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New bi-harmonic superspace formulation of $4D$, $\mathcal{N} = 4$ SYM theory

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Abstract: We develop a novel bi-harmonic $\mathcal{N} = 4$ superspace formulation of the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) in four dimensions. In this approach, the $\mathcal{N} = 4$ SYM superfield constraints are solved in terms of on-shell $\mathcal{N} = 2$ harmonic superfields. Such an approach provides a convenient tool of constructing the manifestly $\mathcal{N} = 4$ supersymmetric invariants and further rewriting them in $\mathcal{N} = 2$ harmonic superspace. In particular, we present $\mathcal{N} = 4$ superfield form of the leading term in the $\mathcal{N} = 4$ SYM effective action which was known previously in $\mathcal{N} = 2$ superspace formulation.

Keywords: Extended Supersymmetry, Superspaces, Supersymmetric Effective Theories, Supersymmetric Gauge Theory

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Dedicated to S. James Gates Jr. on the occasion of his 70th birthday.

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Contents

1 Introduction

The $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theory in four-dimensional Minkowski space exhibits many remarkable properties on both the classical and the quantum levels. Apparently, it is the most symmetric field theoretical model in physics known to date. The model is gauge invariant, has the maximal extended rigid supersymmetry (that is, the maximal spin in the relevant gauge supermultiplet is equal to one) and possesses R-symmetry $\mathrm{SU}(4) \sim \mathrm{SO}(6)$, as well as the whole $\mathrm{PSU}(2,2|4)$ superconformal symmetry. As its most remarkable property, it is a finite quantum field theory free from any anomalies. $\mathcal{N}=4$ SYM theory bears a close relation with string/brane theory and is a key object of the modern AdS/CFT activity. Nowadays, various (classical and quantum) aspects of this theory remain a subject of intensive study. Of particular interest is working out the relevant superspace approaches highlighting one or another side of the rich symmetry structure of this theory.

$\mathcal{N}=4$ SYM theory was originally deduced in a component formulation via dimensional reduction from the ten-dimensional $\mathcal{N}=1$ SYM theory [1] and proceeding from the dual spinor model [2].[1] Subsequently, it was further developed and used by many authors under diverse angles (see, e.g., the reviews [4–11] and books [12–16]). The on-shell field content of the theory amounts to the vector field, six real scalar fields and four Weyl spinor fields, all being in the adjoint representation of the gauge group. These fields can be combined into $\mathcal{N}=1$ superfields encompassing the vector multiplet and three chiral multiplets (see, e.g., [12, 15]), into conventional $\mathcal{N}=2$ superfields or into $\mathcal{N}=2$ harmonic superfields [14, 17] encompassing the relevant vector multiplet and a hypermultiplet. In the component description, all four supersymmetries of the theory under consideration are on-shell and hidden. In the $\mathcal{N}=1$ superfield description, one out of four supersymmetries is manifest and the other three are still hidden and on-shell. In the $\mathcal{N}=2$ conventional superfield description, all supersymmetries are on-shell and two of them are manifest. When $\mathcal{N}=2$ harmonic superfields are used, two supersymmetries are manifest and off-shell, the other two are hidden. The superfield formulation in terms of unconstrained $\mathcal{N}=4$ superfields, with all four supersymmetries being manifest and off-shell, is as yet unknown.[2]

There are at least two large domains of tasks where the manifestly supersymmetric formulations are of crucial importance. The first one is related to quantum calculations in supersymmetric field theories. The manifest supersymmetry provides tools to keep these calculations under an efficient control and to write down all admissible contributions to the quantum effective action of the given theory up to some numerical coefficients. The second circle of problems is associated with the analysis of the low-energy dynamics in the string/brane theory. It is desirable to explicitly know the admissible supersymmetric invariants, in four or higher dimensions, describing the low-energy string/brane interactions.

[1] See also ref. [20] in [2]. The necessary ingredients for constructing the component action are contained in [3] where, in particular, the term “hypermultiplet” was introduced.

[2] There is also an $\mathcal{N}=3$ superfield formulation manifesting 3 out of 4 underlying supersymmetries [18]. It is based upon a somewhat complicated techniques of $\mathcal{N}=3$ harmonic superspace which by now have sill not been enough developed, especially in the quantum domain (see, however, a recent ref. [19]).

In particular, the low-energy D3-brane interactions are described in terms of $\mathcal{N}=4$ SYM theory, so it is useful to have a list of all possible $\mathcal{N}=4$ invariant functionals containing the vector fields. In all cases, we need to be aware of a general technique of constructing supersymmetric invariants. In $\mathcal{N}=4$ SYM theory, this problem is most complicated just because no manifestly $\mathcal{N}=4$ supersymmetric formulation of this theory is available by now.

Taking into account the lack of such a general off-shell description, for setting up $\mathcal{N}=4$ supersymmetric invariants, especially in the context of quantum effective actions, there were worked out the approaches employing various harmonic superspaces with the lesser number of manifest supersymmetries (see the review [10] and reference therein). These approaches allow one to construct *on-shell* $\mathcal{N}=4$ supersymmetric invariants, which display the manifest $\mathcal{N}=2$ supersymmetry and an additional hidden $\mathcal{N}=2$ supersymmetry. In some special cases only the hidden supersymmetry proves to have an on-shell closure, while for checking manifest supersymmetry the equations of motions are not needed.

In this paper we propose a new method to construct the on-shell $\mathcal{N}=4$ superinvariants. As the starting point, we rewrite the standard superspace constraints of $\mathcal{N}=4$ SYM theory [20] in $\mathcal{N}=4$ bi-harmonic superspace involving two independent sets of harmonic SU(2) variables. This form of the $4D,\mathcal{N}=4$ SYM constraints still preserves the manifest $\mathcal{N}=4$ supersymmetry. Our crucial observation is that these constraints can be solved explicitly in terms of few $4D,\mathcal{N}=2$ harmonic superfields subjected to some on-shell constraints. As a result, proceeding from the manifestly $\mathcal{N}=4$ supersymmetric invariants we can express, in a simple way, these invariants in terms of $\mathcal{N}=2$ superfields.[3] Our approach is $4D,\mathcal{N}=4$ counterpart of a similar method worked out in [24] for $\mathcal{N}=(1,1)$ SYM theory in 6 dimensions.

Effective actions play an important role in quantum field theory. The low-energy effective action of $\mathcal{N}=4$ SYM theory is not an exception. According to [16], it can be matched with the effective action of a D3-brane propagating in the AdS_5 background. The bi-harmonic superspace approach can be used to find the low-energy effective action. Using our method, we reconstruct not only the known result for the leading term in the effective action [25, 26], but also present some higher-order supersymmetric invariants which can hopefully be identified with the next parts of the derivative expansion of the effective action.

The paper is organized as follows. In section 2 we recall the basics of $\mathcal{N}=2$ harmonic superspace approach, including the formulation of $\mathcal{N}=4$ SYM theory in this superspace. Also the expression for the leading low-energy effective action is presented, and the method for its calculation is briefly outlined. Sections 3 and 4 are devoted to the definition of basics of bi-harmonic $\mathcal{N}=4$ superspace and its implications in $\mathcal{N}=4$ SYM theory. The constraints of $\mathcal{N}=4$ SYM theory are rewritten in bi-harmonic superspace in section 3 and then are solved in section 4. Different $\mathcal{N}=2$ superfields which specify this solution are matched with the objects defined in section 2. In section 5 it is shown how the bi-

[3] Another type of the bi-harmonic approach to $\mathcal{N}=4$ SYM was worked out in [21]. As distinct from the one used here, it does not allow a direct passing to $\mathcal{N}=2$ superfields. An interpretation of the on-shell $\mathcal{N}=4$ SYM constraints in the harmonic superspace with the $\frac{\mathrm{SU}(4)}{[\mathrm{SU}(2)\times\mathrm{SU}(2)'\times\mathrm{U}(1)]}$ harmonic part [22] was suggested in [23] for finding out the restrictions imposed by superconformal symmetry on some correlation functions of the $\mathcal{N}=4$ SYM superfield strengths.

harmonic superspace approach allows one to construct $\mathcal{N}=4$ supersymmetric invariants in terms of $\mathcal{N}=2$ harmonic superfields. In addition, we show that the low-energy effective action given earlier in $\mathcal{N}=2$ harmonic superspace can be rewritten, in a rather simple form, in terms of the bi-harmonic superspace quantities. The main results of our work are summarized in Conclusions. Appendices A and B collect some technical details.

2 4*D*, $\mathcal{N}=4$ SYM theory in harmonic superspace

The bi-harmonic superspace we are going to deal with is an extension of $\mathcal{N}=2$ harmonic superspace. Therefore, we start by giving here some basic facts about $\mathcal{N}=2$ harmonic superspace and superfields. For more details, see refs. [14, 17].

2.1 Brief survey of $\mathcal{N}=2$ harmonic superspace

The standard $4D$, $\mathcal{N}=2$ superspace amounts to the set of coordinates

$$z^M = (x^m, \theta^\alpha_i, \bar\theta^{\dot\alpha i}), \tag{2.1}$$

where x^m, $m=0,1,2,3$, are the Minkowski space coordinates and θ^α_i, $\bar\theta^{\dot\alpha i}$, $i=1,2$, $\alpha,\dot\alpha=1,2$, are spinor Grassmann coordinates.

In order to pass to harmonic superspace we add to these coordinates the harmonics $u^{\pm i}$ $(u^-_i=(u^{+i})^*,\ u^{+i}u^-_i=1)$ which represent the "harmonic sphere" $\mathrm{SU}(2)_R/\mathrm{U}(1)$, with $\mathrm{SU}(2)_R$ being the R-symmetry group realized on the doublet indices $i,\ k$. The $4D$, $\mathcal{N}=2$ harmonic superspace in the central basis is defined as the enlarged coordinate set

$$Z=(z,u)=(x^m,\theta^\alpha_i,\bar\theta^{\dot\alpha i},u^{\pm i}). \tag{2.2}$$

In the analytic basis it is parametrized by the coordinates

$$Z_{\rm an} = (x^m_{\rm an},\theta^\pm_\alpha,\bar\theta^\pm_{\dot\alpha},u^{\pm i}), \tag{2.3}$$

$$x^m_{\rm an} = x^m - 2i\theta^{(i}\sigma^m\bar\theta^{j)}u^+_iu^-_j\,,\quad \theta^\pm_\alpha=u^\pm_i\theta^i_\alpha,\quad \bar\theta^\pm_{\dot\alpha}=u^\pm_i\bar\theta^i_{\dot\alpha}. \tag{2.4}$$

The crucial feature of the analytic basis is that it manifests the existence of subspace involving only half of the original Grassmann coordinates

$$\zeta=(x^m_{\rm an},\theta^+_\alpha,\bar\theta^+_{\dot\alpha},u^{\pm i}), \tag{2.5}$$

such that it closed under $4D$, $\mathcal{N}=2$ supersymmetry transformations. The set (2.5) parametrizes what is called the "harmonic analytic superspace".

The important ingredients of the harmonic superspace approach are the spinor and harmonic derivatives. In the analytic basis, they are defined as

$$\begin{aligned}
D^+_\alpha &= \frac{\partial}{\partial\theta^{-\alpha}}, & \bar D^+_{\dot\alpha} &= \frac{\partial}{\partial\bar\theta^{-\dot\alpha}},\\
D^-_\alpha &= -\frac{\partial}{\partial\theta^{+\alpha}}+2i\bar\theta^{-\dot\alpha}\partial_{\alpha\dot\alpha}, & \bar D^-_{\dot\alpha} &= -\frac{\partial}{\partial\bar\theta^{+\dot\alpha}}+2i\theta^{-\alpha}\partial_{\alpha\dot\alpha},
\end{aligned}$$

$$\begin{aligned}
D^{++} &= u^{+i}\frac{\partial}{\partial u^{-i}}-2i\theta^{+\alpha}\bar\theta^{+\dot\alpha}\partial_{\alpha\dot\alpha}+\theta^{+\alpha}\frac{\partial}{\partial\theta^{-\alpha}}+\bar\theta^{+\dot\alpha}\frac{\partial}{\partial\bar\theta^{-\dot\alpha}},\\
D^{--} &= u^{-i}\frac{\partial}{\partial u^{+i}}-2i\theta^{-\alpha}\bar\theta^{-\dot\alpha}\partial_{\alpha\dot\alpha}+\theta^{-\alpha}\frac{\partial}{\partial\theta^{+\alpha}}+\bar\theta^{-\dot\alpha}\frac{\partial}{\partial\bar\theta^{+\dot\alpha}}.
\end{aligned} \tag{2.6}$$

They are related to the spinor derivatives in the central basis,

$$D^i_\alpha = \frac{\partial}{\partial\theta^\alpha_i} + i\bar\theta^{\dot\alpha i}\partial_{\alpha\dot\alpha}, \qquad \bar D_{\dot\alpha i} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha i}} - i\theta^\alpha_i\partial_{\alpha\dot\alpha}\,, \tag{2.7}$$

as

$$D^\pm_\alpha = D^i_\alpha u^\pm_i\,, \qquad \bar D^\pm_{\dot\alpha} = \bar D^i_{\dot\alpha}u^\pm_i\,. \tag{2.8}$$

The harmonic derivatives $D^{\pm\pm}$, together with the harmonic U(1) charge operator

$$D^0 = u^{+i}\frac{\partial}{\partial u^{+i}} - u^{-i}\frac{\partial}{\partial u^{-i}} + \theta^{+\alpha}\frac{\partial}{\partial\theta^{+\alpha}} + \bar\theta^{+\dot\alpha}\frac{\partial}{\partial\bar\theta^{+\dot\alpha}} - \theta^{-\alpha}\frac{\partial}{\partial\theta^{-\alpha}} - \bar\theta^{-\dot\alpha}\frac{\partial}{\partial\bar\theta^{-\dot\alpha}}\,,$$

form an SU(2) algebra,

$$[D^{++}, D^{--}] = D^0\,, \quad [D^0, D^{\pm\pm}] = \pm 2D^{\pm\pm}\,. \tag{2.9}$$

In the central basis, the harmonic derivatives are simply

$$D^{\pm\pm} = \partial^{\pm\pm} = u^{\pm i}\frac{\partial}{\partial u^{\mp i}}\,, \quad D^0 = \partial^0 = u^{+i}\frac{\partial}{\partial u^{+i}} - u^{-i}\frac{\partial}{\partial u^{-i}}\,. \tag{2.10}$$

Of course, the (super)algebra of the spinor and harmonic derivatives does not depend on the choice of basis in $\mathcal{N}=2$ harmonic superspace.

The harmonic superfields (as well as the harmonic projections of the spinor covariant derivatives) carry a definite integer harmonic U(1) charges, $D^0\Phi^q(Z) = q\Phi^q(Z)$, $[D^0, D^\pm_{\alpha,\dot\alpha}] = \pm D^\pm_{\alpha,\dot\alpha}$. The harmonic U(1) charge is assumed to be strictly preserved in any superfield action defined on the superspaces (2.3) or (2.5). Just due to this requirement, all invariants are guaranteed to depend just on two parameters of the "harmonic sphere" $\mathrm{SU}(2)_R/\mathrm{U}(1)$.

In addition, we will use the identities

$$\begin{aligned}
&(\theta^\pm)^2 = \theta^{\pm\alpha}\theta^\pm_\alpha, && (\bar\theta^\pm)^2 = \bar\theta^\pm_{\dot\alpha}\bar\theta^{\pm\dot\alpha},\\
&(D^\pm)^2 = D^{\pm\alpha}D^\pm_\alpha, && (\bar D^\pm)^2 = \bar D^\pm_{\dot\alpha}\bar D^{\pm\dot\alpha},\\
&(D^+)^4 = \frac{1}{16}(D^+)^2(\bar D^+)^2, && (D^-)^4 = \frac{1}{16}(D^-)^2(\bar D^-)^2,
\end{aligned} \tag{2.11}$$

and the following definition of the integration measures over the total harmonic superspace and its analytic subspace

$$dud^{12}z = d^4x du (D^+)^4(D^{-4}), \qquad d\zeta^{-4} = d^4x_{\mathrm{an}} du (D^{-4}). \tag{2.12}$$

The "shortness" of the spinor derivatives $D^+_\alpha, \bar D^+_{\dot\alpha}$ in the analytic basis (2.6) reflects the existence of the analytic harmonic subspace (2.5) in the general harmonic superspace (2.3): one can define an analytic $\mathcal{N}=2$ superfield by imposing the proper covariant "Grassmann analyticity" constraints on a general harmonic superfield, viz., $D^+_\alpha\Phi^q(Z) = \bar D^+_{\dot\alpha}\Phi^q(Z) = 0 \;\Rightarrow\; \Phi^q(Z) = \varphi^q(\zeta)$. The harmonic derivative D^{++} commutes with these spinor derivatives and so possesses a unique property of preserving the Grassmann harmonic analyticity: $D^{++}\Phi^q(Z)$ is an analytic superfield if $\Phi^q(Z)$ is.

2.2 $\mathcal{N}=4$ SYM action

When formulated in $\mathcal{N}=2$ harmonic superspace, $\mathcal{N}=4$ vector gauge multiplet can be viewed as a "direct sum" of gauge $\mathcal{N}=2$ superfield $\mathcal{V}^{++}$ and the hypermultiplet superfield $q_A^+=(q^+,-\tilde{q}^+)$. Both these superfields are analytic,

$$D^+_\alpha \mathcal{V}^{++}=\bar{D}^+_{\dot\alpha}\mathcal{V}^{++}=0\,,\quad D^+_\alpha q^+_A=\bar{D}^+_{\dot\alpha}q^+_A=0\,, \tag{2.13}$$

and belong to the same adjoint representation of the gauge group.

$\mathcal{N}=2$ gauge multiplet $\mathcal{V}^{++}$ is described by the classical action [24]

$$S^{\mathcal{N}=2}_{\text{SYM}}=\frac{1}{2}\sum_{n=2}^{\infty}\text{tr}\frac{(-i)^n}{n}\int d^{12}z du_1\dots du_n\frac{\mathcal{V}^{++}(z,u_1)\dots\mathcal{V}^{++}(z,u_n)}{(u_1^+u_2^+)\dots(u_n^+u_1^+)}, \tag{2.14}$$

where $d^{12}z=d^4xd^8\theta$ and the harmonic distributions $1/(u_1^+u_2^+),\cdots$ are defined in [14].

This action yields the following equations of motion

$$(D^+)^2\mathcal{W}=0,\qquad (\bar{D}^+)^2\bar{\mathcal{W}}=0\,, \tag{2.15}$$

where D^+_α and $\bar{D}^+_{\dot\alpha}$ were defined in (2.6) and $\mathcal{W}$, $\bar{\mathcal{W}}$ are the chiral and antichiral gauge superfield strengths ,

$$\mathcal{W}=-\frac{1}{4}(\bar{D}^+)^2\mathcal{V}^{--},\qquad \bar{\mathcal{W}}=-\frac{1}{4}(D^+)^2\mathcal{V}^{--}, \tag{2.16}$$

with $\mathcal{V}^{--}$ being a non-analytic harmonic gauge connection related to $\mathcal{V}^{++}$ by the harmonic flatness condition

$$D^{++}\mathcal{V}^{--}-D^{--}\mathcal{V}^{++}+i[\mathcal{V}^{++},\mathcal{V}^{--}]=0 \iff [\nabla^{++},\,\nabla^{--}]=0\,, \tag{2.17}$$

$$\nabla^{\pm\pm}:=D^{\pm\pm}+i[\mathcal{V}^{\pm\pm},\ .]. \tag{2.18}$$

Note that in the considered "λ" frame, in which the gauge group is represented by transformations with the manifestly analytic gauge parameters, the spinor derivatives D^+_α and $\bar{D}^+_{\dot\alpha}$ require no gauge connection terms as they are gauge-covariant on their own right. The gauge-covariant derivatives ∇^-_α and $\bar\nabla^-_{\dot\alpha}$ are defined as

$$\nabla^-_\alpha:=[\nabla^{--},D^+_\alpha]\,,\qquad \bar\nabla^-_{\dot\alpha}:=[\nabla^{--},\bar{D}^+_{\dot\alpha}]\,. \tag{2.19}$$

Using these definitions and the relation (2.17), one can check that

$$\bar\nabla^-_{\dot\alpha}\mathcal{W}=0\,,\qquad \nabla^-_\alpha\bar{\mathcal{W}}=0\,, \tag{2.20}$$

while the rest of (anti)chirality conditions, $\bar{D}^+_{\dot\alpha}\mathcal{W}=D^+_\alpha\bar{\mathcal{W}}=0$, directly follows from the definition (2.16). It also follows from the definition (2.16) that the superfield strengths $\mathcal{W},\bar{\mathcal{W}}$ satisfy the reality condition

$$(D^+)^2\mathcal{W}=(\bar{D}^+)^2\bar{\mathcal{W}}\,, \tag{2.21}$$

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as well as the conditions of the covariant harmonic independence

$$\nabla^{\pm\pm}\mathcal{W} = \nabla^{\pm\pm}\bar{\mathcal{W}} = 0\,. \tag{2.22}$$

Let us point out that both the (anti)chirality of the superfield strengths and the constraints (2.21), (2.22) hold off shell, as the consequences of the definition (2.16), the flatness condition (2.17) and the analyticity of the gauge connection $\mathcal{V}^{++}$, eq. (2.13). Using (2.21), one can cast the equations of motion (2.15) into an equivalent form

$$F^{++} = 0\,, \quad F^{++} := \frac{1}{16}\left(D^+\right)^2\left(\bar{D}^+\right)^2\mathcal{V}^{--}\,. \tag{2.23}$$

The superfield F^{++} is analytic and satisfies the off-shell constraint $\nabla^{++}F^{++} = 0\,.$

The classical action for the hypermultiplet in the adjoint representation reads [14, 17]

$$S_q = \frac{1}{2}\mathrm{tr}\int d\zeta^{-4} q^+_A \nabla^{++} q^{+A} = \frac{1}{2}\mathrm{tr}\int d\zeta^{-4} q^+_A\left(D^{++}q^{+A} + i[\mathcal{V}^{++}, q^{+A}]\right). \tag{2.24}$$

The action of $\mathcal{N}=4$ SYM theory in $\mathcal{N}=2$ harmonic superspace is the sum of the actions (2.14) and (2.24),

$$S^{\mathcal{N}=4}_{\mathrm{SYM}} = S^{\mathcal{N}=2}_{\mathrm{SYM}} + S_q. \tag{2.25}$$

This action yields the equations of motion

$$\nabla^{++}q^+_A = 0, \qquad F^{++} = -i[q^{+A}, q^+_A]\,, \tag{2.26}$$

where the second equation is just the modification of (2.23) by the hypermultiplet source term.

The total action (2.25) is manifestly $\mathcal{N}=2$ supersymmetric by construction. Also, it is invariant under the hidden $\mathcal{N}=2$ supersymmetry transformations which complement the manifest $\mathcal{N}=2$ supersymmetry to the full $\mathcal{N}=4$ supersymmetry

$$\delta\mathcal{V}^{++} = \left[\epsilon^{A\alpha}\theta^+_\alpha - \bar{\epsilon}^A_{\dot{\alpha}}\bar{\theta}^{+\dot{\alpha}}\right]q^+_A, \quad \delta q^+_A = -\frac{1}{32}(D^+)^2(\bar{D}^+)^2\left[\epsilon^\alpha_A\theta^-_\alpha\mathcal{V}^{--} + \bar{\epsilon}_{A\dot{\alpha}}\bar{\theta}^{-\dot{\alpha}}\mathcal{V}^{--}\right], \tag{2.27}$$

with $\bar{\epsilon}_{A\dot{\alpha}}$ and ϵ^α_A as new anticommuting parameters. The algebra of these transformations is closed modulo terms proportional to the classical equations of motion. Only the manifest $\mathcal{N}=2$ supersymmetry in (2.25) is off-shell closed.

We also note that the actions (2.14), (2.24) and hence their sum (2.25) are manifestly invariant under the automorphism group $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_{PG} \times \mathrm{U}(1)_R$. The group $\mathrm{SU}(2)_{PG}$ acts on the doublet indices of the hypermultiplet superfield q^+_A and commutes with manifest $\mathcal{N}=2$ supersymmetry (but forms a semi-direct product with the hidden supersymmetry (2.27)), while $\mathrm{U}(1)_R$ acts as a phase transformation of the Grassmann variables and covariant spinor derivatives. It forms a semi-direct product with both types of supersymmetry, like the R-symmetry group $\mathrm{SU}(2)_R$.

2.3 The leading low-energy effective action in $\mathcal{N} = 4$ SYM theory

The $\mathcal{N} = 4$ supersymmetric leading low-energy effective action is the exact contribution to the quantum effective action of $\mathcal{N} = 4$ SYM theory in the Coulomb phase (see, e.g., reviews [9, 10] and references therein). From a formal point of view, such an action is some on-shell $\mathcal{N} = 4$ supersymmetric invariant constructed out of the abelian $\mathcal{N} = 2$ superfields $\mathcal{V}^{++}$ and q^+_A belonging to the Cartan subalgebra of the gauge group. All other components of these superfields, being "heavy", in the Coulomb phase can be integrated out in the relevant functional integral and so do not contribute to the effective action. As we will see, such invariants can be written easily enough in terms of bi-harmonic superfields. However, before doing this, we will briefly remind how such an $\mathcal{N} = 4$ invariant is written through $\mathcal{N} = 2$ harmonic superfields, limiting ourselves, for simplicity, to the gauge group SU(2).

Construction of the leading low-energy effective action in $\mathcal{N} = 4$ SYM theory begins, as a starting point, from $\mathcal{N} = 2$ invariant low-energy effective action $\mathcal{S}_{\rm eff}$ written through the non-holomorphic effective potential $\mathcal{H}(W, \bar{W})$ in the form:

$$\mathcal{S}_{\rm eff} = \int d^{12}z du\, \mathcal{H}(W, \bar{W}), \qquad \mathcal{H}(\mathcal{W}, \bar{\mathcal{W}}) = c \ln\left(\frac{\mathcal{W}}{\Lambda}\right) \ln\left(\frac{\bar{\mathcal{W}}}{\Lambda}\right), \tag{2.28}$$

where Λ is an arbitrary scale,[4] the $\mathcal{W}, \bar{\mathcal{W}}$ satisfy the equations of motion (2.15) and c is some constant. The non-holomorphic effective potential was studied and the constant c was calculated in many papers by various methods (see the reviews [8–10] and references therein).

