Clock-dependent spacetime

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ABSTRACT: Einstein's theory of general relativity is based on the premise that the physical laws take the same form in all coordinate systems. However, it still presumes a preferred decomposition of the total kinematic Hilbert space into local kinematic Hilbert spaces. In this paper, we consider a theory of quantum gravity that does not come with a preferred partitioning of the kinematic Hilbert space. It is pointed out that, in such a theory, dimension, signature, topology and geometry of spacetime depend on how a collection of local clocks is chosen within the kinematic Hilbert space.

Contents

1 Introduction

In Newtonian paradigm, physical laws govern the evolution of dynamical degrees of freedom with respect to one universal time. Einstein's theory of relativity demotes time from the absolute status in two ways. First, the notion of simultaneity becomes observer-dependent for events that are spatially separated, and there is no universal sense of past, present and future. Second, time evolution is turned into a gauge transformation, and time as a gauge parameter has no physical meaning by itself [1–4]. One needs to use dynamical variables as clocks to describe the relative evolution of other degrees of freedom [5–7]. The theory only predicts correlations among dynamical degrees of freedom.

In quantum gravity, choosing clocks boils down to dividing the kinematic Hilbert space into a sub-Hilbert space for clocks and the rest for the 'true' physical degrees of freedom. In general relativity, while there is no preferred coordinate system, there is still a preferred way of identifying a sub-Hilbert space for each local clock. This is because the theory is covariant only under the transformations that preserve the integrity of local sites. Under diffeomorphism, points in space are permuted, but the quantum information stored at a point is never spread over multiple points. The preferred set of local kinematic Hilbert spaces is invariant under diffeomorphism, and each local clock variable is chosen from one of the local Hilbert spaces.

Requiring that physical laws are covariant only under the local Hilbert space-preserving transformations may be too restrictive in quantum gravity that has no predetermined notion of locality. In priori, one partitioning of Hilbert space is no better than others. Furthermore, the fact that locality is a dynamical concept in quantum gravity obscures the distinction between local and non-local transformations. Consider a unitary transformation that mixes local Hilbert spaces associated with multiple sites. For states in which those sites are within a short-distance cutoff scale, the transformation can be regarded as local in space, and we may gauge it as an internal symmetry. However, it is no longer local for other states in which the sites affected by the transformation are macroscopically apart. Once we gauge general unitary transformations that do not preserve local kinematic Hilbert spaces, the preferred Hilbert space decomposition is lost.

In this paper, we examine consequences of having no preferred Hilbert space decomposition in a recently proposed model of quantum gravity [8]. In the model, geometric degrees of freedom are collective variables of underlying quantum matter [9–11]. In the absence of a predetermined Hilbert space decomposition, there exists a greater freedom in how clock variables are identified within the kinematic Hilbert space. Instead of choosing local clock variables from predetermined local Hilbert spaces, in this theory local Hilbert spaces are identified from a choice of clock variables. In other words, the notion of local site is derived from clocks. A set of local observers who use a particular set of local clocks constructs a geometry based on the pattern of entanglement present across the local Hilbert spaces associated with the clocks. Inasmuch as patterns of entanglement depend on partitioning of the total Hilbert space, one state can exhibit different geometries with different choices of local clocks. The spacetime that emerges with respect to one choice of clocks is in general different from the spacetime that arises with respect to another set of local clocks. The purpose of the paper is to show that the spacetimes that emerge from different choices of local clocks can exhibit different dimensions, signatures, topologies and geometries.

The rest of the paper is organized as follows. In section 2, we review the theory introduced in ref. [8] as it forms the basis of the present work. In the review, the gauge symmetry, the constraint algebra, and the way an emergent geometry is identified from the underlying microscopic degrees of freedom are emphasized as they are the key ingredients needed in this paper. Section 3 is the main part of the present paper. The objectives of this section are two-folded. The first is to identify a set of clock variables to construct spacetime from the correlation between the clock variables and the remaining physical degrees of freedom in the semi-classical limit. The second is to examine how different choices of local clocks leads to different spacetimes. In section 3.1, a procedure that generates gauge invariant states from a set of first-class constraints is discussed. In section 3.2, the gauge constraints are solved to identify the constraint surface in the semi-classical limit. Section 3.3 discusses a gauge fixing prescription that introduces clock variables and the associated local Hilbert spaces. The section also discusses how the correlation between the remaining physical degrees of freedom and the local clocks determines an emergent spacetime. In section 3.4, two schemes that use different sets of local clocks are applied to one physical state, and extract the spacetimes that emerge from those choices. Section 4 is a summary with open questions.

2 Review: a model of quantum gravity with emergent spacetime

In this section, we review the model introduced in ref. [8]. We recap the main results that are needed in this paper, and refer the readers to the original paper for details.

- *Kinematic Hilbert space*

  The fundamental degree of freedom is a real rectangular matrix with M rows and L columns with $M \gg L \gg 1$: $\Phi^A_{\ i}$ with $A = 1, 2, \ldots, M$ and $i = 1, 2, \ldots, L$. The row (A) labels flavour, and the column (i) labels sites. The full kinematic Hilbert space $\mathbb{H}$ is spanned by the set of basis states, $\left\{ \left| \Phi \right\rangle \middle| -\infty < \Phi^A_{\ i} < \infty \text{ with } 1 \leq A \leq M, 1 \leq i \leq L \right\}$, where $\left| \Phi \right\rangle \equiv \otimes_{i,A} \left| \Phi^A_{\ i} \right\rangle$, and $\left| \Phi^A_{\ i} \right\rangle$ is the eigenstate of $\hat{\Phi}^A_{\ i}$. The inner product between basis states is $\left\langle \Phi' \middle| \Phi \right\rangle = \prod_{i,A} \delta \left(\Phi'^A_{\ i} - \Phi^A_{\ i} \right)$.

- *Flavour (Global) symmetry*

  The global symmetry is $O(M)$. It rotates the flavour index acting as a left multiplication on Φ: $\Phi \to O\Phi$, where $O \in O(M)$. The generator of the $O(M)$ flavour symmetry is $\hat{T}_{\tilde{o}} = \frac{1}{2} \operatorname{tr} \left\{ \left(\hat{\Phi} \hat{\Pi} - \hat{\Pi}^T \hat{\Phi}^T \right) \tilde{o} \right\}$, where $\hat{\Phi}$ represents the operator valued $M \times L$ matrix, $\hat{\Pi}$ is the conjugate momentum that is an $L \times M$ matrix, and $\tilde{o}$ is a real $M \times M$ anti-symmetric matrix.

- *Frame*

  A frame is a decomposition of the total kinematic Hilbert space into a direct product of local Hilbert spaces. For example, $\mathbb{H}$ can be written as

  $$\mathbb{H} = \otimes_i \mathbb{H}_i, \tag{2.1}$$

  where $\mathbb{H}_i$ is the local Hilbert space spanned by $\left\{ \otimes_A \left| \Phi^A_{\ i} \right\rangle \right\}$ at site i. The frame can be rotated with $\mathrm{SL}(L, \mathbb{R})$ transformations that act as right multiplications on Φ: $\Phi \to \Phi g$, where $g \in \mathrm{SL}(L, \mathbb{R})$. New basis states defined by $\left| \Phi \right) \equiv \left| \Phi g \right\rangle$ have the same inner product, $\left(\Phi \middle| \Phi' \right) = \prod_{I,A} \delta \left(\Phi'^A_{\ I} - \Phi^A_{\ I} \right)$. This allows us to write $\left| \Phi \right) = \otimes_{A,I} \left| \Phi^A_{\ I} \right)$. The Hilbert space $\mathbb{H}'_I$ spanned by $\left\{ \otimes_A \left| \Phi^A_{\ I} \right) \right\}$ forms a local Hilbert space for site I in the rotated frame, and the kinematic Hilbert space can be decomposed as

  $$\mathbb{H} = \otimes_I \mathbb{H}'_I. \tag{2.2}$$

  In general, a state that is unentangled in one frame has non-trivial inter-site entanglement in another frame.

- *Gauge symmetry*

  In the limit that the size of matrix becomes large, the sites can collectively form a space manifold. We identify the emergent geometric degrees of freedom from the microscopic degree of freedom based on the algebra that gauge constraints obey. Just as the momentum and Hamiltonian constraints generate spacetime diffeomorphism in general relativity, the present theory comes with generalized momentum and Hamiltonian constraints.

1. *Generalized momentum*

   The $SL(L, \mathbb{R})$ group that rotates frame is taken as the gauge group that generalizes the spatial diffeomorphism in the general relativity. The generalized momentum operator that generates $SL(L, \mathbb{R})$ is

$$\hat{G}_y = \operatorname{tr}\left\{ \hat{G} y \right\}, \tag{2.3}$$

   where $\hat{G}$ is an operator valued rank 2 traceless tensor given by $\hat{G}^i{}_j = \frac{1}{2}\left(\hat{\Pi}^i{}_A \hat{\Phi}^A{}_j + \hat{\Phi}^A{}_j \hat{\Pi}^i{}_A \right) - \frac{1}{2L}\left(\hat{\Pi}^k{}_A \hat{\Phi}^A{}_k + \hat{\Phi}^A{}_k \hat{\Pi}^k{}_A \right)\delta^i_j$, and y is a traceless $L \times L$ real matrix called the shift tensor. Under the $SL(L, \mathbb{R})$ transformation, $\hat{\Phi}$ and $\hat{\Pi}$ transform as $e^{-i\hat{G}_y}\, \hat{\Phi}\, e^{i\hat{G}_y} = \hat{\Phi} g_y$ and $e^{-i\hat{G}_y}\, \hat{\Pi}\, e^{i\hat{G}_y} = g_y^{-1}\hat{\Pi}$, where $g_y = e^{-y}$.

2. *Generalized Hamiltonian*

   The Hamiltonian constraint is written as

$$\hat{H}_v = \operatorname{tr}\left\{ \hat{H} v \right\}, \tag{2.4}$$

   where $\hat{H}$ is an operator valued rank 2 symmetric tensor given by $\hat{H}^{ij} = \frac{1}{2}\Big[\left(-\hat{\Pi}\hat{\Pi}^T + \frac{\tilde{\alpha}}{M^2}\hat{\Pi}\hat{\Pi}^T\hat{\Phi}^T\hat{\Phi}\hat{\Pi}\hat{\Pi}^T \right)^{ij} + \left(-\hat{\Pi}\hat{\Pi}^T + \frac{\tilde{\alpha}}{M^2}\hat{\Pi}\hat{\Pi}^T\hat{\Phi}^T\hat{\Phi}\hat{\Pi}\hat{\Pi}^T \right)^{ji} \Big]$, $\tilde{\alpha}$ is a constant parameter of the theory, and v is an $L \times L$ real symmetric matrix called the lapse tensor. Under the $SL(L, \mathbb{R})$ transformation, v transforms as $e^{-i\hat{G}_y}\, \hat{H}_v\, e^{i\hat{G}_y} = \hat{H}_{v'}$, where $v' = (g_y^{-1})^T v g_y^{-1}$ with $g_y = e^{-y}$.

