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4d $\mathcal{N} = 2$ SCFTs and lisse W-algebras

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ABSTRACT: We continue our studies of the correspondence between 4d $\mathcal{N} = 2$ SCFTs and 2d W-algebras. The purpose of this paper is to study the relationship between 2d *lisse* W-algebras and their 4d SCFT partners. The lisse W-algebra is the W-algebra whose associated Zhu's C_2 algebra is finite dimensional. As the associated variety of Zhu's C_2 algebra is identified with the Higgs branch in the 4d/2d correspondence, the lisse condition is equivalent to the absence of the Higgs branch on the 4d side. We classify 4d $\mathcal{N} = 2$ SCFTs which do not admit Higgs branch, then these theories would give lisse W-algebras through the 4d/2d correspondence. In particular, we predict the existence of a large class of new non-admissible lisse W-algebras, which have not been studied before. The 4d theories corresponding to lisse W-algebra can appear in the Higgs branches of generic 4d $\mathcal{N} = 2$ SCFTs, therefore they are crucial to understand the Higgs branches of $\mathcal{N} = 2$ SCFTs.

KEYWORDS: Supersymmetric Gauge Theory, Supersymmetry and Duality

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1 Introduction

The correspondence between the Schur sector of 4d $\mathcal{N} = 2$ superconformal field theories (SCFTs) and 2d vertex operator algebras (VOAs) is very useful in understanding both 4d and 2d theories [1]. It is now clear that results from one side of the correspondence can provide invaluable insights to the other side. In this paper, we continue our studies between 4d $\mathcal{N} = 2$ Argyres-Douglas (AD) theories and 2d W-algebras (See [2–5] for our previous studies), and provide a systematic description of the so-called lisse W-algebras and 4d $\mathcal{N} = 2$ SCFTs.

Rational VOAs have attracted lots of attentions in mathematics and physics literature [6, 7]. Rational VOAs arise from the study of 2d rational conformal field theories [6], and studies of such VOAs turn out to be quite useful in understanding the general structure of VOAs and conformal field theories. Rational VOAs also have interesting applications in other branches of mathematics and physics [8], i.e. three dimensional topological field theory, condensed matter physics, and etc. Given a VOA, it is in general difficult to determine whether it is rational or not. Zhu constructed a non-commutative algebra called Zhu’s algebra [9] and reduced the study of rationality of VOAs to representation properties of Zhu’s algebra, which is simpler. Zhu also defined a commutative algebra called Zhu’s C_2 algebra, and a VOA is called C_2 co-finite if the corresponding Zhu’s C_2 algebra is finite dimensional. C_2 co-finiteness is closely related to rationality, and is much easier to check in practice. A C_2 co-finite VOA is also called lisse [10].

Now in the context of the 4d/2d correspondence [1], Zhu’s C_2 algebra also plays a crucial role: the associated variety of Zhu’s C_2 algebra is identified with the Higgs branch of 4d $\mathcal{N} = 2$ theories [3, 11, 12]. Since the associated variety of the lisse VOA is trivial, the *lisse* condition of a 2d VOA is just equivalent to the *absence* of the Higgs branch of the corresponding 4d $\mathcal{N} = 2$ theory. While it is in general quite complicated to compute Zhu’s C_2 algebra in the VOA context, it is a lot easier to check whether a 4d $\mathcal{N} = 2$ theory has a Higgs branch by using 4d methods, i.e. the geometry of Coulomb branch.

We perform a systematic search for 4d theories which do not admit a Higgs branch. The two major classes of 4d $\mathcal{N} = 2$ SCFTs are class- $\mathcal{S}$ theories [13, 14] constructed using 6d $(2, 0)$ SCFTs and theories constructed from three dimensional canonical singularities [15]. Class- $\mathcal{S}$ theories always have a Higgs branch if they are constructed using regular singularities only [13], so we are forced to consider AD theories which are constructed by using an irregular singularity and a possible regular singularity [14]. In the paper we first classify irregular singularities which do not contribute to the Higgs branch using the identification between the irregular singularity and the 3-fold singularity [16, 17], then we study theories engineered using an irregular singularity and a regular singularity, and it is much harder to classify the ones without Higgs branches. To accomplish this task, we develop new methods to compute the associated variety of non-admissible affine Kac-Moody (AKM) algebra, then complete the classification of lisse W-algebras arising from 4d theories constructed using 6d $(2, 0)$ theory.

For theories constructed using three fold singularities, the Higgs branch is identified as the crepant resolution of the three dimensional canonical singularity [18]. So the classifi-

cation of theory without Higgs branch is reduced to finding all 3-fold singularity with a $\mathbb{C}^*$ action, and without the crepant resolution. Such singularity having no crepant resolution is called Q-factorial terminal singularity. While it might not be easy to give a classification of such singularities using algebraic geometrical method, it is much easier to check using the input of 4d physics. For example, a Gorenstein terminal singularity is Q-factorial if the corresponding 4d theory has no mass deformation, and the existence of mass deformation can be easily checked using the Coulomb branch solution.

The next step is to classify class $\mathcal{S}$ theories which do not have Higgs branch. To reach such goal, we employ the following strategy.

Strategy. A naive constraint would be that the irregular or regular singularities do not carry mass deformation. Since a mass deformation would imply the existence of Higgs branch operators which would give the conserved current for the corresponding flavor symmetry. These regular singularities are summarized in tables 5, 6 and 7. The classification of such regular singularities is well known [19]. For the irregular singularity, one can classify such singularities by explicitly computing the Coulomb branch spectrum.

However, “no mass deformation” is only a necessary condition, and it is known that class- $\mathcal{S}$ theories with regular punctures only will always have a Higgs branch [19], leaving us with only theories from a sphere with one irregular puncture and at most one regular puncture.

If the irregular singularity is sufficiently large, namely, the irregular singularity itself can be used to define a 4d $\mathcal{N} = 2$ SCFT, the classification is simple. In this case, firstly, we can not have a regular puncture, as regular puncture would give non-trivial Higgs branch [19]. Then we use the isomorphism with the theories engineered using 3-fold singularity [16, 17] and conclude that: the absence of Higgs branch is equivalent to the absence of mass deformation in irregular singularity.

If the irregular singularity is small (the order of pole is less than 2), things are a lot more complicated. To complete the classification, we have to use following new method: first, we consider the regular singularity to be maximal so that the corresponding 4d theory has flavor symmetry G (notice that the irregular singularity is constrained to having no flavor symmetry); We then compute the Higgs branch of above theory which is a nilpotent orbit $\mathcal{O}_u$ of G , here we discovered new physical methods to do it, see subsection 7.2. Finally, for a general regular singularity labeled with nilpotent orbit f , the Higgs branch would be $\mathcal{O}_u \cap S_f$ (Here S_f is the Slowdoy slice of f). After above computation, we conclude: the theory would have no Higgs branch if we choose the regular singularity $f \in \mathcal{O}_u$!

Summary of results. Our conclusion is that all class-S theories without Higgs branch are classified by an irregular puncture without mass deformation and a regular puncture f satisfying $f \in \mathcal{O}$, where f is the label of the nilpotent orbit corresponding to the regular puncture and $\mathcal{O}$ is a nilpotent orbit depends only on the irregular puncture. And we also give a systematic method to obtain $\mathcal{O}$ of a given irregular puncture.

According to the identification between Zhu’s C_2 algebra and Higgs branch, each 4d SCFT without Higgs branch would give a 2d lisse VOA. The list of principal lisse W-

algebras¹ are listed in tables 11, 13, and 16. Some more examples are found by adding one regular singularity to the irregular singularity, see tables 9, 10, 12, 15, 21 and 23. We would like to highlight two interesting aspects of our results:

- Previous studies in VOA literature mainly focuses on studies of so-called admissible lisse W-algebras [20]. Our results show that there are actually a large class of non-admissible lisse W-algebras. It is definitely interesting to study the mathematical and physical consequences of these new lisse W-algebras.
- The 4d theory corresponding to the lisse W-algebra has no Higgs branch, so they can be the IR theory in the Higgs branch deformation of a $\mathcal{N} = 2$ theory. Therefore, the classification of these 4d theories are crucial to understand the Higgs branch behavior of generic 4d $\mathcal{N} = 2$ SCFTs. More physical consequences of these theories on space of 4d $\mathcal{N} = 2$ SCFT would be discussed somewhere else.

This paper is organized as follows: section 2 reviews basic facts of lisse VOAs and their relations to the absence of Higgs branches of 4d $\mathcal{N} = 2$ SCFTs. Section 3 reviews the space of 4d $\mathcal{N} = 2$ SCFTs and how their Higgs branch can be computed. Section 4 classifies 4d theories which do not have a Higgs branch. Section 5 studies 4d theories whose 2d VOAs are admissible lisse W-algebra. Section 6 studies non-admissible lisse W-algebra with classical Lie algebra. Section 7 studies non-admissible lisse W-algebra with exceptional Lie algebra. Finally a conclusion is given in section 8.

2 Review of VOAs and 4d $\mathcal{N} = 2$ SCFTs

2.1 Lisse VOAs

A VOA arises as the chiral part of a 2d conformal field theory (CFT). In math literature, a VOA is a vector space V with following properties (V can be thought as the vacuum module of the chiral part of a 2d CFT) [21]:

- A vacuum vector $|0\rangle$.
- A linear map

$$Y : V \rightarrow \mathcal{F}(V), \quad a \mapsto Y(a, z) = \sum_n a_n z^{-n-1} = a(z), \quad (2.1)$$

where $a_n \in \text{End}(V)$. This is just the state-operator correspondence.² Given an operator $a(z)$, one can recover the corresponding state $|a(z)\rangle = \lim_{z \rightarrow 0} a(z)|0\rangle$.

For our purpose, we need to consider the VOA with a conformal vector ω , which is nothing but the chiral part $T(z)$ of the stress tensor. Modes in the expansion of $T(z) = \sum L_n z^{-n-2}$

¹The principal lisse W-algebra means that we use principal nilpotent element to define our W-algebra.

²In physics literature, the mode expansion of a field takes the form $\sum a_n z^{-n-h}$ with h being the scaling dimension. In VOA literature, however, the above convention of mode expansion without further shift by the scaling dimension h is used so that VOAs without the definition of scaling dimension can be considered.

satisfy the Virasoro algebra (by using the standard contour integral and OPE of $T(z)$)

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c(n^3 - n)}{12}\delta_{n+m,0}. \quad (2.2)$$

Here c is the central charge, an important quantity associated with a VOA. The normal order product of two fields $a(z)$ and $b(z)$ is denoted as $:ab:(z)$, and its modes are

$$(:ab:(z))_m = \sum_{n \leq -h_a} a_n b_{m-n} + \sum_{n > -h_a} b_{m-n} a_n \quad (2.3)$$

In current convention we set $h_a = 1$. Other properties of VOAs can be found in [21].

A VOA is called *rational* if

- V has finite number of irreducible representations M_j .
- The character $\text{ch}_j = \text{tr}_{M_j}(e^{2\pi i\tau(L_0 - \frac{c}{24})})$ converges to a holomorphic function on upper half plane $\mathbb{C}^+$.³
- Functions ch_j span an $\mathfrak{sl}_2(\mathbb{Z})$ invariant space.

A VOA is called *finitely strongly generated* if there is finite number of elements $a_i \in V$, $i = 1, \dots, s$ such that the whole VOA is spanned by normal order products of the following form

$$:\partial^{k_1}a_1 \dots \partial^{k_s}a_s: \quad (2.4)$$

Notice that the choice of generators may not be unique and in general there are relations among the above basis. It is an interesting problem to find a minimal generating set given a finitely strongly generated VOA.

For a VOA V , there exists a Li's filtration [22] which is a decreasing filtration

$$F^0 \supset F^1 \supset F^2 \supset \dots, \quad (2.5)$$

in which each F^p is spanned by following states

$$F^p(V) = \left\{ a_{-n_1-1}^{i_1} a_{-n_2-1}^{i_2} \dots |0\rangle, \quad \sum n_i \geq p \right\}, \quad (2.6)$$

then one can define a graded sum of the VOA

$$\text{Gr}(V) = \bigoplus_p \frac{F^p}{F^{p+1}}. \quad (2.7)$$

It is obvious that $F^0 = V$, and F^1 is generated by $\{a_{-2}b | a \in V, b \in V\}$. The Zhu's C_2 algebra is defined as

$$R_V = \frac{F^0(V)}{F^1(V)}. \quad (2.8)$$

R_V is a Poisson algebra and is finitely generated if and only if V is strongly finitely generated. Moreover the image of generators of V in R_V generates R_V as well. Notice that R_V

³We use ch to denote the character of a VOA and χ to denote the character of a finite Lie algebra.

is in general not reduced, namely the ideal defining it would contain a nilpotent element.⁴ Once we determine R_V , we have now an associated scheme and an associated variety X_V defined from the Zhu's C_2 algebra

$$\tilde{X}_V = \text{spec}(R_V), \quad X_V = \text{spec}((R_V)_{\text{red}}). \quad (2.9)$$

A *lisse* VOA is defined as the VOA such that $\dim(X_V) = 0$, which also implies that Zhu's C_2 algebra is finite dimensional. A rational VOA has to be lisse, but there are examples that lisse VOA is not rational (see [23]). A quasi-lisse VOA is defined as the VOA whose associated variety X_V has finite number of symplectic leaves. Quasi-lisse VOAs have many interesting properties [11, 24]:

- The VOA is strongly finitely generated.
- The Virasoro vector ω_V is nilpotent in R_V .
- There are finite number of ordinary modules, and they transform nicely under modular transformations. A weak V -module (M, Y_M) is called *ordinary* if L_0 acts semi-simply on M , which means that any L_0 -eigenspace M_Δ of M of eigenvalue $\Delta \in \mathbb{C}$ is finite-dimensional, and for any $\Delta \in \mathbb{C}$, $M_{\Delta-n} = 0$ for all sufficiently large $n \in \mathbb{Z}$.
- Characters are solutions of a modular differential equation.

2.1.1 Lisse W-algebras

A class of interesting VOAs are constructed from quantum Drinfeld-Sokolov (qDS) reduction of affine Kac-Moody (AKM) algebras. Namely, we start with an AKM algebra $V^k(\mathfrak{g})$, where k is the level, and $\mathfrak{g}$ is a simple Lie algebra, and then choose a nilpotent element f of $\mathfrak{g}$ to perform the qDS reduction. The W-algebra obtained in this way is labeled as

$$W^k(\mathfrak{g}, f). \quad (2.10)$$

Now an AKM algebra is called principal admissible if the level k takes the following value

$$k = -h^\vee + \frac{p}{q}, \quad p \in \mathbb{Z}, \quad q \in \mathbb{Z}_{>0}, \quad p \geq h^\vee, \quad (p, q) = 1, \quad (p, r^\vee) = 1, \quad (2.11)$$

where $r^\vee$ is the lacity for the Lie algebra $\mathfrak{g}$,⁵ and $h^\vee$ is the dual Coxeter number. If we start with an admissible AKM algebra, the resulting W-algebra after qDS reduction is called admissible W-algebra. If $p = h^\vee$, the AKM and the corresponding W-algebra is called *boundary admissible* [25]. Other cases are called non-admissible.

A very interesting question is to classify all possible lisse W-algebras. The admissible case has been studied in [7, 26].

⁴A nilpotent element x of an ideal is an element not in I but $x^n \in I$ for some n .

⁵ $r^\vee = 1$ for ADE Lie algebra, $r^\vee = 2$ for BCF Lie algebra, and $r^\vee = 3$ for G_2 algebra.

2.2 4d/2d correspondence

It was proposed in [1] that one can get a 2d VOA from the Schur sector of a 4d $\mathcal{N} = 2$ SCFT (see also in [2–5, 11, 12, 17, 24, 27–71] for recent developments), and the basic 4d/2d dictionary used in current paper is [1]:

- There is an AKM sub-algebra ($V^{k_{2d}}(\mathfrak{g})$) in 2d VOA, where $\mathfrak{g}$ is the Lie algebra of 4d flavor symmetry G_F .
- The 2d central charge c_{2d} and the level of AKM algebra k_{2d} are related to the 4d central charge c_{4d} and the flavor central charge k_F as⁶

$$c_{2d} = -12c_{4d}, \quad k_{2d} = -k_F. \quad (2.12)$$

- The (normalized) vacuum character of 2d VOA is the 4d Schur index $\mathcal{I}(q)$.
- The associated variety is the Higgs branch of the 4d $\mathcal{N} = 2$ SCFT [3, 11, 60].

In this paper we use the notation $\mathcal{T}[V]$ to denote the 4d $\mathcal{N} = 2$ SCFT whose 2d corresponding VOA is V .

2.3 Lisse VOAs and the absence of Higgs branches

Given a 2d VOA corresponding to a 4d $\mathcal{N} = 2$ SCFT, it is conjectured that the associated variety is identified with the Higgs branch of the 4d theory [3, 11, 60]. We are particularly interested in the lisse VOA whose Zhu's C_2 algebra is finite dimensional, and would like to explore what the exact meaning of the lisse property of 2d VOAs for 4d $\mathcal{N} = 2$ SCFTs.

A 4d $\mathcal{N} = 2$ SCFT has a bosonic symmetry group $\text{SO}(2, 4) \times \text{SU}(2)_R \times \text{U}(1)_r \times G_F$, where $\text{SO}(2, 4)$ is the 4d conformal group, $\text{SU}(2)_R \times \text{U}(1)_r$ is the R symmetry group which exists for every $\mathcal{N} = 2$ SCFT, and G_F is the flavor symmetry group which might be absent for some theories. The representation theory of 4d $\mathcal{N} = 2$ superconformal algebra is studied in [72], in which short representations (where the primary states in the supermultiplet are annihilated by a fraction of supercharges) were classified. Important half-BPS operators include primary operators of multiplets $\mathcal{E}_r$ and $\hat{\mathcal{B}}_R$.

The moduli space of vacuum of a 4d $\mathcal{N} = 2$ SCFT is extremely rich. It consists of various branches. The Coulomb branch is parameterized by expectation values of the primaries of $\mathcal{E}_r$ multiplets, and the low energy effective theory is described by a Seiberg-Witten (SW) geometry [73, 74]. The set of rational numbers $[r_1, \dots, r_s]$ of $\text{U}(1)_r$ charges of $\mathcal{E}_r$ (unitarity implies that $r_i > 1$) is an important set associated to a 4d $\mathcal{N} = 2$ SCFT. The $\text{U}(1)_r$ symmetry acts non-trivially on the Coulomb branch while the $\text{SU}(2)_R \times G_F$ symmetry acts trivially. The set of Coulomb branch operators $\mathcal{E}_r$ can be found from the SW geometry. $\mathcal{E}_r$ operators form a ring which might be called the Coulomb branch chiral ring, and it is freely generated for almost all theories we know.

The Higgs branch is parameterized by expectation values of the primaries of half-BPS operators $\hat{\mathcal{B}}_R$. These operators also form a ring which might be called the Higgs branch

⁶Our normalization of k_F is half of that of [1, 27].

chiral ring. It is believed that the Higgs branch chiral ring is finitely generated, and there are in general complicated relations among generators. It is in general difficult to determine the full set of $\hat{\mathcal{B}}_R$ operators, and relations among them. The $\hat{\mathcal{B}}_1$ type operators, on the other hand, are easily determined by the SW geometry. The $\hat{\mathcal{B}}_1$ multiplet consists of conserved currents of the flavor symmetry, and one can use them to deform the 4d theory as mass deformations

$$\delta S = m^2 \int d^2\theta \hat{\mathcal{B}}_1 + \text{c.c.}, \quad (2.13)$$

which can be detected from the SW geometry of the Coulomb branch: they are dimension one parameters of the SW curve. Therefore if we find a mass deformation in the SW geometry, we know for sure the existence of at least one $\hat{\mathcal{B}}_1$ type operator, therefore the theory has a Higgs branch. However, if there is no mass deformation in the SW geometry, one can not claim that the theory has no Higgs branch! As it is still quite possible that other $\hat{\mathcal{B}}_R$ type operators with $R > 1$ exist. There are many such examples in class- $\mathcal{S}$ theories as we will review in later sections.

More generally, we could have a branch parameterized by expectation values of both $\mathcal{E}_r$ and $\hat{\mathcal{B}}_R$ type operators. In general, this branch is a direct product of a component parameterized by $\mathcal{E}_r$ type operators and a component parameterized by $\hat{\mathcal{B}}_R$ type operators. This type of branch is called the mixed branch.

Now we discuss the low energy behavior in various branches. The Coulomb branch is parameterized by expectation values of $\mathcal{E}_r$ type operators, and the low energy effective theory at generic point consists of free vector multiplets, free hypermultiplets, and interacting theories whose Coulomb branch is trivial (see [18] for examples of this type). In the literature, one mainly consider theories whose Coulomb branch consists of only free vectormultiplets. More general cases whose Coulomb branch consist of free hypermultiplets and free vector multiplets are considered in [75].

The Higgs branch is parameterized by expectation values of $\hat{\mathcal{B}}_R$ type operators, and the low energy effective theory at generic point would also involve free hypermultiplets, free vector multiplets, and an interacting theory which *does not* have a Higgs branch. In literature, people mainly consider theories whose Higgs branch only consists of free hypermultiplets, and our studies in this paper shows that it is quite common that the Higgs branch of a general 4d $\mathcal{N} = 2$ SCFT consists of an interacting theory and/or free vector multiplets.

Let us further clarify some definitions used in the literature: a branch of $\mathcal{N} = 2$ moduli space is an irreducible component of the variety defined by full chiral ring of an $\mathcal{N} = 2$ theory. A Coulomb branch is called *pure* Coulomb branch if the low energy effective theory on the generic point consists of only abelian gauge theory. A Higgs branch is called *pure* Higgs branch if the low energy effective theory on the generic point consists of only free hypermultiplets.

We now state the exact meaning of the lisse property for 4d $\mathcal{N} = 2$ SCFTs. The corresponding 2d VOA counts Schur operators and in particular count $\hat{\mathcal{B}}_R$ operators, and the reduced Zhu's C_2 algebra is conjectured to be the chiral ring for $\hat{\mathcal{B}}_R$ operators. Assuming that the Higgs branch chiral ring is reduced, we have the following conjecture:

- A 2d VOA corresponding to a 4d $\mathcal{N} = 2$ SCFT is *lisse* if the moduli space of the 4d theory has no Higgs branch, or equivalently the 4d theory has no $\hat{\mathcal{B}}_R$ type operators.

3 Space of $\mathcal{N} = 2$ SCFTs and their Higgs branch

3.1 A review on various constructions

3.1.1 4d $\mathcal{N} = 2$ SCFT from 6d (2,0) theory

One can engineer a large class of 4d $\mathcal{N} = 2$ SCFTs by starting with a 6d (2,0) theory of type $\mathfrak{j} = ADE$ on a sphere with an irregular singularity and at most one regular singularity [13, 14, 16, 17, 76].⁷ Its Coulomb branch is captured by a Hitchin system with singular boundary conditions near singularities. The Higgs field Φ of the Hitchin system near the irregular singularity takes the following form

$$\Phi = \frac{T}{z^{2+\frac{k}{b}}} + \dots \quad (3.1)$$

Here T is determined by a positive principal grading of Lie algebra $\mathfrak{j}$ [77], and is a regular semi-simple element of $\mathfrak{j}$. k is an integer greater than $-b$. Subsequent terms are chosen such that they are compatible with the leading order term (essentially the grading determines the choice of these terms). We call them $J^{(b)}[k]$ type irregular punctures, and summarize allowed values of b in table 1 (k and b are *not* required to be coprime). Theories constructed using the above irregular singularity alone can also be engineered by using a three (complex) dimensional singularity in type IIB string theory which are also summarized in table 1 [15]. One can add another regular singularity which is labeled by a nilpotent orbit f of $\mathfrak{j}$ (We use Nahm labels such that the trivial orbit corresponding to the regular puncture with maximal flavor symmetry). Detailed discussion about these defects can be found in [19]. All in all, the theory we consider could be labeled by a pair $(j^{(b)}[k], f)$ and we denote this theory $\mathcal{T}[j^{(b)}[k], f]$.

