

Helicity amplitudes for QCD with massive quarks

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ABSTRACT: The novel massive spinor-helicity formalism of Arkani-Hamed, Huang and Huang provides an elegant way to calculate scattering amplitudes in quantum chromodynamics for arbitrary quark spin projections. In this note we compute two families of tree-level QCD amplitudes with one massive quark pair and $n - 2$ gluons. The two cases include all gluons with identical helicity and one opposite-helicity gluon being color-adjacent to one of the quarks. Our results naturally incorporate the previously known amplitudes for both quark spins quantized along one of the gluonic momenta. In the all-multiplicity formulae presented here the spin quantization axes can be tuned at will, which includes the case of the definite-helicity quark states.

KEYWORDS: Perturbative QCD, Scattering Amplitudes, Space-Time Symmetries

ARXIV EPRINT: [1802.06730](https://arxiv.org/abs/1802.06730)

Contents

1	Introduction	1
2	Spinor-helicity review	2
2.1	Massless Weyl spinors	3
2.2	Massive Weyl spinors	4
2.3	Dirac spinors and spin	5
3	Four-point amplitudes	7
4	All-multiplicity amplitudes	10
4.1	All-plus amplitudes with two quarks	10
4.2	One-minus amplitudes with two quarks	11
5	Checks	13
6	Summary and discussion	16
A	Massless spinor parametrizations	17
B	Massive spinor parametrizations	17

1 Introduction

The recent advances in the analytic understanding of the scattering amplitudes are often believed to be specific to massless theories, preferably with supersymmetry. It is arguably due to the absence, until recently, of a fully satisfactory spinor-helicity formalism for massive particles. Of course, the massless spinor-helicity formalism [1–6] (popularized e.g. by ref. [7]) has been applied [8–10] to define massive Dirac spinors. However, that construction did not manage to dispel the notion of the on-shell amplitude methods being restricted to the massless case. Recently, however, Arkani-Hamed, Huang and Huang [11] have introduced a complete version of a massive spinor-helicity formalism and used it to reconsider an array of quantum field-theoretic results from the fully on-shell perspective.

This note is about how this massive formalism can be used in one field theory of interest — quantum chromodynamics with heavy quarks. For simplicity, here we only consider the amplitudes with one massive quark-antiquark pair, with the other particles being gluons of definite helicity. The main goals of this note are two-fold:

- We provide new all-multiplicity expressions, eqs. (4.1) and (4.8), for the n -point color-ordered amplitudes with two quarks in case of all gluons of identical helicity and the case of one gluon of opposite helicity color-adjacent to one of the quarks.

- We pay special attention to our conventions so that our results be consistent with the vast QCD literature. That involves flexible transitions between the presented massive formalism, its massless analogue recovered in the high-energy limit, the general Dirac spinors and their realization using the massless Weyl spinors.

In view of the second goal, in section 2 we review the spinor-helicity formalism in an effort to combine brevity with comprehensiveness. We illustrate the introduced methods in section 3, where we show two ways to derive a full color-dressed amplitude for four-particle scattering (corresponding e.g. to non-abelian Compton scattering). We highlight the difference between the Feynman-diagrammatic approach and the on-shell construction, which deals solely with gauge-invariant quantities.

In section 4 we present and prove the aforementioned all-multiplicity amplitudes with two specific gluon-helicity configurations. For that we employ the Britto-Cachazo-Feng-Witten (BCFW) on-shell recursion [12, 13]. The spins of the quark and the antiquark remain unfixed throughout the calculations, which lets us specialize to the specific quark-spin projections considered previously [14] in the massless-spinor-based formalism [8–10]. Hence, in section 5, we give a simple dictionary (5.4) between the two descriptions and thus compare our results with the literature. It also shows that the new formalism easily incorporates the old one, the elegance of which suffered from the loss of the explicit little-group SU(2) symmetry.

We hope that this note will pave the way to more tree- and loop-level calculations in the newly complete spinor-helicity formalism [11], as outlined in section 6.

2 Spinor-helicity review

It is well-known that particles are defined as irreducible unitary representations of the Poincare group [15, 16]. Once the translation operator is diagonalized and the particles are labeled by their momentum p^μ , one is left with the Lorentz SO(1,3) subgroup of the Poincare group. The remaining labels of a one-particle state turn out to belong to a representation of its little group. This subgroup of SO(1,3) is crucial for understanding spin. It is defined through the Lorentz transformations that preserve the momentum p^μ of the particle. It corresponds to SO(2) for massless states or to SO(3) for massive ones.

To include fermions into consideration, one must generalize to the universal covering group SL(2, C) of SO(1,3). The homomorphism between these two groups is implemented by the spinor maps

$$p_{\alpha\dot{\beta}} = p_\mu \sigma_{\alpha\dot{\beta}}^\mu, \quad p^{\dot{\alpha}\beta} = p^\mu \bar{\sigma}_\mu^{\dot{\alpha}\beta}. \quad (2.1)$$

The Pauli matrices¹ $\sigma^\mu = (1, \sigma^1, \sigma^2, \sigma^3)$ and $\bar{\sigma}^\mu = (1, -\sigma^1, -\sigma^2, -\sigma^3)$ here translate Lorentz transformations between the spinorial and vectorial languages:

$$L^\mu_\nu = \frac{1}{2} \text{tr}(\bar{\sigma}^\mu S \sigma_\nu S^\dagger) : \quad p_{\alpha\dot{\delta}} \rightarrow S_\alpha^\beta p_{\beta\dot{\gamma}} (S_\delta^\gamma)^* \quad \Rightarrow \quad p^\mu \rightarrow L^\mu_\nu p^\nu, \quad (2.2)$$

for $L \in \text{SO}(1,3)$ and $S \in \text{SL}(2, \mathbb{C})$. At the same time, the little groups for massless and massive particles are accordingly promoted to U(1) and SU(2).

¹We use $\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, as well as $\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

An important property of the $SL(2, \mathbb{C})$ transformations (and hence the $SU(2)$ ones) is that they preserve the antisymmetric form $\epsilon_{\alpha\beta} = -\epsilon^{\alpha\beta}$, i.e. the spinor product:

$$S_\alpha^\gamma S_\beta^\delta \epsilon_{\gamma\delta} = \frac{1}{2} \epsilon_{\gamma\delta} (\delta_\alpha^\phi \delta_\beta^\psi - \delta_\beta^\phi \delta_\alpha^\psi) S_\phi^\gamma S_\psi^\delta = \frac{1}{2} \epsilon_{\gamma\delta} \epsilon_{\alpha\beta} \epsilon^{\phi\psi} S_\phi^\gamma S_\psi^\delta = \epsilon_{\alpha\beta} \det S = \epsilon_{\alpha\beta}. \quad (2.3)$$

This form allows to raise and lower both the spinor and massive-little-group indices at will.

Now let us explore different spinor types one by one. The massless and massive Weyl spinors comprise the spinor-helicity formalism [1–6, 11], while the Dirac spinors are helpful to connect it to the more traditional approaches.

2.1 Massless Weyl spinors

In the massless case, the on-shell condition $p^2 = \det\{p_{\alpha\dot{\beta}}\} = 0$ means that the degenerate matrix $p_{\alpha\dot{\beta}}$ can be decomposed as a tensor product of two Weyl spinors. That decomposition can be written in various interchangeable ways using the spinor bra-ket notation:

$$\begin{aligned} p_{\alpha\dot{\beta}} &= \lambda_{p\alpha} \tilde{\lambda}_{p\dot{\beta}} \equiv |p\rangle_\alpha [p]_{\dot{\beta}} \\ p^{\dot{\alpha}\beta} &\stackrel{\Downarrow}{=} \tilde{\lambda}_{p\dot{\alpha}} \lambda_p^\beta \equiv [p]^{\dot{\alpha}} \langle p|^\beta \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \not{p} &= |p\rangle [p] + |p\rangle \langle p| \\ p^\mu &\stackrel{\Downarrow}{=} \frac{1}{2} \lambda_p^\alpha \sigma_\mu^{\alpha\dot{\beta}} \tilde{\lambda}_{p\dot{\beta}} \equiv \frac{1}{2} \langle p | \sigma^\mu | p \rangle \\ p_\mu &\stackrel{\Downarrow}{=} \frac{1}{2} \tilde{\lambda}_{p\dot{\alpha}} \bar{\sigma}_\mu^{\dot{\alpha}\beta} \lambda_{p\beta} \equiv \frac{1}{2} [p | \bar{\sigma}_\mu | p]. \end{aligned} \quad (2.4)$$

This notation fits the spinor products [1–6] particularly well:

$$\langle pq \rangle \equiv \lambda_p^\alpha \lambda_{q\alpha} = \lambda_p^\alpha \epsilon_{\alpha\beta} \lambda_q^\beta, \quad [pq] \equiv \tilde{\lambda}_{p\dot{\alpha}} \tilde{\lambda}_{q\dot{\alpha}} = \tilde{\lambda}_{p\dot{\alpha}} \epsilon^{\dot{\alpha}\dot{\beta}} \tilde{\lambda}_{q\dot{\beta}}, \quad \langle pq \rangle [qp] = 2p \cdot q. \quad (2.5)$$

The Lorentz transformations (2.2) act on the Weyl spinors $\lambda_{p\alpha} \equiv |p\rangle_\alpha$ and $\tilde{\lambda}_{p\dot{\alpha}} \equiv [p]^{\dot{\alpha}}$ via $S \in SL(2, \mathbb{C})$, but only up to the little-group $U(1)$ rotations:²

$$\begin{aligned} \lambda_{p\alpha} &\rightarrow S_\alpha^\beta \lambda_{p\beta} = e^{i\phi/2} \lambda_{Lp\alpha}, & \lambda_p^\alpha &\rightarrow \lambda_p^\beta (S^{-1})_\beta^\alpha = e^{i\phi/2} \lambda_{Lp}^\alpha, \\ \tilde{\lambda}_{p\dot{\alpha}} &\rightarrow \tilde{\lambda}_{p\dot{\beta}} (S^\dagger)^{\dot{\beta}}_{\dot{\alpha}} = e^{-i\phi/2} \tilde{\lambda}_{Lp\dot{\alpha}}, & \tilde{\lambda}_{p\dot{\alpha}} &\rightarrow (S^{\dagger-1})^{\dot{\alpha}}_{\dot{\beta}} \tilde{\lambda}_{p\dot{\beta}} = e^{-i\phi/2} \tilde{\lambda}_{Lp\dot{\alpha}}. \end{aligned} \quad (2.6)$$

