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$T\bar{T}$ Deformation of stress-tensor correlators from random geometry

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Abstract: We study stress-tensor correlators in the $T\bar{T}$-deformed conformal field theories in two dimensions. Using the random geometry approach to the $T\bar{T}$ deformation, we develop a geometrical method to compute stress-tensor correlators. More specifically, we derive the $T\bar{T}$ deformation to the Polyakov-Liouville conformal anomaly action and calculate three and four-point correlators to the first-order in the $T\bar{T}$ deformation from the deformed Polyakov-Liouville action. The results are checked against the standard conformal perturbation theory computation and we further check consistency with the $T\bar{T}$-deformed operator product expansions of the stress tensor. A salient feature of the $T\bar{T}$-deformed stress-tensor correlators is a logarithmic correction that is absent in two and three-point functions but starts appearing in a four-point function.

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1 Introduction

Conformal Field Theories (CFTs) describe universal behaviors of quantum field theories (QFTs), independent of model details, at the endpoint(s) of renormalization group (RG) flows and provide an important characterization of universality classes. In this paper, we study a deformation of two dimensional CFTs by the $T\bar{T}$ operator, a bilinear of the stress tensor [1], which exists in any QFTs with a stress tensor and is a model-independent deformation. Since $T\bar{T}$-deformed CFTs are not sensitive to the detail of the UV theories which flow to the parent IR CFTs, they may add new dimensions to the characterization of universality classes. Despite the fact that the $T\bar{T}$ deformation is power-counting non-renormalizable and is an irrelevant deformation in the sense of the RG, quite remarkably, $T\bar{T}$-deformed QFTs turned out to be UV-complete. Moreover, the $T\bar{T}$ deformation preserves integrability of the parent undeformed theory and the energy spectrum problem, for example, can be solved exactly [2, 3].

In contrast to asymptotic safety [4, 5], however, the $T\bar{T}$-deformed theories do not flow to UV fixed points and exhibit signs of non-locality [6, 7] and non-unitarity [8] at short distances set by the scale μ of the $T\bar{T}$ deformation. These peculiar features originate from the fact that the $T\bar{T}$ deformation, being an irrelevant operator, significantly alters the UV behavior of the parent theory. Because of this nature, the study of the $T\bar{T}$-deformed theories may provide a new perspective on the short-distance physics. This idea can be further sharpened by a remarkable dual property of the $T\bar{T}$ deformation: a $T\bar{T}$-deformed QFT $\mathcal{T}[\mu]$ on a 2d space X_0 is equivalent to the undeformed QFT $\mathcal{T}[0]$ on a UV-deformed 2d space X_μ that is importantly state-dependent [7, 9–11].[1] Furthermore, in relation to the deformed UV property, there is evidence that the $T\bar{T}$ deformation is related to 2d quantum gravity and string theory [3, 6, 7, 16–18].

With these field theoretical backgrounds as our motivation as well as the aim to understand dual gravitational implications in the AdS/CFT correspondence [19–21], we further develop Cardy's random geometry approach to the $T\bar{T}$ deformation [22] building on our previous work on the subject [21]. In this paper, we focus on stress-tensor correlators in the $T\bar{T}$-deformed CFTs and develop a new geometrical method to compute stress-tensor correlators. Earlier works studied stress-tensor two-point functions to the second order in the $T\bar{T}$ deformation [25] and three-point functions to the first order [25, 26]. The reference [25] uses the $T\bar{T}$ flow equation and the conformal perturbation theory, whereas the reference [26] combines the random geometry approach with the Ward-Takahashi (WT) identity for the stress tensor. Here we provide a new method that is purely based on the random geometry approach, generalizing the technique developed in our previous work [21] to the stress-tensor correlators. More specifically, we derive the $T\bar{T}$ deformation to the Polyakov-Liouville conformal anomaly action [24] and calculate three and four point correlators to

[1]See [12] for a more recent discussion in the case of curved spaces and its relation to the gravity dual description. This property is not specific to 2d relativistic theories but also present in 1d (non-)relativistic analogues of the $T\bar{T}$ deformation in which a rod picture of particles emerges [13, 14]. This dual property has also been exploited to reveal underlying symmetries in the $T\bar{T}$- and JT-deformed theories [15].

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the first-order in the $T\bar{T}$ deformation from the deformed Polyakov-Liouville action.[2] As we will see, one of the most interesting features of the $T\bar{T}$-deformed stress-tensor correlators is a logarithmic correction that is absent in two- and three-point functions but starts appearing in a four-point function.

The organization of the paper is as follows: in section 2, we first give a brief review of Cardy's random geometry approach [22] to the $T\bar{T}$ deformation and generalize it to curved background spaces. We then apply the result so obtained and compute the $T\bar{T}$ deformation to the Polyakov-Liouville conformal anomaly action, setting up the computation of stress-tensor correlators. In section 3, we develop and detail the algorithm to calculate $T\bar{T}$-deformed stress-tensor correlators. In section 4, as a concrete application, we calculate the 3-point stress-tensor correlators to the first order in the $T\bar{T}$ deformation from the deformed Polyakov-Liouville action, reproducing the known results found by different methods in [25, 26]. In section 5, as a further advanced application, we compute the 4-point stress-tensor correlators to the first order in the $T\bar{T}$ deformation. The results are checked against the standard conformal perturbation theory computation performed in appendix C. In section 6 we discuss the $T\bar{T}$ deformation to the stress tensor operator product expansions (OPEs) and check its consistency with the 4-point function results. In section 7, we comment on the translation of our results into the gravity dual [21] and give discussions on future works. Appendix A summarizes conventions used in this paper, and appendix B contains some details of the computations in the main text. In appendix D, we discuss the contour integral approach, another approach to the $T\bar{T}$ deformation, providing further checks of the correlators in the main text.

2 $T\bar{T}$-deformed Polyakov-Liouville action

2.1 $T\bar{T}$-deformation as random geometry

We work with quantum field theory on a two-dimensional space with metric $g_{ij}(x)$ of Euclidean signature. We define the stress-energy tensor $T^{ij}(x)$ via the variation of the Euclidean action as follows:[3]

$$\delta_g S = \frac{1}{4\pi}\int d^2x\,\sqrt{g}\,T^{ij}\delta g_{ij}. \tag{2.1}$$

where $i, j, \dots = 1, 2$ and $g = \det g_{ij}$. We define the "$T\bar{T}$ operator" $\mathcal{O}_{T\bar{T}}$ by

$$\mathcal{O}_{T\bar{T}} \equiv -\frac{1}{8}\epsilon_{ik}\epsilon_{jl}T^{ij}T^{kl}, \tag{2.2}$$

with $\epsilon_{12} = -\epsilon_{21} = \sqrt{g}$. In the special case of flat space $g_{ij} = \delta_{ij}$ with complex coordinates[4] $z = x^1 + ix^2$, $\bar{z} = x^1 - ix^2$, this becomes

$$\mathcal{O}_{T\bar{T}} = T\bar{T} - \Theta^2 = -\frac{1}{4}\det T_{ij}\,, \tag{2.3}$$

[2] We often refer to the Polyakov-Liouville action as the Liouville action in short in the main text.

[3] The convention in the present paper is related to that in [21] by $T^{ij}_{\text{here}} = -4\pi T^{ij}_{\text{there}}$.

[4] Our convention for complex coordinates is summarized in appendix A.

justifying the name of the operator, where

$$
\begin{aligned}
&T = T_{zz} = \frac{1}{4}(T_{11} - T_{22} - 2iT_{12}), \qquad \bar{T} = T_{\bar{z}\bar{z}} = \frac{1}{4}(T_{11} - T_{22} + 2iT_{12}), \\
&\Theta = T_{z\bar{z}} = \frac{1}{4}(T_{11} + T_{22}).
\end{aligned} \tag{2.4}
$$

However, we will work with general curved spacetime below.

The $T\bar{T}$-deformed theory $\mathcal{T}[\mu]$ of a CFT is characterized by a finite coupling μ of length dimension two. The original, undeformed CFT can be denoted by $\mathcal{T}[0]$. The $T\bar{T}$ deformation is defined through the following incremental change in the action when we go from $\mathcal{T}[\mu]$ to $\mathcal{T}[\mu+\delta\mu]$ with infinitesimal $\delta\mu$:

$$
S[\mu+\delta\mu] = S[\mu] + \frac{\delta\mu}{\pi^2}\int d^2x\sqrt{g}\,\mathcal{O}_{T\bar{T}} \equiv S[\mu] + \delta S\ . \tag{2.5}
$$

Here, the stress tensor T_{ij} entering into the definition of the operator $\mathcal{O}_{T\bar{T}}$ is that of the deformed theory $\mathcal{T}[\mu]$ rather than that of the undeformed theory $\mathcal{T}[0]$. The deformed theory $\mathcal{T}[\mu]$ of a finite coupling μ can be constructed from the undeformed theory $\mathcal{T}[0]$ by iteration of the infinitesimal deformation (2.5).

The idea of the random geometry approach [22] to the $T\bar{T}$ deformation is to split the $T\bar{T}$ operator by a Hubbard-Stratonovich transformation

$$
\exp\left(-\delta S\right) \propto \int [dh] \exp\left[-\frac{1}{8\delta\mu}\int d^2x\sqrt{g}\,\epsilon^{ik}\epsilon^{jl}h_{ij}h_{kl} - \frac{1}{4\pi}\int d^2x\sqrt{g}\,h_{ij}T^{ij}\right]. \tag{2.6}
$$

In view of (2.1), the last term in the exponential has the effect of changing the background metric from g to $g+h$. Therefore, the $T\bar{T}$ deformation can be interpreted as putting the original theory on randomly fluctuating geometries and averaging over them with a Gaussian weight.[5] The fluctuation part of the metric, h, is infinitesimal because the saddle point is at

$$
h^*_{ij} = -\frac{\delta\mu}{\pi}\epsilon_{ik}\epsilon_{jl}T^{kl}. \tag{2.7}
$$

We only have to keep track of up to $\mathcal{O}(\delta\mu)$ quantities and can drop $\mathcal{O}(\delta\mu^2)$ terms, in the $\delta\mu \to 0$ limit we are working in.

So, in this formulation, quantities in the deformed theory $\mathcal{T}[\delta\mu]$ with metric g can be written in terms of random geometry as

$$
\langle\ldots\rangle_{\delta\mu,g} = \mathcal{N}\int [dh]\exp\left[-\frac{1}{8\delta\mu}\int d^2x\sqrt{g}\,\epsilon^{ik}\epsilon^{jl}h_{ij}h_{kl}\right]\langle\ldots\rangle_{0,g+h}, \tag{2.8}
$$

[5] In other words, the $T\bar{T}$-deformed theories can be thought of as an ensemble of the T-deformed theories. This is the viewpoint put forward in our previous work on the gravity dual in [21]. This is somewhat reminiscent of an ensemble interpretation of near-conformal quantum mechanics dual to near-AdS$_2$ JT gravity [27] as suggested in a matrix model description [28]. The UV deformation is also common in both cases. However, it is not clear whether there is any relation between the two at all.

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where "…" represents general insertions,

$$\mathcal{N}^{-1} = \int [dh] \exp\left[-\frac{1}{8\delta\mu}\int d^2x\sqrt{g}\,\epsilon^{ik}\epsilon^{jl}h_{ij}h_{kl}\right] \tag{2.9}$$

is the normalization constant, and $\langle\ldots\rangle_{0,g+h}$ means the path integral in the undeformed theory $\mathcal{T}[0]$ with metric $g+h$.

We parametrize the deformation of the metric, h_{ij}, as

$$h_{ij} = \nabla_i\alpha_j + \nabla_j\alpha_i + 2g_{ij}\Phi, \tag{2.10}$$

where ∇_i is the covariant derivative with respect to the background metric g. This corresponds to the statement that, in two dimensions, any infinitesimal change in the metric can be decomposed into an infinitesimal coordinate transformation $x^i \to x^i + \alpha^i$ and an infinitesimal Weyl transformation $ds^2 \to e^{2\Phi}ds^2 \approx (1+2\Phi)ds^2$. We find it convenient to shift Φ and define ϕ by

$$\Phi = \phi - \frac{1}{2}\nabla_k\alpha^k, \tag{2.11}$$

so that α^i and ϕ represent the traceless and trace parts of h_{ij}, respectively. The Jacobian $\mathrm{Det}(\partial h/\partial(\alpha,\phi))$ in going from $[dh]$ to $[d\alpha][d\phi]$ does not depend on α, ϕ, although it depends on the background metric g. Therefore, in computing the path integral (2.8), we can replace $[dh]$ by $[d\alpha][d\phi]$ because the Jacobian factor cancels against that in the normalization factor. So, (2.8) can also be written as

$$\langle\ldots\rangle_{\delta\mu,g} = \mathcal{N}\int [d\alpha][d\phi] \exp\left[-\frac{1}{8\delta\mu}\int d^2x\sqrt{g}\,\epsilon^{ik}\epsilon^{jl}h_{ij}h_{kl}\right]\langle\ldots\rangle_{0,g+h}, \tag{2.12}$$

where h is given in terms of α^i, ϕ by (2.10) and (2.11), and

$$\mathcal{N}^{-1} = \int [d\alpha][d\phi] \exp\left[-\frac{1}{8\delta\mu}\int d^2x\sqrt{g}\,\epsilon^{ik}\epsilon^{jl}h_{ij}h_{kl}\right]. \tag{2.13}$$

The expression for the h Gaussian action in (2.12) in terms of α^i, ϕ is

$$\int d^2x\sqrt{g}\,\epsilon^{ik}\epsilon^{jl}h_{ij}h_{kl} = 2\int d^2x\sqrt{g}\left[\alpha_i\left(\Box_{\mathrm{v}} + \frac{R}{2}\right)\alpha^i + 4\phi^2\right]. \tag{2.14}$$

Here, R is the scalar curvature for the background metric g, and $\Box_{\mathrm{v}}$ is the vector Laplacian; namely, $\Box_{\mathrm{v}}\alpha^i \equiv \nabla^j\nabla_j\alpha^i$ for a quantity α^i with a vector index i.

In the above, we discussed going from $\mathcal{T}[0]$ to $\mathcal{T}[\delta\mu]$, but going from $\mathcal{T}[\mu]$ to $\mathcal{T}[\mu+\delta\mu]$ is exactly the same. Quantities in theory $\mathcal{T}[\mu+\delta\mu]$ are related to those in theory $\mathcal{T}[\mu]$ as

$$\langle\ldots\rangle_{\mu+\delta\mu,g} = \mathcal{N}\int [d\alpha][d\phi]\, \exp\left[-\frac{1}{8\delta\mu}\int d^2x\sqrt{g}\,\epsilon^{ik}\epsilon^{jl}h_{ij}h_{kl}\right]\langle\ldots\rangle_{\mu,g+h}. \tag{2.15}$$

This will give a differential equation (Burgers equation), upon solving which we can get quantities in the finitely deformed theory $\mathcal{T}[\mu]$.

2.2 Polyakov-Liouville action

Conformal anomaly dictates that the partition function $Z_0[g]$ of a CFT on a two-dimensional curved manifold with metric g_{ij} is completely fixed by g and the central charge c as [23, 24]

$$Z_0[g] = e^{-S_0[g]} Z_0[\delta], \tag{2.16}$$

where $S_0[g]$ is the so-called Liouville action,

$$S_0[g] = \frac{c}{96\pi} \int d^2x \sqrt{g}\, R \,\Box^{-1} R, \tag{2.17}$$

the operator $\Box$ is the scalar Laplacian for the metric g, and $Z_0[\delta]$ is the partition function in flat space, $g_{ij} = \delta_{ij}$. In this paper, we will consider the case where $Z_0[\delta]$ is the partition function on $\mathbb{R}^2$ and set $Z_0[\delta] = 1$.