3. *Constraint Algebra*

   The generalized momentum and Hamiltonian constraints satisfy the first-class quantum algebra:

$$\begin{aligned}
\left[\hat{G}^i{}_j, \hat{G}^k{}_l \right] &= i A^{ikn}_{jlm}\, \hat{G}^m{}_n, \\
\left[\hat{G}^i{}_j, \hat{H}^{kl} \right] &= i B^{ikl}_{jmn}\, \hat{H}^{mn}, \\
\left[\hat{H}^{ij}, \hat{H}^{kl} \right] &= i \hat{C}^{ijkln}_m\, \hat{G}^m{}_n + \frac{i}{M}\hat{D}^{ijkl}_{nm}\, \hat{H}^{mn},
\end{aligned} \tag{2.5}$$

where

$$A^{ikn}_{jlm} = \delta^k_j \delta^i_m \delta^n_l - \delta^i_l \delta^k_m \delta^n_j,$$

$$B^{ikl}_{jmn} = \delta^k_j \delta^{il}_{mn} + \delta^l_j \delta^{ki}_{mn},$$

$$\hat{C}^{ijkln}_m = -4\tilde\alpha \Big[\hat{U}^{n(j}\hat{U}^{i)[l}\delta^{k]}_m - \hat{U}^{n[l}\hat{U}^{k](j}\delta^{i)}_m \Big]$$

$$+ 4\tilde\alpha^2 \Big[-(\hat{U}\hat{Q})^{(j}_{[m}\hat{U}^{\{l[n}\hat{U}^{n']k\}}\delta^{i)}_{m']} + \hat{U}^{(j[n}(\hat{Q}\hat{U})^{\{k}_{[m'}\hat{U}^{n']i)}\delta^{l\}}_{m]}$$

$$+ \hat{U}^{(j[n}(\hat{U}\hat{Q})^{\{l}_{[m}\hat{U}^{n']i)}\delta^{k\}}_{m']} - \hat{U}^{\{l[n}\hat{U}^{n']k\}}(\hat{Q}\hat{U})^{(i}_{[m']}\delta^{j)}_{m]}$$

$$+ \frac{1}{M^2}\Big(M\hat{U}^{(j[n}\hat{U}^{n']\{k}\delta^{l\}}_{[m}\delta^{i)}_{m']} + (M+2)\hat{U}^{(j[n}\hat{U}^{\{li)}\delta^{n']}_{[m}\delta^{k\}}_{m']}$$

$$+ 2\hat{U}^{(j[n}\hat{U}^{n']i)}\delta^{\{l}_{[m}\delta^{k\}}_{m']} - 2\hat{U}^{(j\{k}\hat{U}^{l][n}\delta^{n']}_{[m}\delta^{i)}_{m']}$$

$$- 2\hat{U}^{(j\{k}\hat{U}^{[n'n]}\delta^{l\}}_{[m}\delta^{i)}_{m']} - 2\hat{U}^{(ij)}\hat{U}^{\{k[n}\delta^{n']}_{[m}\delta^{l\}}_{m']}\Big)\Big]\delta^{m'}_{n'},$$

$$\hat{D}^{ijkl}_{nm} = -4i\tilde\alpha\left(\hat{U}^{kl}\delta^{ij}_{nm} - \hat{U}^{ij}\delta^{kl}_{nm} \right) \tag{2.6}$$

with $\hat{U} = \frac{1}{M}(\hat\Pi\hat\Pi^T)$, $\hat{Q} = \frac{1}{M}(\hat\Phi^T\hat\Phi)$ and $\delta^{kl}_{ij} = \frac{1}{2}\left(\delta^k_i\delta^l_j + \delta^l_i\delta^k_j \right)$. While A^{ikn}_{jlm} and B^{ikl}_{jmn} are constant tensors, $\hat{C}^{ijkln}_m$ and $\hat{D}^{ijkl}_{nm}$ are operator valued structure function. In the expression for $\hat{C}^{ijkln}_m$, each pair of indices in (i,j), $[n,n']$, $[m,m']$, $\{k,l\}$ are understood to be symmetrized. In the large M limit, $A^{ikn}_{jlm}, B^{ikl}_{jmn}, \hat{C}^{ijkln}_m, \hat{D}^{ijkl}_{nm} \sim O(1)$, and the term that is proportional to $\hat{H}^{mn}$ in $\left[\hat{H}^{ij}, \hat{H}^{kl} \right]$ is sub-leading. Eq. (2.5) is the exact commutator that remains well-defined at the quantum level. In the large M limit, the algebra reduces to the one that includes the hypersurface embedding algebra of the general relativity. This plays the key role in identifying the emergent geometry in this model.

- *Emergent geometry*

  A coordinate system is a mapping $r : i \to r_i$ from the set of sites in a frame to a manifold $\mathscr{M}$ with a region $R_i \ni r_i$ assigned to site i (see figure 1). In the large L limit in which the image of sites is dense in $\mathscr{M}$, Eq. (2.5) induces an algebra on $\mathscr{M}$. The generators of the induced algebra include the Weyl generator ($\mathscr{D}$), the momentum density ($\mathscr{P}_\mu$), the Hamiltonian density ($\mathscr{H}$) and generators for higher spin gauge transformations. Since v transforms as a rank 2 symmetric tensor under $\mathrm{SL}(L, \mathbb{R})$, one can always choose a frame in which v is diagonal. In this frame, the Hamiltonian constraint can be written as

  $$\hat{H}_v = \int dr\; \mathscr{H}(r)\theta_v(r). \tag{2.7}$$

  Here $\mathscr{H}(r_i) = V_i^{-1}\hat{H}^{ii}$ is the Hamiltonian density, V_i is the coordinate volume of R_i, and $\theta_v(r_i) = v_{ii}$ is the lapse function. $\hat{G}^i_{\ j}$ is now viewed as a bi-local operator defined on $\mathscr{M}$. It can be expanded in the relative coordinate as

  $$\hat{G}^i_{\ j} = \hat{G}^i_{\ i} + \left.\frac{\partial \hat{G}^i_{\ j}}{\partial r^\mu_j}\right|_{j=i} (r^\mu_j - r^\mu_i) + \dots, \tag{2.8}$$
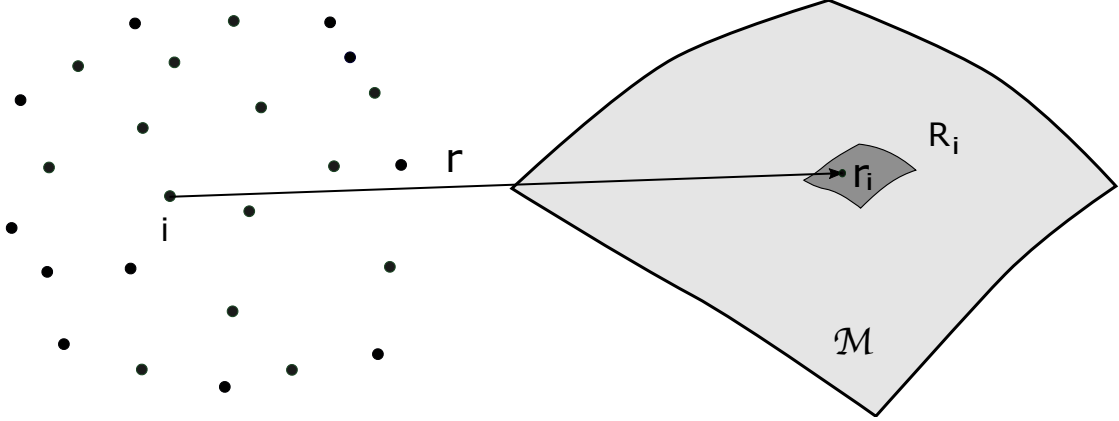
Figure 1. A coordinate system is a mapping from the set of sites in a frame into a manifold $\mathcal{M}$. r_i represents the image of site i, and R_i is a simply connected region allocated to site i such that $r_i \in R_i$, $\mathcal{M} = \cup_i R_i$ and $R_i \cap R_j = \emptyset$ for $i \neq j$.

where the ellipsis represents higher derivative terms. This leads to

$$\hat{G}_y = \int dr \left(\mathscr{D}(r)\zeta_y(r) + \mathscr{P}_\mu(r)\xi_y^\mu(r) + \ldots \right), \tag{2.9}$$

where the derivative expansion is shown to the first order that includes the momentum density. Here $\mathscr{D}(r_i) = V_i^{-1}\hat{G}^i{}_i$ and $\mathscr{P}_\mu(r_i) = V_i^{-1} \left. \frac{\partial \hat{G}^i{}_j}{\partial r_j^\mu} \right|_{j=i}$ are the Weyl generator and the momentum density, respectively. $\zeta_y(r_i) = \sum_j y^j{}_i$ and $\xi_y^\mu(r_i) = \sum_j y^j{}_i(r_j^\mu - r_i^\mu)$ represent the Weyl parameter and the shift vector, respectively.