These spinors also give us the building blocks for the polarization vectors of gauge bosons:

$$\varepsilon_{p+}^\mu = \frac{1}{\sqrt{2}} \frac{\langle q | \sigma^\mu | p \rangle}{\langle qp \rangle} \quad \Leftrightarrow \quad \not{\varepsilon}_p^+ = \sqrt{2} \frac{|q\rangle [p] + |p\rangle \langle q|}{\langle qp \rangle}, \quad (2.7a)$$

$$\varepsilon_{p-}^\mu = -\frac{1}{\sqrt{2}} \frac{[q | \bar{\sigma}^\mu | p \rangle}{[qp]} \quad \Leftrightarrow \quad \not{\varepsilon}_p^- = -\sqrt{2} \frac{|p\rangle [q] + |q\rangle \langle p|}{[qp]}, \quad (2.7b)$$

where q can be any null vector such that $|q\rangle \not\sim |p\rangle$ and $[q] \not\sim [p]$. Indeed, different reference vectors are equivalent up to a pure gauge, e.g.

$$\varepsilon_{p+}^\mu(q') = \varepsilon_{p+}^\mu(q) + \frac{\sqrt{2} \langle q'q \rangle p^\mu}{\langle q'p \rangle \langle pq \rangle}. \quad (2.8)$$

²In the case that the Lorentz transformation L is a pure $SO(2)$ rotation around the momentum axis \hat{p} by the angle ϕ , the little-group phases in eq. (2.6) are unambiguous and precisely equal to $\pm\phi/2$.

Now it is important to note that under a Lorentz transformation (2.6) the polarization vectors do not actually transform as proper vectors. For instance, comparing

$$L^\mu{}_\nu \varepsilon_{p^+}^\nu = \frac{\langle \lambda_q S^{-1} | \sigma^\mu | S^{\dagger-1} \tilde{\lambda}_p \rangle}{\sqrt{2} \langle \lambda_q S^{-1} | S \lambda_p \rangle} = e^{-i\phi} \frac{\langle \lambda_q S^{-1} | \sigma^\mu | \tilde{\lambda}_{Lp} \rangle}{\sqrt{2} \langle \lambda_q S^{-1} | \lambda_{Lp} \rangle} \quad \text{vs.} \quad \varepsilon_{Lp^+}^\mu \equiv \frac{\langle \lambda_q | \sigma^\mu | \tilde{\lambda}_{Lp} \rangle}{\sqrt{2} \langle \lambda_q | \lambda_{Lp} \rangle}, \quad (2.9)$$

we conclude that Lorentz transformations act as

$$\varepsilon_{p^\pm}^\mu \rightarrow L^\mu{}_\nu \varepsilon_{p^\pm}^\nu \sim e^{\mp i\phi} \varepsilon_{Lp^\pm}^\mu \quad (2.10)$$

only up to an additional term proportional to the new momentum $L^\mu{}_\nu p^\nu$. However, up to this caveat, this shows that these polarization vectors can be thought of as conversion coefficients between the off-shell Lorentz transformations and the corresponding on-shell little-group rotations [11]. A similar statement for the Weyl spinors is demonstrated by eq. (2.6) and is also true for the massive case, see eq. (2.13) below.

As a concrete realization of the Weyl spinors, one could use, for instance,

$$\lambda_{p\alpha} = \sqrt{2E} \begin{pmatrix} -e^{-i\varphi} \sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{pmatrix}, \quad \tilde{\lambda}_p^{\dot{\alpha}} = \sqrt{2E} \begin{pmatrix} \cos \frac{\theta}{2} \\ e^{i\varphi} \sin \frac{\theta}{2} \end{pmatrix}, \quad (2.11)$$

for a null momentum expressible as $p^\mu = E(1, \cos \varphi \sin \theta, \sin \varphi \sin \theta, \cos \theta)$. A more practical implementation is given in appendix A.

2.2 Massive Weyl spinors

For a nonzero mass m , we have a non-degenerate matrix $p_{\alpha\dot{\beta}}$ that satisfies $\det\{p_{\alpha\dot{\beta}}\} = m^2$. The Weyl spinors are then introduced [11] by expanding $p_{\alpha\dot{\beta}}$ in terms of two explicitly degenerate matrices $\lambda_{p\alpha}^1 \tilde{\lambda}_{p\dot{\beta}1}$ and $\lambda_{p\alpha}^2 \tilde{\lambda}_{p\dot{\beta}2}$:

$$\begin{aligned} p_{\alpha\dot{\beta}} &= \lambda_{p\alpha}^a \tilde{\lambda}_{p\dot{\beta}a} = \lambda_{p\alpha}^a \epsilon_{ab} \tilde{\lambda}_{p\dot{\beta}}^b, & \det\{\lambda_{p\alpha}^a\} &= m, & \det\{\tilde{\lambda}_{p\dot{\alpha}}^a\} &= m, \\ p^{\dot{\alpha}\beta} &\stackrel{\uparrow}{=} \tilde{\lambda}_{p\dot{\alpha}}^{\dot{\beta}} \lambda_p^{\beta a} = -\tilde{\lambda}_p^{\dot{\alpha}\dot{\beta}} \epsilon_{ab} \lambda_p^{\beta b}, & \lambda_{\alpha a} \lambda^{\beta a} &\stackrel{\downarrow}{=} m \delta_\alpha^\beta, & \tilde{\lambda}^{\dot{\alpha} a} \tilde{\lambda}_{\dot{\beta} a} &\stackrel{\downarrow}{=} m \delta_{\dot{\beta}}^{\dot{\alpha}}. \end{aligned} \quad (2.12)$$

Here we have already indicated that the little-group indices $a, b = 1, 2$ are lowered and raised by the antisymmetric form ϵ_{ab} , preserved by SU(2) rotations. Such little-group transformations follow from the action of the Lorentz group on these spinors:

$$\begin{aligned} \lambda_{p\alpha}^a &\rightarrow S_\alpha^\beta \lambda_{p\beta}^a = \omega^a_b \lambda_{Lp\alpha}^b, & \lambda_p^{\alpha a} &\rightarrow \lambda_p^{\beta a} (S^{-1})_\beta^\alpha = \omega^a_b \lambda_{Lp}^{\alpha b}, \\ \tilde{\lambda}_{p\dot{\alpha}}^a &\rightarrow \tilde{\lambda}_{p\dot{\beta}}^a (S^\dagger)^{\dot{\beta}}_{\dot{\alpha}} = \omega^a_b \tilde{\lambda}_{Lp\dot{\alpha}}^b, & \tilde{\lambda}_p^{\dot{\alpha} a} &\rightarrow (S^{\dagger-1})^{\dot{\alpha}}_{\dot{\beta}} \tilde{\lambda}_p^{\dot{\beta} a} = \omega^a_b \tilde{\lambda}_{Lp}^{\dot{\alpha} b}, \end{aligned} \quad (2.13)$$

where $\omega \in \text{SU}(2)$ correspond to the SO(3) rotations in the rest frame of the massive particle momentum. These transformations are a massive analogue of eq. (2.6). Furthermore, the momentum decomposition (2.12) implies the two-dimensional version of the Dirac equation

$$p^{\dot{\alpha}\alpha} \lambda_{p\alpha}^a = m \tilde{\lambda}_p^{\dot{\alpha} a}, \quad p_{\alpha\dot{\beta}} \tilde{\lambda}_{p\dot{\beta}}^a = m \lambda_{p\alpha}^a. \quad (2.14)$$

For further convenience, let us rewrite the above identities in the spinor bra-ket notation:

$$\begin{aligned}
 |p^a\rangle_\alpha [p_a]_{\dot{\beta}} &= p_{\alpha\dot{\beta}} & p^{\dot{\alpha}\beta} |p^a\rangle_\beta &= m |p^a]_{\dot{\alpha}} \\
 |p^a]_{\dot{\alpha}} \langle p_a|^\beta &= -p^{\dot{\alpha}\beta} & p_{\alpha\dot{\beta}} |p^a]_{\dot{\beta}} &= m |p^a\rangle_\alpha & \langle p^a p^b \rangle &= -m \epsilon^{ab} \\
 |p^a\rangle_\alpha \langle p_a|^\beta &= -m \delta_\alpha^\beta & \langle p^a |^\alpha p_{\alpha\dot{\beta}} &= -m [p^a]_{\dot{\beta}} & [p^a p^b] &= m \epsilon^{ab}. \\
 |p^a]_{\dot{\alpha}} [p_a]_{\dot{\beta}} &= m \delta_{\dot{\alpha}\dot{\beta}} & [p^a]_{\dot{\alpha}} p^{\dot{\alpha}\beta} &= -m \langle p^a |^\beta
 \end{aligned} \tag{2.15}$$

As an explicit spinor realization, one may use [11]

$$\lambda_{p\alpha}^a = \begin{pmatrix} \sqrt{E-P} \cos \frac{\theta}{2} - \sqrt{E+P} e^{-i\varphi} \sin \frac{\theta}{2} \\ \sqrt{E-P} e^{i\varphi} \sin \frac{\theta}{2} \end{pmatrix}, \quad \tilde{\lambda}_{p\dot{\alpha}}^a = \begin{pmatrix} -\sqrt{E+P} e^{i\varphi} \sin \frac{\theta}{2} & -\sqrt{E-P} \cos \frac{\theta}{2} \\ \sqrt{E+P} \cos \frac{\theta}{2} & -\sqrt{E-P} e^{-i\varphi} \sin \frac{\theta}{2} \end{pmatrix}, \tag{2.16}$$

given a massive momentum expressible as $p^\mu = (E, P \cos \varphi \sin \theta, P \sin \varphi \sin \theta, P \cos \theta)$, such that $E^2 - P^2 = m^2$. A more detailed implementation is given in appendix B.

