Vector perturbations of Kerr-AdS$_5$ and the Painlevé VI transcendent

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ABSTRACT: We analyze the Ansatz of separability for Maxwell equations in generically spinning, five-dimensional Kerr-AdS black holes. We find that the parameter μ introduced in [1] can be interpreted as apparent singularities of the resulting radial and angular equations. Using isomonodromy deformations, we describe a non-linear symmetry of the system, under which μ is tied to the Painlevé VI transcendent. By translating the boundary conditions imposed on the solutions of the equations for quasinormal modes in terms of monodromy data, we find a procedure to fix μ and study the behavior of the quasinormal modes in the limit of fast spinning small black holes.

KEYWORDS: Black Holes, Black Holes in String Theory, Integrable Hierarchies

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Contents

1 Introduction

Black holes in higher dimensions [2, 3] are important to understand aspects of the gauge/gravity correspondence, with the ultimate goal a better understanding of theories with non-trivial infrared fixed points. On the other hand, a better understanding of general relativity is an interesting goal *per se*, with a clear view on the generic properties and the features which are special to four dimensions well worth pursuing.

Black hole solutions are particularly distinguished by their integrable structure. The first example is the four-dimensional vacuum solution given by the Kerr geometry, which can be found explicitly even though its isometries — time translation and axial rotation — do not warrant integrability of the equations in the Liouville sense. The solution was generalized to non-zero cosmological constant by Carter [4], and to higher even dimensions $D = 2n$ by Myers and Perry [5], characterized by n conserved charges. In odd dimensions, they were constructed in [6] for the particular case $D = 5$ and then, generically, in [7, 8]. The family of solutions present an integrable set of null congruences, and the integrability of the solutions themselves can be ascribed to the existence of higher-rank tensors, satisfying an analogue of the Killing equation for isometries, the so-called Killing tensors.

This integrable structure, called *hidden symmetries*, allows not only for the construction of the solutions, but also for separability of the scalar and spinor wave equations [9]. For spin 1 fields, however, the situation is murkier. The separation of Maxwell's equations in four dimensions, obtained first by Teukolsky [10], is a result of the existence of a Killing-Yano conserved tensor. In higher dimensions, the separation was achieved by Lunin [1] at the expense of the introduction of an arbitrary parameter μ. This new technique was

dubbed "μ-separability" in [11]. The new parameter is related to the existence of different polarizations of the electromagnetic and Proca fields [12, 13], as well as the higher p-form generalization considered in [14]. Because of this, the treatment of tensor fields in these black hole backgrounds is quite different from the scalar case. The introduction of this extra separability parameter brings in further questions, related to which physical requirements should fix its value, as in the determination of scattering coefficients, angular eigenvalues and the frequency quasinormal modes.

Coming from a different perspective, the separability of the scalar wave equation in the subcase of a five dimensional black hole with a negative cosmological constant — Kerr-AdS$_5$ — was tied to the construction of two flat holomorphic connections in a previous article by the authors [15], related to the solutions of the angular and radial differential equations. There, the purpose was solely "dynamical": flat holomorphic connections have a residual gauge-symmetry which allows for solving the connection problem of the differential equations [16].

The residual gauge symmetry, known as "isomonodromic transformations" in the theory of ordinary differential equations [17], is realized in the angular and radial equations by the presence of an extra singular point in the Fuchsian equations, whose monodromy properties are trivial. This extra apparent singularity can be moved around the complex plane, and the isomonodromy transformation forces a functional dependence between the position of the apparent singularity and the positions of the other singularities, which was found to be the celebrated Painlevé transcendent of the sixth type.

In the scalar case, these extra singularities play an auxiliary role in the actual solution of the problem: quantities such as scattering amplitudes and the quasinormal modes depend solely on the monodromy data. One can then compute them at any point of the isomonodromic flow, with the coincident point where the apparent singularity merges with one of the remaining singularities being particularly convenient.

The purpose of the present article is to study the μ-separability in the particular case of the spin 1 field in a generic Kerr-AdS$_5$ black hole in order to further elucidate the role of the μ parameter. As we will see this is directly related by a Möbius transformation to the Painlevé transcendent, and parametrizes the position of the apparent singularity of both the radial and angular equations. This leads us to the conclusion that the role of μ in higher dimensions is different to that in four dimensions, where it can be eliminated by a change of parametrization in the corresponding equations.

In the case considered here, the trick of "deforming" the Heun equation by adding an extra, apparent singularity is mandatory. We will see, however, that the boundary conditions for angular eigenvalues and quasinormal modes can be written in terms of monodromy, and hence can be thought of as isomonodromy invariants. Assuming this invariance, we are able to fix the parameter μ through a consistency condition of the isomonodromic flow in the radial and angular equations. We then proceed to a short numerical analysis of the solution proposed and close with a short discussion and prospects.

2 Maxwell perturbations on Kerr-AdS₅

The five dimensional, generically rotating, Kerr-AdS₅ metric was given in [6]

$$
ds^2 = -\frac{\Delta}{\rho^2}\left(dt - \frac{a_1\sin^2\theta}{1-a_1^2}d\phi - \frac{a_2\cos^2\theta}{1-a_2^2}d\psi\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \frac{\rho^2}{1-x^2}d\theta^2
$$
$$
+ \frac{(1-x^2)\sin^2\theta\cos^2\theta}{x^2\rho^2}\left[(a_2^2-a_1^2)\,dt + \frac{a_1(r^2+a_1^2)}{1-a_1^2}d\phi - \frac{a_2(r^2+a_2^2)}{1-a_2^2}d\psi\right]^2
$$
$$
+ \frac{a_1^2 a_2^2}{x^2 r^2}\left[dt - \frac{(r^2+a_1^2)\sin^2\theta}{a_1(1-a_1^2)}d\phi - \frac{(r^2+a_2^2)\cos^2\theta}{a_2(1-a_2^2)}d\psi\right]^2, \tag{2.1}
$$

where

$$
\Delta = \frac{1}{r^2}(r^2+a_1^2)(r^2+a_2^2)(1+r^2) - 2M, \qquad x^2 = a_1^2\cos^2\theta + a_2^2\sin^2\theta
$$
$$
\rho^2 = r^2 + x^2 \tag{2.2}
$$

and a_1 and a_2 are two independent rotation parameters. This particular form of the metric allows to define an orthonormal 1-form basis e^A,[1]

$$
e^0 = \sqrt{\frac{\Delta}{r^2+x^2}}\left(dt - \frac{a_1\sin^2\theta}{1-a_1^2}d\phi - \frac{a_2\cos^2\theta}{1-a_2^2}d\psi\right), \tag{2.3a}
$$

$$
e^1 = \sqrt{\frac{r^2+x^2}{\Delta}}dr, \tag{2.3b}
$$

$$
e^2 = \sqrt{\frac{r^2+x^2}{1-x^2}}d\theta, \tag{2.3c}
$$

$$
e^3 = \sqrt{\frac{1-x^2}{r^2+x^2}}\frac{\sin\theta\cos\theta}{x}\left((a_2^2-a_1^2)\,dt + \frac{a_1(r^2+a_1^2)}{1-a_1^2}d\phi - \frac{a_2(r^2+a_2^2)}{1-a_2^2}d\psi\right), \tag{2.3d}
$$

$$
e^4 = \frac{a_1 a_2}{rx}\left(dt - \frac{(r^2+a_1^2)\sin^2\theta}{a_1(1-a_1^2)}d\phi - \frac{(r^2+a_2^2)\cos^2\theta}{a_2(1-a_2^2)}d\psi\right), \tag{2.3e}
$$

which, then, allows us to write

$$
ds^2 = -(e^0)^2 + (e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2. \tag{2.4}
$$

The inverse metric has a similar factorization

$$
g^{\mu\nu}\partial_\mu\partial_\nu = -(e_0)^2 + (e_1)^2 + (e_2)^2 + (e_3)^2 + (e_4)^2, \tag{2.5}
$$

where

$$
e_0 = \sqrt{\frac{1}{\Delta(r^2+x^2)}}\frac{(r^2+a_1^2)(r^2+a_2^2)}{r^2}\left(\partial_t + \frac{a_1(1-a_1^2)}{r^2+a_1^2}\partial_\phi + \frac{a_2(1-a_2^2)}{r^2+a_2^2}\partial_\psi\right), \tag{2.6a}
$$

$$
e_1 = \sqrt{\frac{\Delta}{r^2+x^2}}\partial_r, \tag{2.6b}
$$

---

[1] $e^A = e^A_\mu dx^\mu$, the Lorentz indices run as follows $A = \{0,1,2,3,4\}$.

$$e_2 = \sqrt{\frac{1-x^2}{r^2+x^2}}\,\partial_\theta, \tag{2.6c}$$

$$e_3 = \frac{1}{\sqrt{(1-x^2)(r^2+x^2)}}\frac{\sin\theta\cos\theta}{x}\left(\left(a_1^2-a_2^2\right)\partial_t + \frac{a_1(1-a_1^2)}{\sin^2\theta}\partial_\phi - \frac{a_2(1-a_2^2)}{\cos^2\theta}\partial_\psi\right), \tag{2.6d}$$

$$e_4 = -\frac{a_1 a_2}{rx}\left(\partial_t + \frac{(1-a_1^2)}{a_1}\partial_\phi + \frac{(1-a_2^2)}{a_2}\partial_\psi\right). \tag{2.6e}$$