The commutators between $\mathscr{D}$, $\mathscr{P}_\mu$ and $\mathscr{H}$ are completely determined from eq. (2.5). To the leading order in $1/M$ and the derivative expansion, the commutators read

$$\left[\int dr\zeta_1(r)\mathscr{D}(r), \int dr\zeta_2(r)\mathscr{D}(r) \right] = 0,$$

$$\left[\int dr\xi^\mu(r)\mathscr{P}_\mu(r), \int dr\zeta(r)\mathscr{D}(r) \right] = i \int dr\mathscr{L}_\xi\zeta(r)\mathscr{D}(r),$$

$$\left[\int dr\zeta(r)\mathscr{D}(r), \int dr\theta(r)\mathscr{H}(r) \right] = 2i \int dr\zeta(r)\theta(r)\mathscr{H}(r),$$

$$\left[\int dr\xi_1^\mu(r)\mathscr{P}_\mu(r), \int dr\xi_2^\nu(r)\mathscr{P}_\nu(r) \right] = i \int dr\mathscr{L}_{\xi_1}\xi_2^\mu(r)\mathscr{P}_\mu(r),$$

$$\left[\int dr\xi^\mu(r)\mathscr{P}_\mu(r), \int dr\theta(r)\mathscr{H}(r) \right] = i \int dr\mathscr{L}_\xi\theta(r)\mathscr{H}(r),$$

$$\left[\int dr\theta_1(r)\mathscr{H}(r), \int dr\theta_2(r)\mathscr{H}(r) \right] = i \int dr \left(\hat{F}^\nu(r)\mathscr{D}(r) + \hat{G}^{\mu\nu}\mathscr{P}_\mu(r) \right) (\theta_1\nabla_\nu\theta_2 - \theta_2\nabla_\nu\theta_1),$$

$$\tag{2.10}$$

where $\mathscr{L}_\xi$ represents the Lie derivative with respect to the vector field ξ, and $\hat{F}^\nu(r_m)$

and $\hat{G}^{\mu\nu}(r_m)$ are given by

$$\hat{F}^\nu(r_m) = \frac{1}{2} \sum_{i,k,n} \hat{C}_m^{iikkn} \left(r_k^\nu - r_i^\nu \right), \tag{2.11}$$

$$\hat{G}^{\mu\nu}(r_m) = \frac{1}{2} \sum_{i,k,n} \hat{C}_m^{iikkn} \left(r_n^\mu - r_m^\mu \right) \left(r_k^\nu - r_i^\nu \right). \tag{2.12}$$

The momentum and Hamiltonian densities obey an algebra that generalizes the hyper-surface deformation algebra of the general relativity [12, 13], provided that the metric is identified as the symmetric part of $\hat{G}^{\mu\nu}$,

$$\hat{g}^{\mu\nu}(r_m) = -\frac{\mathcal{S}}{4} \sum_{i,k,n} \hat{C}_m^{iikkn} \left[\left(r_n^\mu - r_m^\mu \right) \left(r_k^\nu - r_i^\nu \right) + \left(r_n^\nu - r_m^\nu \right) \left(r_k^\mu - r_i^\mu \right) \right], \tag{2.13}$$

where $\mathcal{S}$ is the signature of the spacetime direction translated by the Hamiltonian constraint. The overall sign of the spacetime metric can be chosen either way. In the rest of the paper, we choose the convention in which $\mathcal{S} = -1$. $\hat{F}^\nu(r)$ and the anti-symmetric part of $\hat{G}^{\mu\nu}$ represent additional collective fields that generalize the hyper-surface deformation algebra of general relativity.

The contravariant metric in eq. (2.13) is given by a second moment of $\hat{C}_m^{iikkn}$, which measures a multi-point correlation in the system. If the range of entanglement and correlation is large in the coordinate distance, the second moment becomes large accordingly, which results in a small proper distance between points in space. The metric identified from the constraint algebra naturally captures the intuition that two sites that are strongly entangled are physically close [14–23]. On the other hand, the metric captures only a specific pattern of entanglement, and there also exist non-geometric entanglements. For example, there exist finely tuned states in which two points that are infinitely far still have $O(M)$ entanglement through other channels such as the higher-order moments of $\hat{C}_m^{iikkn}$ [8]. In this sense, EPR is strictly 'bigger' than ER in the present theory [24].

States for which there exist coordinate systems with well-defined metric in the large M and L limit form a special set of states, and are referred to have local structures. For a state with a classical local structure, there exists a coordinate system associated with a well-defined manifold such that $\langle \hat{g}^{\mu\nu} \rangle \equiv \dfrac{\langle \Psi | \hat{g}^{\mu\nu} | \Psi \rangle}{\langle \Psi | \Psi \rangle}$ is invertible and smooth on the manifold, and $\left\langle \left(\hat{g}^{\mu\nu} - \langle \hat{g}^{\mu\nu} \rangle \right)^2 \right\rangle \to 0$ in the large M and L limit. The dimension, topology and geometry of the manifold are properties of state.

$\hat{H}_v$ in eq. (2.4) is a non-local Hamiltonian as a quantum operator, but it is relatively local [25] in the following sense.[1] Suppose $|\Psi\rangle$ has a local structure in a frame. To this state, $\hat{H}_v$ with a lapse tensor diagonal in that frame acts as a local Hamiltonian to the leading order in the large M limit, that is,

$$\hat{H}_v |\Psi\rangle \approx \hat{H}_{\text{eff}}^\Psi |\Psi\rangle, \tag{2.14}$$

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[1]We note that this is different from the relative locality introduced in ref. [30].

where $\hat{H}^{\Psi}_{\text{eff}}$ is a Hamiltonian that is local in the manifold associated with the local structure of $|\Psi\rangle$. The discrepancy between $\hat{H}_v$ and $\hat{H}^{\Psi}_{\text{eff}}$ in eq. (2.14) is sub-leading in $1/M$. This can be understood by writing the Hamiltonian with the lapse tensor $v = I$ as

$$\hat{H}_v \approx -\hat{\Pi}^i_A\hat{\Pi}^i_A + \frac{\tilde{\alpha}}{M^2}\langle\hat{\Pi}^i{}_A\hat{\Pi}^j{}_A\rangle\hat{\Phi}^B{}_j\hat{\Phi}^B{}_k\langle\hat{\Pi}^k{}_A\hat{\Pi}^i{}_A\rangle, \tag{2.15}$$

where all repeated indices are summed over. In the large M limit, the fluctuation of $\hat{\Pi}^i{}_A\hat{\Pi}^j{}_A$ is small, and the replacement of the operator with its expectation value is valid to the leading order in the large M limit. The second term in the Hamiltonian can be viewed as a hopping term between sites j and k whose hopping amplitude is proportional to the expectation value of $\langle\hat{\Pi}^i{}_A\hat{\Pi}^j{}_A\rangle\langle\hat{\Pi}^k{}_A\hat{\Pi}^i{}_A\rangle$ in a state. If the two-point function $\langle\hat{\Pi}^i{}_A\hat{\Pi}^j{}_A\rangle$ is short-ranged as a function of $r_i - r_j$ in a manifold, the Hamiltonian effectively behaves as a local Hamiltonian in the manifold. The Hamiltonian acts in a state-dependent manner to the leading order in the large M limit, and the local properties of the effective Hamiltonian are inherited from the state [26].

3 Clocks and emergent spacetime

3.1 Gauge invariant states

The physical Hilbert space is given by the set of gauge invariant states that satisfy

$$\hat{H}_v|\chi\rangle = 0, \quad \hat{G}_y|\chi\rangle = 0 \tag{3.1}$$

for any lapse tensor v and shift tensor y. A gauge invariant state can be constructed by projecting an arbitrary state to the physical Hilbert space. The projection can be implemented with a series of gauge transformations applied to an initial trial state $|\chi\rangle$ as

$$|0_\chi\rangle = \lim_{Z\to\infty}\int\mathscr{D}v\int\mathscr{D}y\; e^{-i\varepsilon\left(\hat{H}_{v^{(1)}}+\hat{G}_{y^{(1)}}\right)}e^{-i\varepsilon\left(\hat{H}_{v^{(2)}}+\hat{G}_{y^{(2)}}\right)}\ldots e^{-i\varepsilon\left(\hat{H}_{v^{(Z)}}+\hat{G}_{y^{(Z)}}\right)}|\chi\rangle, \tag{3.2}$$

where ε is a non-zero constant, $\int\mathscr{D}v \equiv \int\prod^Z_{l=1}Dv^{(l)}$ and $\int\mathscr{D}y \equiv \int\prod^Z_{l=1}Dy^{(l)}$ denote the sum over all possible combinations of the lapse and shift tensors.[2] The resulting state is gauge invariant if it does not vanish (see appendix A).

Although the momentum and Hamiltonian constraints are invariant under the $O(M)$ flavour symmetry, a gauge invariant state may spontaneously break the global symmetry to a smaller group. To simplify the problem of extracting the dynamical information from gauge invariant states, it is convenient to focus on a sector with a definite flavour symmetry group. Let us denote the set of all states (gauge invariant or not) that respect the global symmetry $\Gamma \subset O(M)$ as $\mathscr{V}_\Gamma$. Basis states of $\mathscr{V}_\Gamma$ can be labeled by a set of collective variables. The bigger Γ is, the less collective variables are needed to span $\mathscr{V}_\Gamma$. If Γ is too big, there are too few kinematic collective variables to support non-trivial physical degrees of freedom after the gauge degrees of freedom are removed. One simple

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[2]While eq. (3.2) is equivalent to the state obtained by one projection, $\int DvDy\; e^{-i\left(\hat{H}_v+\hat{G}_y\right)}|\chi\rangle$, eq. (3.2) is more convenient to use in the path integral formalism by taking small ε limit.

choice of Γ that supports a minimal number of non-trivial physical degrees of freedom is $\Gamma^* = S_L^f \times O(N/2) \times O(N/2)$ with $N = M - L$ [8]. Here S_L^f is the permutation group acting on the first L flavours. Two $O(N/2)$ groups generate flavour rotations within the remaining two sets of $N/2$ flavours. Basis states for $\mathscr{V}_{\Gamma^*}$ can be written as

$$|s, t_1, t_2\rangle = \sum_{P \in S_L^f} \int D\Phi \; e^{i\left[\sqrt{N}\sum_{a,a'=1}^{L} s^i{}_a P^a_{a'} \Phi^{a'}{}_i + \sum_{b=L+1}^{L+\frac{N}{2}} t_1^{ij} \Phi^b{}_i \Phi^b{}_j + \sum_{c=L+\frac{N}{2}+1}^{L+N} t_2^{ij} \Phi^c{}_i \Phi^c{}_j\right]} |\Phi\rangle,$$

(3.3)

where s, t_1, t_2 are collective variables that label the basis states; s is $L \times L$ matrix and t_1, t_2 are $L \times L$ symmetric matrices. In eq. (3.3), all repeated site indices (i, j) are understood to be summed over from 1 to L. Due to the sum over the flavour permutations $P \in S_L^f$ in eq. (3.3), $|s, t_1, t_2\rangle = |sP, t_1, t_2\rangle$ for $P \in S_L^f$.

s transforms as $s \to gs$ under $\mathrm{SL}(L, \mathbb{R})$, and as $s \to sO$ under $O(L) \subset O(M)$, where $g \in \mathrm{SL}(L, \mathbb{R})$ and $O \in O(L)$. An invertible s breaks $\mathrm{SL}(L, \mathbb{R})$ down to the subgroup, $\mathscr{I} = \{sPs^{-1} | P \in S_L^f \text{ with } \det P = 1\}$. This follows from $(sPs^{-1})s = sP \sim s$. The unbroken gauge group is related to the even site-permutation group through a similarity transformation. Therefore, s acts as a Stueckelberg field that breaks the generalized spatial diffeomorphism to the discrete permutation group. On the other hand, t_1^{ij} and t_2^{ij} are bi-local fields that generate inter-site entanglement. The mutual information between sites i and j for state in eq. (3.3) is proportional to $-N \sum_c \frac{|t_c^{ij}|^2}{Im t_c^{ii} Im t_c^{jj}} \ln \frac{|t_c^{ij}|^2}{Im t_c^{ii} Im t_c^{jj}}$ to the leading order in the small t_c^{ij} limit [27]. Geometry is determined from the connectivity formed by these bi-local fields. Generic choices of t_c^{ij} would break $\mathrm{SL}(L, \mathbb{R})$ completely. If t_c^{ij} depends only on $r_i - r_j$ in a coordinate system, the global translational symmetry in the manifold remains unbroken.