2.3 Dirac spinors and spin

In this paper, we wish to study massive quarks that are traditionally described in terms of the Dirac spinors. Hence it may be illuminating to consider how the Weyl spinors (2.12) naturally unify into the Dirac spinors.³

$$\begin{aligned}
 u_p^a &= \begin{pmatrix} \lambda_{p\alpha}^a \\ \tilde{\lambda}_{p\dot{\alpha}}^a \end{pmatrix} = \begin{pmatrix} |p^a\rangle_\alpha \\ |p^a]_{\dot{\alpha}} \end{pmatrix} \\
 \bar{u}_p^a &= \begin{pmatrix} -\lambda_p^{\alpha a} \\ \tilde{\lambda}_{p\dot{\alpha}}^a \end{pmatrix} = \begin{pmatrix} -\langle p^a |^\alpha \\ [p^a]_{\dot{\alpha}} \end{pmatrix}
 \end{aligned} \Rightarrow \begin{cases} (\not{p} - m) u_p^a = \bar{u}_p^a (\not{p} - m) = 0, \\ \bar{u}_p^a u_p^b = 2m \epsilon^{ab}, \\ \bar{u}_p^a \gamma^\mu u_p^b = 2p^\mu \epsilon^{ab}, \\ u_p^a \bar{u}_{pa} = u_p^a \epsilon_{ab} \bar{u}_p^b = \not{p} + m. \end{cases} \tag{2.17}$$

This choice of \bar{u}_p^a is consistent with the conjugation property $(u_p^a)^\dagger = \text{sgn}(p^0) \bar{u}_{pa} \gamma^0$. The energy sign here is due to our convention

$$m = \text{sgn}(p^0) \sqrt{p^2}, \tag{2.18}$$

which defines the signs of the energy and mass to be the same. As discussed in appendix B, this lets us cover the u - and v -spinors together simultaneously by eq. (2.17).

We can treat these spinors as quantum-mechanical wavefunctions and compute the expectation values of the spin operator $\vec{\Sigma}/2$, where $\Sigma^i \equiv i\epsilon^{ijk} \gamma_j \gamma_k / 2$. Given the spinor parametrization (2.16), we obtain the three-dimensional spin vector

$$\vec{s}(u_p^a) \equiv \frac{1}{2} \frac{u_p^{a\dagger} \vec{\Sigma} u_p^a}{u_p^{a\dagger} u_p^a} = \frac{\bar{u}_{pa} \vec{\gamma} \gamma^5 u_p^a}{2 \bar{u}_{pa} \gamma^0 u_p^a} = \frac{(-1)^{a-1}}{2} (\cos \varphi \sin \theta, \sin \varphi \sin \theta, \cos \theta) \equiv \frac{(-1)^{a-1}}{2} \hat{p}. \tag{2.19}$$

Therefore, the spinors (2.16) have definite $\pm 1/2$ helicities, i.e. the eigenvalues of the helicity operator $h = \hat{p} \cdot \vec{\Sigma}/2$, which is a conserved quantity for a one-particle state.

³We use the Dirac matrices in the Weyl basis, $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$, hence $\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\vec{\Sigma} = \gamma^0 \vec{\gamma} \gamma^5 = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$.

To delve into the subject of spin a bit further, we rewrite the massive spinor parametrization (2.16) as

$$\lambda_{p\alpha}^a = \sqrt{E+P} \begin{pmatrix} -e^{-i\varphi} \sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{pmatrix}_{\alpha} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}^a + \frac{m}{\sqrt{E+P}} \begin{pmatrix} \cos \frac{\theta}{2} \\ e^{i\varphi} \sin \frac{\theta}{2} \end{pmatrix}_{\alpha} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}^a, \quad (2.20a)$$

$$\tilde{\lambda}_p^{\dot{\alpha}a} = \sqrt{E+P} \begin{pmatrix} \cos \frac{\theta}{2} \\ e^{i\varphi} \sin \frac{\theta}{2} \end{pmatrix}^{\dot{\alpha}} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}^a + \frac{m}{\sqrt{E+P}} \begin{pmatrix} -e^{-i\varphi} \sin \frac{\theta}{2} \\ \cos \frac{\theta}{2} \end{pmatrix}^{\dot{\alpha}} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}^a, \quad (2.20b)$$

which makes obvious the smooth limit of the massive spinors $\lambda_{p\alpha}$ and $\tilde{\lambda}_p^{\dot{\alpha}}$ to their massless homonymes (2.11):

$$\lambda_{p\alpha}^a \xrightarrow{m \rightarrow 0} \lambda_{p\alpha} \zeta_-^a, \quad \tilde{\lambda}_p^{\dot{\alpha}a} \xrightarrow{m \rightarrow 0} \tilde{\lambda}_p^{\dot{\alpha}} \zeta_+^a, \quad (2.21)$$

where $\zeta_-^a \equiv (0, 1)$ and $\zeta_+^a \equiv (1, 0)$. To rephrase this in a more general way, we can introduce two-dimensional spinors $\lambda_{p\alpha}$ and $\eta_{p\alpha}$ such that $\lambda_{p\alpha}^a$ and $\tilde{\lambda}_p^{\dot{\alpha}a}$ decompose as

$$\lambda_{p\alpha}^a = \lambda_{p\alpha} \zeta_-^a + \eta_{p\alpha} \zeta_+^a, \quad \tilde{\lambda}_p^{\dot{\alpha}a} = \tilde{\lambda}_p^{\dot{\alpha}} \zeta_+^a - \tilde{\eta}_p^{\dot{\alpha}} \zeta_-^a, \quad \langle \lambda_p \eta_p \rangle = [\tilde{\eta}_p \tilde{\lambda}_p] = m. \quad (2.22)$$

The massive momentum is now expressed as a sum of two null momenta:

$$p_{\alpha\dot{\alpha}} = \lambda_{p\alpha} \tilde{\lambda}_p^{\dot{\alpha}} + \eta_{p\alpha} \tilde{\eta}_p^{\dot{\alpha}}, \quad (2.23)$$

which gives a link to the massive extension of the massless spinor-helicity formalism used previously in the literature [8–10]. We make this link precise in section 5 below.

Now let us discuss a subtle point concerning spin. Traditional quantum-mechanical spin operators are thought of as acting on the SU(2) indices, which correspond to the little group. The spin of the decomposition (2.16) points along the three-momentum \vec{p} , whereas the little-group vectors ζ_{\pm}^a seem to describe states with spin direction along the z -axis. In other words, the massive Weyl spinors (2.16) convert the physical helicity operator $h = \hat{p} \cdot \vec{\Sigma} / 2 = \begin{pmatrix} \hat{p} \cdot \vec{\sigma} & 0 \\ 0 & \hat{p} \cdot \vec{\sigma} \end{pmatrix} / 2$ to $\sigma^3 / 2$:

$$(\hat{p} \cdot \vec{\sigma})_{\alpha}^{\beta} \lambda_{p\beta}^a = \sigma^3{}^a{}_b \lambda_{p\alpha}^b, \quad (\hat{p} \cdot \vec{\sigma})^{\dot{\alpha}}{}_{\dot{\beta}} \tilde{\lambda}_p^{\dot{\beta}a} = \sigma^3{}^a{}_b \tilde{\lambda}_p^{\dot{\alpha}b}. \quad (2.24)$$

This should be regarded as a nice feature of the parametrization (2.16) rather than an inconsistency. Indeed, the little-group SU(2) transformations correspond to SO(3) rotations in the rest frame of the massive particle, in which $p_{\text{rest}}^{\mu} = (m, \vec{0})$, whereas the spinorial matrices $\vec{\sigma}_{\alpha}^{\beta} = \vec{\sigma}^{\dot{\alpha}}{}_{\dot{\beta}}$ generate rotations in the boosted frame where $p^{\mu} = (E, \vec{p})$. It is therefore convenient that the spinorial $(\hat{p} \cdot \vec{\sigma})$,⁴ taken along the momentum direction, are converted to the simplest of the Pauli matrices, $\sigma^3{}^a{}_b$.

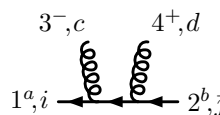
In principle, one can easily break the above property by SU(2)-rotating the spin states. Apart from losing the relatively simple parametrization (2.16), this would mix the pure

⁴In fact, the other two spatial directions corresponding to the little-group matrices $\sigma^1{}^a{}_b$ and $\sigma^2{}^a{}_b$ in the sense of eq. (2.24) turn out to be complex for any nonzero \vec{p} . The corresponding spin-projection operators are thus not hermitian, and there is no unitary intertwining operator between the two representations of the complete spin operator $\vec{\sigma} / 2$. Indeed, such an operator would have to involve a boost transformation to the rest frame, which lies outside the rotational SU(2).

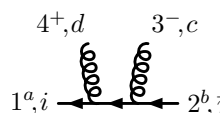
helicity eigenstates and produce wavefunctions with a spin quantization axis other than the momentum, and therefore undetermined helicity. The massive spinor-helicity formalism of ref. [11] reviewed here allows to easily switch that axis, and this is precisely what we do in section 5 in order to compare our results with the literature.

