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Worldsheet (anti)instanton bound states in type II on T^2

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Abstract: The 1/8 BPS $D^6\mathcal{R}^4$ coupling in type II string theory compactified on T^2 receives contributions from worldsheet instantons and anti-instantons wrapping the T^2, up to genus three in string perturbation theory. These involve contributions separately from bound states of instantons and anti-instantons, which are qualitatively similar to such contributions to the 1/2 and 1/4 BPS couplings. At genus two, the $D^6\mathcal{R}^4$ coupling also receives contributions from instanton/anti-instanton bound states unlike the 1/2 and 1/4 BPS couplings, which is a consequence of a T-duality invariant eigenvalue equation a term in the coupling satisfies. We solve this eigenvalue equation to obtain the complete structure of the worldsheet (anti)instanton contributions. In the type IIB theory, strong weak coupling duality leads to certain contributions involving bound states of D string (anti)instantons wrapping the T^2.

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Contents

1 Introduction

Consider type II string theory toroidally compactified on T^2. This maximally supersymmetric theory has a U-duality symmetry group $\mathrm{SL}(2,\mathbb{Z}) \times \mathrm{SL}(3,\mathbb{Z})$. In the type IIB theory, the non-perturbative[1] $\mathrm{SL}(2,\mathbb{Z})_\tau$ S-duality symmetry which is inherited from ten dimensions is contained in $\mathrm{SL}(3,\mathbb{Z})$ of the U-duality group. The perturbative T-duality symmetry group is

$$\mathrm{SL}(2,\mathbb{Z})_T \times \mathrm{SL}(2,\mathbb{Z})_U \tag{1.1}$$

where T and U are the complexified Kahler and complex structure moduli of the T^2 respectively. While $\mathrm{SL}(2,\mathbb{Z})_U$ directly arises as the $\mathrm{SL}(2,\mathbb{Z})$ factor in the U-duality group, the $\mathrm{SL}(2,\mathbb{Z})_T$ is contained in $\mathrm{SL}(3,\mathbb{Z})$. The moduli dependent coefficients of various amplitudes in this theory when expanded around weak string coupling exhibit a rich perturbative as well as non-perturbative structure.

In the string frame, the perturbative part of the amplitude takes the form

$$\sum_g (e^{-2\phi}V)^{1-g} f_g(T,\overline{T};U,\overline{U}) \tag{1.2}$$

where ϕ is the dilaton, and V is the volume of T^2 in the string frame metric. The Kahler modulus is given by

$$T = T_1 + iT_2 = B_N + iV, \tag{1.3}$$

[1] Here perturbative and non-perturbative are with respect to the string coupling.

where B_N is the scalar from the NS-NS sector. In (1.2), the term involving f_g is the genus g amplitude, which involves the T-duality invariant string coupling $e^{-2\phi}V$ as the overall factor. Equality of the IIA and IIB theories on compactifying on T^2 yields that $f_g(T,\overline{T};U,\overline{U}) = f_g(U,\overline{U};T,\overline{T})$. Note that the perturbative contribution (1.2) does not involve states from the Ramond sector.

The non-perturbative contributions arise from D-instantons as well as from (p,q) string instantons[2] wrapping T^2 where $q \neq 0$, and are exponentially suppressed for large τ_2. For the n D-instanton contribution, the exponentially suppressed factor is of the form

$$e^{2\pi i(n\tau_1+i|n|\tau_2)}, \tag{1.4}$$

while for the (p,q) string instanton contribution, it is of the form [1, 2]

$$e^{2\pi i T_{p,q}}, \tag{1.5}$$

where $T_{p,q} = (qB_R+pB_N)+i|p-q\tau|V$, where B_R is the scalar from the R-R sector. While the instantons carry positive NS (or R) charge, the anti-instantons carry negative charge.

Let us consider the perturbative contributions given by (1.2). Though they are perturbative in the string coupling, they can receive contributions which are non-perturbative in α', the inverse string tension. These contributions arise from worldsheet instantons and anti-instantons[3] wrapping T^2. While it is difficult to calculate these contributions for generic interactions, the BPS interactions are amenable to a detailed analysis.