Following [1], to separate the radial and angular equation for the gauge field, we need to construct a special frame with a pair of real null vectors, a pair of complex null vectors and a space-like unit vector orthogonal to all others. They are given as $\ell, n, m, \bar{m}, k$ as follows

$$\begin{aligned}
\ell &= \sqrt{\frac{r^2+x^2}{\Delta}}\,(e_0 + e_1)\\
&= \frac{(r^2+a_1^2)(r^2+a_2^2)}{r^2\Delta}\left(\partial_t + \frac{a_1(1-a_1^2)}{r^2+a_1^2}\partial_\phi + \frac{a_2(1-a_2^2)}{r^2+a_2^2}\partial_\psi\right) + \partial_r,
\end{aligned} \tag{2.7a}$$

$$\begin{aligned}
n &= \frac{1}{2}\sqrt{\frac{\Delta}{r^2+x^2}}\,(e_0 - e_1)\\
&= \frac{(r^2+a_1^2)(r^2+a_2^2)}{2r^2(r^2+x^2)}\left(\partial_t + \frac{a_1(1-a_1^2)}{r^2+a_1^2}\partial_\phi + \frac{a_2(1-a_2^2)}{r^2+a_2^2}\partial_\psi\right) - \frac{\Delta}{2(r^2+x^2)}\partial_r,
\end{aligned} \tag{2.7b}$$

$$\begin{aligned}
m &= \frac{1}{\sqrt{2}}\frac{r-ix}{\sqrt{r^2+x^2}}\,(e_2 + ie_3)\\
&= \frac{\sqrt{1-x^2}}{\sqrt{2}(r+ix)}\left[\partial_\theta + i\frac{\sin\theta\cos\theta}{x(1-x^2)}\left((a_1^2-a_2^2)\partial_t + \frac{a_1(1-a_1^2)}{\sin^2\theta}\partial_\phi - \frac{a_2(1-a_2^2)}{\cos^2\theta}\partial_\psi\right)\right],
\end{aligned} \tag{2.7c}$$

$$\begin{aligned}
\bar{m} = (m)^* &= \frac{1}{\sqrt{2}}\frac{r+ix}{\sqrt{r^2+x^2}}\,(e_2 - ie_3)\\
&= \frac{\sqrt{1-x^2}}{\sqrt{2}(r-ix)}\left[\partial_\theta - i\frac{\sin\theta\cos\theta}{x(1-x^2)}\left((a_1^2-a_2^2)\partial_t + \frac{a_1(1-a_1^2)}{\sin^2\theta}\partial_\phi - \frac{a_2(1-a_2^2)}{\cos^2\theta}\partial_\psi\right)\right],
\end{aligned} \tag{2.7d}$$

$$k = -\frac{a_1 a_2}{rx}\left(\partial_t + \frac{(1-a_1^2)}{a_1}\partial_\phi + \frac{(1-a_2^2)}{a_2}\partial_\psi\right). \tag{2.7e}$$

The first four elements of the list $\ell, n, m, \bar{m}$ are null vectors — a null tetrad — whereas k is orthogonal and space-like unit vector. Now we define the null transformed "light-cone" basis

$$\begin{aligned}
\ell_+ &= \ell, & \ell_- &= -\frac{2(r^2+x^2)}{\Delta}n,\\
m_+ &= \sqrt{2}(r+ix)m, & m_- &= \sqrt{2}(r-ix)\bar{m} = (m_+)^*,
\end{aligned} \tag{2.8}$$

leaving k unchanged. Now (ℓ_+, ℓ_-) do not depend on the polar angle θ, and (m_+, m_-) do not depend on the radial coordinate r.

2.1 Separation of variables for Maxwell equations

In a particular coordinate basis $\{x^\mu\}$, the source-free Maxwell equations for a massless vector field can be written as

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}F^{\mu\nu}\right) = 0, \quad \text{with} \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{2.9}$$

which unfortunately are *not* separable in the background (2.1). Again following [1], we can achieve separability by introducing a parameter μ, and define two classes of solutions, corresponding to two different polarizations, called electric and magnetic modes:

$$\ell_\pm^a A_a^{(el)} = \pm\frac{\mu r}{\mu \mp ir}\ell_\pm^a \nabla_a \Psi, \qquad m_\pm^a A_a^{(el)} = \pm\frac{i\mu x}{\mu \pm x}m_\pm^a \nabla_a \Psi, \qquad k^a A_a^{(el)} = 0; \tag{2.10}$$

$$\ell_\pm^a A_a^{(mgn)} = \pm\frac{1}{r \pm i\mu}\ell_\pm^a \nabla_a \Psi, \quad m_\pm^a A_a^{(mgn)} = \mp\frac{i}{x \pm \mu}m_\pm^a \nabla_a \Psi, \quad k^a A_a^{(mgn)} = \lambda\Psi, \tag{2.11}$$

where Ψ is a scalar function which, as we will see below, satisfies a separable equation. We note that the covariant derivative is always applied to scalars, so they are independent of the Christoffel connection. Writing

$$\Psi = e^{-i\omega t + i\tilde{m}_1\phi + i\tilde{m}_2\psi}\Phi(r)S(x), \tag{2.12}$$

we can express the components of the potential A_a explicitly in the "light-cone" basis (2.8) and (2.10). For instance, for the electric solution $A_a^{(el)}$ ($A_\mu^{(el)} = A_a^{(el)}(\partial_\mu)^a$):

$$A_t^{(el)} = \frac{\Psi}{(r^2 + \mu^2)}\left\{\frac{\mu^2 r\Delta}{(r^2 + x^2)}\frac{\Phi'(r)}{\Phi(r)} + \frac{S'(x)}{S(x)}\frac{\mu^2(r^2 + \mu^2)(1 - x^2)\sqrt{(a_1^2 - x^2)(x^2 - a_2^2)}}{(r^2 + x^2)(x^2 - \mu^2)}\right.$$
$$+ \frac{\mu}{(x^2 - \mu^2)}\left[\omega(r^2 + \mu^2)(x^2 - a_2^2) + \omega(r^2 + a_1^2)(a_2^2 - \mu^2)\right.$$
$$\left.\left. - a_1\tilde{m}_1(a_2^2 - \mu^2) - a_2\tilde{m}_2(a_1^2 - \mu^2)\right]\right\}, \tag{2.13a}$$

$$A_r^{(el)} = \frac{i\Psi}{(r^2 + \mu^2)}\left[r^2\mu\frac{\Phi'(r)}{\Phi(r)} - \frac{\mu^2(r^2 + a_1^2)(r^2 + a_2^2)}{r\Delta}\left(\omega - \frac{a_1\tilde{m}_1}{r^2 + a_1^2} - \frac{a_2\tilde{m}_2}{r^2 + a_2^2}\right)\right], \tag{2.13b}$$

$$A_\theta^{(el)} = \frac{i\Psi}{(x^2 - \mu^2)}\left[\mu x^2\frac{S'(x)}{S(x)} - \frac{\mu^2\sqrt{(a_1^2 - x^2)(x^2 - a_2^2)}}{(1 - x^2)}\left(\omega - \frac{a_1\tilde{m}_1}{(a_1^2 - x^2)} - \frac{a_2\tilde{m}_2}{(a_2^2 - x^2)}\right)\right], \tag{2.13c}$$

$$A_\phi^{(el)} = -\frac{a_1\Psi}{(1 - a_1^2)(r^2 + \mu^2)}\left[\frac{\Delta\mu^2 r(x^2 - a_1^2)}{(r^2 + x^2)(a_2^2 - a_1^2)}\frac{\Phi'(r)}{\Phi(r)} - \frac{\mu^2(r^2 + a_1^2)(r^2 + \mu^2)}{(r^2 + x^2)}\right.$$
$$\frac{(1 - x^2)\sqrt{(a_1^2 - x^2)(x^2 - a_2^2)}}{(a_2^2 - a_1^2)(x^2 - \mu^2)}\frac{S'(x)}{S(x)} + \frac{\mu}{(x^2 - \mu^2)}\left(a_1\tilde{m}_1(\mu^2 - a_2^2)\right.$$
$$+ a_1\tilde{m}_1(r^2 + a_2^2)\frac{(a_2^2 - x^2)}{(a_2^2 - a_1^2)} - a_2\tilde{m}_2(r^2 + a_1^2)\frac{(x^2 - a_1^2)}{(a_2^2 - a_1^2)}$$
$$\left.\left. + \omega(a_2^2 - \mu^2)(r^2 + a_1^2)\frac{(x^2 - a_1^2)}{(a_2^2 - a_1^2)}\right)\right], \tag{2.13d}$$

$$A_\psi^{(\text{el})} = -\frac{a_2\Psi}{(1-a_2^2)(r^2+\mu^2)}\left[\frac{\Delta\mu^2 r(a_2^2-x^2)}{(r^2+x^2)(a_2^2-a_1^2)}\frac{\Phi'(r)}{\Phi(r)} - \frac{\mu^2(r^2+a_2^2)(r^2+\mu^2)}{(r^2+x^2)}\times\right.$$

$$\frac{(1-x^2)\sqrt{(a_1^2-x^2)(x^2-a_2^2)}}{(x^2-\mu^2)(a_1^2-a_2^2)}\frac{S'(x)}{S(x)} + \frac{\mu}{(x^2-\mu^2)}\left(a_2\tilde{m}_2(\mu^2-a_1^2)\right.$$

$$+ a_2\tilde{m}_2(r^2+a_1^2)\frac{(x^2-a_1^2)}{(a_2^2-a_1^2)} - a_1\tilde{m}_1(r^2+a_2^2)\frac{(a_2^2-x^2)}{(a_2^2-a_1^2)}$$

$$\left.\left.- \omega(r^2+a_2^2)(\mu^2-a_1^2)\frac{(a_2^2-x^2)}{(a_2^2-a_1^2)}\right)\right]. \tag{2.13e}$$

