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From spinning primaries to permutation orbifolds

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Abstract: We carry out a systematic study of primary operators in the conformal field theory of a free Weyl fermion. Using SO(4, 2) characters we develop counting formulas for primaries constructed using a fixed number of fermion fields. By specializing to particular classes of primaries, we derive very explicit formulas giving the generating functions for the number of primaries in these classes. We present a duality map between primary operators in the fermion field theory and polynomial functions. This allows us to construct the primaries that were counted. Next we show that these classes of primary fields correspond to polynomial functions on certain permutation orbifolds. These orbifolds have palindromic Hilbert series.

Keywords: AdS-CFT Correspondence, Conformal Field Theory, Duality in Gauge Field Theories

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1 Introduction

The remarkable success of the conformal bootstrap [1–4] suggests that algebraic structures present in conformal field theory (CFT) can profitably be exploited to extract highly non-trivial information about the CFT. In the papers [5, 6] a systematic approach towards manifesting and exploiting some of these algebraic structures was outlined. The key result is that the algebraic structure of CFT defines a two dimensional topological field theory (TFT2) with SO(4, 2) invariance. Crossing symmetry is expressed as associativity of the algebra of local CFT operators. A basic observation which is at the heart of this result, is that the free four dimensional CFT of a scalar field can be formulated as an infinite dimensional associative algebra. This algebra admits a decomposition into linear representations of SO(4, 2), and is equipped with a non-degenerate bilinear product. A concrete application of these ideas has enabled a systematic study of primaries in bosonic free field theories in four dimensions, for scalar, vector and matrix models [7, 8]. For closely related ideas see [9, 10].

We know from the AdS/CFT correspondence [11–13] that strongly coupled CFTs have a dual holographic gravitational description. The combinatorics of the matrix model Feynman diagrams plays an important role in this holography. In this setting the TFT2 structure also appears as a powerful organizing structure, explicating algebraic structures that were not previously appreciated [14–17]. Thus, it seems that the TFT2 idea is rich enough to incorporate the algebraic structure emerging both from the conformal symmetry, and from the color combinatorics.

In this paper we extend the study of [7, 8] by carrying out a systematic study of primaries in free fermion field theories in four dimensions. In section 2 we obtain formulae for the counting of primary fields constructed from n copies of a left handed Weyl fermion, using the characters of representations of $so(4,2)$. For a beautiful discussion of these characters, see [18]. The basic quantity that we are interested in is the generating function

$$G_n(s,x,y) = \sum_{\Delta,j_1,j_2} N_{[\Delta,j_1,j_2]} s^{\Delta} x^{j_1} y^{j_2} \,, \tag{1.1}$$

which counts the number of conformal multiplets (denoted $N_{[\Delta,j_1,j_2]}$) labeled by the quantum numbers Δ, j_1, j_2 of their highest weight state. These quantum numbers are charges of the Cartan of SO$(4,2)$, namely the scaling dimension Δ and the two spins j_1 and j_2 associated to the SO$(4) = $ SU$(2) \times$ SU(2) subgroup of SO$(4,2)$. Although we obtain a concrete expression for $G_n(s,x,y)$, it is not very useful. By specializing to particular classes of primaries, we can make the counting formulae very explicit. These special classes of primaries obey extremality conditions stated using relations between the charges under the Cartan of SO$(4,2)$. The first class of primaries that we consider are the leading twist primaries. Recall that the twist τ is given by $\tau = \Delta - (j_1 + j_2)$. As we explain below, the primary operators constructed using n fields that maximize the twist τ have quantum numbers given by

$$[\Delta, j_1, j_2] = \left[\frac{n(n+2)}{2} + q, \frac{n(n+1)}{4} + \frac{q}{2}, \frac{n(n-1)}{4} + \frac{q}{2}\right] . \tag{1.2}$$

These quantum numbers are not at all obvious. To get some insight into the above list, write the scaling dimension as $\Delta = \frac{n(n-1)}{2} + q + \frac{3}{2}n$. The terms $q + \frac{3}{2}n$ are the expected contribution to the dimension from q derivatives and n fermion fields. Recall that for scalar fields we'd simply have $\Delta = q + n$ for the leading twist primaries. Fermi statistics requires that we antisymmetrize the fermion fields. Since each field has two components, to get a non-zero answer extra derivatives are needed and this leads to the additional contribution of $\frac{n(n-1)}{2}$. We denote the generating function counting this class of primaries by $G_n^{\max}(s,x,y)$ and we find

$$G_n^{\max}(s,x,y) = (s\sqrt{xy})^{\frac{n(n-1)}{2}} (s^{\frac{3}{2}}\sqrt{x})^n \prod_{k=2}^{n} \frac{1}{1 - (s\sqrt{xy})^k} \,. \tag{1.3}$$

Following [7, 8], we consider a second larger class of primaries, called the extremal primary operators in [7, 8]. This class is the set of operators with maximal j_1 spin at given Δ

$$[\Delta, j_1] = \left[\frac{3n}{2} + q, \frac{n}{2} + \frac{q}{2}\right] . \tag{1.4}$$

We denote the corresponding generating function by $G_n^{\text{ext}}(s,x,y)$. Although we do not have a closed formula for $G_n^{\text{ext}}(s,x,y)$ valid for any n, we explain how it can be computed for low values of n, by specializing the general counting formula. As an example we evaluate

$$\begin{aligned} G_3^{\text{ext}}(s,x,y) &= \frac{s^{\frac{13}{2}}x^{\frac{5}{2}}(1+s\sqrt{x}y^{\frac{3}{2}})}{(1-s^4x^2)(1-s^2xy)(1-s^3x^{\frac{3}{2}}y^{\frac{3}{2}})} \\ &= s^{\frac{13}{2}}x^{\frac{5}{2}} + s^{\frac{15}{2}}x^3y^{\frac{3}{2}} + s^{\frac{17}{2}}x^{\frac{7}{2}}y + s^{\frac{19}{2}}x^4y^{\frac{3}{2}} + s^{\frac{19}{2}}x^4y^{\frac{5}{2}} + s^{\frac{21}{2}}x^{\frac{9}{2}} \\ &\quad + s^{\frac{21}{2}}x^{\frac{9}{2}}y^2 + s^{\frac{21}{2}}x^{\frac{9}{2}}y^3 + \cdots . \end{aligned} \tag{1.5}$$

After developing these results for the counting of primary operators, we consider the problem of constructing the primaries that were counted. The construction of primary fields is mapped to a problem of determining multi-variable polynomials subject to a system of algebraic and differential constraints. Each primary operator corresponds to a specific polynomial. This relies on a function space realization of the conformal algebra, which is explained in section 3. The special classes of primary operators that we count above have a natural interpretation in this polynomial construction. Leading twist primaries correspond to holomorphic polynomials in a single complex variable z, while extremal primaries correspond to holomorphic polynomials in two complex variables, z and w. We give concrete examples of polynomials obeying the constraints and the associated primary operators.

Finally, in the last section we verify that the Hilbert series for the counting of extremal primaries are palindromic. The palindromy property of Hilbert series is indicative that the ring being enumerated is Calabi-Yau. It it interesting that palindromic Hilbert series also arise for moduli spaces of supersymmetric vacua of gauge theories, as found in [19, 20].

2 Counting primaries

This section considers the problem of enumerating the $SO(4,2)$ irreducible representations appearing among the composite fields made out of $n = 2, 3, \cdots$ copies of a free chiral fermion field. The chiral fermion is a lowest weight representation with $\Delta = \frac{3}{2}$, $j_1 = \frac{1}{2}$ and $j_2 = 0$. The fermions are Grassman fields, so there is a sign change when two fields are swapped. Consequently, we should be taking the antisymmetric product of the $SO(4,2)$ representations. We will denote the lowest weight representation corresponding to local operators built by taking derivatives of the fermion field by W_+. Enumerating the primaries entails decomposing the antisymmetrized tensor product $\text{Asym}^n(W_+)$ into irreducible representations. We start by deriving a formula for the character of the antisymmetrized tensor product of n copies of the free Weyl fermion representation. We then explain how to express this character as a sum of characters of irreducible representations, achieving the required decomposition. After obtaining a general formula in terms of an infinite product, we specialize to primaries that obey extremality conditions relating their dimension to their spin. For these primaries using results from [21], we find simple explicit formulas for the counting.