First let us consider the 1/2 BPS $\mathcal{R}^4$ interaction, where only the terms involving $g = 0$ and 1 are non-vanishing in (1.2). The worldsheet (anti)instanton contributions are given by [2–4]

$$f_1 = 2\pi \sum_{n=1}^{\infty} \frac{\sigma_1(n)}{n}\Big(e^{2\pi inT}+e^{-2\pi in\overline{T}}\Big), \tag{1.6}$$

where we have ignored all other contributions.[4]

For the 1/4 BPS $D^4\mathcal{R}^4$ interaction, where only terms involving $g = 0$, 1 and 2 are non-vanishing in (1.2), the worldsheet (anti)instanton contributions are given by [4, 5]

$$\begin{aligned} f_1 &= \frac{4}{\pi}E_2(U,\overline{U})\sum_{n=1}^{\infty}\frac{\sigma_3(n)}{n^2}\left(1+\frac{1}{2\pi nT_2}\right)\Big(e^{2\pi inT}+e^{-2\pi in\overline{T}}\Big),\\ f_2 &= \frac{4\pi^2}{3}\sum_{n=1}^{\infty}\frac{\sigma_3(n)}{n^2}\left(1+\frac{1}{2\pi nT_2}\right)\Big(e^{2\pi inT}+e^{-2\pi in\overline{T}}\Big) \end{aligned} \tag{1.7}$$

where the Eisenstein series E_2 is defined by (A.1) and we have ignored all other contributions.[5]

Thus (1.6) and (1.7) both involve an infinite sum of worldsheet (anti)instanton contributions. In fact, each term in the sum results from either instantons or from anti-instantons.

[2] We follow the convention of denoting the fundamental string as the $(1,0)$ state.

[3] In fact, (1.5) also contains such contributions for $p \neq 0$. However, for the sake of simplicity we restrict ourselves to contributions involving no Ramond sector states.

[4] Our normalization is such that $f_0 = \zeta(3)$.

[5] Our normalization is such that $f_0 = \zeta(5)$.

This feature changes qualitatively when we consider the 1/8 BPS $D^6\mathcal{R}^4$ interaction which preserves 4 supercharges. This interaction receives contributions from $g = 0$, 1, 2 and 3 in (1.2). Again, keeping only terms involving the worldsheet (anti)instanton contributions, we have that [4, 6][6]

$$\begin{aligned}
f_1 &= \frac{10}{\pi^2} E_3(U,\overline{U}) \sum_{n=1}^{\infty} \frac{\sigma_5(n)}{n^3} \Big(1 + \frac{3}{2\pi n T_2} + \frac{3}{4\pi^2 n^2 T_2^2}\Big) \Big(e^{2\pi i n T} + e^{-2\pi i n \overline{T}}\Big) \\
&\quad + 2\pi\zeta(3) \sum_{n=1}^{\infty} \frac{\sigma_1(n)}{n} \Big(e^{2\pi i n T} + e^{-2\pi i n \overline{T}}\Big), \\
f_2 &= 2\pi \Big(E_1(U,\overline{U}) + \frac{\pi}{6}\Big) \sum_{n=1}^{\infty} \frac{\sigma_1(n)}{n} \Big(e^{2\pi i n T} + e^{-2\pi i n \overline{T}}\Big) + F(T,\overline{T}), \\
f_3 &= \frac{\pi^3}{9} \sum_{n=1}^{\infty} \frac{\sigma_5(n)}{n^3} \Big(1 + \frac{3}{2\pi n T_2} + \frac{3}{4\pi^2 n^2 T_2^2}\Big) \Big(e^{2\pi i n T} + e^{-2\pi i n \overline{T}}\Big).
\end{aligned} \tag{1.8}$$

In (1.8), $F(T,\overline{T})$ satisfies the eigenvalue equation

$$\Big(\Delta - 12\Big) F(T,\overline{T}) = -6\Big(E_1(T,\overline{T})\Big)^2, \tag{1.9}$$

where

$$\Delta = 4T_2^2 \frac{\partial^2}{\partial T \partial \overline{T}} \tag{1.10}$$

is the $SL(2,\mathbb{Z})_T$ invariant Laplacian. The relevant Eisenstein series that appear in (1.8) are defined by (A.1) and (A.3). The 1/8 BPS couplings have also been analyzed from the worldsheet perspective in [7–11], and from the spacetime point of view in [12–23].

