Twin modular S_4 with SU(5) GUT

Stephen F. King and Ye-Ling Zhou

*School of Physics and Astronomy, University of Southampton,
Southampton SO17 1BJ, U.K.*

E-mail: king@soton.ac.uk, ye-ling.zhou@soton.ac.uk

ABSTRACT: We discuss the SU(5) grand unified extension of flavour models with multiple modular symmetries. The proposed model involves two modular S_4 groups, one acting in the charged fermion sector, associated with a modulus field value τ_T with residual Z_3^T symmetry, and one acting in the right-handed neutrino sector, associated with another modulus field value τ_{SU} with residual Z_2^{SU} symmetry. Quark and lepton mass hierarchies are naturally generated with the help of weightons, which are SM singlet fields, where their non-zero modular weights play the role of Froggatt-Nielsen charges. The model predicts TM_1 lepton mixing, and neutrinoless double beta decay at rates close to the sensitivity of current and future experiments, for both normal and inverted orderings, with suppressed corrections from charged lepton mixing due to the triangular form of its Yukawa matrix.

KEYWORDS: GUT, Neutrino Physics, Discrete Symmetries

ARXIV EPRINT: [2103.02633](https://arxiv.org/abs/2103.02633)

Contents

1	Introduction	1
2	Multiple S_4 modular symmetries	4
3	An $SU(5)$ model with twin S_4 modular symmetries	5
4	Numerical study	11
5	Conclusion	14
A	Modular group and modular forms of level 4	16
B	Vacuum alignments	19

1 Introduction

The Standard Model (SM) of particle physics, though highly successful, does not explain why there are three families of quarks and leptons, and does not account for neutrino mass and mixing. The SM also offers no insight into the pattern of quark and lepton (including neutrino) mass and mixing. Moreover, the three gauge groups of the SM are unrelated, with anomaly cancellation and charge quantisation being unexplained, a seemingly accidental consequence of having complete families of quarks and leptons.

Promising attempts to go beyond the SM in order to address these shortcomings are often based on family symmetry on the one hand, and grand unified theories (GUTs) [1] on the other, where the combination of these two symmetries [2] provides a powerful and constrained framework. Neutrino mass and mixing can be included by the addition of gauge singlet states, the right-handed neutrinos, which together with the type I seesaw mechanism [3–9], provides an elegant understanding of the extreme smallness of neutrino mass compared to charged fermion masses. However understanding the observed approximate tri-bimaximal (TB) nature of neutrino mixing, in which the lepton mixing matrix has a very symmetric structure with $U_{e3} = 0$ and equal non-zero elements in the second and third column, suggests an underlying discrete non-Abelian family symmetry [2, 10]. For example, the group S_4 , with generators S, T, U ,¹ is capable of enforcing partial symmetries in the neutrino and charged lepton sector capable of enforcing TB mixing, with T associated with a Z_3^T subgroup preserved in the charged lepton sector, and S, U associated with $Z_2^S \times Z_2^U$ controlling the neutrino sector.

Following the discovery at reactor experiments that U_{e3} takes a non-zero value [11], exact tri-bimaximal mixing became excluded, while the larger mixing measured initially by

¹We assume the convention $S^2 = T^3 = U^2 = (ST)^3 = (SU)^2 = (TU)^2 = \mathbf{1}$ [10].

atmospheric and solar neutrino experiments, and later by long baseline oscillation experiments, continue to take the tri-bimaximal form. This observation suggests possible relaxed forms of mixing known as trimaximal (TM) mixing, in which the first or second column of the lepton mixing matrix maintains the tri-bimaximal form, while the third column is unconstrained, allowing a non-zero element U_{e3} . There is a phenomenological preference for the first case, called TM_1 mixing [12, 13], in which follows from S_4 where T associated with a Z_3^T subgroup preserved in the charged lepton sector, as usual, while the product SU associated with Z_2^{SU} controlling the neutrino sector.

Despite the motivation from neutrino physics for introducing a discrete non-Abelian family symmetry such as S_4 , this raises the question of the origin of such symmetry, and also the nature of its spontaneous breaking, with particular subgroups preserved in different sectors of the theory, all of which seems to require a large number of flavon fields whose vacuum alignment requires further driving fields, plus *ad hoc* shaping symmetries [12, 13]. The S_4 , for example, could originate from a continuous non-Abelian symmetry [14–21], or have an extra-dimensional origin [22–33], being an accidental symmetry of orbifolding [30, 34–36], or as a subgroup of the so-called modular symmetry [37, 38]. Indeed it has been suggested that a finite subgroup of modular symmetry may be well suited to describe neutrino mass and mixing [39–41], where modular forms play the role of the vacuum alignments of flavon models, but without requiring flavons or driving fields [42].

In the modular symmetry approach [42], finite discrete flavour symmetry groups such as S_4 emerge as the quotient group of modular group $\text{SL}(2, Z)$ over the principal congruence subgroups. Quarks and leptons transform nontrivially under the modular S_4 , for example, and are assigned to have certain modular weights. Then Yukawa and mass structures emerge as modular forms, holomorphic functions of the complex modulus τ , with even (or odd) modular weights. The modular form of level N and integer weight k can be arranged into some modular multiplets of the homogeneous finite modular group $\Gamma_N \equiv \bar{\Gamma}/\bar{\Gamma}(N)$ if k is an even number [42]. This has been studied for $N = 2$ [43–46], $N = 3$ [42–44, 47–71], $N = 4$ [41, 60, 72, 73, 75–79], $N = 5$ [77, 80, 81] and $N = 7$ [82]. We shall be interested in $N = 4$ case where $\Gamma_N \simeq S_4$.

In general the complex modulus τ can take any complex value in the upper half complex plane, but it can also be restricted to take particular values, associated with preserved residual symmetries of the finite modular group, which is referred to as the stabiliser in the framework of finite modular symmetry [51, 60, 73, 74]. However it is clear from the previous discussion that realistic neutrino models require more than one stabiliser to generate interesting models. This implies that more than one modular symmetry must be considered requiring the framework to be extended to multiple modular symmetries [75]. In a recent paper [76] we considered an example with twin modular S_4 symmetries, leading to TM_1 mixing. The idea was that one modular S_4 controls the neutrino sector, associated with a modulus field value τ_{SU} , while another S_4 acts in the charged lepton sector, associated with a modulus field value τ_T . Apart from the predictions of TM_1 mixing, the model led to a novel neutrino mass sum rule with stringent lower bounds on neutrino masses close to current limits from neutrinoless double beta decay experiments and cosmology.

Such models, while interesting, hitherto do not address the question of quark mass and mixing, nor the possibility of extending the framework of modular symmetry to GUTs. In-

deed there has been progress of both questions in the literature, with modular models of both quark and lepton mass and mixing having been proposed [53, 65, 83]. In one of these approaches [65] the quark and lepton mass hierarchies were explained in the framework of modular symmetry without introducing any additional symmetry such as the Froggatt-Nielsen (FN) $U(1)$ [84] which is commonly used to account for charged fermion mass hierarchies. It was pointed out that the role of the FN flavon may be played by a singlet under both the gauge symmetry and the finite modular symmetry which carries a non-zero modular weight, called the weighton [65], leading to natural quark and lepton mass hierarchies.² The question of quark and lepton mass and mixing is necessarily simultaneously addressed in GUTs, and various $SU(5)$ models have been proposed [45, 49, 87–89] which include finite modular symmetries which act as family symmetries. Modular symmetry in the context of $SU(5)$ GUTs was first studied in an $(\Gamma_3 \simeq A_4) \times SU(5)$ model in [49], then $(\Gamma_2 \simeq S_3) \times SU(5)$ [45, 87], and $(\Gamma_4 \simeq S_4) \times SU(5)$ [88]. Most recently a comprehensive study of $(\Gamma_3 \simeq A_4) \times SU(5)$ has been performed [89]. All these models consider a single modular group.

In this paper, we shall propose an $SU(5)$ GUT with Trimaximal TM_1 neutrino mixing arising from twin modular S_4 groups. The resulting model is based on a similar strategy that we developed for our previous lepton model [76], and indeed the resulting neutrino mass and Yukawa matrices are identical to those considered previously. However, the quark and charged lepton sectors are quite different, with both necessarily having a non-diagonal structure in order to account for quark mixing, unlike the previous model in which the charged lepton Yukawa matrix was diagonal. Also, the hierarchy of the charged lepton masses which was previously obtained by the tuning of Yukawa couplings, is now accounted for by the weighton fields, associated with the twin S_4 symmetries, along with the quark mass hierarchies and quark mixing parameters which of course were not considered at all before, since the previous model was purely a model of leptons. Thus we shall construct a flavour model based on $SU(5)$ with two modular symmetries $S_4^F \times S_4^N$, broken to a single S_4 by a bi-triplet scalar, leading to the effective theory invariant under one single S_4 but involving two modulus fields. The modulus fields gain different VEVs, leading to the breaking of S_4 to a residual Z_3^T in the charged fermions sector, and a residual Z_2^{SU} symmetry in the neutrino sector. We analyse the resulting model numerically and show that, in the quark sector, the correct CKM matrix can be achieved, while in the lepton sector, there are small deviations from the TM_1 mixing. These corrections are small, but may be tested in the future high-precision neutrino oscillation experiments.

The layout of the remainder of the paper is as follows. In section 2, we briefly review multiple S modular symmetries as the direct origin of fermion masses and mixing. In section 3, we propose the $SU(5)$ GUT model based on $S_4^F \times S_4^N$ with two moduli fields. We perform a numerical scan in section 4 to see how good the model match with experimental data. Section 5 concludes the paper. The appendix A includes some background information about the modular group and modular forms of level 4, while appendix B gives some details about the vacuum alignment required in our model.

²Alternatively, fermion mass hierarchies can arise from the slight breaking of certain residual modular symmetries [71, 85, 86].

2 Multiple S_4 modular symmetries

We review the approach of using multiple finite modular symmetries to explain flavour mixing, with $\Gamma_4 \simeq S_4$ as the representative in the following discussion.

