Do all $5d$ SCFTs descend from $6d$ SCFTs?

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ABSTRACT: We present examples of $5d$ SCFTs that serve as counter-examples to a recently actively studied conjecture according to which it should be possible to obtain all $5d$ SCFTs by integrating out BPS particles from $6d$ SCFTs compactified on a circle. We further observe that it is possible to obtain these $5d$ SCFTs from $6d$ SCFTs if one allows integrating out BPS strings as well. Based on this observation, we propose a revised version of the conjecture according to which it should be possible to obtain all $5d$ SCFTs by integrating out both BPS particles and BPS strings from $6d$ SCFTs compactified on a circle. We describe a general procedure to integrate out BPS strings from a $5d$ theory once a geometric description of the $5d$ theory is given. We also discuss the consequences of the revised conjecture for the classification program of $5d$ SCFTs.

KEYWORDS: Conformal Field Theory, Field Theories in Higher Dimensions, M-Theory, Supersymmetric Gauge Theory

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1 Introduction and discussion

Recent work on the classification of $5d$ SCFTs was initiated by [1, 2]. In particular, [2] provided significant evidence in support of a conjecture which formed the foundation for the subsequent work on the classification of $5d$ SCFTs [3–9].¹ According to this conjecture, it should be possible to obtain all $5d$ SCFTs by performing RG flows on $6d$ SCFTs compactified on a circle. More specifically, it was assumed that these RG flows take the form of integrating out BPS particles from the $5d$ theories obtained by compactifying $6d$ SCFTs on a circle (often called $5d$ KK theories).

To understand the motivation behind the proposal of this conjecture, consider the example of a $5d$ $\mathcal{N} = 1$ pure gauge theory with gauge algebra $\mathfrak{g}$. Taking the strong coupling limit of this theory leads us to the conformal point of a $5d$ SCFT. This theory can be obtained by integrating out the adjoint hypermultiplet from $5d$ $\mathcal{N} = 1$ gauge theory with gauge algebra $\mathfrak{g}$ and a hypermultiplet transforming in adjoint representation. The latter $5d$ gauge theory can be obtained by compactifying a $6d$ $\mathcal{N} = (2, 0)$ SCFT on a circle of finite radius R with the radius controlling the gauge coupling of the $5d$ gauge theory. The BPS particles being integrated out under this RG flow are the ones produced by the hyper in the adjoint representation once we move onto the Coulomb branch of the gauge theory.

In this work, we will present several examples of $5d$ SCFTs which cannot be obtained from $5d$ KK theories by integrating out BPS particles alone. The mass deformations of these $5d$ SCFTs admit certain Coulomb branch phases that can be described in terms of the following $5d$ $\mathcal{N} = 1$ gauge theories (deformed by their mass parameters and Coulomb branch moduli):

- $\mathfrak{f}_4$ with $1 \leq n \leq 3$ hypers in “fundamental” representation, that is, the irreducible representation of dimension **26**.
- $\mathfrak{e}_6$ with $1 \leq n \leq 4$ hypers in “fundamental” representation, that is, the irreducible representation of dimension **27**.

¹See [10–20] for other related recent work on $5d$ SCFTs and $5d$ gauge theories.

- $\mathfrak{e}_7$ with $1 \leq n \leq 6$ half-hypers in “fundamental” representation, that is, the irreducible representation of dimension **56**.

These gauge theories can be constructed by compactifying M-theory on a Calabi-Yau threefold. We will show in section 2 that there is a point in the Kahler moduli space of the Calabi-Yau threefold where all of the compact 2-cycles and 4-cycles of the Calabi-Yau threefold shrink simultaneously to zero volume, thus giving rise to a $5d$ SCFT decoupled from gravity and other stringy physics. This argument establishes the existence of the above mentioned $5d$ SCFTs.

We claim that these $5d$ SCFTs cannot be obtained by integrating out BPS particles from a $5d$ KK theory. The argument proceeds in two steps:

1. Let us first argue that for a $5d$ SCFT admitting a description as a $5d$ gauge theory, integrating in a BPS particle always corresponds to integrating in a matter hypermultiplet into the corresponding gauge theory.

