Superluminality in DHOST theory with extra scalar

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ABSTRACT: We consider DHOST Ia theory interacting gravitationally with an additional conventional scalar field minimally coupled to gravity. At the linearized level of perturbations about cosmological background, we find that in the presence of a slowly rolling extra scalar field, one of the modes generically propagates at superluminal speed. This result is valid for any stable cosmological background. We identify a subclass of DHOST Ia theories in which this superluminality property is absent, and all modes may propagate (sub)luminally. We discuss possible implications for the interacting DHOST Ia theories.

KEYWORDS: Classical Theories of Gravity, Effective Field Theories

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Contents

1	Introduction	1
2	DHOST Ia with extra scalar	2
3	Rolling scalar and superluminality	4
4	Exceptional DHOST Ia subclass	5
5	Conclusion	6
A	Linearized theory for DHOST Ia + k-essence system	6

1 Introduction

Scalar-tensor theories have become one of the most popular frameworks for addressing cosmological challenges, from the late-time accelerated expansion to the early-time dynamics of the Universe. The most general known type of the scalar-tensor theories, featuring the desired $2 + 1$ degrees of freedom, is the Degenerate Higher-Order Scalar-Tensor (DHOST) theories [1–4] (see also reviews [5, 6]). DHOST theories involve second derivative terms in the Lagrangian and, thus, are superficially described by a set of fourth order differential equations of motion. However, DHOST theories are protected against the Ostrogradsky ghost by a degeneracy property, which ensures that the set of equations of motion can be combined into a system of second order equations.

A complete classification of DHOST theories is given in refs. [2–4]. The most promising from the phenomenological point of view is the so-called “Ia” (or “N-1”) class [3]. Interestingly, DHOST Ia family includes both Horndeski [9, 10] and beyond Horndeski theories (or GLPV) [11, 12] as its special cases. General DHOST Ia family and its Horndeski and beyond Horndeski subclasses provide a rich framework for cosmological model building, especially because they shed new light on various stability problems [7, 8, 13–16]; they also appear promising from the viewpoint of constructing space-times with traversable wormholes [17–19].

Apart from the stability issues, modified gravity and, in particular, DHOST theories may suffer from superluminality.¹ This problem has been addressed from various perspectives, e.g., in refs. [22–26]. Superluminal propagation was argued to be troublesome, since it indicates that a Lorentz-covariant UV-completion is impossible for such a theory [27]. The potential superluminality problem becomes indeed acute in DHOST theories as soon as one adds an extra matter component. This was first illustrated in ref. [25] within the cosmological Genesis scenario based on a specific cubic Horndeski model: the scalar perturbation necessarily became superluminal in some region of phase space upon adding perfect fluid into a formerly non-superluminal (and stable) setup.

¹To avoid confusion, we note that superluminality we consider in this paper is not parametrically suppressed and persists at high spatial momenta, unlike superluminality in conventional EFT of gravity [20] where it is harmless from the causality viewpoint [21].

Another result in this direction has been recently obtained in ref. [28], where beyond Horndeski theories were analysed in cosmological backgrounds from the superluminality viewpoint. It has been shown that adding even the tiniest amount of perfect fluid with the flat-space sound speed equal to that of light, $c_m = 1$, inevitably results in the appearance of a superluminal mode (this does not necessarily happen for c_m substantially smaller than 1). This applies to any stable cosmological background and any beyond Horndeski Lagrangian. This finding has been supported by similar result [29] in the case where instead of perfect fluid, a conventional, minimally coupled scalar field (whose flat-space propagation is luminal, $c_m = 1$) is added to a cosmological setup in beyond Horndeski theory.

The purpose of this paper is to show that superluminality is quite general property of DHOST Ia theories gravitationally interacting with conventional scalar field(s) minimally coupled to gravity. Namely, we prove that unless an additional constraint is imposed on the functions in the DHOST Ia Lagrangian, there exists a superluminal mode in the scalar sector in an arbitrary stable cosmological background with small but non-zero kinetic energy density of the extra scalar. We identify an exceptional subclass of DHOST Ia theories where this property does not hold. It is defined by the constraint (4.1) below; there, all modes may be safely (sub)luminal. Clearly, this subclass is particularly interesting from the viewpoint of cosmological model building.