To get non-simply laced flavor groups, we need to consider the outer-automorphism twist of ADE Lie algebra and its Langlands dual. A systematic study of these AD theories was performed in [17]. Denoting the twisted Lie algebra of $\mathfrak{j}$ as $\mathfrak{g}^\vee$ and its Langlands dual as $\mathfrak{g}$, outer-automorphisms and twisted algebras of $\mathfrak{j}$ are summarized in table 2. The irregular singularity of regular semi-simple type is also classified as in table 3 with the following form,

$$\Phi = \frac{T^t}{z^{2+\frac{k_t}{b_t}}} + \dots \quad (3.2)$$

Here T^t is an element of Lie algebra $\mathfrak{g}^\vee$ or other parts of the decomposition of $\mathfrak{j}$ under the outer automorphism. $k_t > -b_t$, and the novel thing is that k_t could take half-integer value or in thirds ($\mathfrak{g} = G_2$). One can also represent those irregular singularities by 3-fold singularities as in table 3. We could again add a twisted regular puncture labeled also by a nilpotent orbit f in $\mathfrak{g}$.

Some interesting physical properties about these theories are:

⁷See the appendix A of [4] for relations between this construction and other constructions.

j	b	Singularity
A_{N-1}	N	$x_1^2 + x_2^2 + x_3^N + z^k = 0$
	$N-1$	$x_1^2 + x_2^2 + x_3^N + x_3 z^k = 0$
D_N	$2N-2$	$x_1^2 + x_2^{N-1} + x_2 x_3^2 + z^k = 0$
	N	$x_1^2 + x_2^{N-1} + x_2 x_3^2 + z^k x_3 = 0$
E_6	12	$x_1^2 + x_2^3 + x_3^4 + z^k = 0$
	9	$x_1^2 + x_2^3 + x_3^4 + z^k x_3 = 0$
	8	$x_1^2 + x_2^3 + x_3^4 + z^k x_2 = 0$
E_7	18	$x_1^2 + x_2^3 + x_2 x_3^3 + z^k = 0$
	14	$x_1^2 + x_2^3 + x_2 x_3^3 + z^k x_3 = 0$
E_8	30	$x_1^2 + x_2^3 + x_3^5 + z^k = 0$
	24	$x_1^2 + x_2^3 + x_3^5 + z^k x_3 = 0$
	20	$x_1^2 + x_2^3 + x_3^5 + z^k x_2 = 0$

Table 1. Three-fold isolated quasi-homogenous singularities of cDV type corresponding to the $j^{(b)}[k]$ irregular punctures of the regular-semisimple type in [16]. These 3d singularity is very useful in extracting the Coulomb branch spectrum, see [15].

j	A_{2N}	A_{2N-1}	D_{N+1}	E_6	D_4
Outer-automorphism o	Z_2	Z_2	Z_2	Z_2	Z_3
Invariant subalgebra $\mathfrak{g}^\vee$	B_N	C_N	B_N	F_4	G_2
Flavor symmetry $\mathfrak{g}$	$C_N^{(1)}$	B_N	$C_N^{(2)}$	F_4	G_2

Table 2. Outer-automorphisms of simple Lie algebras j , its invariant sub-algebra $g^\vee$ and flavor symmetry g from the Langlands dual of $g^\vee$.

- If there is no mass parameter in the irregular singularity (the constraint on k will be listed in the next section), the corresponding VOA is conjectured to be given by the following W algebra [17]:

$$W^{k_{2d}}(\mathfrak{g}, f), \quad k_{2d} = -h^\vee + \frac{1}{n} \frac{b}{k' + b}, \quad (3.3)$$

where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$, n is the number listed in table 4, and k' is restricted to the value such that no mass parameter is in the irregular singularity. k' is related to the value k in tables 1 and 3 as follows: a) $n = 1$, $k' = k$; b) $n = 2$ or $n = 4$, $k' = 2k + 1$; c) $n = 3$, $k' = 3k \pm 1$. Hence $\mathcal{T}[W^{k_{2d}}(\mathfrak{g}, f)]$ and $\mathcal{T}[\mathfrak{g}^{(b)}[k'], f]$ are just different names of the same theory.

- The SW geometry of these theories is identified with the spectral curve of the corresponding Hitchin system [78]:

$$\det(x - \Phi) = 0, \quad (3.4)$$

and one can read off the Coulomb branch spectrum from an associated Newton polygon [14, 16, 17], which is also reviewed in the appendix B of [4].

j with twist	b_t	SW geometry at SCFT point	$\Delta[z]$
A_{2N}/Z_2	$4N+2$	$x_1^2 + x_2^2 + x^{2N+1} + z^{k+\frac{1}{2}} = 0$	$\frac{4N+2}{4N+2k+3}$
	$2N$	$x_1^2 + x_2^2 + x^{2N+1} + xz^k = 0$	$\frac{2N}{k+2N}$
A_{2N-1}/Z_2	$4N-2$	$x_1^2 + x_2^2 + x^{2N} + xz^{k+\frac{1}{2}} = 0$	$\frac{4N-2}{4N+2k-1}$
	$2N$	$x_1^2 + x_2^2 + x^{2N} + z^k = 0$	$\frac{2N}{2N+k}$
D_{N+1}/Z_2	$2N+2$	$x_1^2 + x_2^N + x_2x_3^2 + x_3z^{k+\frac{1}{2}} = 0$	$\frac{2N+2}{2k+2N+3}$
	$2N$	$x_1^2 + x_2^N + x_2x_3^2 + z^k = 0$	$\frac{2N}{k+2N}$
D_4/Z_3	12	$x_1^2 + x_2^3 + x_2x_3^2 + x_3z^{k\pm\frac{1}{3}} = 0$	$\frac{12}{12+3k\pm 1}$
	6	$x_1^2 + x_2^3 + x_2x_3^2 + z^k = 0$	$\frac{6}{6+k}$
E_6/Z_2	18	$x_1^2 + x_2^3 + x_3^4 + x_3z^{k+\frac{1}{2}} = 0$	$\frac{18}{18+2k+1}$
	12	$x_1^2 + x_2^3 + x_3^4 + z^k = 0$	$\frac{12}{12+k}$
	8	$x_1^2 + x_2^3 + x_3^4 + x_2z^k = 0$	$\frac{8}{12+k}$

Table 3. SW geometry of twisted theories at the SCFT point. Here we also list the scaling dimension of coordinate z . All k 's in this table are integer valued and the power of z coordinate in singularity is equal to k_t used in equation (3.2).

	dimension	h	$h^\vee$	n
A_{N-1}	$N^2 - 1$	N	N	1
B_N	$(2N+1)N$	$2N$	$2N-1$	2
$C_N^{(1)}$	$(2N+1)N$	$2N$	$N+1$	4
$C_N^{(2)}$	$(2N+1)N$	$2N$	$N+1$	2
D_N	$N(2N-1)$	$2N-2$	$2N-2$	1
E_6	78	12	12	1
E_7	133	18	18	1
E_8	248	30	30	1
F_4	52	12	9	2
G_2	14	6	4	3

Table 4. Lie algebra data. h is the Coxeter number and $h^\vee$ is the dual Coxeter number. n is the number which appears in the level of 2d W-algebra, see 3.3.

Remark. Besides the case discussed above, other class- $\mathcal{S}$ theories involving irregular singularities will not lead to a non-trivial IR fixed point [13, 14, 16, 17, 76]. We could also consider class- $\mathcal{S}$ theories constructed using regular singularities only, and such theories are studied extensively in [13].

3.1.2 4d $\mathcal{N} = 2$ SCFTs from 3-fold singularities

One can also engineer a large class of 4d $\mathcal{N} = 2$ SCFTs by putting type IIB string theory on a 3-fold canonical singularity with a $\mathbb{C}^*$ action [15]. We first review the hypersurface case: starting with an isolated singularity which is defined by a polynomial $f : (\mathbb{C}^4, 0) \rightarrow (\mathbb{C}, 0)$, which is required to have a $\mathbb{C}^*$ action satisfying following condition

$$f(\lambda^{q_i} z_i) = \lambda f(z_i), \quad \sum q_i > 1. \quad (3.5)$$

The SW geometry of the theory defined by f is identified with the mini-versal deformation of f , i.e.

$$F(z, \lambda) = f(z) + \sum_{\alpha=1}^{\mu} \lambda_{\alpha} \phi_{\alpha}, \quad (3.6)$$

with ϕ_{α} being a monomial basis of the Jacobi algebra

$$J_f = \frac{\mathbb{C}[z_1, z_2, z_3, z_4]}{\left\{ \frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \frac{\partial f}{\partial z_3}, \frac{\partial f}{\partial z_4} \right\}}, \quad (3.7)$$

and λ_{α} is identified with the Coulomb branch parameters with scaling dimension

$$[\lambda_{\alpha}] = \frac{1 - Q_{\alpha}}{\sum q_i - 1}. \quad (3.8)$$

Here Q_{α} is the charge of ϕ_{α} under the $\mathbb{C}^*$ action. In particular, if $[\lambda_{\alpha}] = 1$, it is a mass parameter. Theories constructed using complete intersection singularity are discussed in [79, 80]. Theories which do not have Coulomb branch are discussed in [18]. Some further physical aspects of those theories are discussed in [81, 82].

3.2 The Higgs branch

As reviewed in the last subsection, one can find the SW geometry for the Coulomb branch once the geometric data specifying the theory is given. The Higgs branch, on the other hand, is more complicated and less studied. However, the basic ideas for studying the Higgs branch were actually already given in [18, 83], and we review those methods first.

3.2.1 4d SCFTs from $(2, 0)$ theories

Higgs branches of 4d SCFTs from 6d $(2, 0)$ theories have been studied in [83]. Taking theories constructed from 6d $A_N (2, 0)$ theory as examples, the low energy effective theory of N M5 branes is a 6d $(2, 0)$ theory with transverse deformations described by 5 real scalars. Then put the $(2, 0)$ theory on a Riemann surface Σ with certain partial topological twist on Σ so that the 4d theory has $\mathcal{N} = 2$ supersymmetry. After the twist, the transverse deformations are described by two complex scalars Φ_1 and Φ_2 defined on Σ . Φ_1 is a section of the canonical bundle on Σ , while Φ_2 is a section of the trivial bundle. The low energy effective theory is described by the following spectral curve

$$\begin{aligned} \det(v - \Phi_1) = 0 &\rightarrow v^N + \sum_{i=2}^N \phi_i(z) x^{N-i} = 0, \\ \det(w - \Phi_2) = 0 &\rightarrow \prod_i (w - c_i) = 0, \end{aligned} \quad (3.9)$$

where ϕ_i is a section of K_{Σ}^i , and c_i are constants. Above equations are independent. However, the two Higgs fields Φ_1 and Φ_2 are actually commuting [84], which give a holomorphic factorization condition on the above curve. Different branches are given by following curves:

$$\begin{aligned} (v^{n_1} + \dots)(v^{n_2} + \dots) \dots (v^{n_s} + \dots) &= 0, \\ (w - c_1)^{n_1}(w - c_2)^{n_2} \dots (w - c_s)^{n_s} &= 0, \end{aligned} \quad (3.10)$$

where $\sum n_i = N$ and $\sum c_i = 0$. Assuming $n_1 \geq n_2 \geq \dots \geq n_s$, the branch is then labeled by a Young Tableaux $[n_1, \dots, n_s]$, although not all Young Tableaux can appear. Here are some further remarks:

- It is always possible to take the Young Tableaux $Y = [N]$ (A single column of N boxes), and all c s are equal to zero, so we have a Coulomb branch.
- The Higgs branch appears if $Y = [1^N]$ (A single row of N boxes), and $v^N = 0$. Notice that this is not always possible.
- If our theory is constructed using regular singularities only, it is always possible to have a Higgs branch or a mixed branch, as we can always turn off all the deformations in v direction, i.e. $v^N = 0$, and it has at least a $N - 1$ dimensional Higgs branch.

So to find a theory without Higgs branch, we have to look at the theory constructed using irregular singularities. Furthermore, the necessary condition is that *there is no holomorphic factorization of the Coulomb branch spectral curve!*

This can be easily checked for the A_{N-1} type theory, as the SW curve at the SCFT point takes the form $x^N + z^k = 0$, and it is easy to see that no holomorphic factorization implies that N and k are coprime. For other Lie algebras, this condition is not easy to check, but luckily it can be translated into a much simpler criteria which will be discussed in the next section. Things become more complicated if there is a regular singularity, as one could have local contribution to Higgs branch. The detailed discussion on regular singularities will also be given in next section.

Example 1. Consider a theory engineered by putting 6d A_1 (2,0) theory on a sphere with the following irregular singularity

$$\Phi = \frac{T}{z^{3+\frac{1}{2}}} + \dots \quad (3.11)$$

We have only one branch

$$\begin{aligned} \text{I: } v^2 &= z^3 + u_1 z + u_2, \\ w^2 &= 0, \end{aligned}$$

which is the usual Coulomb branch. This theory has no Higgs branch.

Example 2. Consider a theory engineered by putting 6d A_1 (2,0) theory on a sphere with the following irregular singularity

$$\Phi = \frac{T}{z^4} + \dots, \quad (3.12)$$

and there are two branches

$$\begin{aligned}
 \text{I: } & v^2 = z^4 + u_1 z^2 + m z + u_2, \\
 & w^2 = 0, \\
 \text{II: } & (v - (z^2 + a z + b))(v + (z^2 + a z + b)) = 0, \\
 & (w - c)(w + c) = 0.
 \end{aligned} \tag{3.13}$$

Branch I is a pure Coulomb branch, while branch II is a mixed branch.

3.2.2 4d SCFTs from 3-fold singularities

For a 4d theory engineered from a 3-fold singularity, its Coulomb branch is described by a mini-versal deformation of the singularity [15]. The Higgs branch, on the other hand, is described by its crepant resolution [18]. A crucial theorem by Kawamata [85] is that every 3-fold canonical singularity has a partial crepant resolution $f : Y \rightarrow X$, where Y is a $\mathbb{Q}$ -factorial terminal singularity. Gorenstein terminal singularity takes the following form,

$$f_{ADE}(x, y, z) + t g(x, y, z, t) = 0. \tag{3.14}$$

Here $f_{ADE}(x, y, z)$ is the familiar 2d ADE singularity. $g(x, y, z, t)$ is a polynomial such that the singularity is isolated. Other terminal singularities are constructed from the quotient of Gorenstein terminal singularities.

The crepant resolution of a three-fold singularity is not unique, but the number of crepant divisor $C(X)$ is constant. Given a partial resolution, the number of free hypermultiplets is given by $b_2(Y)$ which is given by following formula [86]:

$$b_2(Y) = \rho(X) + C(X). \tag{3.15}$$

Here $\rho(X)$ is the rank of local class group of the singularity. In the hypersurface case, $\rho(x)$ is identified with the number of mass parameter in the SW geometry. We are looking for the singularity X with $b_2(Y) = 0$, therefore

- $C(X)$ has to be zero, which implies that X has to be a terminal singularity.
- $\rho(X)$ is zero, which further implies that X is a $\mathbb{Q}$ -factorial terminal singularity.

The conclusion is then *theories without Higgs branch corresponds to $\mathbb{Q}$ -factorial terminal singularity!*

In general $b_3(Y)$ is nonzero, and this means that the low energy theory in the Higgs branch contains free vector multiplets. Moreover, if Y is not smooth, one can have an interacting theory in the IR which is described by the singularity of Y .

Example. Consider the singularity $X: x^3 + y^3 + z^3 + t^n = 0$, and it is possible to do a weighted blow up so that the resolution is a crepant resolution, the new variety has singularity of type $x^3 + y^3 + z^3 + t^{n-3} = 0$. So we have a chain of blow ups

$$X_s \rightarrow X_{s-1} \rightarrow \dots \rightarrow X_1 \rightarrow X \tag{3.16}$$

with $s = [\frac{n}{3}]$. The ending variety X_s is either a non-singular variety if $n \equiv 0$ or $1 \pmod 3$, or a Q-factorial terminal singularity if $n \equiv 2 \pmod 3$. The topology of X_s is then computed as follows. We have $b_2(X_s) = C(X) + \rho(X)$ with

$$C(X) = [\frac{n}{3}] \quad (3.17)$$

and

$$\rho(X) = \begin{cases} 6 & \text{if } n \equiv 0 \pmod 3 \\ 0 & \text{otherwise} \end{cases}. \quad (3.18)$$

The third Betti number $b_3(X_s)$ can be computed as

$$b_3(X_s) = \begin{cases} 2([n/3] - 1) & \text{if } n \equiv 0 \pmod 3 \\ 2([n/3]) & \text{otherwise} \end{cases}. \quad (3.19)$$

All in all we have following interesting situations:

- 1 For $x^3 + y^3 + z^3 + t^2 = 0$, this is a Q-factorial terminal singularity, and there is no Higgs branch!
- 2 For $x^3 + y^3 + z^3 + t^3 = 0$, $b_3(X_s) = 0$ and $b_2(X_s) = 7$. This theory has a pure Higgs branch. This theory actually has a quiver gauge theory description, and the quiver is of the affine E_6 shape [87].
- 3 For $x^3 + y^3 + z^3 + t^n = 0$ with $n > 3$, $b_3(X_s) \neq 0$ and $b_2(X_s) \neq 0$, the theory has a Higgs branch and the low energy effective theory involves free hypermultiplets and free vector multiplets for $n \equiv 1 \pmod 3$, and the low energy effective theory has free hypermultiplets, vector multiplets, and an interacting theory described by the singularity $x^3 + y^3 + z^3 + t^2 = 0$ for $n \equiv 2 \pmod 3$.

The theory in the last example can also be constructed by taking three $D_3[\text{SU}(n)]$ theories⁸ and couple all of them to a vector multiplet of the adjoint of the $\text{SU}(n)$ (see figure 1). If $n \neq 3k$, the corresponding VOA can be given by the coset construction

$$\frac{\mathfrak{su}(n)_{-\frac{2n}{3}} \oplus \mathfrak{su}(n)_{-\frac{2n}{3}} \oplus \mathfrak{su}(n)_{-\frac{2n}{3}}}{\mathfrak{su}(n)_{-2n}}, \quad (3.20)$$

and we take the diagonal coset in the construction.

4 Absence of Higgs branch: classification

Now we would like to classify 4d SCFTs which do not have Higgs branch. As mentioned before, a necessary condition is that the theory does not have a mass deformation, as the existence of mass deformation implies the existence of $\hat{\mathcal{B}}_1$ operators. Therefore we need

⁸The $D_3[\text{SU}(n)]$ theory is another name for the $(A_{n-1}^{(n)}[3-n], \text{trival})$ theory in section 3.1.1. See the appendix A of [4] for more details.

Figure 1. The theory corresponding to singularity $x^3 + y^3 + z^3 + t^n = 0$ can also be constructed by gauging three $D_3[\text{SU}(n)]$ theories.

first find out conditions on absence of mass deformations in both classes of theories we are considering.

For 4d class- $\mathcal{S}$ theories constructed from 6d $(2, 0)$ theories, and classify irregular singularities which do not have mass parameters. We then list those regular singularities which do not have a flavor symmetry and therefore do not have mass deformation. Therefore, a class- $\mathcal{S}$ theory without Higgs branch can only be constructed by compactification of a 6d $(2, 0)$ theory on a Riemann surface with regular and/or irregular singularities without mass parameters because in class- $\mathcal{S}$ theories mass deformations appear only from punctures.

For theories constructed from three fold singularities, the Higgs branch deformation is identified with the crepant resolution of the singularity, and a very powerful theorem by Kawamata reduces the classification of theory without Higgs branch to the classification of $\mathbb{Q}$ -factorial terminal singularity. The classification of quasi-homogeneous *Gorenstein $\mathbb{Q}$ -factorial terminal singularities* actually *coincides* with the classification of special irregular singularity which do not have mass parameter in class $\mathcal{S}$ construction, although in general it is not known that whether a theory constructed from singularity has a class- $\mathcal{S}$ construction as well.

4.1 Irregular singularity: absence of mass deformation

4.1.1 Untwisted theory and $\mathbb{Q}$ factorial Gorenstein terminal singularity

Let's consider class $\mathcal{S}$ theory constructed using 6d $(2, 0)$ theory of type $\mathfrak{g} = ADE$, and f to be a regular nilpotent orbit of $\mathfrak{g}$. The same theory can be then engineered from the three fold singularity listed in table 1. They can be written in the form

$$f_{ADE}(x, y, z) + tg(x, y, z, t) = 0, \quad (4.1)$$

$\mathcal{T}$	no mass	$\mathcal{T}$	no mass
$A_{N-1}^N[k]$	$(k, N) = 1$	$A_{N-1}^{N-1}[k]$	No solution
$D_N^{2N-2}[k]$	$\frac{2N-2}{\gcd(k, 2N-2)}$ is even & $\gcd(k, 2N-2)$ is odd	$D_N^N[k]$	$\frac{N}{\gcd(k, N)}$ is even
$E_6^{12}[k]$	$k \neq 3n$	$E_6^9[k]$	$k \neq 9n$
$E_6^8[k]$	No solution	$E_7^{18}[k]$	$k \neq 2n$
$E_7^{14}[k]$	$k \neq 2n, n > 1$	$E_8^{30}[k]$	$k \neq 30n$
$E_8^{24}[k]$	$k \neq 24n$	$E_8^{20}[k]$	$k \neq 20n$

Table 5. Constraint on k so that irregular singularity denoted by $J^b[k]$ has no mass deformation.

j with twist	SW geometry at SCFT point	no mass
A_{2N}/Z_2	$x_1^2 + x_2^2 + x^{2N+1} + z^{k+\frac{1}{2}} = 0$	$\frac{4N+2}{\gcd(4N+2, 2k+1)}$ is even
	$x_1^2 + x_2^2 + x^{2N+1} + xz^k = 0$	$\frac{2N}{\gcd(2N, k)}$ is even
A_{2N-1}/Z_2	$x_1^2 + x_2^2 + x^{2N} + xz^{k+\frac{1}{2}} = 0$	$\frac{4N-2}{\gcd(4N-2, 2k+1)}$ is even
	$x_1^2 + x_2^2 + x^{2N} + z^k = 0$	$\frac{2N}{\gcd(2N, k)}$ is even
D_N/Z_2	$x_1^2 + x_2^{N-1} + x_2x_3^2 + x_3z^{k+\frac{1}{2}} = 0$	$\frac{2N}{\gcd(2k+1, 2N)}$ is even
	$x_1^2 + x_2^{N-1} + x_2x_3^2 + z^k = 0$	$\frac{2N-2}{\gcd(k, 2N-2)}$ & $\gcd(k, 2N-2)$ are even
D_4/Z_3	$x_1^2 + x_2^3 + x_2x_3^2 + x_3z^{k\pm\frac{1}{3}} = 0$	no constraint
	$x_1^2 + x_2^3 + x_2x_3^2 + z^k = 0$	$k \neq 6n$
E_6/Z_2	$x_1^2 + x_2^3 + x_3^4 + x_3z^{k+\frac{1}{2}} = 0$	no constraint
	$x_1^2 + x_2^3 + x_3^4 + z^k = 0$	$k \neq 12n$
	$x_1^2 + x_2^3 + x_3^4 + x_2z^k = 0$	$k \neq 8n$ and k even

Table 6. Constraint on twisted irregular singularity which does not have mass deformation.

with $f_{ADE}(x, y, z)$ being the famous 2d ADE singularity. The Higgs branch of these theories is conjectured to be described by crepant resolutions of these singularities. So we would like to classify above singularities which do not admit crepant resolutions. The singularity of above type is called Gorenstein terminal singularity, and those which do not admit crepant resolution is called Q-factorial terminal singularity [85], which is actually equivalent to the fact that the corresponding 4d $\mathcal{N} = 2$ SCFT does not admit a mass parameter. Such singularities have already been classified in [2], and the complete list is shown in table 5.

4.1.2 Twisted theory

Now we would like to consider the classification of twisted irregular singularity which does not admit a Higgs branch. We do not know the geometric criteria for them yet, however, we would still like to conjecture that *the absence mass parameters in the Coulomb branch*

$\mathfrak{g}$	No flavor symmetry
A_{N-1}	$Y = [N]$
B_N	$[n_1, \dots, n_s], n_i \neq n_j$ and n_i are all odd
C_N	$[n_1, \dots, n_s], n_i \neq n_j$ and n_i are all even
D_N	$[n_1, \dots, n_s], n_i \neq n_j$ and n_i are all odd
E_6	$E_6, E_6(a_1), E_6(a_3)$
E_7	$E_7, E_7(a_1), E_7(a_2), E_7(a_3), E_7(a_4), E_7(a_5)$
E_8	$E_8, E_8(a_1), E_8(a_2), E_8(a_3), E_8(a_4), E_8(a_5), E_8(a_6), E_8(a_7), E_8(b_4), E_8(b_5), E_8(b_6)$
G_2	$G_2, G_2(a_1)$
F_4	$F_4, F_4(a_1), F_4(a_2), F_4(a_3)$

Table 7. Nilpotent orbits which do not have flavor symmetry. A nilpotent orbit is labeled by a partition for classical groups, and by a Bala-Carter label for exceptional groups.

implies that there is no Higgs branch! The complete list of these irregular singularities is shown in table 6.