We give a brief introduction of S_4 as a finite modular group here. For more details of Γ_4 and its connection with the infinite modular group, see appendix A. In the modular S_4 symmetry, each element γ acting on the complex modulus τ ($\text{Im}(\tau) > 0$) as a linear fractional transformation:

$$\gamma : \tau \rightarrow \gamma\tau = \frac{a\tau + b}{c\tau + d}, \quad (2.1)$$

where a, b, c, d are integers mod 4 and satisfy $ad - bc = 1$. It is convenient to represent each element of the modular S_4 group by a 2×2 matrix, namely,

$$S_4 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} / (\pm \mathbf{1}), \quad a, b, c, d \in \mathbb{Z}_4, \quad ad - bc = 1 \right\}. \quad (2.2)$$

The finite modular group has two generators, S_τ and T_τ , which satisfy $S_\tau^2 = (S_\tau T_\tau)^3 = T_\tau^4 = \mathbf{1}$.³ They acting on the modulus τ take the following forms

$$S_\tau : \tau \rightarrow -\frac{1}{\tau}, \quad T_\tau : \tau \rightarrow \tau + 1, \quad (2.3)$$

respectively. They could be represented by 2×2 matrices as

$$S_\tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T_\tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (2.4)$$

We consider flavour model construction in the modular invariance approach where the modular symmetry is regarded as the direct origin of flavour mixing. Each field ϕ_i , which transforms as an irreducible representation (irrep) I_i of S_4 in the flavour space, has a modular weight $2k_i$. Under the modular transformation,

$$\gamma : \phi_i(\tau) \rightarrow \phi_i(\gamma\tau) = (c\tau + d)^{2k_i} \rho_{I_i}(\gamma) \phi_i(\tau). \quad (2.5)$$

The superpotential in an $\mathcal{N} = 1$ supersymmetry can be generically written in powers of the fields as

$$w(\phi_i; \tau) = \sum_n \sum_{\{i_1, \dots, i_n\}} (Y_{I_Y} \phi_{i_1} \cdots \phi_{i_n})_{\mathbf{1}}, \quad (2.6)$$

where the subscript $\mathbf{1}$ is the singlet contraction of the coupling and fields. Here, the coupling Y_{I_Y} does not have to be a trivial coefficient, but any modular form with the representation contributed to the singlet contraction and the modular weight k_Y must satisfy $k_Y + k_1 + k_2 + \cdots + k_n = 0$. With these requirements, the superpotential is invariant

³Relaxing the last condition $T_\tau^4 = \mathbf{1}$ leads to the infinite modular group $\bar{\Gamma}$. Replacing it with $T_\tau^N = \mathbf{1}$ with $N = 2, 3, 5$ leads to finite modular groups $\Gamma_2 \simeq S_3$, $\Gamma_3 \simeq A_4$ and $\Gamma_5 \simeq A_5$, respectively. For $N > 5$, additional restrictions are required to make Γ_N finite. For example, at $N = 7$, one such restriction is $(S_\tau T_\tau)^4 = \mathbf{1}$, leading to the finite $\Gamma_7 \simeq \Sigma(168)$ [82].

under any transformation of S_4 . Besides, k_Y is required to be an even number, as an intrinsic property of modular forms. It transforms as

$$\gamma : Y_{I_Y}(\tau) \rightarrow Y_{I_Y}(\gamma\tau) = (c\tau + d)^{2k_Y} \rho_{I_Y}(\gamma) Y_{I_Y}(\tau) \quad (2.7)$$

under the action of γ .

Given suitable arrangements for representations and weights for particles in the flavour space, a modular-invariant flavour model can be constructed by including a set of modular forms $\{Y_{I_Y}\}$ which keep the superpotential invariant under the modular transformation. After the modulus field τ gains a vacuum expectation value (VEV), a specific flavour texture of Yukawa couplings is achieved.

We extend the discussion to theories including more modular symmetries. Without loss of generality and for convenience of the latter discussion, we include two, labeled as S_4^F and S_4^N , respectively. Let us assume S_4^F and S_4^N to be independent of each other, and the moduli fields are denoted as τ_F and τ_N , respectively. Any modular transformation $\gamma_F \times \gamma_N$ in $S_4^F \times S_4^N$ takes the form

$$\gamma_F \times \gamma_N : (\tau_F, \tau_N) \rightarrow (\gamma_F \tau_F, \gamma_N \tau_N) = \left(\frac{a_F \tau_F + b_F}{c_F \tau_F + d_F}, \frac{a_N \tau_N + b_N}{c_N \tau_N + d_N} \right). \quad (2.8)$$

The superpotential $w(\phi_i; \tau_F, \tau_N)$ invariant in $S_4^F \times S_4^N$ is in general expressed as

$$w(\phi_i; \tau_F, \tau_N) = \sum_n \sum_{\{i_1, \dots, i_n\}} \left(Y_{(I_{Y,F}, I_{Y,N})} \phi_{i_1} \cdots \phi_{i_n} \right)_{(\mathbf{1}, \mathbf{1})}, \quad (2.9)$$

where the field ϕ_i and coupling $Y_{(I_{Y,F}, I_{Y,N})}$ transform as

$$\begin{aligned} \phi_i(\tau_F, \tau_N) &\rightarrow \phi_i(\gamma_F \tau_F, \gamma_N \tau_N) \\ &= (c_F \tau_F + d_F)^{2k_{i,F}} (c_N \tau_N + d_N)^{2k_{i,N}} \rho_{I_{i,F}}(\gamma_F) \phi_i(\tau_F, \tau_N) \rho_{I_{i,N}}^T(\gamma_N), \\ Y_{(I_{Y,F}, I_{Y,N})}(\tau_F, \tau_N) &\rightarrow Y_{(I_{Y,F}, I_{Y,N})}(\gamma_F \tau_F, \gamma_N \tau_N) \\ &= (c_F \tau_F + d_F)^{2k_{Y,F}} (c_N \tau_N + d_N)^{2k_{Y,N}} \rho_{I_{Y,F}}(\gamma_F) Y_{(I_{Y,F}, I_{Y,N})}(\tau_F, \tau_N) \rho_{I_{Y,N}}^T(\gamma_N), \end{aligned} \quad (2.10)$$

respectively. Here, we have arranged ϕ_i and $Y_{(I_{Y,F}, I_{Y,N})}$ as matrices such that γ_F acts on them vertically and γ_N acts on them horizontally. The modular weights $2k_{Y,F}$ and $2k_{Y,N}$ are even numbers.

In the perspective of model building, as we have shown in [76], including two modular symmetries 1) allows the modular symmetries to be broken into different residual symmetries in the charged lepton sector and neutrino sector, respectively, and 2) leads to new lepton flavour mixing pattern. In the rest of the paper, we extend the multiple modular invariance approach into the quark sector in the framework of grand unification.

3 An $SU(5)$ model with twin S_4 modular symmetries

Below we construct a flavour model with two S_4 modular symmetries. The particle contents and their representation and modular weight assignments are given in table 1. In the gauge

Fields	SU(5)	S_4^F	S_4^N	$2k_F$	$2k_N$
T_1	10	1	1	+4	+2
T_2	10	1	1	+3	+1
T_3	10	1'	1	0	0
F	$\bar{\mathbf{5}}$	3	1	0	+2
N	10	1	3	0	-2
H_5	5	1	1	0	0
$H_{\bar{5}}$	$\bar{\mathbf{5}}$	1	1	0	0
H_{45}	$\overline{\mathbf{45}}$	1	1	0	0
Φ	1	3	3	0	0
ϕ_1	1	1	1	-1	-1
ϕ_2	1	1	1	-3	-1

Table 1. Transformation properties of fields in the gauge symmetry SU(5) and modular symmetries S_4^F and S_4^N .

Yukawas / masses	S_4^F	S_4^N	$2k_F$	$2k_N$
$Y_{\bar{5}}^{(2)}(\tau_F), Y_{45}^{(2)}(\tau_F)$	3'	1	+2	0
$Y_{\bar{5}}^{(4)}(\tau_F), Y_{45}^{(4)}(\tau_F)$	3	1	+4	0
$Y_{\bar{5}}'^{(4)}(\tau_F), Y_{45}'^{(4)}(\tau_F)$	3	1	+4	0
$Y_{\bar{5}}^{(6)}(\tau_F), Y_{45}^{(6)}(\tau_F)$	3	1	+6	0
$Y_{\bar{5}}^{(8)}(\tau_F), Y_{45}^{(8)}(\tau_F)$	3	1	+8	0
$y_{uu}^{(4)}(\tau_F)$	3	1	+4	0
$M_1(\tau_N)$	1	1	0	+4
$M_2(\tau_N)$	1	2	0	+4
$M_3(\tau_N)$	1	3	0	+4

Table 2. Transformation properties of Yukawa couplings and right-handed neutrino masses in modular symmetries $S_4^F \times S_4^N$.

space, the SM matter fields belong to $\bar{\mathbf{5}}$ - and **10**-plets and the latter are represented as

$$F = \begin{pmatrix} d_R^c \\ d_G^c \\ d_B^c \\ e \\ -\nu \end{pmatrix}, \quad T = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & u_B^c & -u_G^c & -u_R & -d_R \\ -u_B^c & 0 & u_R^c & -u_G & -d_G \\ u_G^c & -u_R^c & 0 & -u_B & -d_B \\ u_R & u_G & u_B & 0 & -e^c \\ d_R & d_G & d_B & e^c & 0 \end{pmatrix}. \quad (3.1)$$

The right-handed neutrino field N is introduced to generate neutrino masses. F and N are arranged as triplets in S_4^F and S_4^N , respectively. T_1 , T_2 and T_3 are arranged as singlets

of S_4^F and S_4^N but with different modular weights. The SM Higgs is embedded into heavy Higgs multiplets $\mathbf{5}$, $\bar{\mathbf{5}}$ and $\bar{\mathbf{45}}$ of $\text{SO}(10)$. They are arranged as trivial singlets in the flavour space. We introduce a bi-triplet scalar Φ , which is supposed to break the twin modular S_4^F and S_4^N symmetries into a single S_4 symmetry [76]. We further introduce two scalars ϕ_1 and ϕ_2 . These particles are singlets in both the gauge and flavour space but take non-trivial modular weights, and therefore named as weightons by authors in [65]. In the present work, they will be responsible for hierarchies of quark masses and charged lepton masses.

Superpotential terms to generate down quark and charged lepton masses are $w \supset w_{\bar{5}} + w_{\bar{45}}$ with

$$w_{\bar{5}} = \left[Y_{\bar{5}}^{(6)}(\tau_F) \tilde{\phi}_1 \tilde{\phi}_2^3 + Y_{\bar{5}}^{(4)}(\tau_F) \tilde{\phi}_1^2 \tilde{\phi}_2^2 + Y_{\bar{5}}^{(8)}(\tau_F) \tilde{\phi}_2^4 \right] T_1 F H_{\bar{5}}, \\ + Y_{\bar{5}}^{(4)}(\tau_F) \tilde{\phi}_1 \tilde{\phi}_2^2 T_2 F H_{\bar{5}} + Y_{\bar{5}}^{(2)}(\tau_F) \tilde{\phi}_1^2 T_3 F H_{\bar{5}}, \quad (3.2)$$

where $\tilde{\phi}_i \equiv \phi_i/\Lambda$ with Λ a dimensionful cut-off flavour scale. $w_{\bar{45}}$ takes a similar form as $w_{\bar{5}}$ but $Y_{\bar{5}}^{(2k_F)}$ and $H_{\bar{5}}$ are replaced by $Y_{\bar{45}}^{(2k_F)}$ and $H_{\bar{45}}$, respectively. Here, $Y_{\dots}^{(2)}$, $Y_{\dots}^{(4)}$, $Y_{\dots}^{(6)}$, $\dots$ are triplet (either $\mathbf{3}$ or $\mathbf{3}'$) modular forms of modular weights 2, 4, 6, $\dots$, respectively. All modular forms which are allowed by the modular invariance are listed in table 2. We fix the VEV of τ_F at $\langle \tau_F \rangle = \tau_T \equiv \omega$, which preserve a *residual* Z_3^T symmetry. Then, we obtain

$$Y_{\bar{5}}^{(6)}(\tau_T) = y_{e1,\bar{5}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad Y_{\bar{5}}^{(4)}(\tau_T) = y_{e2,\bar{5}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad Y_{\bar{5}}^{(8)}(\tau_T) = y_{e3,\bar{5}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \\ Y_{\bar{5}}^{(4)}(\tau_T) = y_{\mu,\bar{5}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad Y_{\bar{5}}^{(2)}(\tau_T) = y_{\tau,\bar{5}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad (3.3)$$

where $y_{e1,\bar{5}}$, $y_{e2,\bar{5}}$, $y_{e3,\bar{5}}$, $y_{\mu,\bar{5}}$ and $y_{\tau,\bar{5}}$ are overall factors which are free parameters in the model. Similarly, we obtain $Y_{\bar{45}}^{(2k_F)}(\tau_T)$ up to overall factors. These textures lead to Yukawa coupling matrices for down-type quarks and charged leptons in the LR convention as

