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Symbol alphabets from plabic graphs II: rational letters

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ABSTRACT: Symbol alphabets of n -particle amplitudes in $\mathcal{N} = 4$ super-Yang-Mills theory are known to contain certain cluster variables of $G(4, n)$ as well as certain algebraic functions of cluster variables. The first paper [arXiv:2007.00646](https://arxiv.org/abs/2007.00646) in this series focused on $n = 8$ algebraic letters. In this paper we show that it is possible to obtain all rational symbol letters (in fact all cluster variables) by solving matrix equations of the form $CZ = 0$ if one allows C to be an arbitrary cluster parameterization of the top cell of $G_+(n-4, n)$.

KEYWORDS: Scattering Amplitudes, Supersymmetric Gauge Theory

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1 Introduction

It was observed some time ago [1] that many symbol letters [2] of perturbative n -particle amplitudes scattering amplitudes in planar maximally supersymmetric Yang-Mills (SYM) theory are cluster variables of the $G(4, n)$ cluster algebra [3–8]. Recently there has been renewed interest in the problem of explaining, or possibly even predicting, the symbol alphabet for a given amplitude.¹ Knowledge of symbol alphabets is important input for a bootstrap program that is currently the state-of-the art for high-loop calculations in SYM theory (see [10] for a review). Some information on symbol alphabets can be gleaned [11–15] by analyzing Landau equations, but as noted in those references, this analysis cannot completely determine symbol letters. More recent ‘phenomenological’ approaches to this problem involving tropical fans, polytopes, and plabic graphs associated to Grassmannians have been discussed in [16–21]. Motivation for the renewed interest comes in part from recent calculations of new amplitudes [22, 24] that have provided fresh data on the sets of cluster variables and algebraic functions of cluster variables that appear as symbol letters of certain amplitudes.

In [20, 21] the connection between symbol letters and plabic graphs was explored via matrix equations of the form $CZ = 0$ that arise naturally [25–28] in the Grassmannian formalism for amplitudes in SYM theory. Here C is a $k \times n$ matrix parameterizing a $4k$ -dimensional cell of the totally non-negative Grassmannian $G_+(k, n)$ [29, 30] and Z is an $n \times 4$ matrix of momentum twistors [31] specifying the kinematic data for n -particle scattering. This paper should be read as a close sequel of [20], to which we refer the reader for many important details. A highlight of that paper was the highly nontrivial appearance (also found in [21]), for $k = 2$, of 18 independent *algebraic* functions of $G(4, 8)$ cluster variables that precisely match the 18 algebraic letters of the eight-particle symbol alphabet of [22].

¹The connection between cluster algebras and symbol alphabets is of course an interesting question also beyond SYM theory; see for example [9].

However, these papers left open the question of whether all *rational* symbol letters could be obtained by solving $CZ = 0$.

In this paper we show that it is *not* possible to obtain all rational symbol letters from plabic graphs, but instead we show how to obtain all rational letters by allowing non-plabic C -matrices. If only plabic C -matrices are used, then 16 out of the 180 rational letters of the eight-particle symbol alphabet cannot be obtained (see also [32]). Non-plabic C -matrices can be constructed by starting with a plabic graph C -matrix and performing a “non-square move”, i.e., by mutating on a node of valence greater than four in the quiver that is dual to the graph.² More specifically, we show that if one allows C to be an arbitrary $4(n-4)$ -dimensional cluster parameterization of $G_+(n-4, n)$ (i.e., of the top cell) then *every* cluster variable of $G(4, n)$ (in particular, all rational symbol letters) can be obtained by solving $CZ = 0$. Moreover one will *never* encounter non-cluster variables, which is interesting in light of the observation made in [20, 21] that for some non-top cells, solving $CZ = 0$ yields quantities that are polynomial in Plücker coordinates but are *not* cluster variables.

The plan of the paper is as follows. In section 2 we explain the construction of non-plabic cluster parameterizations by way of an $n = 7$ example and then discuss how the rational letters of the eight-particle symbol alphabet arise from solving $CZ = 0$. In section 3 we establish the claims of the previous paragraph by observing that solving $CZ = 0$ for top cells induces the obvious $G(n-m, n) \cong G(m, n)$ cluster isomorphism.