2.1 Generalities

The basic formula we use in this section states

$$\det(1+tM) = \sum_{n=0}^{\infty} t^n \chi_{(1^n)}(M)\,, \tag{2.1}$$

where $\chi_{(1^n)}(M)$ is the trace over the antisymmetrized product of n copies of M. Below we will use this formula to obtain the character of the antisymmetrized tensor products of n copies of the free Weyl fermion representation. We will reserve the letter χ for characters. The character for the free fermion representation is denoted by $\mathcal{D}_{[\frac{3}{2},\frac{1}{2}]+}$ in [18]. From formula (3.44) of [18] we know the character of a left handed Weyl fermion is

$$\begin{aligned} \chi_{W_+}(s,x,y) &= s^{\frac{3}{2}}(\chi_{\frac{1}{2}}(x) - s\chi_{\frac{1}{2}}(y))P(s,x,y) \\ &= s^{\frac{3}{2}}\sum_{q=0}^{\infty} s^q \chi_{\frac{q+1}{2}}(x)\chi_{\frac{q}{2}}(y) \\ &= \mathrm{Tr}_{W_+}(M)\,, \end{aligned} \tag{2.2}$$

with $M = s^D x^{J_1^3} y^{J_2^3}$ and

$$P(s,x,y) = \frac{1}{(1-s\sqrt{xy})\left(1-s\sqrt{\frac{x}{y}}\right)\left(1-s\sqrt{\frac{y}{x}}\right)\left(1-\frac{s}{\sqrt{xy}}\right)}\,. \tag{2.3}$$

Here J_1^3 is the third component of the $\vec{J}_1$ spin. It is straightforward to verify that for $M = s^D x^{J_1^3} y^{J_2^3}$ we have

$$\det(1+tM) = \prod_{q=0}^{\infty}\prod_{a=-\frac{q+1}{2}}^{\frac{q+1}{2}}\prod_{b=-\frac{q}{2}}^{\frac{q}{2}}(1+ts^{\frac{3}{2}+q}x^a y^b)\,. \tag{2.4}$$

Applying (2.1) we find the generating function of the characters of the antisymmetrized tensor products of the free Weyl fermion representation

$$\mathcal{Z}(t,s,x,y) = \prod_{q=0}^{\infty}\prod_{a=-\frac{q+1}{2}}^{\frac{q+1}{2}}\prod_{b=-\frac{q}{2}}^{\frac{q}{2}}(1+ts^{\frac{3}{2}+q}x^a y^b) = \sum_{n=0}^{\infty} t^n \chi_{(1^n)}(s,x,y)\,. \tag{2.5}$$

By expanding $\mathcal{Z}(t,s,x,y)$ as a series in t we can easily read off $\chi_{(1^n)}(s,x,y)$ as the coefficient of t^n. To be clear, $\chi_{(1^n)}(s,x,y)$ is the character of M in the representation given by the antisymmetrized tensor product $\mathrm{Asym}^n(W_+)$. The next step is to decompose this into a sum of SO(4,2) characters, for irreps of dimension Δ and spins j_1, j_2

$$\chi_{(1^n)}(s,x,y) = \sum_{[\Delta,j_1,j_2]} N_{[\Delta,j_1,j_2]}\chi_{[\Delta,j_1,j_2]}(s,x,y)\,. \tag{2.6}$$

The coefficients $N_{[\Delta,j_1,j_2]}$ count how many times the irreducible representation with lowest weight labeled by $[\Delta,j_1,j_2]$ appears in $\mathrm{Asym}^n(W_+)$. Hence, $N_{[\Delta,j_1,j_2]}$ are non-negative integers. The case that $n=2$ is complicated by the fact that some of the irreducible representations appearing in the above decomposition are short. We will consider $n=2$ separately in detail below. For $n \geq 3$ the character for the irreducible representation with lowest weight $[\Delta,j_1,j_2]$ is given by [18]

$$\chi_{[\Delta,j_1,j_2]}(s,x,y) = \frac{s^{\Delta}\chi_{j_1}(x)\chi_{j_2}(y)}{(1-s\sqrt{xy})\left(1-s\sqrt{\frac{x}{y}}\right)\left(1-s\sqrt{\frac{y}{x}}\right)\left(1-\frac{s}{\sqrt{xy}}\right)}\,. \tag{2.7}$$

It is useful to define

$$Z_n(s,x,y) \equiv \sum_{\Delta,j_1,j_2} N_{[\Delta,j_1,j_2]} s^{\Delta} \chi_{j_1}(x)\chi_{j_2}(y)\,, \tag{2.8}$$

so that

$$Z_n(s,x,y) = (1-s\sqrt{xy})\left(1-s\sqrt{\frac{x}{y}}\right)\left(1-s\sqrt{\frac{y}{x}}\right)\left(1-\frac{s}{\sqrt{xy}}\right)\chi_{(1^n)}(s,x,y)\,. \tag{2.9}$$

The right hand side of (2.8) is a sum of (products of) SU(2) characters. Following [22], it can be simplified by using the orthogonality of SU(2) characters. Towards this end, we introduce the generating function

$$\begin{aligned} G_n(s,x,y) &\equiv \sum_{\Delta,j_1,j_2} N_{[\Delta,j_1,j_2]} s^{\Delta} x^{j_1} y^{j_2} \\ &= \left[\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)Z_n(s,x,y)\right]_{\geq}\,. \end{aligned} \tag{2.10}$$

The subscript $\geq$ is an instruction to keep only non negative powers of x and y.

It is easy to check that this agrees with standard character computations. For example, the expansion

$$\begin{aligned} G_3(s,x,y) \,=\, & s^{\frac{11}{2}} x\sqrt{y} + s^{\frac{13}{2}} x^{\frac{5}{2}} + s^{\frac{13}{2}} x^{\frac{3}{2}} y + s^{\frac{15}{2}} y^{\frac{3}{2}} + s^{\frac{15}{2}} x^3 y^{\frac{3}{2}} + s^{\frac{15}{2}} x^2 y^{\frac{3}{2}} + s^{\frac{17}{2}} x^{\frac{7}{2}} y \\ & + s^{\frac{17}{2}} x^{\frac{3}{2}} y^2 + s^{\frac{17}{2}} x^{\frac{5}{2}} y^2 + s^{\frac{19}{2}} x^4 y^{\frac{3}{2}} + s^{\frac{19}{2}} x y^{\frac{5}{2}} + 2 s^{\frac{19}{2}} x^3 y^{\frac{5}{2}} + s^{\frac{19}{2}} x^4 y^{\frac{5}{2}} + \ldots\,, \end{aligned} \tag{2.11}$$

can be reproduced using characters, as we will now demonstrate. The relevant Schur polynomial for this case is calculated as follows

$$\chi_{(1^3)}(s,x,y) = \frac{1}{6}\left[(\chi_L(s,x,y))^3 - 3\chi_L(s^2,x^2,y^2)\chi_L(s,x,y) + 2\chi_L(s^3,x^3,y^3)\right]\,. \tag{2.12}$$

Using Mathematica, we find the following terms

$$\begin{aligned} \chi_{(1^3)}(s,x,y) \,=\, & \chi_{[\frac{11}{2},1,\frac{1}{2}]}(s,x,y) + \chi_{[\frac{13}{2},\frac{5}{2},0]}(s,x,y) + \chi_{[\frac{13}{2},\frac{3}{2},1]}(s,x,y) \\ & + \chi_{[\frac{15}{2},0,\frac{3}{2}]}(s,x,y) + \chi_{[\frac{15}{2},2,\frac{3}{2}]}(s,x,y) + \chi_{[\frac{15}{2},3,\frac{3}{2}]}(s,x,y) \\ & + \chi_{[\frac{17}{2},\frac{7}{2},1]}(s,x,y) + \chi_{[\frac{17}{2},\frac{3}{2},2]}(s,x,y) + \chi_{[\frac{17}{2},\frac{5}{2},2]}(s,x,y) \\ & + \chi_{[\frac{19}{2},4,\frac{3}{2}]}(s,x,y) + \chi_{[\frac{19}{2},1,\frac{5}{2}]}(s,x,y) + 2\chi_{[\frac{19}{2},3,\frac{5}{2}]}(s,x,y) + \chi_{[\frac{19}{2},4,\frac{5}{2}]}(s,x,y) \\ & + \chi_{[\frac{21}{2},\frac{9}{2},0]}(s,x,y) + \chi_{[\frac{21}{2},\frac{9}{2},2]}(s,x,y) + \chi_{[\frac{21}{2},\frac{3}{2},3]}(s,x,y) + \chi_{[\frac{21}{2},\frac{5}{2},3]}(s,x,y) \\ & + \chi_{[\frac{21}{2},\frac{7}{2},3]}(s,x,y) + \chi_{[\frac{21}{2},\frac{9}{2},3]}(s,x,y) + \ldots\,, \end{aligned} \tag{2.13}$$

in complete agreement with (2.11).

To end this subsection, we will now discuss the case that $n=2$. For this case we must account for the fact that representations that include null states appear in the decomposition. A lowest weight multiplet $[\Delta, j_1, j_2]$ will be short if [23] $\Delta = f(j_1)+f(j_2)$ with $f(j)=0$ if $j=0$ or $f(j)=j+1$ if $j>0$. This does not cover the case of the scalar field ($j_1=j_2=0$), which is short for $\Delta=1$. For $n=2$ the decomposition includes a primary with $\Delta=3$

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and $j_1 = j_2 = 0$ which is not short, as well as primaries with $\Delta = 2j$ $j_1 = (2j-1)/2$ and $j_2 = (2j-3)/2$ which are short representations and hence have null states. For the correct counting, these null states (and their descendants) must be removed. These short representations arise because their primary operators are conserved higher spin currents

$$\partial_\mu J^{\mu\mu_2\cdots\mu_{2j-2}}_{\alpha\dot\beta} = 0\,. \tag{2.14}$$

The subtraction of null states is achieved by removing the $\Delta = 3$ primary that does not need to be subtracted, dividing by $1 - s/\sqrt{xy}$ which removes the null descendents and then putting the original primary back in. The final result

$$\begin{aligned} G_2(s,x,y) &= \left[\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)\left(Z_2(s,x,y)-s^3\right)\frac{1}{1-\frac{s}{\sqrt{xy}}}\right]_{\geq} + s^3 \\ &= \sum_{j=0}^{\infty} s^{3+2j} x^{\frac{3}{2}+j} y^{\frac{1}{2}+j}\,, \end{aligned} \tag{2.15}$$

agrees with [24].