General states in $\mathscr{V}_{\Gamma^*}$ can be written as

$$|\chi\rangle = \int Ds Dt_1 Dt_2 \; |s, t_1, t_2\rangle \chi(s, t_1, t_2),$$

(3.4)

where $\chi(s, t_1, t_2)$ is a wavefunction of the collective variables. In eq. (3.4), the integrations over s, t_1, t_2 are defined along the real axis of each component of the matrices. If we choose the initial state $|\chi\rangle$ from $\mathscr{V}_{\Gamma^*}$, it follows that $|0_\chi\rangle \in \mathscr{V}_{\Gamma^*}$ in eq. (3.2) because $\hat{G}$ and $\hat{H}$ are invariant under the $O(M)$ favour rotation. Therefore, eq. (3.2) can be represented as a path integration over s, t_1, t_2 and their conjugate variables. By taking the small ε limit after the large Z limit is taken first, eq. (3.2) can be written as

$$|0_\chi\rangle = \int Ds^{(0)} Dt^{(0)} \int \mathscr{D}s \mathscr{D}t \mathscr{D}q \mathscr{D}p \mathscr{D}v \mathscr{D}y \; |s^{(\infty)}, t_1^{(\infty)}, t_2^{(\infty)}\rangle e^{iS} \; \chi(s^{(0)}, t_1^{(0)}, t_2^{(0)}).$$

(3.5)

Here S is the action for the collective variables and their conjugate momenta,

$$S = N \int_0^\infty d\tau \; \mathrm{tr}\Bigg\{ -q \partial_\tau s - p_c \partial_\tau t_c - v(\tau) \mathscr{H}[q(\tau), s(\tau), p_1(\tau), t_1(\tau), p_2(\tau), t_2(\tau)]$$

$$- y(\tau) \mathscr{G}[q(\tau), s(\tau), p_1(\tau), t_1(\tau), p_2(\tau), t_2(\tau)] \Bigg\}.$$

(3.6)

$\mathscr{H}$ and $\mathscr{G}$ are the induced Hamiltonian and momentum constraints, respectively,

$$\mathscr{H}[q, s, p_1, t_1, p_2, t_2] = -U + \tilde{\alpha} U Q U + O\left(\frac{1}{N}\right), \tag{3.7}$$

$$\mathscr{G}[q, s, p_1, t_1, p_2, t_2] = \left(sq + 2\sum_c t_c p_c - i\frac{M}{2N}I\right). \tag{3.8}$$

Here $U^{ij} = \left(ss^T + \sum_{c=1}^2 \left[4t_c p_c t_c - it_c\right]\right)^{ij}$ and $Q_{ij} = \left(q^T q + p_1 + p_2\right)_{ij}$. q is a $L \times L$ matrix that is conjugate to s. p_1 and p_2 are symmetric $L \times L$ matrices conjugate to t_1 and t_2, respectively. While s, t_1 and t_2 represent the 'sources', the conjugate variables represent the corresponding operators, $q^a{}_i = \frac{1}{\sqrt{N}} \Phi^a{}_i$ with $1 \le a \le L$, $p_{1,ij} = \frac{1}{N} \sum_{b=L+1}^{L+N/2} \Phi^b{}_i \Phi^b{}_j$ and $p_{2,ij} = \frac{1}{N} \sum_{c=L+N/2+1}^M \Phi^c{}_i \Phi^c{}_j$ [8]. In total, there are $D_k = 2L^2 + 2L(L+1)$ kinematic phase space variables. $\mathscr{D}x \equiv \prod_{l=1}^\infty Dx^{(l)}$ and $x(\tau) = x^{(l)}$ with $\tau = l\varepsilon$ for $x = s, q, t_c, p_c, v, y$. τ is the parameter that labels the evolution of dynamical variables along gauge orbits.

All gauge invariant states have an infinite norm with respect to the inner product of the underlying Hilbert space. The non-normalizability of gauge invariant states is attributed to the fact that gauge orbits defined in the infinite-dimensional kinematic Hilbert space are non-compact [8]. This is fine because the dynamical variables include both clocks and physical degrees of freedom, and a gauge invariant state encodes the information about an entire spacetime history. In the large N limit with $L \gg 1$, the path integration in eq. (3.5) is well approximated by the saddle-point approximation. In this paper, we study the classical dynamics of the theory in the semi-classical limit. In particular, we identify a set of local clocks from the dynamical variables, and construct a spacetime from the correlation between the clocks and the remaining dynamical variables. We will see that different choices of local clocks lead to different spacetimes.

3.2 Constraint surface

From now on, we denote the saddle-point configuration as $\{q, s, t_1, t_2, p_1, p_2\}$, using the same collective variables that appear in the path integration. As an initial state in eq. (3.2), we consider a semi-classical state in which both the collective variables and their conjugate momenta are well defined. An example is the gaussian wavepacket considered in ref. [8]. Let us denote a semi-classical state whose collective variables are peaked at $\{q, s, p_1, t_1, p_2, t_2\}$ as $\left|\Psi_{q,s,p_1,t_1,p_2,t_2}\right\rangle$. Because of the permutation symmetry S_L^f in Γ^*, $\left|\Psi_{q,s,p_1,t_1,p_2,t_2}\right\rangle = \left|\Psi_{Pq,sP^T,p_1,t_1,p_2,t_2}\right\rangle$ for any $P \in S_L^f$. To the leading order in $1/N$, the application of the gauge transformation results in

$$e^{-i\varepsilon\left(\hat{H}_v + \hat{G}_y\right)} \left|\Psi_{q,s,p_1,t_1,p_2,t_2}\right\rangle \approx e^{-i\varepsilon N \, \text{tr}\{\mathscr{H}[q,s,p_1,t_1,p_2,t_2]v + \mathscr{G}[q,s,p_1,t_1,p_2,t_2]y\}} \left|\Psi_{q',s',p_1',t_1',p_2',t_2'}\right\rangle, \tag{3.9}$$

where $x' = x + \varepsilon\left\{x, \, \text{tr}\{\mathscr{G}y\}\right\}_{PB} + \varepsilon\left\{x, \, \text{tr}\{\mathscr{H}v\}\right\}_{PB}$ for $x = \{q, s, t_1, t_2, p_1, p_2\}$, and $\{A, B\}_{PB} = \left(\frac{\partial A}{\partial q^\alpha_i} \frac{\partial B}{\partial s^i_\alpha} - \frac{\partial A}{\partial s^i_\alpha} \frac{\partial B}{\partial q^\alpha_i}\right) + \delta_{ij}^{kl} \left(\frac{\partial A}{\partial p_{c,ij}} \frac{\partial B}{\partial t_c^{kl}} - \frac{\partial A}{\partial t_c^{kl}} \frac{\partial B}{\partial p_{c,ij}}\right)$ is the Poisson bracket. In the large N limit, the semi-classical initial state survives the projections in eq. (3.2) only if the collective variables and conjugate momenta satisfy the momentum and Hamiltonian

constraints *classically*,

$$\text{tr}\left\{\left(sq + 2\sum_c \tilde{t}_c p_c\right) y\right\} = 0, \tag{3.10}$$

$$\text{tr}\left\{(-U + \tilde{\alpha} U Q U)\, v\right\} = 0 \tag{3.11}$$

for arbitrary traceless matrix y (shift tensor) and symmetric matrix v (lapse tensor).[3] Here $\tilde{t}_c = t_c - \frac{i}{8p_c}$ in terms of which U is written as $U = \left(ss^T + \sum_c \left[4\tilde{t}_c p_c \tilde{t}_c + \frac{1}{16}p_c^{-1}\right]\right)$. Eqs. (3.10) and (3.11) give rise to $D_c = (L^2 - 1) + \frac{L(L+1)}{2}$ constraints.

Now we solve these constraints to remove D_c kinematic variables. The momentum constraint in eq. (3.10) is readily solved by expressing s in terms of $\tilde{t}_c$, p_c and q as

$$s = \left(\beta I - 2\sum_c \tilde{t}_c p_c\right) q^{-1} \tag{3.12}$$

for any constant β. For $U \neq 0$, the Hamiltonian constraint in eq. (3.11) is equivalent to

$$\alpha U Q = I. \tag{3.13}$$

Plugging eq. (3.12) into eq. (3.13), we obtain a quadratic matrix equation for $\tilde{t}_1$,

$$\tilde{t}_1 A^2 \tilde{t}_1 + \tilde{t}_1 B + B^T \tilde{t}_1 + C = 0, \tag{3.14}$$

where $A = 2\sqrt{p_1 q^{-1}(q^{-1})^T p_1 + p_1}$,[4] $B = 2p_1 q^{-1}(q^{-1})^T(2p_2\tilde{t}_2 - \beta)$, $C = (2\tilde{t}_2 p_2 - \beta)q^{-1}(q^{-1})^T$ $(2p_2\tilde{t}_2 - \beta) + 4\tilde{t}_2 p_2 \tilde{t}_2 + \frac{1}{16}(p_1^{-1} + p_2^{-1}) - \frac{1}{\tilde{\alpha}}Q^{-1}$. The solution to eq. (3.14) is written as

$$\tilde{t}_1 = -A^{-2}B + A^{-1}O\sqrt{B^T A^{-2}B - C}, \tag{3.15}$$

where O is an orthogonal matrix that should be chosen so that $\tilde{t}_1$ is symmetric. For every orthogonal matrix O that satisfies

$$-A^{-2}B + A^{-1}O\sqrt{B^T A^{-2}B - C} = -B^T A^{-2} + \sqrt{B^T A^{-2}B - C}\, O^T A^{-1}, \tag{3.16}$$

eq. (3.15) is a solution to eq. (3.13). In general, eq. (3.16) admits a discrete set of solutions because it contains $\frac{L(L-1)}{2}$ equations with the same number of unknowns. If A, B, C can be simultaneously diagonalized, $O = \text{diag}(\pm 1, \pm 1, \ldots)$ gives the solutions in the diagonal basis. At least locally in the phase space, the generalized momentum and Hamiltonian constraints are solved by expressing s and $\tilde{t}_1$ in terms of $\{\beta, q, p_1, \tilde{t}_2, p_2\}$. This results in the $(D_k - D_c)$-dimensional constraint surface on which the gauge constraints are satisfied classically.