For this argument, we use the M-theory construction of the $5d$ SCFT.² A BPS particle being integrated into this $5d$ SCFT arises as a compact 2-cycle C living in a *non-compact 4-cycle* N and intersecting all the compact 4-cycles S_i transversely. The flop of C corresponds to the process of integrating in the BPS particle into the $5d$ SCFT, whereas decompactifying C inside N completely integrates it out. On the other hand, the Coulomb branch phase transitions of $5d$ SCFT are implemented by flopping compact 2-cycles living inside *compact 4-cycles* S_i in the Calabi-Yau threefold. Thus, the flops corresponding to addition of a BPS particle and the flops corresponding to Coulomb branch phase transitions commute. Consequently, the addition of a BPS particle into any phase of the $5d$ SCFT can always be thought as the addition of a BPS particle into a gauge-theoretic phase of the $5d$ SCFT described by the corresponding $5d$ gauge theory.

From the point of view of the gauge-theoretic phase, the compact 2-cycle C can be combined with the compact 2-cycles f_i (living inside S_i) corresponding to W-bosons of the gauge algebra $\mathfrak{g}$. Together, these form the BPS particles corresponding to a hypermultiplet charged in a representation R of $\mathfrak{g}$. The Dynkin coefficients of the highest weight of R are identified with the intersection numbers $C \cdot S_i$.

2. We can now use the results of [1] according to which a $5d$ gauge theory obtained after adding a hypermultiplet to any of the following $5d$ gauge theories

$$\mathfrak{f}_4 + 3F \tag{1.1}$$

$$\mathfrak{e}_6 + 4F \tag{1.2}$$

$$\mathfrak{e}_7 + 3F \tag{1.3}$$

$$\mathfrak{e}_7 + \frac{5}{2}F \tag{1.4}$$

²Here we use the M-theory construction only to phrase the argument in a familiar language. One can phrase the argument in a purely field-theoretic language too.

(where F denotes a full hyper in fundamental representation) cannot describe a $5d$ SCFT or a $5d$ KK theory. Similarly, a $5d$ gauge theory obtained after adding a hypermultiplet in a representation other than the fundamental representation to any of the following $5d$ gauge theories

$$\mathfrak{f}_4 + nF \quad \text{for } n < 3 \tag{1.5}$$

$$\mathfrak{e}_6 + nF \quad \text{for } n < 4 \tag{1.6}$$

$$\mathfrak{e}_7 + \frac{n}{2}F \quad \text{for } n < 5 \tag{1.7}$$

cannot describe a $5d$ SCFT or a $5d$ KK theory.

We note that the above argument applies irrespective of whether the classification of $6d$ SCFTs [21, 22]³ is complete or not.

We will show in section 2 that the theories (1.1—1.7) can be obtained from $5d$ KK theories if one allows RG flows integrating out BPS strings as well. Such RG flows correspond to decompactifying compact 4-cycles in the Calabi-Yau threefold used in the corresponding M-theory construction. The BPS strings being integrated out during this process are the ones produced by compactifying M5 branes on the 4-cycles being decompactified. We will discuss the general geometric conditions under which such a decompactification process can consistently take place in section 3.

The theories (1.1—1.7) can be obtained via such a decompactification process applied to $5d$ KK theories obtained by untwisted compactification of following $6d$ SCFTs:⁴

$$\mathfrak{f}_4 \text{ on } -k \text{ curve with } 2 \leq k \leq 5 \tag{1.8}$$

$$\mathfrak{e}_6 \text{ on } -k \text{ curve with } 2 \leq k \leq 6 \tag{1.9}$$

$$\mathfrak{e}_7 \text{ on } -k \text{ curve with } 2 \leq k \leq 8 \tag{1.10}$$

We are thus led to the conjecture that:

Conjecture 1. All $5d$ SCFTs can be obtained (on their mass-deformed Coulomb branch) by consistently integrating out BPS particles and BPS strings from the mass-deformed Coulomb branches of $5d$ KK theories.