Now, the equations for $\Phi(r)$ and $S(x)$ can be written as two separate equations, coupled by a separation constant C_m and the parameter μ

$$\frac{D_r}{r}\frac{d}{dr}\left[\frac{rQ_r^2(\Delta-R)}{D_r}\frac{d\Phi}{dr}\right] + \left\{\frac{2\tilde{\Lambda}}{D_r} + \frac{R^2\tilde{W}_r^2}{r^4Q_r^2(\Delta-R)} - \frac{a_1^2a_2^2D_r}{r^2}\tilde{\Omega}^2 + \mu^2C_mD_r\right\}\Phi(r) = 0, \tag{2.14a}$$

$$\frac{D}{x}\frac{d}{dx}\left[\frac{Q^2H}{xD}\frac{dS}{dx}\right] + \left\{\frac{2\tilde{\Lambda}}{D} - \frac{H\tilde{W}^2}{Q^2x^2} + \frac{a_1^2a_2^2D}{x^2}\tilde{\Omega}^2 + \mu^2C_mD\right\}S(x) = 0, \tag{2.14b}$$

and the functions and constants given by

$$R = (r^2+a_1^2)(r^2+a_2^2), \qquad Q_r^2 = \frac{R(1+r^2)-2Mr^2}{r^2(\Delta-R)}, \tag{2.15}$$

$$D_r = 1 + \frac{r^2}{\mu^2}, \qquad \tilde{W}_r = \omega - \frac{\tilde{m}_1a_1}{r^2+a_1^2} - \frac{\tilde{m}_2a_2}{r^2+a_2^2},$$

$$H = (a_1^2-x^2)(a_2^2-x^2), \qquad Q^2 = 1-x^2, \tag{2.16}$$

$$D = 1 - \frac{x^2}{\mu^2}, \qquad \tilde{W} = \omega - \frac{\tilde{m}_1a_1}{a_1^2-x^2} - \frac{\tilde{m}_2a_2}{a_2^2-x^2}.$$

$$\tilde{\Lambda} = \frac{(a_1^2-\mu^2)(a_2^2-\mu^2)}{\mu^3}\left(\omega - \frac{\tilde{m}_1a_1}{a_1^2-\mu^2} - \frac{\tilde{m}_2a_2}{a_2^2-\mu^2}\right), \qquad \tilde{\Omega} = \omega - \frac{\tilde{m}_1}{a_1} - \frac{\tilde{m}_2}{a_2}. \tag{2.17}$$

The subscript m in the separation constant C_m is an integer index and will be discussed in detail in section 4.

The equations above determine the electric polarization, in the sense described in (2.10), for the potential. The corresponding equations for the magnetic polarizations were also worked out in [1], and it is eventually found that the function Ψ defined through (2.11) also satisfies (2.14a) and (2.14b), provided the separation parameter μ is substituted by $1/\mu$. Given μ, the value for λ in (2.11) is fixed to $\tilde{\Omega}/\mu$. The details can be checked in [1] — although we note the slight change of notation $\omega(\text{there}) = -\omega(\text{here})$, $a_{1,2}(\text{there}) = -a_{1,2}(\text{here})$, $M(\text{there}) = 2M(\text{here})$ and $P_0(\text{there}) = \mu^2C_m(\text{here})$. We also note that, because of the periodicity of the coordinates ϕ and ψ, we have $\tilde{m}_i = (1-a_i^2)m_i$, with m_i integers.

Explicitly, the radial and the angular equations are

$$\frac{r^2 + \mu^2}{r}\frac{d}{dr}\left[\frac{(r^2 - r_0^2)(r^2 - r_-^2)(r^2 - r_+^2)}{r(r^2 + \mu^2)}\frac{d\Phi}{dr}\right] + \left\{-\frac{(r^2 + \mu^2)}{\mu^2 r^2}(a_1 a_2 \omega - (1 - a_1^2)m_1 a_2\right.$$

$$- (1 - a_2^2)m_2 a_1)^2 + \frac{(r^2 + a_1^2)^2(r^2 + a_2^2)^2}{r^2(r^2 - r_0^2)(r^2 - r_-^2)(r^2 - r_+^2)}\left(\omega - \frac{a_1(1 - a_1^2)m_1}{r^2 + a_1^2} - \frac{a_2(1 - a_2^2)m_2}{r^2 + a_2^2}\right)^2$$

$$- \frac{2(a_1^2 - \mu^2)(a_2^2 - \mu^2)}{\mu(r^2 + \mu^2)}\left(\omega - \frac{a_1(1 - a_1^2)m_1}{a_1^2 - \mu^2} - \frac{a_2(1 - a_2^2)m_2}{a_2^2 - \mu^2}\right) + C_m(r^2 + \mu^2)\right\}\Phi(r) = 0,$$

$$(2.18)$$

and

$$\frac{(\mu^2 - x^2)}{x}\frac{d}{dx}\left[\frac{(1 - x^2)(a_1^2 - x^2)(a_2^2 - x^2)}{x(\mu^2 - x^2)}\frac{dS}{dx}\right] + \left\{\frac{(\mu^2 - x^2)}{\mu^2 x^2}(a_1 a_2 \omega - a_2(1 - a_1^2)m_1\right.$$

$$- a_1(1 - a_2^2)m_2)^2 - \frac{(a_1^2 - x^2)(a_2^2 - x^2)}{x^2(1 - x^2)}\left(\omega - \frac{a_1(1 - a_1^2)m_1}{a_1^2 - x^2} - \frac{a_2(1 - a_2^2)m_2}{a_2^2 - x^2}\right)^2$$

$$- \frac{2(a_1^2 - \mu^2)(a_2^2 - \mu^2)}{\mu(\mu^2 - x^2)}\left(\omega - \frac{a_1(1 - a_1^2)m_1}{a_1^2 - \mu^2} - \frac{a_2(1 - a_2^2)m_2}{a_2^2 - \mu^2}\right) + C_m(\mu^2 - x^2)\right\}S(x) = 0$$

$$(2.19)$$

where now the values r_+^2, r_-^2 and r_0^2 are defined, following [16], as the roots of Δ,

$$\Delta = \frac{(1 - r^2)(r^2 + a_1^2)(r^2 + a_2^2)}{r^2} - 2M = \frac{(r^2 - r_0^2)(r^2 - r_+^2)(r^2 - r_-^2)}{r^2}. \tag{2.20}$$

2.2 The radial and angular systems

The radial equation (2.18) can be brought to a standard form by making a Möbius transformation

$$z = \frac{r^2 - r_-^2}{r^2 - r_0^2}, \quad \text{with} \quad z_0 = \frac{r_+^2 - r_-^2}{r_+^2 - r_0^2}, \tag{2.21}$$

followed by introducing a new radial function regular at horizon and the boundary,

$$\Phi(z) = z^{-\alpha_-}(z - 1)^{\alpha_\infty}(z - z_0)^{-\alpha_+}R(z). \tag{2.22}$$

The exponents α_k are related to the monodromy parameters θ_k as

$$\alpha_k = \pm\frac{1}{2}\theta_k, \quad k = +, -, 0 \quad \text{and} \quad \alpha_\infty = \frac{1}{2}\left(1 \pm \sqrt{1 - C_m}\right), \tag{2.23}$$

which in turn are given in terms of the physical parameters by

$$\theta_k = \frac{i}{2\pi}\left(\frac{\omega - m_1\Omega_{k,1} - m_2\Omega_{k,2}}{T_k}\right), \qquad \theta_1 = -\sqrt{1 - C_m}. \tag{2.24}$$

We note that, just like the scalar case [15], and in the four-dimensional Teukolsky master equation [18], the monodromy parameters θ_+ and θ_-, respectively associated to the outer and inner horizons at $r = r_+$ and $r = r_-$ are proportional to the variation of the black

hole entropy as a quantum of energy ω and angular momenta m_1 and m_2 passes through the horizon.