Now in (1.8) all the contributions apart from that involving $F(T,\overline{T})$ are given by an infinite sum of terms involving either worldsheet instantons or anti-instantons. However, while $F(T,\overline{T})$ yields qualitatively similar contributions separately from bound states of instantons or anti-instantons, it receives additional contributions involving bound states of instantons/anti-instantons because of the presence of the source term in the eigenvalue equation (1.9). In this paper, we analyze the content of (1.9) in detail to understand all these contributions at a quantitative level.

2 The analysis of the eigenvalue equation for $F(T,\overline{T})$

We now analyze the eigenvalue equation (1.9) in detail. To start with, we express $F(T,\overline{T})$ as

$$F(T,\overline{T}) = \sum_{n\in\mathbb{Z}} F_n(T_2) e^{2\pi i n T_1}. \tag{2.1}$$

This involves an infinite sum over topologically distinct sectors carrying non-trivial NS charge (the $n = 0$ sector carries no charge).

We shall solve (1.9) along with specific boundary conditions. For large T_2, we have that $F(T,\overline{T}) \sim T_2^2$ simply because this contribution arises at genus two and this is the

[6] Our normalization is such that $f_0 = \zeta(3)^2$.

large volume scaling. For small T_2, the large T_2 behavior along with $\mathrm{SL}(2,\mathbb{Z})_T$ invariance yields that [24]

$$F_n(T_2) \sim T_2^{-1} \tag{2.2}$$

for all n.

2.1 The mode carrying no NS charge

First let us consider the mode $F_0(T_2)$ in (2.1) which carries no NS charge. Using (1.9) and the large T_2 expansion of E_1 given in (A.3), we have that

$$\left(T_2^2\frac{d^2}{dT_2^2} - 12\right)F_0(T_2) = -6\left(\frac{\pi^2}{3}T_2 - \pi\ln T_2\right)^2 - 48\pi^2\sum_{n=1}^\infty \frac{\sigma_1(n)^2}{n^2}e^{-4\pi n T_2}. \tag{2.3}$$

While the solution of the homogeneous equation is given by[7]

$$F_0^H(T_2) = a_0 T_2^{-3}, \tag{2.4}$$

where a_0 is an arbitrary constant, the particular solution is given by [6]

$$F_0^P(T_2) = \frac{\pi^2}{720}\Big(65 - 20\pi T_2 + 48\pi^2 T_2^2\Big) + \pi^2\ln T_2\left(-\frac{\pi}{3}T_2 + \frac{1}{2}\ln T_2 - \frac{1}{12}\right) + \sum_{n=1}^\infty Q_n(T_2)e^{-4\pi n T_2}. \tag{2.5}$$

In (2.5), $Q_n(T_2)$ is given by

$$Q_n(T_2) = -\frac{\sigma_1(n)^2}{224 n^5 \pi T_2^3}\Big[24(x+1)^2 + x^4(2-x) + (x^3-3)^2 + 15 + x^7 e^x \mathrm{Ei}(-x)\Big], \tag{2.6}$$

where $x = 4\pi n T_2$ and $\mathrm{Ei}(-x)$ is the exponential integral function defined in (B.1). Using (B.2), we see that $Q_n(T_2)$ is an infinite series in powers of T_2 in the large T_2 expansion. Since this contribution is weighted by $e^{-4\pi n T_2}$, it follows that it arises from the bound state of worldsheet instantons/anti-instantons carrying equal and opposite NS charge n.

We now fix a_0 in (2.4) using the boundary condition (2.2). Using (B.3) we see that the singular terms in the small T_2 limit in $F_0^P(T_2)$ are given by

$$F_0^P(T_2) = -\frac{3}{14\pi T_2^3}\sum_{n=1}^\infty \frac{\sigma_1(n)^2}{n^5} + \frac{\pi^2}{2}\ln T_2\left(\ln T_2 - \frac{1}{6}\right) + O(T_2^0). \tag{2.7}$$

