Right-handed neutrinos and $U(1)_X$ symmetry-breaking

C. Herbert Clemens^a and Stuart Raby^b

^a*Mathematics Department, Ohio State University,
Columbus, OH 43210, U.S.A.*

^b*Physics Department, Ohio State University,
Columbus, OH 43210, U.S.A.*

E-mail: clemens.43@osu.edu, raby.1@osu.edu

ABSTRACT: In [1] we proposed a model for Heterotic F -theory duality with Wilson line symmetry-breaking and a $4 + 1$ split of the F -theory spectral divisor. One goal of this note is to call attention to the existence of right-handed neutrinos in our F -theory model. As pointed out in section 4 of [2] such existence may be evidence for the $U(1)_X$ -symmetry that remains after the Higgsing of E_8 via

$$E_8 \Rightarrow SU(5)_{\text{gauge}} \oplus [SU(4) \oplus U(1)_X]_{\text{Higgs}}$$

occasioned by the $4 + 1$ split of the spectral divisor. In addition, as a result of the $\mathbb{Z}_2$ -action that supports the Wilson line we argue that the $U(1)_X$ -symmetry is, in fact, broken to $\mathbb{Z}_2$ -matter parity. Finally we identify co-dimension 3 singularities which determine Yukawa couplings for the MSSM matter fields.

KEYWORDS: F-Theory, Superstrings and Heterotic Strings, Supersymmetric Effective Theories

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Contents

1	The geometric model	2
1.1	The second section	2
1.2	The singular locus	2
1.3	Double-cover model	3
1.4	Image of spectral divisor	3
1.5	Resolution of singularities of $\bar{W}_4$	4
1.5.1	Resolution of codimension-one singularities of Δ	4
1.5.2	Resolution of codimension-two singularities of $\Delta^{(2)}$	5
1.5.3	Lifting $S_{\text{GUT}} \subseteq B_3$	5
1.6	Matter and Higgs curves in the spectral divisor $\tilde{\mathcal{C}}_{\text{Higgs}}^{(4)} \subseteq \tilde{W}_4$	6
2	Configuration of components of $\tilde{W}_4 \times_{B_3} S_{\text{GUT}}$	6
2.1	Root configuration over points $p \in S_{\text{GUT}}$	8
2.1.1	Over a general point $p \in \tilde{S}_{\text{GUT}}$	8
2.1.2	Over a general point $p \in \Sigma_{10}^{(4)} = S_{\text{GUT}} \cap \{a_5 = 0\}$	8
2.1.3	Over a general point $p \in S_{\text{GUT}}$ in the image $\Sigma_5^{(44)}$ of $C_5^{(44)}$	9
2.1.4	Over a general point $p \in \Sigma_5^{(41)} = S_{\text{GUT}} \cap \{a_{420} = 0\}$	9
3	$U(1)_X$-charges	11
3.1	$U(1)_X$ -charge on $\Sigma_{10}^{(4)} \subseteq D_4$	11
3.2	Right handed neutrinos	12
4	Yukawa couplings	16
5	$\mathbb{Z}_2$-quotient	16

In [1] we proposed a model for Heterotic F -theory duality with Wilson line symmetry-breaking and a $4+1$ split of the F -theory spectral divisor. One goal of this note is to call attention to the existence of right-handed neutrinos in our F -theory model. As pointed out in section 4 of [2] such existence may be evidence for the $U(1)_X$ -symmetry that remains after the Higgsing of E_8 via

$$E_8 \Rightarrow \text{SU}(5)_{\text{gauge}} \oplus [\text{SU}(4) \oplus U(1)_X]_{\text{Higgs}}$$

occasioned by the $4+1$ split of the spectral divisor. In addition, as a result of the $\mathbb{Z}_2$ -action that supports the Wilson line we argue that the $U(1)_X$ -symmetry is, in fact, broken to $\mathbb{Z}_2$ -matter parity. Finally we identify co-dimension 3 singularities which determine Yukawa couplings for the MSSM matter fields.

1 The geometric model

The Tate form for the Calabi-Yau fourfold W_4/B_3 in our set-up is given by the equation

$$\begin{vmatrix} x^3 + a_4 z w x^2 + a_2 z^3 w^2 x + a_0 z^5 w^3 & 1 \\ w y^2 - (a_5 w x + a_3 z^2 w^2) y & 1 \end{vmatrix} = 0 \quad (1.1)$$

over a Fano threefold

$$B_3 = \mathbb{P}_{[u_0, v_0]} \times D_2$$

where D_2 is a special del Pezzo surface with a $\mathbb{Z}_4$ -symmetry. The pair W_4/B_3 admits an equivariant $\mathbb{Z}_2$ -action with respect to which

$$a_j, \quad z, \quad \frac{y}{x} = t \in H^0(K_{B_3}^{-1})^{[-1]},$$

the skew eigenspace.

1.1 The second section

We impose the condition

$$a_{02345} := a_0 + a_2 + a_3 + a_4 + a_5 = 0 \quad (1.2)$$

from which one checks that (1.1) is satisfied on the locus

$$\begin{aligned} x &= z^2 w \\ y &= z^3 w \end{aligned}$$

so that, besides the tautological section

$$\zeta := \{[w, x, y] = [0, 0, 1]\}$$

of $\tilde{W}_4/B_3$ we have a second section

$$\tau := \{[w, x, y] = [w, z^2 w, z^3 w]\}.$$

These two sections generate a third section obtained for each $b_3 \in B_3$ from the third point of intersection

$$v := \{[w, x, y] = [w, z^2 w, -(z + a_{420}) z^2 w]\}$$

of the line between $\zeta(b_3)$ and $\tau(b_3)$ with the elliptic fiber of $\tilde{W}_4/B_3$ over b_3 . (Again $a_{420} := a_4 + a_2 + a_0$.)

1.2 The singular locus

By Bertini's theorem, the singular locus of the Tate form lies in

$$y^2 - x^3 = \begin{vmatrix} a_4 z w x^2 + a_2 z^3 w^2 x + a_0 z^5 w^3 & 1 \\ -(a_5 w x + a_3 z^2 w^2) y & 1 \end{vmatrix} = 0.$$

Parametrizing $\{y^2 - x^3 = 0\}$ by

$$x = t^2 w, \quad y = t^3 w$$

the pull-back of the Tate form to $\{y^2 - x^3 = 0\}$ is given by

$$a_5 t^5 + a_4 t^4 z + a_3 t^3 z^2 + a_2 t^2 z^3 + a_0 z^5 = \\ (a_5 t^4 + a_{54} t^3 z - a_{20} t^2 z^2 - a_0 z^3 (t + z))(t - z) = 0.$$

The subvariety

$$\mathcal{D}^{(4)} + \mathcal{D}^{(1)} \subseteq \{y^2 - x^3 = 0\}$$

is called the spectral variety. We make the assumption that

$$\{u_0 v_0 = z = 0\} \subseteq \{a_5 = z = 0\}$$

so that

$$\{u_0 v_0 = z = 0\} \subseteq \mathcal{D}^{(4)}.$$

1.3 Double-cover model

As in section 6.1 of [1], we project from the third section v mentioned above and write the birational model $\bar{W}_4/B_3$ of W_4/B_3 as a branched double cover of

$$Q := \mathbb{P}(\mathcal{O}_{B_3} \oplus \mathcal{O}_{B_3}(N))$$

where Q/B_3 has affine fiber coordinate ϑ_0 . The branch locus Δ is then given by the equation

$$0 = ((3z^2 + a_4 z) - \vartheta_0(\vartheta_0 - a_5))^2 \\ - 4((3z^4 + (2a_4 + a_2 - a_5)z^3 - a_5 a_{420} z^2) + (2z^3 + a_{420} z^2)\vartheta_0) \quad (1.3)$$

on the space

$$Q - \{X = 0\} = |\mathcal{O}_{B_3}(N)|,$$

the total space of the line bundle $\mathcal{O}_{B_3}(N)$.