With these definitions, the radial equation becomes

$$\frac{d^2R}{dz^2} + \left[\frac{1-\theta_-}{z} + \frac{1-\theta_1}{z-1} + \frac{1-\theta_+}{z-z_0} - \frac{1}{z-z_\star}\right]\frac{dR}{dz}$$
$$+ \left(\frac{\kappa_+\kappa_-}{z(z-1)} + \frac{z_0(z_0-1)K_0}{z(z-1)(z-z_0)} + \frac{z_\star(z_\star-1)K_\star}{z(z-1)(z-z_\star)}\right)R(z) = 0, \quad (2.25)$$

with the parameters as

$$z_\star = \frac{r_-^2+\mu^2}{r_0^2+\mu^2} \tag{2.26a}$$

$$\kappa_+\kappa_- = \frac{1}{4}((\theta_-+\theta_++\theta_1-1)^2 - \theta_0^2), \tag{2.26b}$$

$$4z_0K_0 = (\theta_-+\theta_++\theta_1-1)^2 - \theta_0^2 - 2\theta_-\theta_1 + 2\theta_1 - 2 - \frac{2(1-\theta_1)\theta_+}{(z_0-1)} + \frac{\omega^2}{(r_-^2-r_0^2)}$$
$$- \frac{a_1^2a_2^2\tilde{\Omega}^2}{\mu^2(r_-^2-r_0^2)} + \frac{2z_\star\theta_+}{(z_0-z_\star)} + \frac{2(z_\star-1)\mu^3\omega}{(r_+^2-r_0^2)(r_-^2-r_0^2)(z_0-z_\star)} + \frac{2z_0(1-\theta_1)}{(z_0-1)}$$
$$+ \frac{2(z_\star-1)a_1^2a_2^2\tilde{\Omega}}{\mu(r_+^2-r_0^2)(r_-^2-r_0^2)(z_0-z_\star)} - \frac{2(z_\star-1)\mu((a_1^2+a_2^2)\omega - a_1(1-a_1^2)m_1 - a_2(1-a_2^2)m_2)}{(r_+^2-r_0^2)(r_-^2-r_0^2)(z_0-z_\star)}$$
$$+ C_m + \frac{(z_0-z_\star)}{(z_0-1)(z_\star-1)}C_m, \tag{2.26c}$$

$$4z_\star K_\star = -\frac{2(z_\star-1)a_1^2a_2^2\tilde{\Omega}}{\mu(r_+^2-r_0^2)(r_-^2-r_0^2)(z_0-z_\star)} - \frac{2(z_\star-1)\mu^3\omega}{(r_+^2-r_0^2)(r_-^2-r_0^2)(z_0-z_\star)} + 2\theta_-$$
$$+ \frac{2(z_\star-1)\mu((a_1^2+a_2^2)\omega - a_1(1-a_1^2)m_1 - a_2(1-a_2^2)m_2)}{(r_+^2-r_0^2)(r_-^2-r_0^2)(z_0-z_\star)} - \frac{2z_\star\theta_+}{(z_0-z_\star)} - \frac{2z_\star(1-\theta_1)}{(z_\star-1)}. \tag{2.26d}$$

The differential equation (2.25) is Fuchsian, with 5 regular singular points at $z = 0, z_0, z_\star, 1, \infty$. It is sometimes called the deformed Heun equation, because, as we will see below, the singular point at $z_\star$ is *apparent*: the indicial coefficients are $\{0, 2\}$ and, due to an algebraic relation between the parameters, there are no logarithmic tails.[2] Then the monodromy property of the solution around this point is trivial. The position of this apparent singularity is related to the parameter μ by a Möbius transformation as it can be seen in (2.26a). Finally, we note that the deformed Heun equation (2.25) depends on μ only through $z_\star, K_0, K_\star$.

The angular equation (2.19) can be brought to the same form (2.25) by the Möbius transformation $u = (x^2 - a_1^2)/(x^2 - 1)$. The resulting equation is again Fuchsian with 5

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[2]The name of K. Heun is usually connected to the Fuchsian equation with 4 regular singular points. The generic differential equation with 5 regular singular points has no widespread name, although it was associated to F. Klein and M. Bôcher in the classic treatise of E. L. Ince [19].

regular singular points, located at

$$u = 0, \qquad u = 1, \qquad u = u_0 = \frac{a_2^2 - a_1^2}{a_2^2 - 1}, \qquad u = u_\star = \frac{\mu^2 - a_1^2}{\mu^2 - 1}, \qquad u = \infty, \quad (2.27)$$

and the characteristic exponents are

$$\beta_0^\pm = \pm\frac{m_1}{2}, \qquad \beta_1^\pm = \frac{1}{2}\left(1 \pm \sqrt{1 - C_m}\right), \qquad \beta_{u_0}^\pm = \pm\frac{m_2}{2}, \quad (2.28)$$

$$\beta_\star = \{0, 2\}, \qquad \beta_\infty^\pm = \frac{1}{2}\varsigma = \frac{1}{2}\left(\omega + a_1 m_1 + a_2 m_2\right). \quad (2.29)$$

We can check that, once more, the point at $u = u_\star$ is an apparent singularity due to an algebraic relation between the parameters.

Finally, the angular equation (2.19) can be brought to a canonical form by the transformation

$$S(u) = u^{m_1/2}(u-1)^{(1-\theta_1)/2}(u-u_0)^{m_2/2}Y(u), \quad (2.30)$$

which leads to the deformed Heun form (3.3),

$$\frac{d^2 Y}{du^2} + \left[\frac{1+m_1}{u} + \frac{1-\theta_1}{u-1} + \frac{1+m_2}{u-u_0} - \frac{1}{u-u_\star}\right]\frac{dY}{du}$$
$$+ \left(\frac{q_+ q_-}{u(u-1)} + \frac{u_0(u_0-1)Q_0}{u(u-1)(u-u_0)} + \frac{u_\star(u_\star-1)Q_\star}{u(u-1)(u-u_\star)}\right)Y(u) = 0, \quad (2.31)$$

with the accessory parameters given by

$$q_+ q_- = \frac{1}{4}((m_1+m_2+1-\theta_1)^2 - (\omega+a_1 m_1+a_2 m_2)^2), \quad (2.32a)$$

$$4u_0 Q_0 = (m_1+m_2+1-\theta_1)^2 - (\omega+a_1 m_1+a_2 m_2)^2 + 2m_1\theta_1 - 2(1-\theta_1) + \frac{2m_2(1-\theta_1)}{(u_0-1)}$$

$$+ \frac{2u_0(1-\theta_1)}{(u_0-1)} - \frac{a_1^2 a_2^2 \tilde{\Omega}^2}{\mu^2(1-a_1^2)} + \frac{\omega^2}{1-a_1^2} - \frac{2u_\star m_2}{(u_0-u_\star)} + \frac{2(u_\star-1)\mu^3\omega}{(1-a_1^2)(1-a_2^2)(u_0-u_\star)}$$

$$+ \frac{2(u_\star-1)a_1^2 a_2^2 \tilde{\Omega}}{\mu(1-a_1^2)(1-a_2^2)(u_0-u_\star)} - \frac{2(u_\star-1)\mu((a_1^2+a_2^2)\omega - a_1(1-a_1^2)m_1 - a_2(1-a_2^2)m_2)}{(1-a_1^2)(1-a_2^2)(u_0-u_\star)}$$

$$+ C_m + \frac{(u_0-u_\star)}{(u_0-1)(u_\star-1)}C_m, \quad (2.32b)$$

$$4u_\star Q_\star = -\frac{2u_\star(1-\theta_1)}{(u_\star-1)} - 2m_1 + \frac{2u_\star m_2}{(u_0-u_\star)} - \frac{2(u_\star-1)\mu^3\omega}{(1-a_1^2)(1-a_2^2)(u_0-u_\star)}$$

$$- \frac{2(u_\star-1)a_1^2 a_2^2 \tilde{\Omega}}{\mu(1-a_1^2)(1-a_2^2)(u_0-u_\star)} + \frac{2(u_\star-1)\mu((a_1^2+a_2^2)\omega - a_1(1-a_1^2)m_1 - a_2(1-a_2^2)m_2)}{(1-a_1^2)(1-a_2^2)(u_0-u_\star)}.$$

$$(2.32c)$$

We see again that the position of the apparent singularity at $u = u_\star$ is related to the parameter μ through a Möbius transformation (2.27) and that the single monodromy parameters m_1, m_2, θ_1 and $\varsigma = \omega + a_1 m_1 + a_2 m_2$ do not depend on μ. The initial proposal of μ-separability in [1] generated some discussion about the interpretation of the parameter [11, 20]. In order to add to that, we need to take a detour and write about isomonodromy.

3 Conditions on the Painlevé VI system

The most natural setting to describe isomonodromy is the theory of flat holomorphic connections. The exposition here follows the monograph by Iwasaki et al., [17], with some additions suited to our purposes. Consider the matricial system of differential equations on a single complex variable

$$\frac{d\Phi(z)}{dz} = \left(\frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_t}{z-t}\right)\Phi(z), \qquad A_\infty = -A_0 - A_1 - A_t = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}. \quad (3.1)$$

Choosing A_∞ diagonal comes at the expense of fixing a basis for the fundamental solution $\Phi(z)$. Let us parametrize it as

$$\Phi(z) = \begin{pmatrix} y_1(z) & y_2(w) \\ w_1(z) & w_2(w) \end{pmatrix}. \quad (3.2)$$

It is a straightforward exercise to see that the differential equation satisfied by the first row of $\Phi(z)$ is

$$y_i''(z) - \left(\operatorname{Tr} A(z) + \frac{A_{12}'(z)}{A_{12}(z)}\right)y_i'(z) + \left(\det A(z) - A_{11}'(z) + A_{11}(z)\frac{A_{12}'(z)}{A_{12}(z)}\right)y_i(z) = 0. \quad (3.3)$$

Furthermore, with A_∞ diagonal, we have

$$A_{12}(z) = \frac{(A_0)_{12}}{z} + \frac{(A_1)_{12}}{z-1} + \frac{(A_t)_{12}}{z-t} = \frac{k(z-\lambda)}{z(z-1)(z-t)}, \quad (3.4)$$

so, for the matricial system (3.1), the associated scalar equation (3.3) will be Fuchsian, with 5 singular points at $z = 0, 1, t, \lambda, \infty$, exactly the type encountered in the radial and angular systems, given by equations (2.25) and (2.31), respectively.