4.1.3 Classification using the positive grading

As described in [16], the irregular singularity described in section 2 is classified by the positive grading of regular semi-simple type,⁹ i.e.

$$g = \bigoplus_{0 \leq j \leq b-1, j \in \mathbb{Z}} g_j. \quad (4.2)$$

The coefficient of the z^{-1} term in Higgs field takes value in g_0 . The coefficient T of the leading order term is chosen as a regular semi-simple element in g_1 , and the number of mass parameter is determined by the dimension of $\{g_0^T\}$ which are all semi-simple elements in g_0 which also commute with T .

Some positive gradings of regular semi-simple type are determined by a nilpotent element e , and the number of mass parameter has a simple description. Each nilpotent element gives a $\mathfrak{sl}_2$ triple ϕ , and g_0^T is given by g^ϕ (this is the sub-algebra which commutes with $\mathfrak{sl}_2$ triple). In particular, if $b = h^\vee$ and $(k, h^\vee) = 1$, the nilpotent element is given by the regular nilpotent element whose flavor symmetry is trivial, so the corresponding irregular singularity does not have any flavor symmetries. This agrees with results shown in table 5, which is derived using the explicit computation of Coulomb branch spectrum.

4.2 Regular singularity: absence of mass deformation

A necessary condition for the absence of Higgs branch is that there is no flavor symmetry in the regular singularity. Such regular singularities can be found from the table listed in [88–95]. Here we reproduce them in table 7.

⁹We take g to be simply-laced here.

4.3 Class- $\mathcal{S}$ theories without Higgs branch

Considering 4d $\mathcal{N} = 2$ SCFTs constructed from 6d $(2,0)$ theory on a sphere with an irregular singularity and a regular singularity discussed in section 3.1.1, the necessary condition for a theory in this class to have no Higgs branch is that the irregular singularity does not admit mass deformation, which provides a constraint on the irregular singularity (such singularities are summarized in tables 5 and 6).

The choice of the regular singularity is subtle on the other hand. Naively, we would think that we have to choose regular singularity which do not have a flavor symmetry, however this is not that simple for following reasons. First of all, there is still possible contribution to Higgs branch from regular singularity even if there is no mass deformation in it, and we have to choose very special regular singularity (whose choice depends on the choice of irregular singularity) to make sure there is no Higgs branch. The main tool to select such special regular singularities is the associated variety of the AKM algebra which corresponds to the Higgs branch of another theory defined by the same irregular singularity and a regular singularity with trivial nilpotent orbit. Second of all, even if a regular puncture carries a flavor symmetry, it is possible that the 4d theory actually does not have such flavor symmetry which is related to the so-called collapsing levels of 2d W-algebra [4].

To specify with regular singularity f , recall that the theory $\mathcal{T}[W^{k_{2d}}(\mathfrak{g}, f)]$ can be obtained by Higgsing the flavor symmetry of the theory with the same irregular singularity and a regular singularity labelled by the trivial nilpotent orbit (whose corresponding VOA is simply $V^{k_{2d}}(\mathfrak{g})$), therefore the Higgs branch of $\mathcal{T}[W^{k_{2d}}(\mathfrak{g}, f)]$ is

$$S_f \cap H_{\mathcal{T}[V^{k_{2d}}(\mathfrak{g})]}, \quad (4.3)$$

where S_f is the Slowdoy slice associated with the nilpotent orbit f and $H_{\mathcal{T}[V^{k_{2d}}(\mathfrak{g})]}$ is the Higgs branch of $\mathcal{T}[V^{k_{2d}}(\mathfrak{g})]$. Moreover, $H_{\mathcal{T}[V^{k_{2d}}(\mathfrak{g})]}$, which is also the associated variety of $V^{k_{2d}}(\mathfrak{g})$, can be identified with the closure of an certain nilpotent orbit of $\mathfrak{g}$,

$$H_{\mathcal{T}[V^{k_{2d}}(\mathfrak{g})]} = \overline{\mathcal{O}}_{V^{k_{2d}}(\mathfrak{g})}. \quad (4.4)$$

$\overline{\mathcal{O}}_{V^{k_{2d}}(\mathfrak{g})}$ depends only with the irregular singularity and should not be confused with the nilpotent orbit corresponds to the regular singularity. Hence “no Higgs branch” is equivalent to $S_f \cap \overline{\mathcal{O}}_{V^{k_{2d}}(\mathfrak{g})} = \emptyset$ which leads to the condition

$$f \in \mathcal{O}_{V^{k_{2d}}(\mathfrak{g})}. \quad (4.5)$$

Now we see that theories without Higgs branch are classified by the irregular singularities without mass deformation and regular singularities satisfies equation (4.5). We need to know the associated variety $\overline{\mathcal{O}}$ for each irregular singularity. We will discuss how to compute these $\overline{\mathcal{O}}$ s in the next few sections and complete a full classification of 4d $\mathcal{N} = 2$ SCFTs which do not have a Higgs branch.

5 Admissible lisse W-algebras

In this section, we focus on the irregular singularity such that the level of the corresponding W-algebra $W^{k_{2d}}(\mathfrak{g}, f)$ is *admissible*, namely, the level takes the following form

$$k_{2d} = -h^\vee + \frac{h^\vee}{b_{\max} + k'} \quad (5.1)$$

Here $(k', h^\vee) = 1$,¹⁰ see equation (3.3) for the convention. The associated variety of the corresponding admissible W-algebra is (defining $u \equiv b_{\max} + k'$)

$$S_f \cap \overline{\mathcal{O}}_u, \quad (5.2)$$

where S_f is the Slowdoy slice associated with the nilpotent orbit f , and $\overline{\mathcal{O}}_u$ is the Higgs branch of the theory defined by the same irregular singularity determined by the pair $(h^\vee, u)$ and a regular puncture labeled by trivial nilpotent orbit (the corresponding VOA is the AKM with level 5.1). So the regular singularity f should be taken as an element in $\mathcal{O}_u$

$$f \in \mathcal{O}_u. \quad (5.3)$$

and the corresponding 4d theory would have no Higgs branch! Actually the nilpotent orbit $\mathcal{O}_u$ depends only on $\mathfrak{g}$ and the parameter $u = b_{\max} + k'$ of the irregular singularity. The explicit results of $\mathcal{O}_u$ are listed in tables of [20],

Once $\mathcal{O}_u$ is determined by the data of irregular singularities, the above condition further restrict the choice of f . We first consider all f 's which do not have mass deformation which are summarized in 8. Dimensions of Coulomb branch operators, Schur indices and other properties of those 4d theories defined using exceptional Lie algebra with $u < h^\vee$ are listed in table 9. Exotic cases where the regular singularity f has mass deformation but the whole theory has no Higgs branch are summarized in tables 10 and 12.

For theories whose VOA is admissible lisse W-algebra, we can compute following things:

- *Coulomb branch spectrum*: One can compute the Coulomb branch spectrum of these theories using the spectral curve of the corresponding Hitchin system. The irregular singularity part can be found from the Newton polygon, and one also need to use the pole structure of regular singularities listed in [88–95].
- *Schur index*: The Schur index (hence the vacuum character of the corresponding W-algebra) takes the following form [5]:

$$\text{ch} = \text{PE} \left[\frac{\sum q^{1+j} \chi_{R_j} - q^u \sum q^{-j} \chi_{R_j}}{(1-q)(1-q^u)} \right]. \quad (5.4)$$

Here R_j is determined by the nilpotent orbit f , which in turn gives a decomposition of the adjoint representation of $\mathfrak{g}$, $\text{adj}_{\mathfrak{g}} \rightarrow \oplus V_j \otimes R_j$. Here V_j is spin j representation of $\text{SU}(2)$ group. In our case $u = q + k'$, and $(u, h^\vee) = 1$, $(u, r) = 1$. Here $h^\vee$ is the dual Coxeter number, and r is the lacety of lie algebra.

¹⁰The W-algebra of A_{2N}/Z_2 class theory given in [17] is not principal admissible, however, using the result of [4], the previous W-algebra is actually isomorphic to an admissible W-algebra.

$\mathfrak{g}$	Exceptional Pair (f, u) , and $\gcd(u, h^\vee) = 1$
A_{N-1}	$([N], u > N)$
B_N	a: $([2N+1], u > 2N)$; b: $([u, s, 1], u)$, u odd
$C_N^{(2)}$	a: $([2N], u > 2N)$; b: $([u-1, s], u)$, u odd
D_N	a: $([2N-1, 1], u > 2N-2)$; b: $([u, s], q)$, u odd
E_6	$(E_6, u > 12), (E_6(a_1), 11), (E_6(a_3), 7)$
E_7	$(E_7, u > 18), (E_7(a_1), 17), (E_7(a_2), 13), (E_7(a_3), 11)$
E_8	$(E_8, u > 30), (E_8(a_1), 29), (E_8(a_2), 23), (E_8(a_3), 19), (E_8(a_4), 17), (E_8(a_5), 13), (E_8(a_6), 11)$
G_2	$(G_2, u > 6), (G_2(a_1), 5)$
F_4	$(F_4, u > 12), (F_4(a_1), 11), (F_4(a_2), 7), (F_4(a_3), 5)$

Table 8. List of lisse VOA with the form $W^{k_{2d}}(\mathfrak{g}, f)$ where f is a nilpotent orbit labelling a regular singularity with *no mass deformation*. u specifies the irregular singularity. The level k_{2d} of the corresponding 2d VOA is $k = -h^\vee + \frac{h^\vee}{u}$. These pairs give rise to 4d $\mathcal{N} = 2$ SCFTs $\mathcal{T}[W^{k_{2d}}(\mathfrak{g}, f)]$ with no Higgs branch.

- *Zhu's C_2 algebra*: Zhu's C_2 algebra can be found from Jacobi algebra of an isolated hypersurface singularity [5].

Here are some further remarks about those 4d theories whose VOA is admissible lisse W-algebras:

- 1 If $u > h^\vee$, the nilpotent orbit O_u is just the regular nilpotent orbit, so f has to be the regular nilpotent orbit. The corresponding W-algebra is just the minimal model of W-algebra listed in table 11. Some of the physical information such as the Coulomb branch, the Schur index, central charge, etc, can be found in [5]. These might be called *admissible principal lisse W-algebra*. These theories (for $\mathfrak{g} = ADE$) can be engineered by following singularity

$$f_{ADE}(x, y, z) + w^k = 0. \quad (5.5)$$

and it is also labeled as (G, A_{k-1}) (or (A_{k-1}, G) theory) theory using the notation of [4]. Here G is the ADE type appearing in above singularity.

- 2 If $u < h^\vee$ and for classical lie algebra, we can actually see from the index that most of them are isomorphic to an admissible principal lisse W-algebra. For example, consider D type case with the regular singularity of the form $[u, s]$ (here u and s are odd integers), and the corresponding W-algebra is

$$W^{-(2N-2)+\frac{2N-2}{u}}(\mathfrak{so}_{2N}, [u, s]) \quad (5.6)$$

Using the index formula in [5], we have

$$\text{ch} = \text{PE} \left[\frac{q^2 + q^4 + \dots + q^{s-1} - q^u(q^{-1} + q^{-3} + \dots + q^{-(s-2)})}{(1-q)(1-q^u)} \right]. \quad (5.7)$$

(f, u)	Coulomb branch	Schur index	Equivalent 4d Theory	Isomorphic W-algebra
$(E_6(a_1), 11)$	$(\frac{24}{11}, \frac{18}{11}, \frac{16}{11}, \frac{15}{11}, \frac{12}{11})$	$\text{PE} \left[\frac{q^2+q^5-(q^7+q^{10})}{(1-q)(1-q^{11})} \right]$	(A_2, D_5)	$\text{Vir}_{3,22} \oplus L(21, 1)$
$(E_6(a_3), 7)$	$(\frac{12}{7}, \frac{9}{7}, \frac{8}{7})$	$\text{PE} \left[\frac{q^2+q^3-(q^5+q^6)}{(1-q)(1-q^7)} \right]$	(A_2, A_3)	$\text{Vir}_{3,14} \oplus L(13, 1)$
$(E_7(a_1), 17)$	$(\frac{54}{17}, \frac{42}{17}, \frac{40}{17}, \frac{36}{17}, \frac{30}{17}, \frac{28}{17}, \frac{24}{17}, \frac{22}{17}, \frac{18}{17})$	$\text{PE} \left[\frac{q^2+q^6-(q^{12}+q^{16})}{(1-q)(1-q^{17})} \right]$	$(D_4/Z_3, k - \frac{1}{3} = \frac{5}{3})$	
$(E_7(a_2), 13)$	$(\frac{22}{13}, \frac{20}{13}, \frac{18}{13}, \frac{16}{13}, \frac{14}{13})$	$\text{PE} \left[\frac{q^2-q^{12}}{(1-q)(1-q^{13})} \right]$	(A_1, A_{10})	$\text{Vir}_{2,13}$
$(E_7(a_3), 11)$	$(\frac{24}{11}, \frac{18}{11}, \frac{16}{11}, \frac{15}{11}, \frac{12}{11})$	$\text{PE} \left[\frac{q^2+q^5-(q^7+q^{10})}{(1-q)(1-q^{11})} \right]$	(A_2, D_5)	$\text{Vir}_{3,22} \oplus L(21, 1)$
$(E_8(a_1), 29)$	$(\frac{30}{29}, \frac{36}{29}, \frac{40}{29}, \frac{42}{29}, \frac{46}{29}, \frac{48}{29}, \frac{52}{29}, \frac{54}{29}, \frac{60}{29}, \frac{66}{29}, \frac{70}{29}, \frac{72}{29}, \frac{76}{29}, \frac{78}{29}, \frac{80}{29}, \frac{96}{29}, \frac{100}{29}, \frac{102}{29}, \frac{120}{29}, \frac{126}{29}, \frac{150}{29})$	$\text{PE} \left[\frac{q^2+q^8+q^{14}-(q^{16}+q^{22}+q^{28})}{(1-q)(1-q^{29})} \right]$	$x^2 + y^3 + z^5 + zw^5 = 0$	
$(E_8(a_2), 23)$	$(\frac{50}{23}, \frac{52}{23}, \frac{54}{23}, \frac{60}{23}, \frac{60}{23}, \frac{70}{23}, \frac{72}{23}, \frac{72}{23}, \frac{72}{23}, \frac{90}{23}, \frac{24}{23}, \frac{30}{23}, \frac{32}{23}, \frac{34}{23}, \frac{36}{23}, \frac{40}{23}, \frac{42}{23})$	$\text{PE} \left[\frac{q^2+q^8-(q^{16}+q^{22})}{(1-q)(1-q^{23})} \right]$	(A_4, E_7)	
$(E_8(a_3), 19)$	$(\frac{48}{19}, \frac{42}{19}, \frac{36}{19}, \frac{32}{19}, \frac{30}{19}, \frac{27}{19}, \frac{26}{19}, \frac{24}{19}, \frac{20}{19}, \frac{14}{19}, \frac{14}{19}, \frac{14}{19}, \frac{19}{19})$	$\text{PE} \left[\frac{q^2+q^9-(q^{11}+q^{18})}{(1-q)(1-q^{19})} \right]$	(A_2, D_9)	$\text{Vir}_{3,38} \oplus L(37, 1)$
$(E_8(a_4), 17)$	$(\frac{18}{17}, \frac{21}{17}, \frac{24}{17}, \frac{25}{17}, \frac{28}{17}, \frac{30}{17}, \frac{33}{17}, \frac{36}{17}, \frac{40}{17}, \frac{45}{17}, \frac{48}{17}, \frac{60}{17}, \frac{60}{17}, \frac{60}{17}, \frac{60}{17}, \frac{60}{17}, \frac{60}{17})$	$\text{PE} \left[\frac{q^2+q^5+q^8-(q^{10}+q^{13}+q^{16})}{(1-q)(1-q^{17})} \right]$	(E_6, A_4)	
$(E_8(a_5), 13)$	$(\frac{30}{13}, \frac{24}{13}, \frac{20}{13}, \frac{18}{13}, \frac{14}{13}, \frac{14}{13})$	$\text{PE} \left[\frac{q^2+q^6-(q^8+q^{12})}{(1-q)(1-q^{13})} \right]$	$(D_4/Z_3, k + \frac{1}{3} = \frac{1}{3})$	$\text{Vir}_{3,26} \oplus L(25, 1)$
$(E_8(a_6), 11)$	$(\frac{12}{11}, \frac{18}{11}, \frac{24}{11}, \frac{30}{11}, (\frac{14}{11})^2, (\frac{20}{11})^2)$	$\text{PE} \left[\frac{q^2+2q^4-(2q^8+q^{10})}{(1-q)(1-q^{11})} \right]$	(D_4, A_4)	
$(G_2(a_1), 5)$	$(\frac{6^3}{5})$	$\text{PE} \left[\frac{3q^2-3q^4}{(1-q)(1-q^5)} \right]$	$3 \times (A_1, A_2)$	$\text{Vir}_{2,5}^{\otimes 3}$
$(F_4(a_1), 11)$	$(\frac{24}{11}, \frac{21}{11}, \frac{18}{11}, \frac{15}{11}, \frac{12}{11}, \frac{16}{11}, \frac{13}{11})$	$\text{PE} \left[\frac{q^2+q^3-(q^9+q^{10})}{(1-q)(1-q^{11})} \right]$	(A_2, A_7)	
$(F_4(a_2), 7)$	$(\frac{10}{7}, \frac{8}{7})$	$\text{PE} \left[\frac{q^2-q^6}{(1-q)(1-q^7)} \right]$	(A_1, A_4)	$\text{Vir}_{2,7}$
$(F_4(a_3), 5)$	$(\frac{6^4}{5})$	$\text{PE} \left[\frac{4q^2-4q^4}{(1-q)(1-q^5)} \right]$	$4 \times (A_1, A_2)$	$\text{Vir}_{2,5}^{\otimes 4}$

Table 9. Lisse VOA $W^{-h^\vee + \frac{h^\vee}{u}}(\mathfrak{g}, f)$ with $u < h^\vee$ and $\mathfrak{g}$ is of EFG type. Again f denotes a regular singularity with *no mass deformation*, and u specifies the irregular singularity. The second and the third columns list dimensions of all Coulomb branch operators of $\mathcal{T}[W^{-h^\vee + \frac{h^\vee}{u}}(\mathfrak{g}, f)]$ and its Schur index (vacuum character) respectively. The fourth column gives another 4d SCFT description of the same theory. The last column gives another VOA (if any) which is isomorphic to $W^{-h^\vee + \frac{h^\vee}{u}}(\mathfrak{g}, f)$ which are obtained by matching the central charge, vacuum character and the corresponding 4d theories. W-algebra isomorphisms and other properties for cases with f subregular were studied in [26]. $\text{Vir}_{p,q}$ is the vacuum module of Virasoro minimal model with label (p, q) , and $L(a, b)$ is a module of the previous minimal model with label (a, b) .

We now recognize that this is actually the vacuum character of $W^{-(s-2) + \frac{s-2}{u}}(\mathfrak{so}_s, f_{\text{prin}})$! In fact, one can check that 4d Coulomb branch spectrum agrees with each other.

- 3 Dimensions of Coulomb branch operators, Schur indices and other properties for exceptional case is listed in tables 9 and 10. One may also find an isomorphic VOA with these data for most of the cases. For example, the theory $\mathcal{T}[W^{-\frac{54}{7}}(\mathfrak{f}_4, F_4(a_2))]$ has Coulomb branch operators with conformal dimension $10/7$ and $8/7$ which are the same as (A_1, A_4) AD theory. Matching of central charge and Schur index also confirm that $\mathcal{T}[W^{-\frac{54}{7}}(\mathfrak{f}_4, F_4(a_2))]$ is the same 4d theory as (A_1, A_4) AD theory, therefore $W^{-\frac{54}{7}}(\mathfrak{f}_4, F_4(a_2))$ is isomorphic to $\text{Vir}_{2,7}$ which is the vacuum module of the $(2, 7)$ Virasoro minimal model.

$(\mathfrak{g}, f, u)$	Coulomb branch	Schur index	Equivalent 4d Theory	Isomorphic W-algebra
$(\mathfrak{e}_6, A_4 + A_1, 5)$	$\frac{6}{5}$	$\text{PE} \left[\frac{q^2 - q^4}{(1-q)(1-q^5)} \right]$	(A_1, A_2)	$\text{Vir}_{2,5}$
$(\mathfrak{e}_7, A_4 + A_2, 5)$	$\frac{6}{5}$	$\text{PE} \left[\frac{q^2 - q^4}{(1-q)(1-q^5)} \right]$	(A_1, A_2)	$\text{Vir}_{2,5}$
$(\mathfrak{e}_8, A_6 + A_1, 7)$	$\frac{10}{7}, \frac{8}{7}$	$\text{PE} \left[\frac{q^2 - q^6}{(1-q)(1-q^7)} \right]$	(A_1, A_4)	$\text{Vir}_{2,7}$

Table 10. Theories without the Higgs branch but whose regular singularity has a flavor symmetry (mass deformation). f denotes the nilpotent orbit of the regular singularity, and u specifies the irregular singularity. The corresponding W algebra is $W^{-h^\vee + \frac{h^\vee}{u}}(\mathfrak{g}, f)$. The second and the third columns list dimensions of all Coulomb branch operators of $\mathcal{T}[W^{-h^\vee + \frac{h^\vee}{u}}(\mathfrak{g}, f)]$ and its Schur index (vacuum character) respectively. The fourth column gives another 4d SCFT description of the same theory. The last column gives another VOA (if any) which is isomorphic to $W^{-h^\vee + \frac{h^\vee}{u}}(\mathfrak{g}, f)$ which are obtained by matching the central charge, vacuum character and the corresponding 4d theories.

	SW geometry	VOA
A_{N-1}	$x_1^2 + x_2^2 + x_3^N + z^k = 0$	$W^{-N + \frac{N}{N+k}}(\mathfrak{sl}_N, f_{\text{prin}})$
D_N	$x_1^2 + x_2^{N-1} + x_2 x_3^2 + z^k = 0$	$W^{-(2N-2) + \frac{2N-2}{2N-2+k}}(\mathfrak{so}_{2N}, f_{\text{prin}})$
E_6	$x_1^2 + x_2^3 + x_3^4 + z^k = 0$	$W^{-12 + \frac{12}{12+k}}(\mathfrak{e}_6, f_{\text{prin}})$
E_7	$x_1^2 + x_2^3 + x_2 x_3^3 + z^k = 0$	$W^{-18 + \frac{18}{18+k}}(\mathfrak{e}_7, f_{\text{prin}})$
E_8	$x_1^2 + x_2^3 + x_3^5 + z^k = 0$	$W^{-30 + \frac{30}{30+k}}(\mathfrak{e}_8, f_{\text{prin}})$
A_{2N-1}/Z_2	$x_1^2 + x_2^2 + x^{2N} + x z^{k+\frac{1}{2}} = 0$	$W^{-(2N-1) + \frac{2N-1}{4N+2k-1}}(\mathfrak{so}_{2N+1}, f_{\text{prin}})$
D_{N+1}/Z_2	$x_1^2 + x_2^N + x_2 x_3^2 + x_3 z^{k+\frac{1}{2}} = 0$	$W^{-N + \frac{N}{2N+2k+1}}(\mathfrak{sp}_{2N-2}, f_{\text{prin}})$
D_4/Z_3	$x_1^2 + x_2^3 + x_2 x_3^2 + x_3 z^{k \pm \frac{1}{3}} = 0$	$W^{-4 + \frac{4}{12+3k \pm 1}}(\mathfrak{g}_2, f_{\text{prin}})$
E_6/Z_2	$x_1^2 + x_2^3 + x_3^4 + x_3 z^{k+\frac{1}{2}} = 0$	$W^{-9 + \frac{9}{18+2k+1}}(\mathfrak{f}_4, f_{\text{prin}})$

Table 11. 4d SCFTs whose VOA is admissible lisse W-algebra. Here k is constrained so that 4d theory has no mass deformation and the level of W-algebra is admissible. Namely, $(k, h^\vee) = 1$ for ADE case, $(2k+1, h^\vee) = 1$ for BCF case, and finally $(3k \pm 1, h^\vee) = 1$ for G_2 case! The corresponding field theory has no exact marginal deformation.