1.4 Image of spectral divisor

As in section 7.4 of [1], to compute the image

$$\mathcal{C}_0^{(4)} + \mathcal{C}_0^{(1)} \subseteq Q$$

of $\mathcal{D}^{(4)} + \mathcal{D}^{(1)}$ we write

$$t = \frac{\vartheta_0 - (z + a_{420})}{2} \\ t - z = \frac{\vartheta_0 - (3z + a_{420})}{2} \\ t + z = \frac{\vartheta_0 + z - a_{420}}{2}$$

so that the equations of the images of the two components are

$$\begin{aligned}
 (\vartheta_0 - (z + a_{420}))^2((a_5(\vartheta_0 - (z + a_{420})) + 2a_4z)(\vartheta_0 + z - a_{420}) - 4a_{420}z^2) \\
 - 8a_0z^3(\vartheta_0 + z - a_{420}) = 0
 \end{aligned} \tag{1.4}$$

and

$$\vartheta_0 - (3z + a_{420}) = 0.$$

In [1] we show that $\mathcal{C}_0^{(4)} + \mathcal{C}_0^{(1)}$ lifts isomorphically to the image of $\mathcal{D}^{(4)} + \mathcal{D}^{(1)}$ in the branched double cover $\bar{W}_4/Q$. (1.4) implies that the map from the spectral divisor to Q is birational so that the inverse image of both $\mathcal{C}_0^{(4)}$ and $\mathcal{C}_0^{(1)}$ in $\bar{W}_4$ are reducible and only one of their two components correspond to the image

$$\bar{\mathcal{C}}_{\text{Higgs}} = \bar{\mathcal{C}}_{\text{Higgs}}^{(4)} + \bar{\mathcal{C}}_{\text{Higgs}}^{(1)} \subseteq \bar{W}_4 \tag{1.5}$$

of $\mathcal{D}^{(4)} + \mathcal{D}^{(1)}$. Here the divisors (ζ) and (τ) given by the two sections each project to the locus $\{\vartheta_0 = \infty\} \subseteq Q$ and $\mathcal{C}_0^{(4)}$ the lift $\bar{\mathcal{C}}_{\text{Higgs}}^{(4)}$ is the component intersecting the proper transform of the section (τ) .

Furthermore, the first two modifications $W_4^{(1)}$ and $W_4^{(2)}$ of $\bar{W}_4/Q$ are induced via fibered product from modifications $Q^{(1)}$ and $Q^{(2)}$ of Q so it will be convenient to describe these modifications in terms of their effect on the branch locus $\Delta \subseteq Q$, modifications that we denote as $\Delta^{(1)}$ and $\Delta^{(2)} = \tilde{\Delta}$.

Also, over the divisor $S_{\text{GUT}} := \{z = 0\} \subseteq B_3$, the spectral locus $(\mathcal{C}_0^{(4)} + \mathcal{C}_0^{(1)}) \times_{B_3} S_{\text{GUT}}$ has affine equations

$$\begin{aligned}
 a_5(\vartheta_0 - a_{420})^4 &= 0 \\
 \vartheta_0 - a_{420} &= 0
 \end{aligned} \tag{1.6}$$

in $Q \times_{B_3} S_{\text{GUT}}$ whereas the affine equation for $\Delta \times_{B_3} S_{\text{GUT}}$ in $Q \times_{B_3} S_{\text{GUT}}$ is

$$\vartheta_0^2(\vartheta_0 - a_5)^2 = 0.$$

Finally we lift S_{GUT} to

$$\bar{S}_{\text{GUT}} := \{z = \vartheta_0 - a_{420} = 0\} \subseteq \bar{\mathcal{C}}_{\text{Higgs}}^{(4)} \subseteq \bar{W}_4.$$

1.5 Resolution of singularities of $\bar{W}_4$

1.5.1 Resolution of codimension-one singularities of Δ

Partial resolution of singularities of Δ is achieved by blowing up smooth loci in Q . The proper transform $\Delta^{(1)}$ of Δ in the first modification $Q^{(1)}$ of Q is given by the equation

$$\begin{aligned}
 0 &= (a_5\vartheta_1 + a_4Z_{14})^2 + 4a_5a_{420}Z_{14}^2 \\
 &+ (-2a_5\vartheta_1^3 - 2a_4\vartheta_1^2Z_{14} + (6a_5 - 4a_{420})\vartheta_1Z_{14}^2 - (2a_4 + 4a_2 - 4a_5)Z_{14}^3)Z_0 \\
 &+ (\vartheta_1^4 - 6\vartheta_1^2Z_{14}^2 - 8Z_{14}^3\vartheta_1 - 3Z_{14}^4)Z_0^2.
 \end{aligned} \tag{1.7}$$

A second modification yields $\Delta^{(2)} \subseteq Q^{(2)} = \tilde{Q}$ given by the equation

$$\begin{aligned} 0 = & (\vartheta_1 \vartheta_2 + a_4 \tilde{Z}_{14})^2 \\ & + ((6a_5 + 4a_{420})\vartheta_2 - (4a_5 + 2a_4 + 4a_2)Z_0 \tilde{Z}_{14}) \tilde{Z}_{14}^2 Z_{23} \\ & - (6\vartheta_2^2 - 8\vartheta_2 Z_0 \tilde{Z}_{14} + 3\tilde{Z}_{14}^2 Z_0^2) \tilde{Z}_{14}^2 Z_{23}^2. \end{aligned} \quad (1.8)$$

However $\Delta^{(2)}$ still has a singular locus, namely a smooth one-dimensional locus that becomes a smooth curve $C_{\tilde{5}}^{(44)}$ of nodal singularities in the double cover $W_4^{(2)}$.

1.5.2 Resolution of codimension-two singularities of $\Delta^{(2)}$

It becomes important at this point that

$$S_{\text{GUT}} := \{z = 0\} \subseteq B_3$$

is a smooth $K3$ -surface on which the $\mathbb{Z}_2$ -action is free, that the imbedded image $\Sigma_{\tilde{5}}^{(44)} \subseteq B_3$ of the curve $C_{\tilde{5}}^{(44)} \subseteq \tilde{Q}$ lies in S_{GUT} , and that $C_{\tilde{5}}^{(44)}$ is disjoint from (the proper transform of) $C_0^{(4)} + C_0^{(1)}$. A choice of either of the two possible small, therefore crepant, resolutions of the nodal locus

$$C_{\tilde{5}}^{(44)} \subseteq W_4^{(2)}$$

yields the crepant resolution $\tilde{W}_4/B_3$ that is our F -theory model.