In two dimensions, we can always bring the metric into the conformal gauge,

$$g_{ij}(x) = e^{2\Omega(x)} \delta_{ij}, \tag{2.18}$$

by an appropriate diffeomorphism. In the conformal gauge, in which $R = -2e^{-2\Omega}\Box\Omega$, eq. (2.17) reduces to

$$Z_0[e^{2\Omega}\delta] = e^{-S_0}, \qquad S_0 = -\frac{c}{24\pi} \int d^2x\, \delta^{ij} \partial_i\Omega\, \partial_j\Omega. \tag{2.19}$$

We will also call (2.19) the Liouville action.

Because the Liouville action S_0 contains complete information about the dependence of the CFT partition on the metric, we can compute arbitrary correlators of the stress tensor T_{ij} by shifting the metric, $g \to g + h$, in the partition function and differentiating it with respect to h_{ij}. This is a straightforward, if tedious, procedure if one uses the covariant form of the Liouville action (2.17). The same stress-tensor correlators can be computed also from the conformal-gauge Liouville action (2.19), which contains the same information as the covariant one (2.17). However, the procedure is slightly more nontrivial, because the shift $g \to g + h$ must be accompanied by a diffeomorphism to bring the metric back to the conformal gauge. We will discuss this procedure in more detail in section 3.

2.3 $T\bar{T}$-deformed Polyakov-Liouville action

The goal here is to apply the general formula (2.12) to the Liouville action $S_0[g]$ in (2.17) to obtain a $T\bar{T}$-deformed Liouville action. The partition function for the deformed theory $\mathcal{T}[\delta\mu]$ is, from the general formula (2.12),

$$Z_{\delta\mu}[g] \equiv e^{-S_{\delta\mu}[g]} = \mathcal{N}^{-1} \int [d\alpha][d\phi] \exp\left[-\frac{1}{8\delta\mu} \int d^2x \sqrt{g}\, \epsilon^{ik}\epsilon^{jl} h_{ij} h_{kl} - S_0[g+h] \right], \tag{2.20}$$

where h is given in terms of α, ϕ as in (2.10). One way to carry this out is to expand $S_0[g+h]$ in h by expanding the quantities appearing in it, such as R and $\Box$, up to quadratic order in h, and perform the Gaussian integral. Here we take a different — although equivalent

— approach. In two dimensions, we can bring the shifted metric $g + h$ into the original metric g by a diffeomorphism up to a Weyl transformation, even for finite h. Namely,

$$(g_{ij}(x) + h_{ij}(x))\, dx^i dx^j = e^{2\Psi(\tilde{x})} g_{ij}(\tilde{x})\, d\tilde{x}^i d\tilde{x}^j, \tag{2.21}$$

where

$$\tilde{x}^i = x^i + A^i(x) \tag{2.22}$$

for some A^i, Ψ. As mentioned before, at linear order in h, these are given by $A^i = \alpha^i, \Psi = \Phi$. The higher order expressions for A^i, Ψ can be obtained by expanding A^i, Φ in powers of h as

$$\begin{aligned} A^i(x) &= \alpha^i(x) + A^i_{(2)}(x) + A^i_{(3)}(x) + \cdots\,, \\ \Psi(\tilde{x}) &= \Phi(\tilde{x}) + \Psi_{(2)}(\tilde{x}) + \Psi_{(3)}(\tilde{x}) + \cdots\,. \end{aligned} \tag{2.23}$$

By substituting this expansion into (2.21) and comparing terms order by order, we can find $A^i_{(n)}, \Psi_{(n)}$ to any order in principle. The explicit form of the second-order terms $A^i_{(2)}, \Psi_{(2)}$ is presented in appendix B. An important thing to note is that the function $g_{ij}(\tilde{x})$ appearing on the right-hand side of (2.21) is the same function as the original metric function $g_{ij}(x)$; we are only plugging $\tilde{x}$ into it instead of x. Namely, it is not the transformed metric function $\tilde{g}_{ij}$ defined by $\frac{\partial \tilde{x}^i}{\partial x^k}\frac{\partial \tilde{x}^j}{\partial x^l}\tilde{g}_{ij}(\tilde{x}) = g_{kl}(x)$.

The Liouville action for the new metric $g'_{ij}(\tilde{x}) \equiv e^{2\Psi(\tilde{x})} g_{ij}(\tilde{x})$ can be found by using the well-known formulas in two dimensions,

$$\begin{aligned} \sqrt{g'(\tilde{x})} &= e^{2\Psi(\tilde{x})}\sqrt{g(\tilde{x})}, \qquad & R_{g'}(\tilde{x}) &= e^{-2\Psi(\tilde{x})}(R_g(\tilde{x}) - 2\tilde{\Box}_g \Psi(\tilde{x})), \\ \tilde{\Box}_{g'} &= e^{-2\Psi(\tilde{x})}\tilde{\Box}_g, & \tilde{\Box}^{-1}_{g'} &= \tilde{\Box}^{-1}_g e^{2\Psi(\tilde{x})}, \end{aligned} \tag{2.24}$$

where $R_g(\tilde{x})$ is the scalar curvature for the metric $g(\tilde{x})$ and $\tilde{\Box}_g$ is the Laplacian for the metric $g(\tilde{x})$. They are identical with $R_g(x)$ and $\Box_g$; we just replace x in them with $\tilde{x}$. The Liouville action $S_0[g + h]$ can be evaluated as

$$\begin{aligned} S_0[g(x) + h(x)] &= S_0[g'(\tilde{x})] \\ &= \frac{c}{96\pi}\int d^2\tilde{x}\sqrt{g'(\tilde{x})}\, R_{g'}(\tilde{x})\, \tilde{\Box}^{-1}_{g'} R_{g'}(\tilde{x}) \\ &= \frac{c}{96\pi}\int d^2\tilde{x}\sqrt{g(\tilde{x})}\, \Big(R_g(\tilde{x}) - 2\tilde{\Box}_g\Psi(\tilde{x})\Big)\, \tilde{\Box}^{-1}_g \Big(R_g(\tilde{x}) - 2\tilde{\Box}_g\Psi(\tilde{x})\Big) \\ &= \frac{c}{96\pi}\int d^2x\sqrt{g(x)}\, (R_g(x) - 2\Box_g\Psi(x))\, \Box^{-1}_g (R_g(x) - 2\Box_g\Psi(x)) \\ &= \frac{c}{96\pi}\int d^2x\sqrt{g}\, \Big(R\Box^{-1}R - 4R\Psi + 4\Psi\Box\Psi\Big)\,, \end{aligned} \tag{2.25}$$

where in the fourth equality we replaced $\tilde{x}$ by x because it is a dummy integration variable. To get to the last line, we integrated by parts, assuming that the relevant fields vanish at infinity sufficiently fast. We will always assume this and freely use integration by parts. Also, we simply wrote $R_g \to R$, $\Box_g \to \Box$ and omitted the argument x. Therefore, the change in the Liouville action due to the $T\bar{T}$ deformation,

$$\delta S[g] \equiv S_{\delta\mu}[g] - S_0[g], \tag{2.26}$$

is given by

$$e^{-\delta S[g]} = \mathcal{N}^{-1} \int [d\alpha][d\phi] \exp\Big[-\frac{1}{4\delta\mu} \int d^2x \sqrt{g} \left(\alpha_i \left(\Box_{\rm v} + \frac{R}{2}\right) \alpha^i + 4\phi^2\right) \\ - \frac{c}{24\pi} \int d^2x \sqrt{g} \left(-R\Phi + \Phi\Box\Phi - R\Psi_{(2)}\right)\Big]. \tag{2.27}$$

Here, we kept terms that are up to quadratic order in α, ϕ, which are relevant in the $\delta\mu \to 0$ limit.

To carry out the integral, let us write the exponent in (2.27) as

$$\int \sqrt{g}(-X^\dagger M X + b^\dagger X + X^\dagger b) = \int \sqrt{g} \left[-(X^\dagger - b^\dagger M^{-1}) M (X - M^{-1}b) + b^\dagger M^{-1} b\right], \tag{2.28}$$

where $M^\dagger = M$ with

$$X = \begin{pmatrix} \alpha^i \\ \phi \end{pmatrix}, \qquad X^\dagger = \begin{pmatrix} \alpha_i & \phi \end{pmatrix}, \qquad M = M_0 + M_1, \qquad b = \frac{c}{96\pi} \begin{pmatrix} \nabla^i R \\ 2R \end{pmatrix}. \tag{2.29}$$

We have split the matrix M into the leading term, $M_0 \propto (\delta\mu)^{-1}$, and the subleading term, $M_1 \propto c(\delta\mu)^0$. The leading term M_0 comes from the first line of (2.27) and is given by

$$(M_0)^I{}_J = \frac{1}{\delta\mu} \begin{pmatrix} \frac{1}{4}\left(\Box_{\rm v} + R/2\right) \delta^i{}_j & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.30}$$

The subleading term M_1 comes from the last two terms in (2.27) and is given by

$$(M_1)^I{}_J = \frac{c}{24\pi} \begin{pmatrix} \partial^i \Box \nabla_j & \partial^i \Box \\ \Box \nabla_j & -\Box \end{pmatrix} + (M_1')^I{}_J\,, \tag{2.31}$$

where M_1' is the contribution from the last $(-R\Psi_{(2)})$ term in (2.27). In the $\delta\mu \to 0$ limit, the saddle-point value of the integral is determined solely by M_0. Explicitly, the saddle-point action is given by

$$\exp\left[\int d^2x \sqrt{g}\, b^\dagger M_0^{-1} b\right] = \exp\left[\left(\frac{c}{48\pi}\right)^2 \delta\mu \int d^2x \sqrt{g}\, R \left(1 - \nabla^k \frac{1}{\Box_{\rm v} + R/2} \nabla_k\right) R\right] \tag{2.32}$$

and the saddle point is at $X = M_0^{-1} b$, namely at

$$\alpha^i = \frac{c\,\delta\mu}{24\pi} \frac{1}{\Box_{\rm v} + R/2} \nabla^i R, \qquad \phi = \frac{c\,\delta\mu}{48\pi} R, \tag{2.33}$$

dropping irrelevant $\mathcal{O}(\delta\mu^2)$ terms.

The subleading term M_1 is important in evaluating the Gaussian fluctuation about the saddle point. Combined with the contribution from the normalization constant $\mathcal{N}^{-1}$, the Gaussian fluctuation gives the extra factor

$$\sqrt{\frac{\operatorname{Det} M_0}{\operatorname{Det}(M_0 + M_1)}} = e^{-\frac{1}{2} \operatorname{Tr} \log(1 + M_0^{-1} M_1)} = e^{-\frac{1}{2} \operatorname{Tr}(M_0^{-1} M_1)} \equiv e^{-\delta S_{\rm fluct}[g]}, \tag{2.34}$$

where we dropped $\mathcal{O}(\delta\mu^2)$ quantities. This fluctuation term contains divergence of the form

$$\mathrm{Tr}[f] = \int d^2x \sqrt{g(x)} \langle x|f|x\rangle = f(x)\,\delta^2(0) \tag{2.35}$$

and requires regularization and renormalization. However, we can argue that it must be renormalized to zero as follows. Note that $\delta S_{\text{fluc}} \sim M_0^{-1}M_1 \sim \mathcal{O}(c\,\delta\mu)$, in contrast to $\delta S_{\text{saddle}} \sim b^\dagger M_0^{-1} b \sim \mathcal{O}(c^2\delta\mu)$. Conformal perturbation theory indicates that the first-order corrections are always of order $\mathcal{O}(c^2\delta\mu)$, which excludes the contributions of order $\mathcal{O}(c\,\delta\mu)$ and thus δS_{fluc} must be renormalized away: $\delta S_{\text{fluc}} = 0$.

To summarize, the $T\bar{T}$-deformed Liouville action at $\mathcal{O}(\delta\mu)$ is

$$\delta S[g] = \delta S_{\text{saddle}}[g], \tag{2.36}$$

where the saddle-point action is given by

$$\delta S_{\text{saddle}}[g] = -\left(\frac{c}{48\pi}\right)^2 \delta\mu \int d^2x \sqrt{g}\, R \left(1 - \nabla^k \frac{1}{\Box_{\text{v}} + R/2} \nabla_k\right) R. \tag{2.37}$$

The fluctuation term $\delta S_{\text{fluct}}[g]$ that can in principle be present vanishes after renormalization and do not contribute to $\delta S[g]$ at order $\delta\mu$.

2.4 Conformal gauge and the flow equation

In the above, we presented in (2.37) the $T\bar{T}$-deformed Liouville action for a generic metric g. Here we discuss the $T\bar{T}$-deformed action in the conformal gauge (2.18) in which the metric is given, in complex coordinates, by

$$ds^2 = e^{2\Omega} dz\, d\bar{z}. \tag{2.38}$$

For this purpose, one could, of course, simply plug the conformal gauge metric (2.38) into the covariant formula (2.37), but here we will run the procedure of integrating out the h field again, as in the previous subsection. The reason is that, because of the simplicity of the conformal gauge, we can derive not only the first-order deformed action but also a flow equation that, in principle, determines the deformed action at a finite coupling μ. The first-order result can then be obtained by the leading order solution to the flow equation.

Let us denote by $S_\mu[e^{2\Omega}\delta]$ the deformed action in the conformal gauge at a finite coupling μ. As in the previous subsection, we can go from μ to $\mu + \delta\mu$ by considering deformations to the metric, $g \to g + h$, and integrating over h, where g is the conformal-gauge metric (2.38). The formula that determines the deformed action at $\mu + \delta\mu$ is given by (2.15). In the conformal gauge, the Hubbard-Stratonovich field h can be parametrized as

$$h_{ij} = \nabla_i\alpha_j + \nabla_j\alpha_i + 2e^{2\Omega}\delta_{ij}\Phi, \qquad \Phi = \phi - \frac{1}{2}\nabla_k\alpha^k = \phi - e^{-2\Omega}(\partial\bar{\alpha} + \bar{\partial}\alpha), \tag{2.39}$$

where $\partial \equiv \partial_z$, $\bar{\partial} \equiv \partial_{\bar{z}}$, $\alpha \equiv \alpha_z$, $\bar{\alpha} \equiv \alpha_{\bar{z}}$. As before, after a compensating diffeomorphism $\tilde{x}^i = x^i + A^i(x)$, we can bring the deformed metric $(g_{ij} + h_{ij})dx^i dx^j$ back into the original (conformal) form up to a Weyl rescaling, as $e^{2\Psi(\tilde{z},\bar{\tilde{z}})} e^{2\Omega(\tilde{z},\bar{\tilde{z}})} d\tilde{z}\, d\bar{\tilde{z}}$, where $\Psi \approx \Phi$

at linear order in h. Therefore, the change in the action appearing in "$\langle\ldots\rangle_{\mu,g+h}$" in the formula (2.15) is

$$\Delta S_\mu \equiv S_\mu[e^{2(\Omega+\Phi)}\delta] - S_\mu[e^{2\Omega}\delta] = \int d^2x\,\frac{\delta S_\mu}{\delta\Omega}\Phi = \int d^2x\,\frac{\delta S_\mu}{\delta\Omega}(\phi - e^{-2\Omega}(\partial\bar{\alpha}+\bar{\partial}\alpha)). \quad (2.40)$$

We also need to find the expression for the Hubbard-Stratonovich action (2.14) in the conformal gauge. In the $(z,\bar{z})$-basis, i.e. for $i,j=z,\bar{z}$, we have

$$\Box_{\rm v}\delta_i{}^j = 4e^{-2\Omega}\begin{pmatrix}\partial\bar{\partial}-2(\partial\Omega)\bar{\partial}-(\partial\bar{\partial}\Omega) & \\ & \partial\bar{\partial}-2(\bar{\partial}\Omega)\partial-(\partial\bar{\partial}\Omega)\end{pmatrix}. \quad (2.41)$$

Because $R=-8e^{-2\Omega}\partial\bar{\partial}\Omega$, this means that

$$\begin{aligned}(\Box_{\rm v}+R/2)\delta_i{}^j &= 4e^{-2\Omega}\begin{pmatrix}\partial\bar{\partial}-2(\partial\Omega)\bar{\partial}-2(\partial\bar{\partial}\Omega) & \\ & \partial\bar{\partial}-2(\bar{\partial}\Omega)\partial-2(\partial\bar{\partial}\Omega)\end{pmatrix} \\ &= 4e^{-2\Omega}\begin{pmatrix}\bar{\partial}e^{2\Omega}\partial & \\ & \partial e^{2\Omega}\bar{\partial}\end{pmatrix}e^{-2\Omega}. \end{aligned} \quad (2.42)$$