The complete leading low-energy $\mathcal{N} = 4$ SYM effective action is an extension of the effective action (2.28) by some hypermultiplet-dependent terms, such that the result is invariant under the hidden $\mathcal{N} = 2$ supersymmetry transformations (2.27). It was computed in a closed form in [25, 26] and reads

$$\Gamma = \int d^{12}z du \left[c \ln\left(\frac{\mathcal{W}}{\Lambda}\right) \ln\left(\frac{\bar{\mathcal{W}}}{\Lambda}\right) + \mathcal{L}\left(-2\frac{q^{+A}q^-_A}{\mathcal{W}\bar{\mathcal{W}}}\right)\right], \tag{2.29}$$

with

$$\mathcal{L}(Z) = c\sum_{n=1}^{\infty} \frac{Z^n}{n^2(n+1)} = c\left[(Z-1)\frac{\ln(1-Z)}{Z} + \mathrm{Li}_2(Z) - 1\right], \tag{2.30}$$

where $\mathrm{Li}_2(Z)$ is the Euler dilogarithm function. The part dependent on the hypermultiplet q^{+A} is fixed, up to the numerical coefficient c, by the requirement that the effective action Γ be invariant under both manifest $\mathcal{N} = 2$ supersymmetry and hidden on-shell $\mathcal{N} = 2$ supersymmetry. As a result, the effective action (2.29) is invariant of the total $\mathcal{N} = 4$ supersymmetry and depends on all fields of the abelian $\mathcal{N} = 4$ vector multiplet. The coefficient c should be the same in both the gauge field sector and the hypermultiplet sector of the low-energy effective action due to $\mathcal{N} = 4$ supersymmetry. This was confirmed in [26] by the direct quantum supergraph calculation. The precise value of c will be of no interest for our further consideration.

[4] In fact, the action does not depend on Λ in virtue of the (anti)chirality of $(\bar{\mathcal{W}})\mathcal{W}$.

The expressions (2.28) and (2.29) will be used in what follows in order to demonstrate the power of $\mathcal{N}=4$ bi-harmonic superspace method which automatically and in a rather simple way yields the on-shell $\mathcal{N}=4$ invariant (2.29), (2.30).

3 $\mathcal{N}=4$ bi-harmonic superspace and superfields

This and subsequent sections deal with the construction and the applications of an extended $\mathcal{N}=4$ bi-harmonic superspace and the relevant bi-harmonic superfields.

3.1 Bi-harmonic superspace

The standard $\mathcal{N}=4$ superspace involves the coordinates

$$z^M = (x^m, \theta^\alpha_I, \bar\theta^{\dot\alpha I}), \tag{3.1}$$

where x^m, $m=0,1,2,3$ are the Minkowski space coordinates, while θ^α_I and $\bar\theta^{\dot\alpha I}$, $I=1,\ldots,4$, $\alpha,\dot\alpha=1,2$ are anticommuting Grassmann coordinates. They transform in the fundamental representation of the $\mathcal{N}=4$ R-symmetry group U(4) acting on the index I.

The spinor derivatives in the central basis are defined as

$$D^I_\alpha = \frac{\partial}{\partial\theta^\alpha_I} + i\bar\theta^{\dot\alpha I}\partial_{\alpha\dot\alpha}, \qquad \bar D_{\dot\alpha I} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha I}} - i\theta^\alpha_I\partial_{\alpha\dot\alpha}\,. \tag{3.2}$$

Like in the $\mathcal{N}=2$ case, passing to the bi-harmonic extension of (3.1) allows one to make manifest Grassmann analyticity with respect to some set of spinorial coordinates.

In order to introduce SU(2) harmonics we reduce the R-symmetry group SU(4) to SU(2) × SU(2) × U(1) in the following way: we substitute the index I by two indices $i, A = 1,2$ according to the rule

$$\begin{aligned} &I=1 \Leftrightarrow i=1, \qquad I=2 \Leftrightarrow i=2, \\ &I=3 \Leftrightarrow A=1, \qquad I=4 \Leftrightarrow A=2\,. \end{aligned} \tag{3.3}$$

The first SU(2) acts on the indices i, j and coincides with $\mathrm{SU(2)}_R$, while the second SU(2) acts on the indices A, B and will be identified with $\mathrm{SU(2)}_{PG}$ of section 2. The indices i or A are raised and lowered according to the ordinary SU(2) rules, using the antisymmetric tensors ϵ^{ij}, ϵ^{AB} and ϵ_{ij}, ϵ_{AB}. The extra U(1) will be identified with $\mathrm{U(1)}_R$ of the previous section. It transforms $\theta^{\alpha\, i}$ and $\hat\theta^{\alpha\, A}$ by the mutually conjugated phase factors.[5]

As the next step we introduce two sets of the harmonic variables $u^\pm_i$ and $v^{\hat\pm}_A$, which parametrize these two SU(2) groups. Respectively, the full set of the $\mathcal{N}=4$ superspace coordinates is extended to

$$\hat Z = \left(x^m, \theta^{\alpha i}, \bar\theta^{\dot\alpha i}, \hat\theta^{\alpha A}, \bar{\hat\theta}^{\dot\alpha A}, u^\pm_i, v^{\hat\pm}_A\right). \tag{3.4}$$

[5]The alternative bi-harmonic superspace of [21] corresponds to the principal embedding of SO(4) ∼ SU(2) × SU(2) in SU(4), such that the Grassmann variables are organized into a complex 4-vector of SO(4) ⊂ SU(4) while the harmonics are still associated with the left and right SU(2) factors. In our case we deal with the diagonal embedding of SU(2) × SU(2) in SU(4).

The analytic basis of this bi-harmonic superspace is defined as the set of the coordinates

$$\hat{Z}_{\rm an} = \left(x^m_{\rm an}, \theta^{\pm}_{\alpha}, \bar{\theta}^{\pm}_{\dot{\alpha}}, \theta^{\hat{\pm}}_{\alpha}, \bar{\theta}^{\hat{\pm}}_{\dot{\alpha}}, u^{\pm}_i, v^{\hat{\pm}}_A\right), \tag{3.5}$$

where[6]

$$\begin{aligned}
\theta^{\pm\alpha} &= \theta^{\alpha i}u^{\pm}_i, \qquad \theta^{\hat{\pm}\alpha} = \hat{\theta}^{\alpha A}v^{\hat{\pm}}_A,\\
\bar{\theta}^{\pm\dot{\alpha}} &= \bar{\theta}^{\dot{\alpha} i}u^{\pm}_i, \qquad \bar{\theta}^{\hat{\pm}\dot{\alpha}} = \bar{\hat{\theta}}^{\dot{\alpha} A}v^{\hat{\pm}}_A,\\
x^m_{\rm an} &= x^m - 2i\theta^{(i}\sigma^m\bar{\theta}^{j)}u^+_i u^-_j - 2i\hat{\theta}^{(A}\sigma^m\bar{\hat{\theta}}^{B)}v^{\hat{+}}_A v^{\hat{-}}_B.
\end{aligned} \tag{3.6}$$

Also, we define the spinor and harmonic derivatives in the central basis like in section 2,

$$\begin{aligned}
&D^{\pm}_{\alpha} = D^i_{\alpha}u^{\pm}_i, \qquad D^{\hat{\pm}}_{\alpha} = \hat{D}^A_{\alpha}v^{\hat{\pm}}_A, \qquad \bar{D}^{\pm}_{\dot{\alpha}} = \bar{D}^i_{\dot{\alpha}}u^{\pm}_i, \qquad \bar{D}^{\hat{\pm}}_{\dot{\alpha}} = \bar{\hat{D}}^A_{\dot{\alpha}}v^{\hat{\pm}}_A,\\
&\partial^{\pm\pm} = u^{\pm i}\frac{\partial}{\partial u^{\mp i}}\,, \qquad \partial^{\hat{\pm}\hat{\pm}} = v^{\hat{\pm}A}\frac{\partial}{\partial v^{\hat{\mp}A}}\,,
\end{aligned} \tag{3.7}$$

where D^i_{α}, $\hat{D}^A_{\alpha}$ and their c.c. are the usual spinor derivatives with respect to θ^i_{α}, $\hat{\theta}^A_{\alpha}$ and their c.c.. They are obtained from (3.2) by splitting the SU(4) index I according to the rule (3.3).

Spinor derivatives in the analytic basis look the same as in the previous section. The difference is that there are now two types of derivatives, with "hat" and without "hat". For example, D^+_{α} and $D^{\hat{+}}_{\alpha}$ read

$$D^+_{\alpha} = \frac{\partial}{\partial\theta^{-\alpha}}, \qquad D^{\hat{+}}_{\alpha} = \frac{\partial}{\partial\theta^{\hat{-}\alpha}}. \tag{3.8}$$

The same is true for harmonic derivatives. For example, D^{++} and $D^{\hat{+}\hat{+}}$ read

$$\begin{aligned}
D^{++} &= u^{+i}\frac{\partial}{\partial u^{-i}} - 2i\theta^{+\alpha}\bar{\theta}^{+\dot{\alpha}}\partial_{\alpha\dot{\alpha}} + \theta^{+\alpha}\frac{\partial}{\partial\theta^{-\alpha}} + \bar{\theta}^{+\dot{\alpha}}\frac{\partial}{\partial\bar{\theta}^{-\dot{\alpha}}},\\
D^{\hat{+}\hat{+}} &= v^{\hat{+}A}\frac{\partial}{\partial v^{\hat{-}A}} - 2i\theta^{\hat{+}\alpha}\bar{\theta}^{\hat{+}\dot{\alpha}}\partial_{\alpha\dot{\alpha}} + \theta^{\hat{+}\alpha}\frac{\partial}{\partial\theta^{\hat{-}\alpha}} + \bar{\theta}^{\hat{+}\dot{\alpha}}\frac{\partial}{\partial\bar{\theta}^{\hat{-}\dot{\alpha}}}.
\end{aligned} \tag{3.9}$$

Since D^+_{α}, $\bar{D}^+_{\dot{\alpha}}$, $D^{\hat{+}}_{\alpha}$, $\bar{D}^{\hat{+}}_{\dot{\alpha}}$ mutually anticommute and all are "short" in the analytic basis, there are three different types of analytic subspaces in $\mathcal{N}=4$ bi-harmonic superspace, in contrast to $\mathcal{N}=2$ harmonic superspace and, correspondingly, three different types of the Grassmann analyticity. These are the "half-analytic" subspace corresponding to nullifying $D^+_{\alpha}, \bar{D}^+_{\dot{\alpha}}$ on the appropriate superfields, the "half-analytic" subspace with nullifying

[6] The action of the standard generalized conjugation $\widetilde{}$ on different objects in the central basis of the bi-harmonic superspace is given by the following rules

$$\widetilde{f^{iA}} = \overline{f^{iA}} = \bar{f}_{iA}, \qquad \widetilde{\theta_{\alpha i}} = \bar{\theta}^i_{\dot{\alpha}}, \qquad \widetilde{\theta_{\alpha A}} = \bar{\theta}^A_{\dot{\alpha}}, \qquad \widetilde{u^{\pm}_i} = u^{\pm i}, \qquad \widetilde{v^{\hat{\pm}}_A} = v^{\hat{\pm}A}.$$

In the analytic basis this operation acts as follows

$$\widetilde{\theta^{\pm}_{\alpha}} = \bar{\theta}^{\pm}_{\dot{\alpha}}, \qquad \widetilde{\bar{\theta}^{\pm}_{\dot{\alpha}}} = -\theta^{\pm}_{\alpha}, \qquad \widetilde{\theta^{\hat{\pm}}_{\alpha}} = \bar{\theta}^{\hat{\pm}}_{\dot{\alpha}}, \qquad \widetilde{\bar{\theta}^{\hat{\pm}}_{\dot{\alpha}}} = -\theta^{\hat{\pm}}_{\alpha}.$$

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$D^{\hat{+}}_{\alpha}, \bar{D}^{\hat{+}}_{\dot{\alpha}}$, and the full analytic subspace, with four independent Grassmann-analyticity constraints. Respectively, they amount to the following sets of coordinates

$$
\begin{aligned}
\zeta_I &= \left(x^m_{\rm an}, \theta^{+\alpha}, \bar{\theta}^{+\dot{\alpha}}, \theta^{\hat{\pm}\alpha}, \bar{\theta}^{\hat{\pm}\dot{\alpha}}, u^{\pm}_i, v^{\hat{\pm}}_A\right),\\
\zeta_{II} &= \left(x^m_{\rm an}, \theta^{\pm\alpha}, \bar{\theta}^{\pm\dot{\alpha}}, \theta^{\hat{+}\alpha}, \bar{\theta}^{\hat{+}\dot{\alpha}}, u^{\pm}_i, v^{\hat{\pm}}_A\right),\\
\zeta_A &= \left(x^m_{\rm an}, \theta^{+\alpha}, \bar{\theta}^{+\dot{\alpha}}, \theta^{\hat{+}\alpha}, \bar{\theta}^{\hat{+}\dot{\alpha}}, u^{\pm}_i, v^{\hat{\pm}}_A\right).
\end{aligned}
\tag{3.10}
$$

All these subspaces are closed under $4D$, $\mathcal{N}=4$ supersymmetry transformations.

3.2 Bi-harmonic superfields of $\mathcal{N}=4$ SYM theory

We start with the gauge-covariant derivatives in the standard $\mathcal{N}=4$ superspace

$$
\nabla^I_\alpha = D^I_\alpha + i\mathcal{A}^I_\alpha, \qquad \bar{\nabla}_{\dot{\alpha}I} = \bar{D}_{\dot{\alpha}I} + i\bar{\mathcal{A}}_{\dot{\alpha}I}, \qquad \nabla_{\alpha\dot{\beta}} = \partial_{\alpha\dot{\beta}} + i\mathcal{V}_{\alpha\dot{\beta}}, \tag{3.11}
$$

where $\mathcal{A}^I_\alpha$, $\bar{\mathcal{A}}_{\dot{\alpha}I}$ and $\mathcal{V}_{\alpha\dot{\beta}}$ are spinor and vector superfield gauge connections. In $\mathcal{N}=4$ SYM theory these derivatives satisfy the constraints [20]

$$
\begin{aligned}
\{\nabla^I_\alpha, \nabla^J_\beta\} &= -2i\epsilon_{\alpha\beta}W^{IJ},\\
\{\bar{\nabla}_{\dot{\alpha}I}, \bar{\nabla}_{\dot{\beta}J}\} &= 2i\epsilon_{\dot{\alpha}\dot{\beta}}\bar{W}_{IJ},\\
\{\nabla^I_\alpha, \bar{\nabla}_{\dot{\beta}J}\} &= -2i\delta^I_J\nabla_{\alpha\dot{\beta}}\,.
\end{aligned}
\tag{3.12}
$$

Here $W^{IJ} = -W^{JI}$ is a real $\mathcal{N}=4$ superfield strength. The reality condition reads

$$
\overline{W^{IJ}} = \bar{W}_{IJ} = \frac{1}{2}\varepsilon_{IJKL}W^{KL}. \tag{3.13}
$$

The gauge connections and the superfield strengths in (3.12) and (3.13) are defined up to gauge transformations

$$
\mathcal{A}'^I_\alpha = -ie^{i\tau}(\nabla^I_\alpha e^{-i\tau}), \qquad W'^{IJ} = e^{i\tau}W^{IJ}e^{-i\tau}, \tag{3.14}
$$

where τ is a real $\mathcal{N}=4$ superfield parameter. Note that the condition (3.13) breaks the U(4) R-symmetry of the "flat" $\mathcal{N}=4$ superspace down to SU(4).

Next we rewrite the constraints (3.12), (3.13) in terms of indices i, A according to the rule (3.3). Using the antisymmetry of the superfield strength W^{IJ} and the reality condition (3.13) we express it in terms of few independent components:

$$
\begin{aligned}
W^{ij} &= \epsilon^{ij}W, & \bar{W}_{ij} &= -\epsilon_{ij}\bar{W},\\
W^{AB} &= \epsilon^{AB}\bar{W}, & \bar{W}_{AB} &= -\epsilon_{AB}W,\\
W^{iA} &= -i\phi^{iA}, & \bar{W}_{iA} &= i\phi_{iA} = i\epsilon_{ij}\epsilon_{AB}\phi^{jB}.
\end{aligned}
\tag{3.15}
$$

Then we plug these expressions back into the constraints (3.12) and obtain

$$
\begin{aligned}
\{\nabla^i_\alpha, \nabla^j_\beta\} &= -2i\epsilon_{\alpha\beta}\epsilon^{ij}W\,, & \{\bar{\nabla}_{\dot{\alpha}i}, \bar{\nabla}_{\dot{\beta}j}\} &= -2i\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{ij}\bar{W}\,,\\
\{\hat{\nabla}^A_\alpha, \hat{\nabla}^B_\beta\} &= -2i\epsilon_{\alpha\beta}\epsilon^{AB}\bar{W}\,, & \{\hat{\bar{\nabla}}_{\dot{\alpha}A}, \hat{\bar{\nabla}}_{\dot{\beta}B}\} &= -2i\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon_{AB}W\,,\\
\{\nabla^i_\alpha, \bar{\nabla}_{\dot{\beta}j}\} &= -2i\delta^i_j\nabla_{\alpha\dot{\beta}}\,, & \{\hat{\nabla}^A_\alpha, \hat{\bar{\nabla}}_{\dot{\beta}B}\} &= -2i\delta^A_B\nabla_{\alpha\dot{\beta}}\,,\\
\{\nabla^i_\alpha, \hat{\nabla}^B_\beta\} &= -2\epsilon_{\alpha\beta}\phi^{iB}\,, & \{\bar{\nabla}_{\dot{\alpha}i}, \hat{\bar{\nabla}}_{\dot{\beta}B}\} &= -2\epsilon_{\dot{\alpha}\dot{\beta}}\phi_{iB}\,,\\
\{\nabla_{\alpha i}, \hat{\bar{\nabla}}_{\dot{\beta}B}\} &= \{\bar{\nabla}_{\dot{\alpha}i}, \hat{\nabla}_{\beta B}\} = 0\,,
\end{aligned}
\tag{3.16}
$$

where the gauge connections are assumed to be rearranged in accord with the rule (3.3):

$$
\begin{aligned}
&\nabla^i_\alpha = D^i_\alpha + i\mathcal{A}^i_\alpha, && \hat{\nabla}^A_\alpha = \hat{D}^A_\alpha + i\hat{\mathcal{A}}^A_\alpha,\\
&\bar{\nabla}_{\dot\alpha j} = \bar{D}_{\dot\alpha j} + i\bar{\mathcal{A}}_{\dot\alpha j}, && \bar{\hat{\nabla}}_{\dot\alpha B} = \bar{\hat{D}}_{\dot\alpha B} + i\bar{\hat{\mathcal{A}}}_{\dot\alpha B},\\
&\nabla_{\alpha\dot\alpha} = \partial_{\alpha\dot\alpha} + i\mathcal{V}_{\alpha\dot\alpha}.
\end{aligned}
\tag{3.17}
$$

The constraints (3.16) imply some important consequences following from the Bianchi identities. E.g., for the mixed-index superfield strength ϕ_{iB} the Bianchi identity implies

$$
\nabla_{\alpha(i}\phi_{j)B} = 0, \quad \hat{\nabla}_{\alpha(A}\phi_{iB)} = 0. \tag{3.18}
$$

Indeed, let us write the Bianchi for ∇^i_α

$$
\{\nabla^j_\gamma\{\nabla^i_\alpha, \hat{\nabla}^B_\beta\}\} + \{\nabla^i_\alpha\{\hat{\nabla}^B_\beta, \nabla^j_\gamma\}\} + \{\hat{\nabla}^B_\beta\{\nabla^j_\gamma, \nabla^i_\alpha\}\} = 0. \tag{3.19}
$$

Substituting the constraints (3.16) into it and symmetrizing over indices $i,\ j$, we obtain

$$
\epsilon_{\alpha\beta}\nabla^{(j}_\gamma\phi^{i)B} + \epsilon_{\gamma\beta}\nabla^{(i}_\alpha\phi^{j)B} = 0 \Rightarrow \nabla_{\alpha(i}\phi_{j)B} = 0. \tag{3.20}
$$

The second equation in (3.18) is derived in a similar way.

As the next step, we define the harmonic projections of the quantities appearing in (3.16), (3.17),

$$
\nabla^{\pm}_{\alpha,\dot\alpha} = \nabla^i_{\alpha,\dot\alpha}u^\pm_i\,, \quad \nabla^{\hat\pm}_{\alpha,\dot\alpha} = \hat{\nabla}^A_{\alpha,\dot\alpha}v^{\hat\pm}_A\,, \quad \phi^{\pm\hat\pm} = \phi^{iA}u^\pm_i v^{\hat\pm}_A\,, \quad \phi^{\pm\hat\mp} = \phi^{iA}u^\pm_i v^{\hat\mp}_A\,, \tag{3.21}
$$

in terms of which the constraints (3.16) can be equivalently rewritten as an extended set:

$$
\begin{aligned}
&(a) && \{\nabla^+_\alpha, \nabla^+_\beta\} = \{\bar\nabla^+_{\dot\alpha}, \bar\nabla^+_{\dot\beta}\} = \{\nabla^+_\alpha, \bar\nabla^+_{\dot\beta}\} = 0,\\
&(b) && \{\nabla^{\hat+}_\alpha, \nabla^{\hat+}_\beta\} = \{\bar\nabla^{\hat+}_{\dot\alpha}, \bar\nabla^{\hat+}_{\dot\beta}\} = \{\nabla^{\hat+}_\alpha, \bar\nabla^{\hat+}_{\dot\beta}\} = 0,\\
&(c) && \{\nabla^+_\alpha, \nabla^{\hat+}_\beta\} = -2\epsilon_{\alpha\beta}\phi^{+\hat+}, \qquad \{\bar\nabla^+_{\dot\alpha}, \bar\nabla^{\hat+}_{\dot\beta}\} = -2\epsilon_{\dot\alpha\dot\beta}\phi^{+\hat+},\\
&(d) && \{\nabla^+_\alpha, \bar\nabla^{\hat+}_{\dot\beta}\} = \{\bar\nabla^+_{\dot\alpha}, \nabla^{\hat+}_\beta\} = 0,\\
&(e) && [\partial^{++}, \nabla^+_\alpha] = [\partial^{\hat+\hat+}, \nabla^+_\alpha] = [\partial^{++}, \nabla^{\hat+}_\alpha] = [\partial^{\hat+\hat+}, \nabla^{\hat+}_\alpha] = 0,\\
&(f) && [\partial^{++}, \bar\nabla^+_{\dot\alpha}] = [\partial^{\hat+\hat+}, \bar\nabla^+_{\dot\alpha}] = [\partial^{++}, \bar\nabla^{\hat+}_{\dot\alpha}] = [\partial^{\hat+\hat+}, \bar\nabla^{\hat+}_{\dot\alpha}] = 0,\\
&(g) && [\partial^{++}, \partial^{\hat+\hat+}] = 0\,.
\end{aligned}
\tag{3.22}
$$

The equivalency can be shown in the following way which is quite common for the harmonic superspace formulations of the extended supersymmetric gauge theories (see [14]). First, from (3.22e) and (3.22f) it follows that $\nabla^\pm_{\alpha,\dot\alpha}$ and $\nabla^{\hat\pm}_{\alpha,\dot\alpha}$ are linear in the harmonics u^+_i and $v^{\hat+}_A$, $\nabla^+_{\alpha,\dot\alpha} = \nabla^i_{\alpha,\dot\alpha}u^+_i$ and $\nabla^{\hat+}_{\alpha,\dot\alpha} = \hat\nabla^A_{\alpha,\dot\alpha}v^{\hat+}_A$. Then, from (3.22a) and (3.22b), the first three lines in the constraints (3.16) follow (e.g., (3.22a) implies $\{\nabla^{(i}_\alpha, \nabla^{j)}_\beta\} = 0$, etc.) From (3.22c) and the proper Bianchi identity (see below) the fourth line in (3.16) follows. At last, (3.22d) implies the fifth line. The negatively charged objects can be obtained from the positively charged ones by the action of the harmonic derivatives $\partial^{--}, \partial^{\hat-\hat-}$.

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The Bianchi identity mentioned above is obtained by commuting the proper spinor and harmonic derivatives with both sides of eq. (3.22c). It implies

$$\nabla^{+}_{\alpha}\phi^{+\hat{+}} = \nabla^{\hat{+}}_{\alpha}\phi^{+\hat{+}} = \bar{\nabla}^{+}_{\dot{\alpha}}\phi^{+\hat{+}} = \bar{\nabla}^{\hat{+}}_{\dot{\alpha}}\phi^{+\hat{+}} = \partial^{++}\phi^{+\hat{+}} = \partial^{\hat{+}\hat{+}}\phi^{+\hat{+}} = 0. \tag{3.23}$$

These relations are equivalent to the identities (3.18). Indeed, given a real superfield $\phi^{+\hat{+}}$ satisfying (3.23), it can be written as $\phi^{iA}u^{+}_{i}v^{\hat{+}}_{A}$ with ϕ^{iA} satisfying (3.18). In particular, the last two relations in (3.23) just imply that $\phi^{+\hat{+}} = \phi^{iA}u^{+}_{i}v^{\hat{+}}_{A}$.

Note that the constraints (3.22) are written in the central basis of $\mathcal{N}=4$ bi-harmonic superspace, with "short" harmonic derivatives $D^{++} = \partial^{++}$ and $D^{\hat{+}\hat{+}} = \partial^{\hat{+}\hat{+}}$. However, their form cannot depend on the choice of the basis, so in what follows we will use the general notation $D^{\pm\pm}$ and $D^{\hat{\pm}\hat{\pm}}$ for the harmonic derivatives.

3.3 The analytic frame

Following the generalities of the harmonic superspace approach, the crucial step now is passing to the analytic frame where it will become possible to solve the constraints (3.22) in terms of the appropriate analytic gauge superfields and to express the superfield strengths $\phi^{+\hat{+}}$, W, $\bar{W}$ in terms of these fundamental objects. If some of the harmonic projections of the gauge-covariant spinor derivatives form an anti-commutative subset, the relevant spinor connections are pure gauge and one can always choose a frame where these derivatives coincide with the "flat" ones, i.e. involve no gauge superconnections. Clearly, such an anticommuting subset of spinor derivatives is in a one-to-one correspondence with the existence of some analytic subspace in the given harmonic superspace.

In our case, because of the constraint $\{\nabla^{+}_{\alpha}, \nabla^{\hat{+}}_{\beta}\} = -2\epsilon_{\alpha\beta}\phi^{+\hat{+}}$, it is impossible to simultaneously make "flat" (having no gauge superconnections) all the positively charged spinor derivatives. Maximum what one can reach is to remove the gauge connections either from $\nabla^{+}_{\alpha,\dot{\alpha}}$ or from $\nabla^{\hat{+}}_{\alpha,\dot{\alpha}}$. Without loss of generality, we will chose the frame in which the derivatives $\nabla^{\hat{+}}_{\alpha}$ and $\bar{\nabla}^{\hat{+}}_{\dot{\alpha}}$ coincide with the flat ones, so that the ζ_{II} analyticity from the sets (3.10) can be made manifest.