From this formulation it seems clear that, from the matricial system (3.1) perspective, the singularity at $z = \lambda$ is a consequence of our choice of gauge $A_\infty = \operatorname{diag}(\kappa_1, \kappa_2)$. As a matter of fact, we can see that there is a residual gauge symmetry that moves λ, as discovered by Jimbo, Miwa and Ueno [21]. Let us introduce the parametrization for the coefficient matrices A_i

$$A_i = \begin{pmatrix} p_i + \hat{\theta}_i & -q_i p_i \\ \frac{p_i + \hat{\theta}_i}{q_i} & -p_i \end{pmatrix}, \quad (3.5)$$

which is the most general for the gauge choice where $\operatorname{Tr} A_i = \hat{\theta}_i$ and $\det A_i = 0$, $i = 0, 1, t$ are fixed. The parameters p_i, q_i are subject to extra constraints. The diagonal terms of $A_\infty = -(A_0 + A_1 + A_t)$ are,

$$\kappa_1 = \frac{\hat{\theta}_\infty - \hat{\theta}_0 - \hat{\theta}_1 - \hat{\theta}_t}{2}, \qquad \kappa_2 = \frac{-\hat{\theta}_\infty - \hat{\theta}_0 - \hat{\theta}_1 - \hat{\theta}_t}{2}. \quad (3.6)$$

Let us now define, along with λ,

$$\eta = A_{11}(z = \lambda) = \frac{p_0 + \hat{\theta}_0}{\lambda} + \frac{p_1 + \hat{\theta}_1}{\lambda - 1} + \frac{p_t + \hat{\theta}_t}{\lambda - t}. \quad (3.7)$$

– 10 –

We will now solve for p_i and q_i in terms of λ and η. The solution will also depend on an extra parameter k, which can be made equal to one by conjugation of all the A_i by a diagonal matrix, the particular value of k will not enter into (3.3). The explicit solutions for p_i and q_i are given as [21]

$$q_0 = \frac{k\lambda}{tp_0}, \qquad q_1 = -\frac{k(\lambda - 1)}{(t-1)p_1}, \qquad q_t = \frac{k(\lambda - t)}{t(t-1)p_t}, \tag{3.8}$$

with k undefined and

$$p_0 = \frac{\lambda}{t\hat{\theta}_\infty}\left(\lambda(\lambda - 1)(\lambda - t)\tilde{\eta}^2 + (\hat{\theta}_1(\lambda - t) + t\hat{\theta}_t(\lambda - 1) - 2\kappa_2(\lambda - 1)(\lambda - t))\tilde{\eta}\right.$$
$$\left. + \kappa_2^2(\lambda - t - 1) - \kappa_2(\hat{\theta}_1 + t\hat{\theta}_t)\right), \tag{3.9a}$$

$$p_1 = -\frac{\lambda - 1}{(t-1)\hat{\theta}_\infty}\left(\lambda(\lambda - 1)(\lambda - t)\tilde{\eta}^2 + ((\hat{\theta}_1 + \hat{\theta}_\infty)(\lambda - t) + t\hat{\theta}_t(\lambda - 1)\right.$$
$$\left. - 2\kappa_2(\lambda - 1)(\lambda - t))\tilde{\eta} + \kappa_2^2(\lambda - t) - \kappa_2(\hat{\theta}_1 + t\hat{\theta}_t) - \kappa_1\kappa_2\right), \tag{3.9b}$$

$$p_t = \frac{\lambda - t}{t(t-1)\hat{\theta}_\infty}\left(\lambda(\lambda - 1)(\lambda - t)\tilde{\eta}^2 + (\hat{\theta}_1(\lambda - t) + t(\hat{\theta}_t + \hat{\theta}_\infty)(\lambda - 1)\right.$$
$$\left. - 2\kappa_2(\lambda - 1)(\lambda - t))\tilde{\eta} + \kappa_2^2(\lambda - 1) - \kappa_2(\hat{\theta}_1 + t\hat{\theta}_t) - t\kappa_1\kappa_2\right), \tag{3.9c}$$

$$\tilde{\eta} = \eta - \frac{\hat{\theta}_0}{\lambda} - \frac{\hat{\theta}_1}{\lambda - 1} - \frac{\hat{\theta}_t}{\lambda - t}. \tag{3.9d}$$

In terms of λ and η, the equation (3.3) is written as

$$y_i''(z) + p(z)y_i'(z) + q(z)y_i(z) = 0,$$
$$p(z) = \frac{1 - \hat{\theta}_0}{z} + \frac{1 - \hat{\theta}_1}{z - 1} + \frac{1 - \hat{\theta}_t}{z - t} - \frac{1}{z - \lambda},$$
$$q(z) = \frac{\kappa_1(\kappa_2 + 1)}{z(z - 1)} + \frac{t(t-1)K}{z(z-1)(z-t)} + \frac{\lambda(\lambda - 1)\eta}{z(z-1)(z-\lambda)}, \tag{3.10}$$

with λ and η as above and

$$K = -H - \frac{\lambda(\lambda - 1)}{t(t-1)}\eta - \frac{\lambda - t}{t(t-1)}\kappa_1 + \frac{\hat{\theta}_0\hat{\theta}_t}{2t} + \frac{\hat{\theta}_1\hat{\theta}_t}{2(t-1)}, \tag{3.11}$$

where H will be relevant to us in the following

$$H = \frac{1}{t}\text{Tr}(A_0 A_t) + \frac{1}{t-1}\text{Tr}(A_1 A_t) - \frac{\hat{\theta}_0\hat{\theta}_t}{2t} - \frac{\hat{\theta}_1\hat{\theta}_t}{2(t-1)}. \tag{3.12}$$

We now note that the singularity at $z = \lambda$ in (3.3) is apparent, and then K, λ and η satisfy an algebraic constraint. Translating this constraint to H,

$$H = \frac{\lambda(\lambda - 1)(\lambda - t)}{t(t-1)}\left(\eta^2 - \left(\frac{\hat{\theta}_0}{\lambda} + \frac{\hat{\theta}_1}{\lambda - 1} + \frac{\hat{\theta}_t}{\lambda - t}\right)\eta + \frac{\kappa_1\kappa_2}{\lambda(\lambda - 1)}\right) + \frac{\hat{\theta}_0\hat{\theta}_t}{2t} + \frac{\hat{\theta}_1\hat{\theta}_t}{2(t-1)}. \tag{3.13}$$

From the gauge field perspective, we can think of $A(z) = [\partial_z \Phi(z)]\Phi(z)^{-1}$ as a flat connection, whose observables are traces of non-contractible Wilson loops. In the language of complex analysis, holonomy is represented by monodromy matrices, M_0, M_t, M_1 and M_∞ associated to loops around each singular point of the systrem (3.1). These matrices are defined up to conjugation and constrained by the fact that the composition of the monodromies over all singular points is a contractible curve:

$$M_\infty M_1 M_t M_0 = \mathbb{1}. \tag{3.14}$$

The gauge-invariant observables are the traces of the matrices $\operatorname{Tr} M_i = 2\cos \pi \hat\theta_i$,[3] and the traces of two of the composite monodromies:

$$\operatorname{Tr} M_0 M_t = 2\cos \pi \sigma_{0t}, \qquad \operatorname{Tr} M_t M_1 = 2\cos \pi \sigma_{1t}. \tag{3.15}$$

The third combination $\operatorname{Tr} M_0 M_1$ is related to these two by a polynomial identity (Fricke-Jimbo relation), involving the $\hat\theta_i$, as can be seen in [22].

The gauge-invariant quantities $\hat\theta_0, \hat\theta_t, \hat\theta_1, \hat\theta_\infty, \sigma_{0t}, \sigma_{1t}$ are called *monodromy data*, associated to the system (3.1) — or, alternatively, to the equation (3.3). Of those, the single monodromy parameters $\hat\theta_i$ can be read directly from the differential equation, whereas σ_{0t} and σ_{1t} are not readily available. On the other hand, they comprise the information needed from the equation to solve the scattering problem [23], or to find quasinormal modes [16]. Therefore, finding them is a problem of interest.

Solving for σ_{0t}, σ_{1t} makes use of a residual gauge symmetry of (3.1), which changes the position of the singularity at $z = t$. The zero curvature condition $\partial_z \partial_t \Phi(z, t) = \partial_t \partial_z \Phi(z, t)$ forces the coefficient matrices A_i to satisfy the *Schlesinger equations*

$$\frac{\partial A_0}{\partial t} = -\frac{1}{t}[A_0, A_t], \qquad \frac{\partial A_1}{\partial t} = -\frac{1}{t-1}[A_1, A_t],$$
$$\frac{\partial A_t}{\partial t} = \frac{1}{t}[A_0, A_t] + \frac{1}{t-1}[A_1, A_t], \tag{3.16}$$

which, when written in terms of λ and η, result in the Painlevé VI transcendent.