3.3 Gauge fixing

Because the constraints obey the first-class algebra, the gauge orbits generated by $\mathscr{H}$ and $\mathscr{G}$ from an initial state in the constraint surface remain within the constraint surface. The

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[3]Otherwise, the fast phase oscillation in eq. (3.9) results in the destructive interference upon integrating over v and y.

[4]The square root of a symmetric matrix can be defined as follows. A real symmetric matrix X can be written as $X = O_X D_X O_X^T$, where D_X is a diagonal matrix and O_X is an orthogonal matrix. Its square root is given by $\sqrt{X} = O_X D_X^{1/2} O_X^T$.
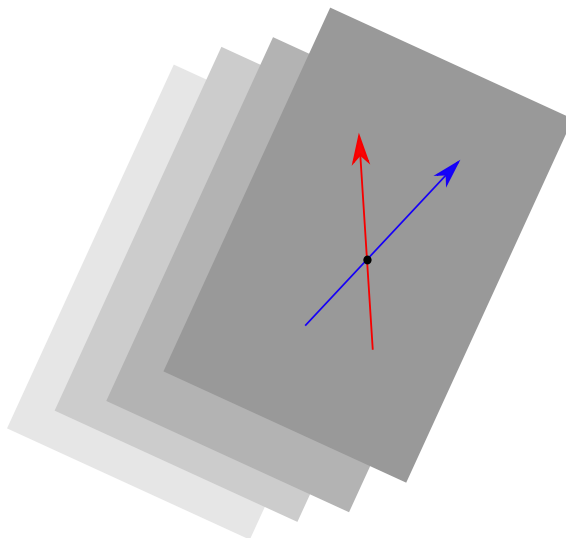
Figure 2. Each sheet represents a D_c-dimensional gauge manifold embedded in the $(D_k - D_c)$-dimensional constraint surface. Any two configurations within a gauge manifold can be transformed to each other through gauge transformations. Different choices of the shift and lapse tensors generate different gauge orbits within a gauge manifold as represented by arrows in the figure.

equation of motion for the gauge orbit reads

$$
\begin{aligned}
\partial_\tau \tilde{t}_c &= -4\tilde{t}_c v \tilde{t}_c - \tilde{\alpha} U v U + \frac{1}{16}\frac{1}{p_c}v\frac{1}{p_c} - y\tilde{t}_c - \tilde{t}_c y^T, \\
\partial_\tau p_c &= 4p_c \tilde{t}_c v + 4v\tilde{t}_c p_c + p_c y + y^T p_c, \\
\partial_\tau s &= -2\tilde{\alpha} U v U q^T - ys, \\
\partial_\tau q &= 2s^T v + qy,
\end{aligned}
\tag{3.17}
$$

where $y(\tau)$ and $v(\tau)$ are the shift and lapse tensors, respectively. Because the shift and lapse tensors comprise D_c independent gauge parameters, the set of configurations generated from the gauge transformations with all possible choices of y and v forms a D_c-dimensional *gauge manifold* (see figure 2). Configurations within a gauge manifold are physically equivalent, and only $L(L+1) + 2$ variables are left to distinguish one gauge manifold from another. These are the physical degrees of freedom. To isolate the physical degrees of freedom, we need to fix the gauge associated with the shift and lapse tensors. This amounts to choosing a set of local clocks and a coordinate system relative to which the dynamical correlation of the remaining physical degrees of freedom is expressed.

3.3.1 Fixing the shift tensor

The momentum constraint generates $\mathrm{SL}(L, \mathbb{R})$ transformations. If eq. (3.17) is evolved for parameter time τ_1 with $v = 0$, one obtains

$$
\begin{aligned}
\tilde{t}_c(\tau_1) &= g(\tau_1)^{-1}\tilde{t}_c(0)(g(\tau_1)^{-1})^T, \\
p_c(\tau_1) &= g(\tau_1)^T p_c(0)g(\tau_1), \\
s(\tau_1) &= g(\tau_1)^{-1}s(0), \\
q(\tau_1) &= q(0)g(\tau_1),
\end{aligned}
\tag{3.18}
$$
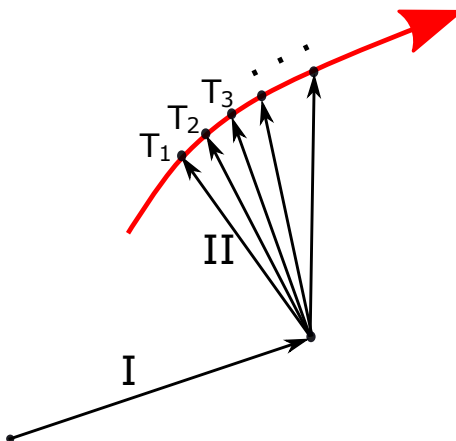
Figure 3. Two stages of gauge transformations that bring an arbitrary initial state to a final state that satisfies the gauge fixing condition in eq. (3.21). In the first stage of gauge transformation (stage I), an initial state is transformed so that the gauge fixing condition, $q = q_d g_f$ is enforced on q. In the second stage of gauge transformation (stage II), the state obtained from the first gauge transformation is brought to the final form in eq. (3.21). The one-parameter family of configurations, denoted as the big red arrow, with varying $p_d = T_1, T_2, \ldots$ describes a spacetime history, where p_d plays the role of time.

where $g(\tau_1) = \bar{\mathscr{P}}_\tau e^{\int_0^{\tau_1} d\tau y(\tau)} \in \mathrm{SL}(L, \mathbb{R})$. $\bar{\mathscr{P}}_\tau$ orders the matrix multiplication so that $e^{d\tau y(\tau)}$ with smaller τ are placed to the left of the terms with larger τ. For q with $\det q \geq 0$, the choice of

$$g(\tau_1) = q_d(0) \ q(0)^{-1} g_f, \tag{3.19}$$

with $q_d(0) = [\det q(0)]^{1/L}$ and $g_f \in \mathrm{SL}(L, \mathbb{R})$ leads to

$$q(\tau_1) = q_d(0) \ g_f. \tag{3.20}$$

For a given g_f, eq. (3.20) completely fixes the gauge freedom associated with $\mathrm{SL}(L, \mathbb{R})$: $g(\tau_1)$ in eq. (3.19) is the only element in $\mathrm{SL}(L, \mathbb{R})$ that satisfies eq. (3.20). This gauge fixing amounts to locking site indices (columns) with reference to the flavour indices (rows). We refer to the frame in which $q = q_d \ g_f$ as g_f-frame. The path that connects an initial configuration to the one that satisfies eq. (3.20) is denoted as path I in figure 3.

3.3.2 Fixing the lapse tensor

In priori, there is no preferred frame, and any g_f can be used in eq. (3.20). Here, we choose a frame in a clock-dependent way. It is natural to use p_1 as our clock variables. Because both p_1 and the lapse tensor have the same number of variables, the freedom associated with the lapse tensor can be fixed with a gauge condition imposed on p_1 up to a potential discrete ambiguity. With p_1 chosen as the clock variable, selecting a particular p_1 along with eq. (3.20) corresponds to choosing a moment of time. Being a symmetric matrix, p_1 can be fixed with $\frac{L(L+1)}{2}$ gauge fixing conditions. We take L eigenvalues of p_1 as the readings of local clocks defined at each site in a frame. The other $\frac{L(L-1)}{2}$ components encode the information on the frames in which p_1 is diagonal. Since the clocks do not create

– 13 –

inter-site entanglement in the frames in which p_1 is diagonal,[5] we choose g_f in eq. (3.20) so that p_1 is diagonal in the g_f-frame.[6] Therefore, specifying a moment of time requires not only the readings of L local clocks but also the information on which part of the kinematic Hilbert space is being used as L local clocks.

Here, the clocks play dual roles. First, the clocks provide a preferred frame dynamically. For different states of the clock, we use different frames to decompose the total kinematic Hilbert space into local Hilbert spaces. In this sense, the notion of local sites is provided by the clocks. Second, the clocks provide a physical time relative to which the evolution of other dynamical variables is tracked. The correlation between p_1 and other degrees of freedom describes the time evolution of the physical degrees of freedom relative to the clocks.

It is instructive to compare the role of clocks in general relativity and the present theory. In general relativity, the four-dimensional spacetime can be sliced into different stacks of three-dimensional spatial manifolds, depending on the choice of the lapse function. To specify a moment of time across the system, one has to fix the lapse function by imposing a gauge fixing condition on a scalar function in space. The scalar function at each position in space plays the role of an internal clock at that position. In the present theory, one needs to specify both frame and diagonal elements of p_1 in that frame to define a moment of time. In other words, one has to specify both the local Hilbert spaces and the readings of local clocks in the chosen local Hilbert spaces. A moment of time chosen in one frame does not correspond to a moment of time in another frame unless p_1 takes diagonal forms in both frames. This is illustrated in figure 4.

The dynamical information of the theory is encoded in the correlation between the clocks and the remaining physical degrees of freedom. The state of physical degrees of freedom given as a function of state of the clocks is a prediction of the theory. To extract this correlation, we impose the following gauge fixing conditions on q and p_1,

$$q = q_d \, g_f, \quad p_1 = p_d \, I, \tag{3.21}$$

where $g_f \in \mathrm{SL}(L,\mathbb{R})$ and q_d, p_d are real variables. $p_{1,ii}$ serves as the local clock at site i in the g_f-frame. Here, the gauge is chosen so that all local clocks run uniformly. In general, one could choose a non-uniform gauge condition such as $p_{1,ij} = p_{d,i}\delta_{ij}$. After the gauge fixing in eq. (3.21), what is left is the $L(L+1)+2$ physical degrees of freedom: $\{p_2, \tilde{t}_2, q_d, \beta\}$. We now ask how the physical degrees of freedom evolve as functions of p_d. This describes the spacetime that emerges for the set of clocks localized in the g_f-frame. Since β is a constant of motion along the gauge orbit [8], we will focus on the evolution of $p_2, \tilde{t}_2, q_d$.

