Entropy of Hawking radiation for two-sided hyperscaling violating black branes

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ABSTRACT: In this paper, we study the von Neumann entropy of Hawking radiation S_R for a $d+2$-dimensional Hyperscaling Violating (HV) black brane which is coupled to two Minkowski spacetimes as the thermal baths. We consider two different situations for the matter fields: first, they are described by a CFT_{d+2} whose central charge c is very large. Second, they are described by a d+2 dimensional HV QFT which has a holographic gravitational theory that is a HV geometry at zero temperature. For both cases, we calculate the Page curve of the Hawking radiation as well as the Page time t_{Page}. For the first case, S_R grows linearly with time before the Page time and saturates after this time. Moreover, t_{Page} is proportional to $\frac{2S_{\text{th}}}{cT}$, where S_{th} and T are the thermal entropy and temperature of the black brane. For the second case, when the hyperscaling violation exponent θ_m of the matter fields is zero, the results are very similar to those for the first case. However, when $\theta_m \neq 0$, the entropy of Hawking radiation grows exponentially before t_{Page} and saturates after this time. Furthermore, the Page time is proportional to $\log\left(\frac{1}{G_{\text{N,r}}}\right)$, where $G_{\text{N,r}}$ is the renormalized Newton's constant. It was also observed that for both cases, t_{Page} is a decreasing and an increasing function of the dynamical exponent z and hyperscaling violation exponent θ of the black brane geometry, respectively. Moreover, for the second case, t_{Page} is independent of z_m, and for $\theta_m \neq 0$, it is a decreasing function of θ_m.

Contents

1 Introduction

It is well known that black holes can emit particles with a thermal spectrum which is called Hawking radiation [1]. In this manner, a black hole loses its mass, and might be disappeared completely. Assuming that the black hole was formed from the collapse of some matter in a pure quantum state leads to the famous Hawking's information paradox [2], since the final state, i.e. the emitted radiation, is in a mixed state. Moreover, the final state cannot be obtained via a S matrix from the initial pure state. On the other hand, the fine-grained (von Neumann) entropy of the Hawking radiation S_R grows linearly in time until the black hole completely disappears. In this manner, it eventually exceeds the coarse-grained (thermodynamic or Bekenstein-Hawking) entropy S_BH of the initial black hole. On the other hand, the von Neumann entropies of the black hole and radiation are equal to each other, i.e. $S_\mathrm{R} = S_\mathrm{black\ hole}$, since the whole system is in a pure state. Moreover, the fine-grained entropy of the black hole has to be less than its coarse-grained entropy, i.e. $S_\mathrm{black\ hole} \leq S_\mathrm{BH}$ [3]. Therefore, one has [3] (See also [4])[1]

$$S_\mathrm{R} \leq S_\mathrm{BH}. \tag{1.2}$$

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[1]Notice that this equation is only valid for one-sided black holes. For two-sided black holes, it has to be modified to [9]

$$S_\mathrm{R} \leq 2S_\mathrm{BH}. \tag{1.1}$$

In this case, S_R saturates at $2S_\mathrm{BH}$ after the Page time and it does not decrease in contrast to that for one-sided black holes.

The time at which the saturation happens is called the "Page time" [3, 5]. Moreover, since the S_{BH} is a decreasing function of time, S_{R} has to start decreasing after the Page time and goes to zero at the end of the evaporation. Consequently, the unitarity requires that the fine-grained entropy of Hawking radiation follows the so called Page curve [3, 5].

Unearthing the unitary evaporation of Black holes is one of the most important puzzles in gravity. Recently, a very sophisticated resolution for information paradox which is called "*island rule*" were introduced [6–9] in the context of the AdS/CFT correspondence [10]. This rule is inspired by the concept of "Quantum Extremal Surfaces" (QES) [11] which are the generalization of the (H)RT surfaces [12, 13] applied in the calculation of holographic entanglement entropy. More precisely, a QES is a classical spacelike codimension-two surface in the bulk spacetime which minimizes the generalized entropy [11, 14] which is composed of an area term and the von Neumann entropy of matter quantum fields in a bulk region enclosed between the QES and the corresponding boundary region. The motivation for the island rule was the observation that the entanglement wedge (EW) of an evaporating black hole at late times does not include all of its interior. Therefore, one might naturally expects that the EW of the radiation region $\mathcal{R}$, which is a spatial region far away from the black hole where the radiation is collected, contains some parts of the black hole interior dubbed "island" $\mathcal{I}$ [6, 8] (See figure 1). Consequently, to calculate the von Neumann entropy of the Hawking radiation, one has to consider the contributions of islands. According to the rule, one should first calculate the generalized entropy S_{gen} as follows [8, 9]

$$S_{\text{gen}}\left(\mathcal{R} \cup \mathcal{I}\right) = \frac{\text{Area}(\partial I)}{4G_N} + S_{\text{matter}}\left(\mathcal{R} \cup \mathcal{I}\right), \tag{1.3}$$

where $\partial \mathcal{I}$ is the boundary of the island and is a codimension-two surface. It is expected that for one-sided black holes $\partial \mathcal{I}$ is located inside the black hole [6–8, 29, 32]. However, for two-sided black holes, it is located outside the black hole [9, 29, 34]. If the matter quantum field theory has a gravitational dual theory, the island is connected to the radiation region through extra dimensions [8]. Moreover, $S_{\text{matter}}\left(\mathcal{R} \cup \mathcal{I}\right)$ is the von Neumann entropy of matter quantum fields in the region $\mathcal{R} \cup \mathcal{I}$. In the following, we denote the endpoints of $\mathcal{R}$ and $\mathcal{I}$ by $b_{\pm}$ and $a_{\pm}$, respectively (See figure 1). Next, one has to extremize S_{gen} over all possible islands. If there is more than an island, one should consider the one which gives the minimum generalized entropy

$$S_{\text{R}} = \min_{\mathcal{I}}\{\text{ext}_{\mathcal{I}}\left[S_{\text{gen}}\right]\}. \tag{1.4}$$

In this case, $\partial \mathcal{I}$ is a minimal QES. Moreover, when there are no islands, the area term in eq. (1.3) vanishes, and hence the entropy of Hawking radiation is simply given by the von Neumann entropy of the matter fields on the region $\mathcal{R}$

$$S_{\text{R}} = S_{\text{matter}}\left(\mathcal{R}\right). \tag{1.5}$$

The island rule were verified for two-dimensional Jackiw-Teitelboim (JT) black holes [15, 16] in refs. [17–19][2] by calculating the Euclidean gravitational path integral via replica trick in gravity [23, 24]. It was observed that the gravitational path integral has two saddle points: first, Hawking saddle which gives the usual entropy for Hawking radiation that grows linearly with time. Second, replica wormhole saddle which is a wormhole connecting various copies of the original black hole. It was observed that the replica saddle leads to the contribution of the islands and enforces the entropy of radiation to saturate. It should be pointed out that the Hawking saddle is dominant before the Page time. However, the replica wormhole saddle is dominant after the Page time.

Furthermore, the island rule have been explored extensively for various black holes in flat and AdS spacetimes such as: JT gravity [9, 17–19, 25–28], two-dimensional dilaton gravity [29–33] , in higher dimensions [33–44], higher derivative gravity theories [45], charged black holes [46–50], pure BTZ black hole microstates[3] [51–53], and black holes coupled to gravitating baths [53–56]. See also [4] for a very nice review on the topic. Furthermore, the rule were applied in de Sitter spacetime [61–66], flat-space cosmology[4] [67] and separate universes [52, 56, 68–70].

In this paper, we study the von Neumann entropy of the Hawking radiation for a two-sided Hyperscaling Violating (HV) black brane which is coupled to two thermal baths that are Minkowski spacetimes. We assume that the matter fields in the black brane geometry as well as in the baths are described by the same QFT. We consider two different scenarios for the QFT: first, it is a d+2 dimensional CFT which does not necessarily have a dual gravity theory. Second, it is a d+2 dimensional QFT which has a dual gravity theory that is a zero-temperature HV geometry (See eq. (4.1)). We dub it "HV QFT". We calculate S_R for the two cases and verify that it obeys eq. (1.1). It is observed that at early times there are no islands. However, at late times there is an island which leads to the saturation of S_R at twice the Bekenstein-Hawking entropy of the black brane. Moreover, we study the Page curve and Page time for different values of the dynamical exponent z and hyperscaling violation exponent θ of the black brane. For the case where the matter is described by a HV QFT, we also study the behavior of the Page time and Page curve with respect to the dynamical and hyperscaling violation exponents of the HV QFT which are shown by z_m and θ_m, respectively. We study the two cases $\theta_m = 0$ and $\theta_m \neq 0$, separately.

The organization of the paper is as follows: in section 2, we briefly review the HV black brane geometry. In section 3, we calculate the entropy of Hawking radiation for the case where the matter fields are described by a CFT_{d+2}. We also find the Page time and study its behavior as a function of the dynamical exponent z and hyperscaling violation exponent θ of the black brane geometry. In section 4, we do the same calculations for the case where the matter fields are described by a d+2 dimensional HV QFT. In section 5, we summarize our results and briefly address a few interesting future directions.

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[2]See also [20, 21] for related discussions. Moreover, a new proof was recently introduced in ref. [22] which is based on minimizing the microcanonical action of an entanglement wedge.

[3]These geometries are obtained by excising a two-sided BTZ black hole with a timelike dynamical brane, dubbed End-of-the-World (ETW) brane [57–59]. The action-complexity of this model was studied in ref [60].

[4]It is a solution of the three dimensional Einstein gravity with no cosmological constant.

2 HV black branes

In this section, we briefly review Hyperscaling Violating (HV) black branes. These are solutions to the Einstein-Maxwell-Dilaton gravity with the following action [71]

$$I_{\text{HV}} = -\frac{1}{16\pi G_N} \int d^{d+2}x\sqrt{-g}\left[R - \frac{1}{2}\left(\partial\phi\right)^2 + V(\phi) - \frac{1}{4}e^{\lambda\phi}F^2\right], \tag{2.1}$$

The gauge field A breaks Lorentz invariance and introduces the dynamical exponent z.[5] Moreover, the non-trivial potential $V(\phi)$ breaks the scaling symmetry and introduces the Hyperscaling Violation exponent θ. The dilaton field ϕ, its potential $V(\phi)$ and the field strength of the gauge field are given by (See ref. [71] for more details)

$$\phi = \phi_0 + \beta\ln r, \qquad V(\phi) = (d_e + z - 2)(d_e + z - 1)e^{\gamma(\phi - \phi_0)},$$

$$F_{rt} = \sqrt{2(z-1)(d_e + z - 1)}e^{\frac{(d_e + \theta_e - 1)\phi_0}{\beta}}\, r^{d_e + z - 2}, \tag{2.2}$$

where the constant parameters are defined as follows

$$\beta = \sqrt{2(d_e - 1)(-\theta_e + z - 1)}, \qquad \lambda = -\frac{2(d_e + \theta_e - 1)}{\beta}, \qquad \gamma = \frac{2\theta_e}{\beta}. \tag{2.3}$$

On the other hand, the metric of the black brane is given by [71][6]

$$ds^2 = r^{-2\theta_e}\left(-r^{2z}f(r)dt^2 + \frac{dr^2}{r^2f(r)} + r^2\sum_{i=1}^{d}dx_i^2\right), \tag{2.4}$$

where the emblankening factor is as follows

$$f(r) = 1 - \left(\frac{r_h}{r}\right)^{d_e + z}. \tag{2.5}$$

Here we defined $\theta_e = \frac{\theta}{d}$ and an effective dimension $d_e = d - \theta$. Furthermore, the null energy condition puts the following constraints on the values of z and θ

$$d_e(d(z-1) - \theta) \geq 0, \qquad (z-1)(d_e + z) \geq 0. \tag{2.6}$$

In the following, we restrict ourselves to the case $z \geq 1$ and $d_e > 0$.[7] It should also be pointed out that for $\theta = 0$ and $z = 1$, the scaling and Lorentz symmetries are restored in the

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[5]It should be pointed out that the action is Lorentz invariant and the solution does not respect the Lorentz symmetry. More precisely, under the coordinate transformation [85]

$$t \to \lambda^z t \qquad x^i \to \lambda x^i \qquad r \to \lambda r$$

the metric (2.4) is covariant, i.e. $ds \to \lambda^{\frac{\theta}{d}}ds$. It is clear that there is an anisotropy among the t and x^i coordinates in the above transformation. Therefore, the Lorentz symmetry is broken when $z \neq 1$, and it is a consequence of having a non-zero gauge field in eq. (2.2) for $z \neq 1$.