2 DHOST Ia with extra scalar

We focus on the quadratic DHOST theories whose Jordan frame action has the following general form:

$$\mathcal{S}_\pi = \int d^4x \sqrt{-g} \left(F(\pi, X) + K(\pi, X) \square \pi + F_2(\pi, X) R + \sum_{i=1}^5 A_i(\pi, X) L_i \right), \quad (2.1)$$

with

$$\begin{aligned} L_1 &= \pi_{;\mu\nu} \pi^{;\mu\nu}, & L_2 &= (\square \pi)^2, & L_3 &= \pi^{;\mu} \pi_{;\mu\nu} \pi^{;\nu} \square \pi, \\ L_4 &= \pi^{;\mu} \pi_{;\mu\nu} \pi^{;\nu\rho} \pi_{;\rho}, & L_5 &= (\pi^{;\mu} \pi_{;\mu\nu} \pi^{;\nu})^2, \end{aligned} \quad (2.2)$$

where π is a scalar field, $X = g^{\mu\nu} \pi_{;\mu} \pi_{;\nu}$, $\pi_{;\mu} = \partial_\mu \pi$, $\pi_{;\mu\nu} = \nabla_\nu \nabla_\mu \pi$, $\square \pi = g^{\mu\nu} \nabla_\nu \nabla_\mu \pi$. In what follows we restrict our analysis to the DHOST Ia class, which is the most phenomenologically viable [7]. In this class, the functions A_2 , A_4 and A_5 are expressed through the independent functions F_2 , A_1 and A_3 as follows [5]:

$$A_2 = -A_1, \quad (2.3a)$$

$$\begin{aligned} A_4 &= \frac{1}{8(F_2 - X A_1)^2} \left[-16X A_1^3 + 4(3F_2 + 16X F_{2X}) A_1^2 - X^2 F_2 A_3^2 \right. \\ &\quad \left. - (16X^2 F_{2X} - 12X F_2) A_3 A_1 - 16F_{2X} (3F_2 + 4X F_{2X}) A_1 \right. \\ &\quad \left. + 8F_2 (X F_{2X} - F_2) A_3 + 48F_2 F_{2X}^2 \right], \end{aligned} \quad (2.3b)$$

$$A_5 = \frac{(4F_{2X} - 2A_1 + X A_3) (-2A_1^2 - 3X A_1 A_3 + 4F_{2X} A_1 + 4F_2 A_3)}{8(F_2 - X A_1)^2}. \quad (2.3c)$$

Other arbitrary functions F and K belong to Horndeski subclass and do not enter these relations.

Our starting point is (perturbed) spatially flat FLRW background with metric

$$ds^2 = (1 + \alpha)^2 dt^2 - \gamma_{ij} (dx^i + \partial^i \beta dt) (dx^j + \partial^j \beta dt) , \quad (2.4)$$

with

$$\gamma_{ij} = a^2(t) e^{2\zeta} \left(\delta_{ij} + h_{ij}^T + \frac{1}{2} h_{ik}^T h_j^{kT} \right) , \quad (2.5)$$

where α , β and ζ are lapse, shift and curvature perturbations, respectively, and h_{ij}^T is traceless and transverse perturbation. It is supported by a rolling DHOST field $\pi = \pi(t)$. We assume that this background is stable, and DHOST perturbations about it are not superluminal. We neither impose any further constraints on the background nor assume any relations other than (2.3) between the Lagrangian functions in (2.1) yet. In the unitary gauge $\delta\pi = 0$, dynamical perturbations in the DHOST sector divide into tensor modes h_{ij}^T and the scalar mode. Obtaining the quadratic action for the tensor mode is straightforward, while in the scalar case one has to deal with constraints. Namely, quadratic action in the scalar sector of pure DHOST Ia theory is written in terms of a dynamical field $\tilde{\zeta}$, which is a linear combination of ζ and α (explicitly defined below), and non-dynamical variables α and β entering the action without temporal derivatives. One integrates out non-dynamical α and β and obtains the unconstrained action for $\tilde{\zeta}$. This is done explicitly in appendix with the result

$$S_\pi^{(2)} = \int dt d^3x a^3 \left[\left(\frac{\mathcal{G}_T}{8} (h_{ik}^T)^2 - \frac{\mathcal{F}_T}{8a^2} (\partial_i h_{kl}^T)^2 \right) + \left(\mathcal{G}_S \tilde{\zeta}^2 - \frac{1}{a^2} \mathcal{F}_S (\partial_i \tilde{\zeta})^2 \right) \right] , \quad (2.6)$$

where

$$\mathcal{G}_T = -2F_2 + 2A_1 X, \quad (2.7a)$$

$$\mathcal{F}_T = -2F_2, \quad (2.7b)$$

$$\mathcal{G}_S = \frac{\tilde{\Sigma} \mathcal{G}_T^2}{\tilde{\Theta}^2} + 3\mathcal{G}_T, \quad (2.7c)$$