3 Four-point amplitudes

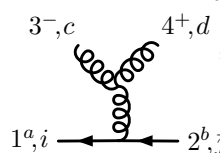
In this section, we demonstrate the use of the various spinors discussed above by dissecting one full color-dressed amplitude. It is convenient to consider the simple case of one massive quark-antiquark pair and two gluons of opposite helicity. Their scattering amplitude has three Feynman diagrams:⁵



$$= -\frac{i}{2} \frac{T_{ik}^c T_{kj}^d}{s_{13} - m^2} (\bar{u}_1^a \not{\epsilon}_3^- (\not{p}_{13} + m) \not{\epsilon}_4^+ u_2^b) \equiv \frac{c_1 n_1}{D_1}, \quad (3.1a)$$



$$= -\frac{i}{2} \frac{T_{ik}^d T_{kj}^c}{s_{14} - m^2} (\bar{u}_1^a \not{\epsilon}_4^+ (\not{p}_{14} + m) \not{\epsilon}_3^- u_2^b) \equiv \frac{c_2 n_2}{D_2}, \quad (3.1b)$$



$$= \frac{i}{2} \frac{\tilde{f}^{cde} T_{ij}^e}{s_{34}} \left\{ (\epsilon_3^- \cdot \epsilon_4^+) (\bar{u}_1^a (\not{p}_3 - \not{p}_4) u_2^b) + 2(p_4 \cdot \epsilon_3^-) (\bar{u}_1^a \not{\epsilon}_4^+ u_2^b) - 2(p_3 \cdot \epsilon_4^+) (\bar{u}_1^a \not{\epsilon}_3^- u_2^b) \right\} \equiv \frac{c_3 n_3}{D_3}. \quad (3.1c)$$

Here u_2^b denotes the Dirac spinor corresponding to negative energy and mass $m_2 = -m_1 = -m$. It is our energy sign convention (2.18) that allows us not to use a separate spinor v_2^b for u_2^b . Now let us recast the above numerators in the spinor-helicity formalism by plugging in the Dirac spinors (2.17) and the polarization vectors (2.7),

$$n_1 = \frac{-i}{[3q_3]\langle 4q_4 \rangle} \left\{ \langle \underline{1}^a 3 \rangle [q_3 | p_{13} | q_4 \rangle [4\bar{2}^b] - [\underline{1}^a q_3] \langle 3 | p_{13} | 4 \rangle \langle q_4 \bar{2}^b \rangle + m \langle \underline{1}^a 3 \rangle [q_3 4] \langle q_4 \bar{2}^b \rangle - m [\underline{1}^a q_3] \langle 3 q_4 \rangle [4\bar{2}^b] \right\}, \quad (3.2a)$$

$$n_2 = \frac{-i}{[3q_3]\langle 4q_4 \rangle} \left\{ \langle \underline{1}^a q_4 \rangle [4 | p_{14} | 3 \rangle [q_3 \bar{2}^b] - [\underline{1}^a 4] \langle q_4 | p_{14} | q_3 \rangle \langle 3 \bar{2}^b \rangle + m \langle \underline{1}^a q_4 \rangle [4 q_3] \langle 3 \bar{2}^b \rangle - m [\underline{1}^a 4] \langle q_4 3 \rangle [q_3 \bar{2}^b] \right\}, \quad (3.2b)$$

$$n_3 = \frac{-i}{[3q_3]\langle 4q_4 \rangle} \left\{ -\frac{1}{2} \langle 3q_4 \rangle [4q_3] (\langle \underline{1}^a | p_3 - p_4 | \bar{2}^b \rangle - [\underline{1}^a | p_3 - p_4 | \bar{2}^b \rangle) - \langle 3 | 4 | q_3 \rangle (\langle \underline{1}^a q_4 \rangle [4\bar{2}^b] - [\underline{1}^a 4] \langle q_4 \bar{2}^b \rangle) + \langle q_4 | 3 | 4 \rangle (\langle \underline{1}^a 3 \rangle [q_3 \bar{2}^b] - [\underline{1}^a q_3] \langle 3 \bar{2}^b \rangle) \right\}, \quad (3.2c)$$

where for brevity we label spinors as $|i\rangle \equiv |p_i\rangle$, etc. We also underline the massive positive-energy spinors and overline the negative-energy ones. The numerators (3.2) may seem

⁵We normalize the group generators to obey $\text{Tr}(T^a T^b) = \delta^{ab}$ and $[T^a, T^b] = \tilde{f}^{abc} T^c$ and regard all particle momenta as outgoing. We use slashed matrices \not{p} to denote either $\gamma^\mu p_\mu$, $\sigma^\mu p_\mu$ or $\bar{\sigma}^\mu p_\mu$, depending on the spinors surrounding them. In expressions like $\langle i | j | k \rangle \equiv \langle i | \not{p}_j | k \rangle = \langle i | p_j | k \rangle$ the slash can be omitted.

complicated, which is due to their explicit gauge dependence on the gluonic reference vectors q_3 and q_4 . Incidentally, one can check that for any such gauge choice they nontrivially satisfy the kinematic-algebra relation $n_1 - n_2 = n_3$, which is color-dual to the commutation relation $c_1 - c_2 = c_3$ [17, 18]. A very beneficial gauge choice is $q_3 = p_4$ and $q_4 = p_3$, for which

$$n_1 = n_2 = \frac{i}{s_{34}} \langle 3|1|4 \rangle (\langle \underline{1}^a 3 \rangle [\bar{2}^b 4] - [\underline{1}^a 4] \langle \bar{2}^b 3 \rangle), \quad n_3 = 0. \quad (3.3)$$

We can thus write simple closed-form expressions for all three color-ordered amplitudes

$$A(\underline{1}^a, \bar{2}^b, 3^-, 4^+) \equiv \frac{n_2}{D_2} - \frac{n_3}{D_3} = \frac{i \langle 3|1|4 \rangle}{(s_{14} - m^2) s_{34}} (\langle \underline{1}^a 3 \rangle [\bar{2}^b 4] - [\underline{1}^a 4] \langle \bar{2}^b 3 \rangle), \quad (3.4a)$$

$$A(\underline{1}^a, \bar{2}^b, 4^+, 3^-) \equiv \frac{n_1}{D_1} + \frac{n_3}{D_3} = \frac{i \langle 3|1|4 \rangle}{(s_{13} - m^2) s_{34}} (\langle \underline{1}^a 3 \rangle [\bar{2}^b 4] - [\underline{1}^a 4] \langle \bar{2}^b 3 \rangle), \quad (3.4b)$$

$$A(\underline{1}^a, 3^-, \bar{2}^b, 4^+) \equiv -\frac{n_1}{D_1} - \frac{n_2}{D_2} = \frac{i \langle 3|1|4 \rangle}{(s_{13} - m^2)(s_{14} - m^2)} (\langle \underline{1}^a 3 \rangle [\bar{2}^b 4] - [\underline{1}^a 4] \langle \bar{2}^b 3 \rangle). \quad (3.4c)$$

These evidently obey the Kleiss-Kuijf relation $A_{1243} + A_{1234} + A_{1324} = 0$ [19], as well as the Bern-Carrasco-Johansson (BCJ) relation [18, 20, 21]

$$(s_{14} - m^2) A(\underline{1}^a, \bar{2}^b, 3^-, 4^+) = (s_{13} - m^2) A(\underline{1}^a, \bar{2}^b, 4^+, 3^-). \quad (3.5)$$

The full color-dressed amplitude can thus be constructed from a single linearly independent color-ordered amplitude as [18, 22]

$$\mathcal{A}(1_i^a, 2_j^b, 3_c^-, 4_d^+) = i \left[\frac{T_{ik}^c T_{kj}^d}{(s_{13} - m^2) s_{34}} + \frac{T_{ik}^d T_{kj}^c}{(s_{14} - m^2) s_{34}} \right] \langle 3|1|4 \rangle (\langle \underline{1}^a 3 \rangle [\bar{2}^b 4] - [\underline{1}^a 4] \langle \bar{2}^b 3 \rangle). \quad (3.6)$$

It is interesting to note [23] that the gluonic color-ordered amplitude (3.4c) is also the correct QED amplitude [11] (up to a factor of -2 due to the color-generator conventions).

Note that the above amplitudes are gauge-invariant and could have been reduced from the numerators (3.2) to the expressions (3.4) for any choice of reference vectors q_3 and q_4 . This illustrates why in general, at least in analytic calculations, it is better to avoid dealing with gauge-dependent objects and compute gauge-invariant quantities directly. Such a way to derive the above amplitudes would be via the BCFW on-shell recursion [12, 13] starting from the three-point amplitudes

$$\left. \begin{aligned} \mathcal{A}(1_i^a, 2_j^b, 3_c^+) &= -\frac{i T_{ij}^c}{\langle 3q \rangle} (\langle \underline{1}^a q \rangle [\bar{2}^b 3] - [\underline{1}^a 3] \langle \bar{2}^b q \rangle) = i T_{ij}^c \frac{\langle \underline{1}^a \bar{2}^b \rangle [3|1|q]}{m \langle 3q \rangle} \\ \mathcal{A}(1_i^a, 2_j^b, 3_c^-) &= \frac{i T_{ij}^c}{[3q]} (\langle \underline{1}^a 3 \rangle [\bar{2}^b q] - [\underline{1}^a q] \langle \bar{2}^b 3 \rangle) = i T_{ij}^c \frac{[\underline{1}^a \bar{2}^b] \langle 3|1|q]}{m [3q]} \end{aligned} \right\} = \begin{array}{c} 3^{\pm, c} \\ \updownarrow \\ 1^a, i \quad \text{---} \quad 2^b, j \end{array} \quad (3.7)$$

These make sense on complex on-shell kinematics and are independent of the gauge-boson reference vector q , despite not looking that way (this feature is explained in ref. [11]).

To reduce the four-point amplitude to the three-point ones, we apply a simple massless-spinor shift

$$|\hat{3}] \equiv |3] - z|4], \quad |\hat{4}] \equiv |4] + z|3], \quad (3.8)$$

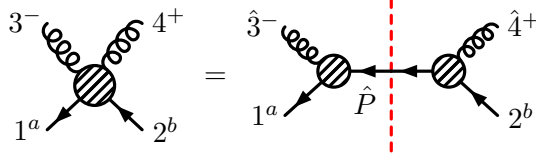


Figure 1. Graphic representation for the BCFW derivation (3.9) of $A(\underline{1}^a, \hat{3}^-, \hat{4}^+, \bar{2}^b)$.