The above revised conjecture has a negative consequence for the program of classifying $5d$ SCFTs based on the analysis of RG flows of $5d$ KK theories. To understand this, let us notice that the RG flows corresponding to removal of BPS particles are *rank-preserving*, that is they do not change the rank of the $5d$ theory, which can be identified with the number of fundamental BPS strings⁵ found on the mass-deformed Coulomb branch of the $5d$ theory. On the other hand, the RG flows corresponding to removal of BPS strings are *rank-lowering*. If all $5d$ SCFTs could be obtained by only integrating out BPS particles from $5d$ KK theories, one would only need to analyze RG flows of $5d$ KK theories of rank

³See [23–27] for other work related to the classification of $6d$ SCFTs.

⁴Here we are denoting the $6d$ SCFTs by the data of their F-theory construction.

⁵We define fundamental BPS strings to be those BPS strings that do not arise as bound states of other BPS strings.

n in order to fully classify $5d$ SCFTs of rank n . This was the logic behind the classification of $5d$ SCFTs upto rank three pursued in [2, 9]. However, since we also need to integrate out BPS strings, we need to actually analyze RG flows of $5d$ KK theories of ranks greater than or equal to n in order to fully classify $5d$ SCFTs of rank n . For example, we might encounter the situation in which a $5d$ SCFT of rank n only arises by removal of a BPS string from a $5d$ SCFT of rank $n + 1$, which only arises by removal of a BPS string from a $5d$ SCFT of rank $n + 2$, and so on until we reach a $5d$ SCFT of rank $n + p$ which only arises by removal of a BPS string from a $5d$ KK theory of rank $n + p + 1$. Thus, *rank* is no longer a good notion to organize the classification of $5d$ SCFTs. To remedy the situation, we make the following conjecture:

Conjecture 2. A rank n $5d$ SCFT can be obtained either by successively integrating out BPS particles from a rank n $5d$ KK theory, or by successively integrating out BPS particles after integrating out a BPS string from a rank $n + 1$ $5d$ KK theory.

Evidence for this conjecture will be provided in [28] where it will be shown that every $5d$ gauge theory (with a simple gauge algebra) that is expected to arise on the mass-deformed Coulomb branch of a $5d$ SCFT (based on the analysis of [1] and a few other conditions) can be obtained from a $5d$ KK theory by using only the RG flows mentioned in **Conjecture 2**.

It would be an interesting future direction to explore if **Conjecture 2** leads to an extension of the classification of $5d$ SCFTs presented in [9]. This would require studying all the possible decompactifications of a single 4-cycle in the Calabi-Yau threefolds associated to $5d$ KK theories upto rank four. The general criteria for decompactifying a compact 4-cycle in a Calabi-Yau threefold is discussed in section 3.

2 Construction of the counter-examples

In this section, we will show that the $5d$ SCFTs described by the $5d$ gauge theories (1.1—1.7) can be obtained by integrating out a BPS string from the $5d$ KK theories obtained by untwisted compactification of $6d$ SCFTs (1.8—1.10). This construction will also allow us to exhibit the existence of a ray in the space of normalizable Kahler parameters for the Calabi-Yau threefolds associated to the theories (1.1—1.7), such that all of the compact 2-cycles and 4-cycles have non-negative volumes along the ray, and moreover at least one of the 4-cycles has strictly positive volume. The existence of such a ray implies that these Calabi-Yau threefolds describe $5d$ SCFTs [2], with the ray becoming a part of the Coulomb branch of the $5d$ SCFTs, and the origin of the ray becoming the conformal point.

We will use the notation detailed in section 5.2.1 of [8] to describe Calabi-Yau threefolds throughout this paper.