$$Y_d = \begin{pmatrix} y_{dd}\epsilon_1\epsilon_2^3 & y_{ds}\epsilon_1^2\epsilon_2^2 & y_{db}\epsilon_2^4 \\ 0 & y_{ss}\epsilon_1\epsilon_2^2 & 0 \\ 0 & 0 & y_{bb}\epsilon_1^2 \end{pmatrix}^*, \\ Y_e = \begin{pmatrix} y_{ee}\epsilon_1\epsilon_2^3 & 0 & 0 \\ y_{\mu e}\epsilon_1^2\epsilon_2^2 & y_{\mu\mu}\epsilon_1\epsilon_2^2 & 0 \\ y_{\tau e}\epsilon_2^4 & 0 & y_{\tau\tau}\epsilon_1^2 \end{pmatrix}^*, \quad (3.4)$$

where $\epsilon_i = \langle \phi_i \rangle / \Lambda$ (for $i = 1, 2$),

$$y_{dd} = y_{e1,\bar{5}}s + y_{e1,\bar{45}}c, \quad y_{ds} = y_{e2,\bar{5}}s + y_{e2,\bar{45}}c, \quad y_{db} = y_{e3,\bar{5}}s + y_{e3,\bar{45}}c, \\ y_{ss} = y_{\mu,\bar{5}}s + y_{\mu,\bar{45}}c, \quad y_{bb} = y_{\tau,\bar{5}}s + y_{\tau,\bar{45}}c, \\ y_{ee} = y_{e1,\bar{5}}s - 3y_{e1,\bar{45}}c, \quad y_{\mu e} = y_{e2,\bar{5}}s - 3y_{e2,\bar{45}}c, \quad y_{\tau e} = y_{e3,\bar{5}}s - 3y_{e3,\bar{45}}c, \\ y_{\mu\mu} = y_{\mu,\bar{5}}s - 3y_{\mu,\bar{45}}c, \quad y_{\tau\tau} = y_{\tau,\bar{5}}s - 3y_{\tau,\bar{45}}c \quad (3.5)$$

with $s = v_5/(v_5^2 + v_{45}^2)^{1/2}$, $c = v_{45}/(v_5^2 + v_{45}^2)^{1/2}$, and $*$ represents the complex conjugation of matrices as we present the result in the left-right notation. The above triangular form of the Yukawa matrices in the LR convention ensures that charged lepton corrections to PMNS mixing are very suppressed, while allowing the down quark matrix to contribute dominantly to Cabibbo mixing at order ϵ_1 .

Superpotential terms to generate up-type quark masses are given by

$$w \supset (y_{uu}\tilde{\phi}_1^2\tilde{\phi}_2^2 + y_{uu}^{(4)}(\tau_F)\tilde{\phi}_2^4)T_1T_1H_5 + y_{cc}\tilde{\phi}_2^2T_2T_2H_5 + y_{tt}T_3T_3H_5 \\ + y_{uc}\tilde{\phi}_1\tilde{\phi}_2^2T_1T_2H_5 + y_{ut}\tilde{\phi}_1\tilde{\phi}_2T_1T_3H_5 + y_{ct}\tilde{\phi}_2T_2T_3H_5, \quad (3.6)$$

where y_{uu} , y_{cc} , y_{tt} , y_{uc} , y_{ut} , and y_{ct} are dimensionless coefficients, and $y_{uu}^{(4)}(\tau_F)$ are singlet modular forms of weight 4 of S_4^F , seeing appendix A. The up-type quark Yukawa coupling matrix is given by

$$Y_u = \begin{pmatrix} y_{uu}\epsilon_1^2\epsilon_2^2 & y_{uc}\epsilon_1\epsilon_2^2 & y_{ut}\epsilon_1\epsilon_2 \\ y_{uc}\epsilon_1\epsilon_2^2 & y_{cc}\epsilon_2^2 & y_{ct}\epsilon_2 \\ y_{ut}\epsilon_1\epsilon_2 & y_{ct}\epsilon_2 & y_{tt} \end{pmatrix}^*, \quad (3.7)$$

where $y_{uu}^{(4)}(\tau_T) = 0$ was used.

In the neutrino sector, the superpotential terms to generate neutrino masses are nothing new but a GUT extension of our former work [76]. They are given by

$$w \supset y_D \frac{\Phi}{\Lambda} (FN)_1 H_5 + M_1(\tau_N)(NN)_1 + M_2(\tau_N)(NN)_2 + M_3(\tau_N)(NN)_3, \quad (3.8)$$

where the subscript of a bracket, e.g., $(NN)_3$, represents the irreducible representation contraction of the fields in the bracket. Φ is a bi-triplet of $S_4^F \times S_4^N$. Its VEV is not responsible for special Yukawa textures for leptons, but used to break two modular S_4 's to a single modular S_4 symmetry [75],

$$S_4^F \times S_4^N \rightarrow S_4. \quad (3.9)$$

The VEV of Φ takes the form

$$\langle \Phi \rangle = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} v_\Phi. \quad (3.10)$$

The technique of how to achieve this VEV structure was first developed in [91] and has been applied to achieve the breaking of multiple modular symmetries in [75]. For details of how to derive it in this model, we refer the reader to appendix B of this paper and [75]. After the breaking $S_4^F \times S_4^N \rightarrow S_4$, the resulted Yukawa coupling between left-handed and right-handed neutrinos is given by

$$M_D = y_D \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \frac{v_\Phi v_5}{\Lambda}. \quad (3.11)$$

On the other hand, the Majorana mass matrix for right-handed neutrinos is straightforwardly obtained from the singlet, doublet and triplet modular forms M_1 , $M_2 = (M_{2,1}, M_{2,2})^T$ and $M_3 = (M_{3,1}, M_{3,2}, M_{3,3})^T$, i.e.,

$$M_R = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & 0 & M_1 \\ 0 & M_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & M_{2,1} & M_{2,2} \\ M_{2,1} & M_{2,2} & 0 \\ M_{2,2} & 0 & M_{2,1} \end{pmatrix} + \begin{pmatrix} 2M_{3,1} & -M_{3,3} & -M_{3,2} \\ -M_{3,3} & 2M_{3,2} & -M_{3,1} \\ -M_{3,2} & -M_{3,1} & 2M_{3,3} \end{pmatrix}. \quad (3.12)$$

By assuming the stabiliser in the neutrino sector at $\langle \tau_N \rangle = \tau_{SU} \equiv -\frac{1}{2} + \frac{i}{2}$. This stabiliser is invariant under the action of SU . Namely, we are left with a *residual* Z_2^{SU} symmetry. Three modular forms M_1 , M_2 and M_3 are left with

$$M_1(\tau_{SU}) = a, \quad M_2(\tau_{SU}) = b \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad M_3(\tau_{SU}) = c \begin{pmatrix} \sqrt{2} \\ \sqrt{2} - \sqrt{3} \\ \sqrt{2} + \sqrt{3} \end{pmatrix}, \quad (3.13)$$

where a , b and c are complex parameters. And the Majorana mass matrix for right-handed neutrinos are written in the form of four matrix patterns, i.e.,

$$M_R = a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} + c \sqrt{2} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} - c \sqrt{3} \begin{pmatrix} 0 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & -2 \end{pmatrix}. \quad (3.14)$$

This is explicitly the same mass matrix as that obtained in [76].

Without considering the correction from the charged lepton sector, a mass texture as in eq. (3.14), together with a Dirac neutrino matrix in eq. (3.11), leads to the trimaximal TM_1 lepton mixing which preserves the first column on the tri-bimaximal mixing matrix,

$$U_{TM_1} = \begin{pmatrix} \frac{2}{\sqrt{6}} & - & - \\ -\frac{1}{\sqrt{6}} & - & - \\ -\frac{1}{\sqrt{6}} & - & - \end{pmatrix}. \quad (3.15)$$

Three equivalent relations are predicted from the TM_1 mixing,

$$\tan \theta_{12} = \frac{1}{\sqrt{2}} \sqrt{1 - 3s_{13}^2} \quad \text{or} \quad \sin \theta_{12} = \frac{1}{\sqrt{3}} \frac{\sqrt{1 - 3s_{13}^2}}{c_{13}} \quad \text{or} \quad \cos \theta_{12} = \sqrt{\frac{2}{3}} \frac{1}{c_{13}}. \quad (3.16)$$

More precisely, we can write U_{TM_1} explicitly in the form

$$U_{TM_1} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} e^{i\alpha'_3} & 0 & 0 \\ 0 & \cos \theta_R e^{i\alpha_1} & \sin \theta_R e^{-i\alpha_2} \\ 0 & -\sin \theta_R e^{i\alpha_2} & \cos \theta_R e^{-i\alpha_1} \end{pmatrix}, \quad (3.17)$$

which includes three free parameters θ_R , α_1 , and α_2 . The α'_3 is not an independent parameter but have a complicated correlation with the rest parameter as seen in [76]. The mixing angles and Dirac-type CP-violating phase are determined to be [13]

$$\begin{aligned}\sin \theta_{13} &= \frac{\sin \theta_R}{\sqrt{3}}, \\ \tan \theta_{12} &= \frac{\cos \theta_R}{\sqrt{2}}, \\ \tan \theta_{23} &= \left| \frac{\cos \theta_R + \sqrt{\frac{2}{3}} e^{i(\alpha_1 - \alpha_2)} \sin \theta_R}{\cos \theta_R - \sqrt{\frac{2}{3}} e^{i(\alpha_1 - \alpha_2)} \sin \theta_R} \right|, \\ \delta &= \arg [(5 \cos 2\theta_R + 1) \cos(\alpha_1 - \alpha_2) - i(\cos 2\theta_R + 5) \sin(\alpha_1 - \alpha_2)].\end{aligned}\quad (3.18)$$

From these equation, we see that the CP-violating phase δ and the angle θ_{23} are correlated via the phase difference $\alpha_1 - \alpha_2$. In the case of $\theta_R \neq 0$ and maximal atmospheric mixing $\theta_{23} = 45^\circ$, the phase $\alpha_1 - \alpha_2$ must equal $\pm 90^\circ$, leading to maximal CP violation $\delta = 90^\circ$ or 270° . We know that the oscillation data shows a small deviation from the maximal atmospheric mixing. Therefore, we expect δ has a small deviation from its maximal CP-violating value. This will be checked numerically in the next section.

The mass matrix in eq. (3.14) further constrains coefficients for the third and fourth mass matrix patterns on the right hand side with the ratio $-\sqrt{2/3}$. This is a feature different from classical flavour models without modular symmetry, e.g., in [13], where all parameters in front of different patterns are arbitrary. We have obtained a neutrino mass sum rule for the light neutrino mass eigenvalues

$$\frac{1}{m_1} = \left| \frac{1}{m_2} \left(\sin^2 \theta_R e^{-i2\alpha_2} + \sin 2\theta_R e^{-i(\alpha_1 + \alpha_2)} \right) + \frac{1}{m_3} \left(\cos^2 \theta_R e^{i2\alpha_1} - \sin 2\theta_R e^{i(\alpha_1 + \alpha_2)} \right) \right|. \quad (3.19)$$

Furthermore, we predicted m_{ee} , appearing as the effective mass in neutrino-less double beta decays, correlated with other mass parameters as

$$\begin{aligned}m_{ee} &= \left| \frac{2m_2 m_3}{3(m_2(\cos^2 \theta_R e^{i2\alpha_1} - \sin 2\theta_R e^{i(\alpha_1 + \alpha_2)}) + m_3(\sin^2 \theta_R e^{-i2\alpha_2} + \sin 2\theta_R e^{-i(\alpha_1 + \alpha_2)}))} \right. \\ &\quad \left. + \frac{1}{3}(m_2 \cos^2 \theta_R e^{2i\alpha_1} + m_3 \sin^2 \theta_R e^{-2i\alpha_2}) \right|,\end{aligned}\quad (3.20)$$

which is beyond those reported in [92].