1.5.3 Lifting $S_{\text{GUT}} \subseteq B_3$

We choose the intersection of the proper transform of $\{\vartheta_0 - a_{420} = 0\}$ in $Q^{(1)}$ to support the lifting $S_{\text{GUT}}^{(1)}$ of $S_{\text{GUT}} \subseteq B_3$ into $Q^{(1)}$. In the notation of [1] the equation of $S_{\text{GUT}}^{(1)}$ can be written as

$$Z_{14} = \vartheta_1 Z_0 - a_{420} = 0.$$

A careful study of the first modification in [1] shows that the first lifting $S_{\text{GUT}}^{(1)}$ of S_{GUT} into $S_{\text{GUT}} \times_{B_3} W_4^{(1)}$ is an isomorphism with residual component $\{Z_0 = a_{420} = 0\}$. Since $S_{\text{GUT}}^{(1)}$ maps into $\{Z_{14} = 0\}$ and meets $\{Z_0 = 0\}$ only over $\{a_{420} = z = 0\}$. That is, it must lie in D_4 where it has equation

$$\vartheta_1 Z_0 - a_{420} = 0$$

and contains liftings of matter curves. Similarly, the proper lifting $\{\{a_5 - \vartheta_1 Z_0\} = Z_{14} = 0\}$ of $\{a_5 - \vartheta_0\}$ has residual component $\{Z_0 = a_5 = 0\}$ meeting $\{Z_0 = 0\}$ only over $\{a_5 = z = 0\}$. $S_{\text{GUT}}^{(1)}$ meets the proper transform $\{a_5 - \vartheta_1 Z_0 = Z_{14} = 0\}$ where

$$a_5 - \vartheta_1 Z_0 = \vartheta_1 Z_0 - a_{420} = Z_{14} = 0,$$

that is, along the entire locus

$$\{a_{420} = a_5 = Z_{14} = 0\}.$$

So we obtain the isomorphic lifting $S_{\text{GUT}}^{(2)} \subseteq W_4^{(2)}$. Finally, since $S_{\text{GUT}}^{(2)}$ maps into $\{\tilde{Z}_{14} = 0\}$ and so never meets the singular curve $C_{\tilde{5}}^{(44)}$, it lifts isomorphically to

$$\tilde{S}_{\text{GUT}} \subseteq D_4 \subseteq \tilde{W}_4.$$

Furthermore

$$a_5 = \vartheta_1 Z_0 + \vartheta_2 Z_{23}. \quad (1.9)$$

1.6 Matter and Higgs curves in the spectral divisor $\tilde{\mathcal{C}}_{\text{Higgs}}^{(4)} \subseteq \tilde{W}_4$

We have let

$$\mathcal{C}_{\text{Higgs}} = \mathcal{C}_{\text{Higgs}}^{(4)} + \mathcal{C}_{\text{Higgs}}^{(1)} \subseteq \tilde{Q} = Q^{(2)}$$

denote the proper (also the total) transform of (1.5) and

$$\tilde{\mathcal{C}}_{\text{Higgs}} \subseteq \tilde{W}_4$$

denote the proper (also the total) transform of $\tilde{\mathcal{C}}_{\text{Higgs}}$.

The matter curves are given by

$$\Sigma_{10}^{(4)} := \{a_5 = \tilde{Z}_{14} = \vartheta_1 \tilde{Z}_0 Z_{23} - a_{420} = 0\} \subseteq \tilde{\mathcal{C}}_{\text{Higgs}}^{(4)} \quad (1.10)$$

where $a_5 = 0$ implies that $Z_0 = \tilde{Z}_0 Z_{23}$ (see next section) and

$$\Sigma_{\tilde{5}}^{(41)} := \{Z_0 = a_{420} = \tilde{Z}_{14} = 0\} \subseteq \tilde{\mathcal{C}}_{\text{Higgs}}^{(4)}. \quad (1.11)$$

The Higgs curve is given by

$$\Sigma_{\tilde{5}}^{(44)} := \left\{ \begin{vmatrix} a_4 & a_5 \\ a_0 + a_3 & -a_3 \end{vmatrix} = \tilde{Z}_{14} = \vartheta_1 Z_0 - a_{420} = 0 \right\} \subseteq \tilde{\mathcal{C}}_{\text{Higgs}}^{(4)}. \quad (1.12)$$

All three curves lie in the surface $\tilde{S}_{\text{GUT}}$ given in D_4 . The identity

$$a_5 = \vartheta_1 Z_0 + \vartheta_2 Z_{23}$$

can be used to give alternative formulas for the matter and Higgs curves by substituting $a_5 - \vartheta_2 Z_{23}$ for $\vartheta_1 Z_0$ in (1.12) or for $\vartheta_1 \tilde{Z}_0 Z_{23}$ in (1.10).

2 Configuration of components of $\tilde{W}_4 \times_{B_3} S_{\text{GUT}}$

Since $W_4^{(2)}/Q^{(2)}$ is a branched double cover we will first describe the three components of $Q^{(2)} \times_{B_3} S_{\text{GUT}}$. These are $\{\tilde{Z}_{14} = 0\}$ that is the proper transform of $Q \times_{B_3} S_{\text{GUT}}$, $\{Z_0 = 0\}$ that is the proper transform of the exceptional locus of the blow-up of the locus $\{\vartheta_0 = z = 0\}$ in Q , and $\{Z_{23} = 0\}$, the exceptional locus of the blow-up of the locus $\{a_5 - \vartheta_1 Z_0 = Z_{14} = 0\}$ in $Q^{(1)}$. Over a general $p \in S_{\text{GUT}}$, the configuration is a chain

$$\{Z_0(p) = 0\} \cup \{\tilde{Z}_{14}(p) = 0\} \cup \{Z_{23}(p) = 0\} \quad (2.1)$$

where the branch locus (1.8) does not intersect $\{\tilde{Z}_{14} = 0\}$ since $\vartheta_1 \vartheta_2$ and $\tilde{Z}_{14}$ have no common zeros, and intersects $\{Z_{23} = 0\}$ when $\left(\vartheta_1 \frac{\vartheta_2}{\tilde{Z}_{14}} + a_4\right)^2 = 0$.

If $a_5 \neq 0$ the centers of the first and second modifications are disjoint, so

$$\{Z_0 = 0\} \cap \{Z_{23} = 0\} \cap \{a_5 \neq 0\} = \emptyset$$

and $\{\vartheta_1 = 0\}$ meets only $\{Z_0 = 0\}$ where it meets transversely as a smooth section disjoint from $\{\tilde{Z}_{14} = 0\} \cup \{Z_{23} = 0\}$. After only the first modification, the equation of the branch

locus restricted to $\{Z_0 = 0\}$ is $\left(a_5 \frac{\vartheta_1}{\tilde{Z}_{14}} + a_4\right)^2 + 4a_5a_{420} = 0$, so, again as long as $a_5 \neq 0$, $\{Z_0 = 0\}$ has a double cover D_0 simply branched at two distinct points that come together when additionally $a_{420} = 0$. Thus if $a_{420} = 0$

$$D_0 = D_{01} + D_{40}$$

indicating attachment of components to D_1 and D_4 respectively.

If $a_5(p) = 0$, the proper transform of $\{Z_0(p) \subseteq Q^{(1)}(p)\}$ does not equal its total transform that becomes $\{\tilde{Z}_0(p)Z_{23}(p) = 0\} \subseteq Q^{(2)}(p)$. Thus from (1.9) we have

$$\vartheta_1(p)\tilde{Z}_0(p)Z_{23}(p) + \vartheta_2(p)Z_{23}(p) = 0$$

giving the functional equation

$$\vartheta_2 = -\vartheta_1\tilde{Z}_0$$

on $Q^{(2)} \times_{B_3} \{a_5 = 0\}$. $\tilde{Z}_0$ divides ϑ_2 when $a_5 = 0$ therefore

$$\{\tilde{Z}_{14}(p) = 0\} \cap \{\tilde{Z}_0(p) = 0\} = \emptyset. \quad (2.2)$$

So the chain configuration (2.1) changes, namely it becomes

$$\{\tilde{Z}_{14}(p) = 0\} \cup \{Z_{23}(p) = 0\} \cup \{\tilde{Z}_0(p) = 0\}.$$