From the discussion in section 3.3.1, we already know that the first condtion in eq. (3.21) can be readily imposed through an $\mathrm{SL}(L,\mathbb{R})$ transformation. To impose the second condition in eq. (3.21), the configuration $\{\tilde{t}_c(\tau_1), p_c(\tau_1), s(\tau_1), q(\tau_1)\}$ in eq. (3.18) with $q(\tau_1) = q_d(\tau_1)g_f$ and a generic $p_1(\tau_1)$ is further evolved with the equation of motion

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[5]Because $p_{1,ij} = \frac{1}{N}\sum_{b=L+1}^{L+N/2} \Phi^b{}_i \Phi^b{}_j$, off-diagonal elements of p_1 generate inter-site entanglement.

[6]There always exist frames in which p_1 is diagonal. Suppose that $p_1 = X$ in the g_f-frame, where X is a general $L \times L$ symmetric matrix. Under a frame rotation, $q' = q_d g_f g$ and $p_1' = g^T X g$, where $g \in \mathrm{SL}(L,\mathbb{R})$. One can always choose g such that $p_1' = p_d I$, where p_d is a real number. Now the clock takes the diagonal form in the g_f'-frame, where $g_f' = g_f g$.
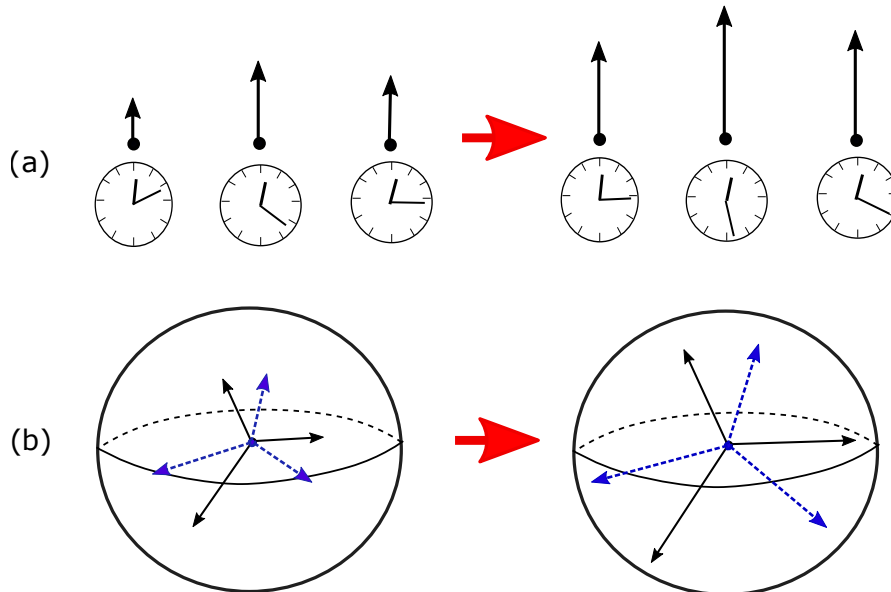
Figure 4. (a) In theories with a preferred set of local Hilbert spaces such as general relativity, a moment of time is determined if a clock variable in each fixed local Hilbert space is specified. The length of the arrow at each site represents the time at that site. The correlation between other physical degrees of freedom and the local clocks describes a time evolution from left to right in the figure. (b) In the present theory, the notion of local Hilbert spaces is determined by frame. A frame is specified by L vectors that form a parallelepiped, where each vector represents a local site. Once a frame is fixed, a local clock can be defined at each site. In this figure, the set of solid (black) arrows represent a moment of time defined by a set of local clocks in one frame, and the set of dashed (blue) arrows represent a moment of time defined by local clocks in another frame. The length of each arrow denotes the reading of the local clock at the corresponding site. The spacetimes that emerge for different sets of local clocks are in general different.

in eq. (3.17). During this evolution, the shift and lapse tensors are chosen so that p_1 at $\tau_2 > \tau_1$ satisfies eq. (3.21). To make sure that the gauge fixing condition for q is maintained along the evolution, the shift is chosen to be

$$y = -2q^{-1}s^T v + 2\langle q^{-1}s^T v\rangle I, \tag{3.22}$$

where $\langle A\rangle \equiv \frac{1}{L}\operatorname{tr}\{A\}$. This guarantees that q is proportional to g_f at all τ irrespective of the lapse tensor. To transform $p_1(\tau_1)$ to the desired form of $p_1(\tau_2) = p_d I$, we write the equation of motion for p_1 as

$$\partial_\tau p_1 = wp_1 + p_1 w^T, \tag{3.23}$$

where $w = v[4\tilde{t}_1 - 2s(q^{-1})^T] + 2\langle q^{-1}s^T v\rangle I$, and choose $v(\tau)$ such that

$$w(\tau)p_1(\tau) + p_1(\tau)w^T(\tau) = 2p_d I - p_1(\tau_1) - p_1(\tau) \tag{3.24}$$

for $\tau_1 \leq \tau \leq \tau_2$. Eq. (3.24) is a set of $\frac{L(L+1)}{2}$ linear equations for $v(\tau)$ at each τ, and admits a unique solution in general. It is straightforward to show that with this choice of the lapse tensor $p_1(\tau_2) = p_d I$ at $\tau_2 = \tau_1 + \ln 2$. This is denoted as path II in figure 3.

For a given initial state, the physical variables obtained at τ_2 depend on p_d and g_f. Therefore, the physical variables at τ_2 can be written as

$$\left\{ p_2(p_d; g_f), \tilde{t}_2(p_d; g_f), q_d(p_d; g_f) \right\}. \tag{3.25}$$

Within the constraint surface, the spatial metric in eq. (2.13) is given by [8]

$$g^{\mu\nu} = -2\tilde{\alpha} \sum_{l,n} U^{nl} U^{lm} \left(r_{nm}^\mu r_{lm}^\nu + r_{nm}^\nu r_{lm}^\mu \right) \tag{3.26}$$

with $r_{nm}^\mu = r_n^\mu - r_m^\mu$ to the leading order in $1/N$. Consequently, eq. (3.26) gives the spatial metric $g^{\mu\nu}(r, p_d; g_f)$ that depends on space (r) and time (p_d) in the g_f-frame. The correlation between the spatial metric and the physical clocks describes a spacetime that emerges for the set of observers who use local clocks chosen in the g_f-frame.

For some p_d, there may be no lapse and shift tensors that brings the initial state to the one that satisfy the gauge fixing condition in eq. (3.21). It is also possible that a constant p_1 surface intersects with a gauge orbits multiple times. In this case, p_1 can not be used as a time variable globally [2, 3]. Here we don't attempt to find a global time variable. We will be content with the fact that p_1 serves as a set of clocks locally in the phase space.

3.4 Multi-fingered internal time

The fact that one can choose any $g_f \in \mathrm{SL}(L, \mathbb{R})$ in eq. (3.21) encodes the freedom in choosing a frame in which local clocks are defined. Under a rotation of frame, a state in a local Hilbert space can be transformed to a linear superposition of states that belong to multiple local Hilbert spaces. As a result, one state can exhibit different local structures in different frames. To illustrate this through a concrete example, let us consider a semi-classical state with

$$p_{2,ij} = p_0 \delta_{i_x, j_x} \delta_{i_y, j_y} + \epsilon \left(\delta_{(i_x - j_x)_{\sqrt{L}}, 1} + \delta_{(i_x - j_x)_{\sqrt{L}}, -1} \right) \delta_{i_y, j_y}, \quad \tilde{t}_2 = 0, \quad p_1 = I, \quad q = I. \tag{3.27}$$

Here, the site index i with $i = 1, 2, \ldots, L$ is labeled in terms of a pair of indices, (i_x, i_y), where $i_x, i_y = 1, 2, \ldots, \sqrt{L}$.[7] $(x)_{\sqrt{L}} = x \bmod \sqrt{L}$. s and $\tilde{t}_1$ are determined from eqs. (3.12) and (3.15). Because eq. (3.27) has the translational invariance under $(i_x, i_y) \to (i_x + 1, i_y)$ for each i_y, all collective variables can be simultaneously diagonalized. Among the possible solutions in eq. (3.15), we choose the branch with $O = +I$. In eq. (3.27), the bi-local collective variable p_2 connects site (i_x, i_y) with sites $(i_x \pm 1, i_y)$. On the other hand, there is no entanglement between sites with different i_y to the leading order in $1/M$. Therefore, the state has an one-dimensional classical local structure. It describes $\sqrt{L}$ copies of one-dimensional manifold with the periodic boundary condition as is shown in figure 5(a). This state breaks $\mathrm{SL}(L, \mathbb{R})$ down to the discrete translation $(i_x, i_y) \to (i_x + 1, i_y)$ and the permutation group that interchanges i_y.

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[7]We assume that L is the square of a whole number.
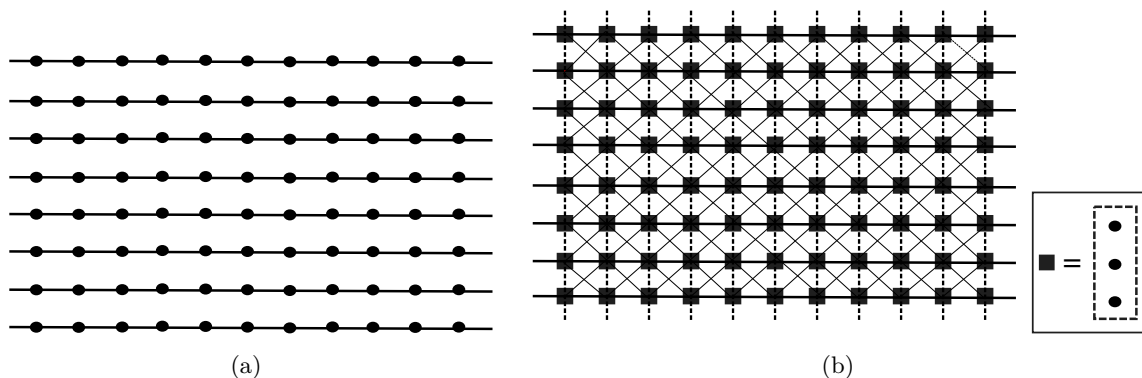
Figure 5. (a) One-dimensional local structure of the state in eq. (3.27). Each dot represents a site in the I-frame in which q is proportional to the identity matrix. Links between sites represent non-zero collective variables ($p_{2,ij}$, $t_{2,ij}$) that create entanglement between the sites. According to eq. (2.13), the state in eq. (3.27) gives rise to $\sqrt{L}$ decoupled one-dimensional manifolds. (b) In the g_o-frame, the state in eq. (3.27) is represented as eq. (3.33). One site (represented as a square) in the g_o-frame is composed of sites that belong to three different chains in the I-frame. As a result, sites are entangled in both x and y directions in the g_o-frame. To avoid clutter in the figure, the links are drawn only for the bi-local fields up to $O(\zeta)$. For this state, eq. (2.13) gives a two-dimensional manifold.