2.2 Leading twist primaries

By restricting to well defined classes of primaries, we can significantly simplify the counting formulas of the previous section. The biggest simplification comes from focusing on the leading twist primaries, which have quantum numbers $[\Delta, j_1, j_2] = [\frac{n(n+2)}{2} + q, \frac{n(n+1)}{4} + \frac{q}{2}, \frac{n(n-1)}{4} + \frac{q}{2}]$. These quantum numbers are not obvious but will be evident in the final answer of this section. In the introduction we motivated these quantum numbers, in the discussion appearing after equation (1.2). Each such leading twist primary operator comes in a complete spin multiplet of $(\frac{n(n+1)}{2} + q + 1)(\frac{n(n-1)}{2} + q + 1)$ operators. Choosing the operator with highest spin corresponds to studying primaries constructed using a single component P_z of the momentum four vector operator. To count the leading twist primaries we will count this highest spin operator in each multiplet. The corresponding generating function is $G_n^{\max}(s,x,y)$. This generating function is obtained after a simple modification of the results of the previous section. First, we replace $\chi_{(1^n)}(s,x,y)$ with a new function $\chi_n^{\max}(s,x,y)$, by keeping only the highest spin state from each multiplet in the product

$$\prod_{q=0}^{\infty}(1 + t s^{\frac{3}{2}+q} x^{\frac{q}{2}+\frac{1}{2}} y^{\frac{q}{2}}) = \sum_{n=0}^{\infty} t^n \chi_n^{\max}(s,x,y)\,. \tag{2.16}$$

The leading twist primaries are constructed using the single component of the momentum that raises left and right spin maximally. Consequently in (2.9) we replace

$$(1 - s\sqrt{xy})\left(1 - s\sqrt{\frac{x}{y}}\right)\left(1 - s\sqrt{\frac{y}{x}}\right)\left(1 - \frac{s}{\sqrt{xy}}\right) \to (1 - s\sqrt{xy})\,. \tag{2.17}$$

Finally, for each spin multiplet we keep only 1 state so there is no longer any need to replace the multiplet of spin states by a single state when we count. The final result is

$$\begin{aligned} G_n^{\max}(s,x,y) &\equiv \sum_{\Delta,j_1,j_2} N^{\max}_{[\Delta,j_1,j_2]} s^{\Delta} x^{j_1} y^{j_2} \\ &= (1 - s\sqrt{xy})\chi_n^{\max}(s,x,y)\,, \end{aligned} \tag{2.18}$$

where $N^{\max}_{[\Delta,j_1,j_2]}$ is the number of leading twist primaries of dimension Δ and spin (j_1, j_2). For the leading twist primaries, once n and the dimension of the operator is specified, the spin of the primary is fixed. Consequently, we need not track the x and y dependence, although we choose to keep this dependence explicit. This leads to the formula

$$\begin{aligned} F(t,s,x,y) &\equiv \frac{1}{(1-s\sqrt{xy})}\sum_{n=0}^{\infty} t^n G_n^{\max}(s,x,y) \\ &= \prod_{q=0}^{\infty}\left(1+ts^{\frac{3}{2}+q}x^{\frac{q}{2}+\frac{1}{2}}y^{\frac{q}{2}}\right). \end{aligned} \tag{2.19}$$

We can obtain explicit expressions for $G_n^{\max}(s,x,y)$ by developing $F(t,s,x,y)$ in a Taylor series. Define

$$f_k(t,s,x,y) = \frac{\partial^k}{\partial t^k}\log F(t,s,x,y). \tag{2.20}$$

A straight forward computation gives

$$f_k(t,s,x,y) = \sum_{q=0}^{\infty}\frac{(-1)^{k+1}(k-1)!s^{\frac{3k}{2}+kq}x^{\frac{kq}{2}+\frac{k}{2}}y^{\frac{kq}{2}}}{(1+ts^{\frac{3}{2}+q}x^{\frac{q}{2}+\frac{1}{2}}y^{\frac{q}{2}})^k}, \tag{2.21}$$

so that we have

$$f_k(0,s,x,y) = (k-1)!(-1)^{k-1}\frac{s^{\frac{3k}{2}}x^{\frac{k}{2}}}{1-s^k x^{\frac{k}{2}}y^{\frac{k}{2}}}. \tag{2.22}$$

Explicit expressions for $G_n^{\max}$ are now easily obtained. For example

$$\begin{aligned} G_3^{\max}(s,x,y) &= \frac{1}{3!}(1-s\sqrt{xy})\frac{\partial^3 F}{\partial t^3}\Big|_{t=0} \\ &= \frac{1}{3!}(1-s\sqrt{xy})(f_3+3f_1f_2+f_1^3) \\ &= \frac{s^{\frac{15}{2}}x^3y^{\frac{3}{2}}}{(1-s^2xy)(1-s^3x^{\frac{3}{2}}y^{\frac{3}{2}})}. \end{aligned} \tag{2.23}$$

Similarly

$$G_4^{\max}(s,x,y) = \frac{s^{12}x^5y^3}{(1-s^2xy)(1-s^3x^{\frac{3}{2}}y^{\frac{3}{2}})(1-s^4x^2y^2)}. \tag{2.24}$$

It is possible to obtain a general closed formula for $G_n^{\max}(s)$. To make the argument as transparent as possible, set $x=1=y$. Evaluate the derivative

$$\frac{\partial^n F}{\partial t^n} = \sum_{n_1,\cdots,n_q}\sum_{k_1,\cdots,k_q}\frac{(n_1k_1+\cdots+n_qk_q)!}{n_1!\cdots n_q!(k_1!)^{n_1}\cdots(k_q!)^{n_q}}f_{k_1}^{n_1}\cdots f_{k_q}^{n_q}\delta_{n,n_1k_1+\cdots n_qk_q}\,F, \tag{2.25}$$

and use the formulas for the f_k's to find

$$\frac{\partial^n F}{\partial t^n}\Big|_{t=0} = \sum_{n_1,\cdots,n_q}\sum_{k_1,\cdots,k_q}\frac{(-1)^{n-\sum_i n_i}n!s^{\frac{3n}{2}}}{n_1!\cdots n_q!\,k_1^{n_1}\cdots k_q^{n_q}}\left(\frac{s^{\frac{3k_1}{2}}}{1-s^{k_1}}\right)^{n_1}\cdots\left(\frac{s^{\frac{3k_q}{2}}}{1-s^{k_q}}\right)^{n_q}\delta_{n,n_1k_1+\cdots n_qk_q}. \tag{2.26}$$

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The sum appearing above can be interpreted as a sum over conjugacy classes of S_n. Recall that a conjugacy class of S_n collects all permutations with n_q k_q-cycles, that is, all permutations with the same cycle structure. This identification of the sum is a consequence of the fact that the coefficient

$$\frac{n!}{n_1!\cdots n_q!\,k_1^{n_1}\cdots k_q^{n_q}} \tag{2.27}$$

is the order of the conjugacy class. Each conjugacy class is weighted by the factor $(-1)^{n-\sum_i n_i}$ which is the signature of the permutation with n_q k_q-cycles. There is a factor of $\frac{s^{\frac{3k}{2}}}{1-s^k}$ for each k-cycle in the permutation. The lowest weight discrete series irreducible representation of SL(2), built on a ground state with dimension $\frac{3}{2}$ has character

$$\chi_1(s) = \mathrm{Tr}_{V_1}(s^{L_0}) = \frac{s^{\frac{3}{2}}}{1-s}\,. \tag{2.28}$$

Denote this irreducible representation by W_1. It then follows that ($P_{[1^n]}$ projects onto the antisymmetric irrep i.e. a single column of n boxes)

$$\frac{1}{n!}\frac{\partial^n F}{\partial t^n}\Big|_{t=0} = \mathrm{Tr}_{W_1}(P_{[1^n]}s^{L_0}) = \frac{s^{\frac{n}{2}(n+2)}}{(1-s)(1-s^2)(1-s^3)\cdots(1-s^n)}\,. \tag{2.29}$$

In writing the last equality above, we have used equation (49) of [21] which studies the SL(2) sector primaries using the language of oscillators. Our final result, for general x and y, is

$$G_n^{\max}(s,x,y) = (s\sqrt{xy})^{\frac{n(n-1)}{2}}(s^{\frac{3}{2}}\sqrt{x})^n\prod_{k=2}^{n}\frac{1}{1-(s\sqrt{xy})^k}\,. \tag{2.30}$$

2.3 Extremal primaries

In this section we will consider the class of primaries with charges

$$\Delta = \frac{3n}{2}+q \quad ; \quad J_1^3 = \frac{n}{2}+\frac{q}{2}\,. \tag{2.31}$$