While the $O(T_2^{-3})$ contribution comes from the worldsheet instanton/anti-instanton sector, the $O(T_2^{-2})$ and $O(T_2^{-1})$ terms that arise from this sector cancel on adding the various contributions. Now using the relation

$$\sum_{n=1}^\infty \frac{\sigma_p(n)\sigma_q(n)}{n^r} = \frac{\zeta(r)\zeta(r-p)\zeta(r-q)\zeta(r-p-q)}{\zeta(2r-p-q)}, \tag{2.8}$$

[7]We neglect the solution T_2^4 as it violates the large T_2 boundary condition.

we see that (2.7) yields

$$F_0^P(T_2) = -\frac{\zeta(3)\zeta(5)}{4\pi T_2^3} + \frac{\pi^2}{2}\ln T_2 \left(\ln T_2 - \frac{1}{6}\right) + O(T_2^0). \tag{2.9}$$

Thus demanding the cancellation of the $O(T_2^{-3})$ terms between $F_0^H(T_2)$ and $F_0^P(T_2)$ for small T_2, we get that

$$a_0 = \frac{\zeta(3)\zeta(5)}{4\pi}. \tag{2.10}$$

This precisely agrees with this result obtained using a different method in [6]. Hence for small T_2, the singularity in $F_0(T_2)$ is only logarithmic, and is weaker than the bound in (2.2).

Thus the complete solution is given by

$$F_0(T_2) = F_0^H(T_2) + F_0^P(T_2) \tag{2.11}$$

on using (2.4), (2.5) and (2.10).

When expanded for large T_2, we see that $F_0(T_2)$ has terms that are power behaved and logarithmic in T_2, as well as terms that are exponentially suppressed in T_2. In the large T_2 expansion of $Q_n(T_2)$ in (2.6), on using (B.2) we see that there are several cancellations at leading orders, which yield the leading contribution

$$-\frac{3\sigma_1(n)^2}{n^4 T_2^2} e^{-4\pi n T_2} \tag{2.12}$$

to $F_0(T_2)$ from the instanton/anti-instanton sector with weight $e^{-4\pi n T_2}$.

2.2 The modes carrying NS charge

We now consider the modes in (2.1) that carry non-vanishing NS charge. We express the mode $F_n(T_2)$ $(n \neq 0)$ which carries n units of NS charge as

$$F_n(T_2) = I_n(T_2) + \sum_{n_i \neq 0, n_1+n_2=n} I_{n_1,n_2}(T_2), \tag{2.13}$$

where $I_n(T_2)$ and $I_{n_1,n_2}(T_2)$ satisfy the differential equations

$$\left(T_2^2 \frac{d^2}{dT_2^2} - 12 - 4\pi^2 n^2 T_2^2\right) I_n(T_2) = -\frac{24\pi^2\sigma_1(n)}{|n|}\left(\frac{\pi}{3}T_2 - \ln T_2\right) e^{-2\pi|n|T_2} \tag{2.14}$$

and

$$\left(T_2^2 \frac{d^2}{dT_2^2} - 12 - 4\pi^2 (n_1+n_2)^2 T_2^2\right) I_{n_1,n_2}(T_2) = -\frac{24\pi^2\sigma_1(n_1)\sigma_1(n_2)}{|n_1 n_2|} e^{-2\pi(|n_1|+|n_2|)T_2} \tag{2.15}$$

respectively.

We now solve (2.14) and (2.15) with appropriate choice of boundary conditions. For large T_2, the solutions $I_n(T_2)$ and $I_{n_1,n_2}(T_2)$ must have a growth no faster than T_2^2 for the same reasons as before.[8] For small T_2 each mode has singular behavior no worse than T_2^{-1} in order to satisfy (2.2).

[8]In fact, we shall see that the solutions are exponentially suppressed, hence exhibiting significantly milder behavior.

2.2.1 The solution for $I_n(T_2)$

We express

$$I_n(T_2) = I_n^H(T_2) + I_n^P(T_2), \tag{2.16}$$

where $I_n^H(T_2)$ is a solution to the homogeneous equation (2.14), while $I_n^P(T_2)$ solves the particular equation (2.14).

Now $I_n^H(T_2)$ is given by [24]

$$I_n^H(T_2) = b_n \sqrt{T_2} K_{7/2}(2\pi|n|T_2) \tag{2.17}$$

where b_n is an arbitrary constant. We ignore the other solution $\sqrt{T_2}I_{7/2}(2\pi|n|T_2)$ since it grows exponentially for large T_2, violating our boundary condition.