Therefore, the Hubbard-Stratonovich action is

$$\begin{aligned}S_{\rm HS} &= \frac{1}{4\delta\mu}\int d^2x\sqrt{g}\left(\alpha_i\left(\Box_{\rm v}+\frac{R}{2}\right)\alpha^i+4\phi^2\right) \\ &= \frac{2}{\delta\mu}\int d^2x\left(\bar{\alpha}e^{-2\Omega}\bar{\partial}e^{2\Omega}\partial e^{-2\Omega}\alpha+\alpha e^{-2\Omega}\partial e^{2\Omega}\bar{\partial}e^{-2\Omega}\bar{\alpha}+\frac{1}{2}e^{2\Omega}\phi^2\right). \end{aligned} \quad (2.43)$$

Combining (2.40) and (2.43), the path integral appearing in the formula (2.15) can be written as

$$\int[d\alpha][d\phi]\,e^{-S_{\rm HS}-\Delta S_\mu} = \int[d\alpha][d\phi]\,\exp\left[\int d^2x(-X^\dagger MX+b^\dagger X+X^\dagger b)\right], \quad (2.44)$$

where

$$\begin{aligned}X &= \begin{pmatrix}\alpha\\ \bar{\alpha}\\ \phi\end{pmatrix}, \qquad X^\dagger = \begin{pmatrix}\bar{\alpha} & \alpha & \phi\end{pmatrix}, \qquad b=\frac{1}{2}\begin{pmatrix}\partial(e^{-2\Omega}(\delta S_\mu/\delta\Omega))\\ \bar{\partial}(e^{-2\Omega}(\delta S_\mu/\delta\Omega))\\ \delta S_\mu/\delta\Omega\end{pmatrix}, \\ M &= \frac{1}{\delta\mu}\begin{pmatrix}2e^{-2\Omega}\bar{\partial}e^{2\Omega}\partial e^{-2\Omega} & & \\ & 2e^{-2\Omega}\partial e^{2\Omega}\bar{\partial}e^{-2\Omega} & \\ & & e^{2\Omega}\end{pmatrix}. \end{aligned} \quad (2.45)$$

Here we ignored $\mathcal{O}(\delta\mu)$ terms, which are irrelevant for computing the saddle-point value.

As in the previous subsection, the result of the path integral (2.44) consists of the saddle-point part $\delta S_\mu^{\rm saddle} = -\int d^2x\,b^\dagger M^{-1}b$ and the fluctuation part $\delta S_\mu^{\rm fluct}$. We will assume that $\delta S_\mu^{\rm fluct}$ vanishes, which is correct at linear order in μ as we argued in the previous subsection. For a finite μ, whether this assumption is valid or not must be independently checked, but for large c this is certainly true because $\delta S_\mu^{\rm fluct}$ is parametrically smaller than $\delta S_\mu^{\rm saddle}$.

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Under the assumption of the vanishing fluctuation part, we can read off the change in the effective action $\delta S_\mu[e^{2\Omega}\delta]$ as follows:

$$\delta S_\mu \equiv \delta S_\mu^{\text{saddle}} = \frac{\delta\mu}{16}\int d^2z\,\frac{\delta S_\mu}{\delta\Omega}\,e^{-2\Omega}\Big(\bar\partial e^{2\Omega}\frac{1}{\partial}e^{-2\Omega}\frac{1}{\bar\partial}e^{2\Omega}\partial e^{-2\Omega} + \partial e^{2\Omega}\frac{1}{\bar\partial}e^{-2\Omega}\frac{1}{\partial}e^{2\Omega}\bar\partial e^{-2\Omega} - 2\Big)\frac{\delta S_\mu}{\delta\Omega}\,. \tag{2.46}$$

This can be rewritten as a differential equation governing the flow of the effective action $S_\mu[e^{2\Omega}\delta]$ as follows:

$$\frac{\partial}{\partial\mu}S_\mu = \frac{1}{16}\int d^2z\,\frac{\delta S_\mu}{\delta\Omega}\,e^{-2\Omega}\Big(\bar\partial e^{2\Omega}\frac{1}{\partial}e^{-2\Omega}\frac{1}{\bar\partial}e^{2\Omega}\partial e^{-2\Omega} + \partial e^{2\Omega}\frac{1}{\bar\partial}e^{-2\Omega}\frac{1}{\partial}e^{2\Omega}\bar\partial e^{-2\Omega} - 2\Big)\frac{\delta S_\mu}{\delta\Omega}\,. \tag{2.47}$$

If we expand $S_\mu[e^{2\Omega}\delta]$ in powers of μ as $S_\mu = \sum_n \frac{\mu^n}{n!}\mathbf{S}_n$, this equation can be recast into a recursive relation:

$$\mathbf{S}_{n+1} = \frac{1}{16}\sum_{k=0}^{n}\binom{n}{k}\int d^2z\,\frac{\delta\mathbf{S}_{n-k}}{\delta\Omega}\,e^{-2\Omega}\Big(\bar\partial e^{2\Omega}\frac{1}{\partial}e^{-2\Omega}\frac{1}{\bar\partial}e^{2\Omega}\partial e^{-2\Omega} + \partial e^{2\Omega}\frac{1}{\bar\partial}e^{-2\Omega}\frac{1}{\partial}e^{2\Omega}\bar\partial e^{-2\Omega} - 2\Big)\frac{\delta\mathbf{S}_k}{\delta\Omega}\,. \tag{2.48}$$

We can find the deformed action at linear order in $\mu \to \delta\mu$ by starting with the conformal-gauge Liouville action (2.19) and its Ω-derivative:

$$\mathbf{S}_0[e^{2\Omega}\delta] = \frac{c}{6\pi}\int d^2x\,\Omega\,\partial\bar\partial\Omega, \qquad \frac{\delta\mathbf{S}_0[e^{2\Omega}\delta]}{\delta\Omega} = \frac{c}{3\pi}\partial\bar\partial\Omega\,. \tag{2.49}$$

By substituting these into (2.48), we immediately find the $T\bar{T}$-deformed Liouville action at linear order,

$$\mathbf{S}_1[e^{2\Omega}\delta] = -\frac{c^2}{72\pi^2}\int d^2z\,\Omega\partial\bar\partial \times\left[1 - \frac{1}{2}e^{-2\Omega}\Big(\bar\partial e^{2\Omega}\frac{1}{\partial}e^{-2\Omega}\frac{1}{\bar\partial}e^{2\Omega}\partial + \partial e^{2\Omega}\frac{1}{\bar\partial}e^{-2\Omega}\frac{1}{\partial}e^{2\Omega}\bar\partial\Big)\right]e^{-2\Omega}\partial\bar\partial\Omega\,. \tag{2.50}$$

One can check that this is the same as what one obtains by plugging the conformal metric (2.38) into the covariant formula (2.37).

By integration by parts, we can further rewrite (2.50) in a form that does not contain nonlocal inverse operators ∂^{-1} and $\bar\partial^{-1}$. As the result, the final expression for the first-order deformation to the Polyakov-Liouville action is given by

$$\delta S_L[e^{2\Omega}\delta] \equiv \delta\mu\,\mathbf{S}_1[e^{2\Omega}\delta] = \frac{c^2\,\delta\mu}{72\pi^2}\int d^2z\,e^{-2\Omega}\Big[-2(\partial\Omega)(\bar\partial\Omega)(\partial\bar\partial\Omega) + (\partial\Omega)^2(\bar\partial\Omega)^2\Big]\,. \tag{2.51}$$

This is one of the key results in this paper and will be applied to the computation of the deformed stress tensor correlators in sections 4 and 5.

2.5 Flow of geometry

It has been observed [7, 9–11] that the $T\bar{T}$-deformed theory can be regarded as the undeformed theory living in a deformed geometry. In the random geometry approach, where observables in the deformed theory with metric g is related to ones in the undeformed theory with metric $g+h$ via (2.12), such flow of geometry is manifest. We parametrized the change in the metric in terms of a diffeomorphism parametrized by α^i and a Weyl transformation parametrized by ϕ as in (2.10), and their saddle point values (2.33) in a curved space can be written in the conformal gauge, $g_{ij}=e^{2\omega}\delta_{ij}$, as

$$\alpha^z=\frac{c\,\delta\mu}{6\pi}\frac{1}{\bar{\partial}}\left[e^{-2\omega}((\bar{\partial}\omega)^2-\bar{\partial}^2\omega)\right], \qquad \phi=-\frac{c\,\delta\mu}{6\pi}e^{-2\omega}\partial\bar{\partial}\omega. \tag{2.52}$$

If we recall the value of the stress tensor in the conformal gauge,[6]

$$\Theta=\frac{c}{6}\partial\bar{\partial}\omega, \qquad T=\frac{c}{6}\left((\partial\omega)^2-\partial^2\omega\right), \qquad \bar{T}=\frac{c}{6}\left((\bar{\partial}\omega)^2-\bar{\partial}^2\omega\right), \tag{2.53}$$

the diffeomorphism can be written as

$$\alpha^z=\frac{\delta\mu}{\pi}\frac{1}{\bar{\partial}}(e^{-2\omega}\bar{T})=\frac{\delta\mu}{\pi}\int_{\bar{X}}^{\bar{z}}d\bar{z}'e^{-2\omega(z')}\bar{T}(\bar{z}'), \tag{2.54}$$

which agrees with the result in [12] derived from a 2d topological gravity [29] and the holographic gravity dual. This is an alternative derivation of the known result [7, 10, 11] and generalizes it to curved spaces [12]. It should, however, be noted that the aforementioned known result is derived in a different "gauge". Namely, specializing to the flat space background, the metric deformation is parametrized by $h_{ij}=\partial_i\alpha_j+\partial_j\alpha_i$ as opposed to (2.39) without introducing the Weyl scalar ϕ. Then from (2.7), for a finite μ, it is easy to find the coordinate transformation $z\mapsto Z^{(\mu)}\equiv z+\alpha^z$ with

$$\alpha^z=\frac{1}{\pi}\int_0^\mu d\mu'\left[\int_{\bar{X}}^{\bar{z}}d\bar{z}'\bar{T}^{(\mu')}(z',\bar{z}')-\int_X^z dz'\Theta^{(\mu')}(z',\bar{z}')\right], \tag{2.55}$$

where the deformed stress tensors are given by

$$\bar{T}^{(\mu)}(z,\bar{z})=\left(\frac{\partial\bar{Z}^{(\mu)}}{\partial\bar{z}}\right)^2\bar{T}(\bar{Z}^{(\mu)})+\frac{c}{12}\{\bar{Z}^{(\mu)},\bar{z}\}, \;\; \Theta^{(\mu)}(z,\bar{z})=-\frac{\mu}{\pi}\left(T^{(\mu)}\bar{T}^{(\mu)}-(\Theta^{(\mu)})^2\right). \tag{2.56}$$

The appearance of the stress tensor trace Θ in the coordinate shift α^z reflects the difference in "gauges". Note also that this is a formal expression and requires point-splitting regularizations as composite operators.

[6] The first equation is the standard conformal anomaly, while T and $\bar{T}$ can be obtained by the conservation law, $\nabla^i T_{ij}=0$. Note that they are the Schwarzian derivatives with $\omega(z,\bar{z})=\frac{1}{2}\ln\left(f'(z)\bar{f}'(\bar{z})\right)$. This also means that our analysis of the flowing geometry here is restricted to the vacuum state.

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3 T^{ij} correlators

3.1 Computing T^{ij} correlators

The correlators of the stress-energy tensor T^{ij} in quantum field theory on a space with metric g_{ij} can be computed from the generating function

$$\begin{aligned}\left\langle e^{-\frac{1}{4\pi}\int d^2x\sqrt{g}\,h_{ij}T^{ij}}\right\rangle_g &= \langle 1\rangle_g - \frac{1}{4\pi}\int d^2x\sqrt{g}\,h_{ij}\langle T^{ij}\rangle_g \\ &\quad + \frac{1}{2(4\pi)^2}\iint d^2x\sqrt{g}\,d^2x'\sqrt{g'}\,h_{ij}\,h'_{kl}\langle T^{ij}T'^{kl}\rangle_g + \cdots \\ &= Z[g+h], \end{aligned} \tag{3.1}$$

where $'$ means that the argument is x'. In the last line, we used the fact that, by the very definition of the stress tensor, this is equal to the partition function of the same theory in the background metric $g+h$.[7]

Now assume that the effective action for the theory in the background metric $g+h$ is known, namely,

$$Z[g+h] = e^{-S_{\rm eff}[g+h]}. \tag{3.2}$$

Then, by comparing this to (3.1), we find that

$$\begin{aligned} S_{\rm eff}[g+h] - S_{\rm eff}[g] &= \frac{1}{4\pi}\int d^2x\sqrt{g}\,h_{ij}\langle T^{ij}\rangle_{g,\rm c} \\ &\quad - \frac{1}{2(4\pi)^2}\iint d^2x\sqrt{g}\,d^2x'\sqrt{g'}\,h_{ij}\,h'_{kl}\langle T^{ij}T'^{kl}\rangle_{g,\rm c} + \cdots, \end{aligned} \tag{3.3}$$

where $\langle\ldots\rangle_{g,\rm c} \equiv \langle\ldots\rangle_{g,\text{connected part}}/\langle 1\rangle_g$. If we know the explicit form of the effective action $S_{\rm eff}[g]$, we can use this relation to compute n-point functions of T^{ij} for any n. For CFT, the effective action is given by the Liouville action $S_0[g]$ in (2.17). For the $T\bar{T}$-deformed theory $\mathcal{T}[\delta\mu]$, the correction to the effective action is (2.36).

In particular, consider the flat space, $g_{ij} = \delta_{ij}$, as the background. From (2.10) and (2.11) the metric deformation h_{ij} is given by

$$h_{ij} = \partial_i\alpha_j + \partial_j\alpha_i + 2\delta_{ij}\Phi, \qquad \Phi = \phi - \frac{1}{2}\partial_k\alpha^k, \tag{3.4}$$

or, in complex coordinates,

$$h_{zz} = 2\partial\alpha, \quad h_{\bar{z}\bar{z}} = 2\bar{\partial}\bar{\alpha}, \quad h_{z\bar{z}} = \phi, \qquad \Phi = \phi - (\bar{\partial}\alpha + \partial\bar{\alpha}), \tag{3.5}$$

[7]Note that the meaning of h_{ij} here is different from that of h_{ij} introduced in the previous section. Here, h_{ij} is introduced so that we can obtain T^{ij} correlators by expanding the generating function $Z[g+h]$ in powers of it. We have to keep up to the h^n terms if we want to compute n-point functions of T^{ij}. On the other hand, h_{ij} in the previous section represents random geometries to be averaged over to produce a $T\bar{T}$-deformed action. Only up to h^2 terms contribute for the Gaussian integral and higher order terms are irrelevant.

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where we introduced a shorthand notation:

$$\partial \equiv \partial_z, \quad \bar{\partial} \equiv \partial_{\bar{z}}, \qquad \alpha \equiv \alpha_z, \quad \bar{\alpha} \equiv \alpha_{\bar{z}}. \tag{3.6}$$

In this case, (3.3) gives

$$\begin{aligned} S_{\text{eff}}[\delta+h] = &\frac{2}{\pi}\int d^2x \left\langle \partial\alpha\bar{T} + \bar{\partial}\bar{\alpha}T + \phi\Theta \right\rangle_{\text{c}} \\ &-\frac{1}{2}\left(\frac{2}{\pi}\right)^2 \iint d^2x\, d^2x' \left\langle (\partial\alpha\bar{T} + \bar{\partial}\bar{\alpha}T + \phi\Theta)(\partial'\alpha'\bar{T}' + \bar{\partial}'\bar{\alpha}'T' + \phi'\Theta') \right\rangle_{\text{c}} + \cdots, \end{aligned} \tag{3.7}$$

where $\langle \ldots \rangle_{\text{c}} \equiv \langle \ldots \rangle_{\delta,\text{c}}$. Therefore, from the coefficients of $\bar{\partial}\bar{\alpha}$, $\partial\alpha$, and ϕ, we can read off the correction functions for T, $\bar{T}$, and Θ, respectively.