Thus, consider the constraints $\{\nabla^{\hat{+}}_{\alpha}, \nabla^{\hat{+}}_{\beta}\} = \{\bar{\nabla}^{\hat{+}}_{\dot{\alpha}}, \bar{\nabla}^{\hat{+}}_{\dot{\beta}}\} = \{\nabla^{\hat{+}}_{\alpha}, \bar{\nabla}^{\hat{+}}_{\dot{\beta}}\} = 0$. Their general solution reads

$$\nabla^{\hat{+}}_{\alpha} = e^{iV}D^{\hat{+}}_{\alpha}e^{-iV}, \qquad \bar{\nabla}^{\hat{+}}_{\dot{\alpha}} = e^{iV}\bar{D}^{\hat{+}}_{\dot{\alpha}}e^{-iV} \implies \mathcal{A}^{\hat{+}}_{\alpha,\dot{\alpha}} = -ie^{iV}(D^{\hat{+}}_{\alpha,\dot{\alpha}}e^{-iV})\,, \tag{3.24}$$

where V is a real "bridge" superfield ($V = \widetilde{V}$) with the following gauge transformation law

$$e^{iV'} = e^{i\tau}e^{iV}e^{i\Lambda}, \tag{3.25}$$

where Λ is "ζ_{II}" analytic superfield, $\Lambda = \Lambda(\zeta_{II})$, and τ is a general real, harmonic-independent ($\partial^{++}\tau = \partial^{\hat{+}\hat{+}}\tau = 0$ in the central basis), $\mathcal{N}=4$ superfield. Now we perform the similarity transformation

$$\begin{aligned} &\nabla^{\hat{+}}_{\alpha} \to e^{-iV}\nabla^{\hat{+}}_{\alpha}e^{iV} = D^{\hat{+}}_{\alpha}, && \bar{\nabla}^{\hat{+}}_{\dot{\alpha}} \to e^{-iV}\bar{\nabla}^{\hat{+}}_{\dot{\alpha}}e^{iV} = \bar{D}^{\hat{+}}_{\dot{\alpha}}, \\ &\nabla^{+}_{\alpha} \to e^{-iV}\nabla^{+}_{\alpha}e^{iV}, && \bar{\nabla}^{+}_{\dot{\alpha}} \to e^{-iV}\bar{\nabla}^{+}_{\dot{\alpha}}e^{iV}, \quad \phi^{+\hat{+}} \to e^{-iV}\phi^{+\hat{+}}e^{iV} \end{aligned} \tag{3.26}$$

and

$$
\begin{aligned}
D^{++} \to \nabla^{++} &= e^{-iV} D^{++} e^{iV} := D^{++} + iV^{++}, \\
D^{\hat{+}\hat{+}} \to \nabla^{\hat{+}\hat{+}} &= e^{-iV} D^{\hat{+}\hat{+}} e^{iV} := D^{\hat{+}\hat{+}} + iV^{\hat{+}\hat{+}}, \qquad (3.27)\\
V^{++} &= -ie^{-iV}\left(D^{++}e^{iV}\right), \quad V^{\hat{+}\hat{+}} = -ie^{-iV}\left(D^{\hat{+}\hat{+}}e^{iV}\right), \qquad (3.28)
\end{aligned}
$$

where V^{++}, $V^{\hat{+}\hat{+}}$ are real bi-harmonic superfields. The transformed spinor and harmonic derivatives satisfy the same algebra (3.22)

$$
\begin{aligned}
\{\nabla^+_\alpha, \nabla^+_\beta\} &= \{\bar\nabla^+_{\dot\alpha}, \bar\nabla^+_{\dot\beta}\} = \{\nabla^+_\alpha, \bar\nabla^+_{\dot\beta}\} = 0, \qquad (3.29)\\
\{D^{\hat{+}}_\alpha, D^{\hat{+}}_\beta\} &= \{\bar D^{\hat{+}}_{\dot\alpha}, \bar D^{\hat{+}}_{\dot\beta}\} = \{D^{\hat{+}}_\alpha, \bar D^{\hat{+}}_{\dot\beta}\} = 0, \qquad (3.30)\\
\{\nabla^+_\alpha, D^{\hat{+}}_\beta\} &= -2\epsilon_{\alpha\beta}\phi^{+\hat{+}}, \qquad \{\bar\nabla^+_{\dot\alpha}, \bar D^{\hat{+}}_{\dot\beta}\} = -2\epsilon_{\dot\alpha\dot\beta}\phi^{+\hat{+}}, \qquad (3.31)\\
\{\nabla^+_\alpha, \bar D^{\hat{+}}_{\dot\beta}\} &= \{\bar\nabla^+_{\dot\alpha}, D^{\hat{+}}_\beta\} = 0, \qquad (3.32)\\
[\nabla^{++}, \nabla^+_\alpha] &= [\nabla^{\hat{+}\hat{+}}, \nabla^+_\alpha] = [\nabla^{++}, D^{\hat{+}}_\alpha] = [\nabla^{\hat{+}\hat{+}}, D^{\hat{+}}_\alpha] = 0, \qquad (3.33)\\
[\nabla^{++}, \bar\nabla^+_{\dot\alpha}] &= [\nabla^{\hat{+}\hat{+}}, \bar\nabla^+_{\dot\alpha}] = [\nabla^{++}, \bar D^{\hat{+}}_{\dot\alpha}] = [\nabla^{\hat{+}\hat{+}}, \bar D^{\hat{+}}_{\dot\alpha}] = 0, \qquad (3.34)\\
[\nabla^{++}, \nabla^{\hat{+}\hat{+}}] &= 0. \qquad (3.35)
\end{aligned}
$$

This is the final form of the $\mathcal{N} = 4$ SYM constraints we will deal with in what follows. It involves two harmonic connections $V^{++}, V^{\hat{+}\hat{+}}$ defined in (3.28) and the spinorial connections $\mathcal{A}^+_\alpha\,, \bar{\mathcal{A}}^+_{\dot\alpha}$ entering the gauge-covariant spinor derivatives

$$
\nabla^+_\alpha = D^+_\alpha + i\mathcal{A}^+_\alpha\,, \quad \bar\nabla^+_{\dot\alpha} = \bar D^+_{\dot\alpha} + i\bar{\mathcal{A}}^+_{\dot\alpha}\,. \qquad (3.36)
$$

It will be convenient to choose the analytic basis in $\mathcal{N} = 4$ bi-harmonic superspace, where $D^{\hat{+}}_\alpha = \partial/\partial\theta^{\hat{-}\alpha}$, $\bar D^{\hat{+}}_{\dot\alpha} = \partial/\partial\bar\theta^{\hat{-}\dot\alpha}$ and the ζ_{II} analyticity is manifest.[7]

The main advantage of the analytic frame and basis is that, in virtue of the relations (3.33), (3.34) (the last two in both chains), the harmonic connections V^{++} and $V^{\hat{+}\hat{+}}$ live on the reduced subspace ζ_{II},

$$
\begin{aligned}
D^{\hat{+}}_\alpha V^{++} = \bar D^{\hat{+}}_{\dot\alpha} V^{++} = 0\,, \qquad & D^{\hat{+}}_\alpha V^{\hat{+}\hat{+}} = \bar D^{\hat{+}}_{\dot\alpha} V^{\hat{+}\hat{+}} = 0\,, \Rightarrow \\
V^{++} = V^{++}(\zeta_{II}), \qquad & V^{\hat{+}\hat{+}} = V^{\hat{+}\hat{+}}(\zeta_{II}), \qquad (3.37)
\end{aligned}
$$

i.e. they *do not depend* on the Grassmann coordinates $\theta^{\hat{-}}_\alpha, \bar\theta^{\hat{-}}_{\dot\alpha}$ in the analytic basis.

The equations (3.23) in the analytic frame are rewritten as

$$
\nabla^+_\alpha \phi^{+\hat{+}} = D^{\hat{+}}_\alpha \phi^{+\hat{+}} = \bar\nabla^+_{\dot\alpha}\phi^{+\hat{+}} = \bar D^{\hat{+}}_{\dot\alpha}\phi^{+\hat{+}} = \nabla^{++}\phi^{+\hat{+}} = \nabla^{\hat{+}\hat{+}}\phi^{+\hat{+}} = 0, \qquad (3.38)
$$

which are also Bianchi identities for the constraints (3.29)–(3.35). We see that

$$
\phi^{+\hat{+}} = \phi^{+\hat{+}}(\zeta_{II})\,,
$$

like the harmonic connections V^{++} and $V^{\hat{+}\hat{+}}$. In the next section we will solve the equations (3.38) and the constraints (3.29)–(3.35).

[7] In the analytic basis, the spinor derivatives $D^+_\alpha, \bar D^+_{\dot\alpha}$ are also "short".

Using the gauge transformations (3.14) and (3.25), one can find the transformation laws of the analytic-frame harmonic and spinor connections, as well as of the superfield strength $\phi^{+\hat{+}}$

$$\delta V^{++} = \nabla^{++}\Lambda(\zeta_{II}), \qquad \delta V^{\hat{+}\hat{+}} = \nabla^{\hat{+}\hat{+}}\Lambda(\zeta_{II}), \tag{3.39}$$

$$\delta\phi^{+\hat{+}} = -i[\Lambda(\zeta_{II}), \phi^{+\hat{+}}], \qquad \delta\mathcal{A}^{+}_{\alpha,\dot{\alpha}} = \nabla^{+}_{\alpha,\dot{\alpha}}\Lambda(\zeta_{II})\,. \tag{3.40}$$

3.4 Gauge fixings

Before solving the constraints (3.29)–(3.35), some preliminary steps are needed. At this stage the harmonic connections V^{++} and $V^{\hat{+}\hat{+}}$ are arbitrary functions of the "hat"-analytic coordinates $\theta^{\hat{+}}_{\alpha}, \bar{\theta}^{\hat{+}}_{\dot{\alpha}}$ and harmonics $v^{\hat{\pm}}_{A}$ (along with the dependence on other coordinates of the analytic subspace ζ_{II}, see (3.10)).

Now we show that the dependence of $V^{\hat{+}\hat{+}}$ on $\theta^{\hat{+}}_{\alpha}, \bar{\theta}^{\hat{+}}_{\dot{\alpha}}$ and $v^{\hat{\pm}}_{A}$ can be reduced by choosing a Wess-Zumino gauge with respect to the transformations (3.39).

It is straightforward to see that the gauge freedom associated with the superfield transformation parameter $\Lambda(\zeta_{II})$ can be partially fixed by casting $V^{\hat{+}\hat{+}}$ in the short form

$$\begin{aligned} V^{\hat{+}\hat{+}} &= -2i\theta^{\hat{+}}_{\alpha}\bar{\theta}^{\hat{+}}_{\dot{\alpha}}\hat{\mathcal{A}}^{\alpha\dot{\alpha}} + (\theta^{\hat{+}})^2\mathcal{W} + (\bar{\theta}^{\hat{+}})^2\bar{\mathcal{W}} + 2(\bar{\theta}^{\hat{+}})^2\theta^{\hat{+}\alpha}\psi^{\hat{-}}_{\alpha} + 2(\theta^{\hat{+}})^2\bar{\theta}^{\hat{+}}_{\dot{\alpha}}\widetilde{\psi}^{\hat{-}\dot{\alpha}} \\ &\quad + 3(\theta^{\hat{+}})^2(\bar{\theta}^{\hat{+}})^2\mathcal{D}^{\hat{-}2}, \end{aligned} \tag{3.41}$$

$$\psi^{\hat{-}}_{\alpha} = \psi^{A}_{\alpha}v^{\hat{-}}_{A}, \quad \widetilde{\psi}^{\hat{-}\dot{\alpha}} = \widetilde{\psi}^{\dot{\alpha}}_{A}v^{\hat{-}A} = -\widetilde{\psi}^{A\dot{\alpha}}v^{\hat{-}}_{A}, \quad \mathcal{D}^{\hat{-}2} = \mathcal{D}^{(AB)}v^{\hat{-}}_{A}v^{\hat{-}}_{B}, \quad \bar{\mathcal{W}} = \widetilde{\mathcal{W}}. \tag{3.42}$$

Here the superfields $\hat{\mathcal{A}}^{\alpha\dot{\alpha}}$, ψ^{A}_{α}, $\mathcal{W}$ and $\mathcal{D}^{(AB)}$ are defined on the coordinate set $(x^m_{\rm an}, \theta^{\pm}_{\alpha}, \bar{\theta}^{\pm}_{\dot{\alpha}}, u^{\pm}_{i})$. While passing to (3.41), the dependence of $\Lambda(\zeta_{II})$ on $(\theta^{\hat{+}}_{\alpha}, \bar{\theta}^{\hat{+}}_{\dot{\alpha}}, v^{\hat{\pm}}_{A})$ has been fully spent, so the residual gauge freedom is connected with the gauge function $\Lambda_{\rm int}(x^m_{\rm an}, \theta^{\pm}_{\alpha}, \bar{\theta}^{\pm}_{\dot{\alpha}}, u^{\pm}_{i})$, $\Lambda(\zeta_{II}) \to \Lambda_{\rm int}$. Below we show that the dependence of $\Lambda_{\rm int}$ on $\theta^{-}_{\alpha}, \bar{\theta}^{-}_{\dot{\alpha}}$ can also be fully spent for a proper gauge choice.

To this end, we need to inspect the structure of the spinor derivative. First, let us examine the spinor part of the relations (3.38), namely

$$\text{(a)}\ D^{\hat{+}}_{\alpha}\phi^{+\hat{+}} = \bar{D}^{\hat{+}}_{\dot{\alpha}}\phi^{+\hat{+}} = 0, \qquad \text{(b)}\ \nabla^{+}_{\alpha}\phi^{+\hat{+}} = \bar{\nabla}^{+}_{\dot{\alpha}}\phi^{+\hat{+}} = 0. \tag{3.43}$$

In this subsection we focus on eq. (3.43a), the consequences of (3.43b) will be discussed later (in subsection 4.1.2). As was mentioned earlier, it follows from (3.43a) that $\phi^{+\hat{+}}$ does not depend on $\theta^{\hat{-}}_{\alpha}$ and $\bar{\theta}^{\hat{-}}_{\dot{\alpha}}$. In addition, using this property in the constraints (3.31) and (3.32) implies that

$$\mathcal{A}^{+}_{\alpha} = A^{+}_{\alpha} + 2i\phi^{+\hat{+}}\theta^{\hat{-}}_{\alpha}, \qquad \bar{\mathcal{A}}^{+}_{\dot{\alpha}} = \bar{A}^{+}_{\dot{\alpha}} + 2i\phi^{+\hat{+}}\bar{\theta}^{\hat{-}}_{\dot{\alpha}}, \tag{3.44}$$

where A^{+}_{α}, $\bar{A}^{+}_{\dot{\alpha}}$ do not depend on $\theta^{\hat{-}}_{\alpha}$ and $\bar{\theta}^{\hat{-}}_{\dot{\alpha}}$ and so can be represented as

$$\begin{aligned} A^{+}_{\beta} &= f^{+}_{\beta} + \theta^{\hat{+}\alpha}f^{+\hat{-}}_{\alpha\beta} + \bar{\theta}^{\hat{+}}_{\dot{\alpha}}g^{+\hat{-}\dot{\alpha}}_{\beta} + (\theta^{\hat{+}})^2 f^{+\hat{-}2}_{\beta} + (\bar{\theta}^{\hat{+}})^2 g^{+\hat{-}2}_{\beta} + \theta^{\hat{+}}_{\alpha}\bar{\theta}^{\hat{+}}_{\dot{\alpha}}f^{+\hat{-}\hat{-}\alpha\dot{\alpha}}_{\beta} \\ &\quad + (\bar{\theta}^{\hat{+}})^2\theta^{\hat{+}\alpha}f^{+\hat{-}3}_{\alpha\beta} + (\theta^{\hat{+}})^2\bar{\theta}^{\hat{+}}_{\dot{\alpha}}g^{+\hat{-}3\dot{\alpha}}_{\beta} + (\theta^{\hat{+}})^2(\bar{\theta}^{\hat{+}})^2 f^{+\hat{-}4}_{\beta}. \end{aligned} \tag{3.45}$$

Note, that $\bar{A}^{+}_{\dot{\alpha}} = -\widetilde{A^{+}_{\alpha}}$. All the coefficients in the expansion (3.45) at this stage are arbitrary functions of the remaining coordinates $(x^m_{\rm an}, \theta^{\pm}_{\alpha}, \bar{\theta}^{\pm}_{\dot{\alpha}}, u^{\pm}_{i}, v^{\hat{\pm}}_{A})$. Below we will show that all terms except the second one can be eliminated either by the constraints or by choosing an additional gauge.

This additional gauge-fixing will be imposed right now and it goes as follows. First note that in the zeroth order in $\theta^{\hat\pm}$ the constraint $[\nabla^{\hat+\hat+}, \nabla^+_{\alpha,\dot\alpha}] = 0$ in eqs. (3.33), (3.34) implies

$$\partial^{\hat+\hat+} f^+_{\alpha,\dot\alpha} = 0\,, \tag{3.46}$$

which means that $f^+_{\alpha,\dot\alpha}$ do not depend on the harmonics $v^{\hat\pm}_A$, i.e. these objects "live" on the coordinate set $(x^m_{\rm an}, \theta^\pm_\alpha, \bar\theta^\pm_{\dot\alpha}, u^\pm_i)$. On the other hand, after substituting (3.44) and (3.45) into the constraint (3.29), we obtain

$$D^+_\alpha f^+_\beta + D^+_\beta f^+_\alpha + i\{f^+_\alpha, f^+_\beta\} = 0\,, \quad \bar D^+_{\dot\alpha} f^+_\beta + D^+_\beta \bar f^+_{\dot\alpha} + i\{f^+_\alpha, \bar f^+_{\dot\beta}\} = 0 \quad \text{(and c.c.)}. \tag{3.47}$$

It stems from (3.47) that $f^+_{\alpha,\dot\alpha} = -ie^{i\tilde v}(D^+_{\alpha,\dot\alpha} e^{-i\tilde v})$, where $\tilde v$ is an additional bridge living on the same coordinate set as the superfields $f^+_{\alpha,\dot\alpha}$. It transforms as $e^{i\tilde v'} = e^{-i\Lambda_{\rm int}} e^{i\tilde v} e^{i\Lambda(\zeta)}$, where $\Lambda_{\rm int}$ was defined after eq. (3.42), while the pregauge freedom parameter $\Lambda(\zeta)$ satisfies the conditions $D^+_\alpha \Lambda = \bar D^+_{\dot\alpha}\Lambda = 0$ and so can be identified with the $\mathcal{N} = 2$ harmonic analytic gauge group parameter. Using the newly introduced bridge, one can pass to the frame where

$$f^+_{\alpha,\dot\alpha} = 0 \tag{3.48}$$

and the residual gauge group is reduced to the standard $\mathcal{N} = 2$ SYM analytic gauge group, $\Lambda_{\rm int} \to \Lambda(\zeta)$. Actually, this passing can be equivalently interpreted as the gauge choice $\tilde v = 0 \;\Rightarrow\; \Lambda_{\rm int} = \Lambda(\zeta)$.

Hereafter we use the spinor connections $\mathcal{A}^+_{\alpha,\dot\alpha}$ in the form (3.44), (3.45) with the condition $f^+_{\alpha,\dot\alpha} = 0$ and the following $\theta^{\hat+}_\alpha, \bar\theta^{\hat+}_{\dot\alpha}$ expansions for the "hat"-analytic superfields $\phi^{+\hat+}$ and V^{++}:

$$\begin{aligned}
\phi^{+\hat+} &= \sqrt{2} q^{+\hat+} + \theta^{\hat+\alpha}\mathcal{W}^+_\alpha + \bar\theta^{\hat+}_{\dot\alpha}\widetilde{\mathcal{W}}^{+\dot\alpha} + (\theta^{\hat+})^2 H^{+\hat-} + (\bar\theta^{\hat+})^2 \widetilde H^{+\hat-} - i\theta^{\hat+}_\alpha\bar\theta^{\hat+}_{\dot\alpha}\beta^{+\hat-\alpha\dot\alpha} \\
&\quad + (\bar\theta^{\hat+})^2\theta^{\hat+\alpha} G^{+\hat-\hat-}_\alpha + (\theta^{\hat+})^2\bar\theta^{\hat+}_{\dot\alpha}\widetilde G^{+\hat-\hat-\dot\alpha} + (\theta^{\hat+})^2(\bar\theta^{\hat+})^2 G^{+\hat-3},
\end{aligned} \tag{3.49}$$

$$\begin{aligned}
V^{++} &= \mathcal{V}^{++} + \theta^{\hat+\alpha} w^{++\hat-}_\alpha + \bar\theta^{\hat+}_{\dot\alpha}\widetilde w^{++\hat-\dot\alpha} + (\theta^{\hat+})^2 w^{++\hat-\hat-} + (\bar\theta^{\hat+})^2 \widetilde w^{++\hat-\hat-} + \theta^{\hat+}_\alpha\bar\theta^{\hat+}_{\dot\alpha} w^{++\hat-\hat-\alpha\dot\alpha} \\
&\quad + (\bar\theta^{\hat+})^2\theta^{\hat+\alpha} w^{++\hat-3}_\alpha + (\theta^{\hat+})^2\bar\theta^{\hat+}_{\dot\alpha}\widetilde w^{++\hat-3\dot\alpha} + (\theta^{\hat+})^2(\bar\theta^{\hat+})^2 w^{++\hat-4}.
\end{aligned} \tag{3.50}$$

The superfield coefficients in these expansions will be shown to be severely constrained. At the moment, they are just $\mathcal{N} = 2$ harmonic superfields with an extra dependence on the harmonics $v^{\hat\pm}_A$, i.e. defined on the set $(x^m_{\rm an}, \theta^\pm_\alpha, \bar\theta^\pm_{\dot\alpha}, u^\pm_i, v^{\hat\pm}_A)$.

4 Solving $\mathcal{N} = 4$ SYM constraints in terms of $\mathcal{N} = 2$ superfields

In this section we will finish solving the constraints (3.29)–(3.38).

4.1 Harmonic equations

4.1.1 Constraint $[\nabla^{\hat+\hat+}, \nabla^{++}] = 0$

We start by showing that V^{++} (3.50) in fact does not depend on the coordinates $\theta^{\hat+}_\alpha, \bar\theta^{\hat+}_{\dot\alpha}, v^{\hat\pm}_A$. This follows from the constraint (3.35) which in a more detailed form reads

$$D^{++}V^{\hat+\hat+} - D^{\hat+\hat+}V^{++} + i[V^{++}, V^{\hat+\hat+}] = 0. \tag{4.1}$$

Substituting the expansions (3.50) and (3.41) in (4.1) and equating to zero the coefficients of the $\theta^{\hat{+}\alpha}$, $\bar{\theta}^{\hat{+}}_{\dot{\alpha}}$ monomials in the resulting expression, we obtain the set of equations

$$\partial^{\hat{+}\hat{+}}\mathcal{V}^{++} = 0\,, \qquad \partial^{\hat{+}\hat{+}}w^{++\hat{-}}_{\alpha} = 0\,, \qquad \partial^{\hat{+}\hat{+}}\widetilde{w}^{++\hat{-}\dot{\alpha}} = 0\,, \tag{4.2}$$

$$\partial^{\hat{+}\hat{+}}w^{++\hat{-}\hat{-}} - D^{++}\mathcal{W} - i[\mathcal{V}^{++},\mathcal{W}] = 0\,, \quad \text{(and c.c.)}, \tag{4.3}$$

$$\partial^{\hat{+}\hat{+}}w^{++\hat{-}\hat{-}\alpha\dot{\alpha}} + 2iD^{++}\hat{\mathcal{A}}^{\alpha\dot{\alpha}} - 2[\mathcal{V}^{++},\hat{\mathcal{A}}^{\alpha\dot{\alpha}}] - 2i\partial^{\alpha\dot{\alpha}}\mathcal{V}^{++} = 0\,, \tag{4.4}$$

$$\partial^{\hat{+}\hat{+}}w^{++\hat{-}3}_{\alpha} - 2D^{++}\psi^{\hat{-}}_{\alpha} - 2i[\mathcal{V}^{++},\psi^{\hat{-}}_{\alpha}] = 0\,, \quad \text{(and c.c.)}, \tag{4.5}$$

$$\partial^{\hat{+}\hat{+}}w^{++\hat{-}4} - 3D^{++}\mathcal{D}^{\hat{-}2} - 3i[\mathcal{V}^{++},\mathcal{D}^{\hat{-}2}] - i[w^{++\hat{-}\hat{-}},\bar{\mathcal{W}}] - i[\widetilde{w}^{++\hat{-}\hat{-}},\mathcal{W}] = 0. \tag{4.6}$$

The last two equations in (4.2) imply

$$w^{++\hat{-}}_{\alpha} = \widetilde{w}^{++\hat{-}}_{\dot{\alpha}} = 0. \tag{4.7}$$

In addition, the first equation implies $\mathcal{V}^{++}$ to bear no dependence on $v^{\hat{\pm}}_{A}$. Inspecting the $\theta^{\pm\alpha}, \bar{\theta}^{\pm}_{\dot{\alpha}}$-independent parts of the first constraints in the chains (3.33) and (3.34), one also observers that the superfield $\mathcal{V}^{++}$ is $\mathcal{N} = 2$ analytic, $D^{+}_{\alpha}\mathcal{V}^{++} = \bar{D}^{+}_{\dot{\alpha}}\mathcal{V}^{++} = 0$, and in fact already at this stage can be identified with the analytic harmonic gauge connection of $\mathcal{N} = 2$ SYM theory.

Eqs. (4.3)–(4.6) further imply

$$w^{++\hat{-}\hat{-}} = \widetilde{w}^{++\hat{-}\hat{-}} = w^{++\hat{-}\hat{-}\alpha\dot{\alpha}} = w^{++\hat{-}3\alpha} = w^{++\hat{-}4} = 0. \tag{4.8}$$

Thus, we found

$$V^{++} \equiv \mathcal{V}^{++}, \quad \Rightarrow \quad \nabla^{++} = D^{++} + i\mathcal{V}^{++} \tag{4.9}$$

Eqs. (4.3)–(4.6) also encode some other consequences appearing in the zeroth order in $v^{\hat{\pm}}_{A}$

$$\nabla^{++}\mathcal{W} = \nabla^{++}\bar{\mathcal{W}} = 0, \tag{4.10}$$

$$\nabla^{++}\hat{\mathcal{A}}^{\alpha\dot{\alpha}} = \partial^{\alpha\dot{\alpha}}\mathcal{V}^{++}, \tag{4.11}$$

$$\nabla^{++}\psi^{A}_{\alpha} = \nabla^{++}\tilde{\psi}^{A}_{\dot{\alpha}} = 0, \qquad \nabla^{++}\mathcal{D}^{(AB)} = 0. \tag{4.12}$$

Note, that (4.11) is equivalent to the vanishing of the commutator

$$[\nabla^{++},\hat{\nabla}^{\alpha\dot{\alpha}}] = 0, \qquad \hat{\nabla}^{\alpha\dot{\alpha}} = \partial^{\alpha\dot{\alpha}} + i\hat{\mathcal{A}}^{\alpha\dot{\alpha}}. \tag{4.13}$$

Thus, the constraint (3.35) has been fully resolved. The validity of the conditions (4.10)–(4.12) on the final solution of all constraints will become clear later, in the end of subsection 4.3.