2 Non-plabic parameterizations

2.1 A motivational example

In order to demonstrate clearly what we mean by a “non-plabic cluster parameterization” of a Grassmannian cell, we start with an example worked out in the style of those considered in [20], to which we refer the reader for all details on notation and conventions. Whereas all of the plabic graphs depicted in [20] had all internal faces bounded by four edges, meaning that one could always apply a square move to mutate to another plabic graph, we now consider a plabic graph which corresponds to the top cell of $G_+(3, 7)$ and has non-square internal faces, shown in figure 1(a). The corresponding boundary measurement is

$$C = \begin{pmatrix} 1 & c_{12} & c_{13} & c_{14} & 0 & c_{16} & 0 \\ 0 & c_{52} & c_{53} & c_{54} & 1 & c_{56} & 0 \\ 0 & c_{72} & c_{73} & c_{74} & 0 & c_{76} & 1 \end{pmatrix} \quad (2.1)$$

²The part of a cluster algebra that can be reached by only performing square moves in the dual graph was called the *accessible* algebra in [23].

where

$$\begin{aligned}
 c_{12} &= f_0 f_2 f_3 f_4 f_5 f_6 f_{11} f_{12} (1 + f_7 + f_9 (1 + f_{10}) + f_7 (1 + f_8) f_9 (1 + f_{10})) \\
 c_{13} &= f_0 f_3 f_4 f_5 f_6 f_{11} f_{12} (1 + f_7 (1 + f_9 (1 + f_8)) + f_9) \\
 c_{14} &= f_0 f_4 f_5 f_6 f_{11} f_{12} (1 + f_7) \\
 c_{16} &= -f_0 f_6 \\
 c_{52} &= f_2 f_3 f_4 f_{11} \\
 c_{53} &= f_3 f_4 f_{11} \\
 c_{54} &= f_4 (1 + f_{11}) \\
 c_{56} &= f_0 f_1 f_2 f_3 f_4 f_6 f_7 f_8 f_9 f_{10} f_{11} f_{12} \\
 c_{72} &= -f_2 f_3 f_4 f_5 f_6 f_{11} f_{12} (1 + f_9 (1 + f_{10})) \\
 c_{73} &= -f_3 f_4 f_5 f_6 f_{11} f_{12} (1 + f_9) \\
 c_{74} &= -f_4 f_5 f_6 f_{11} f_{12} \\
 c_{76} &= f_6
 \end{aligned} \tag{2.2}$$

and $\prod f_i = 1$. For a 7×4 matrix Z , the solution to $CZ = 0$ is

$$\begin{aligned}
 f_0 &= \frac{\langle 1234 \rangle}{\langle 2347 \rangle} & f_1 &= \frac{\langle 2367 \rangle \langle 2456 \rangle}{\langle 6(71)(23)(45) \rangle} \\
 f_2 &= -\frac{\langle 3456 \rangle}{\langle 2456 \rangle} & f_3 &= \frac{\langle 1467 \rangle \langle 2456 \rangle}{\langle 6(71)(23)(45) \rangle} \\
 f_4 &= -\frac{\langle 1567 \rangle}{\langle 1467 \rangle} & f_5 &= -\frac{\langle 2346 \rangle}{\langle 2345 \rangle} \\
 f_6 &= -\frac{\langle 2347 \rangle}{\langle 2346 \rangle} & f_7 &= \frac{\langle 1237 \rangle \langle 2346 \rangle}{\langle 1234 \rangle \langle 2367 \rangle} \\
 f_8 &= -\frac{\langle 2347 \rangle \langle 6(71)(23)(45) \rangle}{\langle 1237 \rangle \langle 2346 \rangle \langle 4567 \rangle} & f_9 &= \frac{\langle 1267 \rangle \langle 2346 \rangle \langle 4567 \rangle}{\langle 1467 \rangle \langle 2367 \rangle \langle 2456 \rangle} \\
 f_{10} &= -\frac{\langle 6(71)(23)(45) \rangle}{\langle 1267 \rangle \langle 3456 \rangle} & f_{11} &= -\frac{\langle 6(71)(23)(45) \rangle}{\langle 1567 \rangle \langle 2346 \rangle} \\
 f_{12} &= -\frac{\langle 1467 \rangle \langle 2345 \rangle \langle 2367 \rangle}{\langle 2347 \rangle \langle 6(71)(23)(45) \rangle}
 \end{aligned} \tag{2.3}$$

where $\langle ijkl \rangle$ are the maximal minors of Z and $\langle a(bc)(de)(fg) \rangle = \langle bade \rangle \langle c afg \rangle - (b \leftrightarrow c)$.