In the present work, the GUT extension induces small corrections from the non-diagonal charged lepton Yukawa coupling Y_e . We will check how large is the deviation from the TM_1 mixing numerically in the next section.

4 Numerical study

We perform a χ^2 analysis and discuss numerical prediction of the model.

The charged fermion Yukawa coupling matrices can be diagonalised via $V_f^T (Y_f Y_f^\dagger) V_f^T = (Y_f^{\text{diag}})^2$ for $f = u, d, e$ where

$$\begin{aligned} Y_u^{\text{diag}} &= \text{diag}\{\tilde{y}_u, \tilde{y}_c, \tilde{y}_t\}, \\ Y_d^{\text{diag}} &= \text{diag}\{\tilde{y}_d, \tilde{y}_s, \tilde{y}_b\}, \\ Y_e^{\text{diag}} &= \text{diag}\{\tilde{y}_e, \tilde{y}_\mu, \tilde{y}_\tau\}. \end{aligned} \quad (4.1)$$

For the eigenvalues of the Yukawa couplings, we use the following numerical results,

$$\begin{aligned} \tilde{y}_u &= (2.92 \pm 1.81) \times 10^{-6}, & \tilde{y}_c &= (1.43 \pm 0.100) \times 10^{-3}, & \tilde{y}_t &= 0.534 \pm 0.0341, \\ \tilde{y}_d &= (4.81 \pm 1.06) \times 10^{-6}, & \tilde{y}_s &= (9.52 \pm 1.03) \times 10^{-5}, & \tilde{y}_b &= (6.95 \pm 0.175) \times 10^{-3}, \\ \tilde{y}_e &= (1.97 \pm 0.0236) \times 10^{-6}, & \tilde{y}_\mu &= (4.16 \pm 0.0497) \times 10^{-4}, & \tilde{y}_\tau &= (7.07 \pm 0.0727) \times 10^{-3}, \end{aligned} \quad (4.2)$$

These values were calculated at the GUT scale from a minimal SUSY breaking scenario, with $\tan\beta = 5$, as done in [53, 65, 93, 94]. Varying $\tan\beta$ does not change these values significantly except $\tan\beta$ takes a very large value. Three mixing angles and one CP-violating phase in the CKM mixing matrix are applied from the same literature,

$$\theta_{12}^q = 13.027^\circ \pm 0.0814^\circ, \quad \theta_{23}^q = 2.054^\circ \pm 0.384^\circ, \quad \theta_{13}^q = 0.1802^\circ \pm 0.0281^\circ, \quad \delta^q = 69.21^\circ \pm 6.19^\circ. \quad (4.3)$$

For neutrino masses and lepton mixing, we take global best-fit (bf) values (without including SK atmospheric data) from NuFIT 5.0 [95, 96] and average the positive and negative 1σ errors.

$$\begin{aligned} \Delta m_{21}^2 &= (7.42 \pm 0.21) \times 10^{-5} \text{eV}^2, & \Delta m_{3l}^2 &= (2.514 \pm 0.028) \times 10^{-3} \text{eV}^2, \\ \theta_{12} &= 33.44^\circ \pm 0.77^\circ, & \theta_{23} &= 49.0^\circ \pm 1.3^\circ, & \theta_{13} &= 8.57^\circ \pm 0.13^\circ, \end{aligned} \quad (4.4)$$

for the normal ordering (NO, i.e., $m_1 < m_2 < m_3$) of neutrino masses and

$$\begin{aligned} \Delta m_{21}^2 &= (7.42 \pm 0.21) \times 10^{-5} \text{eV}^2, & \Delta m_{3l}^2 &= -(2.497 \pm 0.028) \times 10^{-3} \text{eV}^2, \\ \theta_{12} &= 33.45^\circ \pm 0.77^\circ, & \theta_{23} &= 49.3^\circ \pm 1.1^\circ, & \theta_{13} &= 8.61^\circ \pm 0.12^\circ, \end{aligned} \quad (4.5)$$

for the inverted ordering (IO, i.e., $m_2 < m_3 < m_1$), where $\Delta m_{3l}^2 = \Delta m_{31}^2$ (for NO) and Δm_{32}^2 (for IO).

To check the phenomenological viability of the model, we perform a simple χ^2 analysis. In the quark sector, the χ^2 function is defined via

$$\begin{aligned} \chi^2 &= \sum_{q \in \text{Obs}} \left(\frac{q(\text{Para}) - q^{\text{bf}}}{\sigma_q} \right)^2, \\ \text{Para} &= \{y_{uu}, y_{uc}, y_{ut}, y_{cc}, y_{ct}, y_{tt}, y_{dd}, y_{ds}, y_{db}, y_{ss}, y_{bb}, \epsilon_1, \epsilon_2\}, \\ \text{Obs} &= \{\tilde{y}_u, \tilde{y}_c, \tilde{y}_t, \tilde{y}_d, \tilde{y}_s, \tilde{y}_b, \theta_{12}^q, \theta_{23}^q, \theta_{13}^q, \theta^q\}. \end{aligned} \quad (4.6)$$

Figure 1. A scan plot of ϵ_1 and ϵ_2 with $\chi^2 < 1$ (gray) and $\chi^2 < 10$ (light gray), respectively.

Here, in the Para set, ϵ_1 and ϵ_2 are scanned in the range $(0, 0.3)$ and all coefficients $y_{\alpha\beta}$ are scanned with absolute values in the range $(0, 1)$ and phases in $(0, 2\pi)$. Due to the large parameter space, the model can fit the experimental data very well. We have checked that in all points with small χ^2 , a hierarchy with $\epsilon_1 \sim \sqrt{\epsilon_2} \sim O(\theta_C/\sqrt{2})$ is obtained, where $\theta_C \approx 0.2276$ is the Cabibbo angle. A scan plot for ϵ_1 and ϵ_2 with $\chi^2 < 1$ (blue) and $\chi^2 < 10$ (light blue), respectively, is shown in figure 1. Below is a sample of parameters and predictions of observables with $\chi^2 = 0.001$,

$$\begin{aligned}
 y_{uu} &= 0.0681 + 0.3260i, & y_{uc} &= 0.4531 - 0.2645i, & y_{ut} &= -0.2657 - 0.2039i, \\
 y_{cc} &= -0.3886 - 1.0082i, & y_{ct} &= -0.5947 + 0.0148i, & y_{tt} &= -0.4844 - 0.2270i, \\
 y_{dd} &= 0.7593 + 0.5024i, & y_{ds} &= 0.6024 - 0.1092i, & y_{db} &= 0.6070 + 0.1213i, \\
 y_{ss} &= 0.5315 + 0.1975i, & y_{bb} &= 0.2788 + 0.8131i, \\
 \epsilon_1 &= 0.1553, & \epsilon_2 &= 0.03259
 \end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
 \tilde{y}_d &= 4.829 \times 10^{-6}, & \tilde{y}_s &= 9.486 \times 10^{-5}, & \tilde{y}_b &= 6.943 \times 10^{-3}, \\
 \tilde{y}_u &= 2.901 \times 10^{-6}, & \tilde{y}_c &= 1.428 \times 10^{-3}, & \tilde{y}_t &= 0.5357, \\
 \theta_{12}^q &= 13.02^\circ, & \theta_{23}^q &= 2.071^\circ, & \theta_{13}^q &= 0.1807^\circ, & \delta^q &= 69.33^\circ.
 \end{aligned} \tag{4.8}$$

We discuss the phenomenological prediction in the lepton sector. We have recovered the same neutrino mass matrix as in [76]. Such a texture leads to the TM_1 mixing if the charged lepton mass matrix is diagonal. However, as we see in eq. (3.4), some small off-diagonal entries of Y_l are allowed by the modular invariance. Therefore, deviations from the TM_1 mixing are expected. Off-diagonal entries of the unitary matrix V_e , which is used to diagonalise $Y_e Y_e^\dagger$ (cf. eq. (4.1)), are estimated to be $(V_e)_{12} \sim \epsilon_1 \epsilon_2$, $(V_e)_{23} \sim \epsilon_2^6 / \epsilon_1^2$ and $(V_e)_{13} \sim \epsilon_2^7 / \epsilon_1^3$. By taking values of ϵ_1 and ϵ_2 in figure 1, these entries can maximally reach $(V_e)_{12} \sim 4 \times 10^{-3}$, and $(V_e)_{23,13} \sim 10^{-5}$. Therefore, a deviation less than one percent may be induced by $(V_e)_{12}$. We perform a χ^2 analysis to compare the prediction with and

Figure 2. Prediction of lepton mixing angles with $\chi^2 < 10$ in the normal ordering (NO, left panel) and inverted ordering (IO, right panel) of neutrino masses. Results with and without the correction from the charged lepton sector are shown. The latter predicts the TM_1 mixing explicitly, as shown in the dark red/blue curves.

Figure 3. Prediction of the CP-violating phase δ with $\chi^2 < 10$. Same inputs as in figure 2 are used.

without the correction from charged lepton sector. The χ^2 function is defined following eq. (4.6), but the set of Obs and Para are given differently,

$$\begin{aligned} \text{Para} &= \{y_{ee}, y_{\mu\mu}, y_{\tau\tau}, y_{\mu e}, y_{\tau e}, \theta_R, \alpha_1, \alpha_2, \Delta m_{21}^2, \Delta m_{3l}^2, \epsilon_1, \epsilon_2\}, \\ \text{Obs} &= \{\tilde{y}_e, \tilde{y}_\mu, \tilde{y}_\tau, \Delta m_{21}^2, \Delta m_{3l}^2, \theta_{12}, \theta_{23}, \theta_{13}\}. \end{aligned} \quad (4.9)$$