3.4.1 Finger 1

In this section, we consider the spacetime that emerges for a set of local observers who use the diagonal elements of p_1 as local clocks in the I-frame. This frame is defined by the gauge fixing conditions,

$$q = q_d\ I, \quad p_1 = p_d\ I. \tag{3.28}$$

Since eq. (3.27) satisfies eq. (3.28) with $q_d = 1$ and $p_d = 1$, eq. (3.27) is already on the desired gauge orbit. To move along the gauge orbit, we take eq. (3.27) as the initial condition, and evolve it with the lapse and shift tensors that maintain the gauge fixing conditions in eq. (3.28) along the orbit. We choose the lapse tensor,

$$v = \left(4\tilde{t}_1 - 2s(q^{-1})^T\right)^{-1} \tag{3.29}$$

with the shift tensor given in eq. (3.22). With this choice, the equation of motion for p_1 becomes

$$\partial_\tau p_1 = 2\left[I + 2\left\langle q^{-1}s^T\left(4\tilde{t}_1 - 2s(q^{-1})^T\right)^{-1}\right\rangle\right]p_1, \tag{3.30}$$

and eq. (3.28) are satisfied along the trajectory. The physical degrees of freedom are also evolved with the same lapse and shift tensors. This gives the information on how the physical degrees of freedom are correlated with the clock variable p_d. The evolution results in the spacetime history measured by the clocks that are local in the I-frame.

In the I-frame, it is convenient to introduce the one-dimensional coordinate system, $r_i = i_x$ for each decoupled ring. In this coordinate system, $p_{2,ij}$ and $\tilde{t}_2^{ij}$ in eq. (3.27) are short-ranged in $r_i - r_j$. Since $\tilde{t}_1$ and s are determined from $p_2, \tilde{t}_2, q$ from eqs. (3.12)
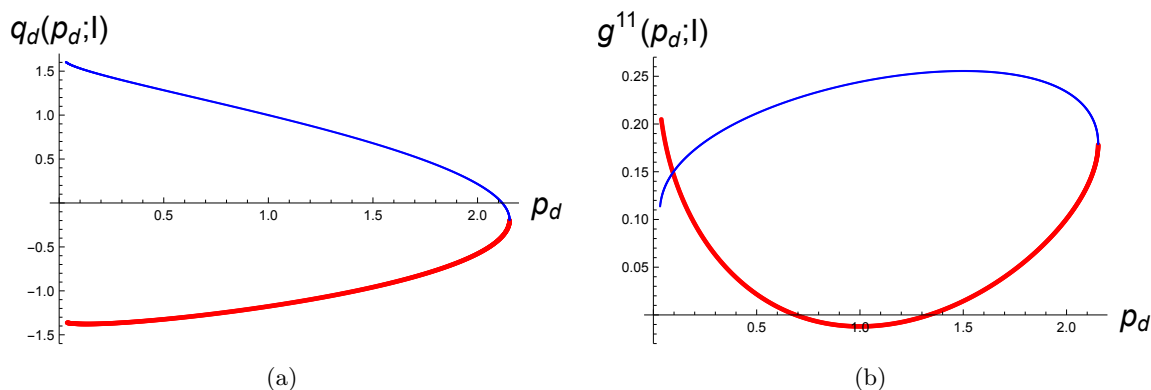
Figure 6. The evolution of q_d and g^{11} as functions of p_d that emerge in the I-frame for the initial condition given by eq. (3.27) with $p_0 = 1$, $\epsilon = 0.2$ for $\tilde{\alpha} = 0.1$, $\beta = 0.2$ and $L = 10^4$. Different chains remain decoupled throughout the evolution, and $g^{22} = 0$ at all time (not plotted here). For each value of p_d within the domain, there are two branches, where one is denoted as thick (red) line and the other as thin (blue) line in both (a) and (b).

and (3.15), $\tilde{t}_1^{ij}$ and $\left(s(q^{-1})^T\right)^{ij}$ also decays exponentially in $r_i - r_j$ in the one-dimensional manifold. Accordingly, the lapse tensor v_{ij} in eq. (3.29) also decays exponentially in $r_i - r_j$. This guarantees that $\hat{H}_v$ is relatively local in the one-dimensional coordinate system [8]. Consequently, the decoupled chains remain decoupled under the Hamiltonian evolution to the leading order in $1/M$. Furthermore, the Hamiltonian acts as a local one-dimensional Hamiltonian within each chain [26]. As a result, the emergent spacetime consists of $\sqrt{L}$ identical two-dimensional spacetimes that remain decoupled throughout the evolution.

Figure 6 shows one copy of the two-dimensional spacetimes that is obtained numerically from the initial condition of eq. (3.27). It shows how q_d and the 'spatial' metric extraced from eq. (3.26) are correlated with the clock variable, p_d. The classical gauge orbit obtained with eq. (3.29) intersects with a constant p_1 surface twice within a finite range of τ considered in the calculation. This results in two branches of solution for each value of p_d. If the wavefunction for the physical variables are constructed conditionally on the outcome of a measurement of the clock variable [28], the physical variables at a fixed p_d are in a linear superposition of macroscopically distinct states. The first branch is denoted as the thick (red) line, while the second branch as the thin (blue) line in figure 6. Near $p_d \approx 0$ in the first branch, the space has $+$ signature, which gives rise to a two-dimensional Lorentzian manifold (in eq. (2.13), the signature of time is chosen to be $-$ as a convention). As p_d increases in the first branch, g^{11} decreases, which results in an expanding universe. At a critical $p_d \approx 0.68$, the spacetime undergoes a phase transition that causes g^{11} to vanish. This is a Lifshitz transition where the second derivative of $U_k = \sum_{i,j} e^{ik(r_i-r_j)} U^{ij}$ with respect to k vanishes at zero momentum.[8] Across the critical point, g^{11} changes the

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[8]According to eq. (2.13), the contravariant metric is given by the second moment of $\hat{C}_m^{iikkn}$. In the presence of the translational invariance, the uniform metric can be written as $g^{\mu\nu} = 4\tilde{\alpha}\left(\frac{\partial U_{\mathbf{k}}}{\partial k_\mu}\frac{\partial U_{\mathbf{k}}}{\partial k_\nu} + U_{\mathbf{k}}\frac{\partial^2 U_{\mathbf{k}}}{\partial k_\mu \partial k_\nu}\right)_{k=0}$ [8]. With the reflection symmetry, $\frac{\partial U_{\mathbf{k}}}{\partial k_\mu}\Big|_{k=0} = 0$, and the metric is given by the second derivative of U_k.

sign, and the spacetime becomes Euclidean. At a later time ($p_d \approx 1.34$), a second Lifshitz transition restores the Lorentzian signature. After the second Lifshitz transition, the space shrinks with increasing p_d until it hits 'the end of time' around $p_d \approx 2.15$. At the end point, the first branch converges with the second branch. In the second branch, the two-dimensional spacetime stays as a Lorentzian manifold throughout the evolution.

3.4.2 Finger 2

Now, let us describe the spacetime that emerges from the same state in eq. (3.27) for observers who use a different set of local clocks. The new local clocks are the diagonal elements of p_1 in a different frame. For concreteness, let us choose clocks that are local in g_o-frame, where

$$(g_o)^i_j = A_\zeta \delta_{i_x j_x} \left[\delta_{i_y j_y} + \zeta \left(\delta_{(i_y - j_y)_{\sqrt{L}}, 1} + \delta_{(i_y - j_y)_{\sqrt{L}}, -1} \right) \right]. \tag{3.31}$$

Here ζ is a constant, and $A_\zeta = \left[\prod_{n=1}^{\sqrt{L}} \left(1 + 2\zeta \cos \left(\frac{2\pi n}{\sqrt{L}} \right) \right) \right]^{-\frac{1}{\sqrt{L}}}$. The new gauge fixing conditions read

$$q = q_d \, g_o, \quad p_1 = p_d \, I. \tag{3.32}$$

The new choice of clocks leads to a different decomposition of the kinematic Hilbert space into local Hilbert spaces. In order to extract the spacetime that emerges in this new frame, we need to apply gauge transformations to eq. (3.27) to enforce the gauge fixing conditions in eq. (3.32). As explained in sections 3.3.1 and 3.3.2, this is done in two stages. First, we apply an $\mathrm{SL}(L, \mathbb{R})$ transformation to enforce the first gauge fixing condition, $q = q_d g_o$. Under the $\mathrm{SL}(L, \mathbb{R})$ transformation that brings q into the form in eq. (3.32), the collective variables become

$$\begin{aligned} p_2(\tau_1) &= (g_o)^T p_2 g_o, \quad \tilde{t}_2(\tau_1) = 0, \\ p_1(\tau_1) &= (g_o)^T p_1 g_o, \quad q(\tau_1) = q_d g_o. \end{aligned} \tag{3.33}$$