Also when $a_5 = 0$, (1.8) becomes

$$\begin{aligned} 0 &= (-\vartheta_1^2\tilde{Z}_0 + a_4\tilde{Z}_{14})^2 \\ &\quad - (4a_{420}\vartheta_1 + (2a_4 + 4a_2)Z_{23}\tilde{Z}_{14})\tilde{Z}_0\tilde{Z}_{14}^2Z_{23} \\ &\quad - (6\vartheta_1^2 + 8\vartheta_1\tilde{Z}_{14} + 3\tilde{Z}_{14}^2)\tilde{Z}_0^2\tilde{Z}_{14}^2Z_{23}^2. \end{aligned} \quad (2.3)$$

Therefore by (2.2) the branch locus $\Delta^{(2)} = \tilde{\Delta}$ does not intersect $\{\tilde{Z}_{14}(p) = 0\}$ and it intersects $\{\tilde{Z}_0(p) = 0\}$ only if additionally $a_4 = 0$, in which case its intersection is the entire fiber $\{\tilde{Z}_0^2(p) = 0\}$. $\Delta^{(2)} = \tilde{\Delta}$ intersects $\{Z_{23}(p) = 0\}$ when $(-\vartheta_1^2\tilde{Z}_0 + a_4\tilde{Z}_{14})^2 = 0$. So if in addition $a_4(p) = 0$ this intersection is given by $Z_{23}(p) = \tilde{Z}_0^2(p) = 0$. From section 7.2.3 of [1] the equation for $W_4^{(2)} \cap \{a_5 = 0\}$ becomes

$$\begin{vmatrix} w_2 - (a_4\tilde{Z}_{14} - \vartheta_1^2\tilde{Z}_0) & \tilde{Z}_{14}^2Z_{23} \\ \tilde{Z}_0 \cdot A' & w_2 + (a_4\tilde{Z}_{14} - \vartheta_1^2\tilde{Z}_0) \end{vmatrix} = 0.$$

We view near $\{w_2 = \tilde{Z}_0 = Z_{23} = 0\}$ as a family of affine quadric surfaces in the coordinates $(w_2, \tilde{Z}_0, Z_{23})$ parametrized by a_4 becoming the quadric cone

$$\begin{vmatrix} w_2 + \vartheta_1^2\tilde{Z}_0 & \tilde{Z}_{14}^2Z_{23} \\ \tilde{Z}_0 \cdot A' & w_2 - \vartheta_1^2\tilde{Z}_0 \end{vmatrix} = 0$$

over $a_4(p) = 0$. The fibers over $\{\tilde{Z}_0 = 0\}$ are pairs of disjoint affine lines lying in the same plane for all $a_4 \neq 0$ and so come from opposite rulings of the a_4 -fiber and coincide

as a double line lying in the $\{w_2 = 0\}$ -plane when $a_4 = 0$. On the other hand, the fibers over $\{Z_{23} = 0\}$ are pairs of distinct intersecting affine lines for all a_4 where the point of intersection converges to $\{w_2 = \tilde{Z}_0 = 0\}$ as a_4 goes to zero. Thus the exceptional curve of either small resolutions will separate the two proper liftings of $\{Z_{23} = 0\}$ entirely when $a_4(p) = 0$. Furthermore the scheme $\{w_2 = \tilde{Z}_0^2 = 0\}$ has intersection number one with the exceptional curve, the point being neither of the points of intersection of the exceptional curve with the two proper liftings of $\{Z_{23} = 0\}$.

2.1 Root configuration over points $p \in S_{\text{GUT}}$

2.1.1 Over a general point $p \in \tilde{S}_{\text{GUT}}$

Since (1.8) has no solution on $\{\tilde{Z}_{14} = 0\}$ therefore $\{\tilde{Z}_{14}(p) = 0\}$ cannot intersect the branch locus $\tilde{\Delta}$. So one concludes that

$$\tilde{W}_4 \times_{B_3} \{\tilde{Z}_{14}(p) = 0\}$$

splits into two disjoint components that we denote as $D_1(p)$ (meeting the section (ζ)) and $D_4(p)$ (meeting the section (τ)).

$\tilde{\Delta} \cap \{Z_{23}(p) = 0\}$ is given by the equation

$$0 = (\vartheta_1 \vartheta_2 + a_4 \tilde{Z}_{14})^2.$$

Thus

$$\tilde{W}_4 \times_{B_3} \{Z_{23} = 0\}$$

consists of two components that cross over the surface $\{\vartheta_1 \vartheta_2 + a_4 \tilde{Z}_{14} = Z_{23} = 0\} \subseteq \tilde{Q}$. We have designated these two components as $D_2(p)$ and $D_3(p)$. Furthermore

$$C_{\tilde{5}}^{(44)} \subseteq D_2 \cap D_3.$$

By (1.8) $\tilde{\Delta} \cap \{Z_0 = 0\}$ is given on $\{Z_0 = 0\}$ by the equation

$$0 = (\vartheta_1 \vartheta_2 + a_4 \tilde{Z}_{14})^2 + 4a_5 a_{420} \tilde{Z}_{14}^2.$$

Thus

$$D_0(p) := \tilde{W}_4 \times_{B_3} \{Z_0(p) = 0\}$$

is an irreducible rational curve meeting $D_1(p)$ and $D_4(p)$, each in a single point. By (1.9), $\{Z_0(p) = 0\}$ does not meet $\{Z_{23}(p) = 0\}$ unless $a_5(p) = 0$, so over a general $p \in \tilde{S}_{\text{GUT}}$, the components of the fiber of $\tilde{W}_4/B_3$ configure themselves as the extended Dynkin diagram of $\text{SU}(5)$.

2.1.2 Over a general point $p \in \Sigma_{10}^{(4)} = S_{\text{GUT}} \cap \{a_5 = 0\}$

On $Q^{(2)} \times_{B_3} \{a_5 = 0\}$.

$$\vartheta_2 = -\vartheta_1 \tilde{Z}_0$$

and the chain configuration becomes

$$\{\tilde{Z}_{14}(p) = 0\} \cup \{Z_{23}(p) = 0\} \cup \{\tilde{Z}_0(p) = 0\}.$$

$\Delta^{(2)} = \tilde{\Delta}$ does not intersect $\{\tilde{Z}_{14}(p) = a_5(p) = 0\}$ and does not intersect $\{\tilde{Z}_0(p) = a_5(p) = 0\}$ since, in general, $a_4 \neq 0$. $\Delta^{(2)} = \tilde{\Delta}$ intersects $\{Z_{23} = 0\}$ when $(\vartheta_1 \vartheta_2 + a_4 \tilde{Z}_{14})^2$, or equivalently $(-\vartheta_1^2 \tilde{Z}_0 + a_4 \tilde{Z}_{14})^2$. Thus $D_2(p)$ meets $D_3(p)$ in such a way that $D_1(p)$ only attaches to $D_2(p)$ and $\{\tilde{Z}_0(p) = 0\} \times_{B_3} \tilde{W}_4$ has two disjoint components, one attaching only to $D_2(p)$ and the other attaching only to $D_3(p)$. So over a general $p \in S_{\text{GUT}} \cap \{a_5 = 0\}$, the components of the fiber of $\tilde{W}_4/B_3$ configure themselves as the extended Dynkin diagram of $\text{SO}(10)$.

2.1.3 Over a general point $p \in S_{\text{GUT}}$ in the image $\Sigma_{\tilde{5}}^{(44)}$ of $C_{\tilde{5}}^{(44)}$

On this locus either of the two small resolutions of the singular locus $C_{\tilde{5}}^{(44)} \subseteq W_4^{(2)}$ inserts an exceptional $\mathbb{P}^1$, one via the specialization

$$D_2(p) \rightsquigarrow D'_2 + \bar{E}_{23}(p)$$

and the other via the specialization

$$D_3(p) \rightsquigarrow D'_3 + E_{23}(p).$$

Both configure the fiber of $\tilde{W}_4/B_3$ over p as the extended Dynkin diagram of $\text{SU}(6)$.