The particular solution $I_n^P(T_2)$ is given by

$$\begin{aligned} I_n^P(T_2) = -\frac{\sigma_1(n)e^{-2\pi|n|T_2}}{16\pi n^4 T_2^3}\Bigg[&-12\Big(2x^2+5x+5\Big)\ln(x/2\pi|n|) \\ &-\frac{4}{|n|}P(x)\ln x + 4\left(1+\frac{1}{|n|}\right)P(-x)e^{2x}\mathrm{Ei}(-2x) \\ &-\Big(26x^2+95x+215\Big) - \frac{4}{|n|}\Big(7x^2+25x+55\Big)\Bigg], \end{aligned} \tag{2.18}$$

where $x = 2\pi|n|T_2$, and $P(x)$ is a polynomial in x defined by

$$P(x) = x^3 + 6x^2 + 15x + 15. \tag{2.19}$$

To determine b_n using the boundary condition at small T_2 mentioned above, we expand both $I_n^H(T_2)$ in (2.17) and $I_n^P(T_2)$ in (2.18) for small T_2. For the solution to the homogeneous equation, we have that

$$I_n^H(T_2) = b_n\left[\frac{15}{16|n|^{7/2}\pi^3 T_2^3}\left(1-\frac{2}{5}\pi^2 n^2 T_2^2\right) + O(T_2)\right]. \tag{2.20}$$

For the particular solution we get

$$I_n^P(T_2) = -\frac{\pi^2\sigma_1(n)}{2|n|}\left[\left(1-\frac{2}{5}\pi^2 n^2 T_2^2\right)\frac{\Psi(n)}{(2\pi|n|T_2)^3} + 4\ln(2\pi|n|T_2)\right] + O(T_2\ln T_2) \tag{2.21}$$

where we have used (B.3), and kept all terms that diverge as $T_2 \to 0$. Here $\Psi(n)$ is given by the expression

$$\Psi(n) = -215 - \frac{220}{|n|} + 60\ln(2\pi|n|) + 60\Big(\gamma + \ln 2\Big)\left(1+\frac{1}{|n|}\right). \tag{2.22}$$

Note that there is no T_2^{-2} term in (2.21). Thus the cancellation of the T_2^{-3} in the small T_2 expansion gives us that

$$b_n = \frac{\pi^2\sigma_1(n)\Psi_{3,n}}{15|n|^{1/2}} \tag{2.23}$$

yielding the complete solution. In fact the T_2^{-1} term also cancels on adding (2.20) and (2.21), and hence the only singular term in $I_n(T_2)$ for small T_2 is given by

$$-2\pi^2\frac{\sigma_1(n)}{|n|}\ln(2\pi|n|T_2). \tag{2.24}$$

Now for large T_2, $I_n(T_2)$ behaves as $e^{-2\pi|n|T_2}$ with the leading contribution being given by

$$I_n(T_2) = 2\pi^2\frac{\sigma_1(n)}{n^2}\ln(2\pi|n|T_2)e^{-2\pi|n|T_2} \tag{2.25}$$

where we have used (B.2). Thus these are contributions from bound states of worldsheet instantons (or anti-instantons) if n is positive (or negative).

2.2.2 The solution for $I_{n_1,n_2}(T_2)$

Like before, we express

$$I_{n_1,n_2}(T_2) = I^H_{n_1,n_2}(T_2) + I^P_{n_1,n_2}(T_2), \tag{2.26}$$

where $I^H_{n_1,n_2}(T_2)$ is a solution to the homogeneous equation (2.15), while $I^P_{n_1,n_2}(T_2)$ solves the particular equation (2.15).

The solution $I^H_{n_1,n_2}(T_2)$ satisfying the large T_2 boundary condition is given by

$$I^H_{n_1,n_2}(T_2) = c_{n_1,n_2}\sqrt{T_2}K_{7/2}(2\pi|n_1+n_2|T_2) \tag{2.27}$$

where c_{n_1,n_2} is an arbitrary constant.