A site with coordinate (i_x, i_y) in the g_o-frame is composed of a linear superposition of sites with $(i_x, i_y - 1), (i_x, i_y), (i_x, i_y + 1)$ in the I-frame. Because one site in the g_o-frame is delocalized across three neighbouring chains of the I-frame, the chains are no longer decoupled in the g_o-frame. Due to the interchain entanglement, eq. (3.33) has a two-dimensional local structure, as is shown in figure 5(b). Now, we apply the second set of gauge transformations to enforce the gauge fixing condition for p_1. As explained in section 3.3.2, this is achieved with the lapse tensor that satisfies eq. (3.24) and the shift tensor given in eq. (3.22). To understand the nature of this second gauge transformation, we use the two-dimensional coordinate system, $r_i = (i_x, i_y)$. This coordinate system makes the two-dimensional local structure manifest. In other words, $p_{c,ij}, \tilde{t}_c^{ij}$ and $\left(s(q^{-1})^T \right)^{ij}$ connect a site with its neighbours in the two-dimensional manifold, and decay exponentially in $r_i - r_j$. As a result, the lapse tensor v_{ij} that satisfies eq. (3.24) also decays exponentially in $r_i - r_j$. This implies that the Hamiltonian acts as a two-dimensional local Hamiltonian along the gauge orbit that connects eq. (3.33) with the one that satisfies the gauge fixing condition in eq. (3.32). Therefore, the state obtained at the end of the second gauge transformation at
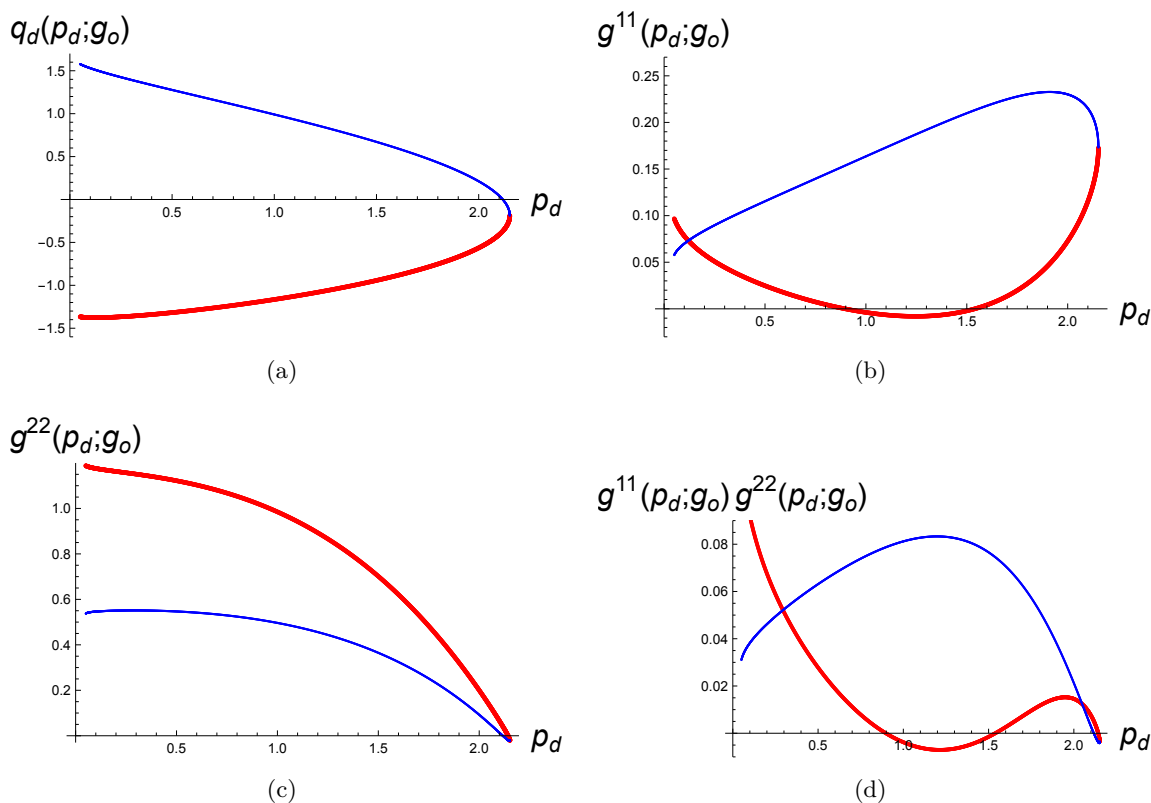
Figure 7. The evolution of q_d and two components of the metric as functions of p_d that emerges in the g_o-frame with $\zeta = 0.1$ for the same initial condition used in figure 6. For each value of p_d, there exist two solutions. The first branch is denoted as thick (red) line, and the second branch as thin (blue) line in all plots.

τ_2 also supports a two-dimensional local structure in the g_o-frame. The physical variables $(q_d, p_2, \tilde{t}_2)$ at τ_2 viewed as functions of p_d describe a three-dimensional spacetime.

Figure 7 shows the evolution of q_d, g^{11} and g^{22} as measured against the physical time p_d. The trajectory is obtained numerically using the same initial condition used in figure 6. Due to the reflection symmetry in each direction in space, $g^{12} = 0$. The signature of the three-dimensional spacetime is give by $\left(-, \mathrm{sgn}\left(g^{11}\right), \mathrm{sgn}\left(g^{22}\right)\right)$. For each choice of p_d, there are two branches of solutions (the first denoted as thick (red) line and the second denoted as thin (blue) line). The evolution of q_d and g^{11} is more or less the same as the one obtained in the I-frame. What is new here is that g^{22} is non-zero and exhibits a non-trivial dynamics because the state has the two-dimensional local structure. For each g^{11} and g^{22}, two Lifshitz transitions occur that flip the signature of each 'spatial' direction from $+$ to $-$ and back to $+$. Because the Lifshitz transitions in the 1 and 2 directions happen at different moments of time, the signature of the spacetime evolves as $(-, +, +) \to (-, -, +) \to (-, +, +) \to (-, +, -) \to (-, +, +)$ as we start from $p_d = 0$ in the first branch, move along the direction of increasing p_d and continue on the second branch. In figure 7(d), the intervals with $g^{11} g^{22} < 0$ correspond to the spacetime with two time

directions. This can be generalized to higher dimensions, and we expect that anisotropic spacetimes close Lifshitz transitions generically exhibit multiple time directions [29].

This example shows that one state can exhibit spacetime manifolds with different dimensions, signatures, topologies and geometries in different frames. This is possible because the enlarged gauge symmetry generated by $\mathrm{SL}(L, \mathbb{R})$ can not only permute sites but also change the very notion of local sites by constructing new sites out of linear superpositions of old sites.

If one chooses local clocks in an arbitrary frame, the state generally does not retain any local structure. Even for a state that has a local structure in one frame, a well-defined spacetime manifold does not emerge if a collection of clocks are chosen in another frame that is related to the first frame through a non-local transformation.[9] The emergence of a well-defined spacetime hinges both on local structure of the state and on the choice of local clocks that are compatible with the local structure of the state.

4 Summary and discussion

In this paper, we consider a theory of quantum gravity that does not have a preferred decomposition of the kinematic Hilbert space into local Hilbert spaces. The theory is covariant under a gauge symmetry larger than diffeomorphism, where the extra gauge symmetry includes transformations that mix local kinematic Hilbert spaces. This gives rise to a greater freedom in choosing a collection of local clocks with respect to which the evolution of other physical degrees of freedom is tracked. It is shown that dimension, signature, topology and geometry of spacetime depend on the choice of local clocks. Just as a gem reveals different facets in different cuts, one state can exhibit different spacetimes with different choices of clocks. We expect that this is a generic feature of theories that do not have a preferred Hilbert space decomposition.

Another consequence of the enlarged gauge symmetry is the presence of extra propagating modes besides the spin 2 gravitational mode. They are represented by the higher-spin fields associated with the bi-local collective fields. Higher-spin gauge fields are Higgsed in states that break $\mathrm{SL}(L, \mathbb{R})$ to the global translation symmetry, as is the case for the states considered in section 3.4[8]. It will be of interest to understand the physical spectrum of the theory.

Acknowledgments

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[9]Equivalently, a state that is short-range entangled in one basis can exhibit long-range entanglement if one chooses non-local basis.

A Gauge invariance of eq. (3.2)

Here we prove that eq. (3.2) is gauge invariant. We write the set of constraints as $\{\hat{C}_1, \hat{C}_2, \ldots, \hat{C}_n\}$, where each element represents one component of $\hat{G}^i{}_j$ or $\hat{H}^{kl}$. Because $\hat{G}$ is traceless and $\hat{H}$ is symmetric, $n = L^2 - 1 + \frac{L(L+1)}{2}$. The associated shift and lapse tensors are written as n gauge parameters, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Eq. (3.2) can be written as

$$|0_\chi\rangle = \int_{\mathbf{x} \in \mathbf{A}} \mathscr{D}\mathbf{x} \, U(\mathbf{x})|0\rangle. \tag{A.1}$$

Here, $U(\mathbf{x}) = \bar{\mathscr{P}}_l \left[\prod_{l=1}^\infty e^{-i\hat{C} \cdot x^{(l)}} \right]$ with $\hat{C} \cdot x \equiv \sum_{i=1}^n \hat{C}_i x_i$. In the definition of $U(\mathbf{x})$, $\bar{\mathscr{P}}_l$ orders the unitary operators so that $e^{-i\hat{C} \cdot x^{(l)}}$ with smaller l are placed to the left of the terms with larger l. $\mathscr{D}\mathbf{x} \equiv \int \prod_{l=1}^\infty Dx^{(l)}$, and A is a set of ordered gauge parameters $A = \left\{ (x^{(1)}, x^{(2)}, \ldots) | x^{(l)} \in \mathbb{R}^n, l = 1, 2, \ldots, \infty \right\}$. ε in eq. (3.2) has been absorbed into the gauge parameter. Now, we consider a state obtained by applying a gauge transformation to eq. (A.1),

$$|0'_\chi\rangle = e^{-i\hat{C} \cdot \tilde{x}} \, |0_\chi\rangle \tag{A.2}$$

for $\tilde{x} \in \mathbb{R}^n$. Eq. (A.2) can be written as

$$|0'_\chi\rangle = \int_{\mathbf{x} \in \mathbf{A}'} D\mathbf{x} \, U(\mathbf{x})|0\rangle, \tag{A.3}$$

where $A' = \left\{ (\tilde{x}, x^{(1)}, x^{(2)}, \ldots) | x^{(l)} \in \mathbb{R}^n, l = 1, 2, \ldots, \infty \right\}$. Now we prove that $W = \left\{ U(\mathbf{x}) | \mathbf{x} \in A \right\}$ and $W' = \left\{ U(\mathbf{x}') | \mathbf{x}' \in A' \right\}$ are the same. For every element $U(\mathbf{x}') \in W'$, there exists $\mathbf{x} = (\tilde{x}, x^{(1)}, x^{(2)}, \ldots)$ in A such that $U(\mathbf{x}) = U(\mathbf{x}')$. This shows that $W' \subset W$. Conversely, for every element $U(\mathbf{x}) \in W$, there exists $\mathbf{x}' = (\tilde{x}, -\tilde{x}, x^{(1)}, x^{(2)}, \ldots)$ in A' such that $U(\mathbf{x}') = U(\mathbf{x})$. This shows that $W \subset W'$. Therefore, $W = W'$. If $|0_\chi\rangle$ does not vanish, $|0_\chi\rangle = |0'_\chi\rangle$.

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