This class of primaries generalizes the higher twist primaries because the charge J_2^3, which is part of $\mathrm{SU}(2)_R$, is not constrained. The extremal primaries fill out complete multiplets of $\mathrm{SU}(2)_R$ and are constructed using two components of the momentum four vector operator which are complex linear combinations of the (hermitian) P_μ. The specific complex linear combinations are determined by the requirement that J_1^3 is maximal. Following the treatment of the last section, we introduce a generating function $G_n^{\mathrm{ext}}(s,x,y)$, given by

$$G_n^{\mathrm{ext}}(s,x,y) = \left[\left(1-\frac{1}{y}\right)Z_n^{\mathrm{ext}}(s,x,y)\right]_{\geq} \tag{2.32}$$

where $Z_n^{\mathrm{ext}}(s,x,y)$ is defined by

$$Z_n^{\mathrm{ext}}(s,x,y) = (1-s\sqrt{xy})(1-s\sqrt{x/y})\chi_n^{\mathrm{ext}}(s,x,y) \tag{2.33}$$

with

$$F^{(2)}(t,s,x,y) \equiv \sum_{n=0}^{\infty} t^n \chi_n^{\text{ext}}(s,x,y)$$
$$= \prod_{q=0}^{\infty} \prod_{b=-\frac{q}{2}}^{\frac{q}{2}} (1 + t s^{\frac{3}{2}+q} x^{\frac{q+1}{2}} y^b)\,. \tag{2.34}$$

It is again possible to derive closed expressions for the generating function $G_n^{\text{ext}}(s,x,y)$. Introduce the functions

$$f_k(t,s,x,y) \equiv \frac{\partial^{k-1}}{\partial t^{k-1}} \log F^{(2)}$$
$$= (-1)^{k-1}(k-1)! \sum_{q=0}^{\infty} \sum_{m=-\frac{q}{2}}^{\frac{q}{2}} \frac{s^{kq+\frac{3k}{2}} x^{\frac{(q+1)k}{2}} y^{km}}{(1 + t s^{q+\frac{3}{2}} x^{\frac{q+1}{2}} y^m)^k}\,. \tag{2.35}$$

It is simple to establish that

$$f_k(0,s,x,y) = (-1)^{k-1}(k-1)! \frac{s^{\frac{3k}{2}} x^{\frac{k}{2}}}{(1 - s^k x^{\frac{k}{2}} y^{\frac{k}{2}})(1 - s^k x^{\frac{k}{2}} y^{-\frac{k}{2}})}\,. \tag{2.36}$$

Exactly as above we have

$$\left.\frac{\partial^n F^{(2)}}{\partial t^n}\right|_{t=0} = \sum_{n_1,\cdots,n_q} \sum_{k_1,\cdots,k_q} \frac{(n_1 k_1 + \cdots + n_q k_q)!}{n_1! \cdots n_q! (k_1!)^{n_1} \cdots (k_q!)^{n_q}} f_{k_1}^{n_1} \cdots f_{k_q}^{n_q} \delta_{n, n_1 k_1 + \cdots n_q k_q}\,. \tag{2.37}$$

Inserting the formulas for the f_k's, expressions for the $Z_n^{\text{ext}}(s,x,y)$ now follow from (2.33). To extract spin multiplets, we need to compute

$$G_n^{\text{ext}}(z,w) = \left[Z_n(s,x,y) \left(1 - \frac{1}{y}\right) \right]_{\geq} = \frac{1}{2\pi i} \oint_C dz \frac{\left(1 - \frac{1}{z^2}\right) Z_n(s,x,z^2)}{z - \sqrt{y}}\,. \tag{2.38}$$

As an example, the generating functions counting the extremal primaries constructed from 3 fields are given by

$$Z_3^{\text{ext}}(s,x,y) = s^{\frac{13}{2}} x^{\frac{5}{2}} y^{-\frac{3}{2}} \frac{(y^{\frac{3}{2}} + s^2 x y^{\frac{3}{2}} + s\sqrt{x}(1+y)(1+y^2))}{(1 - s^2 xy)(1 - s^3 x^{\frac{3}{2}} y^{\frac{3}{2}})(1 - \frac{s^2 x}{y})(1 - \frac{s^3 x^{\frac{3}{2}}}{y^{\frac{3}{2}}})} \tag{2.39}$$

$$G_3^{\text{ext}}(s,x,y) = \frac{s^{\frac{13}{2}} x^{\frac{5}{2}} (1 + s\sqrt{x} y^{\frac{3}{2}})}{(1 - s^4 x^2)(1 - s^2 xy)(1 - s^3 x^{\frac{3}{2}} y^{\frac{3}{2}})}$$
$$= s^{\frac{13}{2}} x^{\frac{5}{2}} + s^{\frac{15}{2}} x^3 y^{\frac{3}{2}} + s^{\frac{17}{2}} x^{\frac{7}{2}} y + s^{\frac{19}{2}} x^4 y^{\frac{3}{2}} + s^{\frac{19}{2}} x^4 y^{\frac{5}{2}} + s^{\frac{21}{2}} x^{\frac{9}{2}}$$
$$+ s^{\frac{21}{2}} x^{\frac{9}{2}} y^2 + s^{\frac{21}{2}} x^{\frac{9}{2}} y^3 + \cdots\,. \tag{2.40}$$

3 Construction

In this section we will explain how the counting of the previous section can be used to derive concrete formulas for the construction of the primary operators in the free fermion CFT. For the leading twist counting this is manifest. For the counting of extremal primaries, we will argue that our formulas can naturally be phrased as counting the multiplicities of symmetric group representations. The quantities being counted are then easily constructed using projectors onto these representations. In this analysis, a polynomial representation of $\mathrm{SO}(4,2)$ will play an important role. This representation is described in the next subsection, after which we describe the construction of leading twist primaries and then extremal primaries.

3.1 Polynomial rep

We use the following representation of $\mathrm{SO}(4,2)$

$$K_\mu = \frac{\partial}{\partial x^\mu}\,, \tag{3.1}$$

$$D = \left(x\cdot\frac{\partial}{\partial x} - \frac{3}{2}\right)\,, \tag{3.2}$$

$$M_{\mu\nu} = x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu} + \mathcal{M}_{\mu\nu}\,, \tag{3.3}$$

$$P_\mu = \left(x^2\frac{\partial}{\partial x^\mu} - 2x_\mu x\cdot\frac{\partial}{\partial x} + 3x_\mu - 2x^\nu \mathcal{M}_{\mu\nu}\right)\,. \tag{3.4}$$

In the formula above we should replace $\mathcal{M}_{\mu\nu}$ by the relevant matrix representing the spin part of the conformal group. In Minkowski spacetime we have (the two possibilities correspond to taking either a left handed $(\frac{1}{2},0)$ or a right handed $(0,\frac{1}{2})$ spinor)

$$\mathcal{M}^{\mu\nu} = \sigma^{\mu\nu}\,, \qquad \text{or} \qquad \bar{\sigma}^{\mu\nu}\,, \tag{3.5}$$

where

$$(\sigma^{\mu\nu})_\alpha{}^\beta = \frac{1}{4}\left(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu\right)_\alpha{}^\beta\,, \tag{3.6}$$

$$(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} = \frac{1}{4}\left(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu\right)^{\dot{\alpha}}{}_{\dot{\beta}}\,, \tag{3.7}$$

and

$$\sigma^\mu{}_{\alpha\dot{\beta}} = (\mathbf{1},\vec{\sigma})\,, \qquad \bar{\sigma}^{\mu\dot{\beta}\alpha} = (\mathbf{1},-\vec{\sigma})\,. \tag{3.8}$$

In Euclidean space we have

$$\mathcal{M}^{\mu\nu} = \sigma^{\mu\nu} \equiv \frac{1}{4}\left(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu\right)\,, \tag{3.9}$$

or

$$\mathcal{M}^{\mu\nu} = \bar{\sigma}^{\mu\nu} \equiv \frac{1}{4}\left(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu\right)\,, \tag{3.10}$$

where now

$$\sigma^\mu = (-i\vec{\sigma},\mathbf{1})\,, \qquad \bar{\sigma}^\mu = (i\vec{\sigma},\mathbf{1})\,. \tag{3.11}$$

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The generators in Minkowski space close the algebra

$$
\begin{aligned}
[M_{\rho\sigma}, M_{\phi\theta}] &= \eta_{\theta\sigma} M_{\phi\rho} + \eta_{\phi\rho} M_{\theta\sigma} - \eta_{\theta\rho} M_{\phi\sigma} - \eta_{\phi\sigma} M_{\theta\rho}\,, \\
[P_\mu, P_\nu] &= 0 = [K_\mu, K_\nu]\,, \qquad\qquad [P_\beta, K_\alpha] = 2\eta_{\alpha\beta} D - 2M_{\alpha\beta}\,, \\
[M_{\beta\rho}, K_\alpha] &= \eta_{\alpha\rho} K_\beta - \eta_{\alpha\beta} K_\rho\,, \qquad [M_{\beta\rho}, P_\alpha] = \eta_{\alpha\rho} P_\beta - \eta_{\alpha\beta} P_\rho\,, \\
[D, P_\mu] &= P_\mu\,, \qquad [D, K_\mu] = -K_\mu\,, \qquad [D, M_{\mu\nu}] = 0\,.
\end{aligned} \tag{3.12}
$$

The Euclidean generators obey the same algebra with $\eta_{\mu\nu}$ replaced with $\delta_{\mu\nu}$.