Let us start by recalling the Calabi-Yau threefold associated to the untwisted compactification of the 6d SCFT carrying $\mathfrak{f}_4$ on $-k$ curve [3, 4, 8]:

$$\begin{array}{c}
 \mathbf{0}_{k-2} \\
 e \downarrow \\
 e \downarrow \\
 \mathbf{4}_{4-k} \xrightarrow{h} \mathbf{3}_{6-k} \xrightarrow{2h \quad e - \sum x_i - \sum y_i} \mathbf{2}_6^{(5-k)+(5-k)} \xrightarrow{h + \sum (f-y_i), f-x_i} \mathbf{6-k} \xrightarrow{e - \sum y_i, f-x_i} \mathbf{1}_8^{(5-k)+(5-k)}
 \end{array}
 \begin{array}{c}
 x_i \frown \quad \smile y_i \\
 \quad \quad 5-k
 \end{array}
 \begin{array}{c}
 x_i \frown \quad \smile y_i \\
 \quad \quad 5-k
 \end{array}
 \quad (2.1)$$

for $2 \leq k \leq 5$, and

$$\mathbf{0}_1 \xrightarrow{h} \mathbf{4}_3 \xrightarrow{h} \mathbf{3}_5 \xrightarrow{2h \quad e - \sum x_i - \sum y_i} \mathbf{2}_6^{4+4} \xrightarrow{h + \sum (f-y_i), f-x_i} \mathbf{5} \xrightarrow{e - \sum y_i, f-x_i} \mathbf{1}_8^{4+4}$$

$$\begin{array}{c}
 x_i \frown \quad \smile y_i \\
 \quad \quad 4
 \end{array}
 \begin{array}{c}
 x_i \frown \quad \smile y_i \\
 \quad \quad 4
 \end{array}
 \quad (2.2)$$

for $k = 1$. These Calabi-Yau threefolds admit a special ray in the space of normalizable Kahler parameters where the Kahler form J can be written as

$$J = \phi (S_0 + 2S_4 + 3S_3 + 2S_2 + S_1) \quad (2.3)$$

with $\phi \geq 0$. The coefficients of S_i in this ray are actually dual Coxeter labels associated to roots of the affine algebra $\mathfrak{f}_4^{(1)}$. Along this ray, all of the compact 2-cycles have non-negative volume and all the compact 4-cycles have zero volume, as the reader can explicitly check.

Now, let us send the volume of the curve f in S_0 of (2.1) to infinity, while keeping the volume of e curve in S_0 finite. This decompactifies S_0 and we obtain the Calabi-Yau threefold

$$\mathbf{4}_{4-k} \xrightarrow{h} \mathbf{3}_{6-k} \xrightarrow{2h \quad e - \sum x_i - \sum y_i} \mathbf{2}_6^{(5-k)+(5-k)} \xrightarrow{h + \sum (f-y_i), f-x_i} \mathbf{6-k} \xrightarrow{e - \sum y_i, f-x_i} \mathbf{1}_8^{(5-k)+(5-k)}$$

$$\begin{array}{c}
 x_i \frown \quad \smile y_i \\
 \quad \quad 5-k
 \end{array}
 \begin{array}{c}
 x_i \frown \quad \smile y_i \\
 \quad \quad 5-k
 \end{array}
 \quad (2.4)$$

which describes the 5d gauge theory $\mathfrak{f}_4 + (5-k)\mathbf{F}$ for $2 \leq k \leq 5$.

Since (2.4) is a limit of (2.1), and (2.1) describes a theory which is UV complete without coupling to dynamical gravity, (2.4) should also describe a theory which is UV complete without coupling to dynamical gravity. In fact, we claim that (2.4) describes a 5d SCFT. To show this, we study the Calabi-Yau threefold (2.4) along the ray

$$J = \phi (2S_4 + 3S_3 + 2S_2 + S_1) \quad (2.5)$$

which is the ray (2.3) with S_0 deleted. First of all, the volume of any 2-cycle or 4-cycle not intersecting S_0 will remain unchanged. Thus only the volumes of 2-cycles e, f in S_4 and the volume of the 4-cycle S_4 will change. It can be checked that both the 2-cycles attain

non-negative volume and S_4 attains strictly positive volume along (2.5). As a consequence, we have shown that $\mathfrak{f}_4 + (5-k)\mathbf{F}$ describes a 5d SCFT for $2 \leq k \leq 5$ as claimed in section 1.