[6]We set the AdS radius to one, i.e. $R = 1$. Moreover, we also set the dynamical scale r_F to one.

[7]As mentioned in ref. [71], this solution is not valid for the case $d_e = 0$. Moreover, for $d > \theta$ the geometry is unstable [85].

dual HV QFT. In this case, the scalar field becomes a constant and the gauge field equals to zero. Moreover, $V(\phi)$ plays the rule of the cosmological constant Λ, and hence the solution reduces to a d+2 dimensional AdS black brane. On the other hand, the temperature and thermal (Bekenstein-Hawking) entropy of the black brane are given by [71]

$$T = \frac{(d_e + z)r_h^z}{4\pi}, \qquad S_{\text{th}} = \frac{V_d r_h^{d_e}}{4G_N}. \tag{2.7}$$

Here V_d is the volume in the transverse directions x^i. On the other hand, the tortoise coordinate is as follows

$$r^*(r) = \int \frac{dr}{r^{z+1}f(r)} = {}_2F_1\left[1, \frac{z}{d_e + z}, 1 + \frac{z}{d_e + z}, \left(\frac{r_h}{r}\right)^{d_e+z}\right]. \tag{2.8}$$

3 Entropy of Hawking radiation: matter CFT

In this section, we study the entropy of Hawking radiation for a two-sided HV black brane geometry. To allow the black brane to evaporate, we couple it to two thermal baths which are two Minkowski spacetimes on the left and right hand side of the Penrose diagram (See figure 1). Similar to ref. [8] we add some matter fields in the bulk spacetime. For convenience, we assume that the matter fields in the HV black brane geometry and inside the two baths are described by the same QFT. Moreover, one can impose transparent boundary conditions [18] for the matter fields on the boundary of the HV black brane geometry. Furthermore, we assume that the matter fields are described by a CFT_{d+2}. Then the whole action in the gravity region (see the purple region in figure 1) is given by

$$I = I_{\text{HV}} + I_{\text{CFT}_{d+2}}^{\text{matter}}. \tag{3.1}$$

The CFT may [8, 25, 27, 29, 34, 37, 40, 46] or may not [9, 35, 45] have a dual gravity theory. Moreover, we assume that its central charge is very large, i.e. $c \gg 1$. Therefore, the contributions of the matter fields of the CFT_{d+2} to the EE are dominant over those of the graviton and the matter fields in I_{HV} [8, 45, 67]. Moreover, we assume that

$$c \ll \frac{V_d r_h^{d_e}}{G_N} \propto S_{\text{th}}, \tag{3.2}$$

to be able to neglect the backreaction of the matter on the HV black brane geometry [45, 67].

Furthermore, it is believed that the generalized entropy is finite and independent of the UV cutoff of the theory [4, 11, 72, 73]. On the other hand, there are some UV divergent terms in the entanglement entropy (EE) of a QFT, and the leading divergent term follows the area law [74, 75]. Notice that our entangling regions are in the shape of strips on a constant time slice such that they are extended in the transverse directions x^i (See figure 1). Moreover, from holographic calculations, it is known that the EE of a strip with width ℓ and lengths L in a holographic CFT_{d+2} with $d \geq 1$ which is in its vacuum state, is given
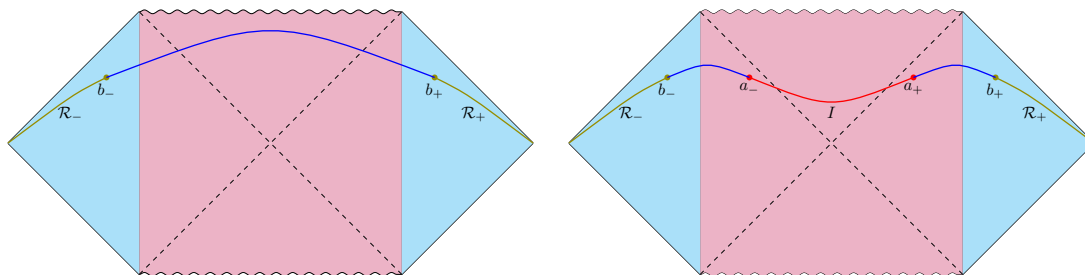
Figure 1. The Penrose diagram of the HV black brane geometry which is shown in purple and two Minkowski spacetimes as the baths shown in cyan. (*Left*) when there are no islands. (*Right*) when there is an island $\mathcal{I}$ indicated in red with endpoints a_- and a_+. The radiation region is $\mathcal{R} = \mathcal{R}_+ \cup \mathcal{R}_-$ and indicated in olive. The endpoints of $\mathcal{R}$ are denoted by $b_\pm$ and are located in the baths. We assume that the state on the full Cauchy slice is pure. Therefore, one can calculate the EE of matter fields $S_{\mathrm{matter}}(\mathcal{R} \cup \mathcal{I})$ on the complement intervals $[b_-, b_+]$ in the left panel and $[b_-, a_-] \cup [a_+, b_+]$ in the right panel which are shown in blue.

by [76][8]

$$S = \frac{1}{2G_N^{d+3}} \left[\frac{2R^{d+1}}{d} \left(\frac{L}{\epsilon} \right)^d - \frac{(2\sqrt{\pi}R)^{d+1}}{d} \left(\frac{\Gamma\left(\frac{d+2}{2(d+1)} \right)}{\Gamma\left(\frac{1}{2(d+1)} \right)} \right) \left(\frac{L}{\ell} \right)^d \right]. \tag{3.3}$$

Here the first term is proportional to the area L^d of the strip and the second term is a constant universal term. Inspired by this observation, one might expect that

$$S_{\mathrm{matter}}(\mathcal{R} \cup \mathcal{I}) = \frac{\mathrm{Area}(\partial\mathcal{I})}{\epsilon^d} + S_{\mathrm{matter}}^{\mathrm{f}}(\mathcal{R} \cup \mathcal{I}), \tag{3.4}$$

where ϵ is the UV cutoff of the matter CFT_{d+2}. Therefore, we renormalize Newton's constant as follows [4, 35, 45, 67, 72][9]

$$\frac{1}{4G_{N,r}} = \frac{1}{4G_N} - \frac{1}{\epsilon^d}, \tag{3.5}$$

to absorb the UV divergent term in eq. (3.4). Consequently, we merely consider the finite part of the EE of the matter fields in the following, and for an arbitrary interval $[m_1, m_2]$ in the bulk spacetime, we denote it by $S_{\mathrm{matter}}^{\mathrm{f}}(m_1, m_2)$. Having said this, we rewrite the generalized entropy in eq. (1.3) as follows [35, 45, 67]

$$S_{\mathrm{gen}} = \frac{\mathrm{Area}(\partial I)}{4G_{N,r}} + S_{\mathrm{matter}}^{\mathrm{f}}(\mathcal{R} \cup \mathcal{I}). \tag{3.6}$$

Another important point is that the metric in eq. (2.4), has translational symmetries along the transverse directions x^i. Therefore, one might expect that the locations of the endpoints

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[8]For spherical entangling regions there are subleading UV divergent terms [76].

[9]It was shown in ref. [72] that in four dimensions, i.e. for d=2, the loop diagrams for matter fields lead to the renormalization of the gravitational coupling G_N as in eq. (3.5). Moreover, this renormalization cancel the area law divergent term in eq. (3.4) such that the generalized entropy is finite. See [73] for a nice review on this topic.

of the island, and hence the generalized entropy are independent of the coordinates x_i. Consequently, one might expect that the problem effectively reduces to two dimensions with coordinates (t, r). Moreover, by compactification of the x_i directions, one might consider the transverse manifold $\mathbb{R}^d$ which is parametrized by x_i's, as a d dimensional sphere with a very large radius. In this manner, one might assume that the s-wave approximation mentioned in refs. [6, 33, 35, 45] can be applied.[10] In other words, after the expansion of the matter fields in terms of the spherical harmonics on the large sphere, one can have both massless and massive Kaluza-Klein modes. Then, each mode behaves as an independent free field in two dimensions (with coordinates t and r) whose mass is proportional to the inverse of the radius of the sphere [6]. Since the radius of the sphere is very large, the masses of the Kaluza-Klein modes have to be very small. Moreover, since the endpoints $b_\pm$ of the radiation region are very far from the endpoints $a_\pm$ of the islands[11] , we assume that only massless modes can reach the radiation region (See also [6, 35, 67]). Therefore, the contribution of the massless modes to $S_{\text{matter}}(\mathcal{R} \cup \mathcal{I})$ is dominant over that of the massive modes.[12] Consequently, one may apply the formula for a two dimensional CFT in its vacuum state to calculate $S_{\text{matter}}(\mathcal{R} \cup \mathcal{I})$, which for an interval of length ℓ is given by [77, 78]

$$S = \frac{c}{3} \log\left(\frac{\ell}{\epsilon}\right), \tag{3.7}$$

where c is the central charge and ϵ is the UV cutoff.

3.1 Kruskal coordinates and the distances

For later convenience, it is better to work in the Kruskal coordinates. The advantage of working with the Kruskal coordinates is that the state on the whole Cauchy slice, i.e. the black brane plus the baths, is the vacuum state [9, 33], and hence one can simply apply the EE formula for the case where the matter QFT is in the vacuum state (See e.g. eq. (3.7)). For the black brane, they are defined as follows[13]

$$U = +e^{-\frac{2\pi}{\beta}(t-r^*(r))}, \qquad\qquad V = -e^{\frac{2\pi}{\beta}(t+r^*(r))}, \qquad \text{the left exterior}$$

$$U = -e^{-\frac{2\pi}{\beta}(t-r^*(r))}, \qquad\qquad V = +e^{\frac{2\pi}{\beta}(t+r^*(r))}, \qquad \text{the right exterior} \tag{3.8}$$

where $\beta = \frac{1}{T}$ and $r^*(r)$ is given by eq. (3.9). In the Kruskal coordinates, the metric of the black brane becomes

$$ds^2 = -\frac{dU dV}{W(r)^2} + r^{2(1-\theta_e)} d\vec{x}_d^2, \qquad\qquad W(r) = \frac{2\pi e^{\frac{2\pi}{\beta} r^*(r)}}{\beta r^{z-\theta_e} \sqrt{f(r)}}, \tag{3.9}$$

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[10]We would like to thank Mohsen Alishahiha and Ali Naseh for their very helpful comments on this topic.