Note that the CP-violating phase δ has been measured to be $195^\circ_{-25^\circ}^{+51^\circ}$ at the best-fit value $\pm 1\sigma$ errors for NO and $286^\circ_{-32^\circ}^{+27^\circ}$ at the best-fit value $\pm 1\sigma$ errors for IO, respectively. We have not included the it in the Obs set as its current global fit is far away from a Gaussian distribution. Instead, we treat δ as a prediction of the model. The lepton sector shares the same weightings and thus have the same parameters ϵ_1 and ϵ_2 in the scan. All the other free parameters are independent of those in the quark sector. In the charged lepton mass matrix, we can rotate phases of all coefficients $\{y_{ee}, y_{\mu\mu}, y_{\tau\tau}, y_{\mu e}, y_{\tau e}\}$ away without loss of generality and scan them in the range $(-1, 1)$. In the neutrino sector, as the mass matrix is explicitly the same as that generated in [76], we follow the same parametrisation therein and left with five free parameters, one angle θ_R , two phases α_1, α_2 , and two mass square differences Δm_{21}^2 and Δm_{3l}^2 . We scan θ_R in $[0, \pi/2)$, $\alpha_{1,2}$ in $[0, 2\pi)$, and two mass square

differences in their 3σ ranges. Points for $\chi^2 < 10$ are abandoned. Note that the two mass square differences appear in both the Para and Obs sets. Points with Δm_{21}^2 and Δm_{3l}^2 in their 3σ ranges but leading to $\chi^2 > 10$ are discarded as required. Prediction of mixing angles θ_{13} and θ_{12} with $\chi^2 < 10$ are shown in figure 2. Predictions without considering corrections from the charged lepton are listed in the figure as a comparison. A deviation of 0.02° from the TM_1 mixing could be predicted. Therefore, the TM_1 mixing is still a very good approximation. The CP-violating phase δ is listed as a prediction of the model, as shown in figure 3. The model in both mass orderings support small deviation from the maximal CP violation, $270^\circ < \delta < 305^\circ$ for $\chi^2 < 10$. Below is a sample of parameters and predictions of observables with $\chi^2 = 4.515$ for the NO of neutrino mass ordering,

$$\begin{aligned} y_{ee} &= 0.366354, & y_{\mu\mu} &= 2.52094, & y_{\tau\tau} &= 0.293129, & y_{\mu e} &= -0.260999, & y_{\tau e} &= -0.828426, \\ \theta_R &= 14.87^\circ, & \alpha_1 &= 119.97^\circ, & \alpha_2 &= 46.94^\circ, \\ \Delta m_{21}^2 &= 7.073 \times 10^{-5} \text{eV}^2, & \Delta m_{3l}^2 &= 2.506 \times 10^{-3} \text{eV}^2, \end{aligned} \quad (4.10)$$

and

$$\begin{aligned} \tilde{y}_e &= 1.97 \times 10^{-6}, & \tilde{y}_\mu &= 4.16 \times 10^{-4}, & \tilde{y}_\tau &= 7.07 \times 10^{-3}, \\ \theta_{12} &= 34.353^\circ, & \theta_{23} &= 48.472^\circ, & \theta_{13} &= 8.522^\circ, \end{aligned} \quad (4.11)$$

where ϵ_1 and ϵ_2 take same values as in eq. (4.7). The CP-violating phase δ in this sample is predicted to be $\delta = 285.55^\circ$.

We also check the prediction of the effective neutrino mass parameter m_{ee} which is under measurements of the neutrinoless double beta decay experiments. The scanned results are shown in figure 2 for both the NO and IO of neutrino mass ordering. In particular, the sample point with inputs in eq. (4.10) predicts $m_{\text{lightest}} = 0.0597 \text{eV}$ and $m_{ee} = 0.0537 \text{eV}$. We have numerically checked that the contribution from charged lepton mixing matrix is negligibly small. Experimental upper limits of m_{ee} , 0.061–0.165, 0.078–0.239, 0.075–0.350, and 0.079–0.180 eV, as measured by KamLAND-Zen [97], EXO-200 [98], CUORE [99] and GERDA [100] at 90% CL, respectively, are shown in the figure by comparison. We also include the cosmological constraints on the neutrino masses from Planck 2018 [101]. The two vertical grey bands refer to the disfavoured region $0.12 \text{eV} < \sum m_i < 0.60 \text{eV}$ and very disfavoured region $\sum m_i > 0.60 \text{eV}$, respectively. Due to neutrino mass sum rule in eq. (3.19), the lightest neutrino mass is constrained to be $m_{\text{lightest}} \gtrsim 0.03 \text{eV}$, leading to most of the parameter space disfavoured by the cosmological constraints.

We emphasise that the sum rule in eq. (3.19) is an unavoidable consequence of the residual Z_2^{SU} modular symmetry, resulting in nearly degenerate masses for right-handed neutrinos, and hence nearly degenerate masses for the light neutrinos via the seesaw mechanism. As a result, a large sum of neutrino masses is predicted by the model, along with relatively large m_{ee} , even for a normal ordering. This special feature makes the model eminently testable by neutrinoless double beta decay experiments.

5 Conclusion

We have constructed an $\text{SU}(5)$ GUT with twin $S_4^F \times S_4^N$ modular symmetries, accompanied by two moduli fields. This is a grand unified extension of our previous work in [76], allowing

Figure 4. Prediction of m_{lightest} vs m_{ee} with $\chi^2 < 10$ for both normal ordering (NO, left panel) and inverted ordering (IO, right panel) of neutrino masses, allowed by the model, where $m_{\text{lightest}} = m_1$ for NO and $m_{\text{lightest}} = m_3$ for IO. The general parameter space of m_{ee} allowed by oscillation data and current upper limit from neutrinoless double beta decay experiments KamLAND-Zen [97], EXO-200 [98], CUORE [99] and GERDA [100], and cosmological constraints from PLANCK 2018 (disfavoured region $0.12 \text{ eV} < \sum m_i < 0.60 \text{ eV}$ and very disfavoured region $\sum m_i > 0.60 \text{ eV}$) [101] are shown for comparison.

quark mass and mixing to be included, while preserving the good predictions in the lepton sector. The two modular symmetries $S_4^F \times S_4^N$, one acting on charged fermions and one on right-handed neutrinos, are broken to a single S_4 symmetry by a bi-triplet scalar, leading to the effective theory invariant under one single S_4 but involving two modulus fields. The two modulus fields gain different VEVs, leading to the breaking of S_4 to a residual modular symmetry Z_3 in the charged fermion sectors, and a residual modular symmetry Z_2 in the neutrino sector, ensuring the leading order TM_1 lepton mixing.

In the neutrino sector, the model has the same flavour structure as that in [76]. In the charged fermion sector, there are two main differences from our former work: 1) charged fermion mass hierarchies are explained due to the coupling to the two weightons; 2) small mixings are induced in the charged fermion mass matrices. The triangular form of the charged lepton and down quark Yukawa matrices plays a special role in this model, ensuring suppressed charged lepton corrections to the PMNS matrix, while allowing the down quark Yukawa matrix to dominantly contribute to Cabibbo mixing, where we have shown that a good fit may be achieved to the CKM matrix. The model predicts TM_1 lepton mixing to very good approximation, and neutrinoless double beta decay at rates close to the sensitivity of current and future experiments, for both normal and inverted orderings.

Acknowledgments

SFK and YLZ acknowledge the STFC Consolidated Grant ST/L000296/1 and the European Union's Horizon 2020 Research and Innovation programme under Marie Skłodowska-Curie grant agreement HIDDeN European ITN project (H2020-MSCA-ITN-2019//860881-HIDDeN).

A Modular group and modular forms of level 4

The modular group $\bar{\Gamma}$ is a set of modular transformations acting on the upper complex plane. Given a modulus viable τ with $\text{Im}(\tau) > 0$, each element γ of $\bar{\Gamma}$ appears as a linear fractional transformation

$$\gamma : \tau \rightarrow \gamma\tau = \frac{a\tau + b}{c\tau + d} \quad (\text{A.1})$$

with a, b, c , and d are any integers satisfying $ad - bc = 1$. As γ can be represented by a 2×2 matrix in eq. (2.1), the group $\bar{\Gamma}$ is expressed as

$$\bar{\Gamma} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} / (\pm \mathbf{1}), \quad a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1 \right\}. \quad (\text{A.2})$$

The modular group is an infinite group. It has two generators, S_τ and T_τ , satisfying $S_\tau^2 = (S_\tau T_\tau)^3 = \mathbf{1}$. These generators act on the modulus τ as,

$$S_\tau : \tau \rightarrow -\frac{1}{\tau}, \quad T_\tau : \tau \rightarrow \tau + 1, \quad (\text{A.3})$$

respectively, which, represented as 2×2 matrices, are given by

$$S_\tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T_\tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \quad (\text{A.4})$$

With the requirement $a, d = 1 \pmod{4}$ and $b, c = 0 \pmod{4}$, a subset of $\bar{\Gamma}$, which is also infinite, is obtained,

$$\bar{\Gamma}(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{Z}), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{4} \right\}. \quad (\text{A.5})$$

Γ_4 is the quotient group $\Gamma_4 = \bar{\Gamma}/\bar{\Gamma}(4)$. It is equivalently obtained by imposing the identity $T_\tau^4 = \mathbf{1}$. As a subgroup of $\bar{\Gamma}$, its elements can also be represented as 2×2 matrices, but the representation matrices are not unique. As the quotient group $\bar{\Gamma}/\bar{\Gamma}(4)$, each element γ of Γ_4 satisfy the equality

$$\eta \begin{pmatrix} 4k_a + a & 4k_b + b \\ 4k_c + c & 4k_d + d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad (\text{A.6})$$

where k_a, k_b, k_c and k_d are integers and satisfy $4k_a k_d + a k_d + d k_a = 4k_b k_c + b k_c + c k_b$ and $\eta = \pm 1$.

The finite modular group Γ_4 is isomorphic to S_4 . The latter is the permutation group of 4 objects, see e.g. [102]. It has 5 irreducible representations, $\mathbf{1}, \mathbf{1}', \mathbf{2}, \mathbf{3}$ and $\mathbf{3}'$. In the studies of flavour symmetries, a common set of generators of S_4 are S, T and U which satisfy $S^2 = T^3 = U^2 = (ST)^3 = (SU)^2 = (TU)^2 = \mathbf{1}$. We work in a widely used basis with representation matrices of S, T and U listed in table 3.

	$\rho(T)$	$\rho(S)$	$\rho(U)$
1	1	1	1
1'	1	1	-1
2	$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
3	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}$	$\frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
3'	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}$	$\frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$	$-\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Table 3. Representation matrices for the S_4 generators T , S and U used in the paper, where $\omega = e^{2\pi i/3}$.

As $S_4 \simeq \Gamma_4$, S , T and U can be represented by S_τ and T_τ , and vice versa,

$$\begin{aligned} T &= S_\tau T_\tau, & S &= T_\tau^2, & U &= T_\tau S_\tau T_\tau^2 S_\tau; \\ T_\tau &= U T S T^2, & S_\tau &= S T S U. \end{aligned} \quad (\text{A.7})$$

Any element of S_4 can be represented by S_τ and T_τ . For example, SU , which is crucial to achieve the special mass structure in the neutrino sector, is given by $SU = S_\tau T_\tau S_\tau T_\tau^{-1} S_\tau$. Given representation matrices of S_τ and T_τ in eq. (A.4), it is straightforward to obtain 2×2 matrices of S , T , U and SU as

$$T = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}, \quad SU = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}. \quad (\text{A.8})$$

We emphasise that the representation matrix for each element is not unique. An alternative representation matrix is obtained by the equality in eq. (A.6).

A main difference of the modular invariance approach from the classical flavour symmetry approach is that the Yukawa couplings are formed as a consequence of modular forms, instead of combinations of a series of flavon VEVs. Modular forms are holomorphic functions of τ under modular transformations.