6 Non-admissible lisse W-algebra: Classical Lie algebra

Now consider 4d $\mathcal{N} = 2$ SCFTs whose associated VOA is non-admissible lisse W-algebra with classical Lie algebra. The 4d theories are constructed from 6d $(2, 0)$ theory of A and D type (We also consider outer automorphism twist so that VOAs also include W-algebra with non-simply-laced classical Lie algebra). The irregular singularity is chosen so that there is no mass parameter in it, as in tables 5 and 6, and if the regular singularity is chosen to be the principal one, the corresponding W-algebra is conjectured to be lisse. Such theories are summarized in table 13.

$\mathfrak{g}$	Nilpotent orbit f	Vacuum character	VOA	Isomorphic principle W-algebra
A_{N-1}	$[u, \dots, u, s], 0 \leq s \leq u-2$	$\text{PE} \left[\frac{\sum_{j=1}^{s-1} (q^{j+1} - q^{u-j})}{(1-q)(1-q^u)} \right]$	$W^{-N+\frac{N}{u}}(\mathfrak{sl}_N, f)$	$W^{-s+\frac{s}{u}}(\mathfrak{sl}_s, f_{\text{prin}})$
B_N	$\underbrace{[u, \dots, u, s]}_{\text{even}}, s \text{ odd}, 0 \leq s \leq u$	$\text{PE} \left[\frac{\sum_{j=1}^{\frac{s-1}{2}} (q^{2j} - q^{u-2j+1})}{(1-q)(1-q^u)} \right]$	$W^{-(2N-1)+\frac{2N-1}{u}}(\mathfrak{so}_{2N+1}, f)$	$W^{-(s-2)+\frac{s-2}{u}}(\mathfrak{so}_s, f_{\text{prin}})$
	$\underbrace{[u, \dots, u, s, 1]}_{\text{odd}}, s \text{ odd}, 1 \leq s \leq u-1$	$\text{PE} \left[\frac{\sum_{j=1}^{\frac{s-1}{2}} (q^{2j} - q^{u-2j+1}) + q^{\frac{s+1}{2}} - q^{u-\frac{s}{2}+\frac{1}{2}}}{(1-q)(1-q^u)} \right]$	$W^{-(2N-1)+\frac{2N-1}{u}}(\mathfrak{so}_{2N+1}, f)$	$W^{-(s-1)+\frac{s-1}{u}}(\mathfrak{so}_{s+1}, f_{\text{prin}})$
C_N	$\underbrace{[u, \dots, u, s]}_{\text{even}}, s \text{ even}, 0 \leq s \leq u-1$	$\text{PE} \left[\frac{\sum_{j=1}^{\frac{s}{2}} (q^{2j} - q^{u-2j+1})}{(1-q)(1-q^u)} \right]$	$W^{-(N+1)+\frac{N+1}{u}}(\mathfrak{sp}_{2N}, f)$	$W^{-(\frac{s+2}{2})+\frac{\frac{s+2}{2}}{u}}(\mathfrak{sp}_s, f_{\text{prin}})$
	$\underbrace{[u, \dots, u, u-1, s]}_{\text{odd}}, s \text{ even}, 1 \leq s \leq u-1$	$\text{PE} \left[\frac{\sum_{j=1}^{\frac{s}{2}} (q^{2j} - q^{u-2j+1}) + q^{u-\frac{s}{2}+\frac{1}{2}} - q^{u+s+\frac{1}{2}}}{(1-q)(1-q^u)} \right]$	$W^{-(N+1)+\frac{N+1}{u}}(\mathfrak{sp}_{2N}, f)$	
D_N	$\underbrace{[u, \dots, u, s]}_{\text{odd}}, s \text{ odd}, 0 \leq s \leq u$	$\text{PE} \left[\frac{\sum_{j=1}^{\frac{s-1}{2}} (q^{2j} - q^{u-2j+1})}{(1-q)(1-q^u)} \right]$	$W^{-(2N-2)+\frac{2N-2}{u}}(\mathfrak{so}_{2N}, f)$	$W^{-(s-2)+\frac{s-2}{u}}(\mathfrak{so}_s, f_{\text{prin}})$
	$\underbrace{[u, \dots, u, s, 1]}_{\text{even}}, s \text{ odd}, 0 \leq s \leq u-1$	$\text{PE} \left[\frac{\sum_{j=1}^{\frac{s-1}{2}} (q^{2j} - q^{u-2j+1}) + q^{\frac{s}{2}+\frac{1}{2}} - q^{u-\frac{s}{2}+\frac{1}{2}}}{(1-q)(1-q^u)} \right]$	$W^{-(2N-2)+\frac{2N-2}{u}}(\mathfrak{so}_{2N}, f)$	$W^{-(s-1)+\frac{s-1}{u}}(\mathfrak{so}_{s+1}, f_{\text{prin}})$

Table 12. Vacuum characters for W-algebra $W^{-h^\vee + \frac{h^\vee}{u}}(\mathfrak{g}, f)$, where $\mathfrak{g}$ is classical lie algebra. u is an odd integer and $(u, h^\vee) = 1$. The last column lists principle W-algebras isomorphic to corresponding lisse W-algebras listed in the fourth column by matching the vacuum character and the central charge.

j with twist	SW geometry at SCFT point	VOA	Constraint	$[z]$
A_{2N}/Z_2	$x_1^2 + x_2^2 + x^{2N+1} + z^{k+\frac{1}{2}} = 0$	$W^{-(N+1)+\frac{n}{2u}}(\mathfrak{sp}_{2N}, f_{\text{prin}})$	n odd and $g = \frac{2N+1}{n}$ odd	$\frac{2n}{u}$
	$x_1^2 + x_2^2 + x^{2N+1} + xz^k = 0$	$W^{-(N+1)+\frac{n}{2u}}(\mathfrak{sp}_{2N}, f_{\text{prin}})$	n odd and $g = \frac{2N}{n}$ even	$\frac{2n}{u}$
A_{2N-1}/Z_2	$x_1^2 + x_2^2 + x^{2N} + xz^{k+\frac{1}{2}} = 0$	$W^{-(2N-1)+\frac{n}{u}}(\mathfrak{so}_{2N+1}, f_{\text{prin}})$	n odd and $g = \frac{2N-1}{n}$ odd	$\frac{2n}{u}$
	$x_1^2 + x_2^2 + x^{2N} + z^k = 0$	$W^{-(2N-1)+\frac{n}{u}}(\mathfrak{so}_{2N+1}, f_{\text{prin}})$	n odd and $g = \frac{2N}{n}$ even	$\frac{2n}{u}$
D_N/Z_2	$x_1^2 + x_2^{N-1} + x_2x_3^2 + x_3z^{k+\frac{1}{2}} = 0$	$W^{-N+\frac{n}{2u}}(\mathfrak{sp}_{2N-2}, f_{\text{prin}})$	n even and $g = \frac{2N}{n}$ odd	$\frac{n}{u}$
	$x_1^2 + x_2^{N-1} + x_2x_3^2 + z^k = 0$	$W^{-N+\frac{n}{2u}}(\mathfrak{sp}_{2N-2}, f_{\text{prin}})$	n even and $g = \frac{2N-2}{n}$ even	$\frac{n}{u}$
D_N	$x_1^2 + x_2^{N-1} + x_2x_3^2 + z^k = 0$	$W^{-(2N-2)+\frac{n}{u}}(\mathfrak{so}_{2N}, f_{\text{prin}})$	n even and $g = \frac{2N-2}{n}$ odd	$\frac{n}{u}$
	$x_1^2 + x_2^{N-1} + x_2x_3^2 + z^kx_3 = 0$	$W^{-(2N-2)+\frac{n}{u}}(\mathfrak{so}_{2N}, f_{\text{prin}})$	n even and $g = \frac{2N}{n}$ even	$\frac{n}{u}$

Table 13. 4d $\mathcal{N} = 2$ SCFTs whose 2d VOA is non-admissible lisse W-algebra defined using classical Lie algebra. u is an odd integer coprime with n .

6.1 A physical derivation of associated variety

Now we consider adding a regular singularity of other type instead of the principal one, and try to find the criteria on f such that the resulting theory has no Higgs branch. As discussed in last section, the crucial input is the associated variety of W-algebra with trivial nilpotent orbit which is actually the AKM

$$V^{k_{2d}}(\mathfrak{g}), \quad k_{2d} = -h^\vee + \frac{n}{ru}. \quad (6.1)$$

For classical groups, proper values of k_{2d} are listed in the VOA column of table 13. The associated variety of the above AKM is the closure of a nilpotent orbit which is denoted as $\overline{\mathcal{O}}_{\mathfrak{g}, n, u}$. The growth function of the Schur index (equivalently the character of vacuum

j with twist	$\mathfrak{g}$	Constraint on n	$d_{\mathfrak{g}}(n)$
A_{2N}/Z_2	$\mathfrak{sp}_{2N}$	n odd and $g = \frac{2N+1}{n}$ odd	$\frac{g}{2}n^2 - \frac{3n}{2} + g$
	$\mathfrak{sp}_{2N}$	n odd and $g = \frac{2N}{n}$ even	$\frac{g}{2}n^2 + g$
A_{2N-1}/Z_2	$\mathfrak{so}_{2N+1}$	n odd and $g = \frac{2N-1}{n}$ odd	$\frac{g}{2}n^2 + \frac{3n}{2} + g$
	$\mathfrak{so}_{2N+1}$	n odd and $g = \frac{2N}{n}$ even	$\frac{g}{2}n^2 + g$
D_N/Z_2	$\mathfrak{sp}_{2N-2}$	n even and $g = \frac{2N}{n}$ odd	$\frac{g}{2}n^2 - \frac{3n}{2} + g$
	$\mathfrak{sp}_{2N-2}$	n even and $g = \frac{2N-2}{n}$ even	$\frac{g}{2}n^2 + g$
D_N	$\mathfrak{so}_{2N}$	n even and $g = \frac{2N-2}{n}$ odd	$\frac{g}{2}n^2 + \frac{3n}{2} + g$
	$\mathfrak{so}_{2N}$	n even and $g = \frac{2N}{n}$ even	$\frac{g}{2}n^2 + g$

Table 14. Data required to compute the growth function $\mathcal{G}$ of VOA $V^{-h^\vee + \frac{n}{ru}}(\mathfrak{g})$, where $\mathcal{G} = \dim(\mathfrak{g}) - \frac{d_{\mathfrak{g}}(n)}{u}$.

module of the above AKM algebra) is proportional to the difference of 4d central charge which can be computed using the Coulomb branch data [5]

$$\mathcal{G} = -48(a_{4d} - c_{4d}). \quad (6.2)$$

In our case the growth function of VOA (6.1) can be summarized by the following formula

$$\mathcal{G}_{\mathfrak{g},n,u} = \dim(\mathfrak{g}) - \frac{d_{\mathfrak{g}}(n)}{u} \quad (6.3)$$

with $d_{\mathfrak{g}}(n)$ summarized in table 14. We found this formula by explicitly computing the 4d central charges.

The lisse condition on 4d theory is equivalent to the following choices of irregular and regular singularity:

- The irregular singularity is restricted to those listed in table 13.
- The regular singularity is chosen such that $f \in \mathcal{O}_{\mathfrak{g},n,u}$, therefore $S_f \cap \overline{\mathcal{O}}_{\mathfrak{g},n,u}$ is trivial.

So the choice of regular singularity is determined once we find the associated variety $\overline{\mathcal{O}}_{\mathfrak{g},n,u}$.

Unlike the admissible case, associated varieties of VOA (6.1) are generally not known in the mathematics literature. Here we will use a simple physical method to compute it. The idea is as the following: our 4d theory is engineered by a 6d (2, 0) theory on a sphere with an irregular and a regular singularity labeled by a trivial nilpotent orbit. Going to the Higgs branch of this theory is equivalent to closing the regular puncture, in other words, changing the type of regular puncture (equivalent to choice a different nilpotent orbit) as sketched in figure 2. The IR theory consists of free hypers and an interacting theory which is described by 6d (2, 0) theory on a sphere with one irregular and one regular puncture labeled by the nilpotent element $\tilde{f}$. We would like to choose the right $\tilde{f}$ such that $\overline{\mathcal{O}}_{\tilde{f}}$ gives the Higgs branch of original theory (the associated variety of the non-admissible AKM). To accomplish that, $\tilde{f}$ has to be chosen as the following:

Figure 2. Left: a 4d theory defined by an irregular singularity and a full regular singularity (the nilpotent orbit f is trivial). Right: a 4d theory defined by an irregular singularity and a regular singularity labeled by f . Going to Higgs branch of this theory is equivalent to closing off the regular puncture, i.e. change the type of regular singularity.

1. The number of Coulomb branch operators in the leading order differential of the corresponding SW curve defined by $\tilde{f}$ should be non-negative. Since otherwise, the right configuration in figure 2 consists of free hypermultiplets, and we can not simply count the hyper-Kahler dimension of $\mathcal{O}_{\tilde{f}}$ as the number of free hypermultiplets. We also imposing the constraint that the Coulomb branch parameters in the leading order differential is as small as possible.
2. Once we use above condition to constrain possible $\tilde{f}$, we choose the maximal among them, namely, there is an order among nilpotent orbit, and we choose the maximal one among the nilpotent orbits satisfying condition 1.

Once such a $\tilde{f}$ is found, we conjecture that the Higgs branch (associated variety) of the left theory (non-admissible AKM) of figure 2 is the closure of $\mathcal{O}_{\tilde{f}}$, in other words $\mathcal{O}_{\mathfrak{g},n,u} = \mathcal{O}_{\tilde{f}}$. Once we know the associated variety $\mathcal{O}_{\mathfrak{g},n,u}$, the number of free hypermultiplets in the IR is $\frac{1}{2} \dim_{\mathbb{C}} \mathcal{O}_{\mathfrak{g},n,u}$. However, the contribution to $a_{4d} - c_{4d}$ from hypers alone may not be enough match the $a_{4d} - c_{4d}$ in the UV, resulting possible vector multiplets or interacting piece without Higgs branch in the IR.

Example 1. First we illustrate the idea for the simple case $u > n$, and in this case, we can completely close off the regular puncture, i.e. we can take the regular puncture to be labeled by the principal nilpotent orbit (which is the maximal regular singularity we can choose), and the remaining theory is defined by an irregular singularity which has no Higgs branch. So we claim that the associated variety in this case is given by the closure of principal nilpotent orbit, $\overline{\mathcal{O}}_{\mathfrak{g},n,u} = \overline{\mathcal{O}}_{\text{prin}}$ for $u > n$.

Example 2. Now consider 4d theory whose VOA is $V^{-N+\frac{N}{u}}(\mathfrak{sl}_N)$, here u is coprime with N so that there is no mass parameter in irregular singularity of corresponding $(2,0)$ construction. We consider the case $u < N$. The SW curve takes the form

$$x^N + \sum_{i=2}^N \phi_i(z) x^{N-i} = 0. \quad (6.4)$$

Given a general regular puncture labeled by a Young Tableaux $[h_1^{r_1}, h_2^{r_2}, \dots, h_s^{r_s}]$, the maximal scaling dimension of Coulomb branch operator in differential ϕ_N is

$$N - \frac{N}{u} s_N, \quad (6.5)$$

where s_N is the height of N th box of the Young Tableaux (we label the Young Tableaux row by row starting with the bottom-left corner), which is equal to h_1 . The maximal one we can choose so that it satisfies condition 1 is $h_1 = u$, and the maximal scaling dimension in ϕ_N is then 0. Using the partial ordering of nilpotent orbit of $\mathfrak{sl}_N$ Lie algebra described in [96], we find that the regular puncture $\tilde{f}$ which satisfies condition 2 is $[u, \dots, u, s]$, $s \leq u$, which agrees with results found by Arakawa [20].

Using the condition 1 and 2, we are able to find associated varieties for non-admissible VOAs listed in table 15.

Alternatively, we notice that we can find the associated variety using the isomorphism found in [4]: our theory can be denoted as $((\mathfrak{g}, n, u), \tilde{f})$, and if $\tilde{f}$ can be chosen so that the above theory is isomorphic to another configuration $((\mathfrak{g}', n', u'), f_{\text{prin}})$ which is conjectured to have no Higgs branch, then the associated variety of original theory is equal to the closure of $\mathcal{O}_{\tilde{f}}$. This method can be used to confirm the result from using previous result, and furthermore we can find the resulting lisse W algebra using this method.

Example. Let us consider following example: consider a four dimensional theory whose 2d VOA is (here n is even, and $g = \frac{2N-2}{n}$ is odd)

$$W^{-(2N-2)+\frac{n}{u}}(\mathfrak{so}_{2N}, [\underbrace{gu, \dots, gu}_{\text{even}}, s, 1]), \quad (6.6)$$

and this is the D_N type theory, see table 15. Using the similar argument of [4], we have following equivalence

$$W^{-(2N-2)+\frac{n}{u}}(\mathfrak{so}_{2N}, [\underbrace{gu, \dots, gu}_{\text{even}}, s, 1]) = W^{-(s-1)+\frac{s-1}{u}}(\mathfrak{so}_{s+1}, f_{\text{prin}}) \quad (6.7)$$

The W algebra $W^{-(s-1)+\frac{s-1}{u}}(\mathfrak{so}_{s+1}, f_{\text{prin}})$ is lisse, which confirms our conjecture of the associated variety of AKM VOA listed in table 15.

6.2 Some details of D type theory

For the untwisted D_N type theory, the irregular singularity takes following form

$$\Phi = \frac{T}{z^{2+\frac{k}{n}}} + \dots \quad (6.8)$$

with $(n, k) = 1$, and n is chosen such that there is no mass parameter in the irregular singularity: either a) n is an even divisor of $2N - 2$ and $\frac{2N-2}{n}$ is odd, or b) n is an even divisor of N . T is chosen to be a regular semi-simple element of D type. We first consider adding a full regular singularity of D_N type besides the irregular singularity. These theories

j with twist	VOA	constraint	associated variety
A_{2N}/Z_2	$V^{-(N+1)+\frac{n}{2u}}(\mathfrak{sp}_{2N})$	n odd and $g = \frac{2N+1}{n}$ odd	$[gu+1, \underbrace{gu, \dots, gu}_{\text{even}}, s], 0 \leq s \leq gu-1, s \text{ even}$ $[gu+1, \underbrace{gu, \dots, gu}_{\text{even}}, gu-1, s], 2 \leq s \leq gu-1, s \text{ even}$
	$V^{-(N+1)+\frac{n}{2u}}(\mathfrak{sp}_{2N})$	n odd and $g = \frac{2N}{n}$ even	$[gu, \dots, gu, s], 0 \leq s \leq gu-1, s \text{ even}$
A_{2N-1}/Z_2	$V^{-(2N-1)+\frac{n}{u}}(\mathfrak{so}_{2N+1})$	n odd and $g = \frac{2N-1}{n}$ odd	$[\underbrace{gu, \dots, gu}_{\text{even}}, s], 0 \leq s \leq gu, s \text{ odd}$ $[\underbrace{gu, \dots, gu}_{\text{odd}}, s, 1], 0 \leq s \leq gu-1, s \text{ odd}$
	$V^{-(2N-1)+\frac{n}{u}}(\mathfrak{so}_{2N+1})$	n odd and $g = \frac{2N}{n}$ even	$[gu+1, \underbrace{gu, \dots, gu}_{\text{even}}, s, 1], 0 \leq s \leq gu-1, s \text{ odd}$ $[gu+1, \underbrace{gu, \dots, gu}_{\text{even}}, gu-1, s], 0 \leq s \leq gu-1, s \text{ odd}$ $[gu+1, \underbrace{gu, \dots, gu}_{\text{even}}, gu-1, gu-1, 1, 1]$
D_N/Z_2	$V^{-N+\frac{n}{2u}}(\mathfrak{sp}_{2N-2})$	n even and $g = \frac{2N}{n}$ odd	$[\underbrace{gu, \dots, gu}_{\text{even}}, s], 0 \leq s \leq gu-1, s \text{ even}$ $[\underbrace{gu, \dots, gu}_{\text{even}}, gu-1, s], 0 \leq s \leq gu-1, s \text{ even}$
	$V^{-N+\frac{n}{2u}}(\mathfrak{sp}_{2N-2})$	n even and $g = \frac{2N-2}{n}$ even	$[gu, \dots, gu, s], 0 \leq s \leq gu-1, s \text{ even}$
D_N	$V^{-(2N-2)+\frac{n}{u}}(\mathfrak{so}_{2N})$	n even and $g = \frac{2N-2}{n}$ odd	$[\underbrace{gu, \dots, gu}_{\text{odd}}, s], 0 \leq s \leq gu, s \text{ odd}$ $[\underbrace{gu, \dots, gu}_{\text{even}}, s, 1], 0 \leq s \leq gu-1, s \text{ odd}$
	$V^{-(2N-2)+\frac{n}{u}}(\mathfrak{so}_{2N})$	n even and $g = \frac{2N}{n}$ even	$[\underbrace{gu, \dots, gu}_{\text{even}}, s, 1], 0 \leq s \leq gu-1, s \text{ odd}$ $[\underbrace{gu, \dots, gu}_{\text{even}}, gu-1, s], 0 \leq s < gu-1, s \text{ odd}$ $[\underbrace{gu, \dots, gu}_{\text{even}}, gu-1, gu-1, 1, 1]$

Table 15. The associated variety of non-admissible lisse W-algebra defined using classical Lie algebra. u is coprime with n , and u is always odd. The associated variety is the closure of the nilpotent orbit of Lie algebra used to define W-algebra.

have a weakly coupled gauge theory descriptions [97] with its quiver shown in figure 3. The corresponding 2d VOA is AKM

$$V^{-(2N-2)+\frac{n}{n+k}}(\mathfrak{so}_{2N}). \quad (6.9)$$

The Schur index can be computed from the index of each individual matter system. We illustrate this by using the quiver listed in top row of figure 3. Define $u = n + k$. Each AD matter T_i^u carries flavor symmetry C_i and D_i (Here C means the symplectic group, and D means orthogonal group). The character of the AD matter is given by the following formula [4]

$$\text{ch}(T_i^u) = \text{PE} \left[\frac{(q - q^u) \chi_{C_i}^{\text{adj}} + (q - q^u) \chi_{D_i}^{\text{adj}} + (q^{\frac{u}{2}} - q^{\frac{u}{2}+1}) (\chi_{C_i}^f \chi_{D_i}^{\bar{f}} + \chi_{C_i}^{\bar{f}} \chi_{D_i}^f)}{(1 - q)(1 - q^u)} \right]. \quad (6.10)$$

Figure 3. Quiver for untwisted D_N type theory defined using irregular singularity listed in 6.8, and a full D type regular singularity. Here we have a) $l = \frac{2N}{n} - 1$ b) $l = \frac{2N-2}{n} - 1$; and $u = n + k$.

Here $\chi_{C_i}^{\text{adj}}$ is the character of the adjoint representation of C_i Lie algebra, and $\chi_{C_i}^f$ is the character of the fundamental representation of C_i , and etc. It is interesting to note that when $u = 1$, the above index reduces to the index of the free bi-fundamental hypermultiplets in C_i and D_i group. So our matter T_i^u can be thought of as an infinite sequence of matter generalizing the usual bi-fundamental matter. Now the vector multiplet (the gauge group is $\mathfrak{g}$) would contribute to the index by

$$\text{ch}(C_i) = \text{PE} \left[\frac{-2q\chi_{\mathfrak{g}}^{\text{adj}}}{1-q} \right]. \quad (6.11)$$

Combine the contribution from matters and vector multiplets, we have the following index formula

$$\begin{aligned} \text{ch} = & \text{PE} \left[\frac{(q - q^u)\chi_{\text{so}(2N)}^{\text{adj}}}{(1-q)(1-q^u)} \right] \\ & \times \int \prod_{i=1}^{\frac{l+1}{2}} dC_i \prod_{i=1}^{\frac{l-1}{2}} dD_i \text{PE} \left[\frac{-2q^u (\sum_{i=1}^{\frac{l+1}{2}} (\chi_{C_i}^{\text{adj}} + \chi_{D_i}^{\text{adj}})) + 2q^{\frac{u}{2}} (\sum_{i=1}^{\frac{l+1}{2}} \chi_{C_i}^f \chi_{D_i}^f + \sum_{i=1}^{\frac{l-1}{2}} \chi_{D_i}^f \chi_{C_{i+1}}^f)}{(1-q^u)} \right]. \end{aligned}$$

Notice that the integral does not involve the flavor group $\text{SO}(2N)$ and the integral actually does depend on fugacities of $\text{SO}(2N)$ flavor group. This formula can be generalized to the bottom quiver in figure 3. The rule is quite simple: $u = 1$ is just the quiver gauge theory with free bi-fundamental hypermultiplets as matter, and we write down the index using the free field realization, and to get the index for general case one simply replaces q by q^u in the full index (a prefactor which is zero for $u = 1$ is also needed).