States in this representation correspond to polynomials in the spacetime coordinates x^μ times a spinor ζ_α, which is independent of x^μ and transforms in the $(\frac{1}{2}, 0)$ if we study the theory of a left handed fermion, or in the $(0, \frac{1}{2})$ if we study a right handed fermion. The 2×2 matrix $\mathcal{M}_{\mu\nu}$ acts on this spinor. Further, ζ_α is Grassman valued to account for the fact that the fermions are anticommuting fields. Concretely, each operator corresponds to a state (by the state operator correspondence) and each state corresponds to a polynomial times the spinor (thanks to the representation we have just described)

$$x^{\mu_1} \cdots x^{\mu_k} \zeta_\alpha\,. \tag{3.13}$$

To deal with operators constructed from a product of n copies of the basic fermion field, we consider a "multiparticle system". When we move to the multiparticle system, we have polynomials on the n particle coordinates x^I_μ, times the n particle spinor, obtained by taking the tensor product of n copies of ζ_α

$$(\zeta \otimes \zeta \otimes \cdots \otimes \zeta)_{\alpha_1 \alpha_2 \cdots \alpha_n}\,. \tag{3.14}$$

To write the generator of the conformal group, for the multiparticle system, we need the matrices

$$\mathcal{M}^{(I)}_{\mu\nu} = \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \mathcal{M}_{\mu\nu} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \tag{3.15}$$

where the matrix $\mathcal{M}_{\mu\nu}$ on the right hand side is the 2×2 matrix we introduced above and it appears as the Ith factor. In total $\mathcal{M}^{(I)}_{\mu\nu}$ has n factors. The n-particle representation of $\mathrm{SO}(4,2)$ includes

$$K_\mu = \sum_{I=1}^{n} \frac{\partial}{\partial x^I_\mu}\,, \tag{3.16}$$

and

$$P_\mu = \sum_{I=1}^{n} \left(x^{I\rho} x^I_\rho \frac{\partial}{\partial x^I_\mu} - 2x^I_\mu x^I \cdot \frac{\partial}{\partial x^I} + 3x^I_\mu - 2x^{I\,\nu} \mathcal{M}^{(I)}_{\mu\nu} \right)\,. \tag{3.17}$$

The representations introduced above all have null states. This is to be expected, since the dimension of the free fermion field saturates the unitarity bound. For the $(\frac{1}{2}, 0)$ field in Minkowski spacetime for example, the null state is exhibited by verifying that

$$\bar{\sigma}^\mu P_\mu \zeta = 0 \tag{3.18}$$

for any choice of ζ.

Let us now spell out the conditions that the polynomial $P_{\mathcal{O}}$ corresponding to an operator $\mathcal{O}$ must obey if the operator $\mathcal{O}$ is a primary operator. The general polynomial $P_{\mathcal{O}}$ will have spinor indices (it is constructed from a tensor product of copies of ζ) as well as four vector indices inherited from the spacetime coordinates. There are three conditions that must be imposed: primaries are annihilated by the special conformal generator K_μ

$$[K_\mu, \mathcal{O}] = 0 \,. \tag{3.19}$$

This implies that the corresponding polynomial is translation invariant

$$\sum_{I=1}^{n} \frac{\partial}{\partial x_\mu^I} P_{\mathcal{O}} = 0 \,. \tag{3.20}$$

Secondly, the equation of motion must be obeyed by each fermionic field. Finally, we require that the polynomials are in the antisymmetric representation of S_n. Since the ζs are Grassman variables, we must impose this condition if we are to get a non-zero primary upon translating back to the language of the fermion field theory.

We do not know how to obtain the complete set of polynomial solutions to the above constraints, corresponding to determing the complete set of primaries. We can however find a class of solutions and these correspond precisely to the leading twist and extremal primaries that we counted above. The fact that these polynomials are to be identified with the leading twist and extremal primaries will be evident in the detailed match between the counting of these solutions (performed in the following subsections) and the counting of the leading twist and extremal primaries. We will now explain how to find a large class of polynomials that solve the equation of motion constraint, leaving the discussion of the remaining two constraints for the subsections which follows. In the remainder of this subsection, we will work in Euclidean space. We use $x_4 = ix_0$ for the Euclidean time coordinate.

Our first observation is simply that any polynomial in the momenta $P(P_\mu)$, acting on the spinor ζ, solves the equation of motion constraint. Indeed, since the different components of momentum commute, we know that

$$\bar{\sigma}^\mu P_\mu \, P(P_\alpha)\zeta = P(P_\alpha)\, \bar{\sigma}^\mu P_\mu \zeta = 0 \,, \tag{3.21}$$

with the last equality following from (3.18). Introduce the complex variables

$$z = x_2 + ix_1, \qquad w = x_3 + ix_4, \tag{3.22}$$

and momenta

$$P_z = P_2 + iP_1, \qquad P_w = P_3 + iP_4 \,. \tag{3.23}$$

Our second observation is that if we specialize to a ζ with maximal J_1^3 eigenvalue, then any polynomial holomorphic in z and w can be translated into a polynomial in P_z and P_w. It is easy to see from a few examples, that $(P_z)^k\zeta \propto z^k\zeta$. When performing this computation use the identity

$$(P_2 + iP_1)\zeta = P_z\zeta = z\zeta \,, \tag{3.24}$$

which holds for the spinor with maximal J_1^3 eigenvalue. For our choices above, this spinor is given by

$$\zeta = \begin{bmatrix} 0 \\ 1 \end{bmatrix} . \tag{3.25}$$

Define the number a_k by the relation

$$(P_z)^k \zeta = a_k z^k \zeta \, . \tag{3.26}$$

Then

$$\begin{aligned}(P_z)^{k+1}\zeta &= P_z a_k z^k \zeta \\ &= -2(k+1) a_k z^{k+1} \zeta \\ &= a_{k+1} z^{k+1} \zeta \, . \end{aligned} \tag{3.27}$$

Thus, we have $a_{k+1} = -2(k+1)a_k$. This recursion together with the intial value $a_1 = -2$, implies that

$$a_k = (-2)^k k! \tag{3.28}$$

Thus we obtain the following translation between polynomials and momenta

$$(P_z)^k \zeta = (-1)^k 2^k k! \, z^k \zeta \, . \tag{3.29}$$

When peforming this computation note that the first term in (3.4) does not contribute because the complex combination we consider assembles the derivative $\partial_{\bar{z}}$ from this first term. The last two terms give $-2z$ for the spinor ζ we are using. Using a very similar argument, we find

$$(P_z)^k (P_w)^l \zeta = (-2)^{k+l} (k+l)! \, z^k \, w^l \, \zeta \tag{3.30}$$

We can now argue that any polynomial in z and w multiplying the spinor ζ with maximal J_1^3 eigenvalue, obeys the equation of motion constraint. It is enough to argue for a single monomial, since any polynomial is a sum of monomials. We argue as follows

$$\begin{aligned}\bar{\sigma}^\mu P_\mu (z^k w^l \zeta) &= \frac{1}{(-2)^{k+l}(k+l)!} \bar{\sigma}^\mu P_\mu (P_z^k P_w^l \zeta) \\ &= \frac{1}{(-2)^{k+l}(k+l)!} P_z^k P_w^l (\bar{\sigma}^\mu P_\mu \zeta) \\ &= 0 \end{aligned} \tag{3.31}$$

which demonstrates the claim.

3.2 Leading twist

Using a counting argument, we will confirm that the leading twist primaries are given by polynomials in a single complex variable z^I, $I = 1, 2, \ldots, n$. Any such polynomial obeys the equation of motion constraint. To solve the translation invariance condition, we work with the hook variables Z^a, $a = 1, 2, \ldots, n-1$ defined by

$$Z^a = \frac{1}{\sqrt{a(a+1)}} \left(z^{(1)} + z^{(2)} + \cdots + z^{(a)} - a z^{(a+1)} \right) . \tag{3.32}$$

These variables fill out the hook representation of S_n, which is labeled by a Young diagram whose first row has $n-1$ boxes and second row has 1 box. We denote the corresponding vector space by V_H, with the subscript H for "hook". Our problem is now reduced to constructing antisymmetric polynomials from the hook variables. By construction, it is clear that the degree k polynomials belong to a subspace of $V_H^{\otimes k}$ of S_n. We can characterize the antisymmetric subspace, that we want to extract, using representation theory. Towards this end, consider the following decomposition in terms of $S_n \times S_k$ irreps

$$V_H^{\otimes k} = \bigoplus_{\Lambda_1 \vdash n,\ \Lambda_2 \vdash k} V_{\Lambda_1}^{(S_n)} \otimes V_{\Lambda_2}^{(S_k)} \otimes V_{\Lambda_1,\Lambda_2}^{Com(S_n\times S_k)} . \tag{3.33}$$