2.1.4 Over a general point $p \in \Sigma_{\tilde{5}}^{(41)} = S_{\text{GUT}} \cap \{a_{420} = 0\}$

Over

$$\{Z_0(p) = 0\} \cup \{\tilde{Z}_{14}(p) = 0\} \cup \{Z_{23}(p) = 0\}$$

the branch locus (1.8) does not intersect $\{\tilde{Z}_{14} = 0\}$, intersects $\{Z_{23} = 0\}$ when $(\vartheta_1 \frac{\vartheta_2}{\tilde{Z}_{14}} + a_4)^2 = 0$, and intersects $\{Z_0 = 0\}$ when $(a_5 \frac{\vartheta_1}{\tilde{Z}_{14}} + a_4)^2 = 0$. So again the fiber of $\tilde{W}_4/B_3$ over p configures as the extended Dynkin diagram of $\text{SU}(6)$.

Lemma 1. *Over each point $p \in \{z = 0, a_5 \neq 0\}$ that does not lie in $\Sigma_{\tilde{5}}^{(44)} \cup \Sigma_{\tilde{5}}^{(41)}$, the fibers*

$$D_0(p) \cup D_1(p) \cup D_2(p) \cup D_3(p) \cup D_4(p)$$

of $\tilde{W}_4 \times_{B_3} S_{\text{GUT}}$ arrange themselves in a cyclic intersection configuration corresponding to the extended Dynkin diagram of $\text{SU}(5)$. If $p \in \Sigma_{\tilde{5}}^{(44)}$, $D_2(p)$ and $D_3(p)$ become separated by a new component lying in the crepant resolution of the locus $C_{\tilde{5}}^{(44)}$ that we declare as the ‘new root’ yielding a configuration corresponding to the extended Dynkin diagram of $\text{SU}(6)$. This new root either emerges as $\bar{E}_{23}(p)$ from the specialization

$$D'_2(p) + \bar{E}_{23}(p)$$

of $D_2(p)$ or as $E_{23}(p)$ from the specialization

$$E(p) + D'_3(p)$$

of $D_3(p)$ depending on the choice of small resolution of $C_{\tilde{5}}^{(44)}$.

If $p \in \Sigma_{\mathbf{5}}^{(41)}$, $D_0(p)$ is split in two by a contracting circle ‘half-way’ between the intersection points $D_0 \cap (D_1 \cup D_4)$ again $C_{\mathbf{5}}^{(44)}$ yielding a cyclic configuration

$$D_{40}(p) \cup D_{01}(p) \cup D_1(p) \cup D_2(p) \cup D_3(p) \cup D_4(p)$$

corresponding to the extended Dynkin diagram of $SU(6)$.

If $p \in \Sigma_{\mathbf{5}}^{(41)} \cap \Sigma_{\mathbf{5}}^{(44)}$, both of the two above splittings occur simultaneously yielding a configuration corresponding to the extended Dynkin diagram of $SU(7)$.

The situation over

$$\Sigma_{\mathbf{10}}^{(4)} := \{z = a_5 = 0\} \subseteq B_3$$

is more complex. $\{Z_0(p) = 0\}$ contains the center of the second modification, that is,

$$Z_0(p) = \tilde{Z}_0(p)Z_{23}(p). \quad (2.4)$$

Furthermore the intersection of the branch locus $\tilde{\Delta} = \Delta^{(2)}$ with $\{Z_0(p) = 0\}$ deposits entirely on the component $\{Z_{23}(p) = 0\}$ as a point of multiplicity two not on either of the other components unless additionally $a_4 = 0$ where it becomes the entire fiber $\{\tilde{Z}_0^2(p) = 0\}$.

Lemma 2.

- i) Over points of $\{z = a_5 = 0\} \cap \{a_4 a_{420} \neq 0\} \subseteq B_3$ the fiber of $\tilde{Q}/B_3$ consists of a chain of three $\mathbb{P}^1$ ’s, with the property that the branch locus $\tilde{\Delta}$ meets only the middle component, namely $\{Z_{23} = 0\}$ which it meets in one point. Thus the fiber of $\tilde{W}_4/B_3$ is a tree of $\mathbb{P}^1$ ’s whose intersection configuration is that of the extended Dynkin diagram of $SO(10)$. The ‘new root’ is obtained by $D_0(p)$ becoming reducible as

$$D_0(p) = D_{02}(p) + D_{03}(p) \quad (2.5)$$

with $D_{03}(p)$ meeting only $D_2(p)$ and $D_{03}(p)$ meeting only $D_3(p)$.

- ii) If $z = a_5 = a_4 = 0$ then the curve $C_{\mathbf{5}}^{(44)} \subseteq \Delta^{(2)}$ passes over $\{z = a_5 = a_4 = 0\} \subseteq B_3$ contributing a last component to the fiber of $\tilde{W}_4/B_3$. This last component meets the component $\{Z_{23} = 0\}$ at the point where $\tilde{Z}_0 = 0$. Thus the fiber of $\tilde{W}_4/B_3$ contains a tree of $\mathbb{P}^1$ ’s configured as the Dynkin diagram of E_6 . In fact the scheme-theoretic fiber, where the ‘short arm’ occurs with multiplicity two is a flat specialization of the tree given by the extended Dynkin diagram of E_6 .
- iii) If $z = a_5 = a_{420} = 0$, then by (1.2) a_3 also vanishes. So the curve $C_{\mathbf{5}}^{(44)} \subseteq \Delta^{(2)}$ in subsection 7.2.3 of [1] passes over $\{z = a_5 = a_3 = 0\} \subseteq B_3$ contributing a last component to the fiber of $\tilde{W}_4/B_3$. This last component meets the component $\{Z_{23} = 0\}$ at the point where $\vartheta_1 \tilde{\vartheta}_2 \tilde{Z}_0 + a_4 \tilde{Z}_{14} = 0$. Thus the fiber of $\tilde{W}_4/B_3$ is a tree of $\mathbb{P}^1$ ’s whose intersection configuration is that of the extended Dynkin diagram of $SO(12)$.

3 $U(1)_X$ -charges

In our model in [1] the $U(1)_X$ -section is derived from the line bundle

$$\mathcal{O}_{W_4}((\zeta) - (\tau))$$

where (τ) is the divisor associated with the section τ and (ζ) is the divisor associated with the section ζ . Lifting $(\zeta) - (\tau)$ to the divisor $(\tilde{\zeta}) - (\tilde{\tau})$ in the desingularization $\tilde{W}_4$ we obtain its $U(1)_X$ -action by normalizing it so as to have zero intersection with all components of a general fiber over S_{GUT} . Since, for $p \in S_{\text{GUT}}$, the section (ζ) meets $D_1(p)$ and the section (τ) meets $D_4(p)$, we obtain the divisor

$$[U(1)_X] : 5((\tilde{\zeta}) - (\tilde{\tau})) + (3D_1 + D_2 - D_3 - 3D_4). \quad (3.1)$$

This divisor is orthogonal to all four of the $D_j(p)$ and therefore supports the $U(1)_X$ -action.