$$\mathcal{F}_S = \frac{1}{a} \frac{d}{dt} \left[\frac{a \mathcal{G}_T (\mathcal{G}_T + \mathcal{D}\dot{\pi} + \mathcal{F}_T \Delta)}{\tilde{\Theta}} \right] - \mathcal{F}_T, \quad (2.7d)$$

with

$$\mathcal{D} = -2A_1 \dot{\pi} + 4F_{2X} \dot{\pi}, \quad (2.8a)$$

$$\Delta = \frac{X}{2\mathcal{G}_T} (2A_1 - 4F_{2X} - A_3 X) . \quad (2.8b)$$

Expressions for $\tilde{\Sigma}$ and $\tilde{\Theta}$ are cumbersome and not illuminating; they are given in appendix, where we derive the above formulas. In these notations the scalar perturbation is

$$\tilde{\zeta} = \zeta - \alpha \Delta .$$

We note in passing that hereafter we do not use the background equations of motion when deriving the action for perturbations.

The stability of the background requires $\mathcal{G}_T, \mathcal{F}_T, \mathcal{G}_S, \mathcal{F}_S > 0$, while the sound speeds squared are

$$c_{S,0}^2 = \frac{\mathcal{F}_S}{\mathcal{G}_S}, \quad c_T^2 = \frac{\mathcal{F}_T}{\mathcal{G}_T}. \quad (2.9)$$

We assume that both speeds are not superluminal, $c_{S,0}, c_T \leq 1$.

We add to the theory another, conventional scalar field χ , which does not interact directly with the DHOST scalar π and has minimal coupling to gravity: $\mathcal{S} = \mathcal{S}_\pi + \mathcal{S}_\chi$, where

$$\mathcal{S}_\chi = \int d^4x \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \chi_{,\mu} \chi_{,\nu} - V(\chi) \right). \quad (2.10)$$

In what follows, it is instructive to consider a somewhat more general case in which the additional scalar field is of k-essence type [30]:

$$S_\chi = \int d^4x \sqrt{-g} P(\chi, Y), \quad Y = g^{\mu\nu} \chi_{,\mu} \chi_{,\nu}. \quad (2.11)$$

In the absence of DHOST field, the stability conditions for k-essence in FLRW background with $Y = \dot{\chi}^2 \neq 0$ have the standard form $P_Y > 0$, $Q \equiv P_Y + 2Y P_{YY} > 0$, while the propagation speed squared of perturbations is

$$c_m^2 = \frac{P_Y}{Q}. \quad (2.12)$$

In the case of a conventional scalar χ described by the action (2.10), which is of primary interest, one has $c_m = 1$.

3 Rolling scalar and superluminality

We now consider DHOST Ia plus extra scalar theory and the background in which both $\dot{\pi}$ and $\dot{\chi}$ do not vanish (in particular, $Y = \dot{\chi}^2 \neq 0$). The expressions for $\mathcal{G}_T$ and $\mathcal{F}_T$ do not get modified, so the tensor perturbations do not become superluminal. On the contrary, the situation in the scalar sector changes dramatically. The non-vanishing background $\dot{\chi}$ induces mixing between the scalars $\tilde{\zeta}$ and $\delta\chi$ [7], so the unconstrained quadratic action in the scalar sector reads (modulo terms with less than two derivatives, see the complete expression in appendix)

$$S_{\pi+\chi}^{(2)\text{scalar}} = \int dt d^3x a^3 \left[G_{AB} \dot{v}^A \dot{v}^B - \frac{1}{a^2} F_{AB} \partial_i v^A \partial_i v^B \right], \quad (3.1)$$

where $A, B = 1, 2$, $v^1 = \tilde{\zeta}$, $v^2 = \delta\chi$. The matrices G_{AB} and F_{AB} have the following forms:

$$G_{AB} = \begin{pmatrix} \mathcal{G}_S + \frac{\mathcal{G}_T^2}{\Theta^2} Y Q & \dot{\chi} Q g \\ \dot{\chi} Q g & Q \end{pmatrix}, \quad F_{AB} = \begin{pmatrix} \mathcal{F}_S & \dot{\chi} P_Y f \\ \dot{\chi} P_Y f & P_Y \end{pmatrix}, \quad (3.2)$$

where

$$g = -\frac{\mathcal{G}_T}{\tilde{\Theta}} \left(1 - 3 \frac{P_Y}{Q} \Delta \right), \quad (3.3a)$$

$$f = -\frac{(\mathcal{G}_T + \mathcal{D}\dot{\pi} + \mathcal{F}_T \Delta)}{\tilde{\Theta}}. \quad (3.3b)$$