4.1.2 Constraints $\nabla^{\hat{+}\hat{+}}\phi^{+\hat{+}} = 0$ and $\nabla^{++}\phi^{+\hat{+}} = 0$

Our task here is to further specify the structure of spinor connection (3.44). It involves the superfield $\phi^{+\hat{+}}$. Consider it in more details. Besides the analyticity conditions (3.43), it satisfies the harmonic equations

$$\text{(a)}\ \nabla^{\hat{+}\hat{+}}\phi^{+\hat{+}} = 0, \qquad \text{(b)}\ \nabla^{++}\phi^{+\hat{+}} = 0. \tag{4.14}$$

We start with (4.14a). Substituting the expansions of $\phi^{+\hat{+}}$ and $V^{\hat{+}\hat{+}}$ from eqs. (3.49) and (3.41), one obtains the set of equations and their solutions

$$
\begin{aligned}
&\partial^{\hat{+}\hat{+}} q^{+\hat{+}} = 0 \implies q^{+\hat{+}} = q^{+A} v^{\hat{+}}_A , && (4.15)\\
&\partial^{\hat{+}\hat{+}} \mathcal{W}^{+}_{\alpha} = \partial^{\hat{+}\hat{+}} \widetilde{\mathcal{W}}^{+\dot{\alpha}} = 0, && (4.16)\\
&\partial^{\hat{+}\hat{+}} H^{+\hat{-}} + \sqrt{2} i [\mathcal{W}, q^{+A}] v^{\hat{+}}_A = 0 \implies H^{+\hat{-}} = -\sqrt{2} i [\mathcal{W},\, q^{+A}] v^{\hat{-}}_A \,, && (4.17)\\
&\partial^{\hat{+}\hat{+}} \beta^{+\hat{-}\alpha\dot{\alpha}} + 2\sqrt{2}\hat{\nabla}^{\alpha\dot{\alpha}} q^{+A} v^{\hat{+}}_A = 0 \implies \beta^{+\hat{-}\alpha\dot{\alpha}} = -2\sqrt{2}\hat{\nabla}^{\alpha\dot{\alpha}} q^{+A} v^{\hat{-}}_A , && (4.18)\\
&\partial^{\hat{+}\hat{+}} G^{+\hat{-}\hat{-}}_{\alpha} + i\hat{\nabla}_{\alpha\dot{\alpha}} \widetilde{\mathcal{W}}^{+\dot{\alpha}} + 2\sqrt{2} i [\psi^{\hat{-}}_{\alpha}, q^{+A}] v^{\hat{+}}_A + i[\bar{\mathcal{W}},\, \mathcal{W}^{+}_{\alpha}] = 0\,, && (4.19)\\
&2\partial^{\hat{+}\hat{+}} G^{+\hat{-}3} - \hat{\nabla}^{\alpha\dot{\alpha}} \beta^{+\hat{-}}_{\alpha\dot{\alpha}} - 2i\{\psi^{\hat{-}\alpha}, \mathcal{W}^{+}_{\alpha}\} - 2i\{\widetilde{\psi}^{\hat{-}}_{\dot{\alpha}}, \widetilde{\mathcal{W}}^{+\dot{\alpha}}\} \\
&+ 6i\sqrt{2}[\mathcal{D}^{AB}, q^{+C}] v^{\hat{-}}_A v^{\hat{-}}_B v^{\hat{+}}_C + 2\sqrt{2}[\bar{\mathcal{W}}, [\mathcal{W}, q^{+A}]] v^{\hat{-}}_A + 2\sqrt{2}[\mathcal{W}, [\bar{\mathcal{W}}, q^{+A}]] v^{\hat{-}}_A = 0\,. && (4.20)
\end{aligned}
$$

Eqs. (4.19), (4.20) constrain the superfields $G^{+\hat{-}\hat{-}}_{\alpha}$ and $G^{+\hat{-}3}$ as

$$
\begin{aligned}
&G^{+\hat{-}\hat{-}}_{\alpha} = G^{+(AB)}_{\alpha} v^{\hat{-}}_A v^{\hat{-}}_B , && G^{+(AB)}_{\alpha} = -\sqrt{2} i [\psi^{(A}_{\alpha}, q^{+B)}], && (4.21)\\
&G^{+\hat{-}3} = G^{+(ABC)} v^{\hat{-}}_A v^{\hat{-}}_B v^{\hat{-}}_C , && G^{+(ABC)} = -i\sqrt{2}[\mathcal{D}^{(AB}, q^{+C)}]. && (4.22)
\end{aligned}
$$

While finding these solutions, we used the relations

$$
v^{\hat{+}}_A v^{\hat{-}}_B = v^{\hat{+}}_{(A} v^{\hat{-}}_{B)} + \frac{1}{2}\epsilon_{AB}, \qquad v^{\hat{-}}_A v^{\hat{-}}_B v^{\hat{+}}_C = v^{\hat{-}}_{(A} v^{\hat{-}}_B v^{\hat{+}}_{C)} + \frac{1}{3}\left(\epsilon_{CA} v^{\hat{-}}_B + \epsilon_{CB} v^{\hat{-}}_A\right). \tag{4.23}
$$

Eq. (4.16) implies the independence of $\mathcal{W}^{+}_{\alpha}, \tilde{\mathcal{W}}^{+}_{\dot{\alpha}}$ from $v^{\hat{\pm}}_A$. The constraint (4.19) and (4.20), in the zeroth and first orders in $v^{\hat{\pm}}_A$, also imply some self-consistency conditions which are quoted in the appendix A. These conditions do not bring any new information, but must be satisfied on the final solution of all constraints (like eqs. (4.10)–(4.12)), and so they provide a good self-consistency check.

Thus, we have completely fixed the $v^{\hat{\pm}}_A$ dependence of the coefficients in the $\theta^{\hat{+}}_{\alpha}, \bar{\theta}^{\hat{+}}_{\dot{\alpha}}$ expansion (3.49) for $\phi^{+\hat{+}}$. The full expression for $\phi^{+\hat{+}}$ at this step reads

$$
\begin{aligned}
\phi^{+\hat{+}} = {} & \sqrt{2} q^{+A} v^{\hat{+}}_A + \theta^{\hat{+}\alpha}\mathcal{W}^{+}_{\alpha} + \bar{\theta}^{\hat{+}}_{\dot{\alpha}} \widetilde{\mathcal{W}}^{+\dot{\alpha}} - \sqrt{2} i (\bar{\theta}^{\hat{+}})^2 \theta^{\hat{+}\alpha} [\psi^{(A}_{\alpha}, q^{+B)}] v^{\hat{-}}_A v^{\hat{-}}_B \\
& - \sqrt{2} i (\theta^{\hat{+}})^2 [\mathcal{W}, q^{+A}] v^{\hat{-}}_A - \sqrt{2} i (\bar{\theta}^{\hat{+}})^2 [\bar{\mathcal{W}}, q^{+A}] v^{\hat{-}}_A + \sqrt{2} i (\theta^{\hat{+}})^2 \bar{\theta}^{\hat{+}}_{\dot{\alpha}} [\widetilde{\psi}^{(A\dot{\alpha}}, q^{+B)}] v^{\hat{-}}_A v^{\hat{-}}_B \\
& + 2i\sqrt{2}\theta^{\hat{+}}_{\alpha}\bar{\theta}^{\hat{+}}_{\dot{\alpha}} \hat{\nabla}^{\alpha\dot{\alpha}} q^{+A} v^{\hat{-}}_A - i(\theta^{\hat{+}})^2(\bar{\theta}^{\hat{+}})^2 [\mathcal{D}^{AB}, q^{+C}] v^{\hat{-}}_A v^{\hat{-}}_B v^{\hat{-}}_C .
\end{aligned} \tag{4.24}
$$

All the coefficients in this expansion are harmonic $\mathcal{N} = 2$ superfields.

Now we are ready to display the constraints imposed by the second harmonic equation (4.14b). In the zeroth order in the "hat"-variables it entails just the equation of motion for the hypermultiplet q^{+}_A

$$
\nabla^{++} q^{+}_A = 0\,. \tag{4.25}
$$

In higher orders, there again appear some extra self-consistency relations to be automatically satisfied on the complete solution of the constraints. Note that the reality of the superfield $\phi^{+\hat{+}}$ implies the reality of q^{+}_A. Indeed,

$$
\phi^{+\hat{+}} = \widetilde{\phi^{+\hat{+}}} \Rightarrow q^{+\hat{+}} = \widetilde{q^{+\hat{+}}} \Rightarrow q^{+A} v^{\hat{+}}_A = \widetilde{q^{+A} v^{\hat{+}}_A} = \tilde{q}^{+}_A v^{\hat{+}A} = -\tilde{q}^{+A} v^{\hat{+}}_A , \tag{4.26}
$$

or, equivalently,

$$
\tilde{q}^A = -q^A \Leftrightarrow \widetilde{q_A} = q^A. \tag{4.27}
$$

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4.1.3 Constraints $[\nabla^{\hat{+}\hat{+}}, \nabla^+_\alpha] = 0$ and $[\nabla^{\hat{+}\hat{+}}, \bar{\nabla}^+_{\dot{\alpha}}] = 0$

Now we can return to the problem of fully fixing the spinor connections $\mathcal{A}^+_\alpha$ and $\bar{\mathcal{A}}^+_{\dot{\alpha}}$. The key role in achieving this is played by the constraints

$$(\text{a})\ [\nabla^{\hat{+}\hat{+}}, \nabla^+_\alpha] = 0, \qquad (\text{b})\ [\nabla^{\hat{+}\hat{+}}, \bar{\nabla}^+_{\dot{\alpha}}] = 0. \tag{4.28}$$

Like the constraint (3.35) for V^{++}, the constraint (4.28a) eliminates all the negatively charged components in the expansion (3.45) of A^+_α, except for the component $f^{+\hat{-}}_{\alpha\beta}$,

$$g^{+\hat{-}\dot{\alpha}}_\beta = f^{+\hat{-}2}_\beta = g^{+\hat{-}2}_\beta = f^{+\hat{-}\hat{-}\alpha\dot{\alpha}}_\beta = f^{+\hat{-}3}_{\alpha\beta} = g^{+\hat{-}3\dot{\alpha}}_\beta = f^{+\hat{-}4}_\beta = 0. \tag{4.29}$$

For $f^{+\hat{-}}_{\alpha\beta}$ we obtain from (4.28a) the harmonic equation

$$\partial^{\hat{+}\hat{+}} f^{+\hat{-}}_{\alpha\beta} + 2i\sqrt{2}\epsilon_{\beta\alpha} q^{+\hat{+}} = 0 \implies f^{+\hat{-}}_{\alpha\beta} = -2\sqrt{2}i\epsilon_{\beta\alpha} q^{+A} v^{\hat{-}}_A. \tag{4.30}$$

We also obtain the set of self-consistency conditions which are listed in appendix A. Here we quote only one important condition which will be needed for the subsequent analysis,

$$D^+_\beta \hat{\mathcal{A}}^{\alpha\dot{\alpha}} + \delta^\alpha_\beta \widetilde{\mathcal{W}}^{+\dot{\alpha}} = 0 \quad (\text{and c.c.}). \tag{4.31}$$

Since (4.28b) is a complex conjugate of eq. (4.28a), the restrictions associated with $\bar{A}^+_{\dot{\alpha}}$ correspond just to conjugating the relations (4.29)–(4.31).

The final form of the spinor connections is obtained by substituting the solution (4.30) into (3.44):

$$\mathcal{A}^+_\alpha = -2\sqrt{2}i\theta^{\hat{+}}_\alpha q^{+A} v^{\hat{-}}_A + 2i\theta^{\hat{-}}_\alpha \phi^{+\hat{+}}, \quad \bar{\mathcal{A}}^+_{\dot{\alpha}} = -2\sqrt{2}i\bar{\theta}^{\hat{+}}_{\dot{\alpha}} q^{+A} v^{\hat{-}}_A + 2i\bar{\theta}^{\hat{-}}_{\dot{\alpha}} \phi^{+\hat{+}}. \tag{4.32}$$

It is the proper place to come back to the analyticity condition (3.43b). Using the exact expressions (4.32) for spinor connections, we draw the following consequences of it

$$D^+_\alpha q^{+A} = 0, \qquad D^+_\alpha \mathcal{W}^+_\beta = -2\epsilon_{\alpha\beta}[q^{+A}, q^+_A], \qquad D^+_\alpha \widetilde{\mathcal{W}}^{+\dot{\alpha}} = 0. \tag{4.33}$$

The first relation and its conjugate, $\bar{D}^+_{\dot{\alpha}} q^{+A} = 0$, are just the $\mathcal{N}=2$ Grassmann analyticity conditions for q^{+A}. As it will become clear later, the other two relations encode the equations of motion for $\mathcal{N}=2$ gauge superfield and the $\mathcal{N}=2$ chirality conditions for the $\mathcal{N}=2$ superfield strengths. We also note that, taking into account (4.33) and the constraint $\nabla^+_{\alpha,\dot{\alpha}} \phi^{+\hat{+}} = 0$ in (3.38) (to be discussed later), one can check that the short connections (4.32) solve the constraints (3.29). The gauge transformation law (3.40), with $\Lambda(\zeta_{II}) \to \Lambda(\zeta)$, is reduced to the homogeneous law $\delta\, \mathcal{A}^+_{\alpha,\dot{\alpha}} = i[\mathcal{A}^+_{\alpha,\dot{\alpha}}, \Lambda(\zeta)]$, taking into account the analyticity of $\Lambda(\zeta)$, that is $D^+_{\alpha\dot{\alpha}}\Lambda(\zeta) = 0$.

4.2 Supersymmetry transformations

In this subsection we discuss the implementation of hidden supersymmetry. In the analytic basis its transformations on the superspace coordinates are as follows

$$\delta\, x^m_{\text{an}} = -2i(\epsilon^{\hat{-}}\sigma^m\bar{\theta}^{\hat{+}} + \theta^{\hat{+}}\sigma^m\bar{\epsilon}^{\hat{-}}), \qquad \delta\,\theta^{\hat{\pm}}_\alpha = \epsilon^{\hat{\pm}}_\alpha = \epsilon^A_\alpha v^{\hat{\pm}}_A, \qquad \delta\,\bar{\theta}^{\hat{\pm}}_{\dot{\alpha}} = \bar{\epsilon}^{\hat{\pm}}_{\dot{\alpha}} = \bar{\epsilon}^A_{\dot{\alpha}} v^{\hat{\pm}}_A. \tag{4.34}$$

In order to preserve Wess-Zumino gauge of the superfield $V^{\hat{+}\hat{+}}$ (4.6), as well as the "short" form of the spinor connections $\mathcal{A}^{+}_{\alpha}$ and $\bar{\mathcal{A}}^{+}_{\dot{\alpha}}$, eqs. (4.32), one needs to add the compensating gauge transformation. So the second supersymmetry transformations (in the "active" form, i.e. taken at the fixed "superpoint") should be

$$\delta\, V^{\hat{+}\hat{+}} = \delta'\, V^{\hat{+}\hat{+}} + \nabla^{\hat{+}\hat{+}}\Lambda^{(\text{comp})}\,, \qquad \delta\, \mathcal{A}^{+}_{\alpha,\dot{\alpha}} = \delta'\, \mathcal{A}^{+}_{\alpha,\dot{\alpha}} + \nabla_{\alpha,\dot{\alpha}}\Lambda^{(\text{comp})}\,, \tag{4.35}$$

where δ' means the variation under the shifts (4.34), e.g., $\delta'\, V^{\hat{+}\hat{+}} = -\delta x^{m}_{\text{an}}\partial_m V^{\hat{+}\hat{+}} + \ldots$, and

$$\Lambda^{(\text{comp})} = \Lambda^{(\text{comp})}_1 + \Lambda^{(\text{comp})}_2, \tag{4.36}$$

where $\Lambda^{(\text{comp})}_1$, $\Lambda^{(\text{comp})}_2$ are chosen, respectively, to preserve (4.6) and (4.32). These composite gauge parameters are easily found to be

$$\begin{aligned}
\Lambda^{(\text{comp})}_1 &= \bar{\theta}^{\hat{+}}_{\dot{\alpha}}\left(2i\epsilon^{\hat{-}}_{\alpha}\hat{\mathcal{A}}^{\alpha\dot{\alpha}} + 2\bar{\epsilon}^{\hat{-}\dot{\alpha}}\bar{\mathcal{W}}\right) - \theta^{\hat{+}\alpha}\left(2i\bar{\epsilon}^{\hat{-}\dot{\alpha}}\hat{\mathcal{A}}_{\alpha\dot{\alpha}} - 2\epsilon^{\hat{-}}_{\alpha}\mathcal{W}\right)\\
&\quad -(\theta^{\hat{+}})^2\bar{\epsilon}^{(A}_{\dot{\alpha}}\widetilde{\psi}^{B)\dot{\alpha}}v^{\hat{-}}_{A}v^{\hat{-}}_{B} + (\bar{\theta}^{\hat{+}})^2\epsilon^{(A\alpha}\psi^{B)}_{\alpha}v^{\hat{-}}_{A}v^{\hat{-}}_{B} - 2\theta^{\hat{+}}_{\alpha}\bar{\theta}^{\hat{+}}_{\dot{\alpha}}\left(\epsilon^{(A\alpha}\widetilde{\psi}^{B)\dot{\alpha}} + \bar{\epsilon}^{(A\dot{\alpha}}\psi^{B)\alpha}\right)v^{\hat{-}}_{A}v^{\hat{-}}_{B}\\
&\quad +\theta^{\hat{+}\alpha}(\bar{\theta}^{\hat{+}})^2\epsilon^{(A}_{\alpha}\mathcal{D}^{BC)}v^{\hat{-}}_{A}v^{\hat{-}}_{B}v^{\hat{-}}_{C} + \bar{\theta}^{\hat{+}}_{\dot{\alpha}}(\theta^{\hat{+}})^2\bar{\epsilon}^{(A\dot{\alpha}}\mathcal{D}^{BC)}v^{\hat{-}}_{A}v^{\hat{-}}_{B}v^{\hat{-}}_{C}\,,
\end{aligned}\tag{4.37}$$

$$\Lambda^{(\text{comp})}_2 = 2\sqrt{2}i\theta^{-\alpha}q^{+A}\epsilon_{A\alpha} + 2\sqrt{2}i\bar{\theta}^{-\dot{\alpha}}q^{+A}\bar{\epsilon}_{A\dot{\alpha}}. \tag{4.38}$$

For the variation of the superfield $\mathcal{V}^{++}$ we obtain

$$\delta\,\mathcal{V}^{++} = 2i(\epsilon^{\hat{-}\alpha}\bar{\theta}^{\hat{+}\dot{\alpha}} + \theta^{\hat{+}\alpha}\bar{\epsilon}^{\hat{-}\dot{\alpha}})\partial_{\alpha\dot{\alpha}}\mathcal{V}^{++} + \nabla^{++}\Lambda^{(\text{comp})}. \tag{4.39}$$

Let us inspect $\nabla^{++}\Lambda^{(\text{comp})}$. Using the relations (4.10)–(4.12) we find

$$\begin{aligned}
\nabla^{++}\Lambda^{(\text{comp})} &= -2i(\epsilon^{\hat{-}\alpha}\bar{\theta}^{\hat{+}\dot{\alpha}} + \theta^{\hat{+}\alpha}\bar{\epsilon}^{\hat{-}\dot{\alpha}})\partial_{\alpha\dot{\alpha}}\mathcal{V}^{++}\\
&\quad + \nabla^{++}\left(2\sqrt{2}i\theta^{-\alpha}q^{+A}\epsilon_{A\alpha} + 2\sqrt{2}i\bar{\theta}^{-\dot{\alpha}}q^{+A}\bar{\epsilon}_{A\dot{\alpha}}\right).
\end{aligned}\tag{4.40}$$

The first term precisely cancels the unwanted term in (4.39) involving $\theta^{\hat{+}\alpha,\dot{\alpha}}$, while the second term, with taking into account the on-shell condition $\nabla^{++}q^{+}_{A} = 0$, yields the already known transformation

$$\delta\,\mathcal{V}^{++} = -\left[2\sqrt{2}i\epsilon^{A\alpha}\theta^{+}_{\alpha} - 2\sqrt{2}i\bar{\epsilon}^{A}_{\dot{\alpha}}\bar{\theta}^{+\dot{\alpha}}\right]q^{+}_{A}. \tag{4.41}$$

Similarly, considering the transformations of the superfield $\phi^{+\hat{+}}$ and using the equations of motion that will be obtained below (eq. (4.77)), one obtains the transformation law of the hypermultiplet q^{+}_{A}

$$\begin{aligned}
\sqrt{2}\delta\, q^{+\hat{+}} &= -\epsilon^{\hat{+}\alpha}\mathcal{W}^{+}_{\alpha} + 2\theta^{-\alpha}\epsilon^{\hat{+}}_{\alpha}[q^{+A}, q^{+}_{A}] + \text{c.c.}\\
&= \frac{i}{8}\epsilon^{\hat{+}\alpha}\left(2D^{+}_{\alpha}(\bar{D}^{+})^2\mathcal{V}^{--} + \theta^{-}_{\alpha}(\bar{D}^{+})^2\mathcal{V}^{--}\right) + \text{c.c.}\\
&= \frac{i}{8}\epsilon^{\hat{+}\alpha}(D^{+})^2(\bar{D}^{+})^2(\theta^{-}_{\alpha}\mathcal{V}^{--}) - \frac{i}{8}\bar{\epsilon}^{\hat{+}}_{\dot{\alpha}}(D^{+})^2(\bar{D}^{+})^2(\bar{\theta}^{-\dot{\alpha}}\mathcal{V}^{--}),
\end{aligned}\tag{4.42}$$

or, equivalently,

$$\delta\, q^{+}_{A} = \frac{1}{16\sqrt{2}}(D^{+})^2(\bar{D}^{+})^2\left[2i\epsilon^{\alpha}_{A}\theta^{-}_{\alpha}\mathcal{V}^{--} - 2i\bar{\epsilon}_{A\dot{\alpha}}\bar{\theta}^{-\dot{\alpha}}\mathcal{V}^{--}\right]. \tag{4.43}$$

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Rescaling the parameters ϵ as

$$\epsilon_{A\alpha} = \frac{i}{2\sqrt{2}}\epsilon'_{A\alpha}, \tag{4.44}$$

we recover the already known realization of the hidden supersymmetry (2.27)

$$\delta\,\mathcal{V}^{++} = \left[\epsilon^{A\alpha}\theta^{+}_{\alpha} - \bar{\epsilon}^{A}_{\dot\alpha}\bar{\theta}^{+\dot\alpha}\right] q^{+}_{A}, \quad \delta\, q^{+}_{A} = -\frac{1}{32}(D^{+})^{2}(\bar{D}^{+})^{2}\left[\epsilon^{\alpha}_{A}\theta^{-}_{\alpha}\mathcal{V}^{--} + \bar{\epsilon}_{A\dot\alpha}\bar{\theta}^{-\dot\alpha}\mathcal{V}^{--}\right]. \tag{4.45}$$

Now it is quite legitimate to identify $\mathcal{N}=2$ superfields q^{+}_{A} and $\mathcal{V}^{++}$ with the hypermultiplet and gauge multiplet superfields from section 2. At this stage, we have expressed all the geometric quantities of $\mathcal{N}=4$ SYM theory in terms of $\mathcal{N}=2$ superfields. It remains to relate the superfield coefficients appearing in (3.41) to the basic $\mathcal{N}=2$ superfields $\mathcal{V}^{++}, q^{+A}$. This can be done in an algebraic way, without solving any differential equations, by requiring that the vector connections and the superfield strengths obtained from the relations with and without "hats" coincide with each other.