which preserves momentum conservation and the on-shell conditions for any complex z . Cauchy's integral theorem then localizes the four-point amplitude $A(\underline{1}^a, \bar{2}^b, 4^+, 3^-)$ on the only pole $z_{13} = \langle 3|1|3\rangle/\langle 3|1|4\rangle$ corresponding to the quark propagator $\hat{P} \equiv p_1 + \hat{p}_3$, see figure 1. Hence the amplitude factorizes as a product of two three-point amplitudes with complex momenta:

$$\begin{aligned}
A(\underline{1}^a, \bar{2}^b, 4^+, 3^-) &= \text{Res}_{z=z_{13}} A(\underline{1}^a, \hat{3}^-, \hat{4}^+, \bar{2}^b) = A(\underline{1}^a, \hat{3}^-, -\hat{P}^c) \frac{1}{s_{13} - m^2} A(\hat{P}^c, \hat{4}^+, \bar{2}^b) \\
&= \frac{1}{(s_{13} - m^2) [\hat{3}q_3] \langle \hat{4}q_4 \rangle} (\langle \underline{1}^a 3 \rangle [-\hat{P}^c | q_3] - [\underline{1}^a q_3] \langle -\hat{P}^c 3 \rangle) (\langle \hat{P}^c q_4 \rangle [\bar{2}^b 4] - [\hat{P}^c 4] \langle \bar{2}^b q_4 \rangle) \\
&= \frac{-i}{(s_{13} - m^2) [34] \langle 43 \rangle} (\langle \underline{1}^a 3 \rangle [4\hat{P}^c] - [\underline{1}^a 4] \langle 3\hat{P}^c \rangle) (\langle \hat{P}^c 3 \rangle [\bar{2}^b 4] - [\hat{P}^c 4] \langle \bar{2}^b 3 \rangle) \\
&= \frac{-i}{(s_{13} - m^2) s_{34}} \left\{ \langle \underline{1}^a 3 \rangle [\bar{2}^b 4] [4\hat{P}^c] \langle \hat{P}^c 3 \rangle - \langle \underline{1}^a 3 \rangle \langle \bar{2}^b 3 \rangle [4\hat{P}^c] [\hat{P}^c 4] \right. \\
&\quad \left. - [\underline{1}^a 4] [\bar{2}^b 4] \langle 3\hat{P}^c \rangle \langle \hat{P}^c 3 \rangle + [\underline{1}^a 4] \langle \bar{2}^b 3 \rangle \langle 3\hat{P}^c \rangle [\hat{P}^c 4] \right\} \\
&= \frac{i \langle 3|1|4 \rangle}{(s_{13} - m^2) s_{34}} (\langle \underline{1}^a 3 \rangle [\bar{2}^b 4] - [\underline{1}^a 4] \langle \bar{2}^b 3 \rangle).
\end{aligned} \tag{3.9}$$

Here we chose the reference vectors as $q_3 = p_4$, $q_4 = p_3$ to remove most z -dependence as early as possible. Otherwise, the spinor products of q_3 and q_4 would cancel anyway, but only after plugging in the specific on-shell solutions for $|\hat{3}\rangle$, $|\hat{4}\rangle$ and \hat{P}^μ and using various Schouten identities. In the last transition of eq. (3.9), we also reduced the sum over the spin label c of the intermediate quark using the completeness relations (2.15).⁶

As a simple check, we verify that the massless limit corresponds to the well-known Parke-Taylor MHV amplitudes [24]:

$$\mathcal{A}(1_i^a, 2_j^b, 3_c^-, 4_d^+) \xrightarrow{m \rightarrow 0} -i \left[\frac{T_{ik}^c T_{kj}^d}{\langle 12 \rangle \langle 24 \rangle \langle 43 \rangle \langle 31 \rangle} + \frac{T_{ik}^d T_{kj}^c}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle} \right] \begin{pmatrix} 0 & \langle 13 \rangle \langle 23 \rangle^3 \\ \langle 13 \rangle^3 \langle 23 \rangle & 0 \end{pmatrix}. \tag{3.10}$$

It is even simpler calculation, either with the Feynman diagrams or via the on-shell recursion, to find the amplitudes with two quarks and two positive-helicity gluons

$$A_{1234} = \frac{-im \langle \underline{1}^a \bar{2}^b \rangle [34]}{(s_{14} - m^2) \langle 34 \rangle}, \quad A_{1243} = \frac{-im \langle \underline{1}^a \bar{2}^b \rangle [34]}{(s_{13} - m^2) \langle 34 \rangle}, \quad A_{1324} = \frac{im \langle \underline{1}^a \bar{2}^b \rangle [34]^2}{(s_{13} - m^2)(s_{14} - m^2)}. \tag{3.11}$$

⁶Due to our mass sign convention (2.18), the quark masses are $m_1 = m$ and $m_2 = -m$, which is important when using the identities (2.15). The mass of the intermediate momentum along the fermion arrow is m .

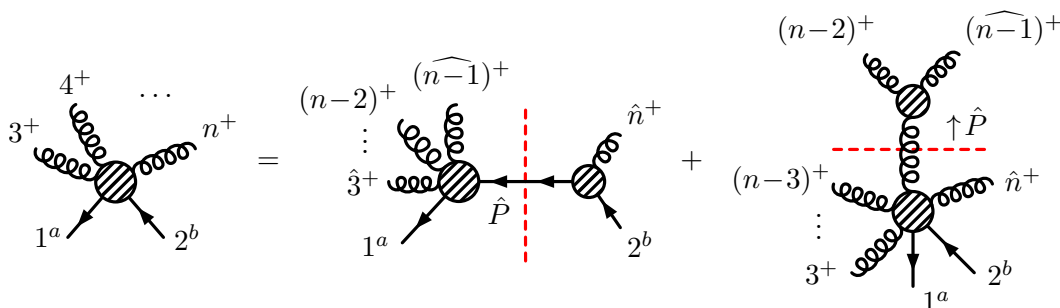


Figure 2. Graphic representation of the BCFW recursion step (4.3) for $A(\underline{1}^a, 3^+, 4^+, \dots, n^+, \bar{2}^b)$.

4 All-multiplicity amplitudes

In this section we turn to the main calculations of this note — two infinite families of color-ordered amplitudes with one massive quark-antiquark pair.

4.1 All-plus amplitudes with two quarks

The n -point amplitude for a quark-antiquark pair and $n - 2$ positive-helicity gluons equals

$$A(\underline{1}^a, 3^+, 4^+, \dots, n^+, \bar{2}^b) = \frac{-im \langle \underline{1}^a \bar{2}^b \rangle [3] \prod_{j=3}^{n-2} \{ \not{p}_{13\dots j} \not{p}_{j+1} + (s_{13\dots j} - m^2) \} |n]}{(s_{13} - m^2)(s_{134} - m^2) \dots (s_{13\dots(n-1)} - m^2) \langle 34 \rangle \langle 45 \rangle \dots \langle n-1 | n \rangle}. \quad (4.1)$$

It is easiest derived using the BCFW recursion [12, 13]. To set up the induction, we check that for $n = 4$ the formula (4.1) visibly reduces to the four-point amplitude A_{1243} in eq. (3.11). For the inductive step, we choose to shift the gluonic spinors

$$|\widehat{n-1}\rangle \equiv |n-1\rangle - z|n\rangle, \quad |\widehat{n}\rangle \equiv |n\rangle + z|n-1\rangle. \quad (4.2)$$

Then well-known arguments [14, 25, 26] guarantee a vanishing boundary behavior at $z \rightarrow \infty$. There are two potential contributions in the on-shell recursion:

$$A(\underline{1}^a, 3^+, 4^+, \dots, n^+, \bar{2}^b) = A(\underline{1}^a, 3^+, \dots, (n-2)^+, (\widehat{n-1})^+, -\hat{P}^c) \frac{1}{s_{2n} - m^2} A(\hat{P}^c, \hat{n}^+, \bar{2}^b) \\ + A((n-2)^+, (\widehat{n-1})^+, -\hat{P}^-) \frac{i}{s_{(n-2)(n-1)}} A(\underline{1}^a, 3^+, \dots, (n-3)^+, \hat{P}^+, \hat{n}^+, \bar{2}^b). \quad (4.3)$$

Any other z -dependent propagator would factorize on a vanishing purely gluonic amplitude with all positive helicities. The second pole in eq. (4.3) is localized on

$$\hat{s}_{(n-2)(n-1)} = 0 \quad \Rightarrow \quad z = \frac{[n-2|n-1]}{[n-2|n]} \quad \Rightarrow \quad |\widehat{n-1}\rangle = |n-2\rangle \frac{[n-1|n]}{[n-2|n]}, \quad (4.4)$$

which sets the three-point $\overline{\text{MHV}}$ amplitude to zero. We then compute the remaining contribution on $z_{2n} = -\langle n|2|n\rangle/\langle n-1|2|n\rangle$:

$$\begin{aligned} A(\underline{1}^a, 3^+, \dots, n^+, \bar{2}^b) &= \frac{\langle \underline{1}^a | -\hat{P}^c \rangle [3 | \prod_{j=3}^{n-3} \{ \not{p}_{13\dots j} \not{p}_{j+1} + (s_{13\dots j} - m^2) \} | \widehat{n-1} \rangle \langle \hat{P}^c \bar{2}^b \rangle [n | \hat{P} | q]}{(s_{13} - m^2) \dots (s_{13\dots(n-2)} - m^2) (s_{2n} - m^2) \langle 34 \rangle \dots \langle n-2 | n-1 \rangle \langle \hat{n} q \rangle} \\ &= -\frac{i m \langle \underline{1}^a \bar{2}^b \rangle [3 | \prod_{j=3}^{n-3} \{ \not{p}_{13\dots j} \not{p}_{j+1} + (s_{13\dots j} - m^2) \} | \widehat{n-1} \rangle \langle n-1 | 2 | n \rangle}{(s_{13} - m^2) \dots (s_{13\dots(n-1)} - m^2) \langle 34 \rangle \dots \langle n-2 | n-1 \rangle \langle n-1 | n \rangle}, \end{aligned} \quad (4.5)$$

where we picked $q = p_{n-1}$ to remove most of the z -dependence. To finish the proof of the closed-form expression (4.1), it now suffices to notice that

$$|\widehat{n-1}\rangle \langle n-1 | 2 | n \rangle = \not{p}_{(n-1)n} \not{p}_2 | n \rangle = \{ \not{p}_{13\dots(n-2)} \not{p}_{n-1} + (s_{13\dots(n-2)} - m^2) \} | n \rangle. \quad (4.6)$$

It is effortless to also write the amplitude for the gluons of negative helicity: we simply exchange the angle and square brackets as in $\langle pq \rangle \leftrightarrow [qp]$ to obtain

$$A(\underline{1}^a, 3^-, 4^-, \dots, n^-, \bar{2}^b) = \frac{(-1)^n i m [\underline{1}^a \bar{2}^b] \langle 3 | \prod_{j=3}^{n-2} \{ \not{p}_{13\dots j} \not{p}_{j+1} + (s_{13\dots j} - m^2) \} | n \rangle}{(s_{13} - m^2) (s_{134} - m^2) \dots (s_{13\dots(n-1)} - m^2) [34][45] \dots [n-1|n]}. \quad (4.7)$$