The growth function can be computed using the following two formulas

$$2a_{4d} - c_{4d} = \frac{1}{4} \sum_{[u] > 1} (2[u] - 1), \quad c_{4d} = -\frac{1}{12} \frac{k_{2d} \dim(\mathfrak{so}_{2N})}{h^\vee + k_{2d}}. \quad (6.12)$$

Here we use the fact that the 2d VOA of above theories are AKM with level $k_{2d} = -h^\vee + \frac{n}{u}$, and the relation $c_{4d} = -\frac{1}{12}c_{2d}$. Since we know the Coulomb branch spectrum, we can compute the central charge a_{4d} and therefore compute the growth function of the character $\mathcal{G} = -48(a_{4d} - c_{4d})$. We find that the growth function takes the following form

$$\mathcal{G} = \dim \mathfrak{so}_{2N} - \frac{d_{\mathfrak{so}_{2N}}(n)}{u}. \quad (6.13)$$

with the function $d_{\mathfrak{so}_{2N}}(n)$ given by:

$$\begin{aligned} l = \frac{2N}{n} - 1, \quad l \text{ odd} : \quad & d_{\mathfrak{so}_{2N}}(n) = \frac{l+1}{2}n^2 + (l+1), \\ l = \frac{2N-2}{n} - 1, \quad l \text{ even} : \quad & d_{\mathfrak{so}_{2N}}(n) = \frac{l+1}{2}n^2 + \frac{3n}{2} + (l+1). \end{aligned}$$

Higgs branch when $u = 1$: now we discuss the associated variety (the Higgs branch) when $u = 1$. When $u = 1$ case the theory is just a linear quiver with Lagrangian description and its branch is just the closure of a nilpotent orbit, which can be read from the quiver shown in figure 3.

To compute the dimension of the Higgs branch we can use the associated quiver tail of a D type nilpotent orbit [98]. For a nilpotent orbit with partition $Y = [h_1, h_2, \dots, h_J]$ (where even h_i appears even times), we attach a quiver tail (See appendix for brane construction):

$$[\mathrm{SO}(2N)] - \mathrm{USp}(r_1) - \mathrm{O}(r_2) - \dots - \mathrm{USp}(r_{J-1}) \quad (6.14)$$

Here J is always even (The end USp gauge group might be rank zero.). The sizes are

$$r_a = \left[\sum_{b=a+1}^J h_b \right]_{+,-}, \quad + : \mathrm{O}, \quad - : \mathrm{USp} \quad (6.15)$$

and $[n]_{+(-)}$ means the smallest (largest) even integer $\geq n$ ($\leq n$). The Higgs branch of the above quiver is the closure of the dual orbit Y^d of the original orbit Y . For our quiver shown in figure 3, we find:

$$\begin{aligned} l = \frac{2N}{n} - 1, \quad l \text{ odd} : \quad & Y = [n+1, \underbrace{n, \dots, n}_{l-1}, n-1], \quad Y^d = [\underbrace{l+1, \dots, l+1}_{n-2}, l, l, 1, 1], \\ l = \frac{2N-2}{n} - 1, \quad l \text{ even} : \quad & Y = [n+1, \underbrace{n, \dots, n}_l, 1], \quad Y^d = [\underbrace{l+1, \dots, l+1}_n, 1, 1]. \end{aligned} \quad (6.16)$$

The results are consistent with associated varieties listed in table 15. The dimension of the Higgs branch is the dimension of the nilpotent orbit $\mathcal{O}_{Y^d}$ which can be computed from Y^d

$$\begin{aligned} l = \frac{2N}{n} - 1, \quad l \text{ odd} : \quad & \dim_{\mathbb{C}} \mathcal{O}_{Y^d} = 2N^2 - N - \frac{1}{2}((l+1)n^2 + 4), \\ l = \frac{2N-2}{n} - 1, \quad l \text{ even} : \quad & \dim_{\mathbb{C}} \mathcal{O}_{Y^d} = 2N^2 - N - \frac{1}{2}((l+1)n^2 + 3n + 2). \end{aligned} \quad (6.17)$$

Now we try to match the $a_{4d} - c_{4d}$ anomaly in UV and IR. Since $u = 1$, the UV $a_{4d} - c_{4d}$ anomaly is computed using $\mathcal{G} = -48(a_{4d} - c_{4d})$ and equation (6.13) with u set to 1. The result is

$$\begin{aligned} l = \frac{2N}{n} - 1, \quad l \text{ odd} : \quad & (a_{4d} - c_{4d})(\mathrm{UV}) = -\frac{1}{48} \left(2N^2 - N - \frac{l+1}{2}n^2 - l - 1 \right), \\ l = \frac{2N-2}{n} - 1, \quad l \text{ even} : \quad & (a_{4d} - c_{4d})(\mathrm{UV}) = -\frac{1}{48} \left(2N^2 - N - \frac{l+1}{2}n^2 - \frac{3n}{2} - l - 1 \right). \end{aligned} \quad (6.18)$$

In the IR, we have $\frac{1}{2} \dim_{\mathbb{C}} \mathcal{O}_{Y^d}$ free hypermultiplets, their contribution to $a_{4d} - c_{4d}$ is $-\frac{1}{48} \dim_{\mathbb{C}} \mathcal{O}_{Y^d}$ and using equation (6.17) for the dimension of Higgs branch we have

$$\begin{aligned} l = \frac{2N}{n} - 1, \quad l \text{ odd} : \quad (a_{4d} - c_{4d})(\text{hyper}) &= -\frac{1}{48} \left(2N^2 - N - \frac{1}{2}((l+1)n^2 + 4) \right), \\ l = \frac{2N-2}{n} - 1, \quad l \text{ even} : \quad (a_{4d} - c_{4d})(\text{hyper}) &= -\frac{1}{48} \left(2N^2 - N - \frac{1}{2}((l+1)n^2 + 3n + 2) \right). \end{aligned} \quad (6.19)$$

Subtracting it from the UV anomaly, the difference is not zero but rather

$$\begin{aligned} l = \frac{2N}{n} - 1, \quad l \text{ odd} : \quad (a_{4d} - c_{4d})(UV) - (a_{4d} - c_{4d})(\text{hyper}) &= \frac{1}{48}(l-1), \\ l = \frac{2N-2}{n} - 1, \quad l \text{ even} : \quad (a_{4d} - c_{4d})(UV) - (a_{4d} - c_{4d})(\text{hyper}) &= \frac{1}{48}l. \end{aligned} \quad (6.20)$$

Recall the contribution to $a_{4d} - c_{4d}$ from one vector multiplet is $\frac{1}{24}$, then it is natural to assume that in the IR there are also $\frac{l-1}{2}$ free vector multiplets for l odd, and $\frac{l}{2}$ free vector multiplets for l even.

Schur index for non-admissible case: We can compute the Schur index or the vacuum character of non-admissible W-algebra using the gauge theory description too. We illustrate for the theory with only irregular singularity, which is in the form of 6.8 with the following constraint:

$$n \text{ even}, \quad l = \frac{2N-2}{n} - 1 \text{ even}, \quad u = n + k \text{ odd}. \quad (6.21)$$

We take $u > n$ for simplicity. Now define an integer number $a = \lfloor \frac{2N}{k} \rfloor$ and $b = l - a$. If $a = 0$, the quiver takes the same form as in figure 3, and the only difference is that T_{l+1}^u has no $\text{SO}(2N)$ flavor symmetry (The matter systems and the gauge groups are different though). The index takes following form

$$\begin{aligned} \text{ch} = \text{PE} &\left[\frac{\sum q^{d_i} - q^{u+1}(\sum q^{-d_i})}{(1-q)(1-q^u)} \right] \\ &\times \int \prod_{i=1}^{\frac{l+1}{2}} dC_i \prod_{i=1}^{\frac{l-1}{2}} dD_i \text{PE} \left[\frac{-2q^u(\sum_{i=1}^{\frac{l+1}{2}} (\chi_{C_i}^{\text{adj}} + \chi_{D_i}^{\text{adj}})) + 2q^{\frac{u}{2}}(\sum_{i=1}^{\frac{l-1}{2}} \chi_{C_i}^f \chi_{D_i}^f + \sum_{i=1}^{\frac{l-1}{2}} \chi_{D_i}^f \chi_{C_{i+1}}^f)}{(1-q^u)} \right]. \end{aligned}$$

Here d_i are Casimirs of $\text{SO}(2N)$ group. If $a \neq 0$, the quiver takes a more complicated form:

$$\begin{aligned} T_1^u - \text{Sp}(n-2) - T_2^u - \text{SO}(2n) - \dots - G_0(bn) - T_{b+1}^u - G_1(ak) - \dots \\ - \text{SO}(2k+1) - T_l^u - \text{Sp}(k-1) - T_{l+1}^u - O(1) \end{aligned} \quad (6.22)$$

if a is even, then $G_0(bn) = \text{Sp}(bn-2)$, $G_1(ak) = \text{SO}(ak+1)$; and if a is odd, then $G_0(bn) = \text{SO}(bn)$, $G_1(ak) = \text{Sp}(ak-1)$. Notice that there is half-hypermultiplet coupled with $\text{Sp}(k-1)$ gauge group. The Schur index takes the following form:

$$\begin{aligned} \text{ch} = \text{PE} &\left[\frac{\sum q^{1+j} - q^u(\sum q^{-j})}{(1-q)(1-q^u)} \right] \\ &\times \int \prod_{i=1}^{\frac{l+1}{2}} dC_i \prod_{i=1}^{\frac{l-1}{2}} dD_i \text{PE} \left[\frac{-2q^u(\sum_{i=1}^{\frac{l+1}{2}} (\chi_{C_i}^{\text{adj}} + \chi_{D_i}^{\text{adj}})) + 2q^{\frac{u}{2}}(\sum_{i=1}^{\frac{l-1}{2}} \chi_{C_i}^f \chi_{D_i}^f + \sum_{i=1}^{\frac{l-1}{2}} \chi_{D_i}^f \chi_{C_{i+1}}^f)}{(1-q^u)} \right]. \end{aligned}$$

Here the set of $\{j\} = \{1, 3, 5, \dots, (b+1)n - ak - 3\}$ for a even, and $\{j\} = \{1, 3, 5, \dots, (b+1)n - ak - 2\}$ for a odd.

Isomorphism: if $n = N$ with N even, then $l = 1$, the theory is actually defined by following singularity: $x^2 + y^{N-1} + yz^2 + zw = 0$, which is actually equivalent to (A_1, A_{2N}) theory.

6.3 Other cases

Twisted D_N type theory. For the twisted D_N type theory, the irregular singularity takes the following form

$$\Phi = \frac{T^t}{z^{2+\frac{k}{n}}} + \dots \quad (6.23)$$

with $(n, k) = 1$. Here n is chosen such that there is no mass parameter in the irregular singularity: either $a)$ n is an even divisor of $2N - 2$ and $\frac{2N-2}{n}$ is even, or $b)$ n is a even divisor of $2N$, and $\frac{2N}{n}$ is odd. T^t is chosen to be a regular semi-simple element of C type Lie algebra. These theories also have a weakly coupled gauge theory description, see figure 4 for the theory defined by the above irregular singularity plus a regular singularity labeled by trivial nilpotent orbit.

The Schur index for the quiver gauge theory shown in figure 4 can be computed using the index for each matter content, and the result is

$$\text{ch} = \text{PE} \left[\frac{(q - q^u) \chi_{sp(2N-2)}^{\text{adj}}}{(1 - q)(1 - q^u)} \right] \text{ch}_{n,u=1}(q^u), \quad (6.24)$$

here $\text{ch}_{n,u=1}(q)$ is the Schur index for the theory defined with $u = 1$, which is a Lagrangian theory. For other theories defined by the same irregular singularity and different regular singularity, we could also write down its Schur index using the weakly coupled gauge theory description, and we leave it for the interested reader.

Let us now discuss a little bit of the Higgs branch of the theory shown in figure 4. If $u = 1$, the theory is just a Lagrangian theory. The idea is: We start from a B_{N-1} partition Y_B and find its associated quiver which has C_{N-1} type flavor symmetry (See appendix for the explanation), we get the higgs branch of the associated quiver by finding its dual partition Y_C^d in its Langlands dual C_{N-1} algebra. For the quiver shown in figure 4 with $u = 1$, we have:

$$\begin{aligned} l = \frac{2N}{n} - 1, \quad l \text{ even} : \quad & Y_B = [\underbrace{n, \dots, n}_l, n-1], \quad Y_C^d = [\underbrace{l+1, \dots, l+1}_{n-2}, l, l], \\ l = \frac{2N-2}{n} - 1, \quad l \text{ odd} : \quad & Y_B = [\underbrace{n, \dots, n}_{l+1}, 1], \quad Y_C^d = [\underbrace{l+1, \dots, l+1}_n]. \end{aligned} \quad (6.25)$$

This agrees with the result shown in table 15. The dimension of the Higgs branch of the quiver in 4 with $u = 1$ is

$$\begin{aligned} l = \frac{2N}{n} - 1, \quad l \text{ even} : \quad & \dim \mathcal{O}_{Y_C^d} = 2N^2 + N - \frac{1}{2}((l+1)n^2 - 3n + 2), \\ l = \frac{2N-2}{n} - 1, \quad l \text{ odd} : \quad & \dim \mathcal{O}_{Y_C^d} = 2N^2 + N - \frac{1}{2}(l+1)n^2. \end{aligned} \quad (6.26)$$

Comparing with the UV growth, we find that in the Higgs branch there are $\frac{l}{2}$ free vector multiplets for l even, and $\frac{l+1}{2}$ free vector multiplets for l odd.

Twisted $\mathfrak{sl}_{2N}$ type theory. For the twisted $\mathfrak{sl}_{2N}$ theory, the Higgs field takes following form

$$\Phi = \frac{T^t}{z^{2+\frac{k+1/2}{n}}} + \dots \quad (6.27)$$

Here n can be chosen as follows: either *a*) n is a odd divisor of $2N$, or *b*) n is a divisor of $2N - 1$. The corresponding quiver description is shown in figure 5. T^t is a regular semi-simple element of C type Lie algebra.

To associate a quiver tail whose Higgs branch is B_N type nilpotent orbit (so the Higgs branch has B_N type flavor symmetry), we actually need to start with a nilpotent orbit of C_N type with partition $Y = [h_1, h_2, \dots, h_J]$ (where odd h_i appears even times), then attach the following quiver tail

$$[\mathrm{SO}(2N+1)] - \mathrm{USp}(r_1) - \mathrm{O}(r_2) - \dots - \mathrm{USp}(r_{J-1}) \quad (6.28)$$

Here J is always even (The end USp gauge group might be rank zero). The sizes are

$$r_a = [1 + \sum_{b=a+1}^J h_b]_{+,-}, \quad + : \mathrm{O}, \quad - : \mathrm{USp} \quad (6.29)$$

and $[n]_+$ means the smallest odd integer $\geq n$, and $[n]_-$ is the largest even integer $\leq n$. The Higgs branch of above quiver is the closure of Langlands dual Y_B^d of nilpotent orbit Y_C . For the quiver shown in figure 5, we have:

$$\begin{aligned} l = \frac{2N}{n} - 1, \quad l \text{ odd} : \quad & Y_{C_N} = [n+1, \underbrace{n, \dots, n}_{l-1}, n-1], \\ & Y_{B_N}^d = [l+2, \underbrace{l+1, \dots, l+1}_{n-3}, l, l, 1, 1], \\ l = \frac{2N-1}{n} - 1, \quad l \text{ even} : \quad & Y_{C_N} = [n+1, \underbrace{n, \dots, n}_l], \\ & Y_{B_N}^d = [\underbrace{l+1, \dots, l+1}_n, 1, 1]. \end{aligned} \quad (6.30)$$

The dimension is

$$\begin{aligned} l = \frac{2N}{n} - 1, \quad l \text{ odd} : \quad & \dim \mathcal{O}_{Y_B^d} = 2N^2 + N - \frac{1}{2}((l+1)n^2 + 4), \\ l = \frac{2N-1}{n} - 1, \quad l \text{ even} : \quad & \dim \mathcal{O}_{Y_B^d} = 2N^2 + N - \frac{1}{2}((l+1)n^2 + 3n + 2). \end{aligned} \quad (6.31)$$

To match the UV growth, the gauge group is not completely Higgsed, and there are $\frac{l+1}{2}$ free vector multiplet for l odd, and $\frac{l}{2}$ free vector multiplet for l even.

Figure 4. Quiver for twisted D_N type theory defined using formula (6.23). Here we have a) $l = \frac{2N}{n} - 1$, b) $l = \frac{2N-2}{n} - 1$, and $u = n + k$.

Figure 5. Quiver for twisted $\mathfrak{sl}_{2N}$ type theory defined using formula (6.27). a) $l = \frac{2N}{n} - 1$ b) $l = \frac{2N-1}{n} - 1$, here $u = 2n + 2k + 1$. When $u = 1$ and l odd, T_1^u is not trivial, but a half hyper.

Figure 6. Quiver for twisted $\mathfrak{sl}_{2N+1}$ type theory defined using formula (6.32). a) $l = \frac{2N}{n} - 1$ b) $l = \frac{2N+1}{n} - 1$, here $u = 2n + 2k + 1$. For l even and $u = 1$, T_1^u is a half hyper for the first gauge group.

Twisted $\mathfrak{sl}_{2N+1}$ type theory. For the twisted $\mathfrak{sl}_{2N+1}$ theory, the Higgs field takes following form

$$\Phi = \frac{T^t}{z^{2+\frac{k+1/2}{n}}} + \dots \quad (6.32)$$

Here n can be chosen as follows: either a) n is an odd divisor of $2N$, or b) n is a divisor of $2N + 1$. The corresponding quiver is shown in figure 6. We can not find the Higgs branch for $u = 1$ theory using previous methods as the quiver can not be realized as the quiver tail of a B type partition, as the quiver constructed from B type partition would have D type and C type quiver nodes, but our quiver shown in figure 6 has B type and C type quiver nodes.

$\mathfrak{g}$	n	VOA
$\mathfrak{e}_6$	12, 9, 6, 3	$W^{-12+\frac{n}{u}}(\mathfrak{e}_6, f_{\text{prin}})$
$\mathfrak{e}_7$	18, 14, 6, 2	$W^{-18+\frac{n}{u}}(\mathfrak{e}_7, f_{\text{prin}})$
$\mathfrak{e}_8$	30, 24, 20, 15, 12, 10, 8, 6, 5, 4, 3, 2	$W^{-30+\frac{n}{u}}(\mathfrak{e}_8, f_{\text{prin}})$
$\mathfrak{g}_2$	4, 2, 1	$W^{-4+\frac{n}{u}}(\mathfrak{g}_2, f_{\text{prin}})$
$\mathfrak{f}_4$	9, 6, 4, 3, 2, 1	$W^{-9+\frac{n}{u}}(\mathfrak{f}_4, f_{\text{prin}})$

Table 16. Lisse W-algebras defined using exceptional Lie algebras. Here u is coprime with n .

7 Non-admissible lisse W-algebra: exceptional Lie algebra

In this section, we are going to classify non-admissible W-algebra whose corresponding four dimensional $\mathcal{N} = 2$ SCFT is engineered from six dimensional $(2, 0)$ theory of exceptional type. The classification of principal lisse W-algebra is equivalent to the classification of E type irregular singularity which do not admit any mass deformation (the regular singularity is given by the principal nilpotent orbit), see tables 5 and 6 for the classification.

The classification of lisse W-algebra from 4d theory engineered using non-trivial regular singularities is more difficult, as little is known in the literature about the associated variety of non-admissible W-algebra with exceptional Lie algebras. However, motivated by considerations of four dimensional physics, we develop a simple method to determine the associated variety of non-admissible AKM, and complete the full classification of those 4d theories whose 2d VOAs are lisse W-algebras.

7.1 Principal lisse W-algebra

First we consider theories engineered using the irregular singularity only. The lisse condition is equivalent to the absence of mass parameter of the theory, and the constraint on irregular singularity is listed in 5 and 6. The corresponding lisse W algebra is listed in table 16. Some of the physical information of these 4d theories are:

1. The W-algebra takes the following form

$$W^{-h^\vee+\frac{n}{u}}(\mathfrak{g}, f_{\text{prin}}) \quad (7.1)$$

with $h^\vee$ being the dual Coxeter number. The allowed value of n is listed in table 16.

2. The 4d theory has no Higgs branch, and the Coulomb branch spectrum can be easily found from the mini-versal deformation of the corresponding 3-fold singularity.
3. We do not know the character of these W-algebras yet, but the growth of the character is given as $\mathcal{G} = -48(a_{4d} - c_{4d})$, which can be calculated using Coulomb branch data, and it takes the following form:

$$\mathcal{G} = \text{rank}(\mathfrak{g}) - \frac{d_{\mathfrak{g}}(n)}{u}, \quad (7.2)$$

Here $\text{rank}(\mathfrak{g})$ is the rank of the lie algebra $\mathfrak{g}$, and $d_{\mathfrak{g}}(n)$ is a constant depending on $\mathfrak{g}$ and n only which are summarized in table 17.

$\mathfrak{g}$	$d_{\mathfrak{g}}(n)$
$\mathfrak{e}_6$	$d(12) = 78, d(9) = 56, d(6) = 36, d(3) = 24$
$\mathfrak{e}_7$	$d(18) = 133, d(14) = 99, d(6) = 39, d(2) = 21$
$\mathfrak{e}_8$	$d(30) = 248, d(24) = 190, d(20) = 156, d(15) = 112, d(12) = 92, d(10) = 72$ $d(8) = 66, d(6) = 40, d(5) = 48, d(4) = 36, d(3) = 32, d(2) = 24$
$\mathfrak{g}_2$	$d(4) = 14, d(2) = 12(u = 3k), d(2) = 4(u \neq 3k), d(1) = 8$
$\mathfrak{f}_4$	$d(9) = 52, d(6) = 30, d(3) = 24(u = 3k - 1), d(3) = 12(u = 3k + 1), d(2) = 18$

Table 17. Data used to compute the growth function $\mathcal{G}$ of VOA $W^{-h^\vee + \frac{n}{u}}(\mathfrak{g}, f_{\text{prin}})$, where $\mathcal{G} = \text{rank}(\mathfrak{g}) - \frac{d_{\mathfrak{g}}(n)}{u}$.

Since there is very little study on non-admissible lisse W-algebras, it is definitely good to know some isomorphism between non-admissible and known lisse W-algebras. We can use the isomorphism between 3-fold singularities to infer isomorphisms between W-algebra. The idea is that a single 3d singularity which defines our 4d theory can be written in different ways, which could have different 6d (2,0) realization. One can read different W-algebra from these different 6d realizations, and therefore we can find non-trivial isomorphism between W-algebras. Some examples are shown in table 18.