4.3 Identifying vector connections

So we are led to explore the superfield vector connections in the sectors with and without "hat". First, we will consider the sector including derivatives with respect to the ordinary coordinates (without "hats"). We define $\bar{\nabla}^{-}_{\dot\alpha}$ in the standard way

$$\bar{\nabla}^{-}_{\dot\alpha} := \bar{D}^{-}_{\dot\alpha} + i\bar{\mathcal{A}}^{-}_{\dot\alpha} = [\nabla^{--}, \bar{\nabla}^{+}_{\dot\alpha}], \quad \bar{\mathcal{A}}^{-}_{\dot\alpha} = \bar{\mathcal{A}}^{-(0)}_{\dot\alpha} - 2i\sqrt{2}\bar{\theta}^{\hat{+}}_{\dot\alpha} q^{-A} v^{\hat{-}}_{A} + 2i\bar{\theta}^{\hat{-}}_{\dot\alpha}\nabla^{--}\phi^{+\hat{+}}, \tag{4.46}$$

$$\bar{\mathcal{A}}^{-(0)}_{\dot\alpha} = -\bar{D}^{+}_{\dot\alpha}\mathcal{V}^{--}, \tag{4.47}$$

where

$$\nabla^{--} = D^{--} + i\mathcal{V}^{--}, \quad q^{-A} := \nabla^{--} q^{+A}\,, \tag{4.48}$$

and $\mathcal{V}^{--}$ is related to $\mathcal{V}^{++}$ via the harmonic zero curvature condition

$$D^{++}\mathcal{V}^{--} - D^{--}\mathcal{V}^{++} + i[\mathcal{V}^{++}, \mathcal{V}^{--}] = 0. \tag{4.49}$$

Accordingly, vector connection is defined in the standard way,

$$\{\nabla^{+}_{\alpha}, \bar{\nabla}^{-}_{\dot\beta}\} = -2i(\partial_{\alpha\dot\beta} + i\mathcal{V}_{\alpha\dot\beta}), \qquad \mathcal{V}_{\alpha\dot\beta} = -\frac{1}{2i}(\nabla^{+}_{\alpha}\bar{\mathcal{A}}^{-}_{\dot\beta} + \bar{D}^{-}_{\dot\beta}\mathcal{A}^{+}_{\alpha}). \tag{4.50}$$

Using the expressions (4.46) for $\bar{\mathcal{A}}^{-}_{\dot\beta}$ and (4.32) for $\mathcal{A}^{+}_{\alpha}$, we find the expression for $\mathcal{V}_{\alpha\dot\beta}$

$$\begin{aligned}\mathcal{V}_{\alpha\dot\beta} = \mathcal{A}_{\alpha\dot\beta} &- \sqrt{2}\,\bar{\theta}^{\hat{+}}_{\dot\beta} D^{+}_{\alpha} q^{-A} v^{\hat{-}}_{A} + \bar{\theta}^{\hat{-}}_{\dot\beta} D^{+}_{\alpha}\nabla^{--}\phi^{+\hat{+}} - \sqrt{2}\theta^{\hat{+}}_{\alpha}\bar{\nabla}^{-(0)}_{\dot\beta} q^{+A} v^{\hat{-}}_{A} \\ &+ \theta^{\hat{-}}_{\alpha}\bar{\nabla}^{-(0)}_{\dot\beta}\phi^{+\hat{+}} + 4\theta^{\hat{+}}_{\alpha}\bar{\theta}^{\hat{+}}_{\dot\beta}[q^{+A}, q^{-B}] v^{\hat{-}}_{A} v^{\hat{-}}_{B} - 2\sqrt{2}\theta^{\hat{+}}_{\alpha}\bar{\theta}^{\hat{-}}_{\dot\beta}[q^{+A}, \nabla^{--}\phi^{+\hat{+}}] v^{\hat{-}}_{A} \\ &- 2\sqrt{2}i\theta^{\hat{-}}_{\alpha}\bar{\theta}^{\hat{+}}_{\dot\beta}[\phi^{+\hat{+}}, q^{-A}] v^{\hat{-}}_{A} + 2\theta^{\hat{-}}_{\alpha}\bar{\theta}^{\hat{-}}_{\dot\beta}[\phi^{+\hat{+}}, \nabla^{--}\phi^{+\hat{+}}]\,,\end{aligned} \tag{4.51}$$

where

$$\bar{\nabla}^{-(0)}_{\dot\beta} = \bar{D}^{-}_{\dot\beta} + i\bar{\mathcal{A}}^{-(0)}_{\dot\beta}\,, \quad \mathcal{A}_{\alpha\dot\beta} = -\frac{1}{2i} D^{+}_{\alpha}\bar{\mathcal{A}}^{-(0)}_{\dot\beta}. \tag{4.52}$$

The $\mathcal{N}=4$ vector connection (4.51) displays a restricted dependence on $\theta^{\hat-}_\alpha, \bar\theta^{\hat-}_{\dot\alpha}$ (only monomials of the first and second orders appear), but includes all $\theta^{\hat+}_\alpha, \bar\theta^{\hat+}_{\dot\alpha}$ monomials. For what follows it will be useful to quote the opposite chirality counterpart of the $\mathcal{N}=2$ spinor covariant derivative (4.52)

$$\nabla^{-(0)}_\beta = D^-_\beta + i\mathcal{A}^{-(0)}_\beta\,, \quad \mathcal{A}^{-(0)}_\beta = -D^+_\beta \mathcal{V}^{--}\,. \tag{4.53}$$

One can perform an analogous construction for the derivatives with "hats". We define the relevant second harmonic connection $V^{\hat-\hat-}$ by the "hat" flatness condition

$$D^{\hat+\hat+}V^{\hat-\hat-} - D^{\hat-\hat-}V^{\hat+\hat+} + i[V^{\hat+\hat+}, V^{\hat-\hat-}] = 0, \tag{4.54}$$

and then define the relevant spinor and vector connections

$$\bar\nabla^{\hat-}_{\dot\alpha} := \bar D^{\hat-}_{\dot\alpha} + i\bar{\mathcal{A}}^{\hat-}_{\dot\alpha} = [\nabla^{\hat-\hat-}, \bar D^{\hat+}_{\dot\alpha}], \qquad \bar{\mathcal{A}}^{\hat-}_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\hat-\dot\alpha}}V^{\hat-\hat-}, \tag{4.55}$$

$$\{D^{\hat+}_\alpha, \bar\nabla^{\hat-}_{\dot\beta}\} = -2i(\partial_{\alpha\dot\beta} + i\hat{\mathcal{V}}_{\alpha\dot\beta}), \qquad \hat{\mathcal{V}}_{\alpha\dot\beta} = -\frac{i}{2}\frac{\partial}{\partial\theta^{\hat-\alpha}}\frac{\partial}{\partial\bar\theta^{\hat-\dot\beta}}V^{\hat-\hat-}, \tag{4.56}$$

where $\nabla^{\hat-\hat-} = D^{\hat-\hat-} + iV^{\hat-\hat-}$.

In order to perform further calculations we need the expression for $V^{\hat-\hat-}$. We parametrize the $\theta^{\hat-}_\alpha, \bar\theta^{\hat-}_{\dot\alpha}$ expansion of $V^{\hat-\hat-}$ in the following way

$$V^{\hat-\hat-} = -2i\theta^{\hat-}_\alpha\bar\theta^{\hat-}_{\dot\alpha}w^{\alpha\dot\alpha} + (\theta^{\hat-})^2 w + (\bar\theta^{\hat-})^2\widetilde{w} + (\bar\theta^{\hat-})^2\theta^{\hat-\alpha}w^{\hat+}_\alpha + (\theta^{\hat-})^2\bar\theta^{\hat-}_{\dot\alpha}\widetilde{w}^{\hat+\dot\alpha} + (\theta^{\hat-})^2(\bar\theta^{\hat-})^2 w^{\hat+2}. \tag{4.57}$$

The $(\theta^{\hat+}_\alpha, \bar\theta^{\hat+}_{\dot\alpha}, v^{\hat+}_A)$-dependence of the coefficients in this expression will be determined from eq. (4.54), with $V^{\hat+\hat+}$ taken in the form (3.41). Possible coefficients of the monomials of first and zeroth orders in $\theta^{\hat-}_\alpha, \bar\theta^{\hat-}_{\dot\alpha}$ can be shown to vanish as a consequence of equations like $\partial^{\hat+\hat+}\omega^{\hat-} = 0 \Rightarrow \omega^{\hat-} = 0$. The process of solving the equation (4.54) is rather tiresome and the solution looks rather bulky. We give it in the appendix B. Here we collect only the information that is needed for further steps, viz., the expressions for the coefficients $w^{\beta\dot\beta}$ and w,

$$w^{\beta\dot\beta} = \hat{\mathcal{A}}^{\beta\dot\beta} - i\theta^{\hat+\beta}\widetilde\psi^{A\dot\beta}v^{\hat-}_A - i\bar\theta^{\hat+\dot\beta}\psi^{A\beta}v^{\hat-}_A + 2i\theta^{\hat+\beta}\bar\theta^{\hat+\dot\beta}\mathcal{D}^{AB}v^{\hat-}_A v^{\hat-}_B, \tag{4.58}$$

$$w = \mathcal{W} - \bar\theta^{\hat+}_{\dot\beta}\widetilde\psi^{A\dot\beta}v^{\hat-}_A + (\bar\theta^{\hat+})^2\mathcal{D}^{AB}v^{\hat-}_A v^{\hat-}_B, \tag{4.59}$$

where we made use of the relations (3.42). Note that one cannot calculate these coefficients directly from eq. (4.54). The procedure of finding them requires the knowledge of all the coefficients obeying a set of coupled harmonic equations.

Now one can determine the vector connection $\hat{\mathcal{V}}_{\alpha\dot\beta}$. Substituting (4.57) into (4.56) and using eq. (4.58) we obtain, in zeroth order in $\theta^{\hat-}_\alpha, \bar\theta^{\hat-}_{\dot\alpha}$,

$$\hat{\mathcal{V}}^{\beta\dot\beta}|_{\theta^{\hat-}_\alpha, \bar\theta^{\hat-}_{\dot\alpha}=0} = \hat{\mathcal{A}}^{\beta\dot\beta} - i\theta^{\hat+\beta}\widetilde\psi^{A\dot\beta}v^{\hat-}_A - i\bar\theta^{\hat+\dot\beta}\psi^{A\beta}v^{\hat-}_A + 2i\theta^{\hat+\beta}\bar\theta^{\hat+\dot\beta}\mathcal{D}^{AB}v^{\hat-}_A v^{\hat-}_B\,. \tag{4.60}$$

Identifying vector connections from both sectors,

$$\hat{\mathcal{V}}_{\alpha\dot\beta} = \mathcal{V}_{\alpha\dot\beta}, \tag{4.61}$$

and, in particular, the two expansions (4.60) and (4.51), in zeroth order in $\theta^{\hat{-}}_\alpha, \bar\theta^{\hat{-}}_{\dot\alpha}$ we get

$$\hat{\mathcal{A}}_{\alpha\dot\beta} = \mathcal{A}_{\alpha\dot\beta}, \quad \mathcal{D}^{AB} = -2i[q^{+(A}, q^{-B)}], \quad \widetilde{\psi}^A_{\dot\beta} = -i\sqrt{2}\bar\nabla^{-(0)}_{\dot\beta} q^{+A}, \quad \psi^A_\beta = i\sqrt{2}\nabla^{-(0)}_\beta q^{+A}. \tag{4.62}$$

Thus, we have succeeded to express a part of $\mathcal{N} = 2$ superfields in (3.41) and (4.24) in terms of the hypermultiplet q^+_A and the gauge superfield $\mathcal{V}^{++}$. However, the superfields $\mathcal{W}^+_\alpha$, $\widetilde{\mathcal{W}}^+_{\dot\alpha}$ and $\mathcal{W}$, $\bar{\mathcal{W}}$ still remain unspecified.

It is easy to obtain the expression for $\mathcal{W}^+_\alpha$, $\widetilde{\mathcal{W}}^+_{\dot\alpha}$. To this end, we substitute $\hat{\mathcal{A}}_{\alpha\dot\alpha}$ from (4.62) into (4.31). As a result, we find

$$\mathcal{W}^+_\alpha = -\frac{i}{4} D^+_\alpha \left(\bar D^+\right)^2 \mathcal{V}^{--}, \qquad \widetilde{\mathcal{W}}^+_{\dot\alpha} = -\frac{i}{4} \bar D^+_{\dot\alpha} \left(D^+\right)^2 \mathcal{V}^{--}. \tag{4.63}$$

The rest of the constraint (4.31) is reduced to $D^+_\alpha \mathcal{A}_{\beta\dot\beta} + D^+_\beta \mathcal{A}_{\alpha\dot\beta} = 0$ (and c.c.), which is satisfied identically since $\mathcal{A}_{\alpha\dot\beta} \sim D^+_\alpha \bar{\mathcal{A}}^{-(0)}_{\dot\beta}$.

More effort is required to determine the superfields $\mathcal{W}$, $\bar{\mathcal{W}}$. One needs to take into account that the superfield strength W, like the vector connection, can de expressed in two ways. First, we can use the relation

$$\{D^{\hat{+}}_\alpha, \nabla^{\hat{-}}_\beta\} = 2i\epsilon_{\alpha\beta}\bar W \quad \text{(and c.c.)}\,, \tag{4.64}$$

where

$$\nabla^{\hat{-}}_\alpha = [\nabla^{\hat{-}\hat{-}}, D^{\hat{+}}_\alpha]. \tag{4.65}$$

It follows from the second line of (3.16) by contracting its both sides with $v^{\hat{+}A}v^{\hat{-}B}$ and then passing to the analytic frame. Substituting (4.65) into (4.64), we obtain

$$\bar W = -\frac{1}{4}(D^{\hat{+}})^2 V^{\hat{-}\hat{-}}, \qquad W = -\frac{1}{4}(\bar D^{\hat{+}})^2 V^{\hat{-}\hat{-}}. \tag{4.66}$$

Note that the definition (4.64) implies, through Bianchi identities, the covariant harmonic independence of W, $\nabla^{\pm\pm}W = \nabla^{\hat{\pm}\hat{\pm}}W = 0$, as well as the reality condition $(D^{\hat{+}})^2\bar W = (\bar D^{\hat{+}})^2 W$.

Alternatively, we can use the relation

$$\{\nabla^+_\alpha, \nabla^-_\beta\} = 2i\epsilon_{\alpha\beta}W \quad \text{(and c.c.)}\,, \tag{4.67}$$

with ∇^-_α being defined in the standard way, $\nabla^-_\alpha = [\nabla^{--}, \nabla^+_\alpha]$. Eq. (4.67) amounts to

$$D^+_\alpha \mathcal{A}^-_\beta + D^-_\beta \mathcal{A}^+_\alpha + i\{\mathcal{A}^+_\alpha, \mathcal{A}^-_\beta\} = 2\epsilon_{\alpha\beta}W \quad \text{(and c.c.)}\,. \tag{4.68}$$

Then, substituting the definition of $\mathcal{A}^+_\beta$ from (4.32), we obtain another expression for the superfield strength W (in zeroth order in $\theta^{\hat\pm}_\alpha, \bar\theta^{\hat\pm}_{\dot\alpha}$)

$$W|_{\theta^{\hat\pm}_\alpha = \bar\theta^{\hat\pm}_{\dot\alpha} = 0} = -\frac{1}{4}\left(D^+\right)^2 \mathcal{V}^{--} \quad \text{(and c.c.)}\,, \tag{4.69}$$

Substituting the expansion (4.57) for $V^{\hat{-}\hat{-}}$ in (4.66) and using that $w|_{\theta^{\hat+}_\alpha = \bar\theta^{\hat+}_{\dot\alpha} = 0} = \mathcal{W}$ as follows from (4.59), we equate (4.69) to (4.66) and obtain the sought expressions for $\mathcal{W}$, $\bar{\mathcal{W}}$

$$\mathcal{W} = -\frac{1}{4}\left(\bar D^+\right)^2 \mathcal{V}^{--}, \qquad \bar{\mathcal{W}} = -\frac{1}{4}\left(D^+\right)^2 \mathcal{V}^{--}. \tag{4.70}$$

These expressions coincide with those defined in subsection 2.2. Eqs. (4.63) can be rewritten as

$$\mathcal{W}^+_\alpha = iD^+_\alpha \mathcal{W}\,, \qquad \widetilde{\mathcal{W}}^+_{\dot\alpha} = i\bar{D}^+_{\dot\alpha}\bar{\mathcal{W}}. \tag{4.71}$$

Thus, we have expressed all superfield components of $V^{\hat{-}\hat{-}}$ in terms of the hypermultiplet and $\mathcal{N} = 2$ gauge superfields q^+_A and $\mathcal{V}^{++}$. Now one can be convinced that the previously deduced conditions (4.10)–(4.13) are indeed satisfied.

To summarize, all the bi-harmonic $\mathcal{N} = 4$ SYM superfields we started with proved to be expressed in terms of the two basic analytic $\mathcal{N} = 2$ superfields involved in the $\mathcal{N} = 2$ harmonic superspace action principle for $\mathcal{N} = 4$ SYM theory considered in section 2.

4.4 Guide to section 4

For reader's convenience, in this subsection we quote the expressions of the basic involved $\mathcal{N} = 4$ superfields in terms of the hypermultiplet and gauge superfields q^+_A and $\mathcal{V}^{++}$, as well as the equations of motion for the latter, as the result of solving the $\mathcal{N} = 4$ SYM constraints (3.29)–(3.35).

The expressions for the $\mathcal{N} = 4$ spinor connections were obtained in subsection 4.1.3

$$\mathcal{A}^+_\alpha = -2\sqrt{2}i\theta^{\hat+}_\alpha q^{+\hat-} + 2i\theta^{\hat-}_\alpha \phi^{+\hat+}, \quad \bar{\mathcal{A}}^+_{\dot\alpha} = -2\sqrt{2}i\bar\theta^{\hat+}_{\dot\alpha} q^{+\hat-} + 2i\bar\theta^{\hat-}_{\dot\alpha}\phi^{+\hat+}. \tag{4.72}$$

The $\mathcal{N} = 2$ vector and spinor connection were found in subsection 4.3

$$\mathcal{A}_{\alpha\dot\alpha} = \frac{1}{2i} D^+_\alpha \bar{D}^+_{\dot\alpha}\mathcal{V}^{--}, \qquad \mathcal{A}^{-(0)}_\alpha = -D^+_\alpha \mathcal{V}^{--}, \qquad \bar{\mathcal{A}}^{-(0)}_{\dot\alpha} = -\bar{D}^+_{\dot\alpha}\mathcal{V}^{--}. \tag{4.73}$$

The expressions for $\mathcal{N} = 2$ superfields entering $V^{\hat+\hat+}$ in WZ gauge (3.41) were obtained in the sections 4.3 in eqs. (4.62) and (4.70):

$$\hat{\mathcal{A}}_{\alpha\dot\alpha} = \mathcal{A}_{\alpha\dot\alpha}\,, \qquad \mathcal{W} = -\frac{1}{4}\left(\bar{D}^+\right)^2\mathcal{V}^{--}, \qquad \bar{\mathcal{W}} = -\frac{1}{4}\left(D^+\right)^2\mathcal{V}^{--}, \tag{4.74}$$

$$\widetilde{\psi}^A_{\dot\beta} = -i\sqrt{2}\bar\nabla^{-(0)}_{\dot\beta} q^{+A}, \qquad \psi^A_\beta = i\sqrt{2}\nabla^{-(0)}_\beta q^{+A}, \qquad \mathcal{D}^{AB} = -2i[q^{+(A}, q^{-B)}]. \tag{4.75}$$

Analyticity of the hypermultiplet q^+_A was shown in subsection 4.1, eq. (4.33).

One of the equations of motion, namely,

$$\nabla^{++} q^+_A = 0, \tag{4.76}$$

was found in subsection 4.1.2, eq. (4.25). To obtain another equation, one uses the relation (4.33) from subsection 4.1.3, as well as the expression (4.63) for $\mathcal{W}^+_\alpha$. After substituting one into another, the second equation of motion follows

$$F^{++} = -i[q^{+A}, q^+_A], \qquad F^{++} = \frac{1}{16}\left(D^+\right)^2\left(\bar{D}^+\right)^2\mathcal{V}^{--}. \tag{4.77}$$

Note that the rest of eq. (4.33), $D^+_\alpha\mathcal{W}^+_\beta + D^+_\beta\mathcal{W}^+_\alpha = 0$ (and c.c), is satisfied identically.

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The definition of the superfield strength $\phi^{+\hat{+}}$ is given by eq. (4.24) of subsection 4.1.2. Substituting the expressions (4.71) and (4.75) into (4.24), we deduce the final expression for $\phi^{+\hat{+}}$

$$\begin{aligned}\phi^{+\hat{+}} = & \sqrt{2}q^{+A}v^{\hat{+}}_A + i\theta^{\hat{+}\alpha}D^{+}_{\alpha}\mathcal{W} + i\bar{\theta}^{\hat{+}}_{\dot{\alpha}}\bar{D}^{+\dot{\alpha}}\bar{\mathcal{W}} + 2(\bar{\theta}^{\hat{+}})^2\theta^{\hat{+}\alpha}[\nabla^{-(0)}_{\alpha}q^{+(A}, q^{+B)}]v^{\hat{-}}_A v^{\hat{-}}_B \\ & -\sqrt{2}i(\theta^{\hat{+}})^2[\mathcal{W}, q^{+A}]v^{\hat{-}}_A - \sqrt{2}i(\bar{\theta}^{\hat{+}})^2[\bar{\mathcal{W}}, q^{+A}]v^{\hat{-}}_A + 2(\theta^{\hat{+}})^2\bar{\theta}^{\hat{+}}_{\dot{\alpha}}[\bar{\nabla}^{-(0)\dot{\alpha}}q^{+(A}, q^{+B)}]v^{\hat{-}}_A v^{\hat{-}}_B \\ & +2\sqrt{2}i\theta^{\hat{+}}_{\alpha}\bar{\theta}^{\hat{+}}_{\dot{\alpha}}\hat{\nabla}^{\alpha\dot{\alpha}}q^{+A}v^{\hat{-}}_A - 2\sqrt{2}(\theta^{\hat{+}})^2(\bar{\theta}^{\hat{+}})^2[[q^{+A}, q^{-B}], q^{+C}]v^{\hat{-}}_A v^{\hat{-}}_B v^{\hat{-}}_C, \end{aligned} \tag{4.78}$$

where the spinor covariant derivatives were defined after eq. (4.51), and the vector covariant derivative was defined in (4.13). They are derivatives with connections (4.73):

$$\hat{\nabla}_{\alpha\dot{\alpha}} = \partial_{\alpha\dot{\alpha}} + \frac{1}{2}D^{+}_{\alpha}\bar{D}^{+}_{\dot{\alpha}}\mathcal{V}^{--}\,, \quad \nabla^{-(0)}_{\alpha} = D^{-}_{\alpha} - iD^{+}_{\alpha}\mathcal{V}^{--}\,, \quad \bar{\nabla}^{-(0)}_{\dot{\alpha}} = \bar{D}^{-}_{\dot{\alpha}} - i\bar{D}^{+}_{\dot{\alpha}}\mathcal{V}^{--}\,. \tag{4.79}$$

Now we have all the necessary ingredients to directly check the on-shell validity of the constraints (3.38) and, hence, of (3.29) (recall the discussion in the end of subsection 4.1.3). When doing so, one should take into account that on shell, with $\nabla^{++}q^{+A} = 0$, the following condition is valid

$$\nabla^{++}q^{-A} = \nabla^{++}\nabla^{--}q^{+A} = q^{+A}\,.$$

The expression for the superfield strength W was found in subsection 4.3

$$\bar{W} = -\frac{1}{4}(D^{\hat{+}})^2 V^{\hat{-}\hat{-}}, \qquad W = -\frac{1}{4}(\bar{D}^{\hat{+}})^2 V^{\hat{-}\hat{-}}. \tag{4.80}$$

Bianchi identity for the superfield strength W follows from its definition (eqs. (4.64), (4.67))

$$\begin{aligned} (D^{\hat{+}})^2\bar{W} &= (\bar{D}^{\hat{+}})^2 W\,, & \nabla^{++}W &= \nabla^{\hat{+}\hat{+}}W = 0 & &(\text{and c.c.})\,, \\ D^{\hat{+}}_{\alpha}W &= -i\nabla^{-}_{\alpha}\phi^{+\hat{+}}, & \nabla^{+}_{\alpha}\bar{W} &= i\nabla^{\hat{-}}_{\alpha}\phi^{+\hat{+}} & &(\text{and c.c.})\,. \end{aligned} \tag{4.81}$$

The chirality conditions are also obvious

$$D^{\hat{+}}_{\alpha}\bar{W} = \nabla^{\hat{-}}_{\alpha}\bar{W} = 0 \quad (\text{and c.c.}). \tag{4.82}$$

The explicit expressions for W and $\bar{W}$ are rather cumbersome, so we will prefer to give them only for abelian case.

4.4.1 Abelian case

This subsection presents some important consequences of constraints in abelian case. In this case everything becomes simpler as all commutators vanish.

The expressions for $\mathcal{N} = 2$ superfields appearing in the definition of $V^{\hat{+}\hat{+}}$ (3.41) become

$$\mathcal{W} = -\frac{1}{4}\left(\bar{D}^{+}\right)^2\mathcal{V}^{--}, \qquad \bar{\mathcal{W}} = \frac{1}{4}\left(D^{+}\right)^2\mathcal{V}^{--}, \tag{4.83}$$

$$\widetilde{\psi}^{A}_{\dot{\beta}} = -i\sqrt{2}\bar{D}^{-}_{\dot{\beta}}q^{+A}, \qquad \psi^{A}_{\beta} = i\sqrt{2}D^{-}_{\beta}q^{+A}, \qquad \mathcal{D}^{AB} = 0. \tag{4.84}$$

The equations of motions from the previous subsection read

$$D^{++}q^{+}_A = 0, \qquad \left(D^{+}\right)^2\left(D^{-}\right)^2\mathcal{V}^{--} = 0. \tag{4.85}$$

The definition of the superfield strength $\phi^{+\hat{+}}$ is given by eq. (4.24). Substituting the expressions (4.83) and (4.84) in it, we obtain

$$\phi^{+\hat{+}} = \sqrt{2}q^{+A}v^{\hat{+}}_{A} + i\theta^{\hat{+}\alpha}D^{+}_{\alpha}\mathcal{W} + i\bar{\theta}^{\hat{+}}_{\dot{\alpha}}\bar{D}^{+\dot{\alpha}}\bar{\mathcal{W}} + 2\sqrt{2}i\theta^{\hat{+}}_{\alpha}\bar{\theta}^{\hat{+}}_{\dot{\alpha}}\partial^{\alpha\dot{\alpha}}q^{+A}v^{\hat{-}}_{A}. \tag{4.86}$$

The expression for the superfield strengths W and $\bar{W}$ were found in subsection 4.3:

$$\bar{W} = -\frac{1}{4}(D^{\hat{+}})^2 V^{\hat{-}\hat{-}}, \qquad W = -\frac{1}{4}(\bar{D}^{\hat{+}})^2 V^{\hat{-}\hat{-}}. \tag{4.87}$$

Taking into account the equations of motion, the $\theta^{\hat{\pm}}_{\alpha}, \bar{\theta}^{\hat{\pm}}_{\dot{\alpha}}$ expansion of these quantities reads

$$\begin{aligned} W &= \bar{\mathcal{W}} + i\sqrt{2}\theta^{\hat{+}\beta}D^{-}_{\beta}q^{+A}v^{\hat{-}}_{A} - i\sqrt{2}\theta^{\hat{-}\beta}D^{-}_{\beta}q^{+A}v^{\hat{+}}_{A} + 2i\theta^{\hat{-}\beta}\bar{\theta}^{\hat{+}\dot{\alpha}}\partial_{\beta\dot{\alpha}}\bar{\mathcal{W}} \\ &\quad -\theta^{\hat{-}\alpha}\theta^{\hat{+}\beta}D^{-}_{\beta}D^{+}_{\alpha}\mathcal{W} + 2\sqrt{2}\theta^{\hat{+}}_{\alpha}\bar{\theta}^{\hat{+}}_{\dot{\alpha}}\partial^{\alpha\dot{\alpha}}\theta^{\hat{-}\beta}D^{-}_{\beta}q^{+A}v^{\hat{-}}_{A}, \end{aligned} \tag{4.88}$$

$$\begin{aligned} \bar{W} &= \mathcal{W} + i\sqrt{2}\bar{\theta}^{\hat{+}}_{\dot{\beta}}\bar{D}^{-\dot{\beta}}q^{+A}v^{\hat{-}}_{A} - i\sqrt{2}\bar{\theta}^{\hat{-}}_{\dot{\beta}}\bar{D}^{-\dot{\beta}}q^{+A}v^{\hat{+}}_{A} - 2i\bar{\theta}^{\hat{-}}_{\dot{\beta}}\theta^{\hat{+}}_{\alpha}\partial^{\alpha\dot{\beta}}\mathcal{W} \\ &\quad -\bar{\theta}^{\hat{-}}_{\dot{\alpha}}\bar{\theta}^{\hat{+}}_{\dot{\beta}}\bar{D}^{-\dot{\beta}}\bar{D}^{+\dot{\alpha}}\bar{\mathcal{W}} + 2\sqrt{2}\theta^{\hat{+}}_{\alpha}\bar{\theta}^{\hat{+}}_{\dot{\alpha}}\partial^{\alpha\dot{\alpha}}\bar{\theta}^{\hat{-}}_{\dot{\beta}}\bar{D}^{-\dot{\beta}}q^{+A}v^{\hat{-}}_{A}. \end{aligned} \tag{4.89}$$

It is instructive to list here a few further properties of the superfield strengths that will be used later. The zero curvature condition (4.54) and the definitions of W and $\bar{W}$ (4.87) imply

$$D^{\hat{+}\hat{+}}W = D^{\hat{-}\hat{-}}W = D^{\hat{+}\hat{+}}\bar{W} = D^{\hat{-}\hat{-}}\bar{W} = 0\,. \tag{4.90}$$

The definitions of W and $\bar{W}$ (4.87) and the conditions listed previously imply the chirality and antichirality of $\bar{W}$ and W in the "hat"-sector

$$D^{\hat{\pm}}_{\alpha}\bar{W} = \bar{D}^{\hat{\pm}}_{\dot{\alpha}}W = 0\,. \tag{4.91}$$

In addition, the expansions (4.88) and (4.89) imply chirality and antichirality of W and $\bar{W}$ in the sector without "hat"

$$D^{\pm}_{\alpha}W = \bar{D}^{\pm}_{\dot{\alpha}}\bar{W} = 0. \tag{4.92}$$

The relations (4.64) and (4.67) entail Bianchi identities relating the superfields W, $\bar{W}$, $\phi^{+\hat{+}}$

$$\begin{aligned} D^{\hat{+}}_{\alpha}W &= -iD^{-}_{\alpha}\phi^{+\hat{+}}\,, \qquad & D^{+}_{\alpha}\bar{W} &= iD^{\hat{-}}_{\alpha}\phi^{+\hat{+}}\,, \\ \bar{D}^{\hat{+}}_{\dot{\alpha}}\bar{W} &= i\bar{D}^{-}_{\dot{\alpha}}\phi^{+\hat{+}}\,, \qquad & \bar{D}^{+}_{\dot{\alpha}}W &= -i\bar{D}^{\hat{-}}_{\dot{\alpha}}\phi^{+\hat{+}}\,. \end{aligned} \tag{4.93}$$

The superfield $\phi^{+\hat{+}}$ also satisfies the conditions

$$\begin{aligned} D^{\hat{+}}_{\alpha}\phi^{+\hat{-}} &= -D^{\hat{-}}_{\alpha}\phi^{+\hat{+}}, \qquad & D^{+}_{\alpha}\phi^{-\hat{+}} &= -D^{-}_{\alpha}\phi^{+\hat{+}}, \\ D^{\hat{-}}_{\alpha}\phi^{\pm\hat{-}} &= 0, \qquad & D^{-}_{\alpha}\phi^{-\hat{\pm}} &= 0, \end{aligned} \tag{4.94}$$

where $\phi^{+\hat{-}} = D^{\hat{-}\hat{-}}\phi^{+\hat{+}}$, $\phi^{-\hat{+}} = D^{--}\phi^{+\hat{+}}$ and $\phi^{-\hat{-}} = D^{--}D^{\hat{-}\hat{-}}\phi^{+\hat{+}}$.