We now consider the particular solution $I^P_{n_1,n_2}(T_2)$. It is convenient to consider the two cases separately:

(i) n_1 and n_2 have same sign (thus $n_1n_2 > 0$), and

(ii) n_1 and n_2 have opposite signs (thus $n_1n_2 < 0$).

For case **(i)**, we have that

$$I^P_{n_1,n_2}(T_2) = -\frac{6\pi^2\sigma_1(n_1)\sigma_1(n_2)}{n_1n_2x^3}e^{-x}(2x^2+5x+5), \tag{2.28}$$

where $x = 2\pi|n_1+n_2|T_2$. Unlike the other cases, there are no contributions involving the exponential integral function.

To determine c_{n_1,n_2}, we demand the cancellation of the T_2^{-3} term in the small T_2 expansion of $I_{n_1,n_2}(T_2)$ as discussed earlier, which gives us that

$$c_{n_1,n_2} = 4\pi^2\frac{|n_1+n_2|^{1/2}}{n_1n_2}\sigma_1(n_1)\sigma_1(n_2). \tag{2.29}$$

This also cancels the T_2^{-1} term in the small T_2 expansion and hence there are no singular terms in $I_{n_1,n_2}(T_2)$ in this limit, as there is no T_2^{-2} term that arises from (2.28).

On expanding $I_{n_1,n_2}(T_2)$ for large T_2, we see that all the terms are suppressed by a factor of $e^{-2\pi|n_1+n_2|T_2}$. Hence they arise from bound states of worldsheet instantons or anti-instantons depending on whether n_1 is positive or negative. In fact, the leading contribution is given by

$$\frac{2\pi^2\sigma_1(n_1)\sigma_1(n_2)}{n_1n_2}e^{-2\pi|n_1+n_2|T_2}. \tag{2.30}$$

Now for case **(ii)**, we have the particular solution

$$\begin{aligned}
I^P_{n_1,n_2}(T_2) = \frac{3e^{-2\pi(|n_1|+|n_2|)T_2}\sigma_1(n_1)\sigma_1(n_2)}{32\pi^2n_1n_2|n_1+n_2|^7T_2^3}\Big[&(\alpha-\beta)\Big(5(|n_1|+|n_2|)R_{13,15}\\
&+10\pi(n_1+n_2)^2R_{3,5}T_2+4\pi^2(n_1+n_2)^2(|n_1|+|n_2|)R_{1,5}T_2^2\Big)\\
&-\frac{\alpha\beta}{2\pi}R_{1,5}\Big(P(2\pi|n_1+n_2|T_2)e^{\alpha T_2}\mathrm{Ei}(-\alpha T_2)\\
&-P(-2\pi|n_1+n_2|T_2)e^{\beta T_2}\mathrm{Ei}(-\beta T_2)\Big)\Big],
\end{aligned} \tag{2.31}$$

where α (>0) and β are defined by

$$\begin{aligned}
\alpha &= 2\pi(|n_1|+|n_2|-|n_1+n_2|),\\
\beta &= 2\pi(|n_1|+|n_2|+|n_1+n_2|),
\end{aligned} \tag{2.32}$$

while $R_{a,b}$ is defined by

$$R_{a,b} = a(n_1+n_2)^2 - b(n_1-n_2)^2. \tag{2.33}$$

Also $P(x)$ is the polynomial defined by (2.19).

In order to determine c_{n_1,n_2}, we cancel the T_2^{-3} term in the small T_2 expansion of $I_{n_1,n_2}(T_2)$ as before. On using (B.3), the small T_2 expansion of (2.31) is given by

$$I^P_{n_1,n_2}(T_2) = \frac{3\sigma_1(n_1)\sigma_1(n_2)}{64\pi^3n_1n_2|n_1+n_2|^7T_2^3}\left[1-\frac{2}{5}\pi^2(n_1+n_2)^2T_2^2\right]\Psi(n_1,n_2)+O(T_2^0) \tag{2.34}$$

where

$$\Psi(n_1,n_2) = -15\alpha\beta R_{1,5}\ln(\alpha/\beta)+\frac{5}{2}(\alpha^2-\beta^2)R_{13,15}. \tag{2.35}$$

We note that the T_2^{-2} term vanishes in (2.34). Thus we have that

$$c_{n_1,n_2} = -\frac{\sigma_1(n_1)\sigma_1(n_2)\Psi(n_1,n_2)}{20n_1n_2|n_1+n_2|^{7/2}}. \tag{2.36}$$

In fact, the T_2^{-1} term also cancels in the small T_2 expansion of $I_{n_1,n_2}(T_2)$ and hence there are no singular terms in this expansion.