3.1 $U(1)_X$ -charge on $\Sigma_{10}^{(4)} \subseteq D_4$

Over $\Sigma_{10}^{(4)} \subseteq B_3$ the new root is given by the fact that $\{Z_0(p) = 0\}$ becomes reducible via the factorization $Z_0 = Z_{23}\tilde{Z}_0$. Furthermore the two branch-points of the irreducible branched double cover $D_0(p) = \{p\} \times_{B_3} \tilde{W}_4 - \sum_{j=1}^4 D_j(p)$ over $\{Z_0(p) = 0\}$ specializes to a single double-point lying on $\{Z_{23} = 0\}$. As we saw in lemma 2 above this splits $D_0(p)$ into two components $D_{02}(p) + D_{03}$. Either of these can be chosen as the ‘new root’. $D_{02}(p)$ has one intersection with D_2 and zero intersection with all other roots and with (ζ) and (τ) since the two sections intersect $D_1(p)$ and $D_4(p)$ respectively. $D_{03}(p)$ has one intersection with D_3 and zero intersection with all other roots and with (ζ) and (τ) .

On the other hand, over the matter curve $\Sigma_{\mathbf{5}}^{(41)}$, $D_0(p)$ again becomes reducible but this time $\{Z_0(p) = 0\}$ does not, rather the branch-points of $D_0(p)$ over $\{Z_0(p) = 0\}$ come together to give

$$D_0(p) = D_{01}(p) + D_{40}(p)$$

where $D_{01}(p)$ has one intersection with D_1 and zero intersection with all other roots and $D_{40}(p)$ has one intersection with D_4 and zero intersection with all other roots. $D_{01}(p)$ also misses (ζ) and (τ) even though $D_{01}(p)$ and (ζ) both meet $D_1(p)$. We designate $D_{01}(p)$ as the new root and so the matter curve $\Sigma_{\mathbf{5}}^{(41)}$ has $U(1)_X$ -charge +3.

Finally over the Higgs curve $\Sigma_{\mathbf{5}}^{(44)}$ the new root $E_{23}(p)$ arises from a splitting of $D_2(p)$ or $D_3(p)$ (depending on which of the two small resolutions of the nodal locus of $W_4^{(2)}$ we choose). If E_{23} is incorporated into $D_3 = D'_3 + E_{23}$ we have

$$E_{23} \cdot [U(1)_X] = E_{23} \cdot (D_2 - E_{23} - D'_3) = +1 - (-2) - 1 = +2$$

and $D'_3 \cdot [U(1)_X] = -2$, whereas if E_{23} is incorporated into $D_2 = D'_2 + \bar{E}_{23}$ we have

$$\bar{E}_{23} \cdot [U(1)_X] = \bar{E}_{23} \cdot (D'_2 + \bar{E}_{23} - D_3) = -1 + (-2) + 1 = -2$$

and $D'_2 \cdot [U(1)_X] = +2$. Here we can choose either small resolution since in either we have a curve with charge +2 and a curve -2, either of which can be designated as the ‘new root’

as needed. We summarize charges as follows:

New root/component	Curve/SU(5) _{gauge} -representation	[U(1) _X]-charge
D_{02} meets D_3	$\Sigma_{\mathbf{10}}^{(4)}$	+1
D_{03} meets D_2	$\Sigma_{\mathbf{10}}^{(4)}$	−1
D_{01} meets D_1	$\Sigma_{\mathbf{5}}^{(41)}$	+3
D_{40} meets D_4	$\Sigma_{\mathbf{5}}^{(41)}$	−3
$D'_3, \bar{E}_{23}$	$\Sigma_{\mathbf{5}}^{(44)}$	−2
D'_2, E_{23}	$\Sigma_{\mathbf{5}}^{(44)}$	+2

Thus there are two possible choices for the assignment of charges to the three curves that are compatible with the restriction of $E_8 \xrightarrow{Ad_{E_8}} \text{Aut}(e_8)$ to $(\text{SU}(5))_{\text{gauge}} \times (\text{SU}(4)_{\text{Higgs}} \times \text{U}(1)_X)$ and with the Yukawa couplings

$$(\mathbf{10}_M, \mathbf{10}_M, \mathbf{5}_H) \quad \text{and} \quad (\mathbf{10}_M, \bar{\mathbf{5}}_M, \mathbf{5}_H).$$

One decomposes as in (88) of [2] into

$$\begin{aligned}
 &(\mathbf{1}, \mathbf{15})_0 \oplus \\
 &(\mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{10}, \mathbf{1})_{-4} \oplus (\bar{\mathbf{10}}, \mathbf{1})_4 \oplus (\mathbf{24}, \mathbf{1})_0 \\
 &\quad \oplus (\mathbf{1}, \mathbf{4})_5 \oplus (\bar{\mathbf{5}}, \mathbf{4})_{-3} \oplus (\mathbf{10}, \mathbf{4})_1 \\
 &\quad \oplus (\mathbf{1}, \bar{\mathbf{4}})_{-5} \oplus (\mathbf{5}, \bar{\mathbf{4}})_3 \oplus (\bar{\mathbf{10}}, \mathbf{4})_{-1} \\
 &\quad \oplus (\mathbf{5}, \mathbf{6})_{-2} \oplus (\bar{\mathbf{5}}, \mathbf{6})_2
 \end{aligned} \tag{3.2}$$

where $u \in \text{U}(1)_X$ acts on the representation space $()_c$ as the character representation

$$u(v) \mapsto u^c \cdot v$$

and the other decomposes into

$$\begin{aligned}
 &(\mathbf{1}, \mathbf{15})_0 \oplus \\
 &(\mathbf{1}, \mathbf{1})_0 \oplus (\mathbf{10}, \mathbf{1})_4 \oplus (\bar{\mathbf{10}}, \mathbf{1})_{-4} \oplus (\mathbf{24}, \mathbf{1})_0 \\
 &\quad \oplus (\mathbf{1}, \mathbf{4})_{-5} \oplus (\bar{\mathbf{5}}, \mathbf{4})_3 \oplus (\mathbf{10}, \mathbf{4})_{-1} \\
 &\quad \oplus (\mathbf{1}, \bar{\mathbf{4}})_5 \oplus (\mathbf{5}, \bar{\mathbf{4}})_{-3} \oplus (\bar{\mathbf{10}}, \mathbf{4})_1 \\
 &\quad \oplus (\mathbf{5}, \mathbf{6})_2 \oplus (\bar{\mathbf{5}}, \mathbf{6})_{-2}.
 \end{aligned} \tag{3.3}$$

3.2 Right handed neutrinos

Now the matter curve $\Sigma_{\mathbf{5}}^{(41)}$ is given by

$$a_{420} = (t + z) = 0$$

lying inside the surface of intersection of $\mathcal{D}^{(4)}$ over $\{a_{420} = 0\} \subseteq B_3$. In fact the restriction of $\mathcal{D}^{(4)}$ factors over $\{a_{420} = 0\}$ and we have the equation

$$\begin{aligned} \frac{a_5 t^4 + a_{54} t^3 z - a_{20} t^2 z^2 - a_0 t^3 (t + z)}{t + \tilde{z}} &= \\ \frac{a_5 t^4 + a_{54} t^3 z + a_4 t^2 z^2 - a_0 t^3 (t + z)}{t + z} &= \\ a_5 t^3 + a_4 t^2 z - a_0 z^3 &= 0 \end{aligned} \quad (3.4)$$

for this residual surface. From section 7.4 in [1] we have the equation

$$t = \frac{y}{x} = \frac{(\vartheta_0 - a_{420}) - z}{2}$$

so that over $\{a_{420} = 0\}$ we have the correspondence

$$\begin{aligned} a_5 t^3 + a_4 t^2 z - a_0 z^3 &= t^2 (a_5 t + a_4 z) - a_0 z^3 = \\ \left(\frac{\vartheta_0 - z}{2} \right)^2 \left(a_5 \left(\frac{\vartheta_0 - z}{2} \right) + a_4 z \right) &- a_0 z^3. \end{aligned}$$