The expansions of W and $\bar{W}$ (4.88), (4.89) also imply the well-known on-shell relations

$$(D^{+})^2W = (D^{\hat{+}})^2W = (D^{+})^2\bar{W} = (D^{\hat{+}})^2\bar{W} = 0. \tag{4.95}$$

All these relations will be employed in section 5, while constructing the expression for invariant effective action.

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5 $\mathcal{N} = 4$ supersymmetric invariants

In this section we apply the bi-harmonic superspace technique to construct examples of the $\mathcal{N} = 4$ supersymmetric invariants which, being rewritten through $\mathcal{N} = 2$ superfields, possess an extra on-shell hidden $\mathcal{N} = 2$ supersymmetry. In particular, we will show how the low-energy effective action (2.29) can be written in terms of $\mathcal{N} = 4$ bi-harmonic superfields, so as to secure, from the very beginning, the hidden second $\mathcal{N} = 2$ supersymmetry. For sake of simplicity, we will focus on the case of Abelian gauge group. As was earlier mentioned, just this case corresponds to the Coulomb branch of $\mathcal{N} = 4$ SYM theory.

5.1 General structure of invariants

We begin with describing a general structure of invariants in bi-harmonic superspace. The simplest expression is

$$\int du\, dv\, d^{20}z\mathcal{L}, \tag{5.1}$$

where $\mathcal{L}$ is some $\mathcal{N} = 4$ superfield and integral goes over the full bi-harmonic superspace. It is invariant under $\mathcal{N} = 4$ supersymmetry due to the presence of integration over all θ's the number of which is twice as bigger than in $\mathcal{N} = 2$ harmonic superspace. The invariant (5.1) can be rewritten, in an obvious way, as an integral over $\mathcal{N} = 2$ harmonic superspace

$$\int du\, dv\, d^{20}z\, \mathcal{L} = \int du\, d^{12}z \left(\int dv\, (D^{\hat{+}})^4 (D^{\hat{-}})^4\, \mathcal{L}\right), \tag{5.2}$$

where the measure $dud^{12}z$ was defined in (2.12). The relation (5.2) allows one to transform the invariants originally written in bi-harmonic superspace to the invariants "living" in the standard $\mathcal{N} = 2$ harmonic superspace.

There are other types of the $\mathcal{N} = 4$ invariants which can be constructed as integrals over various invariant analytic subspaces (3.10) in bi-harmonic superspace:

$$\begin{aligned}
\int du \int dv \int d^4x\, d^4\theta^+\, d^4\theta^{\hat{-}}\, d^4\theta^{\hat{+}} \mathcal{L}^{+4}(\zeta_I) &= \int du \int dv \int d^4x (D^-)^4 (D^{\hat{+}})^4 (D^{\hat{-}})^4 \mathcal{L}^{+4}(\zeta_I),\\
\int du \int dv \int d^4x\, d^4\theta^+\, d^4\theta^-\, d^4\theta^{\hat{+}} \mathcal{L}^{\hat{+}4}(\zeta_{II}) &= \int du \int dv \int d^4x (D^+)^4 (D^-)^4 (D^{\hat{-}})^4 \mathcal{L}^{\hat{+}4}(\zeta_{II}),\\
\int du \int dv \int d^4x\, d^4\theta^+\, d^4\theta^{\hat{+}} \mathcal{L}^{+4\hat{+}4}(\zeta_A) &= \int du \int dv \int d^4x (D^-)^4 (D^{\hat{-}})^4 \mathcal{L}^{+4\hat{+}4}(\zeta_A),
\end{aligned} \tag{5.3}$$

where $\mathcal{L}^{+4},\ \mathcal{L}^{\hat{+}4}, \mathcal{L}^{+4\hat{+}4}$ are (half)analytic $\mathcal{N} = 4$ superfields defined by the constraints

$$\begin{aligned}
D^+_\alpha \mathcal{L}^{+4} &= \bar{D}^+_{\dot{\alpha}} \mathcal{L}^{+4} = 0,\\
D^{\hat{+}}_\alpha \mathcal{L}^{\hat{+}4} &= \bar{D}^{\hat{+}}_{\dot{\alpha}} \mathcal{L}^{\hat{+}4} = 0,\\
D^+_\alpha \mathcal{L}^{+4\hat{+}4} &= \bar{D}^+_{\dot{\alpha}} \mathcal{L}^{+4\hat{+}4} = D^{\hat{+}}_\alpha \mathcal{L}^{+4\hat{+}4} = \bar{D}^{\hat{+}}_{\dot{\alpha}} \mathcal{L}^{+4\hat{+}4} = 0.
\end{aligned} \tag{5.4}$$

The superfield Lagrangian densities in (5.3) are integrated over all those θ's on which they depend. Hence, the expressions (5.3) are invariants of $\mathcal{N} = 4$ supersymmetry. These invariants can also be rewritten as integrals over $\mathcal{N}=2$ harmonic superspace:

$$
\begin{aligned}
\int du \int dv \int d^4x\, d^4\theta^+\, d^4\theta^{\hat{-}}\, d^4\theta^{\hat{+}} \mathcal{L}^{+4} &= \int d\zeta^{-4} \left(\int dv\, (D^{\hat{+}})^4 (D^{\hat{-}})^4 \mathcal{L}^{+4} \right), \\
\int du \int dv \int d^4x\, d^4\theta^+\, d^4\theta^-\, d^4\theta^{\hat{+}} \mathcal{L}^{\hat{+}4} &= \int du\, d^{12}z \left(\int dv\, (D^{\hat{-}})^4 \mathcal{L}^{\hat{+}4} \right), \\
\int du \int dv \int d^4x\, d^4\theta^+\, d^4\theta^{\hat{+}} \mathcal{L}^{+4\hat{+}4} &= \int d\zeta^{-4} \left(\int dv\, (D^{\hat{-}})^4 \mathcal{L}^{+4\hat{+}4} \right),
\end{aligned}
\tag{5.5}
$$

where the eventual integrals go over $\mathcal{N}=2$ harmonic superspace or its analytic subspace and the measure $d\zeta^{-4}$ was defined in (2.12).

In a similar manner, one can construct $\mathcal{N}=4$ invariants as integrals over some other invariant subspaces of $\mathcal{N}=4$ bi-harmonic superspace, e.g., over chiral subspaces.

5.2 From bi-harmonic $\mathcal{N}=4$ superinvariants to $\mathcal{N}=2$ superfields

In this subsection we will consider three examples of the higher-derivative invariants admitting a formulation in bi-harmonic superspace where the whole on-shell $\mathcal{N}=4$ invariance is manifest. They will be transformed to some invariants in $\mathcal{N}=2$ harmonic superspace, where only $\mathcal{N}=2$ supersymmetry is manifest, while the invariance under the second, hidden $\mathcal{N}=2$ supersymmetry requires a non-trivial check. We will deal with the abelian U(1) gauge group as a remnant of SU(2) gauge group in the Coulomb branch of the corresponding $\mathcal{N}=4$ SYM theory.[8]

Let us start with the expression

$$
I = \int du\, dv\, d^{20}z\, (W\bar{W})^2, \tag{5.6}
$$

where the integration goes over the total bi-harmonic superspace. The functional (5.6) is manifestly on-shell $\mathcal{N}=4$ supersymmetric by construction. To transform the expression (5.6) to $\mathcal{N}=2$ harmonic superspace, we substitute the expressions for the superfield strengths W, $\bar{W}$ (4.88), (4.89), then do the integral over Grassmann and harmonic variables with "hat" and finally obtain the expression in terms of harmonic $\mathcal{N}=2$ superfields

$$
I = \int du\, d^{12}z\, \mathcal{L}, \tag{5.7}
$$

where

$$
\begin{aligned}
\mathcal{L} = {} & (\partial\mathcal{W})^2(\partial\bar{\mathcal{W}})^2 - 2i(\partial^{\alpha\dot{\alpha}} D^{-\beta} q^+_A \partial_{\beta\dot{\alpha}} \bar{\mathcal{W}})(\bar{D}^{-\dot{\beta}} q^{+A} \partial_{\alpha\dot{\beta}} \mathcal{W}) \\
& + i(\partial^{\alpha\dot{\alpha}} D^-_\beta q^+_A D^-_\alpha D^{+\beta} \mathcal{W})(\bar{D}^-_{\dot{\beta}} q^{+A} \bar{D}^-_{\dot{\alpha}} \bar{D}^{+\dot{\beta}} \bar{\mathcal{W}}) - \partial^{\beta\dot{\alpha}} \bar{\mathcal{W}} \bar{D}^-_{\dot{\alpha}} \bar{D}^+_{\dot{\beta}} \bar{\mathcal{W}} \partial^{\alpha\dot{\beta}} \mathcal{W} D^-_\alpha D^+_\beta \mathcal{W} \\
& - \frac{2}{3}\partial^2 (D^{-\beta} q^+_A D^-_\beta q^+_B)(\bar{D}^-_{\dot{\beta}} q^{+A} \bar{D}^{-\dot{\beta}} q^{+B}).
\end{aligned}
\tag{5.8}
$$

Here $\mathcal{W}, \bar{\mathcal{W}}$ are the $\mathcal{N}=2$ superfield strengths, $(\partial\mathcal{W})^2 = (\partial_{\alpha\dot{\alpha}}\mathcal{W})(\partial^{\dot{\alpha}\alpha}\mathcal{W})$, and $\partial^2 = \partial_m \partial^m$. The expression (5.8) is manifestly $\mathcal{N}=2$ supersymmetric, while its hidden on-shell $\mathcal{N}=2$ supersymmetry is not evident in advance and requires a rather non-trivial check. However,

[8]The consideration can easily be extended to Cartan subalgebra of any gauge group.

we are guaranteed to have it since we started from the manifestly $\mathcal{N}=4$ supersymmetric expression.

Consider next an invariant of the same dimension containing both the superfields $\phi^{+\hat{+}}$ and $\phi^{-\hat{-}}$. Since the total harmonic charge of such an expression has to be zero, it should simultaneously include $\phi^{+\hat{+}}$ and $\phi^{-\hat{-}}$. For example, let us write the $\mathcal{N}=4$ invariant of the form

$$I' = \int du\ dv\ d^{20}z\,(\phi^{+\hat{+}}\phi^{-\hat{-}})^2. \tag{5.9}$$

After the procedure similar to what we used to derive (5.8), we arrive at the action in $\mathcal{N}=2$ superspace

$$I' = \int du\ d^{12}z\ \mathcal{L}', \tag{5.10}$$

where

$$\begin{aligned}\mathcal{L}' = &-\frac{16}{3}q^{+A}q^{+B}\partial^4\left(q^-_A q^-_B\right) - D^{+\alpha}\mathcal{W}\bar{D}^{+\dot{\alpha}}\bar{\mathcal{W}}\partial^2\left(D^-_\alpha\mathcal{W}\bar{D}^-_{\dot{\alpha}}\bar{\mathcal{W}}\right)\\ &-4iD^{+\alpha}\mathcal{W}\bar{D}^{+\dot{\alpha}}\bar{\mathcal{W}}\partial^2\left(q^{-A}\partial_{\alpha\dot{\alpha}}q^-_A\right).\end{aligned} \tag{5.11}$$

This expression is manifestly $\mathcal{N}=2$ supersymmetric, while its invariance under hidden $\mathcal{N}=2$ supersymmetry is not immediately seen. Note that (5.9) can be evidently rewritten in the central basis, where some additional harmonic projections of ϕ^{iA} can be defined, viz., $\phi^{+\hat{-}}, \phi^{-\hat{+}}$. Using the evident relations like $\phi^{+\hat{-}} = \partial^{++}\phi^{-\hat{-}}, \phi^{-\hat{+}} = \partial^{\hat{+}\hat{+}}\phi^{-\hat{-}}$, etc, one can check that any neutral product of four such projections is reduced to (5.9) via integrating by parts with respect to the harmonic derivatives.

As the last example of invariants of the same dimension, we consider the $\mathcal{N}=4$ invariant including both W and $\phi^{+\hat{+}}$ superfields

$$I'' = \int du\ dv\ d^{20}z\,(\phi^{+\hat{+}}\phi^{-\hat{-}})(W\bar{W}). \tag{5.12}$$

After descending to $\mathcal{N}=2$ superspace, we obtain

$$I'' = \int du\ d^{12}z\ \mathcal{L}'', \tag{5.13}$$

where

$$\begin{aligned}\mathcal{L}'' = &-\frac{8i}{3}\partial^2(q^+_A\partial_{\alpha\dot{\alpha}}q^-_B)(D^{-\alpha}q^{+A}\bar{D}^{-\dot{\alpha}}q^{+B})\\ &+\frac{1}{4}(D^{+\beta}\mathcal{W}\partial_{\alpha\dot{\alpha}}D^-_\beta\mathcal{W})\partial^{\alpha\dot{\gamma}}\bar{\mathcal{W}}\bar{D}^-_{\dot{\gamma}}\bar{D}^{+\dot{\alpha}}\bar{\mathcal{W}}+\frac{1}{4}(\bar{D}^{+\dot{\beta}}\bar{\mathcal{W}}\partial_{\alpha\dot{\alpha}}\bar{D}^-_{\dot{\beta}}\bar{\mathcal{W}})\partial^{\dot{\alpha}\gamma}\mathcal{W}D^-_\gamma D^{+\alpha}\mathcal{W}\\ &-\frac{i}{4}(D^{+\beta}\mathcal{W}\partial_{\alpha\dot{\alpha}}\bar{D}^-_{\dot{\beta}}\bar{\mathcal{W}})D^-_\beta D^{+\alpha}\mathcal{W}\bar{D}^{-\dot{\beta}}\bar{D}^{+\dot{\alpha}}\bar{\mathcal{W}}-i(D^+_\beta\mathcal{W}\partial_{\alpha\dot{\alpha}}\bar{D}^-_{\dot{\beta}}\bar{\mathcal{W}})\partial^{\beta\dot{\alpha}}\mathcal{W}\partial^{\alpha\dot{\beta}}\bar{\mathcal{W}}\\ &+(D^+_\beta\mathcal{W}\partial_{\alpha\dot{\alpha}}q^{-A})\partial^{\beta\dot{\gamma}}D^{-\alpha}q^+_A\bar{D}^-_{\dot{\gamma}}\bar{D}^{+\dot{\alpha}}\bar{\mathcal{W}}+i(D^{+\beta}\mathcal{W}\partial_{\alpha\dot{\alpha}}q^{-A})\partial_{\beta\dot{\gamma}}\bar{D}^{-\dot{\alpha}}q^+_A\partial^{\alpha\dot{\gamma}}\bar{\mathcal{W}}\\ &+(\bar{D}^+_{\dot{\beta}}\bar{\mathcal{W}}\partial_{\alpha\dot{\alpha}}q^{-A})\partial^{\dot{\beta}\gamma}\bar{D}^{-\dot{\alpha}}q^+_A D^-_\gamma D^{+\alpha}\mathcal{W}-i(\bar{D}^{+\dot{\beta}}\bar{\mathcal{W}}\partial_{\alpha\dot{\alpha}}q^{-A})\partial_{\dot{\beta}\gamma}D^{-\alpha}q^+_A\partial^{\dot{\alpha}\gamma}\mathcal{W}.\end{aligned} \tag{5.14}$$

This expression is written in terms of $\mathcal{N}=2$ harmonic superfields. It is on-shell $\mathcal{N}=4$ supersymmetric since it was derived from the manifestly $\mathcal{N}=4$ supersymmetric invariant (5.12). If we would forget about the $\mathcal{N}=4$ superfield origin of (5.14), the proof of its

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hidden on-shell $\mathcal{N}=2$ supersymmetry is a rather involved procedure (though it could be performed, of course). Note that (5.12) is unique among the invariants of this type: the possible invariant $\sim \phi^{+\hat{-}}\phi^{-\hat{+}} W\bar{W}$ is reduced to (5.12) after integrating by parts with respect to harmonic derivatives and taking into account the harmonic independence of $W, \bar{W}$.

Thus we have given three examples of superinvariants in bi-harmonic superspace. All of them are on-shell $\mathcal{N}=4$ supersymmetric by construction. We have shown how they can be equivalently rewritten in $\mathcal{N}=2$ harmonic superspace, where only $\mathcal{N}=2$ supersymmetry remains manifest, while the proof of invariance under additional hidden $\mathcal{N}=2$ supersymmetry is a non-trivial job.

These three examples demonstrate a power of bi-harmonic superspace approach for constructing $\mathcal{N}=4$ supersymmetric invariants. The manifestly $\mathcal{N}=4$ supersymmetric invariants look simple when written in terms of bi-harmonic superspace, however, are converted into the rather complicated expressions after passing to their $\mathcal{N}=2$ harmonic superspace form. Moreover, the inverse problem of promoting these $\mathcal{N}=2$ harmonic superfield densities to their $\mathcal{N}=4$ bi-harmonic prototypes cannot be accomplished in a simple way.

As we saw, the above on-shell $\mathcal{N}=4$ superinvariants admit a unique representation in terms of $\mathcal{N}=2$ harmonic superfields. Since the technique of deriving the component structures of the local functionals defined on $\mathcal{N}=2$ harmonic superspace is well developed, we can in principle calculate the component structure of above superinvariants. All of these component Lagrangians contain the higher derivatives. We suppose that such invariants could arise as some sub-leading contributions to $\mathcal{N}=4$ SYM low-energy effective action. As a simple exercise, we calculated the terms depending on Maxwell field strength. Making in (5.8), (5.11) and (5.14) the substitutions

$$\mathcal{W} \;\Rightarrow\; 2\theta^{+}_{\alpha}\theta^{-}_{\beta}F^{\alpha\beta}\,, \quad \bar{\mathcal{W}} \;\Rightarrow\; 2\bar{\theta}^{-}_{\dot\beta}\bar{\theta}^{+}_{\dot\alpha}\bar{F}^{\dot\alpha\dot\beta}\,, \quad q^{\pm A} \;\Rightarrow\; 0\,, \tag{5.15}$$

and integrating over Grassmann and harmonic variables, we deduce (modulo terms vanishing on the free equations of motion)

$$\begin{aligned}
I \quad &\Longrightarrow \quad 2\int d^4x\left[\partial^2(\bar{F}^2)\partial^2(F^2) + 2\partial^2(\bar{F}_{\dot\gamma\dot\alpha}F_{\gamma\alpha})\partial^2(\bar{F}^{\dot\gamma\dot\alpha}F^{\gamma\alpha})\right],\\
I' \quad &\Longrightarrow \quad 2\int d^4x\left[\partial^2(\bar{F}^2)\partial^2(F^2) + 2\partial^2(\bar{F}_{\dot\gamma\dot\alpha}F_{\gamma\alpha})\partial^2(\bar{F}^{\dot\gamma\dot\alpha}F^{\gamma\alpha})\right],\\
I'' \quad &\Longrightarrow \quad \int d^4x\left[\partial^2(\bar{F}^2)\partial^2(F^2) + 2\partial^2(\bar{F}_{\dot\gamma\dot\alpha}F_{\gamma\alpha})\partial^2(\bar{F}^{\dot\gamma\dot\alpha}F^{\gamma\alpha})\right].
\end{aligned} \tag{5.16}$$

One of the reasons why these expressions proved to be the same is the relation

$$\int du\; dv\; d^{20}z\;\left[(W\bar{W})^2 - 4(\phi^{++\hat{+}})(\phi^{--\hat{-}})(W\bar{W}) + (\phi^{++\hat{+}}\phi^{--\hat{-}})^2\right] = 0, \tag{5.17}$$

which is a consequence of the condition

$$D^{\hat{+}}_{\alpha,\dot\alpha}[(W\bar{W})^2 - 4(\phi^{++\hat{+}})(\phi^{--\hat{-}})(W\bar{W}) + (\phi^{++\hat{+}}\phi^{--\hat{-}})^2] \sim D^{-}_{\alpha,\dot\alpha}[W\bar{W}^2\phi^{++\hat{+}} - \bar{W}(\phi^{++\hat{+}})^2\phi^{--\hat{-}}],$$

following from Bianchi identities (4.91)–(4.94). For the time being it remains unclear to us why *all three* $\mathcal{N}=4$ superfield actions give rise to the same Maxwell higher-derivative action. Perhaps, these invariants are related by the $\mathcal{N}=4$ R-symmetry group SU(4).[9]

5.3 Low-energy effective action in bi-harmonic superspace

Our aim here is to recast the invariant (2.29) in the bi-harmonic superspace.

Let us consider the following functional

$$\Gamma = c\int du\int dv\int d^4x\; d^4\theta^+\; d^4\theta^{\hat+}\,\mathcal{L}^{+4\hat+4}_{\rm eff}\,, \qquad \mathcal{L}^{+4\hat+4}_{\rm eff} = \frac{(\phi^{+\hat+})^4}{(W\bar W)^2}\mathcal{M}(Z), \tag{5.18}$$

where

$$\mathcal{M}(Z) = \sum_{n=0}^{\infty}\frac{(-1)^n(n+1)!^2}{n!(n+4)!}Z^n \tag{5.19}$$

and

$$Z = \frac{\phi^{+\hat+}\phi^{-\hat-}}{W\bar W}. \tag{5.20}$$

The series in (5.19) is summed up into the following expression

$$\mathcal{M}(Z) = \frac{(6+8Z+2Z^2)\ln(1+Z)-6Z-5Z^2}{4Z^4} = \frac{1}{24}+\mathcal{O}(Z)\,. \tag{5.21}$$

Below we check consistency of the integral (5.18) and prove that its part leading in derivatives actually coincides with (2.29).

The expression (5.18) is written as an integral over full analytic subspace. In order to show that it is $\mathcal{N}=4$ supersymmetric, one needs, first of all, to check that the integrand is analytic or at least analytic up to derivative. We should act by the derivatives $D^{\hat+}_\alpha, \bar D^{\hat+}_{\dot\alpha}, D^+_\alpha, \bar D^+_{\dot\alpha}$ on a generic term in series (5.18). To have a feeling what happens we consider in some detail the action of $D^{\hat+}_\alpha$. Using the identities (4.91)–(4.95) and the result of acting on them various harmonic derivatives, we are able to show that

$$\begin{aligned}D^{\hat+}_\alpha\left[\frac{(\phi^{+\hat+})^{n+4}(\phi^{-\hat-})^n}{(W\bar W)^{n+2}}\right] &= -i\frac{n}{n+1}D^-_\alpha\left[\frac{(\phi^{+\hat+})^{n+4}(\phi^{-\hat-})^{n-1}}{W^{n+2}\bar W^{n+1}}\right] - \frac{(\phi^{+\hat+})^{n+4}(\phi^{-\hat-})^n}{W^{n+3}\bar W^{n+2}}(n+2)D^{\hat+}_\alpha W\\ &\quad - \frac{n(n+4)}{n+1}\frac{(\phi^{+\hat+})^{n+3}(\phi^{-\hat-})^{n-1}D^{\hat+}_\alpha W}{W^{n+2}\bar W^{n+1}}.\end{aligned} \tag{5.22}$$

From this generic relation one can deduce that the second and third terms in (5.22) are canceled by the contributions from the adjacent terms in the sum (5.18) and that all such unwanted terms are mutually canceled when acting by $D^{\hat+}_\alpha$ on the whole series (5.18). So finally we obtain

$$D^{\hat+}_\alpha\,\mathcal{L}^{+4\hat+4}_{\rm eff} = D^-_\alpha G^{+5\hat+5}_{(1)}\,, \qquad \bar D^{\hat+}_{\dot\alpha}\,\mathcal{L}^{+4\hat+4}_{\rm eff} = \bar D^-_{\dot\alpha}\widetilde G^{+5\hat+5}_{(1)}, \tag{5.23}$$

$$G^{+5\hat+5}_{(1)} = i\sum_{n=0}^{\infty}\frac{(-1)^{n+1}n(n+1)!}{(n+4)!}\frac{(\phi^{+\hat+})^{n+4}(\phi^{-\hat-})^{n-1}}{W^{n+2}\bar W^{n+1}}\,. \tag{5.24}$$

[9] The superfield realization of the SU(4) R-symmetry in $\mathcal{N}=2$ harmonic superspace was given in [27]. We plan to discuss its implications in bi-harmonic $\mathcal{N}=4$ superspace elsewhere.