Modular forms of level 4 are classified by modular weights. The latter must be even and positive integrals, which we label as $2k$. There are $4k+1$ linearly independent modular forms of level 4 and weight $2k$. They are all decomposed to irreducible representations of S_4 . For $2k=2$, there are 5 modular forms. They are decomposed into a doublet **2** and a triplet **3'** of S_4 ,

$$Y_{\mathbf{2}}^{(2)}(\tau) = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \end{pmatrix}, \quad Y_{\mathbf{3}'}^{(2)}(\tau) = \begin{pmatrix} Y_3(\tau) \\ Y_4(\tau) \\ Y_5(\tau) \end{pmatrix}. \quad (\text{A.9})$$

A non-linear algebra satisfied among three of the modular forms,

$$(Y_3^2 + 2Y_4Y_5)^2 = (Y_4^2 + 2Y_3Y_5)(Y_5^2 + 2Y_3Y_4) \quad (\text{A.10})$$

is satisfied [72]. This constraint is essential to cover the modular space of Γ_4 . Modular forms of higher weights are contracted from $Y_1, \dots, Y_5$. For $2k = 4$, there are 9 linearly independent modular forms, forming a singlet **1**, a doublet **2** and two triplets **3** and **3'** of S_4 ,

$$\begin{aligned} Y_1^{(4)}(\tau) &= Y_1Y_2, & Y_2^{(4)}(\tau) &= \begin{pmatrix} Y_2^2 \\ Y_1^2 \end{pmatrix}, \\ Y_3^{(4)}(\tau) &= \begin{pmatrix} Y_1Y_4 - Y_2Y_5 \\ Y_1Y_5 - Y_2Y_4 \\ Y_1Y_3 - Y_2Y_4 \end{pmatrix}, & Y_{3'}^{(4)}(\tau) &= \begin{pmatrix} Y_1Y_4 + Y_2Y_5 \\ Y_1Y_5 + Y_2Y_4 \\ Y_1Y_3 + Y_2Y_4 \end{pmatrix}. \end{aligned} \quad (\text{A.11})$$

For $2k = 6$, the number is increased to 13. They are decomposed to

$$\begin{aligned} Y_1^{(6)}(\tau) &= Y_1^3 + Y_2^3, & Y_{1'}^{(6)}(\tau) &= Y_1^3 - Y_2^3, \\ Y_2^{(6)}(\tau) &= Y_1^{(4)}Y_2^{(2)}, & Y_{3',1}^{(6)}(\tau) &= Y_1^{(4)}Y_{3'}^{(2)}, \\ Y_3^{(6)}(\tau) &= \begin{pmatrix} Y_2^2Y_4 - Y_1^2Y_5 \\ Y_2^2Y_5 - Y_1^2Y_4 \\ Y_2^2Y_3 - Y_1^2Y_4 \end{pmatrix}, & Y_{3',2}^{(6)}(\tau) &= \begin{pmatrix} Y_2^2Y_4 + Y_1^2Y_5 \\ Y_2^2Y_5 + Y_1^2Y_4 \\ Y_2^2Y_3 + Y_1^2Y_4 \end{pmatrix}. \end{aligned} \quad (\text{A.12})$$

More modular forms with higher modular weights are listed in [72]. In particular, singlet modular forms at weights 8, 10 and 12 are respectively given by

$$\begin{aligned} Y_1^{(8)}(\tau) &= [Y_1^{(4)}]^2, & Y_1^{(10)}(\tau) &= Y_1^{(4)}Y_1^{(6)}, & Y_{1'}^{(10)}(\tau) &= Y_1^{(4)}Y_{1'}^{(6)}, \\ Y_{1,1}^{(12)}(\tau) &= [Y_1^{(6)}]^2, & Y_{1,2}^{(12)}(\tau) &= [Y_{1'}^{(6)}]^2, & Y_{1,3}^{(12)}(\tau) &= [Y_{1'}^{(4)}]^3, & Y_{1'}^{(12)}(\tau) &= Y_1^{(6)}Y_{1'}^{(6)}. \end{aligned} \quad (\text{A.13})$$

A modular form at a stabiliser takes an interesting weight-dependent direction which satisfies [75]

$$\rho_I(\gamma)Y_I(\tau_\gamma) = (c\tau_\gamma + d)^{-2k}Y_I(\tau_\gamma). \quad (\text{A.14})$$

Namely, a modular form at a stabiliser τ_γ is an eigenvector of the representation matrix $\rho_I(\gamma)$ with respective eigenvalue $(c\tau_\gamma + d)^{-2k}$. In the special case $(c\tau_\gamma + d)^{-2k} = 1$, leading to $\rho_I(\gamma)Y_I(\tau_\gamma) = Y_I(\tau_\gamma)$, the residual modular symmetry is reduced to the residual flavour symmetry. Otherwise, the residual modular symmetry is different from the latter.

We discuss which directions triplet modular forms $Y_{\mathbf{3}(\gamma)}^{(2k)}(\tau)$ may take at $\tau = \tau_T$ and τ_{SU} . The eigenvalue $(c\tau_\gamma + d)^{-2k}$ at these stabilisers is given by $(-\tau_T - 1)^{-2k} = \omega^{2k}$ at γ_T and $(2\tau_{SU} + 1)^{-2k} = (-1)^k$ at γ_{SU} , respectively. Given triplet (**3**, **3'**) representation

matrices for S , T and U in table 3, it is straightforward to obtain

$$\begin{aligned} Y_{\mathbf{3}'^{(i)}}^{(2)}(\tau_T), Y_{\mathbf{3}'^{(i)}}^{(8)}(\tau_T) &\propto \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, & Y_{\mathbf{3}'^{(i)}}^{(4)}(\tau_T) &\propto \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \\ Y_{\mathbf{3}'^{(i)}}^{(6)}(\tau_T) &\propto \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, & Y_{\mathbf{3}}^{(2)}(\tau_{SU}), Y_{\mathbf{3}'}^{(4)}(\tau_{SU}) &\propto \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}. \end{aligned} \quad (\text{A.15})$$

These results are directly obtained from the symmetry argument, without knowing explicit expressions of modular forms. However, there are some exceptions of modular forms whose directions cannot be directly obtained from the above argument, e.g., $Y_{\mathbf{3}}^{(4)}(\tau_{SU})$, $Y_{\mathbf{3}'}^{(2)}(\tau_{SU})$, which correspond to eigenvectors of degenerate eigenvalues. Their directions have been calculated in [76] with the help of the identity in eq. (A.10),

$$Y_{\mathbf{3}'}^{(2)}(\tau_{SU}) \propto \begin{pmatrix} 1 \\ 1 - \sqrt{6} \\ 1 + \sqrt{6} \end{pmatrix}, \quad Y_{\mathbf{3}}^{(4)}(\tau_{SU}) \propto \begin{pmatrix} \sqrt{2} \\ \sqrt{2} - \sqrt{3} \\ \sqrt{2} + \sqrt{3} \end{pmatrix}, \quad (\text{A.16})$$

For the singlet modular forms, some of them may vanish at the stabilisers, e.g., $Y_{\mathbf{1}}^{(4)}(\tau_T) = Y_{\mathbf{1}}^{(8)}(\tau_T) = Y_{\mathbf{1}}^{(10)}(\tau_T) = 0$. They do not contribute to fermion masses.

B Vacuum alignments

We discuss how to achieve the required VEVs of the bi-triplet scalar and weightons using the general driving-field approach in the supersymmetry.

The vacuum alignment for the bi-triplet scalar Φ can be realised by introducing two driving fields $\chi_{FN} \sim (\mathbf{3}, \mathbf{3})$ and $\chi_N \sim (\mathbf{1}, \mathbf{3})$ of $S_4^F \times S_4^N$ with zero modular weights. The general superpotential terms for driving fields are given by

$$w_d \supset ((\Phi\Phi)_{(\mathbf{3},\mathbf{3})} + M_\Phi \Phi) \chi_{FN} + (\Phi\Phi)_{(\mathbf{1},\mathbf{3})} \chi_N, \quad (\text{B.1})$$

where M is a mass-dimensional coefficient. Minimisation of the superpotential gives rise to

$$\begin{aligned} (\Phi\Phi)_{(\mathbf{3},\mathbf{3})} + M_\Phi \Phi &= 0, \\ (\Phi\Phi)_{(\mathbf{1},\mathbf{3})} &= 0, \end{aligned} \quad (\text{B.2})$$

As proven in [75], these identities guarantee the VEV of Φ as in eq. (3.10) up to an unphysical basis transformation, thus leading to the breaking $S_4^F \times S_4^N \rightarrow S_4$.

The non-vanishing VEVs of weightons can be realised by introducing two driving fields χ_1 and χ_2 , both of which are trivial singlets of $S_4^F \times S_4^N$ with zero modular weights. Superpotential terms are

$$w_d \supset \chi_1 \left(\sum_{k=0}^4 y_1^{(4+2k,4)}(\tau_F, \tau_N) \frac{\phi_1^k \phi_2^{4-k}}{\Lambda^2} - M_1^2 \right) + \chi_2 \left(\sum_{k=0}^4 y_2^{(4+2k,4)}(\tau_F, \tau_N) \frac{\phi_1^k \phi_2^{4-k}}{\Lambda^2} - M_2^2 \right), \quad (\text{B.3})$$

where M_1 and M_2 are two mass-dimensional parameters, $y_1^{(4+2k,4)}$ and $y_1^{(4+2k,4)}$ are any singlet modular form of modular weights $4+2k$ and 4 in $S_4^F \times S_4^N$ which are allowed by the symmetries. We have ignored higher dimensional terms with dimension ≥ 7 . The superpotential gives rise to

$$\begin{aligned} y_1^{(6,4)}(\tau_T, \tau_{SU}) \frac{\phi_1 \phi_2^3}{\Lambda^2} + y_1^{(12,4)}(\tau_T, \tau_{SU}) \frac{\phi_1^4}{\Lambda^2} - M_1^2 &= 0, \\ y_2^{(6,4)}(\tau_T, \tau_{SU}) \frac{\phi_1 \phi_2^3}{\Lambda^2} + y_2^{(12,4)}(\tau_T, \tau_{SU}) \frac{\phi_1^4}{\Lambda^2} - M_2^2 &= 0, \end{aligned} \quad (\text{B.4})$$

where we have taken $\langle \tau_F \rangle = \tau_T$ and $\langle \tau_N \rangle = \tau_{SU}$ and used $Y_1^{(4)}(\tau_T) = Y_1^{(8)}(\tau_T) = Y_1^{(10)}(\tau_T) = 0$. Eq. (B.4) assures non-trivial vacuum of ϕ_1 and ϕ_2 . However, The correlation $\epsilon_1 \sim \sqrt{\epsilon_2} \sim \theta_C / \sqrt{2}$ cannot be derived from the identities.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] H. Georgi and S.L. Glashow, *Unity of all elementary particle forces*, *Phys. Rev. Lett.* **32** (1974) 438 [[INSPIRE](#)].
- [2] S.F. King, *Unified models of neutrinos, flavour and CP-violation*, *Prog. Part. Nucl. Phys.* **94** (2017) 217 [[arXiv:1701.04413](#)] [[INSPIRE](#)].
- [3] P. Minkowski, $\mu \rightarrow e\gamma$ at a rate of one out of 10^9 muon decays?, *Phys. Lett. B* **67** (1977) 421 [[INSPIRE](#)].
- [4] T. Yanagida, *Horizontal gauge symmetry and masses of neutrinos*, *Conf. Proc. C* **7902131** (1979) 95 [[INSPIRE](#)].
- [5] M. Gell-Mann, P. Ramond and R. Slansky, *Complex spinors and unified theories*, *Conf. Proc. C* **790927** (1979) 315 [[arXiv:1306.4669](#)] [[INSPIRE](#)].
- [6] S.L. Glashow, *The future of elementary particle physics*, *NATO Sci. Ser. B* **61** (1980) 687 [[INSPIRE](#)].
- [7] R.N. Mohapatra and G. Senjanović, *Neutrino mass and spontaneous parity nonconservation*, *Phys. Rev. Lett.* **44** (1980) 912 [[INSPIRE](#)].
- [8] J. Schechter and J.W.F. Valle, *Neutrino masses in $SU(2) \times U(1)$ theories*, *Phys. Rev. D* **22** (1980) 2227 [[INSPIRE](#)].
- [9] J. Schechter and J.W.F. Valle, *Neutrino decay and spontaneous violation of lepton number*, *Phys. Rev. D* **25** (1982) 774 [[INSPIRE](#)].
- [10] S.F. King and C. Luhn, *Neutrino mass and mixing with discrete symmetry*, *Rept. Prog. Phys.* **76** (2013) 056201 [[arXiv:1301.1340](#)] [[INSPIRE](#)].
- [11] PARTICLE DATA GROUP collaboration, *Review of particle physics*, *PTEP* **2020** (2020) 083C01 [[INSPIRE](#)].
- [12] I. de Medeiros Varzielas and L. Lavoura, *Flavour models for TM_1 lepton mixing*, *J. Phys. G* **40** (2013) 085002 [[arXiv:1212.3247](#)] [[INSPIRE](#)].