[11]We are assuming that $a \approx r_h$ and $b \gg r_h$, where a and b are the radial coordinates of the endpoints $a_\pm$ and $b_\pm$, respectively.

[12]We would like to thank Ali Naseh for bringing this point to our attention.

[13]Since the endpoints of the island are located outside the black brane horizon (See figure 1), we do not need the coordinates in the interior regions.

which is conformally flat in the t and r plane. Moreover, $W(r)$ is the warp factor. On the other hand, for each of the two thermal baths which are Minkowski spacetimes, the tortoise coordinate is simply given by

$$r^*(r) = r. \tag{3.10}$$

On the other hand, one might consider the left and right baths as the Rindler wedges of a Minkowski spacetime [9]. In this manner, the Kruskal coordinates can be defined as follows for the two baths (See also refs. [9, 33, 45])

$$U = +e^{-\frac{2\pi}{\beta}(t-r)}, \qquad V = -e^{\frac{2\pi}{\beta}(t+r)}, \qquad \text{Left bath}$$
$$U = -e^{-\frac{2\pi}{\beta}(t-r)}, \qquad V = +e^{\frac{2\pi}{\beta}(t+r)}, \qquad \text{Right bath} \tag{3.11}$$

Thus, the metric of each bath is given by (See also [67])

$$ds^2 = -\frac{dU\,dV}{W_B(r)^2} + r^2 d\vec{x}_d^2, \qquad W_B(r) = \frac{2\pi e^{\frac{2\pi}{\beta}r}}{\beta}. \tag{3.12}$$

To calculate the entropy of Hawking radiation, one also needs to know the distances $d(a, b)$ among the endpoints of the radiation region $\mathcal{R}$ and island $\mathcal{I}$. We indicate the endpoints of the radiation region by $b_\pm$ and those of the island region by $a_\pm$. From figure 1, one can easily find the (t, r) coordinates of the endpoints as follows (See also [28])

$$a_+ : (t_a, a), \qquad a_- : (-t_a + i\beta/2, a),$$
$$b_+ : (t_b, b), \qquad b_- : (-t_b + i\beta/2, b). \tag{3.13}$$

From eqs. (3.9) and (3.12), one can write the distance $d(m_1, m_2)$ between two arbitrary points m_1 and m_2 in the bulk spacetime as follows

$$d^2(m_1, m_2) = \frac{(U(m_2) - U(m_1))\,(V(m_1) - V(m_2))}{W(m_2)W(m_1)}. \tag{3.14}$$

It is straightforward to verify that the distances are given by

$$d(a_+, a_-) = \frac{\beta a^{(z-\theta_e)}\sqrt{f(a)}}{\pi} \cosh\left(\frac{2\pi t_a}{\beta}\right), \tag{3.15}$$

$$d(b_+, b_-) = \frac{\beta}{\pi} \cosh\left(\frac{2\pi t_b}{\beta}\right),$$

$$d^2(a_+, b_+) = d^2(a_-, b_-) = \frac{\beta^2 a^{(z-\theta_e)}\sqrt{f(a)}}{2\pi^2}\left(\cosh\left(\frac{2\pi}{\beta}(b - r^*(a))\right) - \cosh\left(\frac{2\pi}{\beta}(t_a - t_b)\right)\right),$$

$$d^2(a_+, b_-) = d^2(a_-, b_+) = \frac{\beta^2 a^{(z-\theta_e)}\sqrt{f(a)}}{2\pi^2}\left(\cosh\left(\frac{2\pi}{\beta}(b - r^*(a))\right) + \cosh\left(\frac{2\pi}{\beta}(t_a + t_b)\right)\right).$$

3.2 No islands

When there are no islands, there are only the radiation regions $\mathcal{R} = \mathcal{R}_- \cup \mathcal{R}_+$ (See the left panel of figure 1). In this case, the area term in eq. (1.3) vanishes, and hence the entropy of Hawking radiation is simply given by eq. (1.5). Therefore, one has

$$S_{\mathrm{R}} = S_{\mathrm{matter}}^{\mathrm{f}}(\mathcal{R}) = S_{\mathrm{matter}}^{\mathrm{f}}(b_+, b_-), \tag{3.16}$$

where $S(b_+, b_-)$ is the EE of matter fields on the interval $[b_-, b_+]$. In the last equality, we applied the fact that the whole state on the Cauchy slice is pure. Therefore, one can calculate the EE of matter fields on the interval $[b_-, b_+]$ instead of the region $\mathcal{R}$ (See the left panel of figure 1). Next, by applying eqs. (3.7) and (3.15), one can rewrite eq. (3.16) as follows

$$S_{\mathrm{R}} = \frac{c}{3} \log d(b_+, b_-) = \frac{c}{3} \log \left(\frac{\beta}{\pi} \cosh \left(\frac{2\pi t_b}{\beta} \right) \right). \tag{3.17}$$

At early times, i.e. $t_b T \ll 1$, it grows quadratically in time

$$S_{\mathrm{R}} = \frac{c}{3} \left[\frac{(d_e + z)^2 r_h^{2z}}{8} t_b^2 + \log \left(\frac{4}{(d_e + z) r_h^z} \right) \right]. \tag{3.18}$$

On the other hand, at late times, i.e. $t_b T \gg 1$, it grows linearly in time

$$S_{\mathrm{R}} \simeq \frac{c(d_e + z) r_h^z}{6} t_b. \tag{3.19}$$

Therefore, the entropy of Hawking radiation exceeds the coarse-grained entropy of two black branes and information paradox is inevitable. To resolve the issue, we consider the contribution of an island to the entropy of the Hawking radiation in the next section. We will see that in this manner the entropy of Hawking radiation saturates after the Page time.

3.3 With islands

In this section, we consider the effect of islands on the entropy of Hawking radiation. We assume that there is an island $\mathcal{I}$ whose endpoints denoted by a_- and a_+ are located outside the event horizon (See the right panel of figure 1). Since the state on the whole Cauchy slice is pure, one has to calculate the EE of matter fields on two disjoint intervals $[b_-, a_-] \cup [a_+, b_+]$. On the other hand, as mentioned below eq. (3.6), we are using the fact that our calculations are reduced to two dimensions with coordinates (U, V). Therefore, we are effectively working with a two-dimensional matter CFT (See also [35, 45, 67]). As far as we are aware, the calculation of the EE for two disjoint intervals has not been fully understood yet. However, it was calculated for free massless fermions [81–83] and Luttinger liquids, i.e. free compactified bosons [79] (See also [80] for a detailed review). In the following, we assume that the quantum fields of the matter CFT are N free massless Dirac fermions similar to refs. [18, 49]. In this case, for the two intervals $A = [a_1, b_1]$ and

$B = [a_2, b_2]$, one has [81–83][14]

$$S(A \cup B) = \frac{c}{3} \log \left(\frac{(b_1 - a_1)(b_1 - a_2)(b_2 - a_1)(b_2 - a_2)}{\epsilon^2 (a_2 - a_1)(b_2 - b_1)} \right), \tag{3.20}$$

where c is the central charge and ϵ is the UV cutoff. After considering the finite part of eq. (3.20), and noticing that our background is conformally flat, one has[15]

$$S_{\text{matter}}^{\text{f}}(\mathcal{R} \cup \mathcal{I}) = \frac{c}{3} \log \left(\frac{d(a_+, a_-)d(b_+, b_-)d(a_+, b_+)d(a_-, b_-)}{d(a_+, b_-)d(a_-, b_+)} \right). \tag{3.21}$$

Next, by applying eq. (3.15), one obtains

$$\begin{aligned}
S_{\text{matter}}^{\text{f}}(\mathcal{R} \cup \mathcal{I}) = \frac{c}{3} &\left[\log \left(\frac{\beta^2 a^{z - \theta_e} \sqrt{f(a)}}{\pi^2} \cosh \left(\frac{2\pi t_a}{\beta} \right) \cosh \left(\frac{2\pi t_b}{\beta} \right) \right) \right. \\
&\left. + \log \left(\frac{\cosh \left(\frac{2\pi}{\beta}(b - r^*(a)) \right) - \cosh \left(\frac{2\pi}{\beta}(t_a - t_b) \right)}{\cosh \left(\frac{2\pi}{\beta}(b - r^*(a)) \right) + \cosh \left(\frac{2\pi}{\beta}(t_a + t_b) \right)} \right) \right]. \tag{3.22}
\end{aligned}$$

After adding the gravitational part, the generalized entropy becomes

$$\begin{aligned}
S_{\text{gen}} &= \frac{V_d a^{d_e}}{2 G_{N,r}} + S_{\text{matter}}^{\text{f}}(\mathcal{R} \cup \mathcal{I}) \\
&= \frac{V_d a^{d_e}}{2 G_{N,r}} + \frac{c}{3} \left[\log \left(\frac{\beta^2 a^{z - \theta_e} \sqrt{f(a)}}{\pi^2} \cosh \left(\frac{2\pi t_a}{\beta} \right) \cosh \left(\frac{2\pi t_b}{\beta} \right) \right) \right. \\
&\quad \left. + \log \left(\frac{\cosh \left(\frac{2\pi}{\beta}(b - r^*(a)) \right) - \cosh \left(\frac{2\pi}{\beta}(t_a - t_b) \right)}{\cosh \left(\frac{2\pi}{\beta}(b - r^*(a)) \right) + \cosh \left(\frac{2\pi}{\beta}(t_a + t_b) \right)} \right) \right] \\
&= \frac{V_d a^{d_e}}{2 G_{N,r}} + \frac{c}{3} \left[\log \left(\frac{\beta^2 a^{z - \theta_e} \sqrt{f(a)}}{\pi^2} \cosh \left(\frac{2\pi t_a}{\beta} \right) \cosh \left(\frac{2\pi t_b}{\beta} \right) \right) \right. \\
&\quad \left. + \log \left(\frac{1 + e^{-\frac{4\pi}{\beta}(b - r^*(a))} - 2 e^{-\frac{2\pi}{\beta}(b - r^*(a))} \cosh \left(\frac{2\pi}{\beta}(t_a - t_b) \right)}{1 + e^{-\frac{4\pi}{\beta}(b - r^*(a))} + 2 e^{-\frac{2\pi}{\beta}(b - r^*(a))} \cosh \left(\frac{2\pi}{\beta}(t_a + t_b) \right)} \right) \right]. \tag{3.23}
\end{aligned}$$

---

[14]It should be pointed out that in the calculation of the EE of two disjoint intervals in the CFT of Luttinger liquids via the replica trick, there is a function $\mathcal{F}_n(x)$ in $tr(\rho_A^n)$. Here x is the cross-ratio of the four endpoints of the two intervals, n denotes the number of the replication and ρ_A is the reduced density matrix of the two intervals. The function $\mathcal{F}_n(x)$ depends on the full operator content of the QFT and is unknown generally [79]. It has the property, $\mathcal{F}_n(0) = 1$. Moreover, its contribution is not included in eq. (3.21), and hence, the equation has to be modified in this case (See [79, 80] for more discussions). However, at late times, i.e. $t_{a,b} T \gg 1$, the two intervals become very far apart from each other. In this case, from eqs. (3.15), (3.26) and (3.27), one has (See also refs. [9, 32, 35])

$$x = \frac{d(b_-, a_-)d(b_+, a_+)}{d(b_-, a_+)d(a_-, b_+)} = \frac{d^2(b_+, a_+)}{d^2(a_-, b_+)} \approx \frac{e^{\frac{2\pi}{\beta}(b - r^*(a))}}{\cosh \left(\frac{2\pi}{\beta}(t_a + t_b) \right)} \approx \frac{e^{\frac{2\pi}{\beta}(b - r^*(a))}}{e^{\frac{4\pi}{\beta} t_b}} \ll 1.$$

Therefore, $x \to 0$ and $\mathcal{F}_n(x) \to 1$. In other words, for free compactified bosons, eq. (3.21) is still valid at late times [18, 32, 45, 67]. We thank the referee for her/his very helpful comments on this point.