Analogously, one can check that a similar result holds as well for + derivatives

$$D^+_\alpha \mathcal{L}^{+4\hat{+}4}_{\text{eff}} = D^{\hat{-}}_\alpha G^{+5\hat{+}5}_{(2)}, \qquad \bar{D}^+_{\dot\alpha} \mathcal{L}^{+4\hat{+}4}_{\text{eff}} = \bar{D}^{\hat{-}}_{\dot\alpha} \widetilde{G}^{+5\hat{+}5}_{(2)}, \tag{5.25}$$

where $G^{+5\hat{+}5}_{(2)}$ is some expression depending on W, $\bar{W}$, $\phi^{+\hat{+}}, \phi^{-\hat{-}}$. Thus, the integrand in (5.18) is analytic up to derivative,

$$\begin{aligned} \mathcal{L}^{+4\hat{+}4}_{\text{eff}} = \mathcal{L}^{+4\hat{+}4}_{\text{eff}(0)}(\zeta_A) - \theta^{-\alpha} D^{\hat{-}}_\alpha G^{+5\hat{+}5}_{(2)} - \bar{\theta}^{-\dot\alpha} \bar{D}^{\hat{-}}_{\dot\alpha} \widetilde{G}^{+5\hat{+}5}_{(2)} \\ - \theta^{\hat{-}\alpha} D^-_\alpha G^{+5\hat{+}5}_{(1)} - \bar{\theta}^{\hat{-}\dot\alpha} \bar{D}^-_{\dot\alpha} \widetilde{G}^{+5\hat{+}5}_{(1)} + \dots\,, \end{aligned} \tag{5.26}$$

where "dots" stand for terms of higher orders in $\theta^-_{\alpha,\dot\alpha}$ and $\theta^{\hat{-}}_{\alpha,\dot\alpha}$, each involving negatively charged spinor derivatives of the appropriate superfield expressions. Because of the presence of the operators $(D^-)^4(D^{\hat{-}})^4$ in the analytic integration measure in (5.18), all terms in (5.26), except for the first one, do not contribute,

$$\mathcal{L}^{+4\hat{+}4}_{\text{eff}} \Longrightarrow \mathcal{L}^{+4\hat{+}4}_{\text{eff}(0)}(\zeta_A)\,, \tag{5.27}$$

and so (5.26) is indeed an on-shell $\mathcal{N} = 4$ superinvariant (the Bianchi identities (4.91)–(4.95) which were used in deriving (5.22) are valid on shell).

It remains to prove that (5.18) coincides with (2.29). To this end, one first needs to rewrite (5.18) as an integral over $\mathcal{N} = 2$ harmonic superspace. Taking into account (5.27), one can put, from the very beginning, $\theta^{\hat{-}}_\alpha = \bar{\theta}^{\hat{-}}_{\dot\alpha} = 0$ in all objects entering (2.29),

$$\begin{aligned} W \;&\Rightarrow\; \bar{\mathcal{W}} + i\sqrt{2}\theta^{\hat{+}\beta} D^-_\beta q^{+\hat{-}}, \qquad \bar{W} \;\Rightarrow\; \mathcal{W} + i\sqrt{2}\bar{\theta}^{\hat{+}}_{\dot\beta} \bar{D}^{-\dot\beta} q^{+\hat{-}}, \\ \phi^{+\hat{+}} \;&\Rightarrow\; \sqrt{2}q^{+\hat{+}} + i\theta^{\hat{+}\alpha} D^+_\alpha \mathcal{W} + i\bar{\theta}^{\hat{+}}_{\dot\alpha} \bar{D}^{+\dot\alpha} \bar{\mathcal{W}} + 2\sqrt{2} i \theta^{\hat{+}}_\alpha \bar{\theta}^{\hat{+}}_{\dot\alpha} \partial^{\alpha\dot\alpha} q^{+\hat{-}}, \\ \phi^{-\hat{-}} \;&\Rightarrow\; \sqrt{2}q^{-\hat{-}}. \end{aligned} \tag{5.28}$$

Next we consider some special cases, because they can clarify why the expressions (5.18) and (2.29) coincide. The general case is rather involved but it can also be worked out in a similar fashion.

First, consider the case when hypermultiplet $q^{+\hat{+}}$ equals zero. Due to (5.28) $\phi^{-\hat{-}}$ also equals zero. Hence, $Z = 0$ and $\mathcal{M}(Z) \Rightarrow 1/24$. Thus, we have

$$\begin{aligned} \Gamma &= \frac{c}{24} \int du\, dv\, d^4x\, d^4\theta^+\, d^4\theta^{\hat{+}} \frac{(\phi^{+\hat{+}})^4}{(W\bar{W})^2} = \frac{c}{16} \int d\zeta^{-4} \frac{(D^+\mathcal{W})^2(\bar{D}^+\bar{\mathcal{W}})^2}{\mathcal{W}^2\bar{\mathcal{W}}^2} \\ &= c \int du\, d^{12}z \, \ln\left(\frac{\mathcal{W}}{\Lambda}\right) \ln\left(\frac{\bar{\mathcal{W}}}{\Lambda}\right) = \int du\, d^{12}z\, \mathcal{H}(\mathcal{W}, \bar{\mathcal{W}}), \end{aligned} \tag{5.29}$$

where we made use of the equations of motion (2.15), when passing to the last line. Hence, if $q^{+\hat{+}}$ equals zero, (5.18) coincides with (2.29).

As the next step, consider the case when hypermultiplet q^+_A does not depend on x_m and θ's, i.e. all derivatives of q^+_A are equal to zero. This is just the standard requirement to distinguish the leading term in the effective action. Then all terms with $\theta^{\hat{+}}_\alpha$ and $\bar{\theta}^{\hat{+}}_{\dot\alpha}$ prove

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to be located in $\phi^{+\hat{+}}$. Hence, in this case the expression (5.18) equals

$$\begin{aligned}\Gamma &= c\int du\, dv\, d^4x\, d^4\theta^+\, d^4\theta^{\hat{+}}\frac{(\phi^{+\hat{+}})^4}{(\mathcal{W}\bar{\mathcal{W}})^2}\mathcal{M}(Z)\\ &= \frac{c}{16}\int d\zeta^{-4}\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)(2q^{+A}q^-_A)^n(D^+\mathcal{W})^2(\bar{D}^+\bar{\mathcal{W}})^2}{(\mathcal{W}\bar{\mathcal{W}})^{n+2}}\\ &= c\int du\, d^{12}z\left[\ln\left(\frac{\mathcal{W}}{\Lambda}\right)\ln\left(\frac{\bar{\mathcal{W}}}{\Lambda}\right)+\sum_{n=1}^{\infty}\frac{(-1)^n(2q^{+A}q^-_A)^n}{n^2(n+1)(\mathcal{W}\bar{\mathcal{W}})^n}\right]\\ &= \int du\, d^{12}z\left[\mathcal{H}(\mathcal{W},\bar{\mathcal{W}})+\mathcal{L}\left(-2\frac{q^{+A}q^-_A}{\mathcal{W}\bar{\mathcal{W}}}\right)\right], \end{aligned} \tag{5.30}$$

where we used equations of motion (2.15) when passing to the next-to-last line. Hence, if $q^{+\hat{+}}$ is constant, expression (5.18), once again, coincides with (2.29). Note that the approximation just used gives rise to a term F^4 of the fourth order in the Maxwell field strength in the component Lagrangian. The effective action (2.29) also encodes the Wess-Zumino term [28] which was derived in [10] by applying to another limit of (2.29), such that x-derivatives of the scalar fields are retained while all other components of $\mathcal{N}=4$ SYM multiplet are put equal to zero. Using the same background in (5.18) we arrive at the same WZ term in components.

Thus, we expressed the effective actions in bi-harmonic superspace in terms of $\mathcal{N}=4$ superfields (5.18). It is given as an integral over the full analytic subspace of $\mathcal{N}=4$ bi-harmonic superspace. A significant difference of (5.18) from the $\mathcal{N}=2$ superspace effective action (2.29) is that (5.18) is $\mathcal{N}=4$ supersymmetric only on shell, while in (2.29) the equations of motion are required only to prove the invariance under hidden $\mathcal{N}=2$ supersymmetry. This peculiarity seems to be not too essential since after passing to $\mathcal{N}=2$ superfield form of the effective action one can "forget" about its $\mathcal{N}=4$ superfield origin and stop to worry on its manifest $\mathcal{N}=2$ supersymmetry. Anyway, one *needs* to assume the equations of motion, once the hidden supersymmetry is concerned. The same is true for $\mathcal{N}=4$ invariants considered in subsection 5.2. To rephrase this argument, in $\mathcal{N}=4$ formulation both $\mathcal{N}=2$ supersymmetries enter on equal footing and so both are on shell, while after passing to $\mathcal{N}=2$ formulation one $\mathcal{N}=2$ becomes formally manifest, while another one remains hidden and on-shell.

6 Conclusions

Let us summarize the results. We have developed a new superfield method of constructing the on-shell $\mathcal{N}=4$ supersymmetric invariants in $4D\,,\mathcal{N}=4$ SYM theory. To know the precise structure of such superinvariants is of high necessity when calculating the effective action in $\mathcal{N}=4$ SYM quantum field theory formulated in $\mathcal{N}=2$ harmonic superspace and when studying the low-energy limit of string/brane theory. The method is based on the concept of bi-harmonic $\mathcal{N}=4$ superspace, which properly generalizes the notion of $\mathcal{N}=2$ harmonic superspace [14] to an extension of the latter with the double sets of the Grassmann and harmonic coordinates, so that the automorphism group $\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)\subset$

SU(4) remains manifest. Using the formulation of $4D\,,\mathcal{N}=4$ SYM theory in this bi-harmonic superspace it becomes possible to construct the on-shell $\mathcal{N}=4$ superinvariants in a manifestly $\mathcal{N}=4$ supersymmetric fashion and then pass to their equivalent $\mathcal{N}=2$ superfield form by a simple general recipe.

The basic merit of the new formulation is that, within its framework, the defining constraints of $\mathcal{N}=4$ SYM theory in $\mathcal{N}=4$ superspace can be resolved in terms of the basic objects of $\mathcal{N}=2$ harmonic superfield description of this theory, the gauge superfield $\mathcal{V}^{++}$ and the hypermultiplet superfield q^{+A}. The relevant $\mathcal{N}=2$ superfield equations of motion directly follow from the bi-harmonic form of the defining $\mathcal{N}=4$ SYM constraints. Thus, there was established the precise correspondence between the on-shell bi-harmonic $\mathcal{N}=4$ superfields and the superfields underlying the $\mathcal{N}=2$ harmonic superspace formulation of $\mathcal{N}=4$ SYM theory, in which two supersymmetries are manifest and the other two are on-shell and hidden.

As an illustration of how the proposed method works, we have constructed three abelian manifestly on-shell $\mathcal{N}=4$ supersymmetric higher-derivative invariants and rewrote them in terms of $\mathcal{N}=2$ harmonic superfields. In the $\mathcal{N}=2$ superfield formulation, the second $\mathcal{N}=2$ supersymmetry of these invariants looks highly implicit and it would be very hard to guess the structure of these invariants in advance. Also, we showed how the $\mathcal{N}=4$ SYM leading low-energy effective action (2.29) can be recast in the manifestly $\mathcal{N}=4$ supersymmetric form.

The proposed method of constructing the manifestly $\mathcal{N}=4$ supersymmetric on-shell invariants seems general enough to apply it for constructing and analyzing various other invariants in $\mathcal{N}=4$ SYM theory, for instance those which are $\mathcal{N}=4$ completions of the F^6, $F^8,\ldots$ invariants from the gauge field sector by adding the proper hypermultiplet terms. Such invariants can correspond to possible contributions to quantum effective actions in the Coulomb phase from higher loops, generalizing the one-loop F^4 effective action described by the invariant (2.29) (see, e.g., [29–32]). Such terms also arise as the next-to-leading corrections in the one-loop effective action (see, e.g., [33]). Among other interesting problems it is worth to mention a possible stringy extension of $\mathcal{N}=4$ SYM constraints in the form (3.29)–(3.35) and construction of $\mathcal{N}=4$ supergravity in $\mathcal{N}=4$ bi-harmonic superspace.

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A Self-consistency conditions

Self-consistency conditions arise as additional relations on superfields, when solving the constraints. They are to be identically satisfied on the final solution of all constraints. In this appendix we list the self-consistency conditions which come out from the equations in subsection 4.1.

The conditions following from eq. (4.14a):

$$\hat{\nabla}_{\alpha\dot{\alpha}}\widetilde{\mathcal{W}}^{+\dot{\alpha}} + \sqrt{2}[\psi_{A\alpha}, q^{+A}] + [\bar{\mathcal{W}}, \mathcal{W}^{+}_{\alpha}] = 0, \tag{A.1}$$

$$\hat{\nabla}^{\alpha\dot{\alpha}}\mathcal{W}^{+}_{\alpha} + \sqrt{2}[\widetilde{\psi}^{\dot{\alpha}}_{A}, q^{+A}] - [\mathcal{W}, \widetilde{\mathcal{W}}^{+}_{\dot{\alpha}}] = 0, \tag{A.2}$$

$$\begin{aligned}\hat{\nabla}^{\alpha\dot{\alpha}}\hat{\nabla}_{\alpha\dot{\alpha}}q^{+A} - i\{\psi^{A\alpha}, \mathcal{W}^{+}_{\alpha}\} + i\{\widetilde{\psi}^{A}_{\dot{\alpha}}, \widetilde{\mathcal{W}}^{+\dot{\alpha}}\}&\\ -2i[\mathcal{D}^{AB}, q^{+}_{B}] + \sqrt{2}[\bar{\mathcal{W}}, [\mathcal{W}, q^{+A}]] + \sqrt{2}[\mathcal{W}, [\bar{\mathcal{W}}, q^{+A}]] &= 0.\end{aligned} \tag{A.3}$$

The conditions following from eqs. (4.14b):

$$\begin{aligned}&\nabla^{++}\mathcal{W}^{+}_{\alpha} = 0, \qquad \nabla^{++}\hat{\nabla}^{\alpha\dot{\alpha}}q^{+A} = 0, \qquad \nabla^{++}[\widetilde{\psi}^{(A\dot{\alpha}}, q^{+B)}] = 0,\\ &\nabla^{++}[\mathcal{D}^{(AB}, q^{+C)}] = 0, \qquad \nabla^{++}[\bar{\mathcal{W}}, q^{+A}] = 0, \qquad \nabla^{++}[\mathcal{W}, q^{+A}] = 0.\end{aligned} \tag{A.4}$$

The conditions following from eqs. (4.28):

$$D^{+}_{\beta}\mathcal{W} + i\mathcal{W}^{+}_{\beta} = 0, \tag{A.5}$$

$$D^{+}_{\beta}\bar{\mathcal{W}} = 0, \tag{A.6}$$

$$D^{+}_{\beta}\hat{A}^{\alpha\dot{\alpha}} + \delta^{\alpha}_{\beta}\widetilde{\mathcal{W}}^{+\dot{\alpha}} = 0, \tag{A.7}$$

$$D^{+}_{\beta}\psi^{A}_{\alpha} + 2\sqrt{2}i\epsilon_{\beta\alpha}[\bar{\mathcal{W}}, q^{+A}] = 0, \tag{A.8}$$

$$D^{+}_{\beta}\widetilde{\psi}^{A\dot{\alpha}} + 2\sqrt{2}\epsilon_{\beta\alpha}\hat{\nabla}^{\alpha\dot{\alpha}}q^{+A} = 0, \tag{A.9}$$

$$D^{+}_{\beta}\mathcal{D}^{(AB)} + \sqrt{2}[\psi^{(A}_{\beta}, q^{+B)}] = 0. \tag{A.10}$$

The conditions following from eqs. (3.43b):

$$D^{+}_{\beta}\hat{\nabla}^{\alpha\dot{\alpha}}q^{+\hat{-}} - i\delta^{\alpha}_{\beta}[q^{+\hat{-}}, \widetilde{\mathcal{W}}^{+\dot{\alpha}}] = 0, \tag{A.11}$$

$$D^{+}_{\beta}[\mathcal{W}, q^{+A}] - i[q^{+A}, \mathcal{W}_{\beta}] = 0, \tag{A.12}$$

$$D^{+}_{\beta}[\bar{\mathcal{W}}, q^{+A}] = 0, \tag{A.13}$$

$$D^{+}_{\beta}[\psi^{(A}_{\alpha}, q^{+B)}] + 2\sqrt{2}i\epsilon_{\beta\alpha}[q^{+(A}, [\bar{\mathcal{W}}, q^{+B)}]] = 0, \tag{A.14}$$

$$D^{+}_{\beta}[\widetilde{\psi}^{(A\dot{\alpha}}, q^{+B)}] - 2\sqrt{2}\epsilon_{\beta\alpha}[q^{+(A}, \hat{\nabla}^{\alpha\dot{\alpha}}q^{+B)}] = 0, \tag{A.15}$$

$$D^{+}_{\beta}[\mathcal{D}^{(AB}, q^{+C)}] - \sqrt{2}[q^{+(A}, [\psi^{B}_{\beta}, q^{+C)}] = 0. \tag{A.16}$$

All these conditions become identities on the final solution of all constraints (subsection 4.4). For example, consider the relation (A.9)

$$D^{+}_{\beta}\widetilde{\psi}^{A\dot{\alpha}} + 2\sqrt{2}\epsilon_{\beta\alpha}\hat{\nabla}^{\alpha\dot{\alpha}}q^{+A} = 0. \tag{A.17}$$

Substituting into it the expression for $\widetilde{\psi}^{A\dot{\alpha}}$ from (4.75)

$$\widetilde{\psi}^{A}_{\dot{\beta}} = -i\sqrt{2}\bar{\nabla}^{-(0)}_{\dot{\beta}}q^{+A}, \tag{A.18}$$

we obtain

$$i\sqrt{2}D^{+}_{\beta}\bar{\nabla}^{-(0)\dot{\alpha}}q^{+A} - 2\sqrt{2}\epsilon_{\beta\alpha}\hat{\nabla}^{\alpha\dot{\alpha}}q^{+A} = 0. \tag{A.19}$$

Hence, we get the identity,

$$\hat{\nabla}^{\alpha\dot{\alpha}}q^{+A} - \hat{\nabla}^{\alpha\dot{\alpha}}q^{+A} = 0. \tag{A.20}$$

B Calculation of $V^{\hat{-}\hat{-}}$

The non-analytic gauge connection $V^{\hat{-}\hat{-}}$ is defined as a solution of the zero curvature condition

$$D^{\hat{+}\hat{+}}V^{\hat{-}\hat{-}} - D^{\hat{-}\hat{-}}V^{\hat{+}\hat{+}} + i[V^{\hat{+}\hat{+}}, V^{\hat{-}\hat{-}}] = 0, \tag{B.1}$$

where

$$\begin{aligned} V^{\hat{+}\hat{+}} &= -2i\theta^{\hat{+}}_{\alpha}\bar{\theta}^{\hat{+}}_{\dot{\alpha}}\hat{\mathcal{A}}^{\alpha\dot{\alpha}} + (\theta^{\hat{+}})^2\mathcal{W} + (\bar{\theta}^{\hat{+}})^2\bar{\mathcal{W}} + 2(\bar{\theta}^{\hat{+}})^2\theta^{\hat{+}\alpha}\psi^{\hat{-}}_{\alpha} + 2(\theta^{\hat{+}})^2\bar{\theta}^{\hat{+}}_{\dot{\alpha}}\widetilde{\psi}^{\hat{-}\dot{\alpha}} \\ &\quad + 3(\theta^{\hat{+}})^2(\bar{\theta}^{\hat{+}})^2\mathcal{D}^{\hat{-}2}. \end{aligned} \tag{B.2}$$

We parametrize the $\theta^{\hat{-}}_{\alpha}, \bar{\theta}^{\hat{-}}_{\dot{\alpha}}$ expansion of $V^{\hat{-}\hat{-}}$ in the following way

$$\begin{aligned} V^{\hat{-}\hat{-}} &= -2i\theta^{\hat{-}}_{\alpha}\bar{\theta}^{\hat{-}}_{\dot{\alpha}}w^{\alpha\dot{\alpha}} + (\theta^{\hat{-}})^2 w + (\bar{\theta}^{\hat{-}})^2\widetilde{w} + (\bar{\theta}^{\hat{-}})^2\theta^{\hat{-}\alpha}w^{\hat{+}}_{\alpha} + (\theta^{\hat{-}})^2\bar{\theta}^{\hat{-}}_{\dot{\alpha}}\widetilde{w}^{\hat{+}\dot{\alpha}} \\ &\quad + (\theta^{\hat{-}})^2(\bar{\theta}^{\hat{-}})^2 w^{\hat{+}2}. \end{aligned} \tag{B.3}$$

All the coefficient do not depend on $\theta^{\hat{-}}_{\alpha}, \bar{\theta}^{\hat{-}}_{\dot{\alpha}}$ and their $\theta^{\hat{+}}_{\alpha}, \bar{\theta}^{\hat{+}}_{\dot{\alpha}}, v^{\hat{+}}_{A}$ dependence will be determined from eq. (B.1). Possible coefficients of the first and zeroth order monomials in these coordinates are killed by the equations like $\partial^{\hat{+}\hat{+}}\omega^{\hat{-}} = 0 \Rightarrow \omega^{\hat{-}} = 0$. Here we give only the general scheme of finding the solution and the final answer.

The first step consists of finding equations on the coefficients in (B.3). The $\theta^{\hat{-}}_{\alpha}, \bar{\theta}^{\hat{-}}_{\dot{\alpha}}$ expansion (B.1) contains the monomials of the first, second, third and fourth degrees. Equating the corresponding coefficients to zero, we obtain the set of equations

$$i\epsilon_{\alpha\beta}\bar{\theta}^{\hat{+}}_{\dot{\alpha}}w^{\beta\dot{\alpha}} + \theta^{\hat{+}}_{\alpha}w = i\epsilon_{\alpha\beta}\bar{\theta}^{\hat{+}}_{\dot{\alpha}}\hat{\mathcal{A}}^{\beta\dot{\alpha}} + \theta^{\hat{+}}_{\alpha}\mathcal{W} + (\bar{\theta}^{\hat{+}})^2\psi^{\hat{-}}_{\alpha} + 2\theta^{\hat{+}}_{\alpha}\bar{\theta}^{\hat{+}}_{\dot{\alpha}}\widetilde{\psi}^{\hat{-}\dot{\alpha}} + 3\theta^{\hat{+}}_{\alpha}(\bar{\theta}^{\hat{+}})^2\mathcal{D}^{\hat{-}2}, \tag{B.4}$$

$$i\theta^{\hat{+}}_{\alpha}w^{\alpha\dot{\alpha}} + \bar{\theta}^{\hat{+}\dot{\alpha}}\widetilde{w} = i\theta^{\hat{+}}_{\alpha}\hat{\mathcal{A}}^{\alpha\dot{\alpha}} + \bar{\theta}^{\hat{+}\dot{\alpha}}\bar{\mathcal{W}} + 2\bar{\theta}^{\hat{+}\dot{\alpha}}\theta^{\hat{+}\alpha}\psi^{\hat{-}}_{\alpha} + (\theta^{\hat{+}})^2\widetilde{\psi}^{\hat{-}\dot{\alpha}} + 6(\theta^{\hat{+}})^2\bar{\theta}^{\hat{+}\dot{\alpha}}\mathcal{D}^{\hat{-}2}, \tag{B.5}$$

$$\begin{aligned} &\nabla^{\hat{+}\hat{+}}w^{\beta\dot{\beta}} - i\bar{\theta}^{\hat{+}\dot{\beta}}w^{\hat{+}\beta} + i\theta^{\hat{+}\beta}\widetilde{w}^{\hat{+}\dot{\beta}} + 2i\theta^{\hat{+}}_{\alpha}\bar{\theta}^{\hat{+}}_{\dot{\alpha}}\partial^{\beta\dot{\beta}}\hat{\mathcal{A}}^{\alpha\dot{\alpha}} - (\theta^{\hat{+}})^2\partial^{\beta\dot{\beta}}\mathcal{W} - (\bar{\theta}^{\hat{+}})^2\partial^{\beta\dot{\beta}}\bar{\mathcal{W}} \\ &-2(\bar{\theta}^{\hat{+}})^2\theta^{\hat{+}\alpha}\partial^{\beta\dot{\beta}}\psi^{\hat{-}}_{\alpha} - 2(\theta^{\hat{+}})^2\bar{\theta}^{\hat{+}}_{\dot{\alpha}}\partial^{\beta\dot{\beta}}\widetilde{\psi}^{\hat{-}\dot{\alpha}} - 3(\theta^{\hat{+}})^2(\bar{\theta}^{\hat{+}})^2\partial^{\beta\dot{\beta}}\mathcal{D}^{\hat{-}2} = 0, \end{aligned} \tag{B.6}$$

$$\nabla^{\hat{+}\hat{+}}w + \bar{\theta}^{\hat{+}}_{\dot{\alpha}}\widetilde{w}^{\hat{+}\dot{\alpha}} = 0, \tag{B.7}$$

$$\nabla^{\hat{+}\hat{+}}\widetilde{w} + \theta^{\hat{+}\alpha}w^{\hat{+}}_{\alpha} = 0, \tag{B.8}$$

$$\nabla^{\hat{+}\hat{+}}\widetilde{w}^{\hat{+}}_{\dot{\alpha}} + 2\bar{\theta}^{\hat{+}}_{\dot{\alpha}}w^{\hat{+}2} = 0, \tag{B.9}$$

$$\nabla^{\hat{+}\hat{+}}w^{\hat{+}}_{\alpha} + 2\theta^{\hat{+}}_{\alpha}w^{\hat{+}2} = 0, \tag{B.10}$$

$$\nabla^{\hat{+}\hat{+}}w^{\hat{+}2} = 0. \tag{B.11}$$

To solve eqs. (B.4)–(B.11), we expand the corresponding unknowns in (4.57) over $\theta^{\hat{+}}_{\alpha}, \bar{\theta}^{\hat{+}}_{\dot{\alpha}}$ and then fix the $v^{\hat{\pm}}_{A}$ dependence of the coefficients by these equations. We explicitly perform this operation only for eq. (B.11). We write