These expressions are valid for any Y . The two sound speeds squared are eigenvalues of the matrix $G^{-1}F$, i.e., they obey

$$\det \left(F_{AB} - c_S^2 G_{AB} \right) = 0. \quad (3.4)$$

We begin with the general case with $f \neq g$ and take Y to be small. We have to distinguish two situations. (i) If $c_m^2 \neq c_{S,0}^2$ (i.e., $P_Y/Q \neq \mathcal{F}_S/\mathcal{G}_S$), then one of the sound speeds is $c_{S,-} = c_{S,0} + \mathcal{O}(Y)$, while the other is given by

$$c_{S,+}^2 = c_m^2 \left(1 + \frac{Y(f-g)^2}{\mathcal{G}_S(c_m^2 - c_{S,0}^2)} \right) + \mathcal{O}(Y^2). \quad (3.5)$$

This means, in particular, that in the theory of conventional scalar field with $c_m = 1$ and subluminal DHOST Ia with $c_{S,0} < 1$, the mode which is predominantly $\delta\chi$ becomes superluminal at small but non-zero background values of Y . (ii) For $c_m^2 = c_{S,0}^2$, the sound speeds are given by

$$c_{S,\pm}^2 = c_m^2 \left[1 \pm \left(\frac{Y P_Y (f-g)^2}{\mathcal{G}_S} \right)^{1/2} \right] + \mathcal{O}(Y), \quad (3.6)$$

which again shows that in the case of conventional scalar field with $c_m = 1$ one of the modes becomes superluminal at small Y .

4 Exceptional DHOST Ia subclass

Formulas (3.5) and (3.6) suggest that the case $f = g$ is special. Indeed, making use of eq. (3.2) one finds that in this case the matrix $G^{-1}F$ is triangular *for any value of Y* , so that one of the sound speeds remains unmodified, $c_{S,+}^2 = c_m^2 = P_Y/Q$, while another is not necessarily superluminal ($c_{S,-} = c_{S,0} + \mathcal{O}(Y)$ for small Y). For luminal extra scalar with $c_m = 1$, the condition $f = g$ gives a constraint $\mathcal{D}\dot{\pi} = -(3\mathcal{G}_T + \mathcal{F}_T)\Delta$ on the DHOST Lagrangian, or, explicitly,

$$A_3 = \frac{2(A_1 - 2F_{2X})(A_1 X - 2F_2)}{X(3A_1 X - 4F_2)}. \quad (4.1)$$

This is the exceptional subclass of DHOST Ia theories in which adding extra luminal scalar field does not necessarily leads to superluminality. Note that this subclass includes the theory with $A_1 = 2F_{2X}$ and $A_3 = 0$, which is Horndeski, and does *not* include beyond Horndeski (GLPV) theories with $XA_3 = 2A_1 - 4F_{2X}$ (the latter relation is inconsistent with (4.1) for $\mathcal{G}_T \neq 0$, see eq. (2.7a)). This observation is in agreement with ref. [29].

5 Conclusion

We have shown that in a general DHOST Ia theory, even tiny amount of kinetic energy $Y > 0$ of the rolling scalar field χ with the Lagrangian (2.10) and flat-space sound speed $c_m = 1$ immediately induces superluminality of perturbations about the cosmological DHOST Ia background. This result is valid for any stable, spatially flat cosmological background in DHOST Ia theory with any choice of Lagrangian functions in the action (2.1), except for the special subclass of theories defined by eq. (4.1).

If one insists on the absence of superluminality, one has two logical possibilities. One of them is that in scalar-tensor theories with multiple scalars, none of the scalar fields has conventional kinetic term as long as there is at least one field of DHOST Ia type. Another is to stick to the exceptional subclass (4.1) of DHOST Ia theories and allow for conventional scalars. It remains to be understood, however, whether the second class of theories is consistent in non-cosmological backgrounds.

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A Linearized theory for DHOST Ia + k-essence system

In this appendix we present the calculation of the quadratic action for perturbations (3.1) for the system of DHOST Ia + k-essence described by the sum of actions (2.1) and (2.11). Our results here generalize the derivation given in ref. [29] for the beyond Horndeski subclass and agree with the existing results in refs. [7, 8] wherever there is an overlap.