[15]Notice that the metric (3.9) is conformally flat and the distances $d(a_\pm, b_\pm)$, i.e. eq. (3.15), contain the warp factor $W(r)$ introduced in eq. (3.9).

Now we first find the entropy of Hawking radiation at early times, i.e. $t_{a,b}T \ll 1$. We assume that t_a and t_b are of the same order. Moreover, we consider the case where the endpoints of the island are close to the horizon, i.e. $a \approx r_h$ [35].[16] Therefore, one may apply the following approximations (See also [45])

$$e^{\frac{2\pi}{\beta}(b-r^*(a))} \gg e^{-\frac{2\pi}{\beta}(b-r^*(a))} - 2\cosh\left(\frac{2\pi}{\beta}(t_a - t_b)\right),$$

$$e^{\frac{2\pi}{\beta}(b-r^*(a))} \gg e^{\frac{-2\pi}{\beta}(b-r^*(a))} + 2\cosh\left(\frac{2\pi}{\beta}(t_a + t_b)\right). \tag{3.24}$$

In this case, by applying eq. (3.24), one can expand the logarithmic term in (3.23) to second order and rewrite the equation as follows[17]

$$S_{\text{gen}} = \frac{V_d a^{d_e}}{2G_{N,r}} + \frac{c}{3}\left[\log\left(\frac{\beta^2 a^{z-\theta_e}\sqrt{f(a)}}{\pi^2}\cosh\left(\frac{2\pi t_a}{\beta}\right)\cosh\left(\frac{2\pi t_b}{\beta}\right)\right)\right.$$

$$-4e^{-\frac{2\pi}{\beta}(b-r^*(a))}\cosh\left(\frac{2\pi t_a}{\beta}\right)\cosh\left(\frac{2\pi t_b}{\beta}\right)$$

$$\left.+2e^{-\frac{4\pi}{\beta}(b-r^*(a))}\left[\cosh^2\left(\frac{2\pi(t_a+t_b)}{\beta}\right) - \cosh^2\left(\frac{2\pi(t_a-t_b)}{\beta}\right)\right]\right]. \tag{3.25}$$

Next, by extremizing the above expression with respect to a, it is straightforward to verify that there is no real solution for a. Therefore, one may conclude that at early times, there are no islands, and hence S_R is given by eq. (3.17).

Now we consider the entropy of Hawking radiation at late times, i.e. $t_{a,b}T \gg 1$. We again assume that t_a and t_b are of the same order and $a \approx r_h$. In this case, we can apply the following approximations (See also [35, 45])

$$e^{\frac{2\pi}{\beta}(b-r^*(a))} \gg e^{-\frac{2\pi}{\beta}(b-r^*(a))} - 2\cosh\left(\frac{2\pi}{\beta}(t_a - t_b)\right), \tag{3.26}$$

$$2\cosh\left(\frac{2\pi}{\beta}(t_a + t_b)\right) \gg e^{\frac{2\pi}{\beta}(b-r^*(a))} + e^{\frac{-2\pi}{\beta}(b-r^*(a)}. \tag{3.27}$$

Therefore, by applying eqs. (3.26) and (3.27), one has[18]

$$S_{\text{gen}} = \frac{V_d a^{d_e}}{2G_{N,r}} + \frac{c}{3}\left[\log\left(\frac{\beta^2 a^{z-\theta_e}\sqrt{f(a)}}{\pi^2}\cosh\left(\frac{2\pi t_a}{\beta}\right)\cosh\left(\frac{2\pi t_b}{\beta}\right)\right) + \frac{2\pi}{\beta}(b-r^*(a))\right.$$

$$\left.-\log\left(2\cosh\left(\frac{2\pi}{\beta}(t_a+t_b)\right)\right) - 2e^{-\frac{2\pi}{\beta}(b-r^*(a))}\cosh\left(\frac{2\pi}{\beta}(t_a-t_b)\right) + e^{-\frac{4\pi}{\beta}(b-r^*(a))}\right]$$

$$= \frac{V_d a^{d_e}}{2G_{N,r}} + \frac{c}{3}\left[\log\left(\frac{\beta^2 a^{z-\theta_e}\sqrt{f(a)}}{4\pi^2}\right) + \frac{2\pi}{\beta}(b-r^*(a))\right.$$

$$\left.-2e^{-\frac{2\pi}{\beta}(b-r^*(a))}\cosh\left(\frac{2\pi}{\beta}(t_a-t_b)\right) + e^{-\frac{4\pi}{\beta}(b-r^*(a))}\right]. \tag{3.28}$$

---

[16]Notice that when $a \to r_h$, one has $r^*(a) \to -\infty$, and hence one may apply $\cosh\left(\frac{2\pi}{\beta}(b-r^*(a))\right) \approx \frac{1}{2}e^{\frac{2\pi}{\beta}(b-r^*(a))}$.

[17]We keep terms of order $e^{-\frac{2\pi}{\beta}(b-r^*(a))}$ and $e^{-\frac{4\pi}{\beta}(b-r^*(a))}$ and omit higher order terms.

[18]It should be pointed out that the term $e^{-\frac{4\pi}{\beta}(b-r^*(a))}$ in eq. (3.28) gives a correction of order $G_{N,r}^2$ to the entropy of Hawking radiation S_R. In the following, we calculate S_R to order $G_{N,r}^0$ (See e.g. eq. (3.32)), and hence one may omit this term similar to refs [35, 45].
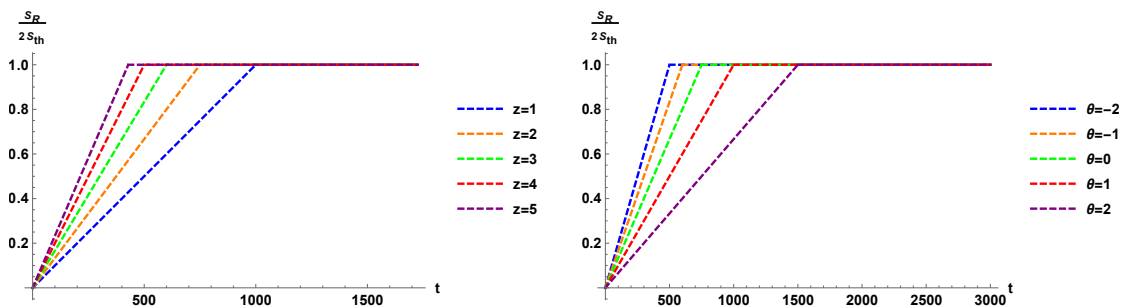
Figure 2. Entropy of Hawking radiation S_R when the matter is described by a CFT_{d+2} or a holographic HV QFT_{d+2} with $\theta_m = 0$ for: (*Left*) $d = 2$, $\theta = -1$ and different values of z. (*Right*) $d = 3$, $z = 1$ and some values of θ. Before the Page time, S_R grows linearly with time. However, it saturates to $2S_{\text{th}}$ after this time. Here we set $c = A = V_d = r_h = 1$ and $G_{N,r} = 0.001$. It should be pointed out we omitted the correction of order $\mathcal{O}(G_{N,r}^0)$ in eqs. (3.32) and (4.29), since we assumed that eq. (3.2) is valid. Furthermore, we restricted ourselves to $d_e > 0$ and $z \geq 1$.

Next, by extremizing S_{gen} with respect to t_a, one has

$$t_a = t_b. \tag{3.29}$$

Then we plug eq. (3.29) into eq. (3.28) and extremize S_{gen} with respect to a. By expanding $\frac{\partial S_{\text{gen}}}{\partial a}$ in powers of $\alpha = \sqrt{\frac{a - r_h}{r_h}}$, one has

$$
\frac{\partial S_{\text{gen}}}{\partial a} = \frac{V_d r_h^{d_e - 1} d_e}{2 G_{N,r}} + \frac{c\left(2(z - \theta_e) - d_e + 2(d_e + z)e^{\gamma - b(d_e + z)r_h^z + \psi\left(\frac{z}{d_e + z}\right)}\right)}{6 r_h}
$$
$$
- \frac{c}{3 r_h \alpha}\sqrt{d_e + z}\, e^{\frac{1}{2}\left(\gamma - b(d_e + z)r_h^z + \psi\left(\frac{z}{d_e + z}\right)\right)} + \mathcal{O}(\alpha) = 0. \tag{3.30}
$$

From which one can easily find the value of a as follows

$$
a = r_h + \left(\frac{2cG_{N,r}}{3V_d d_e}\right)^2 \frac{(d_e + z)}{r_h^{2d_e - 1}} e^{\gamma - b(d_e + z)r_h^z + \psi\left(\frac{z}{d_e + z}\right)} + \mathcal{O}\left(G_{N,r}^3\right), \tag{3.31}
$$

where $\gamma = 0.577$ is Euler's constant and $\psi(\alpha)$ is the digamma function. Next, by plugging eqs. (3.29) and (3.31) into (3.28), one can find the entropy of Hawking radiation as follows

$$
S_R = 2S_{\text{th}} + \frac{c}{6}\left[b(d_e + z)r_h^z - \gamma - \psi\left(\frac{z}{d_e + z}\right) + \log\left(\frac{16}{(d_e + z)^4 r_h^{2(z + \theta_e)}}\right) + \mathcal{O}(G_{N,r})\right], \tag{3.32}
$$

where the first therm is twice the thermodynamic entropy S_{th} of a black brane. Notice that S_R is independent of time. Therefore, at late times the presence of the island leads S_R to saturate. In figure 2, S_R is plotted for different values of the exponents z and θ of the black brane geometry. At late times, before the Page time, S_R has a linear growth (See eq. (3.19)). However, after the Page time, it is independent of time (See eq. (3.32)).

On the other hand, by equating eqs. (3.19) and (3.32), the Page time is obtained as follows

$$
t_{\text{page}} = \frac{3V_d r_h^{d_e - z}}{cG_{N,r}(d_e + z)} + \mathcal{O}\left(G_{N,r}^0\right) = \frac{3}{\pi}\frac{S_{\text{th}}}{cT}. \tag{3.33}
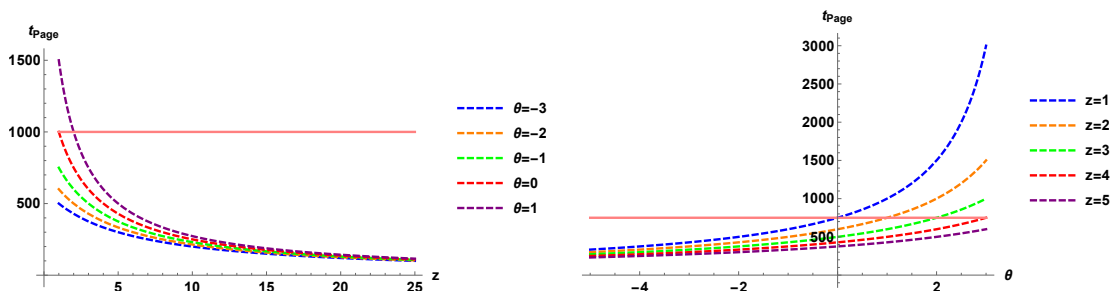$$

Figure 3. Page time when the matter is described by a CFT_{d+2} or a holographic HV QFT_{d+2} with $\theta_m = 0$ as a function of (*Left*) z for $d = 2$ and different values of θ. (*Right*) θ for $d = 3$ and different values of z. The solid pink line indicates t_{Page} for the case $\theta = 0$ and $z = 1$. It is evident that t_{Page} is a decreasing and increasing function of z and θ, respectively. Here we set $c = A = V_d = r_h = 1$ and $G_{N,r} = 0.001$. Moreover, we restricted ourselves to $d_e > 0$ and $z \geq 1$.