In the above expression, $Com(S_n \times S_k)$ is the algebra of linear operators on $V_H^{\otimes k}$ that commute with $S_n \times S_k$, $V_{\Lambda_1}^{(S_n)}$ carries the irreducible representation Λ_1 of S_n, $V_{\Lambda_2}^{(S_k)}$ carries the irreducible representation Λ_2 of S_k and $V_{\Lambda_1,\Lambda_2}^{Com(S_n\times S_k)}$ carries the representation (Λ_1,Λ_2) of $Com(S_n\times S_k)$. This decomposition has been studied in detail in [21]. The Z variables are commuting so that we need to consider the case that $\Lambda_2=[k]$, the symmetric representation given by a Young diagram with a single row of k boxes. The resulting multiplicity is given by the coefficient of q^k in

$$\begin{aligned} Z_{SH}(q;\Lambda_1) &= (1-q)\, q^{\frac{\sum_i c_i(c_i-1)}{2}} \prod_b \frac{1}{(1-q^{h_b})} \\ &= \sum_k q^k Z_{SH}^k(\Lambda_1) . \end{aligned} \tag{3.34}$$

The subscript SH denotes "symmetrized hook" and it refers to the fact that we have taken the symmetrized ($\Lambda_2=[k]$) tensor product of k copies of the hook representation V_H. Here c_i is the length of the i'th column in Λ_1, b runs over boxes in the Young diagram Λ_1 and h_b is the hook length of the box b. Evaluating this formula for the antisymmetric representations, for which Λ_1 is a single column, gives [21]

$$\frac{q^{n(n-1)}}{(1-q^2)\cdots(1-q^n)} . \tag{3.35}$$

After accounting for the dimension of n elementary fermion fields and reinstating x and y, (3.35) is in complete agreement with (2.30) confirming that the number of polynomials in the complex variables z^I matches the number of leading twist primary operators.

Now that we have verified that the number of translation invariant, holomorphic polynomials in the antisymmetric representation of S_n agrees with the counting of leading twist primaries, we can move on to construction formulas for these primaries. Indeed, the relevant polynomials are given by acting with a projector onto the antisymmetric representation, on the hook variables. This polynomial multiplies an anticommuting tensor product of Grassman valued constant spinors. The projector from the tensor product of k copies of the hook onto the antisymmetric representation of S_n is

$$P_{(1^n)} = \frac{1}{n!} \sum_{\sigma\in S_n} \mathrm{sgn}(\sigma)\Gamma_k(\sigma) , \tag{3.36}$$

where $\mathrm{sgn}(\sigma)$ is the signature of permutation σ. When acting on a product of variables, say $Z^{a_1}Z^{a_2}\cdots Z^{a_k}$ we have

$$\Gamma_k(\sigma)=\Gamma_H(\sigma)\otimes\cdots\otimes\Gamma_H(\sigma)\,, \tag{3.37}$$

where on the right hand side we take a tensor product (the usual Kronecker product) of k copies of the matrices of the hook representation of S_n. Our construction formula is

$$\frac{1}{n!}\sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)\Gamma_k(\sigma)_{a_1a_2\cdots a_k,b_1b_2\cdots b_k}Z^{b_1}Z^{b_2}\cdots Z^{b_k}(\zeta_1\otimes\zeta_2\cdots\otimes\zeta_n)_{\alpha_1\cdots\alpha_n}\,. \tag{3.38}$$

The above formula produces an expression of the form $\sum_i \hat{n}_i P_i(Z)$ where $\hat{n}_i$ are unit vectors inside the carrier space of $V_H^{\otimes k}$ and $P_i(Z)$ are the polynomials that correspond to primary operators. To translate polynomials into momenta, the formula [7]

$$z^k \quad\leftrightarrow\quad \frac{(-1)^k P^k}{2^k k!}\,, \tag{3.39}$$

that we derived above, is very useful. We will now give some examples of polynomials obtained from formula (3.38). We will also translate these polynomials into primary operators.

If we consider $n=2$ fields, there is a single hook variable given by $Z=z_1-z_2$. To find a polynomial that is antisymmetric under swapping $1\leftrightarrow 2$, we must raise Z to an odd power. Thus, we find that primaries for the fermion fields correspond to the polynomials

$$(z_1-z_2)^{2s+1}=\sum_{k=0}^{2s+1}\frac{(2s+1)!}{k!(2s-k+1)!}(-1)^k z_1^{2s-k+1}z_2^k\,. \tag{3.40}$$

Translating the polynomial variables into momenta we find the following primary

$$|\psi\rangle=\sum_{k=0}^{2s+1}\frac{(-1)^k}{((2s-k+1)!k!)^2}\,P^k|\frac{3}{2},\frac{1}{2},0\rangle\otimes P^{2s-k+1}|\frac{3}{2},\frac{1}{2},0\rangle\,, \tag{3.41}$$

where, because our fields are fermions, we have

$$|\frac{3}{2},\frac{1}{2},0\rangle_1\otimes|\frac{3}{2},\frac{1}{2},0\rangle_2=-|\frac{3}{2},\frac{1}{2},0\rangle_2\otimes|\frac{3}{2},\frac{1}{2},0\rangle_1\,. \tag{3.42}$$

Thus, our expression for the fermionic primaries built from two fields are

$$\sum_{k=0}^{2s+1}\frac{(-1)^k}{((2s-k+1)!k!)^2}\,(\partial_1+i\partial_2)^k\psi(x)(\partial_1+i\partial_2)^{2s-k+1}\psi(x)\,, \tag{3.43}$$

which exactly matches the form of the higher spin currents [25, 26].

For $n=3$ fields it is easy to see that

$$(z_1-z_2)(z_1-z_3)(z_2-z_3)\,, \tag{3.44}$$

is holomorphic, translation invariant and in the antisymmetric representation of S_3. The corresponding primary operator can be simplified to

$$\psi(x)(\partial_1+i\partial_2)\psi(x)(\partial_1+i\partial_2)^2\psi(x)\,. \tag{3.45}$$

It is not difficult to see that this operator is indeed annihilated by K_μ, as discussed in appendix A.

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3.3 Extremal primaries

In this section we will consider the construction of extremal primaries, which correspond to polynomials in two holomorphic coordinates, z and w. The identification of these polynomials with the extremal primaries is again established by showing agreement of the counting of these polynomials with the counting of extremal primaries. We will characterize these polynomials by two degrees, one for Z and one for W. Polynomials of degree k in Z and of degree l in W belong to a subspace of $V_H^{\otimes k} \otimes V_H^{\otimes l}$ of S_n. The relevant decompositions in terms of $S_n \times S_k$ irreducible representations are

$$\begin{aligned} V_H^{\otimes k} &= \bigoplus_{\Lambda_1 \vdash n, \Lambda_2 \vdash k} V_{\Lambda_1}^{(S_n)} \otimes V_{\Lambda_2}^{(S_k)} \otimes V_{\Lambda_1,\Lambda_2}^{Com(S_n \times S_k)} \\ V_H^{\otimes l} &= \bigoplus_{\Lambda_3 \vdash n, \Lambda_4 \vdash l} V_{\Lambda_3}^{(S_n)} \otimes V_{\Lambda_4}^{(S_l)} \otimes V_{\Lambda_3,\Lambda_4}^{Com(S_n \times S_l)} . \end{aligned} \tag{3.46}$$

The tensor product $V_H^{\otimes k} \otimes V_H^{\otimes l}$ is a representation of

$$\mathbb{C}(S_n) \otimes \mathbb{C}(S_k) \otimes \mathbb{C}(S_n) \otimes \mathbb{C}(S_l) . \tag{3.47}$$

The Z and W variables are commuting so that $\Lambda_2 \otimes \Lambda_4 = [k] \otimes [l]$ is the trivial representation of $S_k \times S_l$. The multiplicity with which a given $S_n \times S_k$ irrep (Λ_1, Λ_2) appears is given by the dimension of the irreducible representation of the commutants $Com(S_n \times S_l)$ in $V_H^{\otimes k}$. Recall that since our polynomials multiply a product of anticommuting Grassman spinors, we want to project to states in $V_H^{\otimes k} \otimes V_H^{\otimes l}$ which are in the totally antisymmetric irreducible representation of the diagonal $\mathbb{C}(S_n)$ in the algebra (3.47). This constrains $\Lambda_3 = \Lambda_1^T$. Thus we find that the number of $S_k \times S_l$ invariants and S_n antisymmetric representations is

$$\sum_{\Lambda_1 \vdash n} \mathrm{Mult}(\Lambda_1^T, [k]; S_n \times S_k) \; \mathrm{Mult}(\Lambda_1, [l]; S_n \times S_l) . \tag{3.48}$$

Thus, for the number of primaries constructed using the variables z_i, w_i we get

$$\sum_{\Lambda_1 \vdash n} Z_{SH}^k(\Lambda_1) Z_{SH}^l(\Lambda_1^T) . \tag{3.49}$$

The above integer gives the number of primaries in the free fermion CFT, of weight $\frac{3n}{2}+k+l$, with spin $(J_1^3, J_2^3) = (\frac{k+l+n}{2}, \frac{k-l}{2})$. The generating function $Z_n^{\rm ext}(s,x,y)$ which encodes all k, l is given by

$$Z_n^{\rm ext}(s,x,y) = s^{\frac{3n}{2}} x^{\frac{n}{2}} \sum_{\Lambda \vdash n} Z_{SH}(s\sqrt{xy}, \Lambda) Z_{SH}\left(s\sqrt{\frac{x}{y}}, \Lambda^T \right) , \tag{3.50}$$

where Λ is a partition of n and we can use the formula (3.34). It is straight forwards to check, for example, that

$$\begin{aligned} Z_n^{\rm ext}(s,x,y) = s^{\frac{9}{2}} x^{\frac{3}{2}} \Bigg(& Z_{SH}(s\sqrt{xy}, \ydiagram{3}) Z_{SH}\left(s\sqrt{\frac{x}{y}}, \ydiagram{1,1,1} \right) + Z_{SH}\left(s\sqrt{xy}, \ydiagram{2,1} \right) Z_{SH}\left(s\sqrt{\frac{x}{y}}, \ydiagram{2,1} \right) \\ & + Z_{SH}\left(s\sqrt{xy}, \ydiagram{1,1,1} \right) Z_{SH}\left(s\sqrt{\frac{x}{y}}, \ydiagram{3} \right) \Bigg) \end{aligned} \tag{3.51}$$

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reproduces (2.39).