3.2 Conformal gauge

The above procedure is applicable if we know the effective action for $S_{\text{eff}}[g]$ for the general metric g. However, in two dimensions, any metric can be brought into conformal gauge, (2.18). Therefore, knowing the effective action for the conformal-gauge metric, $S_{\text{eff}}[e^{2\Omega}\delta]$, is sufficient for computing T^{ij} correlators in any background metric. For example, for CFT, the conformal-gauge Liouville action (2.19) is sufficient. Here, let us discuss in detail how to use such a conformal-gauge effective action to compute T^{ij} correlators.

The procedure to compute the T^{ij} correlator in a given background metric g using a conformal-gauge effective action $S_{\text{eff}}[e^{2\Omega}\delta]$ is as follows:

(i) Rewrite the background metric g in the conformal gauge as $g_{ij} = e^{2\omega}\delta_{ij}$.

(ii) Given the metric shift $h_{ij} = \nabla_i\alpha_j + \nabla_j\alpha_i + 2e^{2\omega}\delta_{ij}\Phi$, find the diffeomorphism $\tilde{x}^i = x^i + A^i(x)$ which brings the shifted metric $g+h$ back into the conformal gauge. Namely, $(e^{2\omega}\delta_{ij} + h_{ij})dx^i dx^j = e^{2\Psi(\tilde{x})}\delta_{ij}d\tilde{x}^i d\tilde{x}^j$.

(iii) Compute the change in the effective action, $\Delta S_{\text{eff}} = S_{\text{eff}}[g+h] - S_{\text{eff}}[g]$. This is possible because in steps (i) and (ii) we have written both g and $g+h$ in the conformal gauge.

(iv) From the expansion of ΔS_{eff} in h (or α, Φ), read off the T^{ij} correlators.

The most non-trivial is step (ii). However, in section 2.3, when we discussed the random-geometry formulation of the $T\bar{T}$ deformation, we have already discussed how to rewrite a metric $g+h$ as a Weyl transformation of the original metric g, after a diffeomorphism. So, we can simply use the results there to find the necessary diffeomorphism.[8]

To be specific, let us focus on the correlators in the flat background metric, $g_{ij} = \delta_{ij}$, i.e. $\omega = 0$. In complex coordinates, $ds^2 = dz\, d\bar{z}$. This is in the conformal gauge, so step (i) is already done. The deformation of the metric, h, can be written in terms of α, Φ as

[8]In section 2.3, we only had to keep up to quadratic order terms in the diffeomorphism and Weyl transformation, because higher order terms are irrelevant in the random geometry approach. However, here, we are interested in higher order terms as well, in principle. See footnote 7.

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in (3.4). For step (ii), we need a diffeomorphism $\tilde{x}^i = x^i + A^i(x)$ for which the following equation is satisfied:

$$dz\, d\bar{z} + 2\big[\partial\alpha(x)\, dz^2 + \bar{\partial}\bar{\alpha}(x)\, d\bar{z}^2 + \phi(x)\, dz\, d\bar{z}\big] = e^{2\Psi(\tilde{x})} d\tilde{z}\, d\tilde{\bar{z}}. \tag{3.8}$$

The functions A^i, Ψ can be worked out using power expansion (2.23). In appendix B, the explicit expressions for $A^i_{(n)}, \Psi_{(n)}$ up to $n = 2$ are given in (B.11)–(B.13), for the conformal-gauge background metric (B.10). For flat background, by setting $\omega = 0$ in those expressions, we obtain

$$A_{(1)} = \alpha, \qquad \bar{A}_{(1)} = \bar{\alpha}, \qquad \Psi_{(1)} = \Phi = \phi - (\partial\bar{\alpha} + \bar{\partial}\alpha), \tag{3.9a}$$

$$A_{(2)} = -\frac{2}{\partial}((\phi - \bar{\partial}\alpha)\partial\alpha), \qquad \bar{A}_{(2)} = -\frac{2}{\bar{\partial}}((\phi - \partial\bar{\alpha})\bar{\partial}\bar{\alpha}), \tag{3.9b}$$

$$\begin{aligned}\Psi_{(2)} = {}& -\phi^2 - 2(\alpha\bar{\partial}\phi + \bar{\alpha}\partial\phi) + (\bar{\partial}\alpha)^2 + 2\alpha\bar{\partial}^2\alpha + (\partial\bar{\alpha})^2 + 2\bar{\alpha}\partial^2\bar{\alpha} \\ & + 2\alpha\partial\bar{\partial}\bar{\alpha} + 2\bar{\alpha}\partial\bar{\partial}\alpha - 2\partial\alpha\bar{\partial}\bar{\alpha} + 2\frac{\partial}{\bar{\partial}}((\phi - \partial\bar{\alpha})\bar{\partial}\bar{\alpha}) + 2\frac{\bar{\partial}}{\partial}((\phi - \bar{\partial}\alpha)\partial\alpha).\end{aligned} \tag{3.9c}$$

where we defined

$$A_{(n)} \equiv A_{(n)z}, \qquad \bar{A}_{(n)} \equiv A_{(n)\bar{z}}. \tag{3.10}$$

Now that the metric being in the conformal gauge in the $(\tilde{z}, \tilde{\bar{z}})$ coordinates, we can plug it into the conformal gauge effective action and evaluate $S_{\text{eff}}[g+h]$ (step (iii)). As a concrete example of the conformal gauge effective action, take the Liouville action (2.19). Evaluated on the metric (3.8), it gives

$$S_0[e^{2\Psi(\tilde{x})}\delta] = -\frac{c}{24\pi}\int d^2\tilde{x}\, \delta^{ij}\, \tilde{\partial}_i\Psi(\tilde{x})\, \tilde{\partial}_j\Psi(\tilde{x}), \tag{3.11}$$

where $\tilde{\partial}_i \equiv \partial/\partial\tilde{x}^i$. To read off correlators from this, we must rewrite this in terms of x, because what we want to compute is the correlator of $T^{ij}(x)$ and not $T^{ij}(\tilde{x})$. So,

$$S_0[e^{2\Psi(\tilde{x})}\delta] = -\frac{c}{24\pi}\int d^2x \left(\det\frac{\partial\tilde{x}}{\partial x}\right)\delta^{ij}\frac{\partial x^k}{\partial\tilde{x}^i}\frac{\partial x^l}{\partial\tilde{x}^j}\,\partial_k\Psi(x + A(x))\,\partial_l\Psi(x + A(x)). \tag{3.12}$$

Plugging the expansions for A^i, Ψ into this expression, computing the coefficients of $\bar{\partial}\bar{\alpha}, \partial\alpha, \phi$, and using the relation (3.7), we can compute the CFT n-point correlators for $T, \bar{T}, \Theta$ for any n (step (iv)).

We will demonstrate this method for CFT and the $T\bar{T}$-deformed theory in the next sections.

4 3-point functions

Having established a general framework for computing correlation functions of the stress tensor in the $T\bar{T}$-deformed theory, we are now going to apply it to the computation of 3- and 4-point functions of T, $\bar{T}$, and Θ to the first order in the $T\bar{T}$ deformation.[9]

[9]The stress tensor 2-point functions receive no correction to the first order in the $T\bar{T}$ deformation [25, 26]. This fact can be seen more explicitly in our formalism in section 4.2.

4.1 Warm-up: CFT 2-point functions

As a warm-up, we first illustrate how this formalism works in the simplest example, namely, the stress tensor two point functions of the undeformed CFT. In this application, we only need to consider the Liouville action $S_0[\delta + h]$ in the conformal gauge in (3.12) to the second order in ϕ, α, and $\bar{\alpha}$:

$$S_0 = -\frac{c}{24\pi}\int d^2x\, \partial_k\Phi(x)\, \partial^k\Phi(x) \qquad \text{with} \qquad \Phi(x) = \phi(x) - \frac{1}{2}\partial_k\alpha^k\,, \tag{4.1}$$

since $\tilde{x} = x$, $A(x) = 0$, and $\Psi(x) = \Phi(x)$ to the lowest order in (2.22) and (2.23). In the complex coordinates $z = x^1 + ix^2$, the lowest-order Liouville action is expressed as

$$\begin{aligned} S_0 &= -\frac{c}{12\pi}\int d^2z\, \partial\left(\phi - \partial\bar{\alpha} - \bar{\partial}\alpha\right)\bar{\partial}\left(\phi - \partial\bar{\alpha} - \bar{\partial}\alpha\right) \\ &= -\frac{c}{12\pi}\int d^2z\left(-\phi\partial\bar{\partial}\phi + 2\phi\partial^2(\bar{\partial}\bar{\alpha}) + 2\phi\bar{\partial}^2(\partial\alpha) - 2(\bar{\partial}\bar{\alpha})\partial\bar{\partial}(\partial\alpha) + \bar{\alpha}\partial^3(\bar{\partial}\bar{\alpha}) + \alpha\bar{\partial}^3(\partial\alpha)\right), \end{aligned} \tag{4.2}$$

where $\int d^2x = \frac{1}{2}\int d^2z$ (see appendix A for our convention).

From (3.7) the variations of the Liouville action yield, for example,

$$\langle T(z,\bar{z})T(0)\rangle = -\frac{\pi^2\delta^2 S_0}{\delta\bar{\partial}\bar{\alpha}(z)\delta\bar{\partial}\bar{\alpha}(0)} = \frac{c}{2z^2}\,, \quad \left\langle \bar{T}(z,\bar{z})\bar{T}(0)\right\rangle = -\frac{\pi^2\delta^2 S_0}{\delta\partial\alpha(z)\delta\partial\alpha(0)} = \frac{c}{2\bar{z}^2}\,. \tag{4.3}$$

This correctly reproduces the standard results. A simple but important technical note is that one needs to use the identities

$$\alpha(z,\bar{z}) = \frac{1}{\partial}(\partial\alpha) = \frac{1}{2\pi}\int d^2z'\frac{\partial'\alpha(z',\bar{z}')}{\bar{z}-\bar{z}'}\,, \quad \bar{\alpha}(z,\bar{z}) = \frac{1}{\bar{\partial}}(\bar{\partial}\bar{\alpha}) = \frac{1}{2\pi}\int d^2z'\frac{\bar{\partial}'\bar{\alpha}(z',\bar{z}')}{z-z'} \tag{4.4}$$

and integration by parts to compute the variations of the action. These identities follow from

$$\bar{\partial}\frac{1}{z} = \partial\frac{1}{\bar{z}} = 2\pi\delta^2(z), \qquad \delta^2(z) = \frac{1}{2}\delta(x^1)\delta(x^2) \tag{4.5}$$

(see appendix A for our convention).

To be complete, we can similarly calculate all the other two point functions of the stress tensor which are only contact terms:

$$\langle\Theta(z,\bar{z})\Theta(0)\rangle = -\frac{\pi c}{6}\partial\bar{\partial}\delta^2(z)\,, \qquad \left\langle T(z,\bar{z})\bar{T}(0)\right\rangle = -\frac{\pi c}{6}\partial\bar{\partial}\delta^2(z)\,, \tag{4.6}$$

$$\langle\Theta(z,\bar{z})T(0)\rangle = \frac{\pi c}{6}\partial^2\delta^2(z)\,, \qquad \left\langle\Theta(z,\bar{z})\bar{T}(0)\right\rangle = \frac{\pi c}{6}\bar{\partial}^2\delta^2(z)\,. \tag{4.7}$$

4.2 $T\bar{T}$-deformation to 3-point functions

The stress tensor 3-point functions have been computed in [25, 26] to the first order in the $T\bar{T}$ deformation. The first paper [25] uses the $T\bar{T}$ flow equation and the conformal perturbation theory, and the second paper [26] combines the random geometry approach with the Ward-Takahashi (WT) identity for the stress tensor. Here we provide an alternative method that is purely based on the random geometry approach, generalizing the technique

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developed in our previous work [21] to the stress-tensor correlators. The advantage of our method is that it becomes straightforward to compute higher-point functions of the stress tensor.

In this subsection, we demonstrate how our formalism can be applied to the stress tensor 3-point functions in the $T\bar{T}$-deformed theory and reproduce the results of [25, 26]. This serves as a nontrivial check of our method, and it also illustrates how it can be applied to the computation of higher-point functions, as we will discuss further in the next section.

As shown in (3.7), all stress-tensor correlators can be computed from the effective action $S_{\rm eff}[g]$ which is the $T\bar{T}$-deformed Liouville action given by

$$S_{\rm eff}[g] = S_0[g] + \delta S_{\rm saddle}[g] + \delta S_{\rm fluc}[g] \qquad \text{with} \qquad g_{ij} = \delta_{ij} + h_{ij} \tag{4.8}$$

where S_0 is the undeformed Liouville action and $\delta S_{\rm saddle} + \delta S_{\rm fluc}$ is the first-order $T\bar{T}$ correction, and their forms are given in (3.12), (2.37), and (2.34), respectively. As commented in section 2.3, the fluctuation action $\delta S_{\rm fluc}$ about the saddle point, although superficially divergent as illustrated in (2.35), is renormalized to zero, $\delta S_{\rm fluc} = 0$, as conformal perturbation theory indicates.

Here, we focus on the first-order $T\bar{T}$ correction $\delta S_{\rm saddle}[g]$ to the 3-point functions and relegate the computation of undeformed CFT 3-point functions to the next section. It is most convenient to work in the conformal gauge (3.8). In order to compute the stress-tensor correlators, we expand the saddle point action $\delta S_{\rm saddle}[g]$ in ϕ, α, and $\bar{\alpha}$. As we have seen in the previous section, the quadratic order is absent, which implies that there is no first-order $T\bar{T}$ correction to the 2-point functions, and the saddle point action starts from the cubic order $\mathcal{O}(\Psi^3)$ in the conformal factor $\Psi(x)$. This means that it suffices to consider the leading order $\tilde{x} = x$, $A(x) = 0$, and $\Psi(x) = \Phi(x)$ for the expansions (2.22) and (2.23). The first-order $T\bar{T}$ correction was computed in (2.51),

$$\delta S_{\rm saddle}[g] = \delta S_L[e^{2\Psi(\tilde{x})}\delta] = \frac{c^2\,\delta\mu}{36\pi^2}\int d^2\tilde{x}\, e^{-2\Psi}\Big[-2(\tilde{\partial}\Psi)(\bar{\tilde{\partial}}\Psi)(\tilde{\partial}\bar{\tilde{\partial}}\Psi) + (\tilde{\partial}\Psi)^2(\bar{\tilde{\partial}}\Psi)^2\Big] \tag{4.9}$$

which reduces to

$$\delta S_L^{(3)}[e^{2\Psi(\tilde{x})}\delta] = -\frac{c^2\delta\mu}{36\pi^2}\int d^2z\, \partial\bar{\partial}\Phi\, \partial\Phi\, \bar{\partial}\Phi \qquad \text{with} \qquad \Phi(x) = \phi(x) - \frac{1}{2}\partial_k\alpha^k \tag{4.10}$$

to the third order in ϕ, α, and $\bar{\alpha}$. We note once again that $\int d^2x = \frac{1}{2}\int d^2z$.

It is now straightforward to calculate the first-order $T\bar{T}$ correction to the stress tensor 3-point functions. As done in the computation of the undeformed CFT 2-point functions, we use the identities (4.4) and integration by parts. The non-contact terms are then found to be

$$\Big\langle \Theta(z_1)T(z_2)\bar{T}(z_3)\Big\rangle = \frac{\pi^3\delta^3(\delta S_L)}{\delta\phi(z_1)\delta\bar{\partial}\bar{\alpha}(z_2)\delta\partial\alpha(z_3)} = -\frac{c^2\delta\mu}{4\pi}\frac{1}{z_{12}^4\bar{z}_{13}^4} \tag{4.11}$$

$$\Big\langle T(z_1)\bar{T}(z_2)\bar{T}(z_3)\Big\rangle = \frac{\pi^3\delta^3(\delta S_L)}{\delta\bar{\partial}\bar{\alpha}(z_1)\delta\partial\alpha(z_2)\delta\partial\alpha(z_3)} = -\frac{c^2\delta\mu}{3\pi}\frac{1}{z_{12}^3\bar{z}_{23}^5} + (z_2 \leftrightarrow z_3) \tag{4.12}$$

where $z_{ij} \equiv z_i - z_j$ and $\bar{z}_{ij} \equiv \bar{z}_i - \bar{z}_j$. These precisely agree with the results in [25] with the identification $\delta\mu_{\rm here} = \pi^2\lambda_{\rm there}$.