We work with the ADM parametrization (2.4), (2.5) for perturbations about isotropic and homogeneous background with spatially flat geometry. The background has $\dot{\pi} \neq 0$ and $\dot{\chi} \neq 0$. We work in the unitary gauge $\delta\pi = 0$, while perturbation of k-essence $\delta\chi$ is non-zero.

We begin with the quadratic part of k-essence action (2.11):

$$S_\chi^{(2)} = \int dt d^3x a^3 \left[YQ\alpha^2 - 2\dot{\chi}Q\alpha\dot{\delta}\chi + 2\dot{\chi}P_Y\delta\chi\frac{\nabla^2\beta}{a^2} + Q\dot{\delta\chi}^2 - P_Y\frac{(\nabla\delta\chi)^2}{a^2} - 6\dot{\chi}P_Y\dot{\zeta}\delta\chi + (P_\chi - 2YP_{\chi Y})\alpha\delta\chi + \Omega\delta\chi^2 \right], \quad (\text{A.1})$$

where

$$Q = P_Y + 2YP_{YY},$$

$$\Omega = P_{\chi\chi}/2 - 3H\dot{\chi}P_{\chi Y} - YP_{\chi\chi Y} - \ddot{\chi}(P_{\chi Y} + 2YP_{\chi YY}).$$

Let us remind that hereafter we do not use the background equations of motion when deriving the terms with derivatives and also terms involving α and β only. However, we did use background equations of motion to obtain the terms without derivatives in (A.1) and, in particular, to see that the term $\zeta \cdot \delta\chi$ vanishes. There are also terms with ζ^2 and $\alpha\zeta$ in (A.1) which are not written because these terms vanish in the complete quadratic action $S_\pi^{(2)} + S_\chi^{(2)}$ upon using the background equations of motion. Anyway, we keep the non-derivative terms for completeness only; they are irrelevant for obtaining our main results.

Now we turn to the quadratic action for DHOST Ia sector, which in the unitary gauge reads:

$$S_\pi^{(2)} = \int dt d^3x a^3 \left[\left(\frac{\mathcal{G}_T}{8} (\dot{h}_{ik}^T)^2 - \frac{\mathcal{F}_T}{8a^2} (\partial_i h_{kl}^T)^2 \right) + (-3\mathcal{G}_T \dot{\zeta}^2 + \mathcal{F}_T \frac{(\nabla\zeta)^2}{a^2} - 2(\mathcal{G}_T + \mathcal{D}\dot{\pi})\alpha \frac{\nabla^2\zeta}{a^2} \right. \\ \left. + \Sigma\alpha^2 + 6\Theta\alpha\dot{\zeta} - 2\Theta\alpha \frac{\nabla^2\beta}{a^2} + 2\mathcal{G}_T\dot{\zeta} \frac{\nabla^2\beta}{a^2} \right) + \left(\Xi \frac{(\nabla\alpha)^2}{a^2} + \Gamma\dot{\alpha} \frac{\nabla^2\beta}{a^2} - 3\Gamma\dot{\alpha}\dot{\zeta} - \frac{3\Gamma^2}{4\mathcal{G}_T} \dot{\alpha}^2 \right) \right], \quad (\text{A.2})$$

where $(\nabla\zeta)^2 = \delta^{ij}\partial_i\zeta\partial_j\zeta$, $\nabla^2 = \delta^{ij}\partial_i\partial_j$, $\mathcal{G}_T$ and $\mathcal{F}_T$ are given by (2.7a) and (2.7b), respectively, and

$$\mathcal{D} = -2A_1\dot{\pi} + 4F_{2X}\dot{\pi}, \quad (\text{A.3a})$$

$$\Gamma = X(-2A_1 + 4F_{2X} + A_3X), \quad (\text{A.3b})$$

$$\Xi = \frac{\Gamma}{2\mathcal{G}_T^2} \left[8F_2^2 - 2F_2(5A_1 + 6F_{2X})X + (A_3F_2 + 16A_1F_{2X})X^2 \right], \quad (\text{A.3c})$$

$$\Theta = (-A_{3X} + A_5)\ddot{\pi}\dot{\pi}X^2 - \dot{\pi}(F_{2\pi} + \ddot{\pi}(-3A_1 + 6F_{2X})) - 2F_2H \\ + X(3A_1 + 2F_{2X})H + X^2 \left(-2A_{1X}H + \frac{3}{2}(4A_{1X} + A_3)H \right) \\ - \dot{\pi}X \left(\ddot{\pi} \left(-2A_{1X} + \frac{3}{2}A_3 - A_4 + 4F_{2XX} \right) + 2F_{2\pi X} + K_X \right), \quad (\text{A.3d})$$