In a seminal paper [22], Jimbo derived asymptotic expansions for the Painlevé VI transcendent in terms of monodromy data, written in a slightly different guise:

$$\frac{\partial}{\partial t} \log \tau(\hat{\vec\theta}, \vec\sigma; t) = H - \frac{1}{2t}\hat\theta_0 \hat\theta_t - \frac{1}{2(t-1)}\hat\theta_1 \hat\theta_t, \tag{3.17}$$

where τ is a function of the monodromy data and of t, called the isomonodromic time. In another big development, [24] gave the full expansion for τ, given generic monodromy arguments, in terms of Nekrasov functions, with the structure

$$\tau(\hat{\vec\theta}, \vec\sigma; t) = \sum_{n\in\mathbb{Z}} C(\hat{\vec\theta}, \sigma_{0t} + 2n)[s(\hat{\vec\theta}, \vec\sigma)]^n t^{\frac{1}{4}\sigma_{0t}^2 + n(\sigma_{0t}+n)} \mathcal{B}(\hat{\vec\theta}, \sigma_{0t} + 2n; t), \tag{3.18}$$

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[3]Here, we have performed a normalization of the solution $\Phi(z)$ so that its determinant is constant equal to one. One can check that this just subtracts from each coefficient matrix A_i its trace.

where the Nekrasov functions $\mathcal{B}$ are analytic in t. We refer to [15] for details. These functions were introduced as the instanton partition function of four-dimensional $\mathcal{N} = 2$ $SU(N)$ Yang-Mills [25] coupled to matter multiplets, and were related to two dimensional conformal blocks by the Alday-Gaiotto-Tachikawa conjecture [26], later proved by Alba, Fateev, Litvinov and Tarnopolsky [27]. The relation then comes full circle to help solve classical field propagation in five dimensional space-times.

With the full expansion given in [24, 28], connection formulas for the expansions at different singular points were given [29]. A Fredholm determinant formulation was constructed [30], which is well-suited to numerical calculations [31].

The monodromy problem, which is the original formulation of the classical Riemann-Hilbert problem, consists in finding the full set of monodromy data from the parameters in the equation (3.10). For our purposes, the latter will consist of the single monodromy data $\{\hat{\theta}_i\}$ and the parameters λ, η — remember that K is related to them by (3.11) and (3.13). These conditions are best written in terms of the ζ function defined in [32] (called $\sigma(\mathrm{t})$ there),

$$\zeta(t) = t(t-1)\frac{\partial}{\partial t}\log\tau(\hat{\vec{\theta}}, \vec{\sigma}; t) = (t-1)\operatorname{Tr} A_0 A_t + t\operatorname{Tr} A_1 A_t - \frac{t-1}{2}\hat{\theta}_0\hat{\theta}_t - \frac{t}{2}\hat{\theta}_1\hat{\theta}_t. \quad (3.19)$$

In terms of $\zeta(t)$, the Schlesinger equations read

$$\frac{d\zeta}{dt}(t) = -\operatorname{Tr}(A_t(A_t + A_\infty)) - \frac{1}{2}(\hat{\theta}_0 + \hat{\theta}_1)\hat{\theta}_t, \qquad \frac{d^2\zeta}{dt^2}(t) = -\frac{\operatorname{Tr}(A_\infty[A_0, A_t])}{t(t-1)}. \quad (3.20)$$

The strategy to recover the monodromy data from (3.10) is now clear: given the differential equation, which is parametrized by particular values for λ_0, η_0, and the monodromy time t_0, one can find the coefficient matrices A_i — up to overall conjugation — by solving for p_i's and q_i's using the formula above. Given (3.11) and (3.13), as well as the formulas for the entries of A_i above, we can then compute the derivatives of ζ at $t = t_0$

$$\zeta(t_0) = \lambda_0(\lambda_0 - 1)(\lambda_0 - t_0)\left(\eta_0^2 - \left(\frac{\hat{\theta}_0}{\lambda_0} + \frac{\hat{\theta}_1}{\lambda_0 - 1} + \frac{\hat{\theta}_t}{\lambda_0 - t_0}\right)\eta_0 + \frac{\kappa_1\kappa_2}{\lambda_0(\lambda_0 - 1)}\right)$$

$$+ \frac{t-1}{2}\hat{\theta}_0\hat{\theta}_t + \frac{t}{2}\hat{\theta}_1\hat{\theta}_t, \quad (3.21a)$$

$$\frac{d\zeta}{dt}(t_0) = -\frac{\lambda_0(\lambda_0 - 1)(\lambda_0 - t_0)^2}{t_0(t_0 - 1)}\left(\eta_0^2 - \left(\frac{\hat{\theta}_0}{\lambda_0} + \frac{\hat{\theta}_1}{\lambda_0 - 1} + \frac{\hat{\theta}_t - \hat{\theta}_\infty}{\lambda_0 - t_0}\right)\eta_0 + \frac{\kappa_1^2}{(\lambda_0 - t_0)^2}\right)$$

$$- \frac{\lambda_0 - 1}{t_0 - 1}\kappa_1\hat{\theta}_0 - \frac{\lambda_0}{t_0}\kappa_1\hat{\theta}_1 - \kappa_1\kappa_2 + \frac{1}{2}(\hat{\theta}_0 + \hat{\theta}_1)\hat{\theta}_t, \quad (3.21b)$$

and the second derivative can be written in terms of $\zeta(t)$ and $\zeta'(t)$, resulting in the "σ-form" of the Painlevé VI equation

$$\zeta'\left(t(t-1)\zeta''\right)^2 + \left[2\zeta'(t\zeta' - \zeta) - (\zeta')^2 - \frac{1}{16}(\hat{\theta}_t^2 - \hat{\theta}_\infty^2)(\hat{\theta}_0^2 - \hat{\theta}_1^2)\right]^2 \quad (3.22)$$

$$= \left(\zeta' + \frac{1}{4}\left(\hat{\theta}_t + \hat{\theta}_\infty\right)^2\right)\left(\zeta' + \frac{1}{4}\left(\hat{\theta}_t - \hat{\theta}_\infty\right)^2\right)\left(\zeta' + \frac{1}{4}\left(\hat{\theta}_0 + \hat{\theta}_1\right)^2\right)\left(\zeta' + \frac{1}{4}(\hat{\theta}_0 - \hat{\theta}_1)^2\right).$$

In principle, the initial conditions for $\zeta(t)$ at $t = t_0$ above determine $\zeta(t)$ uniquely through the differential equation. The function can then be inverted to recover σ_{0t} and σ_{1t}.

We note that the change of the parameters λ and η with respect to t, along the isomonodromy solution is a gauge transformation in the sense that the monodromy data is kept invariant, therefore

$$\delta\lambda = \frac{\partial K}{\partial \eta}\delta t, \qquad \delta\eta = -\frac{\partial K}{\partial \lambda}\delta t, \tag{3.23}$$

is a residual gauge transformation.

4 Formal solution to the radial and angular systems

4.1 Writing the boundary conditions in terms of monodromy data

In terms of the Painlevé VI τ-function, or rather the ζ function defined in (3.19), the parameters of ODE comprise an initial value problem for the ζ function. The case of angular and radial equations above is a little more involved since the parameters are coupled. Let us start with the following identification

	t_0	$\hat{\theta}_0$	$\hat{\theta}_t$	$\hat{\theta}_1$	$\hat{\theta}_\infty$	K	λ_0	η_0
$\zeta_{\mathrm{Rad}}(t)$	z_0	θ_-	θ_+	θ_1	θ_0	$K_0(\mu, C_m)$	$z_\star(\mu)$	$K_\star(\mu)$
$\zeta_{\mathrm{Ang}}(t)$	u_0	$-m_1$	$-m_2$	θ_1	ς	$Q_0(\mu, C_m)$	$u_\star(\mu)$	$Q_\star(\mu)$

where we highlighted the dependence of the parameters of the radial and angular systems on μ and the separation constant C_m. It is a straightforward exercise to show that the corresponding quantities K, λ_0, η_0 and t_0 for the radial and angular systems are not independent, satisfying (3.11) and (3.13). This fact shows that the singularity at λ_0 for both radial and angular systems is apparent, as anticipated in section 2.2.

For the angular system, we want to solve the eigenvalue problem. So, in principle, (3.21) gives a condition on the generic solution of the Painlevé equation (3.22), given by (3.18), in which we read the two monodromy parameters

$$\sigma_{0,u_0;\mathrm{Ang}}(\omega, \mu, C_m), \qquad \text{and} \qquad \sigma_{u_0,1;\mathrm{Ang}}(\omega, \mu, C_m), \tag{4.1}$$

where we omitted the dependence on m_1, m_2, a_1, a_2.

As discussed in [15, 16], the condition that the solutions of the angular differential equation (2.19) are well-behaved both at the North and South poles of the sphere x, ϕ, ψ can be written in terms of the monodromy parameters. Let us now review this construction.

Let $y_{1,2;i}(z)$ be (normalized) solutions of the deformed Heun equation (3.3) associated to a fundamental matrix $\Phi_i(z)$ whose monodromy matrix is diagonal at a chosen regular singular point $z = z_i$:

$$\Phi_i((z - z_i)e^{2\pi i} + z_i) = \Phi_i(z)e^{\pi i \hat{\theta}_i \sigma_3}. \tag{4.2}$$

Note that this implies that the solutions $y_{1,2;i}(z)$ have different behavior asymptoting z_i, with $y_{1;i}(z)/y_{2;i}(z) \propto (z - z_i)^{\theta_i}$ as $z \to z_i$. The analogous solution at a different singular point $z = z_j$, $\Phi_j(z)$, is connected to $\Phi_i(z)$ by a constant matrix E_{ij}

$$\Phi_j(z) = \Phi_i(z)E_{ij}, \tag{4.3}$$

called the *connection matrix* between z_i and z_j. It is straightforward to see that, if a given solution $y(z)$ of (3.3) has a definite behavior, in the sense that it asymptotes one of the solutions at $z = z_i$, say $y_{1,i}(z)$, and one of the solutions at $z = z_j$, say $y_{1,j}(z)$, then the connection matrix E_{ij} must be triangular. This in turn implies that the monodromy matrix of $\Phi_i(z)$ around $z = z_j$, generically of the form $M_j = E_{ij}e^{\pi i \hat{\theta}_j \sigma_3}E_{ij}^{-1}$, is also triangular, and then the composite monodromy parameter σ_{ij} will satisfy

$$2\cos\pi\sigma_{ij} = \text{Tr}\, M_i M_j = 2\cos\pi(\hat{\theta}_i + \hat{\theta}_j) \longrightarrow \sigma_{ij} = \hat{\theta}_i + \hat{\theta}_j + 2m, \quad m \in \mathbb{Z}. \tag{4.4}$$

It is also straightforward to show that the converse is also true: if the composite monodromy σ_{ij} satisfies (4.4), then the connection matrix is triangular.