Therefore, the Page time is proportional to $\frac{S_{\text{th}}}{cT}$ and depends on both the exponents z and θ of the black brane geometry. It should be pointed out that for $\theta = 0$ and $z = 1$ the black brane geometry becomes a d+2 dimensional planar AdS-Schwarzschild black hole. In this case, our results for $d = 2$ reduce to those reported in section 5 of ref. [45].[19] Moreover, in the left panel of figure 3, t_{Page} is plotted as a function of z for different values of θ. It is observed that t_{Page} is a decreasing function of z. Furthermore, in the right panel of figure 3, t_{Page} is plotted as a function of θ for some values of z. It is observed that t_{Page} is an increasing function of θ. Form these diagrams, it is evident that for $\theta \leq 0$, the Page time is always smaller than that for the case $\theta = 0$ and $z = 1$. In other words, the entropy of Hawking radiation for a HV black brane with $\theta \leq 0$, saturates sooner than that for a planar AdS-Schwarzschild black hole. However, for positive values of θ, by decreasing z, the Page time can become larger than that for the case $\theta = 0$ and $z = 1$.

4 Entropy of Hawking radiation: matter HV QFT

In this section, we study the entropy of Hawking radiation for the case where the matter fields are described by a d+2 dimensional HV QFT which has a dual gravity that is a d+3 dimensional HV geometry at zero temperature. The metric of this geometry is simply obtained by setting $r_h = 0$ in eq. (2.4) as follows [84][20,21]

$$ds^2 = r^{-\frac{2\theta_m}{d+1}} \left(-r^{2z_m} dt^2 + \frac{dr^2}{r^2} + r^2 \sum_{i=1}^{d+1} dx_i^2 \right). \tag{4.1}$$

We denote the dynamical and hyperscaling violation exponents of the corresponding HV QFT by z_m and θ_m, respectively to draw a distinction between them and the exponents z

---

[19]Note that in ref. [45], one should turn off all of the higher derivative couplings in the action of the critical gravity to get an AdS-Schwarzschild black hole in Einstein gravity.

[20]Note that to have a d + 3 dimensional dual gravity we also replace d with $d + 1$ in eq. (4.1).

[21]Some measures of quantum entanglement such as holographic entanglement entropy, mutual information and entanglement wedge cross section were studied for this background in the literature. See, e. g. [85, 88–96], which is not an exhaustive list.

and θ of the HV black brane. Moreover, for $z_m = 1$ and $\theta_m = 0$, the metric (4.1) becomes an AdS_{d+3} spacetime which is dual to a CFT_{d+2}. Therefore, in this case, all of the results of this section should reduce to those in the previous section. Another important point is that since the HV QFT has a dual gravity, we can apply the RT formula [12] for the calculation of the EE of matter fields as well as the entropy of Hawking radiation.

4.1 Holographic entanglement entropy in a $d+2$ dimensional HV QFT

Now, we review the holographic entanglement entropy (HEE) for the metric (4.1) which is derived in ref. [85]. We consider an entangling region in the shape of a strip on a constant time slice of the dual QFT whose width in the x_1 direction is ℓ, and denote its lengths along the $x_{i \neq 1}$ directions by L. It was observed that for $d = \theta_m$, the dual QFT has a Fermi surface and the HEE is given by [85]

$$S = \frac{L^d}{2\tilde{G}_N} \log \left(\frac{\ell}{\epsilon} \right), \tag{4.2}$$

where $\tilde{G}_N$ is Newton's constant in d+3 dimensions, and ϵ is the cutoff in the dual QFT.[22] On the other hand, for $d \neq \theta_m$, the HEE is given by [85]

$$S = \frac{L^d}{2\tilde{G}_N (d - \theta_m)} \left[\frac{1}{\epsilon^{d-\theta_m}} - \Upsilon^{d-\theta_m+1} \left(\frac{2}{\ell} \right)^{d-\theta_m} \right], \tag{4.3}$$

where

$$\Upsilon = \frac{\sqrt{\pi} \Gamma \left(\frac{d-\theta_m+2}{2(d-\theta_m+1)} \right)}{\Gamma \left(\frac{1}{2(d-\theta_m)+1} \right)}. \tag{4.4}$$

In the following, we renormalize Newton's constant to absorb the cutoff dependent terms in the HEE. For $d = \theta_m$, from eq. (4.2), one can renormalize Newton's constant as follows

$$\frac{1}{4G_{N,r}} = \frac{1}{4G_N} + \log \epsilon. \tag{4.5}$$

However, for $d \neq \theta_m$, from eq. (4.3), one has[23]

$$\frac{1}{4G_{N,r}} = \frac{1}{4G_N} - \frac{1}{\epsilon^{d-\theta_m}}. \tag{4.6}$$

Notice that for $\theta_m = 0$, it reduces to eq. (3.5). Therefore, we only consider those parts of $S_{\mathrm{matter}}(\mathcal{R} \cup \mathcal{I})$ which are independent of the cutoff ϵ in what follows and again rewrite S_{gen}

---

[22]One might regard the HV QFT as an IR theory which can be obtained by a deformation of a CFT in the UV. In this sense, the cutoff ϵ should be considered as an effective cutoff ϵ_{eff} [86]. When the HV QFT has a Fermi surface, one has $\epsilon_{\mathrm{eff}} \propto k_F^{-1}$, where k_F is the radius of the corresponding Fermi surface [86]. In other words, the gravity with the metric in eq. (4.1) is well defined only above a dynamical scale r_F i.e. $r > r_F$ [85, 86] (Notice that our radial coordinate r is related to that in ref. [85] by $r_{\mathrm{here}} = \frac{1}{r_{\mathrm{there}}}$). In this manner, the cutoff dependent terms in the HEE are IR effects and not divergent in the UV. We would like to thank Edgar Shaghoulian very much for his helpful comments on this topic.

[23]Notice that we set the dynamical scale r_F to one. Otherwise, it is written as $\frac{1}{4G_{N,r}} = \frac{1}{4G_N} - \frac{1}{r_F^{\theta_m} \epsilon^{d-\theta_m}}$.

as eq. (3.6). Moreover, as mentioned in section 3, due to the translational symmetry of the metric (2.4) along the traverse directions x^i, one might effectively work in two dimensions parametrized by the coordinates t and r. Therefore, we need to know the HEE of an interval in a two dimensional HV QFT to calculate the entropy of Hawking radiation.[24] In this case, the cutoff independent parts of the HEE are as follows

- For $\theta_m = 0$, by applying eq. (4.2), one has

$$S^{\mathrm{f}} = \frac{A}{3} \log \ell, \tag{4.7}$$

where[25]

$$A = \frac{3}{2\tilde{G}_N}. \tag{4.8}$$

Notice that, for $\theta_m = 0$ and $z_m = 1$, the HV QFT_2 becomes a holographic CFT_2, and hence "A" is equal to the corresponding central charge c [87].

- For $\theta_m \neq 0$, by applying eq. (4.3), one has

$$S^{\mathrm{f}} = \frac{A}{3\theta_m} \Upsilon_0^{1-\theta_m} \left(\frac{\ell}{2} \right)^{\theta_m}, \tag{4.9}$$

where

$$\Upsilon_0 = \frac{\sqrt{\pi} \Gamma\left(\frac{2-\theta_m}{2(1-\theta_m)} \right)}{\Gamma\left(\frac{1}{2(1-\theta_m)} \right)}. \tag{4.10}$$

Notice that in this case S^{f} is negative (positive) when θ_m is negative (positive). Moreover, we impose the constraint $d + 1 > \theta_m$ which for $d = 0$ leads to

$$\theta_m < 1. \tag{4.11}$$

In the next section, we calculate the entropy of Hawking radiation for the case where the matter QFT is a d+2 dimensional HV QFT. We study the two cases $\theta_m = 0$ and $\theta_m \neq 0$, separately. Moreover, to have a positive entropy of Hawking radiation when there are no islands and $\theta_m \neq 0$, the finite part of the EE of matter has to be positive, i.e. $\theta_m > 0$. Combining this inequality with eq. (4.11) leads to the constraint $0 < \theta_m < 1$.

4.2 Entropy of Hawking radiation for $\theta_m = 0$

In this section, we study the case where the hyperscaling violation exponent θ_m of the matter fields is zero. We first assume that there are no islands and calculate the entropy of Hawking radiation. We will see that it grows linearly in time and violates the unitarity, i.e. eq. (1.1). Therefore, one needs to include the contribution of an island to stop this growth and saturate the entropy of Hawking radiation.

---

[24]Note that to have a two-dimensional HV QFT, one should set $d = 0$ in eq. (4.1).
[25]Recall that we set the AdS radius R to one.

4.2.1 No islands

In this case, there are only the radiation regions $\mathcal{R} = \mathcal{R}_- \cup \mathcal{R}_+$ (See the left panel of figure 1). Similar to the previous section, we assume that the whole state on the Cauchy slice is pure. Therefore, one can calculate the EE of the matter fields on the interval $[b_-, b_+]$ instead of that on the region $\mathcal{R}$, and hence the entropy of Hawking radiation is simply given by

$$S_{\mathrm{R}} = S_{\mathrm{matter}}^{\mathrm{f}}(\mathcal{R}) = S_{\mathrm{matter}}^{\mathrm{f}}(b_+, b_-). \tag{4.12}$$

Next, by applying eqs. (3.15) and (4.7), one can rewrite eq. (4.12) as follows

$$S_{\mathrm{R}} = \frac{A}{3} \log d(b_+, b_-) = \frac{A}{3} \log \left(\frac{\beta}{\pi} \cosh\left(\frac{2\pi t_b}{\beta} \right) \right). \tag{4.13}$$

It grows quadratically in time at early times, i.e. $t_b T \ll 1$,

$$S_{\mathrm{R}} = \frac{A}{3} \left[\frac{(d_e + z)^2 r_h^{2z}}{8} t_b^2 + \log\left(\frac{4}{(d_e + z) r_h^z} \right) \right]. \tag{4.14}$$

On the other hand, it grows linearly in time at late times, i.e. $t_b T \gg 1$,

$$S_{\mathrm{R}} \simeq \frac{A(d_e + z) r_h^z}{6} t_b, \tag{4.15}$$

which again will exceed the coarse-grained entropy of the two black branes. In the next section, we will see that by adding the contribution of an island to the entropy of Hawking radiation, it saturates after the Page time.