On the other hand, an analogous RG flow is not possible for the $k = 1$ case shown in (2.2). Decompactifying f of S_0 while keeping e of S_0 compact in (2.2) necessarily decompactifies h of S_0 which is glued to e of S_4 , thus decompactifying S_4 in the process as well.⁶ Thus, we are unable to obtain an $\mathfrak{f}_4$ gauge theory from (2.2). This was as expected, since according to the analysis of [1], $\mathfrak{f}_4 + 4\mathbf{F}$ does not describe a 5d SCFT or a 5d KK theory. Thus $\mathfrak{f}_4 + 4\mathbf{F}$ either cannot be UV completed or requires coupling to dynamical gravity for consistent UV completion. We would find a contradiction if we were able to decompactify S_0 while keeping the rest of the Calabi-Yau threefold compact.

A similar argument works for $\mathfrak{e}_6$ and $\mathfrak{e}_7$ theories. For $\mathfrak{e}_6$ theories, we start with the Calabi-Yau threefold for untwisted compactification of the 6d SCFT carrying $\mathfrak{e}_6$ on $-k$ curve

$$\begin{array}{c}
 \mathbf{5}_4 \\
 \begin{array}{c} e \\ | \\ h-\sum x_i \\ | \\ \mathbf{4}_{8-k}^{6-k} \end{array} \begin{array}{c} f \\ \diagdown \\ 6-k \end{array} \\
 \begin{array}{c} x_i \\ \diagdown \\ 6-k \end{array} \begin{array}{c} f-x_i \\ \diagdown \\ 6-k \end{array} \begin{array}{c} f-x_i \\ \diagdown \\ 6-k \end{array} \begin{array}{c} x_i-y_i \\ \diagdown \\ \mathbf{1}_{10-k}^{(6-k)+(6-k)} \end{array} \\
 \begin{array}{c} e \\ | \\ h \\ | \\ \mathbf{0}_{k-2} \end{array} \begin{array}{c} e \\ | \\ \mathbf{6}_{k-4} \end{array} \begin{array}{c} e \\ | \\ \mathbf{3}_{6-k} \end{array} \begin{array}{c} h \\ | \\ \mathbf{2}_{8-k}^{6-k} \end{array} \begin{array}{c} e \\ | \\ \mathbf{1}_{10-k}^{(6-k)+(6-k)} \end{array}
 \end{array}
 \quad (2.6)$$

for $2 \leq k \leq 6$. The dual Coxeter labels for $\mathfrak{e}_6^{(1)}$ provide us with the ray

$$J = \phi(S_0 + S_1 + S_5 + 2S_2 + 2S_4 + 2S_6 + 3S_3) \quad (2.7)$$

For $k \geq 2$, it is possible to decompactify S_0 while keeping other S_i compact by decompactifying f of S_0 . We obtain

$$\begin{array}{c}
 \mathbf{5}_4 \\
 \begin{array}{c} e \\ | \\ h-\sum x_i \\ | \\ \mathbf{4}_{8-k}^{6-k} \end{array} \begin{array}{c} f \\ \diagdown \\ 6-k \end{array} \\
 \begin{array}{c} x_i \\ \diagdown \\ 6-k \end{array} \begin{array}{c} f-x_i \\ \diagdown \\ 6-k \end{array} \begin{array}{c} f-x_i \\ \diagdown \\ 6-k \end{array} \begin{array}{c} x_i-y_i \\ \diagdown \\ \mathbf{1}_{10-k}^{(6-k)+(6-k)} \end{array} \\
 \begin{array}{c} h \\ | \\ \mathbf{6}_{k-4} \end{array} \begin{array}{c} e \\ | \\ \mathbf{3}_{6-k} \end{array} \begin{array}{c} h \\ | \\ \mathbf{2}_{8-k}^{6-k} \end{array} \begin{array}{c} e \\ | \\ \mathbf{1}_{10-k}^{(6-k)+(6-k)} \end{array}
 \end{array}
 \quad (2.8)$$

⁶In fact, following this reasoning, we can see that this process will decompactify all the other surfaces as well.

which describes the $5d$ gauge theory $\mathfrak{e}_6 + (6 - k)F$ for $2 \leq k \leq 6$. The ray

$$J = \phi(S_1 + S_5 + 2S_2 + 2S_4 + 2S_6 + 3S_3) \quad (2.9)$$

which implies that the Calabi-Yau threefold (2.8) describes a $5d$ SCFT for $2 \leq k \leq 6$, thus establishing that (1.2) and (1.6) are indeed $5d$ SCFTs.