At the level of the plabic graph shown in figure 1(a), we cannot perform a move on face f_{12} since it is not a square, but we certainly can mutate on node f_{12} in the corresponding quiver shown in figure 1(b). This transforms five of the mutable variables in this cluster according to

$$\begin{aligned}
 f_7 &\mapsto f'_7 = f_7 / (1 + 1/f_{12}), & f_{12} &\mapsto f'_{12} = \frac{1}{f_{12}}, & f_9 &\mapsto f'_9 = f_9 / (1 + 1/f_{12}), \\
 f_8 &\mapsto f'_8 = f_8 (1 + f_{12}), & & & f_{11} &\mapsto f'_{11} = f_{11} (1 + f_{12}),
 \end{aligned} \tag{2.4}$$

leaving the others unchanged. If we perform this transformation on the C -matrix shown in (2.1), we obtain a new matrix C' that is not the boundary measurement of any plabic

Figure 1. (a) A plabic graph corresponding to the top cell of $G_+(3,7)$ and (b) the associated dual quiver, associated to a cluster of the $G(3,7)$ cluster algebra.

graph, but perfectly well parameterizes the top cell of $G_+(3,7)$ as the f 's range over $\mathbb{R}^+$. Moreover, it is a cluster parameterization in the sense that the f 's are cluster variables of $G(3,7)$ (they belong to the cluster obtained by mutating figure 1(b) on node f_{12}). This exemplifies what we mean by a “non-plabic cluster parameterization”, or (more simply) a “non-plabic C -matrix”.

At the level of the solution to $CZ = 0$, the transformation (2.4) has the effect of introducing one additional symbol letter not already present as a multiplicative factor in (2.3). Specifically, by computing

$$1 + f_{12} = -\frac{\langle 2346 \rangle \langle 7(61)(23)(45) \rangle}{\langle 2347 \rangle \langle 6(71)(23)(45) \rangle} \quad (2.5)$$

we see that the new factor is $\langle 7(61)(23)(45) \rangle$.

2.2 Rational eight-particle symbol letters

The symbol alphabet for the two-loop NMHV octagon contains [22] 180 cluster variables of $G(4,8)$:

- 68 Plücker coordinates of the form $\langle a \ a+1 \ b \ c \rangle$,
- 8 cyclic images of $\langle 12\bar{4} \cap \bar{7} \rangle$,
- 40 cyclic images of $\langle 1(23)(45)(78) \rangle$, $\langle 1(23)(56)(78) \rangle$, $\langle 1(28)(34)(56) \rangle$, $\langle 1(28)(34)(67) \rangle$, $\langle 1(28)(45)(67) \rangle$,
- 48 dihedral images of $\langle 1(23)(45)(67) \rangle$, $\langle 1(23)(45)(68) \rangle$, $\langle 1(28)(34)(57) \rangle$,
- 8 cyclic images of $\langle \bar{2} \cap (245) \cap \bar{8} \cap (856) \rangle$,
- and 8 distinct dihedral images of $\langle \bar{2} \cap (245) \cap \bar{6} \cap (681) \rangle$.

We see that 96 are quadratic in Plücker coordinates and the last 16 are cubic. (The $G(4,8)$ cluster algebra has, in total, 120 quadratic and 174 cubic cluster variables [33, 34].) By applying

Figure 2. A plabic graph associated to the 12-dimensional cell in $G_+(3,8)$ labeled by the decorated permutation $\{4, 5, 7, 6, 9, 8, 11, 10\}$. The cubic symbol letter $\langle \bar{4} \cap (467) \cap \bar{2} \cap (278) \rangle$ appears in the solution of $CZ = 0$ after performing a non-square move mutation on face f_8 .

the algorithm described in [20, 21] to all plabic graphs associated to $4k$ -dimensional cells of $G_+(k,8)$, with $1 \leq k \leq 4$ and including all members of each cyclic class, one encounters all of the Plücker coordinates and quadratic cluster variables (in addition, of course, to numerous non-cluster variables, similar to the examples described in [20], as well as the 18 algebraic symbol letters). However, the cubic symbol letters on the above list are missing (see also [32]).

We find that the cubic letters are obtainable if one allows non-plabic C -matrices as described in the previous subsection. For example, the first type of cubic letter can be obtained from the (non-top cell) $G_+(3,8)$ plabic graph shown in figure 2 by applying a mutation on face f_8 . We spare the details of writing down the C -matrix for this graph and the solution for all face variables; it suffices to display

$$1 + f_8 = \frac{\langle 3567 \rangle \langle \bar{4} \cap (467) \cap \bar{2} \cap (278) \rangle}{\langle 2378 \rangle \langle 4567 \rangle \langle 3(12)(45)(67) \rangle} \quad (2.6)$$

which contains a cubic letter in the same cyclic class as $\langle \bar{2} \cap (245) \cap \bar{8} \cap (856) \rangle$.