Now consider the large T_2 expansion of $I_{n_1,n_2}(T_2)$. For fixed n_1 and n_2, the leading contribution comes from the homogeneous solution and is of the form $e^{-2\pi|n_1+n_2|T_2}$. Thus the leading contribution is given by

$$-\frac{\sigma_1(n_1)\sigma_1(n_2)}{40n_1n_2(n_1+n_2)^4}\Psi(n_1,n_2)e^{-2\pi|n_1+n_2|T_2}. \tag{2.37}$$

The particular solution is exponentially suppressed by an additional factor of $e^{-\alpha T_2}$, and the leading contribution is given by

$$- \frac{3\sigma_1(n_1)\sigma_1(n_2)}{2n_1^2 n_2^2 T_2^2} e^{-2\pi(|n_1|+|n_2|)T_2} \tag{2.38}$$

on using (B.2).

Contributions of this kind that are exponentially suppressed at large T_2 arise from bound states of worldsheet instantons and anti-instantons.

Thus the above expressions yield the complete data needed to evaluate (2.13). Now in (2.13), the contributions arising from $n_1 n_2 < 0$ yield an infinite sum given by

$$2 \sum_{n_1 \geq n+1} I_{n_1, n-n_1} \tag{2.39}$$

and hence it is worthwhile to check the convergence of this sum. For this, we focus on the large n_1 behavior of the various terms while keeping n fixed. The contribution arising from the particular solution (2.31) is exponentially damped in this limit, hence convergence is trivial. To analyze the contributions that arise from the homogeneous solution (2.27) consider the large n_1 limit of $c_{n_1,n-n_1}$ in (2.36). This is given by

$$c_{n_1,n-n_1} \to \frac{4\pi^2 \sigma_1(n_1)^2 |n|^{7/2}}{35|n_1|^5} \tag{2.40}$$

as several leading contributions cancel. Using the inequality [25]

$$\sigma_1(n) < e^{\gamma} n \mathrm{lnln} n + \frac{0.6483n}{\mathrm{lnln} n} \tag{2.41}$$

for $n \geq 3$, it follows that the sum over n_1 is convergent.[9]

3 S-duality and an elementary consequence for D string instanton contributions

The worldsheet instanton contributions under S-duality get mapped to D string instanton contributions [1, 27]. Given the exact expressions for the worldsheet instanton contributions, though it takes work to implement S-duality in order to obtain the complete D string instanton contributions, it is elementary to implement strong weak coupling duality to obtain a part of the D string instanton contributions, which we now illustrate.

[9] A related inequality is given by [26]

$$\sigma_1(n) < H_n + e^{H_n} \mathrm{ln} H_n,$$

for $n > 1$, where H_n is the nth harmonic number. Using the asymptotic expansion for H_n given by

$$H_n = \mathrm{ln} n + \gamma + \frac{1}{2n} - \sum_{m=1}^{\infty} \frac{B_{2m}}{2m \cdot n^{2m}}$$

where B_m are the Bernoulli numbers, we again see that the sum over n_1 in (2.40) is convergent.

As a simple example, consider the worldsheet instanton contribution to the $\mathcal{R}^4$ coupling given by

$$2\pi \sum_{n=1}^{\infty} \frac{\sigma_1(n)}{n} e^{2\pi i n T}, \tag{3.1}$$

which follows from (1.6). In the background where $\tau_1 = 0$, strong weak coupling duality yields

$$\tau_2 \to \frac{1}{\tau_2}, \quad V \to \tau_2 V, \quad B_N \to B_R, \quad B_R \to -B_N. \tag{3.2}$$