The proper transform of the surface in $Q^{(1)} \times_{B_3} \{a_{420} = 0\}$ defined by this last polynomial has equation

$$Z_0 = a_5 \left(\frac{\vartheta_1 - Z_{14}}{2} \right)^3 + a_4 Z_{14} \left(\frac{\vartheta_1 - Z_{14}}{2} \right)^2 - a_0 Z_{14}^3 = 0 \quad (3.5)$$

that does not meet $\{Z_{14} = 0\}$ unless a_5 also vanishes since ϑ_1 and Z_{14} have no common zeros. Now for $p \in \{z = a_{420} = 0\}$, $D_0(p) = D_{01}(p) + D_{40}(p)$ neither component of which connect with $D_2(p) + D_3(p)$ unless $a_5(p)$ also vanishes. By (1.7), $D_{01}(p)$ crosses $D_{40}(p)$ when $\frac{\vartheta_1}{Z_{14}} = -\frac{a_4}{a_5}$ whereas the locus (3.5) is given by the locus

$$Z_0 = \frac{a_5}{8} \left(\frac{\vartheta_1}{Z_{14}} - 1 \right)^3 + \frac{a_4}{4} \left(\frac{\vartheta_1}{Z_{14}} - 1 \right)^2 - a_0 = 0.$$

Since the cubic (3.5) lies in the spectral divisor, it lifts into D_{40} and meets no other component of the fiber as long as $a_5 \neq 0$. We denote the closure of this locus in D_{40} as

$$\Gamma_0 \subseteq \{z = a_{420} = 0\} \times_{B_3} D_{04} \subseteq W_4^{(1)}.$$

If additionally $a_5(p) = 0$ then, by allowably general choices of the forms a_j , $a_4(p) \neq 0$ and solutions to (3.5) specialize to

$$\begin{aligned} Z_0(p) &= Z_{14}(p) = 0 \\ Z_0(p) &= 0, \quad \left(\frac{\vartheta_1}{Z_{14}}(p) - 1 \right)^2 = \frac{4a_0(p)}{a_4(p)}. \end{aligned}$$

Thus Γ_0 meets $\{Z_{14} = 0\}$ simply over $a_5(p) = 0$.

Now the transform of (3.5) in $Q^{(2)} \times_{B_3} \{Z_0 = a_{420} = 0\}$ becomes

$$\left(\frac{-\tilde{Z}_{14}Z_{23}}{2}\right)^2 \left(a_5 \left(\frac{-\tilde{Z}_{14}Z_{23}}{2}\right) + a_4 \tilde{Z}_{14}Z_{23}\right) - a_0 \tilde{Z}_{14}^3 Z_{23}^3 = 0$$

or alternatively

$$\left(\frac{a_5 - (\tilde{Z}_{14} + \vartheta_2) Z_{23}}{2}\right)^2 \left(a_5 \left(\frac{a_5 - (\tilde{Z}_{14} + \vartheta_2) Z_{23}}{2}\right) + a_4 \tilde{Z}_{14}Z_{23}\right) - a_0 \tilde{Z}_{14}^3 Z_{23}^3 = 0.$$

Also on $\{Z_0 = 0\}$ we can use the second form of the equation to obtain a curve

$$\left\{ Z_0 = a_{420} = \left(\frac{a_5 - (\tilde{Z}_{14} + \vartheta_2) Z_{23}}{2}\right)^2 \left(a_5 \left(\frac{a_5 - (\tilde{Z}_{14} + \vartheta_2) Z_{23}}{2}\right) + a_4 \tilde{Z}_{14}Z_{23}\right) - a_0 \tilde{Z}_{14}^3 Z_{23}^3 = 0 \right\}$$

whose lifting into $D_{04} \subseteq W_4^{(2)}$ we designate as $\tilde{\Gamma}_0$. Now $\tilde{\Gamma}_0$ is disjoint from the support of $[U(1)_X]$ except over the point $\{z(p) = a_5(p) = a_{420}(p) = 0\}$ where it meets the remaining components over $\{\tilde{Z}_{14}Z_{23} = 0\}$. Depending on the choice of small resolution over the Higgs curve, define the right handed neutrino curve as

$$\tilde{\Gamma} \equiv \tilde{\Gamma}_0 + D_4(p) + D_3(p) + E_{23}(p)$$

or

$$\tilde{\Gamma} \equiv \tilde{\Gamma}_0 + D_4(p) + D'_3(p)$$

where $p \in \{z = a_{420} = a_5 = 0\} \subseteq S_{\text{GUT}}$ and $\tilde{\Gamma}_0$ meets D_4 simply above the point $\{a_5 = a_{420} = \tilde{Z}_0 = \tilde{Z}_{14} = 0\} \subseteq Q^{(2)}$.

We have the following table of intersection numbers with (3.1):

$\tilde{\Gamma}$ -component	Intersection number with $[U(1)_X]$
$\tilde{\Gamma}_0$	-3
$D_4(p)$	$-5 + (-2)(-3) + (-1) = 0$
$D_3(p)$	$-3 + 2 + 1 = 0$
$\bar{E}_{23}(p), D'_3(p)$	-2

Recall now that the choice in the last entry in the table depends on the choice of small resolution of $C_{\tilde{5}}^{(44)}$ over $p \in \{z = a_5 = a_{420} = 0\}$. For either of the choices

$$\tilde{\Gamma} \cdot [U(1)_X] = -5$$

and, following section 4 of [2] the curve $\tilde{\Gamma}$ is the candidate for the support of right-handed neutrinos. However the geometric construction forces a Yukawa coupling

$$(\mathbf{1}_{\tilde{\Gamma}}, \bar{\mathbf{5}}_M, \mathbf{5}_H)$$

since the curves $\Sigma_{\mathbf{5}}^{(41)}$ and $\Sigma_{\mathbf{5}}^{(44)}$ lie in D_4 . Thus we are forced to assign charge +3 to $\Sigma_{\mathbf{5}}^{(41)}$ as in (3.3).

The fact that $\tilde{\Gamma}$ is reducible will force a coupling

$$(\mathbf{1}_{\tilde{\Gamma}}, \mathbf{1}_{\tilde{\Gamma}}, \mathbf{1}_{\tilde{\Lambda}})$$

involving a curve $\tilde{\Lambda}$ with $U(1)_X$ -charge +10. To construct this curve, consider the curve $\tilde{\Lambda}_0 = D_{01}(p)$ for $p \in S_{\text{GUT}}$ such that $a_{420}(p) = a_5(p) = 0$. Then

$$[U(1)_X] \cdot \tilde{\Lambda}_0 = (-5)(\tilde{\zeta}) \cdot \tilde{\Lambda}_0 + 3D_1 \cdot \tilde{\Lambda}_0 = 8$$

and so

$$[U(1)_X] \cdot (\tilde{\Lambda}_0 + D_1(p) + D'_2(p)) = 10$$

and

$$[U(1)_X] \cdot (\tilde{\Lambda}_0 + D_1(p) + D_2(p) + E_{23}(p)) = 10.$$

Now the choice between

$$\tilde{\Lambda} = \tilde{\Lambda}_0 + D_1(p) + D'_2(p)$$

or

$$\tilde{\Lambda} = \tilde{\Lambda}_0 + D_1(p) + D_2(p) + E_{23}(p)$$

must be coordinated with the choice between

$$\tilde{\Gamma} = \tilde{\Gamma}_0 + D_4(p) + D_3(p) + \bar{E}_{23}(p)$$

or

$$\tilde{\Gamma} = \tilde{\Gamma}_0 + D_4(p) + D'_3(p)$$

so that $\tilde{\Lambda}$ and $\tilde{\Gamma}$ pass through a common point.