Coming back to the angular system, the condition that the solutions are well-behaved at the North pole ($x = a_2^2$, or $u = 0$) and at the South pole ($x = a_1^2$, or $u = u_0$) means

$$\sigma_{0,u_0;\text{Ang}}(\omega, \mu, C_m) = -m_1 - m_2 - 2m, \quad m \in \mathbb{Z} \tag{4.5}$$

which defines the separation constant as an integer family of functions of μ and ω: $C_m(\mu, \omega)$. We will overlook issues of existence and uniqueness for the purposes of this exposition.

Now, for the radial system, again the solution of the isomonodromic equation (3.22) with the initial conditions (3.21) will define the corresponding two composite monodromy parameters, associated to paths encircling the singularities at $z = 0, z_0$ and $z = z_0, 1$, respectively

$$\sigma_{0,z_0;\text{Rad}}(\omega, \mu, C_m), \qquad \sigma_{z_0,1;\text{Rad}}(\omega, \mu, C_m), \tag{4.6}$$

where we omit the dependence on the other physical parameters M, a_1, a_2, m_1, m_2. It is customary to substitute in this condition the expression for the separation constant C_m obtained from the angular equation. We will postpone this step for now.

As an aside, note that the interest in the radial and angular systems, apart from finding the actual form of the radial wavefunctions — whose local expansions can be obtained from Frobenius method — usually consists of the scattering problem and the quasinormal modes problem. They can both be cast in terms of the monodromy parameters problem, with now the relevant composite monodromy parameter involving the outer horizon $r = r_+$, or $z = z_0$ and the conformal boundary at $r = \infty$, or $z = 1$. The transmission coefficient, for instance, is [33]

$$|\mathcal{T}|^2 = \left| \frac{\sin\pi\theta_+ \sin\pi\theta_0}{\sin\frac{\pi}{2}(\sigma_{z_0,1;\text{Rad}} - \theta_+ + \theta_1)\sin\frac{\pi}{2}(\sigma_{z_0,1;\text{Rad}} + \theta_+ - \theta_1)} \right|, \tag{4.7}$$

which poses an interpretation problem for μ: since in principle the electric and magnetic modes (2.10) exhaust the 3 polarizations of the photons in five dimensions — with 1 of them electric and 2 magnetic as argued in [1] — the fact that the scattering coefficient depends on the extra parameter μ seems spurious.

This redundancy also arises in the calculation of quasinormal modes from the radial system, whose method of solution mirrors that of the angular eigenvalue. The requirement that the radial wavefunction is "purely outgoing" at $r = r_+$ and "purely ingoing" at

$r = \infty$ can be cast, by the same arguments put forward above, in terms of the quantization condition for the composite monodromy parameter

$$\sigma_{z_0,1;\mathrm{Rad}}(\omega, \mu) = \theta_+(\omega) - \sqrt{1 - C_m(\omega, \mu)} + 2n, \qquad n \in \mathbb{Z}. \tag{4.8}$$

This condition defines implicitly the modes $\omega_{n,m}(\mu)$ as a function of the radial and azimuthal quantum numbers n and m. Again, it seems rather unphysical that there will be a 1-parameter family of vector quasinormal modes in the space-time.

The redundancy is solved by the intertwining of both radial and angular systems. Note that, in both angular and radial equations, (2.31) and (2.25), the function that parametrizes the position of the apparent singularity, represented by λ in the generic deformed Heun equation (3.10) is essentially, up to a global conformal transformation, the parameter μ:

$$\lambda_{\mathrm{Ang}} = u_\star = \frac{\mu^2 - a_1^2}{\mu^2 - 1}, \qquad \lambda_{\mathrm{Rad}} = z_\star = \frac{r_-^2 + \mu^2}{r_0^2 + \mu^2}. \tag{4.9}$$

Using the properties of the Painlevé VI equation, the function λ can be computed using the ζ function defined in (3.19) by resorting to the demonstration in [32],

$$\frac{1}{\lambda - t} = -\frac{1}{2}\left(\frac{1}{t} + \frac{1}{t-1}\right) \tag{4.10}$$
$$+ \frac{\hat{\theta}_\infty t(t-1)\zeta'' + (\zeta' + \frac{1}{4}(\hat{\theta}_t^2 - \hat{\theta}_\infty^2))((2t-1)\zeta' - 2\zeta + \frac{1}{4}(\hat{\theta}_0^2 - \hat{\theta}_1^2)) + \frac{1}{4}\hat{\theta}_\infty^2(\hat{\theta}_0^2 - \hat{\theta}_1^2)}{2t(t-1)(\zeta' + \frac{1}{4}(\hat{\theta}_t - \hat{\theta}_\infty)^2)(\zeta' + \frac{1}{4}(\hat{\theta}_t + \hat{\theta}_\infty)^2)},$$

using the Hamiltonian properties of the isomonodromic system. Equation (4.10), as well as analogues for the other Painlevé transcendents, can be found in [24].

With the help of (4.10), we can lift the ambiguity, using the procedure we can now describe. The angular and radial systems define four monodromy parameters as stated above. Of these, the quantization condition for the angular solutions and the quasinormal modes will set two,

$$\sigma_{0,u_0;\mathrm{Ang}}(\omega, \mu, C_m), \qquad \sigma_{z_0,1;\mathrm{Rad}}(\omega, \mu, C_m) \tag{4.11}$$

which can be used to implicitly define $\omega(\mu)$ and $C_m(\mu)$ as functions of the redundancy parameter μ. Now, with this substitution, one can write the two remaining monodromy parameters as functions of μ

$$\sigma_{u_0,1;\mathrm{Ang}}(\omega(\mu), \mu, C_m(\mu)), \qquad \sigma_{0,z_0;\mathrm{Rad}}(\omega(\mu), \mu, C_m(\mu)). \tag{4.12}$$

These four one-parameter families of monodromy parameters can now be fed into (3.18) to define two one-parameter families of ζ functions, one for the angular system and one for the radial system. Calling $\lambda_{\mathrm{Ang}}(\mu)$ and $\lambda_{\mathrm{Rad}}(\mu)$ the respective functions defined by the right-hand side of (4.10), we find an extra condition by requiring that the parameter μ defined by both systems is equal:

$$\mu^2 = \frac{\lambda_{\mathrm{Ang}}(\mu) - a_1^2}{\lambda_{\mathrm{Ang}}(\mu) - 1} = \frac{r_-^2 - \lambda_{\mathrm{Rad}}(\mu)r_0^2}{\lambda_{\mathrm{Rad}}(\mu) - 1}, \tag{4.13}$$

which can be seen as a consistency condition for both isomonodromic systems defined by the angular and radial equations. One can rephrase this property in different ways, and we will find below that using the expression for the second derivative of ζ given by (3.20) as a fifth condition, along with the values of the monodromy parameters above is a more computationally efficient approach. This condition, along with the corresponding quantization conditions for the angular (4.5) and radial system (4.8) are sufficient to determine all the separation constants, as well as the frequencies for the quasi-normal modes. This procedure is in stark contrast to the role of μ in the four-dimensional case studied previously [13], where it can be eliminated by a simple change of variables.

We should point out, however, that the implicit definitions for the quantities ω, C_m, μ presented above may not be single valued, which will then allow for orbits of μ with disconnected components. In particular, one can indeed have more than one solution to (4.13). In the next section, we will take the solution closest to u_0, due to the nature of the Nekrasov expansion. It is an open question whether a different choice will lead to different physics. We will leave the study of these subtleties for future work.

4.2 Quasinormal modes from the radial system

The conditions put forth in the last section provide an exact, procedural solution to the quasinormal modes for the vector perturbations. However, the analytical treatment of these conditions is of very limited scope at this moment, given the five transcendental equations one has to deal with. One may, however, resort to numerical implementations of the τ function.

In this section we are going to describe the numerical treatment of eigenfrequencies for ultraspinning $a_1 \to 1$ black holes. This amounts to solving numerically in this limit the equations listed previously for $\{\omega, \sigma_{0,z_0;Rad}, \sigma_{z_0,1;Rad}, \mu, C_m, \sigma_{u_0,1;Ang}, \sigma_{0,u_0;Ang}\}$. For $a_1 \to 1$, the angular system is better served by the expansion of the τ function around $u_0 = 1$, which was given in [24]. Recently, in [34] a similar analysis was performed for four dimensional Kerr-Newman and Kerr-Sen black holes. Nevertheless, our problem presents an extra difficulty related to the presence of μ in the radial and angular equations, which is not the case in four dimensions where the separation parameter μ can be absorbed by a redefinition of the separation constant C_m, as can be seen in [13, 20]. See also [35] for a similar discussion.