For $\mathfrak{e}_7 + nF$ theories, we start with the Calabi-Yau threefold for the untwisted compactification of the $6d$ SCFT carrying $\mathfrak{e}_7$ on $-k$ curve

$$(2.10)$$

for $k = 2m$ and $1 \leq m \leq 4$. Decompactifying f of S_0 leads to the Calabi-Yau threefold

$$(2.11)$$

which describes the $5d$ gauge theory $\mathfrak{e}_7 + (4 - m)F$ for $1 \leq m \leq 4$. The dual Coxeter labels for $\mathfrak{e}_7^{(1)}$ suggest studying the Calabi-Yau threefold along

$$J = \phi(S_1 + 2S_2 + 2S_6 + 2S_7 + 3S_3 + 3S_5 + 4S_4) \quad (2.12)$$

where we find that the Calabi-Yau threefold describes a $5d$ SCFT for all $1 \leq m \leq 4$.

Let $\mathcal{M}$ be the Mori cone of S_i and let $\mathcal{C}_0$ be the sub-cone generated by non-negative linear combinations of the curves C_{ij}^α and C_i^α . Now, consider a curve C in $\mathcal{C}_0$. It will in general admit multiple decompositions of the form

$$C = \sum n_\mu C_\mu \quad (3.1)$$

where C_μ are generators of the Mori cone of S_i and $n_\mu \geq 0$. Since the volume of all C_μ must remain non-negative throughout the process and the volume of C must remain finite, it follows that the volume of any C_μ appearing in any of the above decompositions of C (with $n_\mu \neq 0$) must remain finite. In this way, considering all $C \in \mathcal{C}_0$, we find a set $\mathcal{S}_0$ of Mori cone generators which must remain compact but which are not in $\mathcal{C}_0$.

Let $\mathcal{C}_1$ be the sub-cone of $\mathcal{M}$ generated by the curves in $\mathcal{C}_0$ and $\mathcal{S}_0$. We again study decompositions of the form (3.1) for every curve $C \in \mathcal{C}_1$. This leads us to a set of Mori cone generators $\mathcal{S}_1$ which must remain compact but are not in $\mathcal{C}_1$. Joining $\mathcal{C}_1$ and $\mathcal{S}_1$, we obtain another sub-cone $\mathcal{C}_2$ of the Mori cone. This process will converge at some step r where $\mathcal{S}_r$ will be empty. Then, $\mathcal{C}_r \subseteq \mathcal{M}$ is the sub-cone containing all the curves that must remain compact during the decompactification process.

If $\mathcal{C}_r = \mathcal{M}$, then it is not possible to decompactify S_i while keeping the rest of the Calabi-Yau threefold compact. If $\mathcal{C}_r$ is a proper subset of $\mathcal{M}$, then we can decompactify the curves in $\mathcal{M} - \mathcal{C}_r$ thus decompactifying the surface S_i .

3.1 Examples

Let us study some examples of decoupling surfaces using the above criteria:

1. First, let us consider decoupling S_0 in (2.2) whose Mori cone is generated by e and f . The only gluing curve is h which can be written as

$$h = e + f \quad (3.2)$$

implying that both the generators must remain compact, and thus it is impossible to decompactify S_0 while keeping the rest of the Calabi-Yau threefold (2.2) compact.