4.2.2 With islands

In this section, we assume that there is an island $\mathcal{I}$ whose endpoints are located outside the event horizon (See the right panel of figure 1). By assuming that the state on the whole Cauchy slice is pure, one needs to calculate the EE of matter fields on the two disjoint intervals $[b_-, a_-] \cup [a_+, b_+]$. Since the matter HV QFT has a holographic gravitational theory, one may apply the holographic prescription of ref. [97] to calculate the EE of the two disjoint intervals. In this case, it is given by

$$S_{[b_-, a_-] \cup [a_+, b_+]} = \mathrm{Min}\left(S_{\mathrm{con}}, S_{\mathrm{dis}} \right), \tag{4.16}$$

where S_{con} and S_{dis} are the HEE of the connected and disconnected configurations for the corresponding RT surfaces, respectively. Recall that in the connected configuration, each RT surface starts from an endpoint of one of the intervals and ends on an endpoint of another interval (See the left panel of figure 4). However, in the disconnected configuration, each RT surface starts from an endpoint of one interval and ends on another endpoint of the same interval (See the right panel of figure 4). From figure 4, it is evident that for the intervals $A = [b_-, a_-]$ and $B = [a_+, b_+]$, one can find

$$\begin{aligned} S_{\mathrm{con}} &= S(a_-, a_+) + S(b_-, b_+), \\ S_{\mathrm{dis}} &= S(b_-, a_-) + S(a_+, b_+). \end{aligned} \tag{4.17}$$
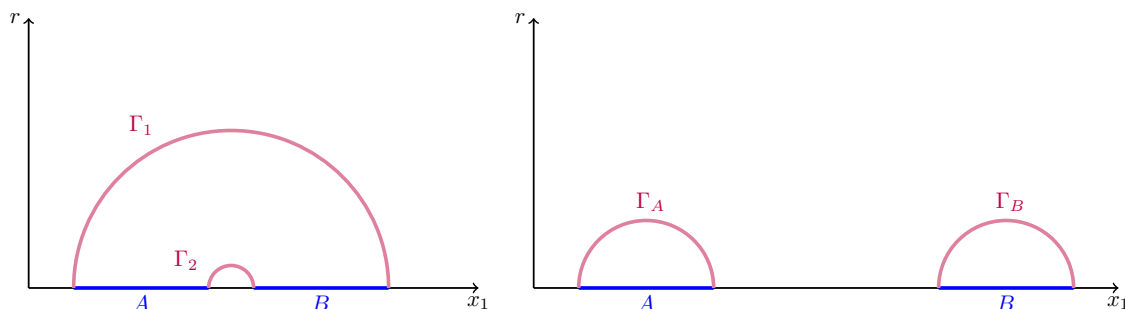
Figure 4. An illustration of the Ryu-Takayanagi (RT) surfaces indicated in purple for the calculation of HEE when the entangling region is composed of two disjoint intervals A and B along the spacial direction x_1. Here r is the radial coordinate in the dual gravity. (*Left*) Connected configuration: each RT surface starts from an endpoint of one of the intervals and ends on an endpoint of another interval. (*Right*) Disconnected configuration: each RT surface starts from one endpoint of an interval and ends on another endpoint of the same interval. It should be emphasized that the RT surfaces are not necessarily half circles and this diagram is schematic.

In the following, we consider two regimes: early times and late times. At early times, i.e. $t_{a,b}T \ll 1$, the two endpoints $a_\pm$ of the island are close to each other. In this case, the lengths of the two intervals $[b_-, a_-]$ and $[a_+, b_+]$ are much larger than the distance between them. Therefore, the EE is given by that of the connected RT surfaces (See the left panel of figure 4)

$$S_{\text{matter}}(\mathcal{R} \cup \mathcal{I}) = S_{\text{con}} = S_{\text{matter}}(a_-, a_+) + S_{\text{matter}}(b_-, b_+). \tag{4.18}$$

Next, by plugging eq. (3.15) and (4.7) into (4.18), one can find the finite part of the EE of the matter as follows

$$
\begin{aligned}
S_{\text{matter}}^{\text{f}}(\mathcal{R} \cup \mathcal{I}) &= \frac{A}{3} \log\left(d(a_-, a_+) d(b_-, b_+) \right) \\
&= \frac{A}{3} \log\left(\frac{\beta^2 a^{z-\theta_e} \sqrt{f(a)}}{\pi^2} \cosh\left(\frac{2\pi t_a}{\beta} \right) \cosh\left(\frac{2\pi t_b}{\beta} \right) \right).
\end{aligned} \tag{4.19}
$$

Therefore the generalized entropy is given by

$$
\begin{aligned}
S_{\text{gen}} &= \frac{V_d a^{d_e}}{2G_{N,r}} + S_{\text{matter}}^{\text{f}}(\mathcal{R} \cup \mathcal{I}) \\
&= \frac{V_d a^{d_e}}{2G_{N,r}} + \frac{A}{3} \log\left(\frac{\beta^2 a^{z-\theta_e} \sqrt{f(a)}}{\pi^2} \cosh\left(\frac{2\pi t_a}{\beta} \right) \cosh\left(\frac{2\pi t_b}{\beta} \right) \right).
\end{aligned} \tag{4.20}
$$

Now, one can extremize S_{gen} with respect to a. If one assumes that $a \approx r_h$ and expands $\frac{\partial S_{\text{gen}}}{\partial a}$ in powers of $\alpha = \sqrt{\frac{a-r_h}{r_h}}$, one obtains

$$\frac{\partial S_{\text{gen}}}{\partial a} = \frac{V_d d_e r_h^{d_e-1}}{2G_{N,r}} + \frac{A(3z - 1 - d_e - 4\theta_e)}{12 r_h} + \frac{A}{6 r_h \alpha^2} + \mathcal{O}(\alpha^2) = 0. \tag{4.21}$$

Then, it can be easily solved to find

$$a = r_h - \frac{A G_{N,r}}{3 d_e V_d r_h^{d_e-1}} + \mathcal{O}(G_{N,r}^2). \tag{4.22}$$

Next, by plugging eq. (4.22) into (4.20) and extremizing S_{gen} with respect to t_a, one simply obtains $t_a = 0$. It is straightforward to check that for these values of a and t_a, the generalized entropy is imaginary. Therefore, one may conclude that at early times there are no islands and the entropy of Hawking radiation is given by eq. (4.13).

On the other hand, at late times, i.e. $t_{a,b}T \gg 1$, the two endpoints of the island are very far from each other. Therefore, the lengths of the two intervals $[b_-, a_-]$ and $[a_+, b_+]$ are much smaller than their distance. In this case, the EE is given by that of the disconnected RT surfaces (See the right panel of figure 4)[26]

$$S_{\text{matter}}(\mathcal{R} \cup \mathcal{I}) = S_{\text{dis}} = S_{\text{matter}}(a_+, b_+) + S_{\text{matter}}(a_-, b_-). \tag{4.23}$$

Next, by plugging eqs. (3.15) and (4.7) into (4.23), one has

$$S^{\text{f}}_{\text{matter}}(\mathcal{R} \cup \mathcal{I}) = \frac{A}{3} \log\left(d(a_+, b_+) d(a_-, b_-)\right) \tag{4.24}$$

$$= \frac{A}{3} \log\left[\frac{\beta^2 a^{z-\theta_e} \sqrt{f(a)}}{2\pi^2} \left(\cosh\left(\frac{2\pi}{\beta}(b - r^*(a))\right) - \cosh\left(\frac{2\pi}{\beta}(t_a - t_b)\right)\right)\right].$$

At late times, by assuming that $a \approx r_h$ and $t_a \approx t_b$, one can apply the approximation given in eq. (3.26). After adding the gravitational part, one has

$$S_{\text{gen}} = \frac{V_d a^{d_e}}{2G_{N,r}} + \frac{A}{3}\left[\log\left(\frac{\beta^2 a^{z-\theta_e}\sqrt{f(a)}}{4\pi^2}\right) + \frac{2\pi}{\beta}(b - r^*(a))\right.$$

$$\left. + e^{\frac{2\pi}{\beta}(r^*(a)-b)}\left(e^{\frac{2\pi}{\beta}(r^*(a)-b)} - 2\cosh\left(\frac{2\pi}{\beta}(t_a - t_b)\right)\right)\right]. \tag{4.25}$$

Next, by extremizing the generalized entropy with respect to t_a, one easily finds

$$t_b = t_a. \tag{4.26}$$

Then one can extremize S_{gen} with respect to a and expand $\frac{\partial S_{\text{gen}}}{\partial a}$ in powers of $\alpha = \sqrt{\frac{a-r_h}{r_h}}$ as follows

$$\frac{\partial S_{\text{gen}}}{\partial a} = \frac{V_d r_h^{d_e-1} d_e}{2G_{N,r}} + \frac{A\left(2(z-\theta_e) - d_e + 2(d_e+z)e^{\gamma - b(d_e+z)r_h^z + \psi\left(\frac{z}{d_e+z}\right)}\right)}{6r_h}$$

$$- \frac{A}{3r_h\alpha}\sqrt{d_e + z}\, e^{\frac{1}{2}\left(\gamma - b(d_e+z)r_h^z + \psi\left(\frac{z}{d_e+z}\right)\right)} + \mathcal{O}(\alpha) = 0. \tag{4.27}$$

The above equation can be easily solved to find the value of a as follows

$$a = r_h + \left(\frac{2AG_{N,r}}{3V_d d_e}\right)^2 \frac{(d_e+z)}{r_h^{2d_e-1}} e^{\gamma - b(d_e+z)r_h^z + \psi\left(\frac{z}{d_e+z}\right)} + \mathcal{O}\left(G_{N,r}^3\right). \tag{4.28}$$

---

[26]It should be pointed out that this prescription were also applied in refs. [9, 17, 18, 32] for two-dimensional eternal black holes at late times.

At the end, by plugging eqs. (4.26) and (4.28) into (4.25), one can obtain the entropy of Hawking radiation as follows

$$S_{\mathcal{R}} = 2S_{\text{th}} + \frac{A}{6}\left[b(d_e+z)r_h^z - \gamma - \psi\left(\frac{z}{d_e+z}\right) + \log\left(\frac{16}{(d_e+z)^4 r_h^{2(z+\theta_e)}}\right) + \mathcal{O}\left(G_{N,r}\right)\right]. \quad (4.29)$$

The first term is twice the thermal entropy S_{th} of the black brane. Moreover, S_{R} is independent of time. Therefore, the presence of the island at late times leads S_{R} to become a constant.

Furthermore, by equating eqs. (4.15) and (4.29), one can obtain the Page time as follows

$$\begin{aligned} t_{\text{page}} &= \frac{3V_d r_h^{d_e-z}}{AG_{N,r}(d_e+z)} + \mathcal{O}\left(G_{N,r}^0\right), \\ &= \frac{3}{\pi}\frac{S_{\text{th}}}{AT}. \end{aligned} \quad (4.30)$$

Therefore, the Page time is proportional to $\frac{S_{\text{th}}}{AT}$ and depends on both the exponents z and θ of the black brane geometry. Moreover, it is independent of the exponent z_m of the matter fields. It should be pointed out that for $z_m = 1$ and $\theta_m = 0$, the HV QFT_{d+2} reduces to a CFT_{d+2}. In this case, $A = \frac{3R}{2G_N}$ becomes equal to the central charge c of the CFT. Therefore, it is expected that all of the results in this section reduces to those in section 3, if one sets $z_m = 1$. On the other hand, we observed that the EE of matter fields, and hence S_{R} are independent of the dynamical exponent z_m of the matter fields. Therefore, all of the results in this section for $z_m \neq 1$, are the same as those for the matter CFT if one replaces A with c. Therefore, the Page curve and Page time are again given by figures 2 and 3, respectively if one replaces c with A.