$$\Sigma = 6F_2H^2 - 2\ddot{\pi}(A_{5\pi}X + A_{5XX}\ddot{\pi})\dot{\pi}^8 - \dot{\pi}^7(2A_{5X}\ddot{\pi} + 3A_{3\pi X}H + 6A_{5X}\ddot{\pi}H) \\ + 6\dot{\pi}[F_{2\pi}H - 2\ddot{\pi}(A_1 - 2F_{2X})H] + \dot{\pi}^6[-2(A_{3\pi X} + A_{4\pi X} + 4A_{5\pi})\ddot{\pi} \\ - (2A_{3XX} + 2A_{4XX} + 13A_5X)\ddot{\pi}^2 - 3A_{3X}\dot{H} - 3(4A_{1XX} + 3A_{3X})H^2] \\ + \dot{\pi}^2[-3(A_3 + A_4)\ddot{\pi}^2 - 12\dot{H}(-A_1 + 2F_{2X}) + F_X - 42F_{2X}H^2 - K_\pi] \\ + \dot{\pi}^4[-6(A_{3\pi} + A_{4\pi})\ddot{\pi} - (9A_{3X} + 9A_{4X} + 12A_5)\ddot{\pi}^2 - 3\dot{H}(-2A_{1X} + 3A_3 + 4F_{2XX}) + 2F_{XX} \\ - 36A_{1X}H^2 - 27A_3H^2 - 24F_{2XX}H^2 - K_{\pi X}] - \dot{\pi}^3\ddot{\pi}[6(A_3 + A_4)\ddot{\pi} \\ + 3\ddot{\pi}(2A_{1X} + 3A_3 + 6A_4 - 4F_{2XX})H + 6H(-2A_{1\pi} - F_{2\pi X} - 2K_X)] \\ - \dot{\pi}^5[2(A_{3X} + A_{4X} + 4A_5) + 3(A_{3X} + 2A_{4X} + 8A_5)\ddot{\pi}H + 3H(-2A_{1\pi X} + 3A_{3\pi} - 2K_{XX})]. \quad (\text{A.3e})$$

Here the functions A_4 and A_5 are given by (2.3b) and (2.3c), respectively. For the same reason as before we do not write terms with ζ^2 and $\alpha\zeta$: they cancel out in the total quadratic action upon using background equations of motion.

We keep the notation of the coefficients in (A.2) similar to that in ref. [29] to compare and contrast the quadratic actions for Horndeski and beyond Horndeski theories on the one hand, and DHOST Ia theories on the other. In particular, the terms in the last parentheses in eq. (A.2) are new as compared to the beyond Horndeski case, in which the coefficients Ξ and Γ vanish, cf. refs. [6, 28, 29].

Clearly, adding the scalar field χ to DHOST theory does not give anything qualitatively new for the tensor sector, so from now on we focus on the scalar sector. As a consequence of the degeneracy feature of DHOST Ia theory, the kinetic matrix for the scalar DOFs in the action (A.2) has vanishing determinant and, hence, the terms $\dot{\zeta}^2$, $\dot{\alpha}\dot{\zeta}$ and $\dot{\alpha}^2$ combine into perfect square. The latter property enables one to introduce, instead of ζ , a new variable, which is the dynamical scalar DOF in DHOST Ia:

$$\tilde{\zeta} = \zeta - \Delta\alpha, \quad (\text{A.4})$$

where

$$\Delta = -\frac{\Gamma}{2\mathcal{G}_T}, \quad (\text{A.5})$$

and its explicit expression is given by (2.8b). In terms of the new variable $\tilde{\zeta}$, the scalar part of the action (A.2) reads

$$\begin{aligned} \mathcal{S}_\pi^{(2)} = \int dt d^3x a^3 \bigg[& -3\mathcal{G}_T \dot{\tilde{\zeta}}^2 + \mathcal{F}_T \frac{(\nabla \tilde{\zeta})^2}{a^2} - 2(\mathcal{G}_T + \mathcal{D}\dot{\pi} + \mathcal{F}_T \Delta) \alpha \frac{\nabla^2 \tilde{\zeta}}{a^2} \\ & + \tilde{\Sigma} \alpha^2 + 6\tilde{\Theta} \alpha \dot{\tilde{\zeta}} - 2\tilde{\Theta} \alpha \frac{\nabla^2 \beta}{a^2} + 2\mathcal{G}_T \dot{\tilde{\zeta}} \frac{\nabla^2 \beta}{a^2} \bigg], \end{aligned} \quad (\text{A.6})$$