4.2 One-minus amplitudes with two quarks

In this section we again use the on-shell recursion to derive an all-multiplicity expression

$$\begin{aligned} A(\underline{1}^a, 3^-, 4^+, \dots, n^+, \bar{2}^b) &= -\frac{i \langle 3 | 1 | 2 | 3 \rangle (\langle \underline{1}^a 3 \rangle [\bar{2}^b | 1 + 2 | 3] - \langle \bar{2}^b 3 \rangle [\underline{1}^a | 1 + 2 | 3])}{s_{12} \langle 34 \rangle \dots \langle n-1 | n \rangle \langle 3 | 1 | 2 | n \rangle} \\ &\quad - \sum_{k=4}^{n-1} \frac{i m \langle 3 | \not{p}_1 \not{p}_{3\dots k} | 3 \rangle (\langle \underline{1}^a \bar{2}^b \rangle \langle 3 | \not{p}_1 \not{p}_{3\dots k} | 3 \rangle + \langle \underline{1}^a 3 \rangle \langle \bar{2}^b 3 \rangle s_{3\dots k})}{s_{3\dots k} (s_{13\dots k} - m^2) \dots (s_{13\dots(n-1)} - m^2) \langle 34 \rangle \dots \langle k-1 | k \rangle \langle 3 | \not{p}_1 \not{p}_{3\dots k} | k \rangle} \\ &\quad \times \frac{\langle 3 | \not{p}_{3\dots k} \prod_{j=k}^{n-2} \{ \not{p}_{13\dots j} \not{p}_{j+1} + (s_{13\dots j} - m^2) \} | n \rangle}{\langle 3 | \not{p}_1 \not{p}_{3\dots k} | k+1 \rangle \langle k+1 | k+2 \rangle \dots \langle n-1 | n \rangle} \end{aligned} \quad (4.8)$$

for the amplitude with two quarks and $n-2$ gluons. Here we assume negative helicity of the gluon 3 color-adjacent to the quark $\underline{1}$, with all other gluon helicities positive, while the quark helicities are still left arbitrary. In principle, this color-ordered amplitude is enough to reconstruct the full color-dressed one-minus amplitude via the BCJ relations [18, 20], such as the four-point one in eq. (3.5). Indeed, the BCJ relations allow to fix the position of any gluon to be color-adjacent to the quark, with permutations acting on the remaining gluons, hence one may choose to fix the position of the minus-helicity gluon. Moreover, one-plus amplitudes can also be retrieved from eq. (4.8) via the conjugation rule $\langle pq \rangle \leftrightarrow [qp]$.

To prove the above formula, we use the same “[34]” shift as in eq. (3.8), which gives only two non-vanishing contributions at each step of the recursion,

$$A(\underline{1}^a, 3^-, 4^+, \dots, n^+, \bar{2}^b) = A(\underline{1}^a, \hat{3}^-, -\hat{P}^c) \frac{1}{s_{13} - m^2} A(\hat{P}^c, \hat{4}^+, 5^+, \dots, \hat{n}^+, \bar{2}^b) \quad (4.9a)$$

$$+ A(\underline{1}^a, \hat{3}^-, \hat{P}^+, 6^+, \dots, n^+, \bar{2}^b) \frac{i}{s_{45}} A(-\hat{P}^-, \hat{4}^+, \hat{5}^+). \quad (4.9b)$$

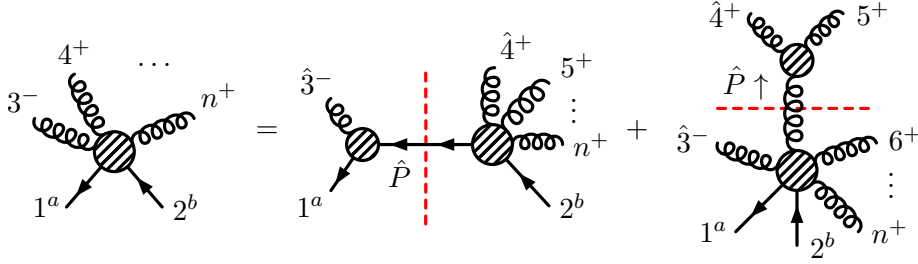


Figure 3. Graphic representation of the BCFW recursion step (4.9) for $A(\underline{1}^a, 3^-, 4^+, \dots, n^+, \bar{2}^b)$.

The two residues are evaluated on the following pole kinematics:

$$z_{13} = \frac{\langle 3|1|3\rangle}{\langle 3|1|4\rangle}: \quad |\hat{3}\rangle = -\frac{|1\rangle\langle 3|34\rangle}{\langle 3|1|4\rangle}, \quad |\hat{4}\rangle = \frac{|3+4\rangle\langle 1|3\rangle}{\langle 3|1|4\rangle}, \quad \hat{P}^\mu = p_1^\mu - \frac{\langle 3|\sigma^\mu|1|3|4\rangle}{2\langle 3|1|4\rangle}; \quad (4.10a)$$

$$z_{45} = -\frac{\langle 45\rangle}{\langle 35\rangle}: \quad |\hat{3}\rangle = \frac{|3+4\rangle\langle 5\rangle}{\langle 35\rangle}, \quad |\hat{4}\rangle = |5\rangle \frac{\langle 34\rangle}{\langle 35\rangle}, \quad |\hat{P}\rangle = |5\rangle, \quad |\hat{P}] = \frac{|4+5\rangle\langle 3\rangle}{\langle 53\rangle}. \quad (4.10b)$$

The residue at z_{13} is computed immediately for any n using the all-plus expression (4.1),

$$\begin{aligned} & A(\underline{1}^a, \hat{3}^-, -\hat{P}^c) \frac{1}{s_{13} - m^2} A(\hat{P}^c, \hat{4}^+, 5^+, \dots, \hat{n}^+, \bar{2}^b) \\ &= \frac{[\underline{1}^a | -\hat{P}^c] \langle \hat{P}^c \bar{2}^b \rangle \langle 3|1|q\rangle [4 | \prod_{j=4}^{n-2} \{ \not{p}_{P4\dots j} \not{p}_{j+1} + (\hat{s}_{P4\dots j} - m^2) \} |n]}{(s_{13} - m^2)(\hat{s}_{P4} - m^2) \dots (\hat{s}_{P4\dots(n-1)} - m^2) [\hat{3}q] \langle 45 \rangle \langle 56 \rangle \dots \langle n-1|n\rangle} \\ &= \frac{im \langle 3|1|4\rangle [4 | \prod_{j=4}^{n-2} \{ \not{p}_{13\dots j} \not{p}_{j+1} + (s_{13\dots j} - m^2) \} |n] (\langle \underline{1}^a \bar{2}^b \rangle \langle 3|1|4\rangle + \langle \underline{1}^a 3 \rangle \langle \bar{2}^b 3 \rangle [34])}{(s_{13} - m^2)(s_{134} - m^2) \dots (s_{13\dots(n-1)} - m^2) [34] \langle 3|1|3+4|5\rangle \langle 56 \rangle \dots \langle n-1|n\rangle}. \end{aligned} \quad (4.11)$$

Here the q -dependent factors explicitly canceled after using eq. (4.10a), and we also simplified

$$[\underline{1}^a | -\hat{P}^c] \langle \hat{P}^c \bar{2}^b \rangle = -i[\underline{1}^a | \hat{P} | \bar{2}^b \rangle = \frac{im}{\langle 3|1|4\rangle} (\langle \underline{1}^a \bar{2}^b \rangle \langle 3|1|4\rangle + \langle \underline{1}^a 3 \rangle \langle \bar{2}^b 3 \rangle [34]). \quad (4.12)$$

The contribution (4.11) from z_{13} in fact coincides with the $k = 4$ term of the $\sum_{k=4}^{n-1}$ sum in the full formula (4.8). To see that, we only need to rewrite

$$\begin{aligned} & \frac{\langle 3|\not{p}_1 \not{p}_{34}|3\rangle}{s_{34} \langle 34 \rangle \langle 3|\not{p}_1 \not{p}_{34}|4\rangle} (\langle \underline{1}^a \bar{2}^b \rangle \langle 3|\not{p}_1 \not{p}_{34}|3\rangle + \langle \underline{1}^a 3 \rangle \langle \bar{2}^b 3 \rangle s_{34}) \langle 3|\not{p}_{34} \\ &= -\frac{\langle 3|1|4\rangle}{(s_{13} - m^2)[34]} (\langle \underline{1}^a \bar{2}^b \rangle \langle 3|1|4\rangle + \langle \underline{1}^a 3 \rangle \langle \bar{2}^b 3 \rangle [34]) [4]. \end{aligned} \quad (4.13)$$

Now we turn to the residue (4.9b) at z_{45} . First of all, we observe that the right-hand three-gluon amplitude is invariably

$$\frac{i}{s_{45}} A(-\hat{P}^-, \hat{4}^+, \hat{5}^+) = \frac{i}{s_{45}} \frac{-i[45]^4}{[-\hat{P}|4][45][5|-\hat{P}]} = -\frac{[45]^2}{\langle 45 \rangle [4\hat{P}][5\hat{P}]} = \frac{\langle 35 \rangle}{\langle 34 \rangle \langle 45 \rangle}. \quad (4.14)$$

Furthermore, in the left-hand amplitude of eq. (4.9b), $A(\underline{1}^a, \hat{3}^-, \hat{P}^+, 6^+, \dots, n^+, \bar{2}^b)$, most momentum sums (and their squares) are simply shifted:

$$p_{3\dots(k-1)} \rightarrow \hat{p}_{3P6\dots k} = \hat{p}_3 + \hat{p}_4 + p_5 + p_6 + \dots + p_k = p_{3\dots k}, \quad s_{3\dots(k-1)} \rightarrow s_{3\dots k}. \quad (4.15)$$