- [13] C. Luhn, *Trimaximal TM_1 neutrino mixing in S_4 with spontaneous CP-violation*, *Nucl. Phys. B* **875** (2013) 80 [[arXiv:1306.2358](#)] [[INSPIRE](#)].
- [14] I. de Medeiros Varzielas, S.F. King and G.G. Ross, *Tri-bimaximal neutrino mixing from discrete subgroups of $SU(3)$ and $SO(3)$ family symmetry*, *Phys. Lett. B* **644** (2007) 153 [[hep-ph/0512313](#)] [[INSPIRE](#)].
- [15] Y. Koide, *S_4 flavor symmetry embedded into $SU(3)$ and lepton masses and mixing*, *JHEP* **08** (2007) 086 [[arXiv:0705.2275](#)] [[INSPIRE](#)].
- [16] T. Banks and N. Seiberg, *Symmetries and strings in field theory and gravity*, *Phys. Rev. D* **83** (2011) 084019 [[arXiv:1011.5120](#)] [[INSPIRE](#)].
- [17] C. Luhn, *Spontaneous breaking of $SU(3)$ to finite family symmetries: a pedestrian's approach*, *JHEP* **03** (2011) 108 [[arXiv:1101.2417](#)] [[INSPIRE](#)].
- [18] A. Merle and R. Zwicky, *Explicit and spontaneous breaking of $SU(3)$ into its finite subgroups*, *JHEP* **02** (2012) 128 [[arXiv:1110.4891](#)] [[INSPIRE](#)].
- [19] Y.-L. Wu, *$SU(3)$ gauge family symmetry and prediction for the lepton-flavor mixing and neutrino masses with maximal spontaneous CP-violation*, *Phys. Lett. B* **714** (2012) 286 [[arXiv:1203.2382](#)] [[INSPIRE](#)].
- [20] B.L. Rachlin and T.W. Kephart, *Spontaneous breaking of gauge groups to discrete symmetries*, *JHEP* **08** (2017) 110 [[arXiv:1702.08073](#)] [[INSPIRE](#)].
- [21] S.F. King and Y.-L. Zhou, *Spontaneous breaking of $SO(3)$ to finite family symmetries with supersymmetry — an A_4 model*, *JHEP* **11** (2018) 173 [[arXiv:1809.10292](#)] [[INSPIRE](#)].
- [22] T. Asaka, W. Buchmüller and L. Covi, *Gauge unification in six-dimensions*, *Phys. Lett. B* **523** (2001) 199 [[hep-ph/0108021](#)] [[INSPIRE](#)].
- [23] G. Altarelli, F. Feruglio and Y. Lin, *Tri-bimaximal neutrino mixing from orbifolding*, *Nucl. Phys. B* **775** (2007) 31 [[hep-ph/0610165](#)] [[INSPIRE](#)].
- [24] T. Kobayashi, H.P. Nilles, F. Ploger, S. Raby and M. Ratz, *Stringy origin of non-Abelian discrete flavor symmetries*, *Nucl. Phys. B* **768** (2007) 135 [[hep-ph/0611020](#)] [[INSPIRE](#)].
- [25] G. Altarelli, F. Feruglio and C. Hagedorn, *A $SUSY$ $SU(5)$ grand unified model of tri-bimaximal mixing from A_4* , *JHEP* **03** (2008) 052 [[arXiv:0802.0090](#)] [[INSPIRE](#)].
- [26] A. Adulpravitchai, A. Blum and M. Lindner, *Non-Abelian discrete flavor symmetries from T^2/Z_N orbifolds*, *JHEP* **07** (2009) 053 [[arXiv:0906.0468](#)] [[INSPIRE](#)].
- [27] T.J. Burrows and S.F. King, *A_4 family symmetry from $SU(5)$ $SUSY$ GUTs in 6d*, *Nucl. Phys. B* **835** (2010) 174 [[arXiv:0909.1433](#)] [[INSPIRE](#)].
- [28] A. Adulpravitchai and M.A. Schmidt, *Flavored orbifold GUT — an $SO(10) \times S_4$ model*, *JHEP* **01** (2011) 106 [[arXiv:1001.3172](#)] [[INSPIRE](#)].
- [29] T.J. Burrows and S.F. King, *$A_4 \times SU(5)$ $SUSY$ GUT of flavour in 8d*, *Nucl. Phys. B* **842** (2011) 107 [[arXiv:1007.2310](#)] [[INSPIRE](#)].
- [30] F.J. de Anda and S.F. King, *An $S_4 \times SU(5)$ $SUSY$ GUT of flavour in 6d*, *JHEP* **07** (2018) 057 [[arXiv:1803.04978](#)] [[INSPIRE](#)].
- [31] T. Kobayashi, S. Nagamoto, S. Takada, S. Tamba and T.H. Tatsuishi, *Modular symmetry and non-Abelian discrete flavor symmetries in string compactification*, *Phys. Rev. D* **97** (2018) 116002 [[arXiv:1804.06644](#)] [[INSPIRE](#)].

- [32] F.J. de Anda and S.F. King, $SU(3) \times SO(10)$ in 6d, *JHEP* **10** (2018) 128 [[arXiv:1807.07078](#)] [[INSPIRE](#)].
- [33] A. Baur, H.P. Nilles, A. Trautner and P.K.S. Vaudrevange, *Unification of flavor, CP, and modular symmetries*, *Phys. Lett. B* **795** (2019) 7 [[arXiv:1901.03251](#)] [[INSPIRE](#)].
- [34] T. Kobayashi, Y. Omura and K. Yoshioka, *Flavor symmetry breaking and vacuum alignment on orbifolds*, *Phys. Rev. D* **78** (2008) 115006 [[arXiv:0809.3064](#)] [[INSPIRE](#)].
- [35] Y. Olguin-Trejo, R. Pérez-Martínez and S. Ramos-Sánchez, *Charting the flavor landscape of MSSM-like Abelian heterotic orbifolds*, *Phys. Rev. D* **98** (2018) 106020 [[arXiv:1808.06622](#)] [[INSPIRE](#)].
- [36] A. Mütter, E. Parr and P.K.S. Vaudrevange, *Deep learning in the heterotic orbifold landscape*, *Nucl. Phys. B* **940** (2019) 113 [[arXiv:1811.05993](#)] [[INSPIRE](#)].
- [37] S. Ferrara, D. Lüst, A.D. Shapere and S. Theisen, *Modular invariance in supersymmetric field theories*, *Phys. Lett. B* **225** (1989) 363 [[INSPIRE](#)].
- [38] S. Ferrara, D. Lüst and S. Theisen, *Target space modular invariance and low-energy couplings in orbifold compactifications*, *Phys. Lett. B* **233** (1989) 147 [[INSPIRE](#)].
- [39] G. Altarelli and F. Feruglio, *Tri-bimaximal neutrino mixing, A_4 and the modular symmetry*, *Nucl. Phys. B* **741** (2006) 215 [[hep-ph/0512103](#)] [[INSPIRE](#)].
- [40] R. de Adelhart Toorop, F. Feruglio and C. Hagedorn, *Finite modular groups and lepton mixing*, *Nucl. Phys. B* **858** (2012) 437 [[arXiv:1112.1340](#)] [[INSPIRE](#)].
- [41] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto and T.H. Tatsuishi, *New A_4 lepton flavor model from S_4 modular symmetry*, *JHEP* **02** (2020) 097 [[arXiv:1907.09141](#)] [[INSPIRE](#)].
- [42] F. Feruglio, *Are neutrino masses modular forms?*, in *From my vast repertoire...: Guido Altarelli's legacy*, A. Levy, S. Forte and G. Ridolfi eds., **World Scientific**, Singapore (2019) [[arXiv:1706.08749](#)] [[INSPIRE](#)].
- [43] T. Kobayashi, K. Tanaka and T.H. Tatsuishi, *Neutrino mixing from finite modular groups*, *Phys. Rev. D* **98** (2018) 016004 [[arXiv:1803.10391](#)] [[INSPIRE](#)].
- [44] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto, T.H. Tatsuishi and H. Uchida, *Finite modular subgroups for fermion mass matrices and baryon/lepton number violation*, *Phys. Lett. B* **794** (2019) 114 [[arXiv:1812.11072](#)] [[INSPIRE](#)].
- [45] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto and T.H. Tatsuishi, *Modular S_3 -invariant flavor model in $SU(5)$ grand unified theory*, *PTEP* **2020** (2020) 053B05 [[arXiv:1906.10341](#)] [[INSPIRE](#)].
- [46] H. Okada and Y. Orikasa, *Modular S_3 symmetric radiative seesaw model*, *Phys. Rev. D* **100** (2019) 115037 [[arXiv:1907.04716](#)] [[INSPIRE](#)].
- [47] J.C. Criado and F. Feruglio, *Modular invariance faces precision neutrino data*, *SciPost Phys.* **5** (2018) 042 [[arXiv:1807.01125](#)] [[INSPIRE](#)].
- [48] T. Kobayashi, N. Omoto, Y. Shimizu, K. Takagi, M. Tanimoto and T.H. Tatsuishi, *Modular A_4 invariance and neutrino mixing*, *JHEP* **11** (2018) 196 [[arXiv:1808.03012](#)] [[INSPIRE](#)].
- [49] F.J. de Anda, S.F. King and E. Perdomo, *$SU(5)$ grand unified theory with A_4 modular symmetry*, *Phys. Rev. D* **101** (2020) 015028 [[arXiv:1812.05620](#)] [[INSPIRE](#)].