For $n = 3$ fields, it is easy to see that the polynomials

$$w_3(z_2 - z_1) + w_2(z_1 - z_3) + w_1(z_3 - z_2) \tag{3.52}$$

and

$$\begin{aligned}&2w_1w_2z_1^2 - w_2^2z_1^2 - 2w_1w_3z_1^2 + w_3^2z_1^2 - 2w_1^2z_1z_2 + 2w_2^2z_1z_2 + 4w_1w_3z_1z_2 - 4w_2w_3z_1z_2\\&+w_1^2z_2^2 - 2w_1w_2z_2^2 + 2w_2w_3z_2^2 - w_3^2z_2^2 + 2w_1^2z_1z_3 - 4w_1w_2z_1z_3 + 4w_2w_3z_1z_3 - 2w_3^2z_1z_3\\&+4w_1w_2z_2z_3 - 2w_2^2z_2z_3 - 4w_1w_3z_2z_3 + 2w_3^2z_2z_3 - w_1^2z_3^2 + w_2^2z_3^2 + 2w_1w_3z_3^2 - 2w_2w_3z_3^2\end{aligned} \tag{3.53}$$

are holomorphic, translation invariant and in the antisymmetric representation of S_3. To translate these polynomials into primary operators, we use the dictionary

$$z^k w^l \quad \leftrightarrow \quad \frac{(-1)^{k+l} P_z^k P_w^l}{2^{k+l}(k+l)!}\,. \tag{3.54}$$

After a little work we finally obtain the following two primary operators

$$\psi_1 = \psi(0) P_z \psi(0) P_w \psi(0) \tag{3.55}$$

and

$$\begin{aligned}\psi_2 = &\frac{1}{3} P_w P_z^2 \psi(0) P_w \psi(0) \psi(0) + \frac{1}{3} P_z \psi(0) P_w^2 P_z \psi(0) \psi(0)\\&+\frac{1}{4} P_w^2 \psi(0) P_z^2 \psi(0) \psi(0) + 2 P_w P_z \psi(0) P_z \psi(0) P_w \psi(0)\,.\end{aligned} \tag{3.56}$$

In the appendix we verify that these operators are annihilated by the special conformal transformations.

4 Geometry

In this section we comment on the permutation orbifolds relevant for the combinatorics of the fermion primaries. The leading twist primaries are holomorphic polynomials in n complex variables. We mod out by translations and restrict to the antisymmetric representation of S_n, so that the leading twist primaries correspond to holomorphic polynomial functions on

$$(\mathbb{C})^n/(\mathbb{C} \times S_n)\,. \tag{4.1}$$

A very similar argument shows that extremal primaries correspond to holomorphic polynomial functions on

$$(\mathbb{C})^{2n}/(\mathbb{C}^2 \times S_n)\,. \tag{4.2}$$

We will now argue that the Hilbert series of the fermionic primaries are counted by palindromic Hilbert series, suggesting that they are Calabi-Yau. We leave a more detailed study of these issues for the future. A palindromic Hilbert series obeys

$$Z_n^{\text{ext}}(q_1^{-1}, q_2^{-1}) = (q_1 q_2)^{n-1} Z_n^{\text{ext}}(q_1, q_2)\,. \tag{4.3}$$

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Our Hilbert series $Z_n^{\text{ext}}(q_1,q_2)$ enjoy this transformation property. To demonstrate this, our starting point is the formula

$$Z_n^{\text{ext}}(q_1,q_2) = s^{\frac{3n}{2}} x^{\frac{n}{2}} \sum_{\Lambda \vdash n} Z_{SH}(q_1,\Lambda) Z_{SH}(q_2,\Lambda^T)\,, \tag{4.4}$$

where we have introduced the variables $q_1 = s\sqrt{xy}, q_2 = s\sqrt{x/y}$. This has the property $Z_n^{\text{ext}}(q_1,q_2) = Z_n^{\text{ext}}(q_2,q_1)$. This follows because exchange of q_1, q_2 amounts to the inversion of y, and by using the identity [7]

$$Z_{SH}(q^{-1},\Lambda) = (-q)^{n-1} Z_{SH}(q,\Lambda^T)\,. \tag{4.5}$$

Using this result we find

$$\begin{aligned} Z_n^{\text{ext}}(q_1^{-1},q_2^{-1}) &= s^n (q_1 q_2)^{n-1} \sum_{\Lambda \vdash n} Z_{SH}(q_1,\Lambda^T) Z_{SH}(q_2,\Lambda) \\ &= s^n (q_1 q_2)^{n-1} \sum_{\Lambda \vdash n} Z_{SH}(q_1,\Lambda) Z_{SH}(q_2,\Lambda^T) \\ &= (q_1 q_2)^{n-1} Z_n^{z,w}(q_1,q_2)\,. \end{aligned} \tag{4.6}$$

The results of section (4.3) of [7] now imply that the Hilbert series $G_n^{\text{ext}}(s,x,y)$ also exhibit the palindromy property.

5 Summary and outlook

Previous studies [7] have explained how to map the algebraic problem of constructing primary fields in the quantum field theory of a free scalar field ϕ in four dimensions to one of finding polynomial functions on $(\mathbb{R}^4)^n$ that are harmonic, translation invariant and which are in the trivial representation of S_n. In this article, we have extended this construction to describe primary fields in the free quantum field theory of a single Weyl fermion. Concrete results achieved with this new point of view include a counting formula for the complete set primary fields, explicit counting formulas (Hilbert series) for counting special classes of primaries, as well as detailed construction formulas for these primary operators. We have also established the palindromy of the Hilbert series

One weak point in our analysis, that warrants further study, is the treatment of the constraint coming from the equation of motion. We have simply demonstrated that polynomials holomorphic in the complex variable z and w, times the spinor ζ with maximal J_1^3 eigenvalue, solve the equation of motion constraint. Our results have been further verified by checking that the numbers of polynomials constructed from a singe complex variable match the numbers of leading twist primaries, that the number of polynomials constructed from two complex variables match the number of extremal primaries and further that when the polynomials are translated back into the operator language, that we do indeed obtain operators annihilated by K_μ. It would however be nice to perform a detailed analysis of the equation of motion constraint, which has to be carried out before the complete class of primaries can be constructed.

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There is an immediate generalization of our study which should be tackled. CPT invariance implies we need both right handed and left handed fermions. As a concrete example, the Hilbert space for a single Weyl fermion is

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathrm{Asym}^n(W_+ \oplus W_-) \tag{5.1}$$

Can the methods developed in this article be used to study the above Hilbert space? In this situation $M = s^D x^{J_1^3} y^{J_2^3} = M_+ \oplus M_-$, with M_+ associated to W_+ and M_- associated to W_-. The basic identity we are using becomes

$$\begin{aligned}\det(1 + t_+ M_+ \oplus t_- M_-) &= \sum_{n_+,n_-=0}^{\infty} t_+^{n_+} t_-^{n_-} \chi_{n_+,n_-}(M) \\ &= \prod_{q_+,q_-=0}^{\infty} \prod_{a=-\frac{q+1}{2}}^{\frac{q+1}{2}} \prod_{b=-\frac{q}{2}}^{\frac{q}{2}} (1 + t_+ s^{\frac{3}{2}+q} x^a y^b)(1 + t_- s^{\frac{3}{2}+q} x^b y^a)\end{aligned} \tag{5.2}$$

where $\chi_{n_+,n_-}(M)$ is the antisymmetrized product of n_+ copies of M_+ and n_- copies of M_-. By expressing $\chi_{n_+,n_-}(M)$ as a sum of characters of irreducible representations of the conformal group, we learn what primaries can be constructed from a product of n_+ left handed and n_- right handed Weyl fermions. To achieve this decomposition, the methods of section 2 can be employed. To obtain simple and explicit results, one can again consider restricting the resulting counting formulas to special classes of primaries. For a given pair of integers (n_+, n_-), we can define both the leading twist (we maximize both J_1^3 and J_2^3 at a given dimension) and the extremal (we maximize J_1^3 or J_2^3 at a given dimension) classes of primaries. For these classes, it would be very interesting to see if a symmetric group interpretation of the counting can be developed, along the lines of sections 3.2 and 3.3. We are currently exploring this promising possibility and hope to return to it in the near future. A symmetric group interpretation of the counting would immediately suggest detailed construction formulas for the associated primaries.