5 4-point functions

To the best of our knowledge, the stress tensor 4-point functions in the $T\bar{T}$-deformed theory have never been computed. Here, as an application of our method, we calculate the 4-point functions to the first order in the $T\bar{T}$ deformation. As we will see, the most interesting result may be the logarithmic correction that appears in one of the 4-point functions:

$$\Big\langle T(z_1)T(z_2)\bar{T}(z_3)\bar{T}(z_4)\Big\rangle = \frac{4c^2\delta\mu}{\pi z_{12}^5\bar{z}_{34}^5}\ln\left|\frac{z_{13}z_{24}}{z_{14}z_{23}}\right|^2 + \cdots . \tag{5.1}$$

Note that the argument in the logarithm is a cross-ratio.

One of the most subtle points in our formalism is to carefully take into account the effect of the coordinate transformation, $x^i \mapsto \tilde{x}^i$, to conformal gauge (3.8). In other words, it is very important to include all the corrections discussed in (2.22), (2.23), and (3.9). For the undeformed CFT 2-point functions and the first-order $T\bar{T}$ correction to the 3-point functions in section 4, it was rather special and simpler, and the effect of the coordinate transformation, $x^i \mapsto \tilde{x}^i$, was absent to the relevant order. For the computation of the 4-point functions and the undeformed CFT 3-point function, however, it is essential to take into account the contributions that come from this subtle effect.

5.1 Warm-up: CFT 3-point functions

In order to illustrate the aforementioned subtlety, we first compute the undeformed CFT 3-point functions as a warm-up exercise. The Liouville action in the conformal gauge is given in (3.12). In the absence of the effect of the coordinate transformation, $x^i \mapsto \tilde{x}^i$, the Liouville action is only of quadratic order $\mathcal{O}(\Phi^2)$ in the conformal factor Φ and thus in ϕ, α, and $\bar{\alpha}$. This would have meant vanishing 3-point functions, but it obviously cannot be true. Indeed, all the contributions to the undeformed CFT 3-point functions come from the subtle effect of the coordinate transformation, $x^i \mapsto \tilde{x}^i$. In order to make this point more explicit and demonstrate how it works, using (2.22), (2.23), and (3.9), we expand the Liouville action (3.12) in the conformal gauge (3.8)

$$S_0 = \frac{c}{6\pi}\int d^2z(\bar{\partial}\bar{\alpha})\Big[\partial^2(\partial\bar{\alpha})^2 - \partial^2(\bar{\alpha}\partial^2\bar{\alpha}) - \partial^3\left(\bar{\alpha}\partial\bar{\alpha}\right) - \partial^2(\partial\bar{\alpha})^2 + \partial^3\bar{A}_{(2)} + \cdots\Big] \tag{5.2}$$

to the third order in ϕ, α, and $\bar{\alpha}$, where the explicit form of $\bar{A}_{(2)}$ is given by

$$\bar{A}_{(2)}(z,\bar{z}) = -\frac{1}{\pi}\int d^2z'\frac{1}{z-z'}(\phi(z') - \partial'\bar{\alpha}(z'))\bar{\partial}'\bar{\alpha}(z') . \tag{5.3}$$

What is explicitly shown in (5.2) are only the terms of order $\mathcal{O}(\bar{\alpha}^3)$ relevant to the holomorphic 3-point function $\langle TTT\rangle$, and we omitted the quadratic terms (4.2). In addition to these terms, there are also the complex conjugate terms of order $\mathcal{O}(\alpha^3)$ relevant to the anti-holomorphic 3-point function $\langle\bar{T}\bar{T}\bar{T}\rangle$ as well as the terms that yield the contact terms. Once again, using

$$\bar{\alpha}(z,\bar{z}) = \frac{1}{\bar{\partial}}(\bar{\partial}\bar{\alpha}) = \frac{1}{2\pi}\int d^2z'\frac{\bar{\partial}'\bar{\alpha}(z',\bar{z}')}{z-z'} \tag{5.4}$$

and integration by parts, it is straightforward to find

$$\langle T(z_1)T(z_2)T(z_3)\rangle = \frac{\pi^3\delta^3 S_0}{\delta\bar{\partial}\bar{\alpha}(z_1)\delta\bar{\partial}\bar{\alpha}(z_2)\delta\bar{\partial}\bar{\alpha}(z_3)} = \frac{c}{z_{12}^2 z_{23}^2 z_{13}^2} \ . \tag{5.5}$$

This correctly reproduces the standard CFT 3-point function of the stress tensor.

5.2 $T\bar{T}$-deformation to 4-point functions

We now apply our method to compute the first-order $T\bar{T}$-correction to the 4-point stress-tensor correlators. Here, we only focus on the non-contact terms. All the contributions come from the saddle point action $\delta S_{\text{saddle}}[g] = \delta S_L[e^{2\Psi(\tilde{x})}\delta]$, but there are two distinct types of contributions:

$$\delta S_L[e^{2\Psi(\tilde{x})}\delta] \quad \supset \quad \delta S_L^{(4)}[e^{2\Psi(\tilde{x})}\delta] = S_{\Phi^4} + S_{\Phi^3} \ . \tag{5.6}$$

The first term in (5.6) is the quartic action of order $\mathcal{O}(\Phi^4)$ without the subtle effect of the coordinate transformation, $x^i \mapsto \tilde{x}^i$, to conformal gauge. On the other hand, the second term in (5.6) is the cubic action of order $\mathcal{O}(\Phi^3)$ with nontrivial contributions from the coordinate transformation. By expanding the saddle point action (4.9) in Φ, we obtain the quartic action

$$\begin{aligned} S_{\Phi^4} &= \frac{c^2\delta\mu}{72\pi^2}\int d^2z\,\left[4\Phi\,\partial\Phi\,\bar{\partial}\Phi\,\partial\bar{\partial}\Phi + (\partial\Phi)^2(\bar{\partial}\Phi)^2\right] \\ &= -\frac{c^2\delta\mu}{36\pi^2}\int d^2z\left[-2\Phi\,\partial\Phi\,\bar{\partial}\Phi\,\partial\bar{\partial}\Phi + \left(\partial\Phi\,\partial\bar{\partial}\Phi\,\frac{1}{\partial\bar{\partial}}\left(\bar{\partial}\Phi\,\partial\bar{\partial}\Phi\right) + \text{c.c.}\right)\right] \end{aligned} \tag{5.7}$$

where in the second line we rewrote the action in a non-local form in terms of 2d Green's function,

$$\frac{1}{\partial\bar{\partial}}f(z,\bar{z}) = \frac{1}{2\pi}\int d^2z'\,\ln|z-z'|^2 f(z',\bar{z}') \ . \tag{5.8}$$

The second expression manifests the aforementioned logarithmic correction in (5.1). Meanwhile, using (2.22), (2.23), and (3.9), the cubic action to the fourth order in ϕ, α, and $\bar{\alpha}$ can be found as

$$\begin{aligned} S_{\Phi^3} &= -\frac{c^2\delta\mu}{36\pi^2}\int d^2\tilde{z}\,\tilde{\partial}\tilde{\bar{\partial}}\Psi\,\tilde{\partial}\Psi\,\tilde{\bar{\partial}}\Psi \\ &= -\frac{c^2\delta\mu}{36\pi^2}\int d^2z\Big[-2\alpha\,\bar{\partial}\Phi\,\partial^2\Phi\,\bar{\partial}^2\Phi + \underbrace{2\alpha\,\bar{\partial}\Phi(\partial\bar{\partial}\Phi)^2}_{\text{contact}} + \text{c.c.} + \delta\Phi\,\partial^2\Phi\,\bar{\partial}^2\Phi - \underbrace{\delta\Phi(\partial\bar{\partial}\Phi)^2}_{\text{contact}}\Big] \end{aligned} \tag{5.9}$$

where we defined

$$\delta\Phi \equiv \Psi_{(2)} + \alpha^k\partial_k\Phi = -\Big[\phi^2 - (\partial\bar{\alpha})^2 - (\bar{\partial}\alpha)^2 + 2\partial\alpha\,\bar{\partial}\bar{\alpha} + \partial\bar{A}_{(2)} + \bar{\partial}A_{(2)}\Big] \ , \tag{5.10}$$

where the explicit expression for $\bar{A}_{(2)}$ is given in (5.3) and $A_{(2)}$ is its complex conjugate. We note that in the last line of (5.9), the first three terms, including the complex conjugate terms denoted by c.c., are the quartic action induced by the Jacobian of the coordinate transformation $x^i \mapsto \tilde{x}^i$ to conformal gauge, whereas the last two terms are induced by the second-order correction to the Weyl factor $\delta\Phi$.

Since we are only interested in non-contact terms in correlators, we can disregard the terms in the action that can only yield contact terms. For 4-point functions, the non-contact terms can only arise from the terms in the action which involve four integrals. This means that since there is one common z-integral, the only relevant terms are those for which it is mandatory to use the identities (4.4) at least three times in order to mold $\partial\alpha$ or $\bar{\partial}\bar{\alpha}$ from α or $\bar{\alpha}$ by introducing $1/\partial$ or $1/\bar{\partial}$. Since $\Phi = \phi - \bar{\partial}\alpha - \partial\bar{\alpha}$, the factor $\partial\bar{\partial}\Phi$ does not necessitate the use of the identities (4.4). Thus the terms indicated as "contact" in (5.7) and (5.9) only involve at most three integrals and can only yield contact terms.

The computation is tedious but straightforward. As stressed several times before, the only point to note is that we make repeated use of the identities (4.4) and integration by parts whenever necessary. As it turns out, there are three types of 4-point correlators which receive the first-order non-contact term corrections. As the computation is straightforward, we will not show the detail. The final results are given as follows:

$$\langle T(z_1)T(z_2)\bar{T}(z_3)\Theta(z_4)\rangle = -\frac{\pi^4\delta^4(\delta S_L)}{\delta\bar{\partial}\bar{\alpha}(z_1)\delta\bar{\partial}\bar{\alpha}(z_2)\delta\partial\alpha(z_3)\delta\phi(z_4)} = -\frac{c^2\delta\mu}{2\pi}\frac{1}{z_{41}^2 z_{42}^2 z_{12}^2 \bar{z}_{34}^4}\,, \quad (5.11)$$

where the minus sign in the variation of the saddle point action comes from the expansion (3.7). For the remaining two correlators, we will not write down the form of the variations as it should be clear by now. One of the other two is found to be

$$\langle T(z_1)T(z_2)T(z_3)\bar{T}(z_4)\rangle = \frac{c^2\delta\mu}{6\pi}\left[\frac{1}{z_{12}^2 z_{13}^3 z_{23}^2} + \frac{1}{z_{12}^3 z_{13}^2 z_{23}^2}\right]\frac{1}{\bar{z}_{14}^3} + \mathrm{perm}(z_1,z_2,z_3)\,, \quad (5.12)$$

where 'perm(z_1, z_2, z_3)' denotes five more terms obtained by permutations. The most interesting of the three may be the correlator that involves two Ts and $\bar{T}$s:

$$\langle T(z_1)T(z_2)\bar{T}(z_3)\bar{T}(z_4)\rangle = \frac{2c^2\delta\mu}{\pi z_{12}^5 \bar{z}_{34}^5}\left[\frac{z_{12}}{z_{31}} + \frac{\bar{z}_{34}}{\bar{z}_{13}} + 2\ln|z_{13}|^2\right] + (z_1 \leftrightarrow z_2, z_3 \leftrightarrow z_4). \quad (5.13)$$

where '$(z_1 \leftrightarrow z_2, z_3 \leftrightarrow z_4)$' denotes three more terms obtained by exchanging z_i. As advertized at the beginning of this section, the most notable feature is the logarithmic correction. Note that a similar logarithmic correction appears in matter correlators [11, 21, 25].

In appendix C and D, as a check, we reproduce (5.12) and (5.13) from conformal perturbation theory.[10] It is a straightforward exercise, which turns out to be easier than the computation from the deformed Liouville action. However, we hasten to say that this does not mean that the random geometry approach is less useful than conformal perturbation theory; although conformal perturbation theory is useful in concrete computations, the random geometry approach has advantage in formal operations. For example, it allowed us to derive all-order recursion equations, (2.47) and (2.48). Also, if we can regulate the fluctuation part, the random geometry approach will allow us to go straightforwardly to

[10] The 4-point function (5.11) can be reproduced by using the flow equation $\Theta = -\frac{\delta\mu}{\pi}T\bar{T}$.

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higher orders in a covariant way, which is presumably not so straightforward in conformal perturbation theory. Moreover, it was shown in [21] that the random geometry approach is a useful language to understand the gravity dual of the $T\bar{T}$-deformed CFTs in the framework of AdS/CFT. Appendix D also provides a novel use of the contour integral representation of the $T\bar{T}$-deformation [11] in conformal perturbation theory.

6 $T\bar{T}$-deformed OPE

The deformation of the 3-point correlators (4.11) and (4.12) can be interpreted as the $T\bar{T}$-deformed operator product expansions (OPE) of the stress tensor. It will provide us with a better perspective on the nature of the $T\bar{T}$ deformation as well as a better understanding of the form of the 4-point correlators in section 5.2. In this section, we read off the deformed OPE from the deformed 3-point functions and discuss consistency of the deformed 4-point functions with the OPE.

Let us recall the first-order $T\bar{T}$-correction to the 3-point functions in (4.11) and (4.12):

$$\Big\langle \Theta(z_1)T(z_2)\bar{T}(z_3)\Big\rangle = -\frac{c^2\delta\mu}{4\pi}\frac{1}{z_{12}^4\bar{z}_{13}^4}\,, \tag{6.1}$$

$$\Big\langle T(z_1)\bar{T}(z_2)\bar{T}(z_3)\Big\rangle = -\frac{c^2\delta\mu}{3\pi}\frac{1}{z_{12}^3\bar{z}_{23}^5} + (z_2\leftrightarrow z_3)\,. \tag{6.2}$$

The first 3-point function (6.1) suggests that

$$\Theta(z)T(w) \sim -\frac{c\,\delta\mu}{2\pi}\frac{\bar{T}(z)}{(z-w)^4} + \cdots\,, \qquad \Theta(z)\bar{T}(w) \sim -\frac{c\,\delta\mu}{2\pi}\frac{T(z)}{(\bar{z}-\bar{w})^4} + \cdots \tag{6.3}$$

which, together with $T(\zeta)T(w)\sim c/(2(\zeta-w)^4)+\cdots$ (or its complex conjugate), reproduces the deformed 3-point function (6.1).