$$\begin{aligned} w^{\hat{+}2} &= r^{\hat{+}2} + \theta^{\hat{+}\alpha}r^{\hat{+}}_{\alpha} + \bar{\theta}^{\hat{+}}_{\dot{\alpha}}\widetilde{r}^{\hat{+}\dot{\alpha}} + (\theta^{\hat{+}})^2 r + (\bar{\theta}^{\hat{+}})^2\widetilde{r} \\ &\quad + (\bar{\theta}^{\hat{+}})^2\theta^{\hat{+}\alpha}r^{\hat{-}}_{\alpha} + (\theta^{\hat{+}})^2\bar{\theta}^{\hat{+}}_{\dot{\alpha}}\widetilde{r}^{\hat{-}\dot{\alpha}} - 2i\theta^{\hat{+}}_{\alpha}\bar{\theta}^{\hat{+}}_{\dot{\alpha}}r^{\alpha\dot{\alpha}} + (\theta^{\hat{+}})^2(\bar{\theta}^{\hat{+}})^2 r^{\hat{-}2}, \end{aligned} \tag{B.12}$$

and obtain the following set of equations and their solutions

$$\partial^{\hat{+}\hat{+}} r^{\hat{+}2} = 0 \quad \Rightarrow \quad r^{\hat{+}2} = r^{AB} v^{\hat{+}}_A v^{\hat{+}}_B, \tag{B.13}$$

$$\partial^{\hat{+}\hat{+}} r^{\hat{+}}_{\alpha} = 0 \quad \Rightarrow \quad r^{\hat{+}}_{\alpha} = r^{A}_{\alpha} v^{\hat{+}}_A, \tag{B.14}$$

$$\partial^{\hat{+}\hat{+}} \tilde{r}^{\hat{+}\dot{\alpha}} = 0 \quad \Rightarrow \quad \tilde{r}^{\hat{+}\dot{\alpha}} = -\tilde{r}^{A\dot{\alpha}} v^{\hat{+}}_A, \tag{B.15}$$

$$\partial^{\hat{+}\hat{+}} r^{\alpha\dot{\alpha}} + \hat{\nabla}^{\alpha\dot{\alpha}} r^{+2} = 0 \quad \Rightarrow \quad r^{\alpha\dot{\alpha}} = r^{\alpha\dot{\alpha}}_0 - \hat{\nabla}^{\alpha\dot{\alpha}} r^{(AB)} v^{\hat{+}}_A v^{\hat{-}}_B, \tag{B.16}$$

$$\partial^{\hat{+}\hat{+}} r + i[\mathcal{W}, r^{\hat{+}2}] = 0 \quad \Rightarrow \quad r = r_0 - i[\mathcal{W}, r^{(AB)} v^{\hat{+}}_A v^{\hat{-}}_B], \tag{B.17}$$

$$\partial^{\hat{+}\hat{+}} \tilde{r} + i[\bar{\mathcal{W}}, r^{\hat{+}2}] = 0 \quad \Rightarrow \quad \tilde{r} = \tilde{r}_0 - i[\bar{\mathcal{W}}, r^{(AB)} v^{\hat{+}}_A v^{\hat{-}}_B], \tag{B.18}$$

$$\partial^{\hat{+}\hat{+}} r^{\hat{-}}_{\alpha} + i\hat{\nabla}_{\alpha\dot{\alpha}} \tilde{r}^{\hat{+}\dot{\alpha}} + 2i[\psi^{\hat{-}}_{\alpha}, r^{\hat{+}2}] + [\bar{\mathcal{W}}, r^{\hat{+}}_{\alpha}] = 0, \tag{B.19}$$

$$\partial^{\hat{+}\hat{+}} \tilde{r}^{\hat{-}\dot{\alpha}} - i\hat{\nabla}^{\alpha\dot{\alpha}} r^{\hat{+}}_{\alpha} + 2i[\tilde{\psi}^{\hat{-}}_{\alpha}, r^{\hat{+}2}] + i[\mathcal{W}, \tilde{r}^{\hat{+}\dot{\alpha}}] = 0, \tag{B.20}$$

$$\begin{aligned} \partial^{\hat{+}\hat{+}} r^{\hat{-}2} - \hat{\nabla}^{\alpha\dot{\alpha}} r_{\alpha\dot{\alpha}} + 3i[\mathcal{D}^{\hat{-}2}, r^{\hat{+}2}] - 2i\{\psi^{\hat{-}\alpha}, r^{\hat{+}}_{\alpha}\} & \\ -2i\{\tilde{\psi}^{\hat{-}}_{\dot{\alpha}}, \tilde{r}^{\hat{+}\dot{\alpha}}\} + i[\mathcal{W}, \tilde{r}] + i[\bar{\mathcal{W}}, r] &= 0. \end{aligned} \tag{B.21}$$

Eqs. (B.19)–(B.21) are rather cumbersome. Their solutions are given below

$$\begin{aligned} \text{(B.19)} \quad \Rightarrow \quad r^{\hat{-}}_{\alpha} = & -i[\bar{\mathcal{W}}, r^{A}_{\alpha}] v^{\hat{-}}_A + i\hat{\nabla}_{\alpha\dot{\alpha}} \tilde{r}^{A\dot{\alpha}} v^{\hat{-}}_A - \frac{4i}{3}[\psi_{A\alpha}, r^{(AB)}] v^{\hat{-}}_B \\ & - i[\psi^{(A}_{\alpha}, r^{BC)}] v^{\hat{+}}_A v^{\hat{-}}_B v^{\hat{-}}_C, \end{aligned} \tag{B.22}$$

$$\begin{aligned} \text{(B.20)} \quad \Rightarrow \quad \tilde{r}^{\hat{-}\dot{\alpha}} = & \, i[\mathcal{W}, \tilde{r}^{A\dot{\alpha}}] v^{\hat{-}}_A + i\hat{\nabla}^{\alpha\dot{\alpha}} r^{A}_{\alpha} v^{\hat{-}}_A + \frac{4i}{3}[\tilde{\psi}^{\dot{\alpha}}_{A}, r^{(AB)}] v^{\hat{-}}_B \\ & + i[\tilde{\psi}^{(A\dot{\alpha}}, r^{BC)}] v^{\hat{+}}_A v^{\hat{-}}_B v^{\hat{-}}_C, \end{aligned} \tag{B.23}$$

$$\begin{aligned} \text{(B.21)} \quad \Rightarrow \quad r^{\hat{-}2} = & \frac{1}{2}\Big(- \hat{\nabla}^{\alpha\dot{\alpha}} \hat{\nabla}_{\alpha\dot{\alpha}} r^{(AB)} + i\{\psi^{A\alpha}, r^{B}_{\alpha}\} + i\{\tilde{\psi}^{A}_{\dot{\alpha}}, \tilde{r}^{B\dot{\alpha}}\} \\ & - [\mathcal{W}, [\bar{\mathcal{W}}, r^{(AB)}]] - [\mathcal{W}, [\bar{\mathcal{W}}, r^{(AB)}]]\Big) v^{\hat{-}}_A v^{\hat{-}}_B - \frac{3i}{2}[\mathcal{D}^{A}_{C}, r^{CB}] v^{\hat{-}}_A v^{\hat{-}}_B \qquad \text{(B.24)} \\ & - i[\mathcal{D}^{(AB}, r^{CD)}] v^{\hat{-}}_{(A} v^{\hat{-}}_B v^{\hat{-}}_C v^{\hat{+}}_{D)}. \end{aligned} \tag{B.25}$$

Eqs. (B.13)–(B.18) and (B.22), (B.23) and (B.25) include definitions of the coefficients in the expansion of $w^{\hat{+}2}$.

The remaining equations can be solved analogously. Instead of giving the analogs of eqs. (B.13)–(B.21,) we will present at once the full solution for the coefficients in (B.3)

$$\begin{aligned} w^{\hat{+}2} = \; & \mathcal{D}^{\hat{+}2} + \theta^{\hat{+}\alpha}\left(-i\hat{\nabla}_{\alpha\dot{\alpha}} \tilde{\psi}^{A\dot{\alpha}} + i[\psi^{A}_{\alpha}, \mathcal{W}]\right) v^{\hat{+}}_A - \bar{\theta}^{\hat{+}}_{\dot{\alpha}}\left(i\hat{\nabla}^{\alpha\dot{\alpha}} \psi^{A}_{\alpha} + i[\tilde{\psi}^{A\dot{\alpha}}, \bar{\mathcal{W}}]\right) v^{\hat{+}}_A \\ & + \frac{1}{4}(\theta^{\hat{+}})^2\Big(- 2\hat{\nabla}^{\alpha\dot{\alpha}} \hat{\nabla}_{\alpha\dot{\alpha}} \mathcal{W} - 2[\mathcal{W}, [\bar{\mathcal{W}}, \mathcal{W}]] + i\{\tilde{\psi}^{\dot{\alpha}}_{A}, \tilde{\psi}^{A}_{\dot{\alpha}}\} - 4i[\mathcal{W}, \mathcal{D}^{\hat{+}\hat{-}}]\Big) \\ & + \frac{1}{4}(\bar{\theta}^{\hat{+}})^2\Big(- 2\hat{\nabla}^{\alpha\dot{\alpha}} \hat{\nabla}_{\alpha\dot{\alpha}} \bar{\mathcal{W}} + 2[\bar{\mathcal{W}}, [\bar{\mathcal{W}}, \mathcal{W}]] + i\{\psi_{\alpha A}, \psi^{A\alpha}\} - 4i[\bar{\mathcal{W}}, \mathcal{D}^{\hat{+}\hat{-}}]\Big) \\ & + (\bar{\theta}^{\hat{+}})^2 \theta^{\hat{+}\alpha}\Big(- [\bar{\mathcal{W}}, \hat{\nabla}_{\alpha\dot{\alpha}} \tilde{\psi}^{A\dot{\alpha}}] v^{\hat{-}}_A + [\bar{\mathcal{W}}, [\psi^{A}_{\alpha}, \mathcal{W}]] v^{\hat{-}}_A - 2\hat{\nabla}_{\alpha\dot{\alpha}} \hat{\nabla}^{\beta\dot{\alpha}} \psi^{A}_{\beta} v^{\hat{-}}_A \\ & - \hat{\nabla}_{\alpha\dot{\alpha}} [\tilde{\psi}^{A\dot{\alpha}}, \bar{\mathcal{W}}] v^{\hat{-}}_A - \frac{4i}{3}[\psi_{A\alpha}, \mathcal{D}^{(AB)}] v^{\hat{-}}_B - i[\psi^{(A}_{\alpha}, \mathcal{D}^{BC)}] v^{\hat{+}}_A v^{\hat{-}}_B v^{\hat{-}}_C\Big) \\ & + (\theta^{\hat{+}})^2 \bar{\theta}^{\hat{+}}_{\dot{\alpha}}\Big(- [\mathcal{W}, \hat{\nabla}^{\alpha\dot{\alpha}} \psi^{A}_{\alpha}] v^{\hat{-}}_A - [\mathcal{W}, [\tilde{\psi}^{A\dot{\alpha}}, \bar{\mathcal{W}}]] v^{\hat{-}}_A + \hat{\nabla}^{\alpha\dot{\alpha}} \hat{\nabla}_{\alpha\dot{\beta}} \tilde{\psi}^{A\dot{\beta}} v^{\hat{-}}_A \end{aligned}$$

$$
\begin{aligned}
&-\hat{\nabla}^{\alpha\dot{\alpha}}[\psi^A_\alpha, \mathcal{W}]v^{\hat{-}}_A + \frac{4i}{3}[\widetilde{\psi}^{\dot{\alpha}}_A, \mathcal{D}^{(AB)}]v^{\hat{-}}_B + i[\psi^{(A\alpha}, \mathcal{D}^{BC)}]v^{\hat{+}}_A v^{\hat{-}}_B v^{\hat{-}}_C\Big) \\
&-i\theta^{\hat{+}}_\alpha\bar{\theta}^{\hat{+}}_{\dot{\alpha}}\Big(i[\hat{\nabla}^{\alpha\dot{\alpha}}\bar{\mathcal{W}}, \mathcal{W}] - i[\bar{\mathcal{W}}, \hat{\nabla}^{\alpha\dot{\alpha}}\mathcal{W}] - \{\psi^\alpha_A, \widetilde{\psi}^{A\dot{\alpha}}\} - 2\hat{\nabla}^{\alpha\dot{\alpha}}\mathcal{D}^{\hat{+}\hat{-}} \\
&+\hat{\nabla}^\alpha{}_{\dot{\beta}}\left(2\partial_\beta{}^{(\dot{\beta}}\hat{\mathcal{A}}^{\beta\dot{\alpha})} + i[\hat{\mathcal{A}}_\beta{}^{\dot{\beta}}, \hat{\mathcal{A}}^{\beta\dot{\alpha}}]\right)\Big) \\
&+\frac{1}{2}(\theta^{\hat{+}})^2(\bar{\theta}^{\hat{+}})^2\Big(\big(-\hat{\nabla}^{\alpha\dot{\alpha}}\hat{\nabla}_{\alpha\dot{\alpha}}\mathcal{D}^{(AB)} + \{\psi^{A\alpha}, \hat{\nabla}_{\alpha\dot{\alpha}}\widetilde{\psi}^{B\dot{\alpha}}\} - \{\psi^{A\alpha}, [\psi^B_\alpha, \mathcal{W}]\} \\
&-\{\widetilde{\psi}^A_{\dot{\alpha}}, \hat{\nabla}^{\alpha\dot{\alpha}}\psi^B_\alpha\} - \{\widetilde{\psi}^A_{\dot{\alpha}}, [\widetilde{\psi}^{B\dot{\alpha}}, \bar{\mathcal{W}}]\} - [\mathcal{W}, [\bar{\mathcal{W}}, \mathcal{D}^{(AB)}]] - [\mathcal{W}, [\bar{\mathcal{W}}, \mathcal{D}^{(AB)}]]\big)v^{\hat{-}}_A v^{\hat{-}}_B \\
&-3i[\mathcal{D}^A_C, \mathcal{D}^{CD}]v^{\hat{-}}_A v^{\hat{-}}_B\Big),
\end{aligned}
\tag{B.26}
$$

$$
\begin{aligned}
w^{\hat{+}}_\beta = &-\psi^A_\beta v^{\hat{+}}_A + \theta^{\hat{+}}_\beta\left(i[\bar{\mathcal{W}}, \mathcal{W}] - 2\mathcal{D}^{\hat{+}\hat{-}}\right) + \theta^{\hat{+}\alpha}\left(2\partial_{(\alpha\dot{\alpha}}\hat{\mathcal{A}}_{\beta)}{}^{\dot{\alpha}} + i[\hat{\mathcal{A}}_{\alpha\dot{\alpha}}, \hat{\mathcal{A}}_\beta{}^{\dot{\alpha}}]\right) \\
&+2i\bar{\theta}^{\hat{+}\dot{\alpha}}\hat{\nabla}_{\beta\dot{\alpha}}\bar{\mathcal{W}} - i(\theta^{\hat{+}})^2\hat{\nabla}_{\beta\dot{\alpha}}\widetilde{\psi}^{A\dot{\alpha}}v^{\hat{-}}_A + i(\bar{\theta}^{\hat{+}})^2[\bar{\mathcal{W}}, \psi^A_\beta]v^{\hat{-}}_A \\
&+(\bar{\theta}^{\hat{+}})^2\theta^{\hat{+}\alpha}\left(\sqrt{2}i\epsilon_{\beta\alpha}[\bar{\mathcal{W}}, \mathcal{D}^{(AB)}]v^{\hat{-}}_A v^{\hat{-}}_B + i\{\psi^{(A}_\alpha, \psi^{B)}_\beta\}v^{\hat{-}}_A v^{\hat{-}}_B\right) \\
&-(\theta^{\hat{+}})^2\bar{\theta}^{\hat{+}}_{\dot{\alpha}}\left(i\{\widetilde{\psi}^{(A\dot{\alpha}}, \psi^{B)}_\beta\}v^{\hat{-}}_A v^{\hat{-}}_B + 2i\epsilon_{\beta\alpha}\hat{\nabla}^{\alpha\dot{\alpha}}\mathcal{D}^{AB}v^{\hat{-}}_A v^{\hat{-}}_B\right) \\
&-2i\theta^{\hat{+}}_\alpha\bar{\theta}^{\hat{+}}_{\dot{\alpha}}\left(\hat{\nabla}^{\alpha\dot{\alpha}}\psi^A_\beta v^{\hat{-}}_A - \delta^\alpha_\beta\left(\hat{\nabla}^{\gamma\dot{\alpha}}\psi^A_\gamma + [\widetilde{\psi}^{A\dot{\alpha}}, \bar{\mathcal{W}}]\right)v^{\hat{-}}_A\right) \\
&+\frac{5i}{3}(\theta^{\hat{+}})^2(\bar{\theta}^{\hat{+}})^2[\mathcal{D}^{(AB}, \psi^{C)}_\beta]v^{\hat{-}}_A v^{\hat{-}}_B v^{\hat{-}}_C,
\end{aligned}
\tag{B.27}
$$

$$
\begin{aligned}
\widetilde{w}^{\hat{+}}_{\dot{\beta}} = &\;\widetilde{\psi}^A_{\dot{\beta}}v^{\hat{+}}_A - \bar{\theta}^{\hat{+}}_{\dot{\beta}}\left(i[\bar{\mathcal{W}}, \mathcal{W}] + 2\mathcal{D}^{\hat{+}\hat{-}}\right) + \bar{\theta}^{\hat{+}\dot{\alpha}}\left(2\partial_{\alpha(\dot{\alpha}}\hat{\mathcal{A}}^\alpha{}_{\dot{\beta})} + i[\hat{\mathcal{A}}_{\alpha\dot{\alpha}}, \hat{\mathcal{A}}_\beta{}^{\dot{\alpha}}]\right) \\
&+2i\theta^{\hat{+}\alpha}\hat{\nabla}_{\alpha\dot{\beta}}\mathcal{W} + i(\bar{\theta}^{\hat{+}})^2\hat{\nabla}_{\alpha\dot{\beta}}\psi^{A\alpha}v^{\hat{-}}_A - i(\theta^{\hat{+}})^2[\bar{\mathcal{W}}, \widetilde{\psi}^A_{\dot{\beta}}]v^{\hat{-}}_A \\
&+(\bar{\theta}^{\hat{+}})^2\theta^{\hat{+}\alpha}\left(2i\hat{\nabla}_{\alpha\dot{\beta}}\mathcal{D}^{AB}v^{\hat{-}}_A v^{\hat{-}}_B - i\{\psi^{(A}_\alpha, \widetilde{\psi}^{B)}_{\dot{\beta}}\}v^{\hat{-}}_A v^{\hat{-}}_B\right) \\
&-(\theta^{\hat{+}})^2\bar{\theta}^{\hat{+}}_{\dot{\alpha}}\left(2i\delta^{\dot{\alpha}}_{\dot{\beta}}[\mathcal{W}, \mathcal{D}^{(AB)}]v^{\hat{-}}_A v^{\hat{-}}_B - i\{\widetilde{\psi}^{(A\dot{\alpha}}, \widetilde{\psi}^{B)}_{\dot{\beta}}\}v^{\hat{-}}_A v^{\hat{-}}_B\right) \\
&+2i\theta^{\hat{+}}_\alpha\bar{\theta}^{\hat{+}}_{\dot{\alpha}}\left(\hat{\nabla}^{\alpha\dot{\alpha}}\widetilde{\psi}^A_{\dot{\beta}}v^{\hat{-}}_A - \delta^{\dot{\alpha}}_{\dot{\beta}}\left(\hat{\nabla}^{\alpha\dot{\gamma}}\widetilde{\psi}^A_{\dot{\gamma}} + [\psi^{A\alpha}, \mathcal{W}]\right)v^{\hat{-}}_A\right) \\
&-\frac{5i}{3}(\theta^{\hat{+}})^2(\bar{\theta}^{\hat{+}})^2[\mathcal{D}^{(AB}, \widetilde{\psi}^{C)}_{\dot{\beta}}]v^{\hat{-}}_A v^{\hat{-}}_B v^{\hat{-}}_C,
\end{aligned}
\tag{B.28}
$$

$$
w = \mathcal{W} - \bar{\theta}^{\hat{+}}_{\dot{\beta}}\widetilde{\psi}^{A\dot{\beta}}v^{\hat{-}}_A + (\bar{\theta}^{\hat{+}})^2\mathcal{D}^{\hat{-}2},
\tag{B.29}
$$

$$
\widetilde{w} = \bar{\mathcal{W}} + \theta^{\hat{+}\beta}\psi^A_\beta v^{\hat{-}}_A + (\theta^{\hat{+}})^2\mathcal{D}^{\hat{-}2},
\tag{B.30}
$$

$$
w^{\beta\dot{\beta}} = \hat{\mathcal{A}}^{\beta\dot{\beta}} - i\theta^{\hat{+}\beta}\widetilde{\psi}^{A\dot{\beta}}v^{\hat{-}}_A - i\bar{\theta}^{\hat{+}\dot{\beta}}\psi^{A\beta}v^{\hat{-}}_A + i\theta^{\hat{+}\beta}\bar{\theta}^{\hat{+}\dot{\beta}}\mathcal{D}^{\hat{-}2}.
\tag{B.31}
$$

Substituting all these expressions in (B.3), one can obtain the full expression for $V^{\hat{-}\hat{-}}$.

In the Abelian case the expressions for the superfield coefficients are essentially simplified

$$
\begin{aligned}
w^{\hat{+}2} = &\;\mathcal{D}^{\hat{+}2} - i\theta^{\hat{+}\alpha}\partial_{\alpha\dot{\alpha}}\widetilde{\psi}^{A\dot{\alpha}}v^{\hat{+}}_A - i\bar{\theta}^{\hat{+}}_{\dot{\alpha}}\partial^{\alpha\dot{\alpha}}\psi^A_\alpha v^{\hat{+}}_A \\
&-\frac{1}{2}(\theta^{\hat{+}})^2\partial^{\alpha\dot{\alpha}}\partial_{\alpha\dot{\alpha}}\mathcal{W} - \frac{1}{2}(\bar{\theta}^{\hat{+}})^2\partial^{\alpha\dot{\alpha}}\partial_{\alpha\dot{\alpha}}\bar{\mathcal{W}} \\
&-(\bar{\theta}^{\hat{+}})^2\theta^{\hat{+}\alpha}\partial_{\alpha\dot{\alpha}}\partial^{\beta\dot{\alpha}}\psi^A_\beta v^{\hat{-}}_A + (\theta^{\hat{+}})^2\bar{\theta}^{\hat{+}}_{\dot{\alpha}}\partial^{\alpha\dot{\alpha}}\partial_{\alpha\dot{\beta}}\widetilde{\psi}^{A\dot{\beta}}v^{\hat{-}}_A \\
&-2i\theta^{\hat{+}}_\alpha\bar{\theta}^{\hat{+}}_{\dot{\alpha}}\Big(-\partial^{\alpha\dot{\alpha}}\mathcal{D}^{\hat{+}\hat{-}} + \partial^\alpha{}_{\dot{\beta}}\partial_\beta{}^{(\dot{\beta}}\hat{\mathcal{A}}^{\beta\dot{\alpha})}\Big) - \frac{1}{2}(\theta^{\hat{+}})^2(\bar{\theta}^{\hat{+}})^2\partial^{\alpha\dot{\alpha}}\partial_{\alpha\dot{\alpha}}\mathcal{D}^{(AB)}v^{\hat{-}}_A v^{\hat{-}}_B,
\end{aligned}
\tag{B.32}
$$

$$w^{\hat+}_\beta = -\psi^A_\beta v^{\hat+}_A - 2\theta^{\hat+}_\beta \mathcal{D}^{\hat+\hat-} + 2\theta^{\hat+\alpha}\partial_{(\alpha\dot\alpha}\mathcal{A}_{\beta)}{}^{\dot\alpha} + 2i\bar\theta^{\hat+\dot\alpha}\partial_{\beta\dot\alpha}\bar{\mathcal{W}} - i(\theta^{\hat+})^2\partial_{\beta\dot\alpha}\widetilde{\psi}^{A\dot\alpha}v^{\hat-}_A \tag{B.33}$$
$$-2i(\theta^{\hat+})^2\bar\theta^{\hat+}_{\dot\alpha}\epsilon_{\beta\alpha}\partial^{\alpha\dot\alpha}\mathcal{D}^{AB}v^{\hat-}_A v^{\hat-}_B - i\theta^{\hat+}_\alpha\bar\theta^{\hat+}_{\dot\alpha}\left(\partial^{\alpha\dot\alpha}\psi^A_\beta v^{\hat-}_A - \delta^\alpha_\beta\partial^{\gamma\dot\alpha}\psi^A_\gamma v^{\hat-}_A\right),$$
$$\widetilde{w}^{\hat+}_{\dot\beta} = \widetilde{\psi}^A_{\dot\beta}v^{\hat+}_A - 2\bar\theta^{\hat+}_{\dot\beta}\mathcal{D}^{\hat+\hat-} + 2\bar\theta^{\hat+\dot\alpha}\partial_{\alpha(\dot\alpha}\mathcal{A}^\alpha{}_{\dot\beta)} + 2i\theta^{\hat+\alpha}\partial_{\alpha\dot\beta}\mathcal{W} + i(\bar\theta^{\hat+})^2\partial_{\alpha\dot\beta}\psi^{A\alpha}v^{\hat-}_A \tag{B.34}$$
$$+2i(\bar\theta^{\hat+})^2\theta^{\hat+\alpha}\partial_{\alpha\dot\beta}\mathcal{D}^{AB}v^{\hat-}_A v^{\hat-}_B + 2i\theta^{\hat+}_\alpha\bar\theta^{\hat+}_{\dot\alpha}\left(\partial^{\alpha\dot\alpha}\widetilde{\psi}^A_{\dot\beta}v^{\hat-}_A - \delta^{\dot\alpha}_{\dot\beta}\partial^{\alpha\dot\gamma}\widetilde{\psi}^A_{\dot\gamma}v^{\hat-}_A\right),$$
$$w = \mathcal{W} - \bar\theta^{\hat+}_{\dot\beta}\widetilde{\psi}^{A\dot\beta}v^{\hat-}_A + (\bar\theta^{\hat+})^2\mathcal{D}^{\hat-2}, \tag{B.35}$$
$$\widetilde{w} = \bar{\mathcal{W}} + \theta^{\hat+\beta}\psi^A_\beta v^{\hat-}_A + (\theta^{\hat+})^2\mathcal{D}^{\hat-2}, \tag{B.36}$$
$$w^{\beta\dot\beta} = \hat{\mathcal{A}}^{\beta\dot\beta} - i\theta^{\hat+\beta}\widetilde{\psi}^{A\dot\beta}v^{\hat-}_A - i\bar\theta^{\hat+\dot\beta}\psi^{A\beta}v^{\hat-}_A + 2i\theta^{\hat+\beta}\bar\theta^{\hat+\dot\beta}\mathcal{D}^{\hat-2}. \tag{B.37}$$

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