Charges associated to the $U(1)_X$ -action are then given by the following table:

New root/component	Curve/ $SU(5)_{\text{gauge}}$ -representation	$[U(1)_X]$ -charge
D_{03} meets D_3	$\Sigma_{\mathbf{10}}^{(4)}$	−1
D_{01} meets D_1	$\Sigma_{\mathbf{5}}^{(41)}$	+3
$D'_3, \bar{E}_{23}$	$\Sigma_{\mathbf{5}}^{(44)}$	−2
D'_2, E_{23}	$\Sigma_{\mathbf{5}}^{(44)}$	+2
Right-handed neutrinos	$\tilde{\Gamma}$	−5
	$\tilde{\Lambda}$	+10

4 Yukawa couplings

(1.2) then implies that at those points we also have $a_3 = 0$ since our second section required $-a_{420} = a_{53}$. So we conclude that over the twelve points $\{a_5 = a_{420} = z = 0\} \subseteq B_3$ both matter curves, the Higgs curve and the curve that supports the right-handed neutrinos all intersect.

We next recall from section 6.2.1 of [3] that the curve $\{a_5 = z = 0\} \subseteq B_3$ contains two disjoint rational curves given by $\{u_0 v_0 = 0\}$ that are interchanged under the $\mathbb{Z}_2$ -action and each meets the residual curve in two points $\{a_5 = a_4 = z = 0\}$. It is over these crossing points of components that we find the ‘top’ $(\mathbf{10}_M, \mathbf{10}_M, \mathbf{5}_H)$ -couplings associated with the $U(1)_X$ -action. Furthermore, since a_3 also vanishes when $a_5 = a_{420} = 0$ allowing ‘bottom’ $(\mathbf{10}_M, \bar{\mathbf{5}}_M, \bar{\mathbf{5}}_H)$ -couplings associated with the $U(1)_X$ action over $\{a_5 = a_0 = 0\}$. Next points where the neutrino curve meets the $\bar{\mathbf{5}}_M$ -curve and the $\mathbf{5}_H$ -curve give $(\mathbf{1}_{\tilde{\Gamma}}, \bar{\mathbf{5}}_M, \mathbf{5}_H)$ -couplings. Finally the crossing point of the components $\tilde{\Gamma}_0$ and $D_1(p)$ of the neutrino curve $\tilde{\Gamma}$ over $\{z = a_5 = a_{420} = 0\}$ allows a $(\mathbf{1}_{\tilde{\Gamma}}, \mathbf{1}_{\tilde{\Gamma}}, \mathbf{1}_{\tilde{\Lambda}})$ -coupling. We have the following table:

Coupling	Charge
$(\mathbf{10}_M, \mathbf{10}_M, \mathbf{5}_H)$	$(-1) + (-1) + (+2) = 0$
$(\mathbf{10}_M, \bar{\mathbf{5}}_M, \bar{\mathbf{5}}_H)$	$(-1) + (+3) + (-2) = 0$
$(\mathbf{1}_{\tilde{\Gamma}}, \bar{\mathbf{5}}_M, \mathbf{5}_H)$	$(-5) + (+3) + (+2) = 0$
$(\mathbf{1}_{\tilde{\Gamma}}, \mathbf{1}_{\tilde{\Gamma}}, \mathbf{1}_{\tilde{\Lambda}})$	$(-5) + (-5) + (+10) = 0$

5 $\mathbb{Z}_2$ -quotient

The $U(1)_X$ -charge c acts on the representations (3.2) as the character $u \mapsto u^c$. Under the $\mathbb{Z}_2$ -action, the section τ is interchanged with the inherited section ζ , so the line bundle $\mathcal{O}_{W_4}((\tau) - (\zeta))$ is carried to its inverse. Roots are sent to their negatives by the reversal of the choice of positive Weyl chamber but that reversal is undone by the first factor of the composite $\mathbb{Z}_2$ -action that sends y to $-y$ but does *not* interchange τ with ζ . It is the second factor of the $\mathbb{Z}_2$ -action, namely the fiberwise translation by $\text{image}(\tau) - \text{image}(\zeta)$ that carries the line bundle $\mathcal{O}_{W_4}((\tau) - (\zeta))$ to its inverse. (See sections 9 and 10.1 of [1].) Therefore

$$[U(1)_X] \Rightarrow -[U(1)_X].$$

Since the composite $\mathbb{Z}_2$ -action on the $U(1)_X$ lying in the maximal torus of the complex algebraic group $E_8^{\mathbb{C}}$ is by complex conjugation, the only possible non-zero $U(1)_X$ -charges c that are $\mathbb{Z}_2$ -invariant are those such that

$$u^c = u^{-c}.$$

Therefore requiring that

$$u^c = u^{-c} \tag{5.1}$$

implies

$$u^{2c} = 1,$$

that is

$$u^c = e^{m\pi i} = \pm 1.$$

Said otherwise, the requirement (5.1) breaks $U(1)_X$ -symmetry to $\mathbb{Z}_2$ -symmetry. That is, as mentioned in (5.1) above, descent to the $\mathbb{Z}_2$ -quotient breaks $U(1)_X$ -symmetry to $\mathbb{Z}_2$ -symmetry. Thus the right-handed neutrinos can obtain a Majorana mass at the compactification scale.

Furthermore, the $\mathbb{Z}_2$ -action moves the spectral divisor

$$\tilde{\mathcal{C}}_{\text{Higgs}}^{(4)} \subseteq \tilde{W}_4$$

to the opposite component of

$$\tilde{\mathcal{C}}_{\text{Higgs}}^{(4)} \times_{\tilde{Q}} \tilde{W}_4,$$

namely to the component whose intersection with $\tilde{W}_4 \times_{B_3} S_{\text{GUT}}$ lies in D_1 . But this ‘opposite’ component is, in fact, the spectral divisor with respect to the opposite choice of positive Weyl chamber and its associated opposite (or ‘flopped’) Brieskorn-Grothendieck equivariant crepant resolution as explained in [4]. Since $\tilde{\Gamma}$ and $\tilde{\Lambda}$ are defined entirely with respect to their relationship with the spectral variety for their respective Brieskorn-Grothendieck equivariant crepant resolutions, both $\tilde{\Gamma}$ and $\tilde{\Lambda}$ will be taken to themselves under the $\mathbb{Z}_2$ -action.

In summary, the $(4+1)$ -spectral equation breaks E_8 -symmetry to

$$SU(5)_{\text{gauge}} \times U(1)_X \subseteq SO(10)$$

and the $\mathbb{Z}_2$ -action breaks $U(1)_X$, leaving only $SU(5)_{\text{gauge}} \times \mathbb{Z}_2$ -symmetry before wrapping the Wilson line. In particular, the right-handed neutrinos discussed above can in principle obtain a large Majorana mass near the GUT/compactification scale. Thus the theory of [1] contains three families of quarks and leptons and one pair of Higgs doublets (after GUT symmetry breaking via the Wilson line). Note, the theory also includes a complete MSSM twin sector. The $\mathbb{Z}_2$ -symmetry is identified as matter parity. However, in addition as explained in [3], the theory has an asymptotic $\mathbb{Z}_4^{\mathbf{R}}$ -symmetry which forbids dimension 4 and 5 baryon-violating operators as well as the Higgs μ term. Finally the theory has non-trivial Yukawa couplings located at co-dimension 3 singularities.

Further analysis is necessary to determine the 3×3 Yukawa coupling matrices. We also want to analyze the MSSM twin world in order to address the question of relative scales between the visible and twin sectors and to determine whether or not there are any possible portals to the twin sector. Finally the issue of supersymmetry breaking must be addressed.

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