Next, we may infer from the second 3-point function (6.2) that[11]

$$\bar{T}(z)\bar{T}(w) \sim -\frac{c\,\delta\mu}{\pi^2}\frac{1}{(\bar{z}-\bar{w})^5}\int d^2z'\ln(z-z')\bar{\partial}'T(z') + (z\leftrightarrow w) + \cdots\,, \tag{6.4}$$

where we dropped the zeroth-order CFT part of the OPE. The $T(z)T(w)$ OPE is given by a complex conjugate of this form. To elaborate on it, assuming the OPE (6.4), one can check that the 3-point function $\langle T\bar{T}\bar{T}\rangle$ reads

$$\begin{aligned}\langle T(z_1)\underbrace{\bar{T}(z_2)\bar{T}(z_3)}_{\text{OPE}}\rangle &\sim -\frac{c\,\delta\mu}{\pi^2}\frac{1}{\bar{z}_{23}^5}\int d^2z'\ln(z_2-z')\bar{\partial}'\langle T(z_1)T(z')\rangle + (z_2\leftrightarrow z_3)\\ &= -\frac{c^2\delta\mu}{2\pi^2}\frac{1}{\bar{z}_{23}^5}\int d^2z'\ln(z_2-z')\bar{\partial}'\frac{1}{(z_1-z')^4} + (z_2\leftrightarrow z_3)\\ &= -\frac{c^2\delta\mu}{3\pi}\frac{1}{\bar{z}_{23}^5 z_{12}^3} + (z_2\leftrightarrow z_3)\end{aligned} \tag{6.5}$$

[11] Since $\bar{\partial}\partial\int d^2z'\ln(z-z')\bar{\partial}'T(z') = 2\pi\bar{\partial}T(z)$, we can express $\int d^2z'\ln(z-z')\bar{\partial}'T(z')$ by a contour integral $2\pi\int^z dz'T(z')$.

where we used $\bar{\partial}' \frac{1}{(z-z')^4} = -\frac{\pi}{3}\partial'^3 \delta^2(z-z')$ and integration by parts. This precisely agrees with the deformed 3-point function (6.2). In particular, the $\bar{T}(z_2)\bar{T}(z_3)$ OPE correctly reproduces the $\bar{z}_{23}=0$ singularity of the 3-point function as required. It is worth commenting on an ambiguity in the form of the OPE: the 3-point function could have been correctly reproduced even if $\ln(z_2-z')$ was replaced by $\ln|z_2-z'|^2$. However, as we will see below, the consistency with the 4-point function (5.13) fixes the form to be the one in (6.4).

Furthermore, the second 3-point function (6.2) also indicates that

$$T(z)\bar{T}(w) \sim \frac{c\,\delta\mu}{6\pi}\frac{\bar{\partial}\bar{T}(w)}{(z-w)^3} - \frac{c\,\delta\mu}{6\pi}\frac{\partial T(z)}{(\bar{z}-\bar{w})^3} + \cdots\,. \tag{6.6}$$

Assuming this OPE, we calculate the 3-point function $\langle T\bar{T}\bar{T}\rangle$:

$$\Big\langle T(z_1)\bar{T}(z_2)\bar{T}(z_3)\Big\rangle = \langle \underbrace{T(z_1)\bar{T}(z_2)}_{\text{OPE}}\bar{T}(z_3)\rangle \sim \frac{c\,\delta\mu}{6\pi z_{12}^3}\bar{\partial}_2\Big\langle \bar{T}(\bar{z}_2)\bar{T}(\bar{z}_3)\Big\rangle = -\frac{c^2\delta\mu}{3\pi z_{12}^3\bar{z}_{23}^5}\,. \tag{6.7}$$

This correctly reproduces the $z_{12}=0$ singularity of the 3-point function (6.2) as $z_1\to z_2$. Since the OPE (6.6) only captures the singularity as $z\to w$, it suffices for the $T(z_1)\bar{T}(z_2)$ OPE to reproduce $z_{12}=0$ singularity of the 3-point function. The fact that the $z_2\leftrightarrow z_3$ exchange term is missing is not a bug but a feature since the singular part of the $T(z_1)\bar{T}(z_2)$ OPE cannot account for the $z_{13}=0$ singularity.

We now discuss the consistency of the deformed 4-point functions with the deformed OPEs. Using the OPE (6.3), we can calculate the 4-point function (5.11) as

$$\langle T(z_1)T(z_2)\underbrace{\Theta(z_4)\bar{T}(z_3)}_{\text{OPE}}\rangle \sim -\frac{c\,\delta\mu}{2\pi}\frac{\langle T(z_1)T(z_2)T(z_4)\rangle}{\bar{z}_{34}^4} = -\frac{c^2\delta\mu}{2\pi}\frac{1}{z_{12}^2 z_{14}^2 z_{24}^2\bar{z}_{34}^4}\,. \tag{6.8}$$

This precisely agrees with the r.h.s. of (5.11). In particular, the $\Theta(z_4)\bar{T}(z_3)$ OPE correctly reproduces the $\bar{z}_{34}=0$ singularity of the 4-point function as required.

Next, using the OPE (6.6), the $\bar{z}_{34}=0$ singularity of the 4-point function (5.12) can be calculated as

$$\langle T(z_1)T(z_2)\underbrace{T(z_3)\bar{T}(z_4)}_{\text{OPE}}\rangle \sim -\frac{c\,\delta\mu}{6\pi\bar{z}_{34}^3}\partial_3\langle T(z_1)T(z_2)T(z_3)\rangle = -\frac{c^2\delta\mu}{3\pi\bar{z}_{34}^3 z_{12}^2}\left[\frac{1}{z_{13}^3 z_{23}^2}+\frac{1}{z_{13}^2 z_{23}^3}\right]. \tag{6.9}$$

This indeed precisely reproduces the $\bar{z}_{34}=0$ singularity of the 4-point function (5.12).

Finally, we check the consistency of the 4-point function (5.13) with the $\bar{T}(z_3)\bar{T}(z_4)$ OPE (6.4). On top of the $\delta\mu$-deformation of the OPE, the zeroth-order CFT part must be taken into account. The $\bar{z}_{34}=0$ singularity of the 4-point function $\langle TT\bar{T}\bar{T}\rangle$ can be calculated as

$$\begin{aligned}\langle T(z_1)T(z_2)\underbrace{\bar{T}(z_3)\bar{T}(z_4)}_{\text{OPE}}\rangle &\sim \frac{2}{\bar{z}_{34}^2}\langle T(z_1)T(z_2)\bar{T}(z_4)\rangle + \frac{1}{\bar{z}_{34}}\bar{\partial}_4\langle T(z_1)T(z_2)\bar{T}(z_4)\rangle \\ &\quad -\frac{c\,\delta\mu}{\pi^2\bar{z}_{34}^5}\int d^2z' \ln(z_3-z')\bar{\partial}'\langle T(z_1)T(z_2)T(z')\rangle + (z_3\leftrightarrow z_4) \\ &= -\frac{c^2\delta\mu}{3\pi z_{12}^5}\left[\frac{2}{\bar{z}_{41}^3\bar{z}_{34}^2}-\frac{3}{\bar{z}_{41}^4\bar{z}_{34}}\right]+\frac{2c^2\delta\mu}{\pi z_{12}^5\bar{z}_{34}^5}\left[-\frac{z_{12}z_{34}}{z_{31}z_{41}}+2\ln\frac{z_{31}}{z_{41}}\right]+(z_1\leftrightarrow z_2)\end{aligned} \tag{6.10}$$

where we dropped the zeroth-order term $\frac{c^2}{4z_{12}^4\bar{z}_{34}^4}$. In performing the integration, we used $\bar{\partial}'\frac{1}{(z-z')^2} = -2\pi\partial'\delta^2(z-z')$ and integration by parts. This is to be compared to (5.13):

$$\begin{aligned}\langle T(z_1)T(z_2)\bar{T}(z_3)\bar{T}(z_4)\rangle &= \frac{2c^2\delta\mu}{\pi z_{12}^5\bar{z}_{34}^5}\left[\frac{\bar{z}_{34}}{\bar{z}_{13}}+\frac{\bar{z}_{34}}{\bar{z}_{14}}-\frac{z_{12}z_{34}}{z_{31}z_{41}}+2\ln\left|\frac{z_{13}}{z_{14}}\right|^2\right]+(z_1\leftrightarrow z_2) \\ &\sim \frac{2c^2\delta\mu}{\pi z_{12}^5\bar{z}_{34}^5}\left[-\frac{\bar{z}_{34}^3}{3\bar{z}_{41}^3}+\frac{\bar{z}_{34}^4}{2\bar{z}_{41}^4}-\frac{z_{12}z_{34}}{z_{31}z_{41}}+2\ln\frac{z_{13}}{z_{14}}\right]+(z_1\leftrightarrow z_2)\end{aligned} \tag{6.11}$$

where we expanded $\bar{z}_{31}=\bar{z}_{41}+\bar{z}_{34}$ for a small $\bar{z}_{34}$. Indeed, the two precisely agree.

As a final remark in this section, we note that the deformed OPE (6.4) contains a manifestly non-local operator via the logarithmic correction in contrast to the OPEs of local field theory. This might be a sign of nonlocality of the $T\bar{T}$-deformed theory.

7 Discussions

In this paper, we investigated the stress-energy sector of general $T\bar{T}$-deformed theories using the random geometry approach, and developed technique to compute stress-energy correlation functions. More specifically, we considered the Polyakov-Liouville conformal anomaly action of CFT and computed its $T\bar{T}$ deformation to first order in the deformation parameter. It is remarkable that we obtained the deformed action in a closed, nonlocal form, as written down in (2.37). Using this deformed action, one can compute arbitrary stress-energy correlators as we have explicitly demonstrated with concrete examples. In the conformal gauge, as in (2.51), the deformed action can be written in a "local" form in the conformal factor exponent Ω in the sense that the inverse of derivative operators do not explicitly appear.

An obvious but important problem is to generalize our geometrical method to higher orders in the $T\bar{T}$ coupling $\delta\mu$. One idea is to iterate the Hubbard-Stratonovich transformation, using (2.8) or (2.12). Another idea is trying to find a partial differential equation similar to the one obtained for the matter correlators [11, 21]. For the partition function, it is not difficult to write down such an equation as we found in section 2.4. Instead, one may want to find an equation which directly gives the flow of correlation functions. In particular, it is of great interest to compute the 2-point function $G_\Theta(|z_{12}|)\equiv\langle\Theta(z_1)\Theta(z_2)\rangle$ to all orders in the $T\bar{T}$ deformation parameter μ. It is expected that this 2-point function is positive, $G_\Theta(|z_{12}|)>0$, at long distances $|z_{12}|\gg\sqrt{|\mu|}$ but becomes negative, $G_\Theta(|z_{12}|)<0$, at short distances $|z_{12}|\ll\sqrt{|\mu|}$, signaling an appearance of a negative norm and indicating a violation of unitarity at short distances [8]. Since this is a very basic property (or pathology) of the $T\bar{T}$-deformed theories, it is clearly very important to better understand it.

In the random geometry approach, the deformed action involves the saddle point part and, in addition, the fluctuation part which is divergent and must be renormalized. At first order in $\delta\mu$, the fluctuation part actually vanishes after renormalization, as the conformal perturbation theory and the contour integral approach suggest. However, at higher order, this might no longer be the case and we might have to confront the task of properly defining the divergent trace. For that, it will presumably be useful to look more carefully at the first-order expression and see why it is renormalized to zero. Note, however, for large c, the fluctuation part is parametrically smaller than the saddle point part and can be safely dropped, even at higher order. This means that, as far as the dual classical gravity is concerned, the fluctuation part can be always dropped.

As discussed in [21], the random geometry description of the $T\bar{T}$-deformed CFTs can be straightforwardly translated into the AdS/CFT framework. The gravity dual is an ensemble of AdS_3 with a "Gaussian" average over boundary metrics. (This, we believe, is equivalent to the nonlinear mixed boundary proposal of Guica and Monten [20] but differs from the cutoff AdS proposal of McGough, Mezei and Verlinde [19].[12]) In our gravity dual description, since the conformal anomaly can be derived via holographic renormalization in AdS/CFT [30, 31], by averaging over the boundary metric with the Hubbard-Stratonovich "Gaussian" weight, we can obtain, holographically, the deformed Liouville action. Thus taking variations of the deformed anomaly action so obtained with respect to the boundary background metric, we can calculate the $T\bar{T}$-deformed stress-tensor correlators in the gravity dual and will find exactly the same answer as we found in the field theory. The word "random geometry" might sound vacuous, because at first order in $\delta\mu$ the contribution from the fluctuation vanishes and the saddle point gives the exact answer. However, as already mentioned above, this will no longer be the case at higher order and random fluctuations of geometry will be important.

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A Conventions and formulas

Our convention for the complex coordinates $z = x^1 + ix^2, \bar{z} = x^1 - ix^2$ is

$$d^2z = 2dx^1dx^2, \qquad \delta^2(z) = \frac{1}{2}\delta(x^1)\delta(x^2), \tag{A.1}$$

so that

$$\int d^2z\, \delta^2(z) = 1. \tag{A.2}$$

[12]In a similar way to [20], by reinterpreting the gravity dual of the $T\bar{T}$-deformed BTZ black holes obtained in [21], we can explicitly show that a "cutoff" surface emerges as a kind of mirage. In this sense, there is a relation between our gravity dual and the cutoff AdS. However, it is hard to regard this surface as a real rigid cutoff in a literal sense.

Also,

$$\partial \frac{1}{\bar{z}} = \bar{\partial} \frac{1}{z} = 2\pi \delta^2(z). \tag{A.3}$$

The Riemann tensor $R^i{}_{jmn}$, the Ricci tensor R_{ij}, and the scalar curvature R are defined as

$$R^i{}_{jmn} = \partial_m \Gamma^i{}_{nj} - \partial_n \Gamma^i{}_{mj} + \Gamma^i{}_{mk} \Gamma^k{}_{nj} - \Gamma^i{}_{nk} \Gamma^k{}_{mj}, \tag{A.4a}$$

$$[\nabla_m, \nabla_n] V^{i_1 i_2 \cdots} = R^{i_1}{}_{jmn} V^{j i_2 \cdots} + R^{i_2}{}_{jmn} V^{i_1 j i_3 \cdots} + \cdots, \tag{A.4b}$$

$$R_{ij} = R^k{}_{ikj}, \qquad R = R^i{}_i. \tag{A.4c}$$

In two dimensions, the following identities hold:

$$R_{ij} = \frac{1}{2} g_{ij} R, \qquad R_{ijkl} = \frac{1}{2}(g_{ik} g_{jl} - g_{il} g_{jk}) R. \tag{A.5a}$$

B Explicit form of the compensating diffeomorphism

As discussed in section 2.3, in two dimensions, we can bring the varied metric $g + h$ into the original metric g by a diffeomorphism up to a Weyl transformation, even for finite h. Namely, we can choose $A^i(x), \Psi(\tilde{x})$ appropriately so that

$$(g_{ij}(x) + h_{ij}(x))\, dx^i dx^j = e^{2\Psi(\tilde{x})} g_{ij}(\tilde{x})\, d\tilde{x}^i d\tilde{x}^j \tag{B.1}$$

holds, where $\tilde{x}^i \equiv x^i + A^i(x)$. We parametrize h by α, Φ (or α, ϕ) as in (2.10) and (2.11), and find A^i, Ψ by expanding them in powers of α, Φ as

$$A^i(x) = \sum_{n=1}^{\infty} A^i_{(n)}(x), \qquad \Psi(\tilde{x}) = \sum_{n=1}^{\infty} \Psi_{(n)}(\tilde{x}). \tag{B.2}$$

At linear order, we clearly have

$$A^i_{(1)} = \alpha^i, \qquad \Psi_{(1)} = \Phi. \tag{B.3}$$

The equation to determine the quadratic order quantities $A^i_{(2)}, \Psi_{(2)}$ is

$$2\Psi_{(2)} g_{kl} + \nabla_k A^{(2)}_l + \nabla_l A^{(2)}_k + Y_{kl} = 0, \tag{B.4}$$

where

$$\begin{aligned} Y_{kl} = {} & 2\Phi^2 g_{kl} + 2(\alpha^i \partial_i \Phi) g_{kl} + 2\Phi(\nabla_k \alpha_l + \nabla_l \alpha_k) \\ & + g_{ij} \partial_k \alpha^i \partial_l \alpha^j + \alpha^p (\partial_p g_{ki}\, \partial_l \alpha^i + \partial_p g_{li}\, \partial_k \alpha^i) + \frac{1}{2} \alpha^p \alpha^q \partial_p \partial_q g_{kl} \end{aligned} \tag{B.5a}$$

$$\begin{aligned} = {} & 2\Phi^2 g_{kl} + 2(\alpha^i \partial_i \Phi) g_{kl} + 2\Phi(\nabla_k \alpha_l + \nabla_l \alpha_k) \\ & + \nabla_k \alpha^i \nabla_l \alpha_i + (\Gamma_{lpi} \nabla_k \alpha^i + \Gamma_{kpi} \nabla_l \alpha^i) \alpha^p \\ & + \frac{1}{4} \Big[g_{ki} \Big(\partial_m \Gamma^i{}_{nl} - \Gamma^i{}_{mj} \Gamma^j{}_{nl} + (m \leftrightarrow n) \Big) \\ & \qquad + g_{li} \Big(\partial_m \Gamma^i{}_{nk} - \Gamma^i{}_{mj} \Gamma^j{}_{nk} + (m \leftrightarrow n) \Big) \Big] \alpha^m \alpha^n. \end{aligned} \tag{B.5b}$$