Example. Consider a three-fold singularity $x_1^2 + x_2^3 + x_3^4 + z^2 = 0$. This singularity can be written in three different ways:

$$\begin{aligned}
 \text{I: } & (x_1^2 + z^2 + x_2^3) + x_3^4 = 0, \\
 \text{II: } & (x_1^2 + z^2 + x_3^4) + x_2^3 = 0, \\
 \text{III: } & (x_1^2 + x_2^3 + x_3^4) + z^2 = 0.
 \end{aligned} \tag{7.3}$$

The first one can be realized by the 6d A_2 (2,0) theory, the second by the 6d A_3 (2,0) theory, and the third one by 6d E_6 (2,0). We then get the following isomorphisms among W-algebras

$$W^{-3+\frac{3}{7}}(\mathfrak{sl}_3, f_{\text{prin}}) = W^{-4+\frac{4}{7}}(\mathfrak{sl}_4, f_{\text{prin}}) = W^{-12+\frac{6}{7}}(\mathfrak{e}_6, f_{\text{prin}}) \tag{7.4}$$

7.2 Adding regular singularity

Now consider 4d theories engineered using an irregular singularity used in the last subsection and a general regular singularity labeled by a nilpotent orbit f of $\mathfrak{g}$. We would like to find f such that the corresponding W-algebra $W^{-h^\vee + \frac{n}{u}}(\mathfrak{g}, f)$ is lisse. To find such f , we follow the strategy which is similar to what was done in the admissible case:

- First consider the theory constructed from using a trivial nilpotent orbit f , so that the 4d theory has a flavor symmetry group G . The corresponding VOA is the non-admissible AKM $V^{-h^\vee + \frac{n}{u}}(\mathfrak{g})$, and we *assume* that its associated variety $\overline{\mathcal{O}}_{\mathfrak{g},n,u}$ is irreducible (see the discussion in [99]) and is the closure of a nilpotent orbit $\mathcal{O}_{\mathfrak{g},n,u}$.

Singularity	Isomorphism	W-algebras
$x_1^2 + x_2^3 + x_3^4 + z^2 = 0$	$(A_2, A_3) = (E_6, A_1)$	$W^{-3+\frac{3}{7}}(\mathfrak{sl}_3, f_{\text{prin}}) = W^{-12+\frac{8}{7}}(\mathfrak{e}_6, f_{\text{prin}})$
$x_1^2 + x_2^3 + x_3^4 + z^5 = 0$	$(E_6, A_4) = (E_8, A_3)$	$W^{-12+\frac{12}{17}}(\mathfrak{e}_6, f_{\text{prin}}) = W^{-30+\frac{15}{17}}(\mathfrak{e}_8, f_{\text{prin}})$
$x_1^2 + x_2^3 + x_3^4 + x_3 z^2 = 0$	$(D_5, A_2) = (E_6, b=9, k=2)$	$W^{-8+\frac{8}{11}}(\mathfrak{so}_{10}, f_{\text{prin}}) = W^{-12+\frac{9}{11}}(\mathfrak{e}_6, f_{\text{prin}})$
$x_1^2 + x_2^3 + x_2 x_3^3 + z^5 = 0$	$(E_7, A_4) = (E_8, b=20, k=3)$	$W^{-23+\frac{18}{23}}(\mathfrak{e}_7, f_{\text{prin}}) = W^{-30+\frac{20}{23}}(\mathfrak{e}_8, f_{\text{prin}})$
$x_1^2 + x_2^3 + x_3^5 + z^2 = 0$	$(A_2, A_4) = (E_8, A_1)$	$W^{-3+\frac{3}{8}}(\mathfrak{sl}_3, f_{\text{prin}}) = W^{-30+\frac{15}{16}}(\mathfrak{e}_8, f_{\text{prin}})$
$x_1^2 + x_2^3 + x_3^5 + z^3 = 0$	$(D_4, A_4) = (E_8, A_2)$	$W^{-6+\frac{6}{11}}(\mathfrak{so}_8, f_{\text{prin}}) = W^{-30+\frac{10}{11}}(\mathfrak{e}_8, f_{\text{prin}})$
$x_1^2 + x_2^3 + x_3^5 + x_3 z^2 = 0$	$(D_6, A_2) = (E_8, b=24, k=2)$	$W^{-10+\frac{10}{13}}(\mathfrak{so}_{12}, f_{\text{prin}}) = W^{-30+\frac{12}{13}}(\mathfrak{e}_6, f_{\text{prin}})$

Table 18. Here we first wrote the equivalence of the 3-fold singularity in the notation of (G, G') , where G denotes the corresponding ADE singularity of type G , then we listed the equivalence between W-algebras.

- Now for the W-algebra $W^{-h^\vee+\frac{n}{u}}(\mathfrak{g}, f)$, its associated variety is

$$S_f \cap \overline{\mathcal{O}}_{\mathfrak{g},n,u} \quad (7.5)$$

so if we choose $f \in \mathcal{O}_{\mathfrak{g},n,u}$, the above associated variety would be trivial, and the corresponding W-algebra is lisse.

So the crucial part of finding f is to determine the associated variety of non-admissible AKM, which has not been studied much yet. It might be possible to directly compute it using the method developed in [20], here we use a method which is motivated from four dimensional physics.

To find the associated variety of a non-admissible AKM from using the data of four dimensional $\mathcal{N} = 2$ SCFT, we need to use following facts:

- First we need to use *anomaly matching*. The difference of central charges $a_{4d} - c_{4d}$ can be computed from the Coulomb branch data [100, 101], and is related to the growth function of W-algebra vacuum character as follows

$$\mathcal{G} = -48(a_{4d} - c_{4d}). \quad (7.6)$$

a_{4d} and c_{4d} can be explicitly calculated using the method developed in [16, 17], and it takes the following simple form

$$\mathcal{G}_{\text{UV}}(V^{-h^\vee+\frac{n}{u}}(\mathfrak{g})) = \dim(\mathfrak{g}) - \frac{d_{\mathfrak{g}}(n)}{u} \quad (7.7)$$

Here the data $d_{\mathfrak{g}}(n)$ is given in table 17.

Now when we turn on expectation values of Higgs branch operators $\hat{\mathcal{B}}_R$, the IR theory could consist of n_h free hypermultiplets, n_v free vector multiplets and an interacting theory T_{IR} which does not have a Higgs branch. To match the anomaly [102], we have the following formula:

$$\mathcal{G}_{\text{UV}} = 2n_h - 2n_v + \mathcal{G}_{\text{IR}}. \quad (7.8)$$

Here $\mathcal{G}_{\text{IR}}$ is the growth function of the IR theory $\mathcal{T}_{\text{IR}}$. Since the Higgs branch is assumed to be the closure of a nilpotent orbit $\mathcal{O}_{\mathfrak{g},n,u}$, $2n_h$ has to equal to the dimension of $\mathcal{O}_{\mathfrak{g},n,u}$, so we have

$$\dim(\mathcal{O}_{\mathfrak{g},n,u}) = \mathcal{G}_{\text{UV}} - \mathcal{G}_{\text{IR}} + 2n_v. \quad (7.9)$$

One simple consequence of formula (7.9) is that if $\mathcal{G}_{\text{UV}}$ is fractional, then $\mathcal{T}_{\text{IR}}$ can not be trivial, as the contribution of free hypermultiplet and free vector multiplet has to be an integer. The theory $\mathcal{T}_{\text{IR}}$ is the four dimensional theory whose Higgs branch is trivial and therefore whose 2d associated W-algebra is lisse.

To determine the dimension of associated variety, we need to have some information of n_v and $\mathcal{G}_{\text{IR}}$ in the IR. Furthermore, even if we determine the dimension of $\mathcal{O}_{\mathfrak{g},n,u}$, it is still possible that there are more than one nilpotent orbit with same dimension. We need to use more four dimensional input.

- The $\mathcal{T}_{\text{IR}}$ in the Higgs branch can actually be identified as follows: it is just the theory defined by adding a regular singularity f_{IR} because of the fact that going to the Higgs branch of a class S theory is equivalent to closing the puncture. We determine f_{IR} using the Coulomb branch data:
 1. If the UV growth $\mathcal{G}_{\text{UV}}$ is fractional, then the IR interacting theory $\mathcal{T}_{\text{IR}}$ has to be non-trivial and has Coulomb branch operator(s). We choose maximal f_{IR} (using the order of nilpotent orbit) such that there are non-trivial Coulomb branch operators. To ensure that there are no free hypermultiplets for the theory defined using f_{IR} , we need to ensure that the leading order differential of the SW curve of corresponding 6d configuration has non-negative Coulomb branch operators.
 2. If the UV growth $\mathcal{G}_{\text{UV}}$ is integral, we can still find the maximal f_{IR} such that there are non-trivial Coulomb branch operators. However, there is a further possibility that the interacting theory is trivial, and we actually need to consider larger f_{IR} , and there is no simple way to determine f due to the possibility of existence of free vector multiplets. We will then need to use Coulomb branch data to further constrain f_{IR} .

In practice, we can first try f_{IR} whose dimension is just below the growth function, and gradually increase f_{IR} . Our assumption is that the orbit of f_{IR} which gives $\mathcal{T}_{\text{IR}}$ is $\mathcal{O}_{\mathfrak{g},n,u}$

$$\mathcal{O}_{f_{\text{IR}}} = \mathcal{O}_{\mathfrak{g},n,u}. \quad (7.10)$$

- We cannot completely determine f_{IR} and n_v using a single equation (7.9), so we need another equation. There is one further constraint from the Coulomb branch: The dimension of Hitchin moduli space $d_{f_{\text{IR}}}$ should be equal to the number of free vector multiplets and the rank of $\mathcal{T}_{\text{IR}}$

$$d_{f_{\text{IR}}} = r_{\text{IR}} + n_v. \quad (7.11)$$

On the other hand, the dimension of the Hitchin moduli space defined using the regular singularity f is

$$d_{f_{\text{IR}}} = r_{\text{UV}} + \frac{1}{2} \dim(\mathcal{O}_{f_{\text{IR}}^d}) - \frac{1}{2} (\dim(\mathfrak{g}) - \text{rank}(\mathfrak{g})), \quad (7.12)$$

where r_{UV} is the Coulomb branch dimension of UV theory, and f_{IR}^d is the dual Hitchin label for the puncture. Hence we have the constraint

$$r_{\text{IR}} + n_v = r_{\text{UV}} + \frac{1}{2} \dim(\mathcal{O}_{f_{\text{IR}}^d}) - \frac{1}{2} (\dim(\mathfrak{g}) - \text{rank}(\mathfrak{g})). \quad (7.13)$$

In practice, using equation (7.9) and (7.13), one can determine the associated variety of AKM with exceptional Lie algebra as in tables 19, 20, and 22. Once we determine the associated variety $\mathcal{O}_{\mathfrak{g},n,u}$ of AKM $V^{-h^\vee + \frac{n}{u}}(\mathfrak{g})$, we can find $f \in \mathcal{O}_{\mathfrak{g},n,u}$ such that $W^{-h^\vee + \frac{n}{u}}(\mathfrak{g}, f)$ is lisse. Full results are listed in tables 21 and 23.

Example 1. Consider the theory $\mathcal{T}[V^{-12+\frac{9}{8}}(\mathfrak{e}_6)]$ engineered by an irregular singularity of the type $\mathfrak{g} = \mathfrak{e}_6, n = 9, u = 8$, and a regular singularity labeled by trivial nilpotent orbit. The growth function is $\mathcal{G}_{\text{UV}} = 78 - 56/8 = 71$, we can try $f_{\text{IR}} = E_6(a_1)$ whose dimension is 70 which is just below the growth function, and the theory has Coulomb branch spectrum $\frac{15}{8}, \frac{9}{8}, \frac{5}{4}, \frac{3}{2}$, which can actually be engineered by the singularity $F_{\text{IR}} : x^2 + y^2 + z^3 + w^5 = 0$. Theory with f_{IR} has no Higgs branch, so we conclude that the associated variety of the $V^{-12+9/8}(\mathfrak{e}_6)$ is just $E_6(a_1)$. The growth function of IR interacting theory is $\mathcal{G}_{\text{IR}} = 1$, so using formula 7.9, we conclude that $n_v = 0$, and there is no free vector multiplets in the generic point of Higgs branch. One can also check that formula (7.13) is also satisfied: we have $r_{\text{UV}} = 29$, and $r_{\text{IR}} = 4$, and the dual of $E_6(a_1)$ is A_1 orbit, whose dimension is 22.

Example 2. Starting from the theory $\mathcal{T}[V^{-12+\frac{3}{2}}(\mathfrak{e}_6)]$ engineered by an irregular singularity of the type $\mathfrak{g} = \mathfrak{e}_6, n = 3, u = 2$, and a regular singularity labeled by trivial nilpotent orbit. The growth function of $\mathcal{T}[V^{-12+\frac{3}{2}}(\mathfrak{e}_6)]$ is $\mathcal{G} = 78 - 24/2 = 66$. Changing the regular singularity to $f_1 = E_6(a_3)$ whose dimension is equal to 66 leads to another theory $\mathcal{T}[W^{-12+\frac{3}{2}}(\mathfrak{e}_6, f_1)]$. The Coulomb branch spectrum of the IR theory $\mathcal{T}_{\text{IR}} = \mathcal{T}[W^{-12+\frac{3}{2}}(\mathfrak{e}_6, f_1)]$ is $\{3, 2, (\frac{3}{2})^4\}$, indicating that $\mathcal{T}_{\text{IR}}$ also has a gauge theory description which is just an $\text{SU}(3)$ vector multiplet coupled with four copies of $D_2(\text{SU}(3))$ theory. This theory can also be engineered by a complete intersection singularity defined with two polynomials. The complete intersection singularity is not terminal, and has a crepant resolution, so it has a Higgs branch. In fact, the Higgs branch of the theory $\mathcal{T}_{\text{IR}}$ has $n_h = 1, n_v = 1$, and no further interacting theory. This implies that

$$\dim(\mathcal{O}_{\mathfrak{e}_6,3,2}) = \mathcal{G}_{\text{UV}} - \mathcal{G}_{\text{IR}} + 2n_v = 68, \quad (7.14)$$

so the associated variety $\overline{\mathcal{O}}_{\mathfrak{e}_6,3,2}$ is just closure of nilpotent orbit with the label D_5 , as this is the unique E_6 nilpotent orbit whose dimension is 68. The Higgs branch of the original theory therefore has 34 free hypermultiplets, and one free vector multiplet.

Example 3. Now consider a more complicated example. Consider the theory $\mathcal{T}[V^{-30+\frac{5}{3}}(\mathfrak{e}_8)]$ engineered by an irregular singularity of the type $\mathfrak{g} = \mathfrak{e}_8, n = 5, u = 3$, we also have the regular singularity with label f trivial. The growth of this theory is $\mathcal{G} = 248 - \frac{48}{3} = 232$, which is an integer and we are not sure whether there would be an interacting piece in the Higgs branch. To find the effective theory on the Higgs branch, we first choose $f_1 = E_8(a_4)$ whose dimension is 232, and the theory $\mathcal{T}_{\text{IR}} = \mathcal{T}[W^{-20+\frac{5}{3}}(\mathfrak{e}_8, E_8(a_4))]$ has Coulomb branch spectrum $\{5, 4, (\frac{10}{3})^3, 3, (\frac{7}{3})^3, 2, (\frac{5}{3})^4, (\frac{4}{3})^3\}$. It is interesting that this theory has the same Coulomb branch spectrum as the theory engineered using three-fold singularity $x^3 + y^3 + z^3 + w^5 = 0$, which actually has a Higgs branch whose IR theory consists of $n_h = 1, n_v = 1$, together with an interacting theory engineered by the singularity $F_{\text{IR}} : x^3 + y^3 + z^3 + w^2 = 0$. The theory engineered by F_{IR} has no Higgs branch, so there is no further Higgs branch deformation. Using the formula (7.9), we see that (We use the fact $\mathcal{G}_{\text{IR}} = 0$):

$$\dim(\mathcal{O}_{\mathfrak{e}_8,5,3}) = \mathcal{G}_{\text{UV}} - \mathcal{G}_{\text{IR}} + 2n_v = 234, \quad (7.15)$$

and there is a unique nilpotent orbit $E_8(a_3)$ whose dimension is equal to 234, so the associated variety $\overline{\mathcal{O}}_{\mathfrak{e}_8,5,3}$ is the closure of $E_8(a_3)$ orbit. The IR theory on the Higgs branch consists of free hypermultiplets, free vector multiplet, and an interacting theory.

Example 4. For $\mathcal{T}[V^{-9}(\mathfrak{e}_6)]$ with $\mathfrak{e}_6, n = 3, u = 1$, we have $\mathcal{G}_{\text{UV}} = 54$, so we first take $f_1 = 2A_2 + A_1$ whose dimension is 54, the Coulomb branch spectrum of the theory $\mathcal{T}[W^{-9}(\mathfrak{e}_6, 2A_2 + A_1)]$ is $\{2, 3\}$. To match the $a - c$ anomaly, the theory should be $\mathcal{N} = 4$ SU(3) SYM. We can further Higgs $\mathcal{N} = 4$ SYM, and in the IR, there are $n_v = 2$ and $n_h = 2$, and no interacting theory. So the associated variety of original theory should have dimension 58, and there is a unique E_6 nilpotent orbit with this dimension $D_4(a_1)$, and we conclude that this is the associated variety of the original theory.

Example 5. Now we give a more detailed explanation on the associated variety of $V^{-27}(\mathfrak{e}_8)$ with $\mathfrak{g} = E_8, n = 3, u = 1$ and $\mathcal{G}_{\text{UV}} = 216$. We first take $f_1 = E_6(a_1)$, and the Coulomb branch spectrum of $\mathcal{T}_{\text{IR}} = \mathcal{T}[W^{-27}(\mathfrak{e}_8, E_6(a_1))]$ is $\{2^4, 3^7\}$. $\mathcal{T}_{\text{IR}}$ is actually isomorphic to $A_2(2,0)$ theory on a genus two Riemann surface with one regular puncture. It is known that one can further Higgs this theory, and the Higgs branch is not a pure Higgs branch, and we have $n_v = 4$ with no interacting part left. Using formula (7.9), we can see that the dimension of the associated variety of original theory should be 224, and there are two choices for it: $E_8(a_6)$ and $E_7(a_2)$. Using formula (7.13), we can conclude the associated variety of $V^{-27}(\mathfrak{e}_8)$ should be $E_8(a_6)$.

7.3 A proposal for the character

Now we discuss the character of lisse W-algebra with exceptional Lie algebra. We first consider the case when the regular puncture has full flavor symmetry group (the nilpotent orbit f is trivial). The corresponding VOA is the affine Kac-Moody algebra $V^k(\mathfrak{g})$, with the level k takes the following form

$$k = -h^\vee + \frac{n}{u}, \quad (7.16)$$

G_2	u	$\overline{\mathcal{O}}_u$	F_4	u	$\overline{\mathcal{O}}_u$
$n = 4$	> 6	G_2	$n = 9$	> 12	F_4
	5	$G_2(a_1)$		11	$F_4(a_1)$
$n = 2$	> 4	G_2		7	$F_4(a_2)$
	3	$G_2(a_1)$		5	$F_4(a_3)$
	1^*	$G_2(a_1)$	$n = 6$	> 6	F_4
$n = 1$	> 2	G_2		5^*	$F_4(a_1)$
	$2^\#$ ($n_v = 1$)	G_2		1^*	$\tilde{A}_1$
	$1^\#$ ($n_v = 2$)	$G_2(a_1)$	$n = 3$	> 3	F_4
				$2^\#$ ($n_v = 1$)	B_3
				1^*	$F_4(a_3)$
			$n = 2$	> 2	F_4
				$1^\#$ ($n_v = 3$)	$F_4(a_3)$

Table 19. Associated variety for G_2 and F_4 type theory. The rank of Coulomb branch of G_2 type theory: $r = \frac{4}{n}u - 1$. The rank of Coulomb branch of F_4 type theory: $r = \frac{18u}{n} - 2$. The one with $\#$ implies that the IR theory has free vector multiplets in the Higgs branch, and n_v is the number of free vector multiplets. The one with $*$ means that the theory has pure Higgs branch, i.e. the IR theory on the Higgs branch consists of free hypermultiplets only.

Here $(n, u) = 1$ and n is chosen such that there is no mass parameter encoded in irregular singularity. In principle, the vacuum character can be computed using the Kazhdan-Lusztig formula, however, the computation is quite involved. Here we propose a much simpler formula inspired by physics and some observations on known formula found in the classical group case.

The formula we are going to propose for $V^{-h^\vee + \frac{n}{u}}(\mathfrak{g})$ is

$$\text{ch}_{\mathfrak{g}, n, u}(q) = \text{PE} \left[\frac{(q - q^u) \chi_{\mathfrak{g}}^{\text{adj}}}{(1 - q)(1 - q^u)} \right] \chi_{\mathfrak{g}, n, u=1}(q^u), \quad (7.17)$$

where $\chi_{\mathfrak{g}, n, u=1}(q)$ is the character for the theory defined at $u = 1$. This formula is confirmed for the classical group case by using the weakly coupled gauge theory description, and we propose that this is true for arbitrary case. This also works for the case $n = h^\vee$ as the $u = 1$ theory is trivial so the index is just the PE term which recovers the general formula found in [5].

If this general proposal is correct, the computation of index of AKM is reduced to find the index for the theory for $u = 1$, which is known for many cases. Firstly, let us set up some notations. For a semi-simple Lie algebra $\mathfrak{g}$ with rank r , define

$$\eta^{\mathfrak{g}}(\mathbf{a}) = \prod_{w \in \text{adj } \mathfrak{g}} (\mathbf{a}^w q; q), \quad (7.18)$$

with $\mathbf{a}^w = \prod_{i=1}^r a_i^{w_i}$. The character, or Schur polynomial, of the representation λ of $\mathfrak{g}$ is denoted as $\chi_{\lambda}^{\mathfrak{g}}(\mathbf{a})$ while the q -dimension of the representation λ is just $\chi_{\lambda}^{\mathfrak{g}}(q^\rho)$ with ρ being the Weyl vector (half sum of positive roots) of $\mathfrak{g}$.

E_6	u	$\overline{\mathcal{O}}_u$	E_7	u	$\overline{\mathcal{O}}_u$
$n = 12$	> 12	E_6	$n = 18$	> 18	E_7
	11	$E_6(a_1)$		17	$E_7(a_1)$
	7	$E_6(a_3)$		13	$E_7(a_2)$
	5	$A_4 + A_1$		11	$E_7(a_3)$
$n = 9$	> 9	E_6		7^*	A_6
	$7^*, 8$	$E_6(a_1)$		5	$A_4 + A_2$
	5	$E_6(a_3)$	$n = 14$	> 14	E_7
	4^*	$D_5(a_1)$		$11^*, 13$	$E_7(a_1)$
	2^*	$A_2 + 2A_1$		9^*	$E_7(a_2)$
	1^*	A_1		5	$E_7(a_5)$
$n = 6$	> 6	E_6		3^*	$A_3 + A_2 + A_1$
	5	$E_6(a_1)$		1^*	A_1
	1^*	A_2	$n = 6$	> 6	E_7
$n = 3$	> 3	E_6		5	$E_7(a_1)$
	$2^\#$ ($n_v = 1$)	D_5		1^*	$D_4(a_1)$
	$1^\#$ ($n_v = 2$)	$D_4(a_1)$	$n = 2$	> 2	E_7
				$1^\#$ ($n_v = 3$)	$E_6(a_1)$

Table 20. Associated variety for E_6 and E_7 type AKM $V^{-h^\vee + \frac{n}{u}}(\mathfrak{g})$. The one with star implies that theory has a pure Higgs branch. The one with # implies that the IR theory has free vector multiplets in the Higgs branch, and n_v is the number of free vector multiplets. The one with * means that the theory has pure Higgs branch, i.e. the IR theory on the Higgs branch consists of free hypermultiplets only. The IR theory in the Higgs branch of other theories has an interacting piece, which is listed in table 21. The rank of Coulomb branch of E_6 type theory: $r_{UV} = \frac{36}{n}u - 3$. The rank of Coulomb branch of E_7 type theory: $r_{UV} = \frac{1}{2}(\frac{126u}{n} - 7)$.