We can then compute the residue at z_{45} as

$$\begin{aligned}
 & A(\underline{1}^a, \hat{3}^-, \hat{P}^+, 6^+, \dots, n^+, \bar{2}^b) \frac{i}{s_{45}} A(-\hat{P}^-, \hat{4}^+, \hat{5}^+) \\
 &= \left\{ - \frac{i \langle 3|1|2|3 \rangle (\langle \underline{1}^a 3 \rangle [\bar{2}^b |1+2|3] - \langle \bar{2}^b 3 \rangle [\underline{1}^a |1+2|3])}{s_{12} \langle 3\hat{P} \rangle \langle \hat{P}6 \rangle \langle 67 \rangle \dots \langle n-1|n \rangle \langle 3|1|2|n \rangle} \right. \\
 &\quad - \frac{i m \langle 3|\not{p}_1 \not{p}_{345}|3 \rangle (\langle \underline{1}^a \bar{2}^b \rangle \langle 3|\not{p}_1 \not{p}_{345}|3 \rangle + \langle \underline{1}^a 3 \rangle \langle \bar{2}^b 3 \rangle s_{345})}{s_{345} (s_{1345} - m^2) \dots (s_{13\dots(n-1)} - m^2) \langle 3\hat{P} \rangle \langle 3|\not{p}_1 \not{p}_{345}|\hat{P} \rangle} \\
 &\quad \times \frac{\langle 3|\not{p}_{345} \{ \not{p}_{1345} \not{p}_6 + (s_{1345} - m^2) \} \prod_{j=6}^{n-2} \{ \not{p}_{13\dots j} \not{p}_{j+1} + (s_{13\dots j} - m^2) \} |n \rangle}{\langle 3|\not{p}_1 \not{p}_{345}|6 \rangle \langle 67 \rangle \dots \langle n-1|n \rangle} \\
 &\quad - \sum_{k=6}^{n-1} \frac{i m \langle 3|\not{p}_1 \not{p}_{3\dots k}|3 \rangle (\langle \underline{1}^a \bar{2}^b \rangle \langle 3|\not{p}_1 \not{p}_{3\dots k}|3 \rangle + \langle \underline{1}^a 3 \rangle \langle \bar{2}^b 3 \rangle s_{3\dots k})}{s_{3\dots k} (s_{13\dots k} - m^2) \dots (s_{13\dots(n-1)} - m^2) \langle 3\hat{P} \rangle \langle \hat{P}6 \rangle \langle 67 \rangle \dots \langle k-1|k \rangle \langle 3|\not{p}_1 \not{p}_{3\dots k}|k \rangle} \\
 &\quad \times \left. \frac{\langle 3|\not{p}_{3\dots k} \prod_{j=k}^{n-2} \{ \not{p}_{13\dots j} \not{p}_{j+1} + (s_{13\dots j} - m^2) \} |n \rangle}{\langle 3|\not{p}_1 \not{p}_{3\dots k}|k+1 \rangle \langle k+1|k+2 \rangle \dots \langle n-1|n \rangle} \right\} \frac{\langle 35 \rangle}{\langle 34 \rangle \langle 45 \rangle}
 \end{aligned} \tag{4.16a}$$

$$\begin{aligned}
 &= - \frac{i \langle 3|1|2|3 \rangle (\langle \underline{1}^a 3 \rangle [\bar{2}^b |1+2|3] - \langle \bar{2}^b 3 \rangle [\underline{1}^a |1+2|3])}{s_{12} \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 67 \rangle \dots \langle n-1|n \rangle \langle 3|1|2|n \rangle} \\
 &\quad - \sum_{k=5}^{n-1} \frac{i m \langle 3|\not{p}_1 \not{p}_{3\dots k}|3 \rangle (\langle \underline{1}^a \bar{2}^b \rangle \langle 3|\not{p}_1 \not{p}_{3\dots k}|3 \rangle + \langle \underline{1}^a 3 \rangle \langle \bar{2}^b 3 \rangle s_{3\dots k})}{s_{3\dots k} (s_{13\dots k} - m^2) \dots (s_{13\dots(n-1)} - m^2) \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 67 \rangle \dots \langle k-1|k \rangle \langle 3|\not{p}_1 \not{p}_{3\dots k}|k \rangle} \\
 &\quad \times \frac{\langle 3|\not{p}_{3\dots k} \prod_{j=k}^{n-2} \{ \not{p}_{13\dots j} \not{p}_{j+1} + (s_{13\dots j} - m^2) \} |n \rangle}{\langle 3|\not{p}_1 \not{p}_{3\dots k}|k+1 \rangle \langle k+1|k+2 \rangle \dots \langle n-1|n \rangle}.
 \end{aligned} \tag{4.16b}$$

Here we were able to integrate the second term in the bracket (4.16a), which corresponds to $k = \hat{P}$, into the $\sum_{k=6}^{n-1}$ sum as that for $k = 5$. Since the $k = 4$ term, missing from eq. (4.16), is provided by the residue at z_{13} , this concludes the proof of the formula (4.8).

5 Checks

As the first simple check of our all-multiplicity formulae (4.1) and (4.8), we evaluate their massless limits. The former explicitly vanishes, as it should, whereas the latter reduces to the massless MHV amplitudes with two quarks:

$$A(\underline{1}^a, 3^-, 4^+, \dots, n^+, \bar{2}^b) \xrightarrow{m \rightarrow 0} \frac{-i}{\langle 13 \rangle \langle 34 \rangle \langle 45 \rangle \dots \langle n-1|n \rangle \langle n|2 \rangle \langle 21 \rangle} \begin{pmatrix} 0 & \langle 13 \rangle \langle 23 \rangle^3 \\ \langle 13 \rangle^3 \langle 23 \rangle & 0 \end{pmatrix}. \tag{5.1}$$

This analytic check, however, is only sensitive to a single term in eq. (4.8) that is not multiplied by the mass. As another partial check, we happened to have a six-point Feynman-diagrammatic calculation at easy access, with which we found numerical agreement to ten significant digits for both helicity configurations. Needless to say, the Feynman diagrams were much lengthier before evaluation than the three-term amplitude generated by the formula (4.8). The all-plus formula (4.1) can also be independently verified via the

fundamental BCJ relation [18, 20, 21, 27]

$$\sum_{i=2}^{n-1} (p_{13\dots i} \cdot p_n) A(\underline{1}^a, \underline{3}^+, \dots, i^+, n^+, (i+1)^+, \dots, (n-1)^+, \underline{2}^b) = 0, \quad (5.2)$$

which is non-trivially satisfied by a linear combination of its permutations. We checked it numerically to ten significant digits for $n = 4, 5, \dots, 12$. Let us now turn to even more stringent checks.

In ref. [14] Schwinn and Weinzierl used a massive extension [8–10] of the massless spinor-helicity formalism to compute QCD amplitudes with the same gluon polarizations that we compute here. In that formalism, the massive spinors are introduced by expanding the massive momentum p_μ in terms of its massless “flat” version p_μ^b and another null vector q_μ :

$$p_\mu = p_\mu^b + \frac{m^2}{2(p \cdot q)} q_\mu, \quad u_p^+(q) = \frac{(\not{p} + m)|q\rangle}{\langle p^b q \rangle}, \quad u_p^-(q) = \frac{(\not{p} + m)|q\rangle}{[p^b q]}. \quad (5.3)$$

The reference momentum q_μ determines the spin quantization axis, since the resulting spin vector is $\vec{s}^\pm = \pm \{\hat{p}/2 - m^2 \vec{q}/4|\vec{p}|p \cdot q\}$. This shows that unless $m \neq 0$ the quark labels \pm in the spinors (5.3) are not helicities but rather general spin labels.

The versatility of the massive spinor-helicity formalism [11] reviewed in section 2 lets us effortlessly pick any quantization axis in the sense of eq. (5.3). We simply need to switch from the helicity parametrization (2.16) to

$$\begin{aligned} u_p^{a=1} &= \begin{pmatrix} |p^1\rangle \equiv \frac{m|q\rangle}{\langle p^b q \rangle} \\ |p^1\rangle \equiv |p^b\rangle \end{pmatrix} = u_p^+(q), & \bar{u}_{p,a=1} &= \begin{pmatrix} \langle p^2| \equiv \langle p^b| \\ -[p^2| \equiv \frac{m[q|}{[qp^b]} \end{pmatrix} = \bar{u}_p^+(q), \\ u_p^{a=2} &= \begin{pmatrix} |p^2\rangle \equiv |p^b\rangle \\ |p^2\rangle \equiv \frac{m|q\rangle}{[p^b q]} \end{pmatrix} = u_p^-(q), & \bar{u}_{p,a=2} &= \begin{pmatrix} -\langle p^1| \equiv \frac{m\langle q|}{\langle qp^b\rangle} \\ [p^1| \equiv [p^b| \end{pmatrix} = \bar{u}_p^-(q). \end{aligned} \quad (5.4)$$

Now if we take the same reference vector q^μ for both quarks, this dictionary gives us

$$\langle \underline{1}^a \underline{2}^b \rangle = \begin{pmatrix} 0 & -m \frac{\langle q 2^b \rangle}{\langle q 1^b \rangle} \\ -m \frac{\langle 1^b q \rangle}{\langle 2^b q \rangle} & \langle 1^b 2^b \rangle \end{pmatrix}, \quad \langle \underline{1}^a \underline{3} \rangle \langle \underline{2}^b \underline{3} \rangle = \begin{pmatrix} 0 & -\frac{m\langle q 3 \rangle}{\langle q 1^b \rangle} \langle 2^b 3 \rangle \\ \langle 1^b 3 \rangle \frac{m\langle q 3 \rangle}{\langle q 2^b \rangle} & \langle 1^b 3 \rangle \langle 2^b 3 \rangle \end{pmatrix}, \quad (5.5)$$

where the signs follow from our convention $m_2 = -m_1 = -m$.