To solve eq. (B.4), let us introduce $\phi_{(2)}$ by

$$\Psi_{(2)} \equiv \phi_{(2)} - \frac{1}{2}\nabla_p A^p_{(2)}. \tag{B.6}$$

First, by looking at the trace part of (B.4), we find

$$\phi_{(2)} = -\frac{1}{4}Y, \qquad Y \equiv g^{kl}Y_{kl}. \tag{B.7}$$

If we plug this back into (B.4), we find

$$\nabla_k A^{(2)}_l + \nabla_l A^{(2)}_k - \nabla_p A^p_{(2)} g_{kl} + \tilde{Y}_{kl} = 0, \qquad \tilde{Y}_{kl} \equiv Y_{kl} - \frac{1}{2}Y g_{kl}. \tag{B.8}$$

If we act with ∇^k on (B.8) and use the relations (A.4b), (A.5a), we find

$$\left(\Box_{\rm v} + \frac{R}{2}\right)A^{(2)}_l + \nabla^k \tilde{Y}_{kl} = 0, \qquad \text{therefore} \quad A^{(2)}_l = -\frac{1}{\Box_{\rm v} + R/2}\nabla^k \tilde{Y}_{kl}. \tag{B.9}$$

In particular, if the background metric g is in conformal gauge,

$$ds^2 = g_{ij}(x)dx^i dx^j = e^{2\omega(x)}dz d\bar{z}, \tag{B.10}$$

Eqs. (B.7) and (B.9) give

$$\phi_{(2)} = -e^{-2\omega}Y_{z\bar{z}}, \tag{B.11a}$$

$$A^{(2)}_z = -\frac{1}{2}e^{2\omega}\frac{1}{\partial}(e^{-2\omega}Y_{zz}), \qquad A^{(2)}_{\bar{z}} = -\frac{1}{2}e^{2\omega}\frac{1}{\bar{\partial}}(e^{-2\omega}Y_{\bar{z}\bar{z}}), \tag{B.11b}$$

$$\Psi_{(2)} = -e^{-2\omega}Y_{z\bar{z}} + \frac{1}{2}\bar{\partial}e^{2\omega}\frac{1}{\partial}(e^{-2\omega}Y_{zz}) + \frac{1}{2}\partial e^{2\omega}\frac{1}{\bar{\partial}}(e^{-2\omega}Y_{\bar{z}\bar{z}}), \tag{B.11c}$$

where

$$Y_{zz} = 4(e^{2\omega}\Phi + \partial\bar{\alpha} + 2\bar{\partial}\omega\,\alpha)\,\partial(e^{-2\omega}\alpha), \tag{B.12a}$$

$$\begin{aligned} Y_{z\bar{z}} = \Phi^2 e^{2\omega} &+ 2\Big[\partial(\Phi\bar{\alpha}) + \bar{\partial}(\Phi\alpha)\Big] \\ &+ 2e^{-2\omega}\bigg\{\Big[2\left(\bar{\partial}\omega\,\bar{\partial}\alpha - \partial\omega\,\bar{\partial}\bar{\alpha}\right)\alpha + \left(\bar{\partial}^2\omega - 2(\bar{\partial}\omega)^2\right)\alpha^2\Big] + \text{c.c.} \\ &\qquad + 2\left(2\partial\omega\bar{\partial}\omega + \partial\bar{\partial}\omega\right)\alpha\bar{\alpha} + \partial\alpha\bar{\partial}\bar{\alpha} + \bar{\partial}\alpha\partial\bar{\alpha}\bigg\} \end{aligned} \tag{B.12b}$$

Or, in terms of $\phi, \alpha, \bar{\alpha}$,

$$Y_{zz} = 4e^{2\omega}\left(\phi - \bar{\partial}(e^{-2\omega}\alpha)\right)\partial(e^{-2\omega}\alpha), \tag{B.13a}$$

$$\begin{aligned} Y_{z\bar{z}} = e^{2\omega}\phi^2 &+ 2(\bar{\partial}\phi\,\alpha + \partial\phi\,\bar{\alpha}) \\ &+ e^{-2\omega}\bigg\{\Big[2\left(-2(\bar{\partial}\omega)^2 + \bar{\partial}^2\omega\right)\alpha^2 + (8\bar{\partial}\alpha\,\bar{\partial}\omega - 2\bar{\partial}^2\alpha)\alpha \\ &\qquad - (\bar{\partial}\alpha)^2 - 2\left(\partial\bar{\partial}\bar{\alpha} - 2\bar{\partial}\omega\,\partial\bar{\alpha} + 2\partial\omega\,\bar{\partial}\bar{\alpha}\right)\alpha\Big] + \text{c.c.} \\ &\qquad + 4\left(2\partial\omega\,\bar{\partial}\omega + \partial\bar{\partial}\omega\right)\alpha\bar{\alpha} + 2\partial\alpha\,\bar{\partial}\bar{\alpha}\bigg\}. \end{aligned} \tag{B.13b}$$

C 4-point functions from conformal perturbation theory

As a check of the results in section 5.2, we reproduce (5.12) and (5.13) from conformal perturbation theory.

To the first order in conformal perturbation theory, bringing down the $T\bar{T}$-operator from the exponent in $e^{-\delta S}$, the 4-point function $\langle TTT\bar{T}\rangle_{\delta\mu}$ reads

$$\langle T(z_1)T(z_2)T(z_3)\bar{T}(z_4)\rangle_{\delta\mu} = -\frac{\delta\mu}{2\pi^2}\int d^2z \langle T(z)T(z_1)T(z_2)T(z_3)\bar{T}(z)\bar{T}(z_4)\rangle\,, \tag{C.1}$$

where we adopt a notation that $\langle\cdots\rangle_{\delta\mu}$ is the vev in the $T\bar{T}$-deformed CFT and $\langle\cdots\rangle$ without subscript is that in CFT. Since the vev on the r.h.s. is the one in CFT, we have

$$\begin{aligned}\langle T(z)T(z_1)T(z_2)T(z_3)\bar{T}(z)\bar{T}(z_4)\rangle &= \langle T(z)T(z_1)T(z_2)T(z_3)\rangle\langle\bar{T}(z)\bar{T}(z_4)\rangle \\ &= -\langle T(z)T(z_1)T(z_2)T(z_3)\rangle\bar{\partial}\frac{c}{6(\bar{z}-\bar{z}_4)^3} \\ &\simeq \frac{c}{6(\bar{z}-\bar{z}_4)^3}\bar{\partial}\langle T(z)T(z_1)T(z_2)T(z_3)\rangle\,,\end{aligned} \tag{C.2}$$

where the last line is an equality up to a total derivative term. Using an explicit expression for the CFT stress tensor 4-point function,

$$\langle T(z)T(z_1)T(z_2)T(z_3)\rangle_{\text{CFT}} = \frac{2c}{(z-z_1)^2z_{12}^2(z-z_3)^2z_{23}^2} - \frac{2c}{(z-z_1)(z-z_2)^2z_{12}(z-z_3)z_{13}^2z_{23}}\,,$$

we find that

$$\begin{aligned}&\langle T(z_1)T(z_2)T(z_3)\bar{T}(z_4)\rangle_{\delta\mu} \\ &= -\frac{c^2\delta\mu}{3\pi}\int d^2z\frac{1}{(\bar{z}-\bar{z}_4)^3}\Bigg[-\left(\frac{\partial\delta^2(z-z_1)}{(z-z_3)^2}+\frac{\partial\delta^2(z-z_3)}{(z-z_1)^2}\right)\frac{1}{z_{12}^2z_{23}^2} \\ &\qquad -\left(\frac{\delta^2(z-z_1)}{(z-z_2)^2(z-z_3)}+\frac{\delta^2(z-z_3)}{(z-z_2)^2(z-z_1)}-\frac{\partial\delta^2(z-z_2)}{(z-z_1)(z-z_3)}\right)\frac{1}{z_{12}z_{13}^2z_{23}}\Bigg] \\ &= -\frac{c^2\delta\mu}{3\pi}\Bigg[-\left(\frac{2}{z_{13}^3\bar{z}_{14}^3}+\frac{2}{z_{31}^3\bar{z}_{34}^3}\right)\frac{1}{z_{12}^2z_{23}^2} \\ &\qquad -\left(\frac{1}{z_{12}^2z_{13}\bar{z}_{14}^3}+\frac{1}{z_{23}^2z_{31}\bar{z}_{34}^3}+\frac{1}{\bar{z}_{24}^3}\partial_2\frac{1}{z_{21}z_{23}}\right)\frac{1}{z_{12}z_{13}^2z_{23}}\Bigg] \\ &= \frac{c^2\delta\mu}{6\pi}\left(\frac{1}{z_{12}^2z_{13}^3}+\frac{1}{z_{12}^3z_{13}^2}\right)\frac{1}{z_{23}^2\bar{z}_{14}^3} + \text{perm}(z_1,z_2,z_3)\,.\end{aligned} \tag{C.3}$$

This indeed reproduces the result (5.12) computed from the deformed Liouville action.

Similarly, the 4-point function $\langle TT\bar{T}\bar{T}\rangle_{\delta\mu}$ can be computed, up to contact terms, as

$$
\begin{aligned}
&\langle T(z_1)T(z_2)\bar{T}(z_3)\bar{T}(z_4)\rangle_{\delta\mu} \\
&= -\frac{\delta\mu}{2\pi^2}\int d^2z \langle T(z)T(z_1)T(z_2)\rangle\langle\bar{T}(z)\bar{T}(z_3)\bar{T}(z_4)\rangle \\
&= -\frac{c^2\delta\mu}{2\pi^2}\int d^2z \frac{1}{(z-z_1)^2(z-z_2)^2z_{12}^2}\frac{1}{(\bar{z}-\bar{z}_3)^2(\bar{z}-\bar{z}_4)^2\bar{z}_{34}^2} \\
&= \frac{c^2\delta\mu}{2\pi^2 z_{12}^4\bar{z}_{34}^2}\int \frac{d^2z}{(\bar{z}-\bar{z}_3)^2(\bar{z}-\bar{z}_4)^2}\partial\left[\frac{1}{z-z_1}+\frac{1}{z-z_2}+\frac{2\ln\left|\frac{z_1-z}{z_2-z}\right|^2}{z_{12}}\right] \\
&= -\frac{c^2\delta\mu}{2\pi^2 z_{12}^4\bar{z}_{34}^2}\int d^2z\,\partial\frac{1}{(\bar{z}-\bar{z}_3)^2(\bar{z}-\bar{z}_4)^2}\left[\frac{1}{z-z_1}+\frac{1}{z-z_2}+\frac{\ln\left|\frac{z_1-z}{z_2-z}\right|^4}{z_{12}}\right] \\
&= \frac{c^2\delta\mu}{\pi z_{12}^4\bar{z}_{34}^2}\int d^2z\left[\frac{1}{z-z_1}+\frac{1}{z-z_2}+\frac{\ln\left|\frac{z_1-z}{z_2-z}\right|^4}{z_{12}}\right]\left[\frac{\bar{\partial}\delta^2(z-z_3)}{(\bar{z}-\bar{z}_4)^2}+\frac{\bar{\partial}\delta^2(z-z_4)}{(\bar{z}-\bar{z}_3)^2}\right] \\
&= \frac{2c^2\delta\mu}{\pi z_{12}^5\bar{z}_{34}^5}\left[\frac{z_{12}z_{43}}{z_{31}z_{41}}+\frac{z_{12}z_{43}}{z_{32}z_{42}}+\frac{\bar{z}_{34}\bar{z}_{21}}{\bar{z}_{13}\bar{z}_{23}}+\frac{\bar{z}_{34}\bar{z}_{21}}{\bar{z}_{14}\bar{z}_{24}}+2\ln\left|\frac{z_{13}z_{24}}{z_{14}z_{23}}\right|^2\right]. \qquad \text{(C.4)}
\end{aligned}
$$

This precisely agrees with the result (5.13) computed from the deformed Liouville action. Note that we made a choice of an integration constant in the third line:

$$
\partial\left[\left(\frac{1}{z-z_1}+\frac{1}{z-z_2}\right)+\frac{2\ln\left|\frac{z_1-z}{z_2-z}\right|^2}{z_{12}}\right]=\partial\left[\left(\frac{1}{z-z_1}+\frac{1}{z-z_2}\right)+\frac{\ln\left(\frac{z_1-z}{z_2-z}\right)^2}{z_{12}}\right]. \qquad \text{(C.5)}
$$

This choice was made to avoid a branch cut in order to justify dropping the boundary contribution at infinity when integration by parts is used.

D Contour integral approach for the $T\bar{T}$ deformation

Here we discuss another approach to the $T\bar{T}$ deformation, by rewriting the $T\bar{T}$ perturbation in terms of a contour integral. This approach was developed by Cardy [11] for finite deformation parameter μ, but here we apply the idea to the first-order perturbation in $\delta\mu$, for which we can use the CFT operators $T(z), \bar{T}(\bar{z})$. This clarifies some issues in Cardy's original discussion [11], as well as provides checks of correlation functions computed in the main text using different approaches.

D.1 $T\bar{T}$-deformed correlators in terms of contour integrals

At first order in $\delta\mu$, the $T\bar{T}$ deformation can be written in terms of the CFT stress energy tensor $T(z), \bar{T}(\bar{z})$, as a contour integral, as

$$
\begin{aligned}
\delta S &= \frac{\delta\mu}{2\pi^2}\int_R d^2z\, T(z)\bar{T}(\bar{z}) = \frac{i\,\delta\mu}{2\pi^2}\int_R dz\wedge d\bar{z}\, T(z)\bar{T}(\bar{z}) \\
&= \frac{i\,\delta\mu}{2\pi^2}\int_R d\left[-aT(z)\bar{\chi}(\bar{z})dz+(1-a)\chi(z)\bar{T}(\bar{z})d\bar{z}\right] \\
&= \frac{i\,\delta\mu}{2\pi^2}\int_{\partial R}\left[-aT(z)\bar{\chi}(\bar{z})dz+(1-a)\chi(z)\bar{T}(\bar{z})d\bar{z}\right], \qquad \text{(D.1)}
\end{aligned}
$$

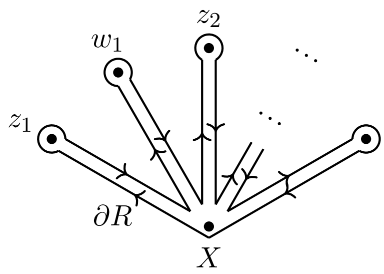


Figure 1. The contour ∂R.

where we defined $\chi, \bar{\chi}$ by

$$\partial\chi(z) = T(z), \qquad \bar{\partial}\bar{\chi}(\bar{z}) = \bar{T}(\bar{z}) \tag{D.2}$$

and used the holomorphicity (anti-holomorphicity) of $T(z), \chi(z)$ ($\bar{T}(\bar{z}), \bar{\chi}(\bar{z})$). The domain of integration R is $\mathbb{R}^2$ with possible singularities excluded, while a is an arbitrary number which any final result must be independent of.