4.3 Entropy of Hawking radiation for $\theta_m \neq 0$

In this section, we study the case where the hyperscaling violation exponent θ_m of matter fields is non-zero. We first study the case where there is no island. Next, we consider the effect of the presence of an island on the entropy of Hawking radiation.

4.3.1 No islands

When there are no islands, by applying eq. (4.9), one can write

$$S_{\mathcal{R}} = S_{\text{matter}}^f(b_-, b_+) = \frac{A}{3\theta_m}\Upsilon_0^{1-\theta_m}\left(\frac{\beta}{2\pi}\cosh\left(\frac{2\pi t_b}{\beta}\right)\right)^{\theta_m}, \quad (4.31)$$

where θ_m is the hyperscaling violating exponent of the matter fields and Υ_0 is defined by eq. (4.10). Notice that for negative values of θ_m, $S_{\mathcal{R}}$ is negative. Therefore, in the following we restrict ourselves to the case $0 < \theta_m < 1$. At early times, one has

$$S_{\mathcal{R}} = \frac{A}{3\theta_m}\Upsilon_0^{1-\theta_m}\left(\frac{2}{(d_e+z)r_h^z}\right)^{\theta_m}\left(1 + \frac{1}{8}\theta_m(d_e+z)^2 r_h^{2z} t_b^2\right). \quad (4.32)$$

Thus, $S_\mathcal{R}$ grows quadratically in time. On the other hand, at late times, one obtains

$$S_\mathcal{R} = \frac{A}{3\theta_m} \Upsilon_0^{1-\theta_m} \left(\frac{1}{(d_e+z)r_h^z} \right)^{\theta_m} e^{\frac{(d_e+z)\theta_m r_h^z t_b}{2}}. \tag{4.33}$$

Notice that $d_e + z > 0$ and $0 < \theta_m < 1$, and hence the entropy of Hawking radiation grows exponentially in time. This behavior is in contrast to the usual linear growth which were previously observed for flat or AdS black holes in the literature, e.g. refs. [35, 45].

4.3.2 With islands

As mentioned before, the EE of the matter fields at early times is given by that of the connected RT surfaces (See the left panel of figure 4). By applying eqs. (3.15) and (4.9), one has

$$
\begin{aligned}
S_\text{matter}^\text{f} &= S_\text{matter}^\text{f}(a_-, a_+) + S_\text{matter}^\text{f}(b_-, b_+) \\
&= \frac{A\Upsilon_0^{1-\theta_m}}{3\theta_m} \left[\left(\left(\frac{\beta a^{z-\theta_e}\sqrt{f(a)}}{2\pi} \right) \cosh\left(\frac{2\pi t_a}{\beta} \right) \right)^{\theta_m} + \left(\left(\frac{\beta}{2\pi} \right) \cosh\left(\frac{2\pi t_b}{\beta} \right) \right)^{\theta_m} \right].
\end{aligned} \tag{4.34}
$$

After adding the gravitational part, the generalized entropy is given by

$$
\begin{aligned}
S_\text{gen} &= \frac{V_d a^{d_e}}{2G_{N,r}} + \frac{A\Upsilon_0^{1-\theta_m}}{3\theta_m} \left[\left(\left(\frac{\beta a^{z-\theta_e}\sqrt{f(a)}}{2\pi} \right) \cosh\left(\frac{2\pi t_a}{\beta} \right) \right)^{\theta_m} \right. \\
&\quad \left. + \left(\left(\frac{\beta}{2\pi} \right) \cosh\left(\frac{2\pi t_b}{\beta} \right) \right)^{\theta_m} \right] \\
&= \frac{V_d a^{d_e}}{2G_{N,r}} + \frac{A\Upsilon_0^{1-\theta_m}}{3\theta_m} \left[\left(\frac{\beta a^{z-\theta_e}\sqrt{f(a)}}{2\pi} \right)^{\theta_m} \left(1 + \frac{2\pi^2\theta_m t_a^2}{\beta^2} \right) \right. \\
&\quad \left. + \left(\frac{\beta}{2\pi} \right)^{\theta_m} \left(1 + \frac{2\pi^2\theta_m t_b^2}{\beta^2} \right) \right].
\end{aligned} \tag{4.35}
$$

Next, by extremizing the above expression with respect to t_a, one obtains $t_a = 0$. On the other hand, after extermization with respect to a one arrives at

$$\frac{\partial S_\text{gen}}{\partial a} = \frac{d_e V_d r_h^{d_e-1}}{2G_N} + \frac{A r_h^{-\theta_m\left(\theta_e+\frac{1}{2}\right)}}{3(d_e+z)^{\frac{\theta_m}{2}}} \left(\frac{\Upsilon_0}{2} \right)^{1-\theta_m} (a-r_h)^{\frac{\theta_m}{2}-1} + \mathcal{O}\left((a-r_h)^{\frac{\theta_m}{2}} \right), \tag{4.36}$$

which gives the following expression for a

$$a = r_h + \left(-\frac{A G_{N,r} 2^{\theta_m} \Upsilon_0^{1-\theta_m}}{3V_d d_e (d_e+z)^{\frac{\theta_m}{2}} r_h^{\left(d_e-1+\theta_m\left(\theta_e+\frac{1}{2}\right)\right)}} \right)^{\frac{2}{2-\theta_m}}. \tag{4.37}$$

It is straightforward to verify that for these values of t_a and a, the entropy of Hawking radiation is imaginary. Therefore, one might conclude that there are no islands at early times.

On the other hand, at late times, the EE of matter fields is given by that of the disconnected RT surfaces. In this case, by plugging eq. (4.9) into eq. (4.23), one has

$$
\begin{aligned}
S_{\text{matter}}^{\text{f}} &= S_{\text{matter}}^{\text{f}}(a_+, b_+) + S_{\text{matter}}^{\text{f}}(a_-, b_-) \\
&= \frac{A\,(2\Upsilon_0)^{1-\theta_m}}{3\theta_m} d(a_+, b_+)^{\theta_m} \\
&= \frac{A\,(2\Upsilon_0)^{1-\theta_m}}{3\theta_m} \left[\frac{\beta^2 a^{(z-\theta_e)}\sqrt{f(a)}}{2\pi^2} \left(\cosh\left(\frac{2\pi}{\beta}(b - r^*(a))\right) - \cosh\left(\frac{2\pi}{\beta}(t_a - t_b)\right) \right) \right]^{\frac{\theta_m}{2}}
\end{aligned}
$$

(4.38)

At late times, by applying the approximation given in eq. (3.26), one obtains

$$
\begin{aligned}
S_{\text{gen}} = \frac{V_d a^{d_e}}{2G_{N,r}} &+ \frac{A\,(2\Upsilon_0)^{1-\theta_m}}{3\theta_m} \left(\frac{\beta^2 a^{(z-\theta_e)}\sqrt{f(a)}}{4\pi^2} e^{\frac{2\pi}{\beta}(b-r^*(a))} \right)^{\frac{\theta_m}{2}} \\
&\times \left[1 + \frac{\theta_m}{2} e^{\frac{4\pi}{\beta}(r^*(a)-b)} - \theta_m e^{\frac{2\pi}{\beta}(r^*(a)-b)} \cosh\left(\frac{2\pi}{\beta}(t_a - t_b)\right) \right].
\end{aligned}
$$

(4.39)

Next, by extremizing S_{gen} with respect to t_a, one finds

$$
t_b = t_a.
$$

(4.40)

On the other hand, by plugging eq. (4.40) into eq. (4.39) and extremizing S_{gen} with respect to a, one has

$$
\frac{\partial S_{\text{gen}}}{\partial a} = B_1 + \frac{B_2}{\alpha} + \mathcal{O}(\alpha) = 0,
$$

(4.41)

where $\alpha = \sqrt{\frac{a - r_h}{r_h}}$ and

$$
\begin{aligned}
B_0 &= \gamma - b\, r_h^z (d_e + z) + \psi\left(\frac{z}{d_e + z}\right), \\
B_1 &= \frac{V_d r_h^{d_e-1} d_e}{2G_N} + \frac{A e^{-\frac{\theta_m}{4}B_0} \Upsilon_0^{1-\theta_m}}{6 d\theta_m (d_e + z)^{\theta_m} r_h^{1+\frac{\theta_m}{2}(z+\theta_e)}} \\
&\quad \times \left[2d(d_e + z)\theta_m e^{B_0} + 2d(z + 1 - d_e) - \theta_m\left(2\theta + d(d_e - 2z)\right) \right], \\
B_2 &= -\frac{A(d_e + z)^{\frac{1}{2}-\theta_m}}{3 r_h^{1+\frac{\theta_m}{2}(z+\theta_e)}} \Upsilon_0^{1-\theta_m} e^{\frac{(2-\theta_m)B_0}{4}}.
\end{aligned}
$$

(4.42)

From eq. (4.41), one simply obtains

$$
a = r_h + \left(\frac{2AG_{N,r}}{3V_d d_e}\right)^2 \frac{(d_e + z)^{1-2\theta_m}\, e^{\frac{(2-\theta_m)}{2}\left(\gamma - b\, r_h^z(d_e+z) + \psi\left(\frac{z}{d_e+z}\right)\right)}}{\Upsilon_0^{2(\theta_m-1)} r_h^{2d_e-1+\theta_m(z+\theta_e)}} + \mathcal{O}\left(G_{N,r}^3\right).
$$

(4.43)

At the end, by plugging eqs. (4.40) and (4.43) into eq. (4.39), the entropy of Hawking radiation is obtained as follows

$$
S_{\mathcal{R}} = 2S_{\text{th}} + \frac{2A\, e^{-\frac{\theta_m}{4}\left(\gamma - b(d_e+z)r_h^z + \psi\left(\frac{z}{d_e+z}\right)\right)}}{3\theta_m (d_e + z)^{\theta_m} r_h^{\frac{1}{2}(z+\theta_e)\theta_m}} \left(\frac{\Gamma\left(\frac{1}{2(1-\theta_m)}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{(\theta_m-2)}{2(\theta_m-1)}\right)} \right)^{\theta_m-1},
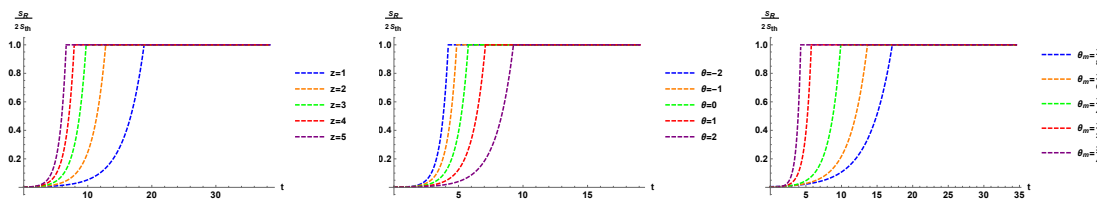$$

(4.44)