- [50] H. Okada and M. Tanimoto, *CP violation of quarks in A_4 modular invariance*, *Phys. Lett. B* **791** (2019) 54 [[arXiv:1812.09677](#)] [[INSPIRE](#)].
- [51] P.P. Novichkov, S.T. Petcov and M. Tanimoto, *Trimaximal neutrino mixing from modular A_4 invariance with residual symmetries*, *Phys. Lett. B* **793** (2019) 247 [[arXiv:1812.11289](#)] [[INSPIRE](#)].
- [52] T. Nomura and H. Okada, *A modular A_4 symmetric model of dark matter and neutrino*, *Phys. Lett. B* **797** (2019) 134799 [[arXiv:1904.03937](#)] [[INSPIRE](#)].
- [53] H. Okada and M. Tanimoto, *Towards unification of quark and lepton flavors in A_4 modular invariance*, *Eur. Phys. J. C* **81** (2021) 52 [[arXiv:1905.13421](#)] [[INSPIRE](#)].
- [54] T. Nomura and H. Okada, *A two loop induced neutrino mass model with modular A_4 symmetry*, *Nucl. Phys. B* **966** (2021) 115372 [[arXiv:1906.03927](#)] [[INSPIRE](#)].
- [55] G.-J. Ding, S.F. King and X.-G. Liu, *Modular A_4 symmetry models of neutrinos and charged leptons*, *JHEP* **09** (2019) 074 [[arXiv:1907.11714](#)] [[INSPIRE](#)].
- [56] H. Okada and Y. Orikasa, *A radiative seesaw model in modular A_4 symmetry*, [arXiv:1907.13520](#) [[INSPIRE](#)].
- [57] T. Nomura, H. Okada and O. Popov, *A modular A_4 symmetric scotogenic model*, *Phys. Lett. B* **803** (2020) 135294 [[arXiv:1908.07457](#)] [[INSPIRE](#)].
- [58] T. Kobayashi, Y. Shimizu, K. Takagi, M. Tanimoto and T.H. Tatsuishi, *A_4 lepton flavor model and modulus stabilization from S_4 modular symmetry*, *Phys. Rev. D* **100** (2019) 115045 [Erratum *ibid.* **101** (2020) 039904] [[arXiv:1909.05139](#)] [[INSPIRE](#)].
- [59] T. Asaka, Y. Heo, T.H. Tatsuishi and T. Yoshida, *Modular A_4 invariance and leptogenesis*, *JHEP* **01** (2020) 144 [[arXiv:1909.06520](#)] [[INSPIRE](#)].
- [60] G.-J. Ding, S.F. King, X.-G. Liu and J.-N. Lu, *Modular S_4 and A_4 symmetries and their fixed points: new predictive examples of lepton mixing*, *JHEP* **12** (2019) 030 [[arXiv:1910.03460](#)] [[INSPIRE](#)].
- [61] D. Zhang, *A modular A_4 symmetry realization of two-zero textures of the Majorana neutrino mass matrix*, *Nucl. Phys. B* **952** (2020) 114935 [[arXiv:1910.07869](#)] [[INSPIRE](#)].
- [62] T. Nomura, H. Okada and S. Patra, *An inverse seesaw model with A_4 -modular symmetry*, *Nucl. Phys. B* **967** (2021) 115395 [[arXiv:1912.00379](#)] [[INSPIRE](#)].
- [63] X. Wang, *Lepton flavor mixing and CP-violation in the minimal type-(I+II) seesaw model with a modular A_4 symmetry*, *Nucl. Phys. B* **957** (2020) 115105 [[arXiv:1912.13284](#)] [[INSPIRE](#)].
- [64] T. Kobayashi, T. Nomura and T. Shimomura, *Type II seesaw models with modular A_4 symmetry*, *Phys. Rev. D* **102** (2020) 035019 [[arXiv:1912.00637](#)] [[INSPIRE](#)].
- [65] S.J.D. King and S.F. King, *Fermion mass hierarchies from modular symmetry*, *JHEP* **09** (2020) 043 [[arXiv:2002.00969](#)] [[INSPIRE](#)].
- [66] G.-J. Ding and F. Feruglio, *Testing moduli and flavon dynamics with neutrino oscillations*, *JHEP* **06** (2020) 134 [[arXiv:2003.13448](#)] [[INSPIRE](#)].
- [67] H. Okada and M. Tanimoto, *Quark and lepton flavors with common modulus τ in A_4 modular symmetry*, [arXiv:2005.00775](#) [[INSPIRE](#)].
- [68] T. Nomura and H. Okada, *A linear seesaw model with A_4 -modular flavor and local $U(1)_{B-L}$ symmetries*, [arXiv:2007.04801](#) [[INSPIRE](#)].

- [69] H. Okada and M. Tanimoto, *Spontaneous CP-violation by modulus τ in A_4 model of lepton flavors*, *JHEP* **03** (2021) 010 [[arXiv:2012.01688](#)] [[INSPIRE](#)].
- [70] C.-Y. Yao, J.-N. Lu and G.-J. Ding, *Modular invariant A_4 models for quarks and leptons with generalized CP symmetry*, [arXiv:2012.13390](#) [[INSPIRE](#)].
- [71] F. Feruglio, V. Gherardi, A. Romanino and A. Titov, *Modular invariant dynamics and fermion mass hierarchies around $\tau = i$* , [arXiv:2101.08718](#) [[INSPIRE](#)].
- [72] J.T. Penedo and S.T. Petcov, *Lepton masses and mixing from modular S_4 symmetry*, *Nucl. Phys. B* **939** (2019) 292 [[arXiv:1806.11040](#)] [[INSPIRE](#)].
- [73] P.P. Novichkov, J.T. Penedo, S.T. Petcov and A.V. Titov, *Modular S_4 models of lepton masses and mixing*, *JHEP* **04** (2019) 005 [[arXiv:1811.04933](#)] [[INSPIRE](#)].
- [74] I. de Medeiros Varzielas, M. Levy and Y.-L. Zhou, *Symmetries and stabilisers in modular invariant flavour models*, *JHEP* **11** (2020) 085 [[arXiv:2008.05329](#)] [[INSPIRE](#)].
- [75] I. de Medeiros Varzielas, S.F. King and Y.-L. Zhou, *Multiple modular symmetries as the origin of flavor*, *Phys. Rev. D* **101** (2020) 055033 [[arXiv:1906.02208](#)] [[INSPIRE](#)].
- [76] S.F. King and Y.-L. Zhou, *Trimaximal TM_1 mixing with two modular S_4 groups*, *Phys. Rev. D* **101** (2020) 015001 [[arXiv:1908.02770](#)] [[INSPIRE](#)].
- [77] J.C. Criado, F. Feruglio and S.J.D. King, *Modular invariant models of lepton masses at levels 4 and 5*, *JHEP* **02** (2020) 001 [[arXiv:1908.11867](#)] [[INSPIRE](#)].
- [78] X. Wang and S. Zhou, *The minimal seesaw model with a modular S_4 symmetry*, *JHEP* **05** (2020) 017 [[arXiv:1910.09473](#)] [[INSPIRE](#)].
- [79] X. Wang, *Dirac neutrino mass models with a modular S_4 symmetry*, *Nucl. Phys. B* **962** (2021) 115247 [[arXiv:2007.05913](#)] [[INSPIRE](#)].
- [80] P.P. Novichkov, J.T. Penedo, S.T. Petcov and A.V. Titov, *Modular A_5 symmetry for flavour model building*, *JHEP* **04** (2019) 174 [[arXiv:1812.02158](#)] [[INSPIRE](#)].
- [81] G.-J. Ding, S.F. King and X.-G. Liu, *Neutrino mass and mixing with A_5 modular symmetry*, *Phys. Rev. D* **100** (2019) 115005 [[arXiv:1903.12588](#)] [[INSPIRE](#)].
- [82] G.-J. Ding, S.F. King, C.-C. Li and Y.-L. Zhou, *Modular invariant models of leptons at level 7*, *JHEP* **08** (2020) 164 [[arXiv:2004.12662](#)] [[INSPIRE](#)].
- [83] J.-N. Lu, X.-G. Liu and G.-J. Ding, *Modular symmetry origin of texture zeros and quark lepton unification*, *Phys. Rev. D* **101** (2020) 115020 [[arXiv:1912.07573](#)] [[INSPIRE](#)].
- [84] C.D. Froggatt and H.B. Nielsen, *Hierarchy of quark masses, Cabibbo angles and CP-violation*, *Nucl. Phys. B* **147** (1979) 277 [[INSPIRE](#)].
- [85] H. Okada and M. Tanimoto, *Modular invariant flavor model of A_4 and hierarchical structures at nearby fixed points*, *Phys. Rev. D* **103** (2021) 015005 [[arXiv:2009.14242](#)] [[INSPIRE](#)].
- [86] P.P. Novichkov, J.T. Penedo and S.T. Petcov, *Fermion mass hierarchies, large lepton mixing and residual modular symmetries*, [arXiv:2102.07488](#) [[INSPIRE](#)].
- [87] X. Du and F. Wang, *SUSY breaking constraints on modular flavor S_3 invariant SU(5) GUT model*, *JHEP* **02** (2021) 221 [[arXiv:2012.01397](#)] [[INSPIRE](#)].
- [88] Y. Zhao and H.-H. Zhang, *Adjoint SU(5) GUT model with modular S_4 symmetry*, *JHEP* **03** (2021) 002 [[arXiv:2101.02266](#)] [[INSPIRE](#)].

- [89] P. Chen, G.-J. Ding and S.F. King, *SU(5) GUTs with A_4 modular symmetry*, [arXiv:2101.12724](#) [[INSPIRE](#)].
- [90] I. de Medeiros Varzielas, T. Neder and Y.-L. Zhou, *Effective alignments as building blocks of flavor models*, *Phys. Rev. D* **97** (2018) 115033 [[arXiv:1711.05716](#)] [[INSPIRE](#)].
- [91] S.F. King and Y.-L. Zhou, *Littlest mu-tau seesaw*, *JHEP* **05** (2019) 217 [[arXiv:1901.06877](#)] [[INSPIRE](#)].
- [92] S.F. King, A. Merle and A.J. Stuart, *The power of neutrino mass sum rules for neutrinoless double beta decay experiments*, *JHEP* **12** (2013) 005 [[arXiv:1307.2901](#)] [[INSPIRE](#)].
- [93] S. Antusch and V. Maurer, *Running quark and lepton parameters at various scales*, *JHEP* **11** (2013) 115 [[arXiv:1306.6879](#)] [[INSPIRE](#)].
- [94] F. Björkeröth, F.J. de Anda, I. de Medeiros Varzielas and S.F. King, *Towards a complete $A_4 \times \text{SU}(5)$ SUSY GUT*, *JHEP* **06** (2015) 141 [[arXiv:1503.03306](#)] [[INSPIRE](#)].
- [95] I. Esteban, M.C. Gonzalez-Garcia, M. Maltoni, T. Schwetz and A. Zhou, *The fate of hints: updated global analysis of three-flavor neutrino oscillations*, *JHEP* **09** (2020) 178 [[arXiv:2007.14792](#)] [[INSPIRE](#)].
- [96] *NuFIT 5.0*, <http://www.nu-fit.org/>, (2020).
- [97] KAMLAND-ZEN collaboration, *Search for Majorana neutrinos near the inverted mass hierarchy region with KamLAND-Zen*, *Phys. Rev. Lett.* **117** (2016) 082503 [Addendum *ibid.* **117** (2016) 109903] [[arXiv:1605.02889](#)] [[INSPIRE](#)].
- [98] EXO-200 collaboration, *Search for neutrinoless double- β decay with the complete EXO-200 dataset*, *Phys. Rev. Lett.* **123** (2019) 161802 [[arXiv:1906.02723](#)] [[INSPIRE](#)].
- [99] CUORE collaboration, *Improved limit on neutrinoless double-beta decay in ^{130}Te with CUORE*, *Phys. Rev. Lett.* **124** (2020) 122501 [[arXiv:1912.10966](#)] [[INSPIRE](#)].
- [100] GERDA collaboration, *Final results of GERDA on the search for neutrinoless double- β decay*, *Phys. Rev. Lett.* **125** (2020) 252502 [[arXiv:2009.06079](#)] [[INSPIRE](#)].
- [101] PLANCK collaboration, *Planck 2018 results. VI. Cosmological parameters*, *Astron. Astrophys.* **641** (2020) A6 [[arXiv:1807.06209](#)] [[INSPIRE](#)].
- [102] J.A. Escobar and C. Luhn, *The flavor group $\Delta(6n^2)$* , *J. Math. Phys.* **50** (2009) 013524 [[arXiv:0809.0639](#)] [[INSPIRE](#)].