2. Now, let us consider the Calabi-Yau threefold associated to the untwisted compactification of the 6d SCFT carrying $\mathfrak{sp}(1)$ on -1 curve

$$\mathbf{0}_0 \xrightarrow{2e+f \quad 2h-\sum x_i} \mathbf{1}_1^{10} \quad (3.3)$$

Flop n blowups in S_1 to obtain

$$\mathbf{0}_0^n \xrightarrow{2e+f-\sum x_i \quad 2h-\sum x_i} \mathbf{1}_1^{10-n} \quad (3.4)$$

We claim that we cannot decouple S_0 until $n \geq 4$. To see this, notice that we can write the gluing curve as

$$2e + f - \sum x_i = e + \left(e + f - \sum x_i \right) \quad (3.5)$$

Since

$$\left(e + f - \sum x_i\right)^2 \geq -1 \quad (3.6)$$

for $n < 4$, $e + f - \sum x_i$ exists in the Mori cone and hence (3.5) is a valid decomposition, from which we learn that e must remain compact. Now notice that

$$e = (e - x_i) + x_i \quad (3.7)$$

for all blowups x_i , implying that $e - x_i$ and x_i must remain compact. For $n = 3$, we can also write the gluing curve as

$$2e + f - \sum x_i = (e - x_1) + (e - x_2) + (f - x_3) \quad (3.8)$$

We can also permute x_1, x_2, x_3 on the right hand side of the above equation. Thus, $f - x_i$ for all x_i must also remain compact. Similarly, the reader can show that all $f - x_i$ must remain compact for the $n = 1, 2$ cases as well, and f must remain compact for the $n = 0$ case. This exhausts all the generators of the Mori cone and we find that it is impossible to decouple S_0 for $n < 4$.

For $n = 4$, we must keep $e + f - x_i - x_j - x_k$ compact for any three distinct blowups x_i, x_j, x_k . We must also keep $e - x_i$ compact for all x_i . But there are no restrictions on $f - x_i$ or x_j and we can decompactify all of them. This leads us to the Calabi-Yau threefold

$$\mathbf{1}_1^6 \quad (3.9)$$

which describes the $5d$ gauge theory $\mathfrak{su}(2) + 6\mathbf{F}$. The compact curves $e + f - x_i - x_j - x_k$ and $e - x_i$ do not intersect the gluing curve $2e + f - \sum x_i$. Thus they are infinitely far separated from the $\mathfrak{su}(2) + 6\mathbf{F}$ theory.⁸ That is, they give rise to some decoupled massive states.

This decompactification process can also be understood as the ungauging of a gauge algebra. To see this, notice that the Calabi-Yau threefold (3.4) for $n = 4$ admits an isomorphic description⁹ as

$$\mathbf{0}_0^4 \xrightarrow{f \quad f - \sum x_i} \mathbf{1}_0^{2+4} \quad (3.10)$$

which describes the $5d$ gauge theory with gauge algebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$, a hyper in bifundamental and four hypers in fundamental of each $\mathfrak{su}(2)$. Then the decompactification process discussed above is simply the tuning of the gauge coupling of the $\mathfrak{su}(2)$ described by S_0 to zero, and the BPS string being decoupled is the BPS monopole associated to $\mathfrak{su}(2)$. The decoupled massive BPS states can be identified with the four hypers of the $\mathfrak{su}(2)$ which is being ungauged.

⁸More concretely, the curves connecting the decoupled compact curves and the gluing curve go to infinite volume. For example, x_i connects $2e_f - \sum x_i$ to $e - x_i$ since it intersects both at a single point.

⁹See [9, 18] for more details on such isomorphisms.

Thus, according to our analysis, we cannot obtain $\mathfrak{su}(2) + (10 - n)F$ for $0 \leq n \leq 3$ by integrating out a BPS string from the 5d KK theory obtained by an untwisted compactification of the 6d SCFT carrying $\mathfrak{sp}(1)$ on -1 curve. This is consistent since it is known that $\mathfrak{su}(2) + mF$ is UV complete only for $m \leq 8$.