Let us use the above expression to evaluate the first-order perturbation in the correlator,

$$I(z_i, w_i) \equiv \Big\langle \prod_i T(z_i) \prod_j \bar{T}(\bar{w}_j) \Big\rangle_{\delta\mu} = -\frac{\delta\mu}{2\pi^2} \Big\langle \int_R d^2z\, T(z)\bar{T}(\bar{z}) \prod_i T(z_i) \prod_j \bar{T}(\bar{w}_j) \Big\rangle. \tag{D.3}$$

Using (D.1), this can be rewritten as a contour integral as

$$I = \frac{\delta\mu}{2\pi^2}[aI' + (1-a)I''], \tag{D.4a}$$

$$I' = i \int_{\partial R} dz \Big\langle T(z) \prod_i T(z_i) \Big\rangle \Big\langle \bar{\chi}(\bar{z}) \prod_j \bar{T}(\bar{w}_j) \Big\rangle, \tag{D.4b}$$

$$I'' = -i \int_{\partial R} d\bar{z} \Big\langle \chi(z) \prod_i T(z_i) \Big\rangle \Big\langle \bar{T}(\bar{z}) \prod_j \bar{T}(\bar{w}_j) \Big\rangle. \tag{D.4c}$$

Because $\chi, \bar{\chi}$ are not single-valued (χ is not single-valued around T insertions, while $\bar{\chi}$ is not single-valued around $\bar{T}$ insertions), we must consider cuts and take the boundary ∂R to go around the cuts. If we take the cuts to be the paths connecting z_i, w_i with some reference point X, the contour ∂R can be taken as in figure 1.

Let us take the part of the contour ∂R that connects X and z_1 (call it ∂R_{z_1}), and study its contribution to I' and I''.

First, in I', because $\bar{\chi}$ is single-valued around $T(z_1)$, there actually is no cut for I' along ∂R_{z_1}. So, the relevant contour is just a small circle going around $z = z_1$. So, the contribution to I' from the contour ∂R_{z_1} is

$$\begin{aligned} I'_{z_1} &= -i \oint_{z_1} dz \Big\langle T(z)T(z_1) \prod_{i\neq 1} T(z_i) \Big\rangle \Big\langle \chi(\bar{z}) \prod_j \bar{T}(\bar{w}_j) \Big\rangle \\ &= -i \oint_{z_1} dz \Big\langle \Big(\frac{c}{2(z-z_1)^4} + \frac{2T(z_1)}{(z-z_1)^2} + \frac{\partial T(z_1)}{z-z_1} + (\text{regular}) \Big) \prod_{i\neq 1} T(z_i) \Big\rangle \Big\langle \bar{\chi}(\bar{z}) \prod_j \bar{T}(\bar{w}_j) \Big\rangle. \end{aligned} \tag{D.5}$$

Here $\oint_{z_1}$ is the counter-clockwise contour integral around $z = z_1$ (and not clockwise as ∂R in figure 1). As we will show later (section D.3), after regularization, a z-contour integral involving both z and $\bar{z}$ can be evaluated as

$$\oint_{z=\alpha} \frac{dz}{(z-\alpha)^n} \bar{f}(\bar{z}) = 2\pi i\, \bar{f}(\bar{\alpha})\, \delta_{n,1} \tag{D.6}$$

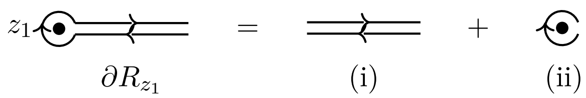
where $\bar{f}(\bar{z})$ is an arbitrary anti-holomorphic function regular at $\bar{z} = \bar{\alpha}$. This means that only the third term in (D.5) contributes, giving

$$I'_{z_1} = 2\pi \Big\langle \partial T(z_1) \prod_{i\neq 1} T(z_i) \Big\rangle \Big\langle \bar{\chi}(\bar{z}_1) \prod_j \bar{T}(\bar{w}_j) \Big\rangle. \tag{D.7}$$

Although $\bar{\chi}(\bar{z})$ is single-valued around $z = z_1$ because there is no $\bar{T}$ insertion there, it is multi-valued globally and we really have to specify the path along which we integrate $\bar{T}(\bar{z})$ to get $\bar{\chi}(\bar{z}_1)$. By taking the path to be one going from the reference point X to z_1 (note that $\bar{\chi}$ is single-valued along this path), we can write the above result as

$$I'_{z_1} = 2\pi \Big\langle \partial T(z_1) \prod_{i\neq 1} T(z_i) \Big\rangle \int_{\bar{X}}^{\bar{z}_1} d\bar{z} \Big\langle \bar{T}(\bar{z}) \prod_j \bar{T}(\bar{w}_j) \Big\rangle. \tag{D.8}$$

Let us turn to the contribution to I'' from ∂R_{z_1}. In the presence of $T(z_1)$, $\chi(z)$ is not single-valued around $z = z_1$. We can split the contour into (i) the piece that connects X and z_1, and (ii) a small circle going around z_1 as follows:



(D.9)

The contribution from part (i) comes from the discontinuity of $\chi(z)$ across the cut. Namely,

$$\begin{aligned} I''_{z_1,\mathrm{(i)}} &= -i \int_{\mathrm{(i)}} d\bar{z} \left\langle \chi(z) T(z_1) \prod_{i\neq 1} T(z_i) \right\rangle \Big\langle \bar{T}(\bar{z}) \prod_j \bar{T}(\bar{w}_j) \Big\rangle \\ &= -i \int_{\bar{X}}^{\bar{z}_1} d\bar{z} \left\langle \Big[\chi(z)|_{\text{below}} - \chi(z)|_{\text{above}}\Big] T(z_1) \prod_{i\neq 1} T(z_i) \right\rangle \Big\langle \bar{T}(\bar{z}) \prod_j \bar{T}(\bar{w}_j) \Big\rangle. \end{aligned} \tag{D.10}$$

If we recall $T(z)T(z_1) \supset \frac{\partial T(z_1)}{z-z_1}$ and thus $\chi(z)T(z_1) \supset \partial T(z_1) \log(z - z_1)$, we see that what this discontinuity does is $[\chi(z)|_{\text{below}} - \chi(z)|_{\text{above}}]\, T(z_1) \to 2\pi i \partial T(z_1)$. Therefore,

$$I''_{z_1,\mathrm{(i)}} = 2\pi \int_{\bar{X}}^{\bar{z}_1} d\bar{z} \left\langle \partial T(z_1) \prod_{i\neq 1} T(z_i) \right\rangle \Big\langle \bar{T}(\bar{z}) \prod_j \bar{T}(\bar{w}_j) \Big\rangle = I'_{z_1}. \tag{D.11}$$

One can show that the contribution from part (ii) vanishes after regularization, namely, $I''_{z_1,\mathrm{(ii)}} = 0$; see the discussion around (D.20).

Substituting the above results (D.8), (D.11) into (D.4), we see that the contribution from ∂R_{z_1} to I is independent of the parameter a. The contribution from ∂R_{z_i}, $i \neq 1$ are similar. Furthermore, the result being independent of a means that the contribution from

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∂R_{w_j} that goes around $\bar{T}$ insertions at $z = w_j$ is obtained by taking the complex conjugate of this result. So, summing up all contributions, we find

$$\begin{aligned} I &= \sum_i I_{z_i} + \sum_j I_{w_j}, \\ I_{z_i} &= -\frac{\delta\mu}{\pi} \Big\langle \partial T(z_i) \prod_{i' \neq i} T(z_i') \Big\rangle \int_{\bar{X}}^{\bar{z}_i} d\bar{z} \Big\langle \bar{T}(\bar{z}) \prod_j \bar{T}(\bar{w}_j) \Big\rangle, \\ I_{w_j} &= -\frac{\delta\mu}{\pi} \Big\langle \bar{\partial}\bar{T}(\bar{w}_j) \prod_{j' \neq j} \bar{T}(\bar{w}_j') \Big\rangle \int_X^{w_j} dz \, \Big\langle T(z) \prod_i T(z_i) \Big\rangle. \end{aligned} \tag{D.12}$$

Here $\langle \ldots \rangle$ is the connected part that is proportional to c. So, the correction I is always $\mathcal{O}(c^2 \delta\mu)$, which implies that the fluctuation part δS_{fluc} of the deformed Liouville action must be renormalized away as argued in section 2.3. Eq. (D.12) implies that the operator $T(z)$ gets deformed as[13]

$$T(z) \to T(z) + \delta T(z), \qquad \delta T(z) = -\frac{\delta\mu}{\pi} \partial T(z) \int_{\bar{X}}^{\bar{z}} d\bar{z}' \, \bar{T}(\bar{z}'). \tag{D.13}$$

Based on this, we can derive the deformed OPEs derived in the main text.

D.2 Examples

Here we apply the general formula (D.12) and reproduce some $T\bar{T}$-deformed correlators computed in the main text.

For example, the 3-point function $\langle T\bar{T}\bar{T} \rangle$ is

$$\begin{aligned} \langle T_1 \bar{T}_2 \bar{T}_3 \rangle_{\delta\mu} &= \frac{\delta\mu}{\pi} \left[\underbrace{\langle \partial T_1 \rangle}_{=0} \int_{\bar{X}}^{\bar{z}_1} dz \, \langle \bar{T} \bar{T}_2 \bar{T}_3 \rangle + \Big\langle \bar{\partial} \bar{T}_2 \bar{T}_3 \Big\rangle \int_X^{z_2} dz \, \langle T T_1 \rangle + \underbrace{\Big\langle \bar{T}_2 \bar{\partial} \bar{T}_3 \Big\rangle}_{=-\langle \bar{\partial} \bar{T}_2 \bar{T}_3 \rangle} \int_X^{z_3} dz \, \langle T T_1 \rangle \right] \\ &= -\frac{\delta\mu}{\pi} \Big\langle \bar{\partial} \bar{T}_2 \bar{T}_3 \Big\rangle \int_{z_2}^{z_3} dz \, \langle T T_1 \rangle \\ &= -\frac{\delta\mu}{\pi} \bar{\partial}_2 \Big(\frac{c}{2 \bar{z}_{23}^4} \Big) \int_{z_2}^{z_3} dz \, \frac{c}{2(z - z_1)^4} = \frac{c^2 \delta\mu}{3\pi} \frac{1}{\bar{z}_{23}^5} \left(-\frac{1}{z_{12}^3} + \frac{1}{z_{13}^3} \right). \end{aligned} \tag{D.14}$$

This correctly reproduces (4.12). Note that the dependence on the reference point X

[13]The deformation in (D.12) is twice as large as what one would get from eq. (3.28) of [11] at $\mathcal{O}(\delta\mu)$, because that paper only took account of I'' and missed the contribution from I' ($a = 1/2$ there). Eq. (3.28) of [11] seems to agree with eq. (3.6) there, but the latter equation also contains an error and must be multiplied by 2.

canceled out. Likewise, the 4-point function $\langle TT\bar{T}\bar{T}\rangle$ can be computed as:

$$
\begin{aligned}
\langle T_1T_2\bar{T}_3\bar{T}_4\rangle_{\delta\mu} &= \frac{\delta\mu}{\pi}\left[\langle\partial T_1T_2\rangle\int_{\bar{X}}^{\bar{z}_1}d\bar{z}\,\langle\bar{T}\bar{T}_3\bar{T}_4\rangle + \underbrace{\langle T_1\partial T_2\rangle}_{=-\langle\partial T_1T_2\rangle}\int_{\bar{X}}^{\bar{z}_2}d\bar{z}\,\langle\bar{T}\bar{T}_3\bar{T}_4\rangle + \overline{(12\leftrightarrow 34)}\right]\\
&= -\frac{\delta\mu}{\pi}\langle\partial T_1T_2\rangle\int_{\bar{z}_1}^{\bar{z}_2}d\bar{z}\,\langle\bar{T}\bar{T}_3\bar{T}_4\rangle + \overline{(12\leftrightarrow 34)}\\
&= -\frac{\delta\mu}{\pi}\partial_1\left(\frac{c}{2z_{12}^4}\right)\int_{\bar{z}_1}^{\bar{z}_2}d\bar{z}\frac{c}{(\bar{z}-\bar{z}_3)^2(\bar{z}-\bar{z}_4)^2\bar{z}_{34}^2} + \overline{(12\leftrightarrow 34)}\\
&= \frac{2c^2\delta\mu}{\pi}\left(\frac{1}{z_{12}^5\bar{z}_{34}^4}\left(-\frac{1}{\bar{z}_{23}}+\frac{1}{\bar{z}_{13}}-\frac{1}{\bar{z}_{24}}+\frac{1}{\bar{z}_{14}}\right)\right.\\
&\qquad\left.+\frac{1}{z_{12}^4\bar{z}_{34}^5}\left(\frac{1}{z_{14}}-\frac{1}{z_{13}}+\frac{1}{z_{24}}-\frac{1}{z_{23}}\right)+\frac{2}{z_{12}^5\bar{z}_{34}^5}\log\left|\frac{z_{24}z_{13}}{z_{23}z_{14}}\right|^2\right). \qquad \text{(D.15)}
\end{aligned}
$$

This reproduces (5.13). Other correlators can be computed in a similar way. In the contour integral approach, the issue of avoiding branch cuts we saw in the conformal perturbation theory in appendix C has already been taken care of, and computations are quite straightforward. There is no problem in computing higher-point functions; it only gets more cumbersome.

D.3 Some formulas

Let us close some loose ends by showing relations that we used above.

First, let us show (D.6). Because $\bar{f}(\bar{z})$ is assumed to be regular at $z = \alpha$, we can expand the left-hand side as

$$\oint_{z=\alpha}\frac{dz}{(z-\alpha)^n}\bar{f}(\bar{z}) = \sum_{m\geq 0}c_m\oint\frac{dy}{y^n}\bar{y}^m, \qquad \text{(D.16)}$$

where we set $z-\alpha =: y$ and expanded $\bar{f}(\bar{z})$ in powers of $\bar{y}$. If we set $y = \epsilon e^{i\theta}$ with small ϵ,

$$i\sum_{m\geq 0}c_m\epsilon^{1-n+m}\int_0^{2\pi}e^{i(1-n-m)\theta}d\theta = 2\pi i\epsilon^{2(1-n)}c_{1-n} \qquad \text{(D.17)}$$

where the only surviving term has $m = 1-n$. Because $m \geq 0$, this means that $1-n \geq 0$, namely $n = 0, 1$. On the other hand, for (D.17) to be non-vanishing in the $\epsilon \to 0$ limit, we need $1-n \leq 0$, namely $n \geq 1$. Therefore, the only non-vanishing case is $n = 1$, and

$$\oint_{z=\alpha}\frac{dz}{z-\alpha}\bar{f}(\bar{z}) = 2\pi i c_0 = 2\pi i\bar{f}(\bar{z}=\bar{\alpha}). \qquad \text{(D.18)}$$

This completes the proof of (D.6).

By completely analogous computations, we can show the following formulas:

$$\oint_{z=\alpha}\frac{d\bar{z}}{(z-\alpha)^n}\bar{f}(\bar{z}) = 0, \qquad \oint_{z=\alpha}d\bar{z}\,\bar{f}(\bar{z})\log(z-\alpha) = 0. \qquad \text{(D.19)}$$

where $\bar{f}(\bar{z})$ is regular at $\bar{z} = \bar{\alpha}$.

Finally, we want to prove that the contribution from the circular part of the contour ∂R_{z_1} does not contribute to I''_{z_1}, as mentioned below (D.11). So, we are interested in

$$I''_{z_1,\mathrm{(ii)}} = -i \int_{\mathrm{(ii)}} d\bar{z} \left\langle \chi(z) T(z_1) \prod_{i\neq 1} T(z_i) \right\rangle \left\langle \bar{T}(\bar{z}) \prod_{j} \bar{T}(\bar{w}_j) \right\rangle. \tag{D.20}$$

where the contour is the second term in the figure (D.9). As $z \to z_1$, the OPE

$$T(z)T(z_1) = \frac{c}{2(z-z_1)^4} + \frac{2T(z_1)}{(z-z_1)^2} + \frac{\partial T(z_1)}{z-z_1} + (\text{regular}) \tag{D.21}$$

implies the behavior

$$\chi(z)T(z_1) = -\frac{c}{6(z-z_1)^3} - \frac{2T(z_1)}{z-z_1} + \partial T(z_1)\log(z-z_1) + (\text{regular}). \tag{D.22}$$

In the $\epsilon \to 0$ limit, the $\bar{z}$ integral vanishes because of (D.19). This completes the proof.

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