(f, n, u)	Coulomb branch	Equiv. 4d Description	Isomorphic W-algebra
$(G_2(a_1), 2, 3)$	$(2, (\frac{4}{3})^3)$	$x^3 + y^3 + z^3 + 2^2 = 0$	$W^{-6 + \frac{2}{3}}(\mathfrak{so}_8, f_{\text{prin}})$
$(E_6(a_1), 9, 8)$	$\frac{15}{8}, \frac{9}{8}, \frac{5}{4}, \frac{3}{2}$	(A_2, A_4)	
$(E_6(a_3), 9, 5)$	$(\frac{6}{5})^2$	$(A_1, A_2)^2$	$\text{Vir}_{2,5}^{\otimes 2}$
$(E_6(a_1), 6, 5)$	$(\frac{6}{5})^2$	$(A_1, A_2)^2$	$\text{Vir}_{2,5}^{\otimes 2}$
$(E_7(a_1), 14, 13)$	$\{\frac{22}{13}, \frac{20}{13}, \frac{18}{13}, \frac{30}{13}, \frac{16}{13}, \frac{28}{13}, \frac{14}{13}, \frac{38}{13}, \frac{24}{13}\}$		
$(E_7(a_5), 14, 5)$	$(\frac{6}{5})^3$	$(A_1, A_2)^3$	$\text{Vir}_{2,5}^{\otimes 3}$
$(E_7(a_1), 6, 5)$	$(\frac{6}{5})^3$	$(A_1, A_2)^3$	$\text{Vir}_{2,5}^{\otimes 3}$

Table 21. The dimensions of Coulomb branch operators, equivalent 4d descriptions and possible isomorphic VOAs of 4d theories whose 2d VOA is $W^{-h^\vee + \frac{n}{u}}(\mathfrak{g}, f)$. Last two columns (if any) are obtained by matching Coulomb branch operators and central charges.

n	u	$\overline{\mathcal{O}}_u$
$n = 30$	> 30	E_8
	29	$E_8(a_1)$
	23	$E_8(a_2)$
	19	$E_8(a_3)$
	17	$E_8(a_4)$
	13	$E_8(a_5)$
	11	$E_8(a_6)$
	7	$A_6 + A_1$
$n = 24$	> 24	E_8
	23, 19*	$E_8(a_1)$
	17	$E_8(a_2)$
	13	$E_8(a_4)$
	11	$E_8(b_4)$
	7	$E_8(b_6)$
	5*	$D_6(a_1)$
	1*	A_1
$n = 20$	> 20	E_8
	19, 17	$E_8(a_1)$
	13*	$E_8(a_2)$
	11	$E_8(a_4)$
	9	$E_8(b_4)$
	7	$E_8(a_6)$
	3*	$A_4 + A_2 + A_1$
	1*	$2A_1$
$n = 15$	> 15	E_8
	14	$E_8(a_1)$
	13	$E_8(a_1)$
	11	$E_8(a_2)$
	8	$E_8(a_4)$
	7*	$E_8(a_4)$
	4*	$E_8(b_6)$
	2*	$A_4 + 2A_1$
	1*	$A_2 + A_1$

n	u	$\overline{\mathcal{O}}_u$
$n = 12$	> 12	E_8
	11	$E_8(a_1)$
	7	$E_8(a_3)$
	5	$E_8(a_5)$
	1*	$2A_2$
$n = 10$	> 10	E_8
	9	$E_8(a_1)$
	7	$E_8(a_2)$
	3*	$E_8(a_6)$
	1*	$D_4(a_1) + A_1$
$b = 8$	> 8	E_8
	7	$E_8(a_1)$
	5	$E_8(a_3)$
	3	$E_8(b_5)$
	$1^\#(n_v = 1)$	$D_4(a_1) + A_2$
$n = 6$	> 6	E_8
	5	$E_8(a_1)$
	1*	$E_8(a_7)$
$b = 5$	> 5	E_8
	$4^\#(n_v = 1)$	$E_8(a_1)$
	$3^\#(n_v = 1)$	$E_8(a_3)$
	$2^\#(n_v = 4)$	$E_8(a_4)$
	$1^\#(n_v = 4)$	$E_8(a_7)$
$n = 4$	> 4	E_8
	3	$E_8(a_2)$
	$1^\#((n_v = 1))$	$E_6(a_1)$
$n = 3$	> 3	E_8
	$2^\#(n_v = 2)$	$E_8(a_2)$
	$1^\#(n_v = 4)$	$E_8(a_6)$
$n = 2$	> 2	E_8
	$1^\#(n_v = 4)$	$E_8(a_4)$

Table 22. Associated variety for E_8 type AKM. The rank of the Coulomb branch is: $r = \frac{120}{n}u - 4$. The one with star implies that the theory has a pure Higgs branch. The one with $\#$ implies that the IR theory in the Higgs branch has free vector multiplets, and n_v is the number. For $n = 24, u = 5$, there are actually two possible choices of nilpotent orbits ($D_6(a_1)$ and A_6) which satisfy our formulas (7.9) and (7.13) (with $n_v = 0$); To pick $D_6(a_1)$, we use the following closure relation: $D_5 + A_1$ orbit is in the closure of $D_6(a_1)$ but is not in closure of A_6 , now we can compute the Coulomb branch operator of the theory defined using $D_5 + A_1$ which is non-empty, and we can further Higgs this particular theory. This observation then implies that we have to choose $D_6(a_1)$ as the Higgs branch of theory defined using regular puncture with trivial nilpotent orbit!

(f, n, u)	Coulomb branch	Equivalent 4d Theory	Isomorphic W-algebra
$(E_8(a_1), 24, 23)$	$\frac{114}{23}, \frac{96}{23}, \frac{90}{23}, \frac{78}{23}, \frac{76}{23}, \frac{72}{23}, \frac{66}{23}, \frac{60}{23}, \frac{58}{23}, \frac{54}{23}, \frac{52}{23}, \frac{48}{23}, (\frac{42}{23})^2$		
$(E_8(a_2), 24, 17)$	$\frac{30}{17}, \frac{28}{17}, \frac{26}{17}, \frac{24}{17}, \frac{22}{17}, \frac{20}{17}, \frac{18}{17}$	$x^2 + y^2 + z^2 + w^{15} = 0$	$\text{Vir}_{2,15}$
$(E_8(a_4), 24, 13)$	$\frac{30}{13}, \frac{27}{13}, \frac{24}{13}, \frac{21}{13}, \frac{20}{13}, \frac{18}{13}, \frac{17}{13}, \frac{15}{13}, \frac{14}{13}, \frac{13}{13}$	$x^2 + y^2 + z^3 + w^{10} = 0$	
$(E_8(b_4), 24, 11)$	$\frac{12}{11}, \frac{14}{11}, \frac{16}{11}, \frac{18}{11}$	$x^2 + y^2 + z^2 + w^9 = 0$	$\text{Vir}_{2,9}$
$(E_8(b_6), 24, 7)$	$\frac{8}{7}, \frac{9}{7}, \frac{12}{7}$	$x^2 + y^2 + z^3 + w^4 = 0$	$\text{Vir}_{3,14} \oplus L(13, 1)$
$(E_8(b_6), 24, 7)$	$\frac{8}{7}, \frac{9}{7}, \frac{12}{7}$	$x^2 + y^2 + z^3 + w^4 = 0$	$\text{Vir}_{3,14} \oplus L(13, 1)$
$(E_8(a_1), 20, 19)$	$\frac{90}{19}, \frac{4}{19}, \frac{70}{19}, \frac{62}{19}, \frac{60}{19}, \frac{56}{19}, \frac{50}{19}, \frac{48}{19}, \frac{46}{19}, \frac{42}{19}, \frac{40}{19}, \frac{36}{19}, \frac{34}{19}, \frac{32}{19}, \frac{30}{19}, \frac{28}{19}, \frac{26}{19}, \frac{22}{19}, \frac{20}{19}$	$x^2 + y^3 + yz^3 + zw^5 = 0$	
$(E_8(a_1), 20, 17)$	$\frac{30}{17}, \frac{28}{17}, \frac{26}{17}, \frac{24}{17}, \frac{22}{17}, \frac{20}{17}, \frac{18}{17}$	$x^2 + y^2 + z^2 + w^{15} = 0$	$\text{Vir}_{2,15}$
$(E_8(a_4), 20, 11)$	$\frac{30}{11}, \frac{25}{11}, \frac{24}{11}, \frac{20}{11}, \frac{19}{11}, \frac{18}{11}, \frac{15}{11}, \frac{14}{11}, \frac{13}{11}, \frac{12}{11}, \frac{11}{11}$	$x^2 + y^2 + z^5 + w^6 = 0$	
$(E_8(b_4), 20, 9)$	$\frac{10}{9}, \frac{4}{9}, \frac{14}{9}$	$x^2 + y^2 + z^2 + w^7 = 0$	$\text{Vir}_{2,7}$
$(E_8(a_6), 20, 7)$	$(\frac{10}{7})^3, (\frac{8}{7})^3$	$3 \times (x^2 + y^2 + z^2 + w^5 = 0)$	$3 \times \text{Vir}_{2,7}$
$(E_8(a_1), 15, 14)$	$\frac{15}{14}, \frac{8}{14}, \frac{9}{14}, \frac{3}{14}, \frac{11}{14}, \frac{12}{14}, \frac{25}{14}, \frac{27}{14}, \frac{15}{14}, \frac{31}{14}, \frac{33}{14}, \frac{18}{14}, \frac{20}{14}, \frac{3}{14}, \frac{45}{14}, \frac{51}{14}, \frac{30}{14}$	$x^2 + y^3 + z^4 + zw^5 = 0$	
$(E_8(a_1), 15, 13)$	$\frac{30}{13}, \frac{27}{13}, \frac{24}{13}, \frac{21}{13}, \frac{20}{13}, \frac{18}{13}, \frac{17}{13}, \frac{15}{13}, \frac{14}{13}, \frac{13}{13}$	$x^2 + y^2 + z^3 + w^{10} = 0$	
$(E_8(a_2), 15, 11)$	$(\frac{9}{8}, \frac{5}{4}, \frac{3}{2}, \frac{15}{8})^2$	$x^2 + y^2 + z^3 + w^5 = 0$	
$(E_8(a_1), 12, 11)$	$\frac{42}{11}, \frac{36}{11}, (\frac{30}{11})^2, \frac{28}{11}, (\frac{24}{11})^2, 2, (\frac{18}{11})^3, (\frac{16}{11})^2, (\frac{12}{11})^2$		
$(E_8(a_3), 12, 7)$	$\frac{12}{7}, \frac{9}{7}, \frac{8}{7}$	$x^2 + y^2 + z^3 + w^4 = 0$	$\text{Vir}_{3,14} \oplus L(13, 1)$
$(E_8(a_5), 12, 5)$	$(\frac{6}{5})^4$	$4 \times (x^2 + y^2 + z^2 + w^3 = 0)$	$4 \times \text{Vir}_{2,5}$
$(E_8(a_1), 10, 9)$	$\frac{10}{9}, \frac{10}{9}, \frac{4}{3}, \frac{4}{3}, \frac{14}{9}, \frac{16}{9}, \frac{16}{9}, 2, \frac{20}{9}, \frac{20}{9}, \frac{22}{9}, \frac{26}{9}, \frac{10}{3}$		
$(E_8(a_2), 10, 7)$	$(\frac{10}{7})^3, (\frac{8}{7})^3$	$3 \times (x^2 + y^2 + z^2 + w^7 = 0)$	$3 \times \text{Vir}_{2,9}$
$(E_8(a_1), 8, 7)$	$\frac{18}{7}, \frac{16}{7}, 2, (\frac{12}{7})^2, (\frac{10}{7})^3, (\frac{8}{7})^2$	$D_7/Z_2 : x^2 + y^6 + yz^2 + w^2 = 0$	
$(E_8(a_3), 8, 5)$	$\frac{12}{5}, 2, (\frac{8}{5})^2, \frac{7}{5}, (\frac{6}{5})^2$	$D_5/Z_2 : x^2 + y^4 + yz^2 + w^2 = 0$	
$(E_8(b_5), 8, 3)$	$2, (\frac{4}{3})^3$	$x^3 + y^3 + z^3 + w^2 = 0$	
$(E_8(a_1), 6, 5)$	$(\frac{6}{5})^5$	$5 \times (x^2 + y^2 + z^2 + w^3 = 0)$	$5 \times \text{Vir}_{2,5}$
$(E_8(a_3), 5, 3)$	$2, (\frac{4}{3})^3$	$x^3 + y^3 + z^3 + w^2 = 0$	
$(E_8(a_2), 4, 3)$	$\frac{10}{3}, (\frac{8}{3})^2, 2^4, (\frac{4}{3})^5$		

Table 23. Dimensions of Coulomb branch operators, equivalent 4d descriptions and possible isomorphic VOAs for 4d theories whose VOA is $W^{-h^\vee + \frac{n}{u}}(\mathfrak{e}_8, f)$. Here f denotes regular singularity. Again, last two columns (if any) are obtained by matching Coulomb branch operators and central charges.

E_6 : $n = 9$, $u = 1$, E_6 MN theory. This is realized by the 6d A_2 (2,0) theory on a sphere with three full punctures, and the Schur index is

$$\mathcal{I}^{E_6, n=9} = \frac{(q^2; q)(q^3; q)}{\eta^{\mathfrak{su}_3}(\mathbf{a})\eta^{\mathfrak{su}_3}(\mathbf{b})\eta^{\mathfrak{su}_3}(\mathbf{c})} \sum_{\lambda} \frac{\chi_{\lambda}^{\mathfrak{su}_3}(\mathbf{a})\chi_{\lambda}^{\mathfrak{su}_3}(\mathbf{b})\chi_{\lambda}^{\mathfrak{su}_3}(\mathbf{c})}{\chi_{\lambda}^{\mathfrak{su}_3}(q^{\rho})}. \quad (7.19)$$

The sum is over all highest weight representations of $\mathfrak{su}_3$. Although not manifest, the above formula has an E_6 symmetry in the end.

E_7 : $n = 14$, $u = 1$, E_7 MN theory. It is realized by the 6d A_3 (2,0) theory on a sphere with Young tableaux $[1^4]$, $[1^4]$ and $[2^2]$ (two full punctures and one partially closed puncture). The Schur index is

$$\mathcal{I}^{E_7, n=14} = \frac{(q^3; q)(q^4; q)}{\eta^{\mathfrak{su}_4}(\mathbf{a})\eta^{\mathfrak{su}_4}(\mathbf{b})\eta^{[2^2]}(c)} \sum_{\lambda} \frac{\chi_{\lambda}^{\mathfrak{su}_4}(\mathbf{a})\chi_{\lambda}^{\mathfrak{su}_4}(\mathbf{b})\chi_{\lambda}^{\mathfrak{su}_4}(\tilde{\mathbf{c}})}{\chi_{\lambda}^{\mathfrak{su}_4}(q^{\rho})}, \quad (7.20)$$

with

$$\eta^{[2^2]}(c) = (q; q)(qc^{\pm}; q)(q^2; q)(q^2c^{\pm}; q) = \prod_{w \in \text{adj } \mathfrak{su}(2)} (qc^w; q)(q^2c^w; q) \quad (7.21)$$

and

$$\tilde{\mathbf{c}} = (q^{\frac{1}{2}}c, q, q^{\frac{1}{2}}c). \quad (7.22)$$

We use the shorthand notation $(qa^{\pm}; q) = (qa; q)(qa^{-1}; q)$. Again the sum is over all highest weight representations of $\text{SU}(4)$ and the result has an E_7 symmetry.

E_8 : $n = 24$, $u = 1$, E_8 MN theory. It is realized by the 6d A_5 (2,0) theory on a sphere with Young tableaux $[1^6]$, $[3^2]$ and $[2^3]$ (one full puncture and two partially closed punctures). The Schur index is

$$\mathcal{I}^{E_8, n=24} = \frac{(q^4; q)(q^5; q)(q^6; q)}{(q^3; q)\eta^{\mathfrak{su}_6}(\mathbf{a})\eta^{[3^2]}(\mathbf{b})\eta^{[2^3]}(c)} \sum_{\lambda} \frac{\chi_{\lambda}^{\mathfrak{su}_6}(\mathbf{a})\chi_{\lambda}^{\mathfrak{su}_6}(\tilde{\mathbf{b}})\chi_{\lambda}^{\mathfrak{su}_6}(\tilde{\mathbf{c}})}{\chi_{\lambda}^{\mathfrak{su}_6}(q^{\rho})} \quad (7.23)$$

where

$$\begin{aligned} \eta^{[3^2]}(\mathbf{b}) &= \prod_{w \in \text{adj } \mathfrak{su}_3} (q\mathbf{b}^w; q)(q^2\mathbf{b}^w; q), \\ \eta^{[2^3]}(c) &= \prod_{w \in \text{adj } \mathfrak{su}_2} (qc^w; q)(q^2c^w; q)(q^3c^w; q), \end{aligned} \quad (7.24)$$

and

$$\begin{aligned} \tilde{\mathbf{b}} &= (q^{\frac{1}{2}}b_1, qb_2, q^{\frac{3}{2}}, qb_2^{-1}, q^{\frac{1}{2}}b_1^{-1}), \\ \tilde{\mathbf{c}} &= (q^{\frac{1}{2}}c, q, q^{\frac{1}{2}}). \end{aligned} \quad (7.25)$$

Now the sum is over all highest weight representations of $\text{SU}(6)$. Again, there is an enhancement to E_8 symmetry.

8 Conclusion

We continue our studies of the correspondence between 2d W-algebras and 4d $\mathcal{N} = 2$ SCFTs. Lisse W-algebras are defined as those W-algebras whose corresponding Zhu's C_2 algebra are finite dimensional. As the associated variety of Zhu's C_2 algebra is identified with the Higgs branch in the 4d/2d correspondence, the lisse condition is equivalent to the absence of Higgs branch of the 4d theory. We classify 4d $\mathcal{N} = 2$ SCFTs which do not admit a Higgs branch, and the corresponding 2d W-algebra of these theories should be lisse. In particular, we predict the existence of a large class of new non-admissible lisse W-algebra which has not been found before (see few examples considered in [103]). These 4d theories can appear in the IR theory of the Higgs branch of a general 4d $\mathcal{N} = 2$ SCFT, and therefore they are crucial to understand the behavior of Higgs branch of general 4d $\mathcal{N} = 2$ SCFTs.

In our study of 4d $\mathcal{N} = 2$ theory with no Higgs branch, we encounter the following generic situation. Consider 4d $\mathcal{N} = 2$ SCFT constructed from 6d $(2, 0)$ theory on a sphere with one irregular singularity and one regular singularity labeled by trivial nilpotent orbit. We call this theory *UV* theory, and the corresponding VOA is just the AKM algebra. It is easy to study the IR theory of its Higgs branch: one simply change the type of regular puncture to the one labeled by the nilpotent element f , whose orbit leads to the associated variety of the AKM. The IR theory generically consists of free hypermultiplets, free vectormultiplets, and an interacting theory which has no Higgs branch. This suggests that the UV AKM might be related to the VOA of free hypermultiplets and vectormultiplets, plus a lisse W-algebra. Indeed such realizations are discussed in [68, 104–107]. Our study reveals that such constructions are much more general and we can identify all the IR pieces in the Higgs branch using $(2, 0)$ construction. Moreover, our results suggest that it is also possible to have free vectormultiplets in the IR. It would be interesting to further study the relations between these W-algebras.

We found new non-admissible lisse W-algebras, which deserve further studies from physical and mathematical point of view. We give a formula for vacuum characters for these W-algebras in the classical Lie algebra case, and it would be interesting to verify them using 2d methods. The module structure is also very interesting to study, and some results will appear in [108].

The lisse W-algebra in our context has interesting connection to three dimensional $\mathbb{Q}$ -factorial terminal singularity. In our paper, we mainly study the Gorenstein $\mathbb{Q}$ -terminal singularity, which gives the results in table 5. The non-Gorenstein ones needs further study, and might be related to the lisse W-algebra with non-simply laced Lie algebra, and it is interesting to further study them as well.

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A Review on the Higgs branch of a quiver with classical gauge groups

This is a review of associating a quiver to a nilpotent orbit of a classical Lie algebra which were discussed in [109]. This construction was used to find the Higgs branch of quiver gauge theories discussed in section 6. The basic idea is the following: the quiver has a flavor symmetry whose Lie algebra $\mathfrak{g}$ is a classical semi-simple one, and its Higgs branch is assumed to be the closure of nilpotent orbit e^d in $\mathfrak{g}$. To find e^d , we first find a nilpotent orbit e in the Langlands dual $\mathfrak{g}^\vee$ Lie algebra, whose associated quiver is just the quiver we started. The nilpotent e^d can be found from e by using the Spaltenstein map.

First consider $\mathfrak{g} = A_{N-1}$, and its Langlands dual is just itself. A nilpotent orbit in A_{N-1} Lie algebra is specified by a partition $[h_1, \dots, h_s]$, and to find its associated quiver, we first construct a $D3 - D5$ brane system as in figure 7. The number of D3 branes suspended between D5 branes are $r_1 = N - h_1$, and $r_2 = N - h_1 - h_2$, and etc. The quiver gauge theory is found by doing S duality on above brane configuration: $D3$ branes stay as $D3$ branes, while $D5$ branes become $NS5$ branes. The quiver gauge theory can be easily read from the $NS5 - D3$ brane system. The process was summarized in figure 7.

For other classical Lie algebras, a nilpotent orbit is also labeled by a partition $[h_1, \dots, h_s]$, but with following constraints: for (B_N, D_N) , even parts appear even time; for C_N , odd parts appear even times. To engineer gauge theories with orthogonal and symplectic gauge groups, we need to use O3 planes. There are a total of four kinds of O3 planes, and we call them B , D , and C , C' type. The name of such O3 planes are clear: if we put D3 branes on certain type of O3 planes, we would get corresponding gauge theory types (the number of D3 branes on corresponding O3 plane is then constrained by the type, i.e. there are $2r$ for D type, $2r+1$ for B type, and $2r$ for C type). S duality exchange those O3 planes as follows:

$$D \leftrightarrow D, \quad B \leftrightarrow C, \quad C \leftrightarrow B, \quad C' \leftrightarrow C' \quad (\text{A.1})$$

Now we construct a quiver from a D type or B type partition, where the even parts appear even times. We first start with a $D3 - O3 - D5$ brane system, and there are two new facts:

- The type of O3 planes are alternating between $D3$ branes. Here we have D type and B type alternating.
- There are only an odd number of D3 branes ending on a half D5 brane.

For the B type or D type partition where parts are all odd, we can construct a $D3 - O3 - D5$ brane system using the same rule as A type partition. The quiver gauge theory can be

Figure 7. Upper row: We construct a $D5 - D3$ brane system for a A type partition. Middle row: We perform S duality on the brane system in upper row, which turns a $D5$ brane to a $NS5$ brane, and a $D3$ brane to a $D3$ brane. Bottom row: We read a quiver gauge theory from the middle row. The rule is following, we associate a $U(r)$ gauge group for r $D3$ branes suspending between $NS5$ brane, and each $NS5$ brane contributes a hypermultiplet in bi-fundamental representation.

found from S dual brane configuration, see figure 8. To deal with a partition with even parts, we need to do following modification:

- We add one more $D3$ brane to $O3$ brane for the segment which violates the corresponding rule for $O3$ plane. For example: if there are $2r$ $D3$ brane on a B type $O3$ plane, we add one more $D3$ brane, or if there are $2r + 1$ $D3$ brane on D type $O3$ plane, we add one more $D3$ brane.

After this modification, it is straightforward to find the quiver gauge theory by doing S duality on $D3 - O3 - D5$ brane system. Notice that the corresponding quiver gauge theory involves only D type and C type gauge groups (the flavor group is D type or C type).

Now we consider how to construct a quiver from a C type partition, where the even parts appear even times. We first start with a $D3 - O3 - D5$ brane system, and there are two new facts

- The type of $O3$ planes are alternating between $D3$ branes. Here we have C type and C' type alternating.

Figure 8. Upper row: We construct a $D5 - D3$ brane system for a D type partition and B type partition respectively. There should be even number of $D3$ branes for a D type $O3$ plane, and odd number of $D3$ branes for a B type $O3$ plane. For a general D type or B type partition, the brane configuration would violate the rule for D type or B type $O3$ planes, and we add one more $D3$ brane so that the number is consistent with the corresponding $O3$ plane. Middle: We perform S duality on brane system in upper row, which turn a $D5$ brane to a $NS5$ brane, and $D3$ brane to a $D3$ brane. Bottom row: We read a quiver gauge theory from brane configuration of middle row.

- There are only an even number of $D3$ branes ending on a half $D5$ brane.

For the C type partition where the parts are all even, we can construct a $D3 - O3 - D5$ brane system using the same rule as A type partition. The quiver gauge theory can be found from S dual brane configuration, see figure 9. To deal with a partition with even parts, we need to do following modification:

- We add one more $D3$ brane to $O3$ brane for the segment which violates the corresponding rule for $O3$ plane. For example: if there are $2r + 1$ $D3$ brane on a C type or C' $O3$ plane, we add one more $D3$ brane.