where

$$\tilde{\Theta} = \Theta - \mathcal{G}_T \dot{\Delta}, \quad (\text{A.7a})$$

$$\tilde{\Sigma} = \Sigma + 3\mathcal{G}_T \dot{\Delta}^2 + 6\tilde{\Theta} \dot{\Delta} - \frac{3}{a^3} \frac{d}{dt} \left[a^3 \left(\tilde{\Theta} + \mathcal{G}_T \dot{\Delta} \right) \Delta \right], \quad (\text{A.7b})$$

and Θ and Σ are explicitly given by (A.3d) and (A.3e). The coefficients $\tilde{\Theta}$ and $\tilde{\Sigma}$ play similar roles as the coefficients Θ and Σ , respectively, in (beyond) Horndeski case.

Now we combine DHOST Ia and k-essence components, and use the variable $\tilde{\zeta}$ instead of ζ in $S_\chi^{(2)}$, eq. (A.1). Variables α and β enter the quadratic action without time derivatives (the first term in the second line in (A.1), written in terms of $\tilde{\zeta}$, involves $\dot{\alpha}$, but this is taken care of by integration by parts), so the variation of the total quadratic action $S_\pi^{(2)} + S_\chi^{(2)}$ with respect to β and α gives two constraint equations:

$$\alpha = \frac{\mathcal{G}_T \dot{\tilde{\zeta}} + \dot{\chi} P_Y \delta\chi}{\tilde{\Theta}}, \quad (\text{A.8a})$$

$$\begin{aligned} \frac{(\nabla^2 \beta)}{a^2} = \frac{1}{\tilde{\Theta}} \bigg(& \left[\tilde{\Sigma} + Y Q \right] \alpha - (\mathcal{G}_T + \mathcal{D}\dot{\pi} + \mathcal{F}_T \Delta) \frac{(\nabla^2 \tilde{\zeta})}{a^2} + 3\tilde{\Theta} \dot{\tilde{\zeta}} \\ & - \left(1 - 3\frac{P_Y}{Q} \Delta \right) \dot{\chi} Q \delta\chi + \frac{1}{2} (P_\chi - 2Y P_{\chi Y} - 6\frac{d}{dt} (P_Y \dot{\chi}) \Delta) \delta\chi \bigg). \end{aligned} \quad (\text{A.8b})$$

Plugging these solutions for α and β back into the action $S_\pi^{(2)} + S_\chi^{(2)}$, we obtain the unconstrained quadratic action in terms of two dynamical variables $\tilde{\zeta}$ and $\delta\chi$:

$$S_{\pi+\chi}^{(2)} = \int dt d^3x a^3 \left[G_{AB} \dot{v}^A \dot{v}^B - \frac{1}{a^2} F_{AB} \nabla_i v^A \nabla^i v^B + \Psi_1 \dot{\tilde{\zeta}} \delta\chi + \Psi_2 \delta\chi^2 \right], \quad (\text{A.9})$$

where notations are the same as in (3.1), i.e., $A, B = 1, 2$ and $v^1 = \tilde{\zeta}$, $v^2 = \delta\chi$, while matrices G_{AB} and F_{AB} are given by (3.2), with $\mathcal{G}_S$, $\mathcal{F}_S$ defined in (2.7c) and (2.7d). Even though the coefficients Ψ_1 and Ψ_2 in (A.9) are irrelevant for the analysis of possible ghost and gradient instabilities, as well as (super)luminality, we give their explicit form here for completeness:

$$\Psi_1 = \frac{\mathcal{G}_T}{\tilde{\Theta}^2} \left[2\dot{\chi} P_Y (\tilde{\Sigma} + YR) + \tilde{\Theta} \left(P_\chi - 2Y P_{\chi Y} + \frac{6\Delta}{a^3} \frac{d}{dt} [a^3 P_Y \dot{\chi}] \right) \right], \quad (\text{A.10a})$$

$$\begin{aligned} \Psi_2 = & \Omega + \frac{\dot{\chi} P_Y}{\tilde{\Theta}^2} \left[\dot{\chi} P_Y (\tilde{\Sigma} + YR) + \tilde{\Theta} \left(P_\chi - 2Y P_{\chi Y} + \frac{6\Delta}{a^3} \frac{d}{dt} [a^3 P_Y \dot{\chi}] \right) \right] \\ & + \frac{6\Delta}{a^3} \frac{d}{dt} \left[\frac{Y P_Y (R - 3P_Y \Delta)}{\tilde{\Theta}} \right], \end{aligned} \quad (\text{A.10b})$$

where Ω is defined in (A.1).