Given that the counting for a Weyl fermion and a scalar field have been carried out, it is natural to ask if one can assemble this counting to give the counting of superconformal primaries. The simplest starting point would be a free boson plus fermion theory, where the counting of this paper for the fermion and of [7] for the boson, would be directly applicable. Other generalizations of the current work would include studies of CFTs which include gauge fields. Note that early constructions of primary fields in the SL(2) sector (leading twist primaries) were performed in the context of deep inelastic scattering in QCD (see for example [27]), suggesting that the free limit of QCD maybe a good starting point. Another natural question is the explicit enumeration and construction of superconformal primary fields in $\mathcal{N} = 4$ SYM, which will give a better understanding of the dual $AdS_5 \times S^5$ background. Finally, our results maybe useful when considering correlators involving the extremal primary fields at the fixed point of the Gross-Neveu model in $2 + \epsilon$ dimensions. In particular, one could attempt to determinae the anomalous dimensions of these fields, using the techniques of [28–32].

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A Primaries examples

In this appendix we will collect a few details on the translation from polynomials to primary operators and then test, for a few examples, that the primaries obtained are indeed annihilated by K_μ.

A.1 Dictionary

We first show that the appropriate way to translate between polynomials and operators is given by (3.54). We again make use of the (Euclidean) representation

$$P_\mu = x^2\partial_\mu - 2x_\mu x\cdot\partial + 3x_\mu - 2x_\nu\mathcal{M}_{\mu\nu}\,. \tag{A.1}$$

We consider a polynomial in $P_z = P_2 + iP_1 = \epsilon_z\cdot P$ and $P_w = P_3 - iP_4 = \epsilon_w\cdot P$ acting on the constant spinor ζ. The ϵ's obey the following identities

$$\begin{aligned}\epsilon_z\cdot\epsilon_z &= 0 = \epsilon_w\cdot\epsilon_w = \epsilon_z\cdot\epsilon_w\\ \epsilon_z\cdot x &= x_2 + ix_1 = z\,, \qquad \epsilon_w\cdot x = x_3 + ix_4 = w\,.\end{aligned} \tag{A.2}$$

The analysis of this appendix is for the leading twist and extremal primaries. In this case we have fixed the left spin to a maximal value, corresponding to choosing the spinor ζ with spin up. Useful formulas to bear in mind are

$$\begin{aligned}(\epsilon_z)_\mu(3x_\mu - 2x_\nu\mathcal{M}_{\mu\nu})\zeta &= 2z\zeta\,,\\ (\epsilon_w)_\mu(3x_\mu - 2x_\nu\mathcal{M}_{\mu\nu})\zeta &= 2w\zeta\,.\end{aligned} \tag{A.3}$$

Finally, we will also make use of the fact that

$$P_w^k P_z^l\psi(0) \leftrightarrow (-2)^{k+l}(k+l)!w^k z^l\zeta\,. \tag{A.4}$$

A.2 $n = 2$ example

Consider the operators given by (3.41). Introduce $K_z = K_2 - iK_1$. It is straightforward to verify that $[K_{\bar z}, P_z] = 0 = [P_{\bar z}, K_z]$ and

$$\begin{aligned}[D, P_z] &= P_z\,,\\ [D, K_z] &= -K_z\,,\\ [K_z, P_z] &= -4D + 4iM_{21} \qquad (A.5)\\ [M_{21}, K_z] &= -iK_z\\ [M_{21}, P_z] &= iP_z\,. \qquad (A.6)\end{aligned}$$

We will now argue that $K_{\bar{z}}$ annihilates (3.41). Using the above algebra we easily find

$$K_z P_z^m|\frac{3}{2},\frac{1}{2},0\rangle = -4P_z^{m-1}(m^2 - m(1-D+iM_{21}))|\frac{3}{2},\frac{1}{2},0\rangle \tag{A.7}$$

Consequently the action of K_z on the state (3.41) yields

$$\begin{aligned} K_z|\psi\rangle &= -4\sum_{k=1}^{2s+1}\frac{(-1)^k k^2}{((2s-k+1)!k!)^2}\,P^{k-1}|\frac{3}{2},\frac{1}{2},0\rangle \otimes P^{2s-k+1}|\frac{3}{2},\frac{1}{2},0\rangle \\ &\quad -4\sum_{k=0}^{2s}\frac{(-1)^k(2s-k+1)^2}{((2s-k+1)!k!)^2}\,P^{k}|\frac{3}{2},\frac{1}{2},0\rangle \otimes P^{2s-k}|\frac{3}{2},\frac{1}{2},0\rangle \\ &= 0 \end{aligned} \tag{A.8}$$

Next, it is straight forward to verify that

$$\begin{aligned} K_3 P_z^m|0\rangle &= imP_z^{m-1}(M_{31}-iM_{32})|0\rangle \\ K_4 P_z^m|0\rangle &= imP_z^{m-1}(M_{41}-iM_{42})|0\rangle. \end{aligned} \tag{A.9}$$

The operators $M_{31}-iM_{32}$ and $M_{41}-iM_{42}$ are raising operators for the right spin. Since the state $|\frac{3}{2},\frac{1}{2},0\rangle$ has vanishing right spin, we have

$$\begin{aligned} iM_{21}\,|\frac{3}{2},\frac{1}{2},0\rangle &= iM_{34}\,|\frac{3}{2},\frac{1}{2},0\rangle = \frac{1}{2}\,|\frac{3}{2},\frac{1}{2},0\rangle \\ (M_{31}-iM_{32})\,|\frac{3}{2},\frac{1}{2},0\rangle &= 0 = (M_{41}-iM_{42})\,|\frac{3}{2},\frac{1}{2},0\rangle. \end{aligned} \tag{A.10}$$

It now follows that K_3 and K_4 annihilate (3.41), completing the demonstration that (3.41) is indeed a primary operator.

A.3 $n = 3$ examples

We will show that the operators (3.55) and (3.56) are annihilated by the special conformal generators. Define $K_w = K_3 + iK_4$. It is straightforwards to evaluate

$$\begin{aligned} [K_w, P_z] &= 2\,(M_{32}+iM_{31}-(M_{41}-iM_{42})) \equiv 4iM_{wz} \\ [M_{wz}, P_w] &= iP_z \\ [M_{wz}, P_z] &= 0 \end{aligned} \tag{A.11}$$

and

$$\begin{aligned} [K_z, P_w] &= 2(M_{41}+iM_{42}-(M_{32}-iM_{31})) \equiv 4iM_{zw} \\ [M_{zw}, P_z] &= iP_w \\ [M_{zw}, P_w] &= 0 \end{aligned} \tag{A.12}$$

To interpret these commutators, note that P_z has spin $(\frac{1}{2},\frac{1}{2})$ and P_w has spin $(\frac{1}{2},-\frac{1}{2})$. Thus, M_{wz} and M_{zw} are raising/lowering operators of the right spin. Since our fermion field has vanishing right spin it is clear that

$$M_{zw}|\frac{3}{2},\frac{1}{2},0\rangle = M_{wz}|\frac{3}{2},\frac{1}{2},0\rangle = 0 \tag{A.13}$$

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which implies the identities

$$
\begin{aligned}
K_z P_w^n P_z^m|\frac{3}{2},\frac{1}{2},0\rangle &= -4(nm+m^2)P_w^n P_z^{m-1}|\frac{3}{2},\frac{1}{2},0\rangle \\
K_w P_w^n P_z^m|\frac{3}{2},\frac{1}{2},0\rangle &= -4(nm+n^2)P_w^{n-1} P_z^{m}|\frac{3}{2},\frac{1}{2},0\rangle
\end{aligned} \tag{A.14}
$$

It now follows that

$$
\begin{aligned}
K_z\psi_1 &= -4\psi(0)\psi(0)P_w\psi(0) = 0 \\
K_w\psi_1 &= -4\psi(0)P_z\psi(0)\psi(0) = 0
\end{aligned} \tag{A.15}
$$

where we used the Grassman statistics of the field. For the action on ψ_2 we find

$$
\begin{aligned}
-\frac{1}{4}K_z\psi_2 &= \frac{1}{3}(6)P_wP_z\psi(0)P_w\psi(0)\psi(0) + \frac{1}{3}(3)P_z\psi(0)P_w^2\psi(0)\psi(0) \\
&\quad + \frac{1}{4}(4)P_w^2\psi(0)P_z\psi(0)\psi(0) + 2P_wP_z\psi(0)\psi(0)P_w\psi(0) = 0 \\
-\frac{1}{4}K_w\psi_2 &= \frac{1}{3}(3)P_z^2\psi(0)P_w\psi(0)\psi(0) + \frac{1}{3}(6)P_z\psi(0)P_wP_z\psi(0)\psi(0) \\
&\quad + \frac{1}{4}(4)P_w\psi(0)P_z^2\psi(0)\psi(0) + 2P_wP_z\psi(0)P_z\psi(0)\psi(0) = 0
\end{aligned} \tag{A.16}
$$

again after using the Grassman nature of the field. This completes the demonstration that ψ_1 and ψ_2 are primary operators.

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