Figure 5. Entropy of Hawking radiation S_R when the matter is described by a HV QFT_{d+2} with $\theta_m \neq 0$ as a function of t: (*Left*) when $d = 2$, $\theta = 1$ and $\theta_m = \frac{1}{3}$ for some values of z. (*Middle*) when $d = 3$, $z = 2$ and $\theta_m = \frac{1}{2}$ for different values of θ. (*Right*) when $d = 3$, $z = 1$ and $\theta = -1$ for some values of θ_m. Here we set $A = V_d = r_h = 1$ and $G_{N,r} = 0.001$. It should be pointed out that we omit the correction of order $\mathcal{O}(G_{N,r}^0)$ in eq. (4.44).
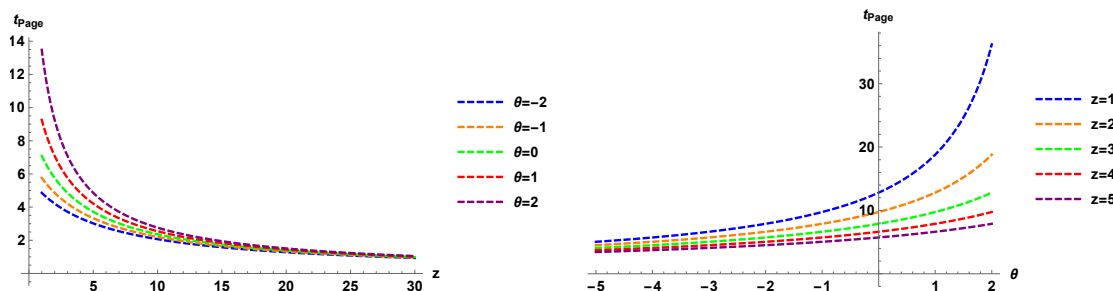
Figure 6. Page time when the matter is described by a HV QFT_{d+2} with $\theta_m \neq 0$ as a function of: (*Left*) z for $d = 3$, $\theta_m = \frac{1}{2}$ and different values of θ. (*Right*) θ for $d = 2$, $\theta_m = \frac{1}{3}$ and different values of z. Here we set $A = V_d = r_h = 1$ and $G_{N,r} = 0.001$.

which is constant in time. Therefore, S_R saturates when there is an island. In figure 5, the entropy of Hawking radiation is plotted as a function of z, θ and θ_m. At late times, before the Page time, S_R shows an exponential growth in time (See eq. (4.33)). However, after the Page time, it saturates (See eq. (4.44)).

On the other hand, by equating eqs. (4.33) and (4.44), the Page time is simply obtained as follows

$$t_{\text{Page}} = \frac{2}{\theta_m(d_e + z)r_h^z} \log \left(\frac{3\theta_m(d_e + z)^{\theta_m} V_d \, r_h^{(d_e + z\theta_m)} \Upsilon_0^{\theta_m - 1}}{2AG_{N,r}} \right). \qquad (4.45)$$

Therefore $t_{\text{page}} \propto \log \left(\frac{1}{G_{N,r}} \right)$, which is a consequence of the exponential growth of S_R with time before reaching the page time (See eq. (4.33)). In figure 6, the Page time is plotted as a function of z and θ of the black brane. It is observed that t_{Page} is a decreasing function of the exponent z. However, it is an increasing function of θ. On the other hand, in figure 7, the Page time is plotted in terms of θ_m which shows that it is a decreasing function of θ_m.

5 Discussion

In this paper we studied the Page curve for a two-sided Hyperscaling Violating (HV) black brane in $d + 2$ dimensions. We assumed that the matter fields in the black brane geometry
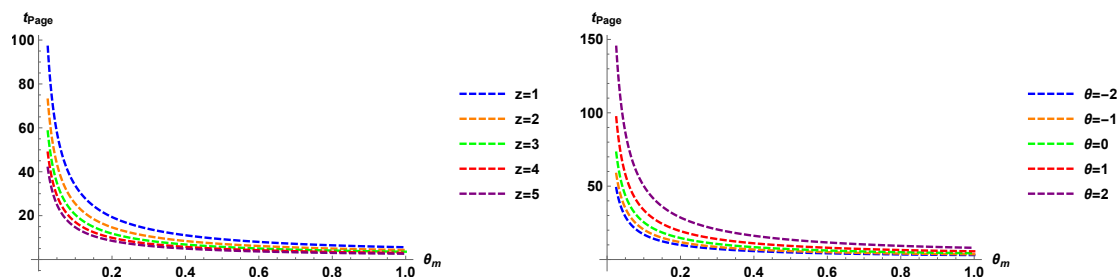
Figure 7. Page time when the matter is described by a HV QFT_{d+2} with $\theta_m \neq 0$ as a function of θ_m: (*Left*) for $d = 2$, $\theta = 0$ and different values of z. (*Right*) for $d = 3$, $z = 1$ and different values of θ. Here we set $A = V_d = r_h = 1$ and $G_{N,r} = 0.001$.

and inside the baths are the same. Moreover, we considered two different situations for the matter fields: first, they are described by a CFT_{d+2}. Second, they are described by a d+2 dimensional HV QFT which is dual to a d+3 dimensional gravity that is a HV geometry at zero temperature (See eq. (4.1)). In both cases, it was observed that at very early times there are no islands. However at late times the presence of an island is necessary to obtain a correct Page curve for the entropy of the Hawking radiation.

For matter CFT, at early times there are no islands. In this case, the entropy of Hawking radiation S_R shows a quadratic growth in time (See eq. (3.18)). Next, it grows linearly with time (See eq. (3.19)). If there are no islands, S_R exceeds twice the coarse-grained (thermodynamic) entropy S_{th} of the black brane. However, when there is an island, S_R becomes a constant (See eq. (3.32)). The corresponding Page curve is plotted in figure 2 for different values of the exponents z and θ of the black brane. On the other hand, the Page time is proportional to $\frac{S_{th}T}{c}$ (See eq. (3.33)), where S_{th} and T are the thermal entropy and temperature of the black brane, and c is the central charge of the matter CFT. Moreover, in figure 3, the Page time is plotted as a function of z and θ. It was observed that t_{Page} is a decreasing function of z. On the other hand, it is an increasing function of θ. Moreover, for $\theta \leq 0$, the Page time is always smaller than that for the case $z = 1$ and $\theta = 0$. In other words, the entropy of Hawking radiation for HV black branes saturates sooner than that for planar AdS-Schwarzschild black holes, if one has $\theta \leq 0$. However, for positive values of θ, the Page time becomes larger than that for the case $z = 1$ and $\theta = 0$, if one decreases z.

It should be emphasized that for $z = 1$ and $\theta = 0$, the black brane geometry reduces to a $d+2$ dimensional planar AdS-Schwarzschild black hole. Therefore all of our results can be applied for this type of black hole, if one sets $z = 1$ and $\theta = 0$. We verified that for $d = 2$, our results are consistent with those for a four dimensional planar AdS-Schwarzschild black hole in the critical gravity model reported in ref. [45], if one sets all of the higher derivative couplings in the action to zero.

On the other hand, we studied the case where the matter is described by a d+2 dimensional HV QFT. In this case one can assign two exponents z_m and θ_m to the matter. Moreover, since the EE of matter is independent of the exponent z_m, the entropy of Hawking radiation is also independent of z_m. In other words, it only depends on the exponent θ_m. We examined the two cases $\theta_m = 0$ and $\theta_m \neq 0$ separately. Moreover, since the HV

QFT has a dual gravity, we applied the holographic prescription of ref. [97] to calculate the EE of the matter fields on the two disjoint intervals $[b_-, a_-] \cup [a_+, b_+]$ when there is an island $\mathcal{I}$ (See figure 4). In other words, at early times when the two intervals $[b_-, a_-]$ and $[a_+, b_+]$ are close to each other, we considered the connected RT surfaces. However, at late times when the two intervals are very far from each other, we applied the disconnected RT surfaces. It was observed that: for $\theta_m = 0$, the behaviors of the Page curve and Page time are the same as those for matter CFT. In other words, at early times there are no islands. In this case, S_R shows a quadratic growth in time (See eq. (4.14)). Next, it grows linearly with time (See eq. (4.15)). If there are no islands, S_R again exceeds twice the coarse-grained entropy of the black brane. However, when there is an island, S_R becomes a constant (See eq. (4.44)). Moreover, the Page time is proportional to $\frac{S_{\text{th}} T}{A}$ (See eq. (4.30)) where $A = \frac{3R}{2G_N}$.

It should be pointed out that the EE of matter fields, and hence the entropy of Hawking radiation is independent of z_m. On the other hand, for $z_m = 1$ and $\theta_m = 0$, the HV QFT_{d+2} becomes a CFT_{d+2}. Therefore, all of the results for $\theta_m = 0$ should be the same as those for the case where the matter fields are described by a CFT_{d+2}. In particular, the plots of the Page curve and Page time are again given by figures 2 and 3, if one replaces A with c in the formulas. This observation shows that by applying the holographic prescription of ref. [97], the entropy of Hawking radiation obeys the expected Page curve. Furthermore, t_{Page} is a decreasing function of the exponent z and an increasing function of θ (See figure 3).

For $\theta_m \neq 0$, it was verified that at early times there are no islands. In the absence of an island, at the beginning S_R again shows a quadratic behavior with time (See eq. (4.32)). Next, it grows exponentially with time (see eq. (4.33)), which is in contrast to the usual linear growth for flat and AdS black holes with matter CFT. If there are no islands, again the entropy of Hawking radiation exceeds twice the coarse-grained entropy of the black brane. However, when there is an island, S_R becomes independent of time (See eq. (4.44)). This behavior is a consequence of the exponential growth of the entropy of Hawking radiation before the Page time. Moreover, the corresponding Page curve is plotted in figure 5 for different values of the exponent z, θ and θ_m. It was also observed that the Page time is proportional to $\log\left(\frac{1}{G_{N,r}}\right)$ (See eq. (4.45)). Furthermore, similar to the case $\theta_m = 0$, the Page time is a decreasing function of z and an increasing function of θ (See figure 6). Moreover, t_{Page} is independent of z_m and is a decreasing function of θ_m (See figure 7).

As mentioned before, for $\theta_m \neq 0$, the EE of radiation grows exponentially before the Page time (see eq. (4.33)). As long as we know, this rate of growth for the EE is surprising.[27] More precisely, the rate of growth of the EE for Vaidya black branes with Hyperscaling Violation when the entangling region is in the shape of a strip with width l and lengths L were explored in refs. [98, 99]. It was observed that there are three phases for the growth of the EE: first, early times, i.e. $t \ll \rho_h^z \propto \frac{1}{T}$ where ρ_h and T are the horizon radius and temperature of the black brane, it obeys a power law behavior and is given by

$$\Delta S = \frac{L^{d-1} m}{8 G_N (z+1)} (zt)^{1+\frac{1}{z}}, \tag{5.1}$$

---

[27]We would like to thank the referee for her/his enlightening comments on this point.

where ΔS is the difference of the EE with that of the vacuum and m is related to the mass of the black brane. Notice that it is independent of θ. For $z = 1$ and $\theta = 0$, where the HV QFT becomes a CFT, it is quadratic in time as it is expected for holographic CFTs [100, 101]. Moreover, for very large z, it becomes linear in time. Second, at intermediate times, i.e. $\rho_h^z \lesssim t \lesssim \frac{l}{2}\rho_h^{z-1}$, it grows linearly in time

$$\Delta S = \frac{L^{d-1}}{2G_N \rho_h^{d_e+z-1}} v_E t, \qquad (5.2)$$

where v_E is the entanglement velocity and given by

$$v_E = \frac{(\eta - 1)^{\frac{\eta-1}{2}}}{\eta^{\frac{\eta}{2}}}, \qquad \eta = \frac{2(d_e + z - 1)}{d_e + z}. \qquad (5.3)$$

Third, it saturates to a constant value at late times. Therefore, this exponential growth is a new feature and it would be very interesting to investigate it further and to explore whether or not this growth rate is consistent with unitarity.

At the end, it would also be interesting to do these calculations for one-sided HV black branes and two-sided charged HV black branes and study the effects of the exponents z and θ on the Page curve and Page time.

Acknowledgments

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