Thus performing the S-duality transformation (3.2) on (3.1), we get the D string instanton contribution[10]

$$2\pi \sum_{n=1}^{\infty} \frac{\sigma_1(n)}{n} e^{2\pi i n S}, \tag{3.3}$$

where

$$S = S_1 + i S_2 = B_R + i\tau_2 V. \tag{3.4}$$

Similarly for the $D^6\mathcal{R}^4$ coupling the S-duality transformations (3.2) yield partial contributions to the D string instanton contributions using the various expressions for the worldsheet instanton contributions we have analyzed. For example, from (2.6) we see that the contribution from the bound states of D string instantons/anti-instantons carrying no net RR charge is given by

$$\sum_{n=1}^{\infty} \widetilde{Q}_n(S_2) e^{-4\pi n S_2}, \tag{3.5}$$

where $\widetilde{Q}_n(S_2)$ is given by

$$\widetilde{Q}_n(S_2) = -\frac{3\sigma_1(n)^2}{n^4 S_2^2}\left[1 - \frac{4}{y} + \frac{1}{168}\sum_{m=0}^{\infty}\frac{(-1)^m (m+7)!}{y^{m+2}}\right], \tag{3.6}$$

where $y = 4\pi n S_2$ and we have performed a weak coupling (large τ_2) expansion using (B.2). While the overall S_2 dependence must arise from the structure of zero modes in the instanton/anti-instanton background, we see that the infinite sum is an expansion in $y \sim e^{-\phi}$, the open string coupling. Note that performing (3.2) on $I_n^P(T_2)$ in (2.18) yields contributions having factors of $\ln\tau_2$, which arise from non-local interactions logarithmic in the external momenta in the string frame, on converting to the Einstein frame. This is precisely what is expected from the structure of the U-duality invariant eigenvalue equation that arises for the $D^6\mathcal{R}^4$ coupling [4, 6], as the source term contains $\ln\tau_2$ that arises from the $\mathcal{R}^4$ coupling [2].

[10] In fact this is the complete answer from the sum over the $(0, n)$ D string instantons which follows from the U-duality invariant expression for the $\mathcal{R}^4$ coupling [2, 3].

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A The SL$(2,\mathbb{Z})$ invariant non-holomorphic Eisenstein series

The non-holomorphic Eisenstein series $E_s(T,\overline{T})$ is given by the expression

$$E_s(T,\overline{T}) = 2\zeta(2s)T_2^s + 2\sqrt{\pi}T_2^{1-s}\frac{\Gamma(s-1/2)}{\Gamma(s)}\zeta(2s-1) \\ + \frac{4\pi^s\sqrt{T_2}}{\Gamma(s)}\sum_{n\neq 0}\frac{\sigma_{2s-1}(n)}{|n|^{s-1/2}}K_{s-1/2}(2\pi|n|T_2)e^{2\pi i n T_1} \tag{A.1}$$

on expanding around large T_2. Here the divisor function $\sigma_m(n)$ is defined by

$$\sigma_m(n) = \sum_{d|n,d>0} d^m, \tag{A.2}$$

where the sum is over the positive divisors of n. The case $s = 1$ has to be regularized and is given by

$$E_1(T,\overline{T}) = -\pi \ln\Big(T_2|\eta(T)|^4\Big) \\ = \frac{\pi^2}{3}T_2 - \pi\ln T_2 + 2\pi\sum_{n\neq 0}\frac{\sigma_1(n)}{|n|}e^{2\pi i(nT_1+i|n|T_2)}. \tag{A.3}$$

B The exponential integral function

The exponential integral function $\mathrm{Ei}(-x)$ is given by the integral representation

$$e^x\mathrm{Ei}(-x) = -\frac{1}{x} + \int_0^\infty dt\frac{e^{-t}}{(t+x)^2}, \quad x > 0. \tag{B.1}$$

Thus we see that $e^x\mathrm{Ei}(-x)$ is a polynomial in $1/x$ of the form

$$e^x\mathrm{Ei}(-x) = -\frac{1}{x} + \sum_{n=0}^\infty \frac{(-1)^n(n+1)!}{x^{n+2}} \tag{B.2}$$

for large x. On the other hand, for small x, the series expansion is given by

$$\mathrm{Ei}(-x) = \gamma + \ln x + \sum_{n=1}^\infty \frac{(-x)^n}{n\cdot n!}, \quad x > 0. \tag{B.3}$$

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