It is also possible to obtain these cubic letters from non-plabic cluster parameterizations of the top cell of $G_+(4,8)$. The second type of cubic letter ($\langle \bar{2} \cap (245) \cap \bar{6} \cap (681) \rangle$ and its images) can *only* be obtained from non-plabic cluster parameterizations of the top cell. In the next section we observe that *all* cluster variables of $G(n-m,n)$ can be obtained from the top cell of $G_+(m,n)$ if one allows consideration of non-plabic cluster parameterizations.

3 A cluster isomorphism for top cells

In [20, 21] it was noted that in some cases, the solution to $CZ = 0$ (for C a boundary measurement of a plabic graph corresponding to an mk -dimensional cell of $G_+(k,n)$ and Z an $n \times m$ matrix) involves *non*-cluster variables of $G(m,n)$. Moreover, we noted in the previous section that it is impossible to obtain all cluster variables of $G(m,n)$ in this way.

Here we point out that if we take $k = n-m$, which means we restrict our attention to the top cell of $G_+(k,n)$, but allow arbitrary *non*-plabic C -matrices, then solving $CZ = 0$

will always produce all cluster variables of $G(m, n)$ (which might be infinite in number), and will never produce non-cluster variables.

To begin with let us consider the simple case $k = 2$. We will demonstrate that the map induced by solving $CZ = 0$ maps Plücker coordinates of $G(2, n)$ to those of $G(n-2, n)$. We do not need to choose any particular parameterization, and simply denote the entries of the $2 \times n$ matrix C by $c_{1,i}, c_{2,i}$ for $i \in [n]$, and let Z be an $n \times (n-2)$ matrix with rows Z_i . We can choose to solve $CZ = 0$ for $2n-4$ of the c variables in terms of the remaining 4. (Here is the step where it is important that C parameterizes a top cell; otherwise there would not be enough independent variables to solve for!). For example, we can express

$$c_{a,i} = (-1)^{n-i-1} \left(c_{a,n-1} \frac{\tilde{\Delta}(i, n)}{\tilde{\Delta}(n-1, n)} + c_{a,n} \frac{\tilde{\Delta}(i, n-1)}{\tilde{\Delta}(n-1, n)} \right) \quad a \in [2], i \in [n-2]$$

in terms of the Plücker coordinates on $G(n-2, n)$, where $\tilde{\Delta}(i, j)$ denotes the determinant of Z with rows i and j removed i.e. $\tilde{\Delta}(i, j) = \epsilon_{i,j,\{1,\dots,n\} \setminus \{i,j\}} \langle \{1 \dots n\} \setminus \{i, j\} \rangle$. Evaluating the Plücker coordinates of $G(2, n)$ on the solution yields

$$\Delta(i, j) = \begin{vmatrix} c_{1i} & c_{1j} \\ c_{2i} & c_{2j} \end{vmatrix} = \tilde{\Delta}(i, j) \frac{\Delta(n-1, n)}{\tilde{\Delta}(n-1, n)}. \quad (3.1)$$

It is worth pointing out that the form of this equation is independent of our choice of solved and unsolved variables, and can be written more invariantly as

$$\frac{\Delta(i, j)}{\Delta(k, l)} = \frac{\tilde{\Delta}(i, j)}{\tilde{\Delta}(k, l)} \quad (3.2)$$

for all $i < j$ and $k < l \in [n]$. Therefore, up to a single irrelevant overall factor, we can simply say that $\Delta(i, j) \mapsto \tilde{\Delta}(i, j)$. This is the sense in which we say that solving $CZ = 0$ maps the Plücker coordinates of $G(2, n)$ to those of $G(n-2, n)$.

The story is similar for arbitrary k . Let C be a $k \times n$ matrix with entries c_{ij} and let Z be an $(n-k) \times n$ matrix. In order to solve $CZ = 0$ let us choose to solve for

$$\begin{pmatrix} c_{1\ k+1} & \dots & c_{1\ n} \\ \vdots & \dots & \vdots \\ c_{k\ k+1} & \dots & c_{k\ n} \end{pmatrix}$$

in terms of $\begin{pmatrix} c_{11} & \dots & c_{1k} \\ \vdots & \dots & \vdots \\ c_{k1} & \dots & c_{kk} \end{pmatrix}$. The solution is

$$c_{ab} = (-1)^{b-k} \sum_{i \in U} \frac{\langle i, \{k+1, \dots, n\} \setminus b \rangle}{\langle k+1, \dots, n \rangle} c_{ai} \quad (3.3)$$