3. In our final example, we will discuss a 5d KK theory which can flow to other 5d KK theories upon integrating out BPS strings. In the above two examples, the theory obtained after the flow was a 5d SCFT rather than a 5d KK theory. The 5d KK theory we will discuss arises via untwisted compactification of the 6d SCFT carrying $\mathfrak{so}(7)$ on -1 curve. The associated Calabi-Yau threefold (in one of its flop frames) is

$$\begin{array}{ccccc}
 & & \mathbf{1}_7^{6+2} & & \\
 & \nearrow^{f-y_i} & \downarrow e & \nwarrow^{f-x_i} & \\
 & 2 & & 6 & \\
 f-x_i & & h+2f & & f-x_i \\
 \mathbf{0}_1^2 & \xleftarrow{e} & \mathbf{2}_1 & \xrightarrow{2h} & \mathbf{3}_6^6
 \end{array} \quad (3.11)$$

We would like to decompactify S_1 , which is not possible in this flop frame. To see it, notice that the sum of the gluing curves e , $f - x_i$, $f - y_i$ is

$$h + f - \sum x_i - \sum y_i = \left(h - \sum x_i - \sum y_i \right) + f \quad (3.12)$$

which implies that the curves $h - \sum x_i - \sum y_i$ and f must remain compact. The compactness of f implies that all of the $f - x_i$ and x_j must remain compact. Thus all the Mori cone generators must remain compact, and the surface S_1 cannot be decompactified. This is good because if S_1 could be decompactified, we would obtain the Calabi-Yau threefold

$$\mathbf{0}_1^2 \xleftarrow{e} \mathbf{2}_1 \xrightarrow{2h} \mathbf{3}_6^6 \quad (3.13)$$

which describes the 5d gauge theory $\mathfrak{so}(7) + 2F + 6S$ [18] (where S denotes a hyper in spinor representation). However, this 5d gauge theory exceeds the bounds presented in [1] and hence cannot describe a 5d SCFT or a 5d KK theory.

However, if we flop an x_i living in S_3 of (3.11) to obtain

$$\begin{array}{ccccc}
 & & \mathbf{1}_7^{5+2+2} & & \\
 & \nearrow^{f-y_i} & \downarrow e & \nwarrow^{f-x_i, f-\sum z_i} & \\
 & 2 & & 6 & \\
 f-x_i & & h+2f & & f-x_i, f \\
 \mathbf{0}_1^2 & \xleftarrow{e} & \mathbf{2}_1 & \xrightarrow{2h} & \mathbf{3}_6^5
 \end{array} \quad (3.14)$$

then we can decompactify S_1 by decompactifying all x_i, y_i, z_i living in S_1 and obtain

$$0_1^2 \xrightarrow[e]{\quad\quad\quad} 2_1 \xrightarrow[2h]{\quad\quad\quad} 3_6^5 \quad (3.15)$$

which describes the $5d$ gauge theory $\mathfrak{so}(7) + 2F + 5S$ which should be a $5d$ KK theory according to [1]. Indeed, it is shown in [28] that the KK theory obtained by untwisted compactification of $6d$ SCFT carrying $\mathfrak{g}_2$ on -1 curve is described by the $5d$ gauge theory $\mathfrak{so}(7) + 2F + 5S$.

We can also flop an x_i living in S_0 of (3.11) to obtain

$$\begin{array}{ccccc}
 & & 1_7^{6+1+2} & & \\
 & \nearrow^{f-y, f-\sum z_i} & \downarrow e & \nwarrow^{f-x_i} & \\
 & 2 & & 6 & \\
 f-x, f & & h+2f & & f-x_i \\
 0_1^1 & \xrightarrow[e]{\quad\quad\quad} & 2_1 & \xrightarrow[2h]{\quad\quad\quad} & 3_6^6
 \end{array} \quad (3.16)$$

in which we can decompactify S_1 to obtain

$$0_1^1 \xrightarrow[e]{\quad\quad\quad} 2_1 \xrightarrow[2h]{\quad\quad\quad} 3_6^6 \quad (3.17)$$

which describes the $5d$ gauge theory $\mathfrak{so}(7) + F + 6S$ which is equivalent to the $5d$ KK theory obtained by twisting the $6d$ SCFT carrying $\mathfrak{su}(4)$ on -1 curve by the outer automorphism of $\mathfrak{su}(4)$ [28].

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