After this modification, it is straightforward to find the quiver gauge theory by doing S duality on $D3 - O3 - D5$ brane system. Notice that the corresponding quiver gauge theory involves only B type and C type gauge groups (flavor group is of C type). Notice that if we fix the flavor group to be C type, we do not get a quiver with alternating D type and C type gauge groups by using this method.

Given a nilpotent orbit e in classical algebra g , and one can find its dual in $g^\vee$ using the Spaltenstein map. The rule is

- For A type: the partition of e^d is given by the transpose of partition of e .
- For D type: the partition of e^d is given as follows: first find the transpose of partition e , and then do D collapse to get a D partition.

Figure 9. Upper row: we construct a $D5 - D3$ brane system for a C type partition. There should be even number of D3 branes for C type or C' type O3 planes. For a general C type partition, the brane configuration would violate the rule for O3 planes, and we add one more D3 brane so that the number is consistent with the O plane type. Middle: we perform S duality on brane system in upper row, which turn a D5 brane to a NS5 brane, and D3 brane to a D3 brane. Bottom row: we read a quiver gauge theory from brane configuration of middle row.

- For B type: the partition of e^d is given as follows: first subtract one box from last row of e , and then do the transpose, and then do C collapse to get a C partition.
- For C type: the partition of e^d is given as follows: first add a box to first row of e and find its transpose, and then do B collapse to get a B partition.

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References

- [1] C. Beem, M. Lemos, P. Liendo, W. Peelaers, L. Rastelli and B.C. van Rees, *Infinite chiral symmetry in four dimensions*, *Commun. Math. Phys.* **336** (2015) 1359 [[arXiv:1312.5344](https://arxiv.org/abs/1312.5344)] [[INSPIRE](#)].

- [2] D. Xie, W. Yan and S.-T. Yau, *Chiral algebra of the Argyres-Douglas theory from M5 branes*, *Phys. Rev. D* **103** (2021) 065003 [[arXiv:1604.02155](#)] [[INSPIRE](#)].
- [3] J. Song, D. Xie and W. Yan, *Vertex operator algebras of Argyres-Douglas theories from M5-branes*, *JHEP* **12** (2017) 123 [[arXiv:1706.01607](#)] [[INSPIRE](#)].
- [4] D. Xie and W. Yan, *W algebras, cosets and VOAs for 4d $\mathcal{N} = 2$ SCFTs from M5 branes*, *JHEP* **04** (2021) 076 [[arXiv:1902.02838](#)] [[INSPIRE](#)].
- [5] D. Xie and W. Yan, *Schur sector of Argyres-Douglas theory and W-algebra*, *SciPost Phys.* **10** (2021) 080 [[arXiv:1904.09094](#)] [[INSPIRE](#)].
- [6] G.W. Moore and N. Seiberg, *Classical and quantum conformal field theory*, *Commun. Math. Phys.* **123** (1989) 177 [[INSPIRE](#)].
- [7] V. Kac and M. Wakimoto, *On rationality of W-algebras*, *Transf. Groups* **13** (2007) 671.
- [8] E. Witten, *Quantum field theory and the Jones polynomial*, *Commun. Math. Phys.* **121** (1989) 351 [[INSPIRE](#)].
- [9] Y. Zhu, *Modular invariance of characters of vertex operator algebras*, *J. Amer. Math. Soc.* **9** (1996) 237.
- [10] T. Arakawa, *A remark on the c_2 -cofiniteness condition on vertex algebras*, *Math. Z.* **270** (2010) 559.
- [11] C. Beem and L. Rastelli, *Vertex operator algebras, Higgs branches, and modular differential equations*, *JHEP* **08** (2018) 114 [[arXiv:1707.07679](#)] [[INSPIRE](#)].
- [12] T. Arakawa, *Representation theory of W-algebras and Higgs branch conjecture*, in *International Congress of Mathematicians*, 12, 2017 [[arXiv:1712.07331](#)] [[INSPIRE](#)].
- [13] D. Gaiotto, *$\mathcal{N} = 2$ dualities*, *JHEP* **08** (2012) 034 [[arXiv:0904.2715](#)] [[INSPIRE](#)].
- [14] D. Xie, *General Argyres-Douglas theory*, *JHEP* **01** (2013) 100 [[arXiv:1204.2270](#)] [[INSPIRE](#)].
- [15] D. Xie and S.-T. Yau, *4d $\mathcal{N} = 2$ SCFT and singularity theory. Part I: classification*, [[arXiv:1510.01324](#)] [[INSPIRE](#)].
- [16] Y. Wang and D. Xie, *Classification of Argyres-Douglas theories from M5 branes*, *Phys. Rev. D* **94** (2016) 065012 [[arXiv:1509.00847](#)] [[INSPIRE](#)].
- [17] Y. Wang and D. Xie, *Codimension-two defects and Argyres-Douglas theories from outer-automorphism twist in 6d (2,0) theories*, *Phys. Rev. D* **100** (2019) 025001 [[arXiv:1805.08839](#)] [[INSPIRE](#)].
- [18] B. Chen, D. Xie, S.S.T. Yau, S.-T. Yau and H. Zuo, *4d $\mathcal{N} = 2$ SCFT and singularity theory. Part III: rigid singularity*, *Adv. Theor. Math. Phys.* **22** (2018) 1885 [[arXiv:1712.00464](#)] [[INSPIRE](#)].
- [19] O. Chacaltana, J. Distler and Y. Tachikawa, *Nilpotent orbits and codimension-two defects of 6d $\mathcal{N} = (2,0)$ theories*, *Int. J. Mod. Phys. A* **28** (2013) 1340006 [[arXiv:1203.2930](#)] [[INSPIRE](#)].
- [20] T. Arakawa, *Associated varieties of modules over Kac-Moody algebras and C_2 -cofiniteness of W-algebras*, *Int. Math. Res. Not.* (2015) 11605.
- [21] V.G. Kac, *Vertex algebras for beginners*, American Mathematical Society, U.S.A. (1998).
- [22] H. Li, *Abelianizing vertex algebras*, *Commun. Math. Phys.* **259** (2005) 391.

- [23] T. Creutzig and T. Gannon, *Logarithmic conformal field theory, log-modular tensor categories and modular forms*, *J. Phys. A* **50** (2017) 404004 [[arXiv:1605.04630](#)] [[INSPIRE](#)].
- [24] T. Arakawa and K. Kawasetsu, *Quasi-lisse vertex algebras and modular linear differential equations*, [arXiv:1610.05865](#) [[INSPIRE](#)].
- [25] V.G. Kac and M. Wakimoto, *A remark on boundary level admissible representations*, *Compt. Rend. Math.* **355** (2017) 128.
- [26] T. Arakawa and J. van Ekeren, *Rationality and fusion rules of exceptional W -algebras*, [arXiv:1905.11473](#).
- [27] C. Beem, W. Peelaers, L. Rastelli and B.C. van Rees, *Chiral algebras of class S* , *JHEP* **05** (2015) 020 [[arXiv:1408.6522](#)] [[INSPIRE](#)].
- [28] M. Lemos and W. Peelaers, *Chiral algebras for trinion theories*, *JHEP* **02** (2015) 113 [[arXiv:1411.3252](#)] [[INSPIRE](#)].
- [29] M. Buican and T. Nishinaka, *Argyres-Douglas theories, S^1 reductions, and topological symmetries*, *J. Phys. A* **49** (2016) 045401 [[arXiv:1505.06205](#)] [[INSPIRE](#)].
- [30] C. Cordova and S.-H. Shao, *Schur indices, BPS particles, and Argyres-Douglas theories*, *JHEP* **01** (2016) 040 [[arXiv:1506.00265](#)] [[INSPIRE](#)].
- [31] M. Buican and T. Nishinaka, *Argyres-Douglas theories, the Macdonald index, and an RG inequality*, *JHEP* **02** (2016) 159 [[arXiv:1509.05402](#)] [[INSPIRE](#)].
- [32] J. Song, *Superconformal indices of generalized Argyres-Douglas theories from 2d TQFT*, *JHEP* **02** (2016) 045 [[arXiv:1509.06730](#)] [[INSPIRE](#)].
- [33] M. Lemos and P. Liendo, *Bootstrapping $\mathcal{N} = 2$ chiral correlators*, *JHEP* **01** (2016) 025 [[arXiv:1510.03866](#)] [[INSPIRE](#)].
- [34] S. Cecotti, J. Song, C. Vafa and W. Yan, *Superconformal index, BPS monodromy and chiral algebras*, *JHEP* **11** (2017) 013 [[arXiv:1511.01516](#)] [[INSPIRE](#)].
- [35] M. Lemos and P. Liendo, *$\mathcal{N} = 2$ central charge bounds from 2d chiral algebras*, *JHEP* **04** (2016) 004 [[arXiv:1511.07449](#)] [[INSPIRE](#)].
- [36] T. Nishinaka and Y. Tachikawa, *On 4d rank-one $\mathcal{N} = 3$ superconformal field theories*, *JHEP* **09** (2016) 116 [[arXiv:1602.01503](#)] [[INSPIRE](#)].
- [37] M. Buican and T. Nishinaka, *Conformal manifolds in four dimensions and chiral algebras*, *J. Phys. A* **49** (2016) 465401 [[arXiv:1603.00887](#)] [[INSPIRE](#)].
- [38] C. Cordova, D. Gaiotto and S.-H. Shao, *Infrared computations of defect Schur indices*, *JHEP* **11** (2016) 106 [[arXiv:1606.08429](#)] [[INSPIRE](#)].
- [39] M. Lemos, P. Liendo, C. Meneghelli and V. Mitev, *Bootstrapping $\mathcal{N} = 3$ superconformal theories*, *JHEP* **04** (2017) 032 [[arXiv:1612.01536](#)] [[INSPIRE](#)].
- [40] F. Bonetti and L. Rastelli, *Supersymmetric localization in AdS_5 and the protected chiral algebra*, *JHEP* **08** (2018) 098 [[arXiv:1612.06514](#)] [[INSPIRE](#)].
- [41] J. Song, *Macdonald index and chiral algebra*, *JHEP* **08** (2017) 044 [[arXiv:1612.08956](#)] [[INSPIRE](#)].
- [42] T. Creutzig, *W -algebras for Argyres-Douglas theories*, [arXiv:1701.05926](#) [[INSPIRE](#)].
- [43] L. Fredrickson, D. Pei, W. Yan and K. Ye, *Argyres-Douglas theories, chiral algebras and wild Hitchin characters*, *JHEP* **01** (2018) 150 [[arXiv:1701.08782](#)] [[INSPIRE](#)].

- [44] C. Cordova, D. Gaiotto and S.-H. Shao, *Surface defect indices and 2d-4d BPS states*, *JHEP* **12** (2017) 078 [[arXiv:1703.02525](#)] [[INSPIRE](#)].
- [45] C. Cordova, D. Gaiotto and S.-H. Shao, *Surface defects and chiral algebras*, *JHEP* **05** (2017) 140 [[arXiv:1704.01955](#)] [[INSPIRE](#)].
- [46] M. Dedushenko, S. Gukov and P. Putrov, *Vertex algebras and 4-manifold invariants*, [arXiv:1705.01645](#) [[INSPIRE](#)].
- [47] M. Buican and T. Nishinaka, *On irregular singularity wave functions and superconformal indices*, *JHEP* **09** (2017) 066 [[arXiv:1705.07173](#)] [[INSPIRE](#)].
- [48] M. Buican, Z. Laczko and T. Nishinaka, *$\mathcal{N} = 2$ S-duality revisited*, *JHEP* **09** (2017) 087 [[arXiv:1706.03797](#)] [[INSPIRE](#)].
- [49] Y. Pan and W. Peelaers, *Chiral algebras, localization and surface defects*, *JHEP* **02** (2018) 138 [[arXiv:1710.04306](#)] [[INSPIRE](#)].
- [50] M. Fluder and J. Song, *Four-dimensional lens space index from two-dimensional chiral algebra*, *JHEP* **07** (2018) 073 [[arXiv:1710.06029](#)] [[INSPIRE](#)].
- [51] M. Buican and Z. Laczko, *Nonunitary Lagrangians and unitary non-Lagrangian conformal field theories*, *Phys. Rev. Lett.* **120** (2018) 081601 [[arXiv:1711.09949](#)] [[INSPIRE](#)].
- [52] J. Choi and T. Nishinaka, *On the chiral algebra of Argyres-Douglas theories and S-duality*, *JHEP* **04** (2018) 004 [[arXiv:1711.07941](#)] [[INSPIRE](#)].
- [53] C. Kozçaz, S. Shakirov and W. Yan, *Argyres-Douglas theories, modularity of minimal models and refined Chern-Simons*, [arXiv:1801.08316](#) [[INSPIRE](#)].
- [54] K. Costello and D. Gaiotto, *Vertex operator algebras and 3d $\mathcal{N} = 4$ gauge theories*, *JHEP* **05** (2019) 018 [[arXiv:1804.06460](#)] [[INSPIRE](#)].
- [55] B. Feigin and S. Gukov, *VOA[M₄]*, *J. Math. Phys.* **61** (2020) 012302 [[arXiv:1806.02470](#)] [[INSPIRE](#)].
- [56] V. Niarchos, *Geometry of Higgs-branch superconformal primary bundles*, *Phys. Rev. D* **98** (2018) 065012 [[arXiv:1807.04296](#)] [[INSPIRE](#)].
- [57] T. Creutzig, *Logarithmic W-algebras and Argyres-Douglas theories at higher rank*, *JHEP* **11** (2018) 188 [[arXiv:1809.01725](#)] [[INSPIRE](#)].
- [58] M. Dedushenko, S. Gukov, H. Nakajima, D. Pei and K. Ye, *3d TQFTs from Argyres-Douglas theories*, *J. Phys. A* **53** (2020) 43LT01 [[arXiv:1809.04638](#)] [[INSPIRE](#)].
- [59] F. Bonetti, C. Meneghelli and L. Rastelli, *VOAs labelled by complex reflection groups and 4d SCFTs*, *JHEP* **05** (2019) 155 [[arXiv:1810.03612](#)] [[INSPIRE](#)].
- [60] T. Arakawa, *Chiral algebras of class S and Moore-Tachikawa symplectic varieties*, [arXiv:1811.01577](#) [[INSPIRE](#)].
- [61] K. Costello, T. Creutzig and D. Gaiotto, *Higgs and Coulomb branches from vertex operator algebras*, *JHEP* **03** (2019) 066 [[arXiv:1811.03958](#)] [[INSPIRE](#)].
- [62] T. Nishinaka, S. Sasa and R.-D. Zhu, *On the correspondence between surface operators in Argyres-Douglas theories and modules of chiral algebra*, *JHEP* **03** (2019) 091 [[arXiv:1811.11772](#)] [[INSPIRE](#)].
- [63] P. Agarwal, S. Lee and J. Song, *Vanishing OPE coefficients in 4d $\mathcal{N} = 2$ SCFTs*, *JHEP* **06** (2019) 102 [[arXiv:1812.04743](#)] [[INSPIRE](#)].

- [64] C. Beem, *Flavor symmetries and unitarity bounds in $\mathcal{N} = 2$ superconformal field theories*, *Phys. Rev. Lett.* **122** (2019) 241603 [[arXiv:1812.06099](#)] [[INSPIRE](#)].
- [65] K. Kiyoshige and T. Nishinaka, *OPE selection rules for Schur multiplets in 4D $\mathcal{N} = 2$ superconformal field theories*, *JHEP* **04** (2019) 060 [[arXiv:1812.06394](#)] [[INSPIRE](#)].
- [66] M. Buican and Z. Laczko, *Rationalizing CFTs and anyonic imprints on Higgs branches*, *JHEP* **03** (2019) 025 [[arXiv:1901.07591](#)] [[INSPIRE](#)].
- [67] Y. Pan and W. Peelaers, *Schur correlation functions on $S^3 \times S^1$* , *JHEP* **07** (2019) 013 [[arXiv:1903.03623](#)] [[INSPIRE](#)].
- [68] C. Beem, C. Meneghelli and L. Rastelli, *Free field realizations from the Higgs branch*, *JHEP* **09** (2019) 058 [[arXiv:1903.07624](#)] [[INSPIRE](#)].
- [69] J. Oh and J. Yagi, *Chiral algebras from Ω -deformation*, *JHEP* **08** (2019) 143 [[arXiv:1903.11123](#)] [[INSPIRE](#)].
- [70] M. Dedushenko and M. Fluder, *Chiral algebra, localization, modularity, surface defects, and all that*, *J. Math. Phys.* **61** (2020) 092302 [[arXiv:1904.02704](#)] [[INSPIRE](#)].
- [71] S. Jeong, *SCFT/VOA correspondence via Ω -deformation*, *JHEP* **10** (2019) 171 [[arXiv:1904.00927](#)] [[INSPIRE](#)].
- [72] F.A. Dolan and H. Osborn, *On short and semi-short representations for four-dimensional superconformal symmetry*, *Annals Phys.* **307** (2003) 41 [[hep-th/0209056](#)] [[INSPIRE](#)].
- [73] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory*, *Nucl. Phys. B* **426** (1994) 19 [Erratum *ibid.* **430** (1994) 485] [[hep-th/9407087](#)] [[INSPIRE](#)].
- [74] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*, *Nucl. Phys. B* **431** (1994) 484 [[hep-th/9408099](#)] [[INSPIRE](#)].
- [75] P. Argyres, M. Lotito, Y. Lü and M. Martone, *Geometric constraints on the space of $\mathcal{N} = 2$ SCFTs. Part I: physical constraints on relevant deformations*, *JHEP* **02** (2018) 001 [[arXiv:1505.04814](#)] [[INSPIRE](#)].
- [76] D. Gaiotto, G.W. Moore and A. Neitzke, *Wall-crossing, Hitchin Systems, and the WKB Approximation*, [arXiv:0907.3987](#) [[INSPIRE](#)].
- [77] M. Reeder, P. Levy, J.-K. Yu and B. H. Gross, *Gradings of positive rank on simple Lie algebras*, *Transf. Groups* **17** (2012) 1123.
- [78] N. Hitchin et al., *Stable bundles and integrable systems*, *Duke Math. J.* **54** (1987) 91
- [79] B. Chen, D. Xie, S.-T. Yau, S.S.T. Yau and H. Zuo, *4D $\mathcal{N} = 2$ SCFT and singularity theory. Part II: complete intersection*, *Adv. Theor. Math. Phys.* **21** (2017) 121 [[arXiv:1604.07843](#)] [[INSPIRE](#)].
- [80] Y. Wang, D. Xie, S.S.T. Yau and S.-T. Yau, *4d $\mathcal{N} = 2$ SCFT from complete intersection singularity*, *Adv. Theor. Math. Phys.* **21** (2017) 801 [[arXiv:1606.06306](#)] [[INSPIRE](#)].
- [81] D. Xie and S.-T. Yau, *Semicontinuity of 4d $N = 2$ spectrum under renormalization group flow*, *JHEP* **03** (2016) 094 [[arXiv:1510.06036](#)] [[INSPIRE](#)].
- [82] S. Li, D. Xie and S.-T. Yau, *Seiberg-Witten differential via primitive forms*, *Commun. Math. Phys.* **367** (2019) 193 [[arXiv:1802.06751](#)] [[INSPIRE](#)].

- [83] D. Xie and K. Yonekura, *The moduli space of vacua of $\mathcal{N} = 2$ class $\mathcal{S}$ theories*, *JHEP* **10** (2014) 134 [[arXiv:1404.7521](#)] [[INSPIRE](#)].
- [84] D. Xie and K. Yonekura, *Generalized Hitchin system, spectral curve and $\mathcal{N} = 1$ dynamics*, *JHEP* **01** (2014) 001 [[arXiv:1310.0467](#)] [[INSPIRE](#)].
- [85] Y. Kawamata, *Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces*, *Ann. Math.* **127** (1988) 93.
- [86] M. Caibar, *On the divisor class group of 3-fold singularities*, *Int. J. Math.* **14** (2003) 105.
- [87] M. Del Zotto, C. Vafa and D. Xie, *Geometric engineering, mirror symmetry and $6d_{(1,0)} \rightarrow 4d_{(\mathcal{N}=2)}$* , *JHEP* **11** (2015) 123 [[arXiv:1504.08348](#)] [[INSPIRE](#)].
- [88] O. Chacaltana and J. Distler, *Tinkertoys for Gaiotto duality*, *JHEP* **11** (2010) 099 [[arXiv:1008.5203](#)] [[INSPIRE](#)].
- [89] O. Chacaltana and J. Distler, *Tinkertoys for the D_N series*, *JHEP* **02** (2013) 110 [[arXiv:1106.5410](#)] [[INSPIRE](#)].
- [90] O. Chacaltana, J. Distler and A. Trimm, *Tinkertoys for the twisted D -series*, *JHEP* **04** (2015) 173 [[arXiv:1309.2299](#)] [[INSPIRE](#)].
- [91] O. Chacaltana, J. Distler and A. Trimm, *Tinkertoys for the E_6 theory*, *JHEP* **09** (2015) 007 [[arXiv:1403.4604](#)] [[INSPIRE](#)].
- [92] O. Chacaltana, J. Distler and A. Trimm, *Tinkertoys for the twisted E_6 theory*, [arXiv:1501.00357](#) [[INSPIRE](#)].
- [93] O. Chacaltana, J. Distler and A. Trimm, *Tinkertoys for the Z_3 -twisted D_4 theory*, [arXiv:1601.02077](#) [[INSPIRE](#)].
- [94] O. Chacaltana, J. Distler, A. Trimm and Y. Zhu, *Tinkertoys for the E_7 theory*, *JHEP* **05** (2018) 031 [[arXiv:1704.07890](#)] [[INSPIRE](#)].
- [95] O. Chacaltana, J. Distler, A. Trimm and Y. Zhu, *Tinkertoys for the E_8 theory*, [arXiv:1802.09626](#) [[INSPIRE](#)].
- [96] D. Collingwood and W. McGovern, *Nilpotent orbits in semisimple Lie algebra*, Routledge, London U.K. (2017).
- [97] D. Xie and K. Ye, *Argyres-Douglas matter and S -duality: Part II*, *JHEP* **03** (2018) 186 [[arXiv:1711.06684](#)] [[INSPIRE](#)].
- [98] F. Benini, Y. Tachikawa and D. Xie, *Mirrors of 3d Sicilian theories*, *JHEP* **09** (2010) 063 [[arXiv:1007.0992](#)] [[INSPIRE](#)].
- [99] T. Arakawa and A. Moreau, *On the irreducibility of associated varieties of W -algebras*, [arXiv:1608.03142](#).
- [100] A.D. Shapere and Y. Tachikawa, *Central charges of $N = 2$ superconformal field theories in four dimensions*, *JHEP* **09** (2008) 109 [[arXiv:0804.1957](#)] [[INSPIRE](#)].
- [101] D. Xie and P. Zhao, *Central charges and RG flow of strongly-coupled $N = 2$ theory*, *JHEP* **03** (2013) 006 [[arXiv:1301.0210](#)] [[INSPIRE](#)].
- [102] H. Shimizu, Y. Tachikawa and G. Zafrir, *Anomaly matching on the Higgs branch*, *JHEP* **12** (2017) 127 [[arXiv:1703.01013](#)] [[INSPIRE](#)].
- [103] T. Arakawa and A. Moreau, *Sheets and associated varieties of affine vertex algebras*, [arXiv:1601.05906](#).

- [104] A.M. Semikhatov, *The MFF singular vectors in topological conformal theories*, *JETP Lett.* **58** (1993) 860 [[hep-th/9311180](#)] [[INSPIRE](#)].
- [105] D. Adamovic, *A Construction of admissible $A_1^{(1)}$ modules of level $-4/3$* , [math/0401023](#) [[INSPIRE](#)].
- [106] D. Adamovic, *A realization of certain modules for the $N = 4$ superconformal algebra and the affine Lie algebra $A_2^{(1)}$* , [arXiv:1407.1527](#) [[INSPIRE](#)].
- [107] D. Adamovic, *Realizations of simple affine vertex algebras and their modules: the cases $\hat{sl}(2)$ and $osp(1,2)$* , [arXiv:1711.11342](#).
- [108] P. Shan, D. Xie and W. Yan, *Moduli space of Hitchin system and modules of vertex operator algebra*, in preparation.
- [109] D. Gaiotto and E. Witten, *S-duality of boundary conditions in $N = 4$ super Yang-Mills theory*, *Adv. Theor. Math. Phys.* **13** (2009) 721 [[arXiv:0807.3720](#)] [[INSPIRE](#)].