where $U = \{1, \dots, k\}$ and the $\langle \dots \rangle$ represent the Plücker coordinates of $G(n-k, n)$. The Plücker coordinates of $G(k, n)$, computed from the C -matrix, evaluate to

$$\Delta(b_1, \dots, b_k) = \epsilon_{b_1, \dots, b_k, \{1, \dots, n\} \setminus \{b_1, \dots, b_k\}} \frac{\langle \{1, \dots, n\} \setminus \{b_1, \dots, b_k\} \rangle}{\langle k+1, \dots, n \rangle} \Delta(1, \dots, k). \quad (3.4)$$

If we choose to label the Plücker coordinates of $G(n-k, n)$ by the missing k columns and define

$$\langle a_1 \dots a_{n-k} \rangle \equiv \tilde{\Delta}(\{1, \dots, n\} \setminus \{a_1, \dots, a_{n-k}\}), \quad (3.5)$$

we can rewrite (3.4) as

$$\frac{\Delta(b_1, \dots, b_k)}{\Delta(1, \dots, k)} = \frac{\tilde{\Delta}(b_1, \dots, b_k)}{\tilde{\Delta}(1, \dots, k)} \quad (3.6)$$

analogous to (3.1). Let us recall that the map $\Delta(b_1, \dots, b_k) \rightarrow \tilde{\Delta}(b_1, \dots, b_k)$ induces the ‘obvious’ isomorphism between the $G(k, n)$ and $G(n-k, n)$ cluster algebras, in that it maps clusters of one to the other and commutes with mutation.

To conclude: we have seen that when C is any cluster parameterization of the top cell of $G_+(n-m, n)$, then the map induced by solving $CZ = 0$ is the natural cluster isomorphism $G(n-m, n) \mapsto G(m, n)$. In other words, if C is a top-cell parameterization associated to some cluster of $G(n-m, n)$, then the letters appearing in the solution to $CZ = 0$ will be the cluster variables of the image of that cluster in $G(m, n)$ under the replacement $\Delta \mapsto \tilde{\Delta}$. Therefore it is also manifest that we will never encounter non-cluster variables of $G(m, n)$, in contrast to what happens for lower-dimensional cells, such as the examples encountered in [20, 21]. Figure 1 of [20] illustrates the one-to-one correspondence between ‘input’ clusters of $G(n-m, n)$ and ‘output’ clusters of $G(m, n)$ for the case $m = 4, n = 6$.

4 Discussion

The connection between symbol alphabets of scattering amplitudes in SYM theory and solutions of matrix equations of the form $CZ = 0$ was investigated in [20, 21]. Here we have pointed out that if one allows C to be an arbitrary cluster parameterization of the top cell of $G_+(n-m, n)$, then *all* cluster variables of $G(m, n)$ can be obtained from such solutions; moreover, *no* non-cluster variables arise in this way. Since all (currently known) rational symbol letters are cluster variables of $G(4, n)$, we see that they can all be obtained from the top cell of $G_+(n-4, n)$.

One of the main points of [20, 21] was that since the number of plabic graphs is manifestly finite (for any given n), they might select ‘preferred’ finite sets of cluster variables to serve as candidate symbol alphabets. This is in line with general expectations [15, 16] that for every n , there is a finite n -particle symbol alphabet that serves to express all n -particle amplitudes to any finite order in perturbation theory. We (and [32]) have found that for $n = 8$ (and presumably, also for any higher n), this finite set is *too small* — it is missing 16 of the rational symbol letters of [22]. In this paper we overcame that problem at the expense of introducing a new one: by allowing arbitrary cluster parameterizations C , we obtain all of the infinitely many cluster variables of $G(4, n \geq 8)$, not a finite subset. However, we note that while these 16 letters are related to non-plabic cluster parameterizations, the diagrams from which these letters arise in the Landau analysis are still planar [11–15].

It would be interesting to investigate some possibility in the middle — something small enough to give finite sets of cluster variables, but large enough to include all known symbol

letters. We could speculate that our story bears some superficial resemblance to the tropical fans and polytopes studied in [16–18]. There, one has the freedom to select from a menu of tropical fans or dual polytopes of various amounts of fineness, with different choices being associated to different symbol alphabets. In the $CZ = 0$ story one could imagine restricting attention to various classes of parameterizations C , in between the extremes of ‘only plabic graphs’ (too few) and ‘all cluster parameterizations’ (too many). It would be interesting to investigate this further.

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