psi$ and $\partial\tilde{\Psi}$, must correspond

to other strong generators of the $\chi[\mathfrak{su}(N)]$ VOA by (4.3). Therefore, we have the stronger result that

$$|\chi[(6, N)]| \leq |\chi[\mathfrak{su}(N)]| - 5 < |\chi[\mathfrak{su}(N)]| - 4, \quad (4.10)$$

and we have proved our result. \square

5 Conclusions

In this paper, we have shown that a surprisingly simple index relation between two very different theories — the $(3, 2)$ AD theory and $\mathfrak{su}(2)$ $\mathcal{N} = 4$ SYM — has a mathematical explanation in terms of an exact GVSI. We’ve argued that physically this relation is a (distant) cousin of spontaneous symmetry breaking. It would be interesting to understand this physical perspective better, perhaps making contact with and generalizing the large charge literature as in [32, 33].

It will also be interesting to understand how general the above phenomena are. As we have mentioned, we will find infinitely many generalizations of some of the above arguments in an upcoming work [26] (though note that the index discussion from the introduction referencing [17] does not directly apply to the case $n \neq 3$).

In addition, we have seen that the VOA associated with the $(3, 2)$ SCFT, $\mathcal{A}(6)$, saturates two bounds on the number of strong generators. It would be interesting to understand if this saturation is related to the fact that the $(3, 2)$ theory lacks a Higgs branch. Moreover, the saturation of these bounds might imply that this theory can be targeted in interesting ways with the bootstrap. Finally, it would be interesting to understand our construction in a more four-dimensional language.

Acknowledgments

We thank H. Jiang for discussions and J. Shafiq for collaboration on the early stages of this project. M.B.’s work is supported by the Royal Society under the grants “New Constraints and Phenomena in Quantum Field Theory” and “New Aspects of Conformal and Topological Field Theories Across Dimensions”. T.N.’s work is supported by JSPS Grant-in-Aid for Early-Career Scientists 18K13547.

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