Let us briefly discuss the stability conditions of the scalar sector in the combined system of DHOST Ia and k-essence, without assuming that the background value of Y is small. These conditions are somewhat more involved than those for the pure DHOST Ia, cf. eq. (2.6), and amount to positive definiteness of both kinetic matrices G_{AB} and F_{AB} in (A.9). Explicitly, one has

$$\begin{aligned} \mathcal{G}_S &> 0, & \mathcal{F}_S &> 0, \\ P_Y &> 0, & Q &> 0, \\ 1 + \Lambda\Delta &> 0, & \mathcal{F}_S - Y P_Y \frac{(\mathcal{G}_T + \mathcal{D}\dot{\pi} + \mathcal{F}_T \Delta)^2}{\tilde{\Theta}^2} &> 0, \end{aligned} \quad (\text{A.11})$$

where

$$\Lambda = \frac{6\mathcal{G}_T^2 Y P_Y}{\tilde{\Theta}^2 \mathcal{G}_S} \left(1 - \frac{3}{2} \frac{P_Y}{Q} \Delta \right).$$

The third line in (A.11) is characteristic of the combined theory, while the first two lines coincide with the stability conditions for separate DHOST Ia theory and k-essence, respectively. Note that similar results for beyond Horndeski subclass + k-essence in ref. [29] are recovered upon setting $\Delta = 0$ (see eq. (A.5) and recall that Γ vanishes in (beyond) Horndeski theories).

Finally, let us compare the scalar sound speeds $c_{S,\pm}^2$ in the general DHOST Ia + k-essence system with those in beyond Horndeski theory [29], still without assuming that Y is small. Sound speeds in DHOST Ia + k-essence theory are obtained from eq. (3.4):

$$c_{S\pm}^2 = \frac{1}{1 + \Lambda\Delta} \left(\frac{1}{2} (c_m^2 + \mathcal{A}) \pm \frac{1}{2} \sqrt{(c_m^2 - \mathcal{A})^2 + \mathcal{B}} \right), \quad (\text{A.12})$$

where

$$\mathcal{A} = \frac{\mathcal{F}_S}{\mathcal{G}_S} - \frac{Y P_Y}{\tilde{\mathcal{G}}_S} \frac{\mathcal{G}_T(\mathcal{G}_T + 2\mathcal{D}\dot{\pi} - \Delta\mathcal{M})}{\tilde{\Theta}^2}, \quad (\text{A.13a})$$

$$\mathcal{B} = 4c_m^2 \left(\frac{Y P_Y}{\mathcal{G}_S} \frac{(\mathcal{D}\dot{\pi} + \mathcal{F}_T\Delta)^2 + \Delta\mathcal{G}_T(\mathcal{M} + 2\mathcal{F}_T)}{\tilde{\Theta}^2} - \Lambda\Delta \left[\frac{\mathcal{F}_S}{\mathcal{G}_S} - \frac{Y P_Y}{\tilde{\mathcal{G}}_S} \frac{(\mathcal{G}_T + \mathcal{D}\dot{\pi} + \mathcal{F}_T\Delta)^2}{\tilde{\Theta}^2} \right] \right), \quad (\text{A.13b})$$

and

$$\mathcal{M} = 6(\mathcal{G}_T + \mathcal{D}\dot{\pi} + \mathcal{F}_T\Delta)c_m^2 - 2\mathcal{F}_T. \quad (\text{A.14})$$

As we mentioned above, to restore the beyond Horndeski case it is sufficient to set $\Delta = 0$ in eqs. (A.13). We see that there is a considerable difference between the beyond Horndeski and general DHOST Ia cases: in the former, the pre-factor in (A.12) is equal to 1, while the coefficient $\mathcal{B}$ reads (cf. ref. [29])

$$\mathcal{B} = 4c_m^2 \frac{Y P_Y}{\mathcal{G}_S} \frac{(\mathcal{D}\dot{\pi})^2}{\Theta^2}, \quad (\text{A.15})$$

i.e., it is manifestly positive for stable and rolling background ($\mathcal{G}_S, P_Y > 0$ and $Y > 0$); then eq. (A.12) immediately gives $c_{S+}^2 > c_m^2$, which implies superluminality for $c_m = 1$. For DHOST Ia the situation is less transparent: even if the stability conditions are satisfied, the coefficient $\mathcal{B}$ in eq. (A.13b) is not necessarily positive and the pre-factor in (A.12) is different from 1. So, for finite values of Y , the (super)luminality issue is not straightforwardly analyzed.

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