In figure 1, we display the results, with the rotation parameter $a_2 = 0.001$ and the size of the outer horizon $r_+ = 0.05$. The ultraspinning black hole regime considered is $0.99 \le a_1 \le 0.99999$, and the values of the quantum numbers are set at $\ell = 2, m_1 = m_2 = 0$. The frequencies found are stable and increase monotonically with a_1. The value of μ has a more complicated behavior with a_1, but it should be kept in mind that the change comes in the tenth decimal place and may be affected by numerical errors. Even with this caveat, the approximation of the angular τ function improves as $a_1 \to 1$ and the expansion of the radial τ function should be valid as long as $|\theta_+ z_0| \ll 1$, with typical values in the range $|\theta_+ z_0| \sim 10^{-2}$. In our analysis, we have used the Nekrasov expansion (3.18) truncating the number of channels $n \le N_c = 7$ and the number of levels in the conformal block $\mathcal{B}$ to $m \le N_b = 7$. For $z_0 \approx 10^{-2}$, we estimate at least 10 digits accuracy.
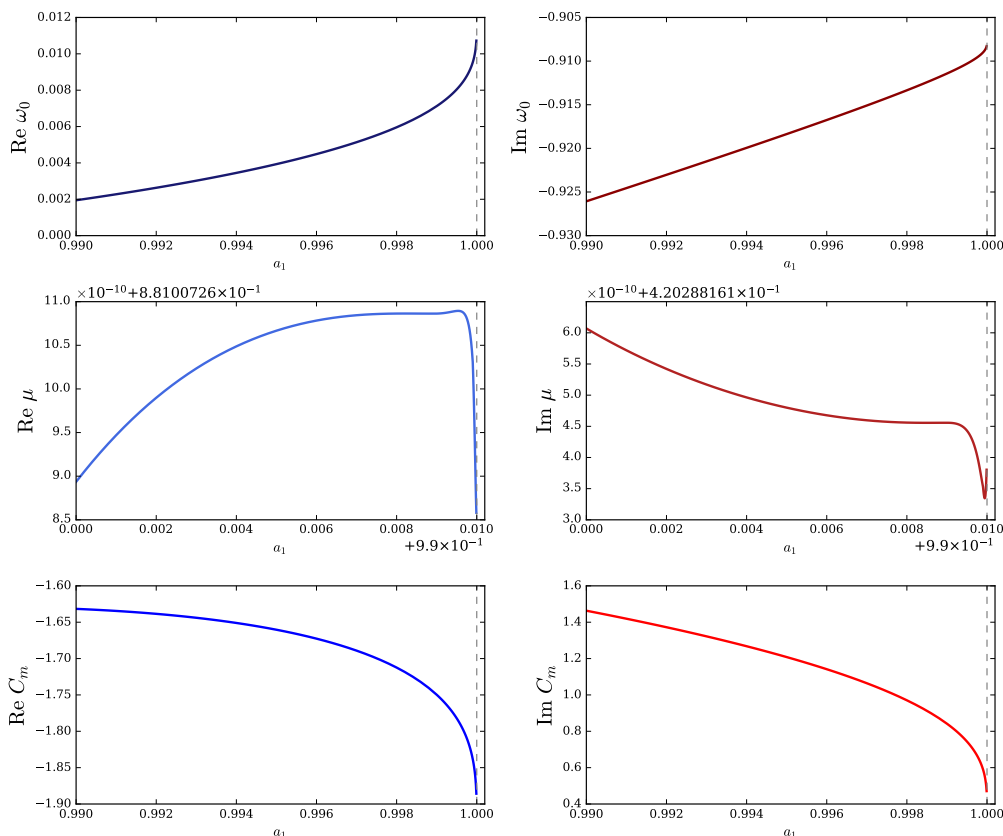
Figure 1. Quasinormal frequencies ω_0, separation constant C_m and the parameter μ for the electric polarization in the ultraspinning limit. Note that the change of the separation parameter μ is very small, in the range of variation of a_1 considered.

Note that the quasinormal frequency ω_0 increases until $a_1 = 0.999$, where it shows an asymptote, and possibly a non polynomial behavior in $1 - a_1$. A better understanding of the physics behind this analysis is indeed well-deserved, particularly the prospect of superradiant modes [16]. We will, however, leave these aspects to future work.

As a preliminary test of the results above, we have constructed the angular eigenfunctions $Y(u)$ using the values for $\{\omega, C_m, \mu\}$ obtained using isomonodromy for a few values of a_1 and plotted in figure 2. The construction is the standard Frobenius method, where expansions for $Y(u)$ at both points $u = 0$ and $u = u_0 \approx 1$ and matching at middle point are performed for the value of the function as well as 15 derivatives. The asymptotic behavior in figure 2 is as expected, and we could verify the values of the parameters obtained to at least 10 digits. Unfortunately, the construction for the radial eigenfunctions is much more computationally demanding and a detailed analysis is also postponed to future work.

5   Discussion

In this work we studied the role of the separation parameter μ, introduced in [1] to allow for the separation of Maxwell equations in a five-dimensional Kerr-(anti) de Sitter background.
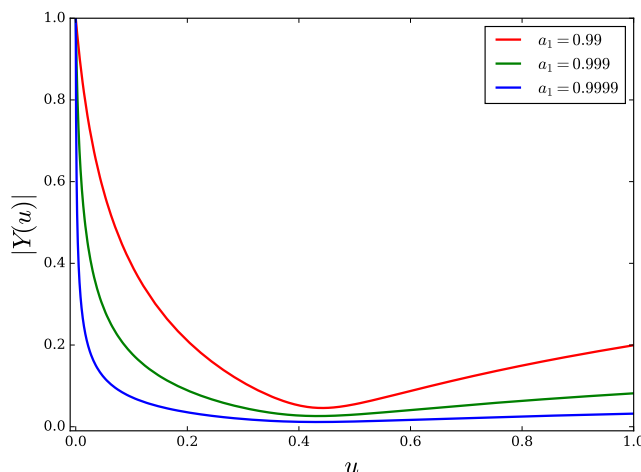
Figure 2. Numerical eigenfunctions of the angular equation (2.31) for different values of a_1. These were obtained by matching the Frobenius expansion at two of the singular points, $u = 0$ and $u = u_0$, with 16 terms. The values of μ were chosen by requiring consistency with the radial system.

We saw that μ is related to an apparent singularity of the resulting angular and radial differential equations. Specifically, the position of the apparent singularity is related to μ by a simple Möbius transformation. After translating the boundary conditions for the radial equation (4.8) and angular equation (4.5) in terms of monodromy data, we could outline a method to find the separation constant and the quasinormal modes frequencies. The separation parameter μ is fixed by a consistency condition between the τ functions for the radial and angular systems (4.13). We have then checked the procedure numerically by considering small $r_+ \ll 1$ and ultraspinning black holes $a_1 = 0.001, a_2 \lesssim 1$.

We should point out now that, unlike the scalar case studied in [16], the use of the τ function and the isomonodromy method for the vector case is not just a numerically more efficient way for computing the quasinormal modes. The monodromy language allows us to define the quantities involved in a way independent of μ and hence to decouple the conditions necessary to solve the problem. It is interesting to notice that the introduction of the apparent singularity mirrors the remark by Poincaré that apparent singularities are necessary in ordinary differential equations if we want to solve the problem of finding the parameters of the ordinary differential equation whose solutions are associated to generic monodromy parameters [17]. We hope that the results in this paper can help to elucidate the geometrical structure behind the introduction of μ. We also expect that the method presented here will help with the solution for the Proca and p-form fields in the same background, which were shown to lead to separable equations in [14].

One can deduce from the analysis that the separation parameter μ in higher dimensions plays a more prominent role than in the four dimensional case, where it can be eliminated by a suitable change of variables. The case studied here, that of "electric" polarizations, as defined in [1] is related to the "magnetic" polarizations by an inversion $\mu \to 1/\mu$. Although this inversion could be interpreted as a gauge transformation of the electric mode, we note

that it nevertheless modifies the asymptotic behavior of the field, so a more careful analysis is in order. At any rate, the results above found that μ should assume a discrete set of values, at least to allow solutions for the quasi-normal modes. In turn, the latter fact opens up the possibility of studying the different polarizations by exploring the symmetries of the Painlevé system. Another outstanding problem is the relation among the different definitions of polarizations in the literature [12]. We leave the exploration of these issues for future work.

The fact that the separation parameter μ parametrizes the position of the extra apparent singularity for both the angular and the radial equations seems an indication that perhaps the matrix system (3.1) plays a more prominent role than previously thought. That the particular choice of parameters, including μ, is able to decouple the equations mirrors the treatment of conformal blocks of conformal field theories in higher ($D > 2$) dimensions [36] where there exists a particular set of coordinates which factorizes the partial wave expansion as a product of two hypergeometric functions (at least in $D = 4$, see [37]).

Given the holographic aspect of perturbations of AdS$_5$ space, perhaps the last point is more than an analogy. One may hope that the results here can be of use for the study of conformal bootstrap and conformal perturbation theory in four dimensions. On a more immediate direction, a systematic study of the quasinormal modes for generic black hole parameters as well as the near extremal case, with the prospect of instabilities, is also necessary.

Acknowledgments

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