Ref. [14] actually sets q^μ to the momentum of the minus-helicity gluon 3. This allowed for BCFW shifts involving this pair of massive and massless momenta, and thus set up a recursion to compute the amplitudes in question. Let us translate the results of ref. [14]

to the conventions of the present paper:

$$A(\underline{1}^1, 3^-, 4^+, \dots, n^+, \bar{2}^1) = 0, \quad (5.6a)$$

$$A(\underline{1}^1, 3^-, 4^+, \dots, n^+, \bar{2}^2) = \frac{i\langle 2^b 3 \rangle}{\langle 1^b 3 \rangle \langle 34 \rangle \dots \langle n-1 | n \rangle} \sum_{k=4}^n \frac{\langle 3 | \not{p}_1 \not{p}_{3\dots k} | 3 \rangle^2}{s_{3\dots k} \langle 3 | \not{p}_1 \not{p}_{3\dots k} | k \rangle} \\ \times \left\{ \delta_{k=n} + \delta_{k \neq n} \frac{m^2 \langle k | k+1 \rangle \langle 3 | \not{p}_{3\dots k} \prod_{j=k+1}^{n-1} \{ (s_{13\dots j} - m^2) - \not{p}_j \not{p}_{13\dots j} \} | n \rangle}{(s_{13\dots k} - m^2) \dots (s_{13\dots(n-1)} - m^2) \langle 3 | \not{p}_1 \not{p}_{3\dots k} | k+1 \rangle} \right\}, \quad (5.6b)$$

$$A(\underline{1}^2, 3^-, 4^+, \dots, n^+, \bar{2}^1) = \frac{i\langle 1^b 3 \rangle}{\langle 2^b 3 \rangle \langle 34 \rangle \dots \langle n-1 | n \rangle} \sum_{k=4}^n \frac{\langle 3 | \not{p}_1 \not{p}_{3\dots k} | 3 \rangle^2}{s_{3\dots k} \langle 3 | \not{p}_1 \not{p}_{3\dots k} | k \rangle} \\ \times \left\{ \delta_{k=n} + \delta_{k \neq n} \frac{m^2 \langle k | k+1 \rangle \langle 3 | \not{p}_{3\dots k} \prod_{j=k+1}^{n-1} \{ (s_{13\dots j} - m^2) - \not{p}_j \not{p}_{13\dots j} \} | n \rangle}{(s_{13\dots k} - m^2) \dots (s_{13\dots(n-1)} - m^2) \langle 3 | \not{p}_1 \not{p}_{3\dots k} | k+1 \rangle} \right\}, \quad (5.6c)$$

$$A(\underline{1}^2, 3^-, 4^+, \dots, n^+, \bar{2}^2) = \frac{-i\langle 1^b 2^b \rangle}{m \langle 34 \rangle \dots \langle n-1 | n \rangle} \sum_{k=4}^n \frac{\langle 3 | \not{p}_1 \not{p}_{3\dots k} | 3 \rangle^2}{s_{3\dots k} \langle 3 | \not{p}_1 \not{p}_{3\dots k} | k \rangle} \left[1 + \frac{s_{3\dots k} \langle 3 2^b \rangle}{\langle 3 | \not{p}_{3\dots k} \not{p}_1 | 2^b \rangle} \right] \\ \times \left\{ \delta_{k=n} + \delta_{k \neq n} \frac{m^2 \langle k | k+1 \rangle \langle 3 | \not{p}_{3\dots k} \prod_{j=k+1}^{n-1} \{ (s_{13\dots j} - m^2) - \not{p}_j \not{p}_{13\dots j} \} | n \rangle}{(s_{13\dots k} - m^2) \dots (s_{13\dots(n-1)} - m^2) \langle 3 | \not{p}_1 \not{p}_{3\dots k} | k+1 \rangle} \right\}. \quad (5.6d)$$

To make a direct comparison easier, here we rewrite our result (4.8) as

$$A(\underline{1}^a, 3^-, 4^+, \dots, n^+, \bar{2}^b) \\ = \frac{-i}{\langle 34 \rangle \dots \langle n-1 | n \rangle} \left\{ \frac{\langle 3 | \not{p}_1 \not{p}_{3\dots n} | 3 \rangle}{m s_{3\dots n} \langle 3 | \not{p}_1 \not{p}_{3\dots k} | n \rangle} (\langle \underline{1}^a \bar{2}^b \rangle \langle 3 | \not{p}_1 \not{p}_{3\dots n} | 3 \rangle + \langle \underline{1}^a 3 \rangle \langle \bar{2}^b 3 \rangle s_{3\dots n}) \right. \\ \left. + m \sum_{k=4}^{n-1} \frac{\langle k | k+1 \rangle \langle 3 | \not{p}_1 \not{p}_{3\dots k} | 3 \rangle \langle 3 | \not{p}_{3\dots k} \prod_{j=k+1}^{n-1} \{ (s_{13\dots(j+1)} - m^2) - \not{p}_{j+1} \not{p}_{13\dots(j+1)} \} | n \rangle}{s_{3\dots k} (s_{13\dots k} - m^2) \dots (s_{13\dots(n-1)} - m^2) \langle 3 | \not{p}_1 \not{p}_{3\dots k} | k \rangle \langle 3 | \not{p}_1 \not{p}_{3\dots k} | k+1 \rangle} \right. \\ \left. \times (\langle \underline{1}^a \bar{2}^b \rangle \langle 3 | \not{p}_1 \not{p}_{3\dots k} | 3 \rangle + \langle \underline{1}^a 3 \rangle \langle \bar{2}^b 3 \rangle s_{3\dots k}) \right\}, \quad (5.7)$$

where we massaged the product $\prod_{j=k}^{n-2}$ into the form of eq. (5.6) using the anticommutator identity

$$\not{p}_{13\dots j} \not{p}_{j+1} + (s_{13\dots j} - m^2) = (s_{13\dots(j+1)} - m^2) - \not{p}_{j+1} \not{p}_{13\dots j}. \quad (5.8)$$

Now we can see with a naked eye that our formula (5.7) exactly reproduces eqs. (5.6a), (5.6b) and (5.6c), where all the terms containing $\langle \underline{1}^a 3 \rangle \langle \bar{2}^b 3 \rangle$ vanish due to the choice $q = p_3$ for both quarks. To match the last amplitude (5.6d), for which $|\underline{1}^{a=2}\rangle = |1^b\rangle$, $|\bar{2}^{b=2}\rangle = |2^b\rangle$, we observe that

$$\langle 1^b 2^b \rangle \langle 3 | \not{p}_1 \not{p}_{3\dots k} | 3 \rangle + \langle 1^b 3 \rangle \langle 2^b 3 \rangle s_{3\dots k} = \langle 1^b 2^b \rangle \langle 3 | \not{p}_1 \not{p}_{3\dots k} | 3 \rangle \left[1 + \frac{s_{3\dots k} \langle 3 2^b \rangle}{\langle 3 | \not{p}_{3\dots k} \not{p}_1 | 2^b \rangle} \right]. \quad (5.9)$$

To conclude, we note that ref. [14] also computed the analogue of the all-plus amplitude in the massless spinor-helicity formalism (5.3) through its relation to the massive scalar amplitude of refs. [28, 29] via a supersymmetric Ward identity [30–32] with an unfixed reference vector q^μ . The same eqs. (5.5) and (5.8) allow to easily verify that these results are incorporated in our formula (4.1).

6 Summary and discussion

In this note we have computed two infinite families of tree-level amplitudes with two quarks of arbitrary spin and any number of gluons with specified helicities. For that we have used the new massive spinor-helicity formalism of ref. [11]. In order to check the consistency of our results with the literature, we have also established straightforward transition rules between our approach and the more traditional ones.

We hope to have demonstrated that the new massive formalism is a analytic tool well-suited for QCD computations. It is a logical extension of the massless spinor-helicity formalism [1–6, 11], which in the last decades has become indispensable for scattering-amplitude calculations. Of course, the scope of the formalism is much more general than QCD, as shown by the recent applications to gravitational scattering [33, 34] and the Standard Model as a whole [11, 35]. It can be used streamline the consideration of all unitarity-compliant three-point [35–37] and four-point [11] interactions. It can also be related to much earlier off-shell reformulations of QED [38, 39] and other theories [40–42] using two-component spinor fields.

The presented formalism has potential to facilitate many QCD calculations, both analytically and numerically. Through its analytic simplicity, it may provide a way to explicit expressions for tree amplitudes with more general gluon helicity configurations [43–46] and more quark-antiquark lines [17, 18], as already achieved [47, 48] for the massless QCD amplitudes with up to three quark-antiquark pairs. For example, it would be interesting to find an analytic expression even for an amplitude with two quarks and one negative-helicity gluon in an arbitrary position, provided that it is more compact than its BCJ relation [18, 20] that involves various permutations of the formula computed in this note.

New loop amplitudes could also be calculated using the presented formalism. Indeed, loops can be obtained from generalized unitarity cuts [49–55] that are constructed from tree amplitudes.⁷ It would also be interesting to investigate, in the spirit of refs. [58, 59], if the massive on-shell formalism could speed up numerical evaluation of tree-level amplitudes. This would be beneficial for computing real-emission radiative QCD corrections to a vast array of elementary-particle scattering processes.

Acknowledgments

The author would like to thank Nikolay Gulitskiy and Achilleas Lazopoulos for their valuable comments on the manuscript. This project has received funding from the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement 746138.

⁷A relevant example is the recent calculation [56] of one-loop amplitudes with external massive fermions via tree amplitudes in six dimensions. One extra dimension was needed to parametrize the loop dependence in dimensional regularization, and another one to account for the quark mass. Since the massive four-dimensional spinor helicity can be viewed as massless but five-dimensional, the amplitudes considered in this note could be embedded into a higher-dimensional formalism, such as the six-dimensional one of ref. [57].

A Massless spinor parametrizations

For this note to be more self-contained, we give a realization of the massless spinors [1–6]. In terms of the light-cone momentum components

$$p_+ = p^0 + p^3, \quad p_- = p^0 - p^3, \quad p_\perp = p^1 + ip^2, \quad \bar{p}_\perp = p^1 - ip^2, \quad (\text{A.1})$$

satisfying $p_+ p_- = p_\perp \bar{p}_\perp$, an explicit solution for the Weyl spinors is (adapted from ref. [60])

$$\lambda_{p\alpha} = \begin{pmatrix} -\bar{p}_\perp/\sqrt{p_+} \\ \sqrt{p_+} \end{pmatrix}, \quad \tilde{\lambda}_{p\dot{\alpha}} = \begin{pmatrix} -p_\perp/\sqrt{p_+} \\ \sqrt{p_+} \end{pmatrix}, \quad (\text{A.2})$$

which is a rewrite of eq. (2.11). If p_+ happens to vanish, an equivalent solution may be used:

$$\lambda_{p\alpha} = \begin{pmatrix} -\sqrt{p_-} \\ p_\perp/\sqrt{p_-} \end{pmatrix}, \quad \tilde{\lambda}_{p\dot{\alpha}} = \begin{pmatrix} -\sqrt{p_-} \\ \bar{p}_\perp/\sqrt{p_-} \end{pmatrix}. \quad (\text{A.3})$$

In the complex-valued case where $p_\pm = 0$ and the momentum equals $(0, p^1, \pm ip^1, 0)$, a valid choice is

$$\lambda_{p\alpha} = \frac{1}{\sqrt{2p^1}} \begin{pmatrix} -\bar{p}_\perp \\ p_\perp \end{pmatrix}, \quad \tilde{\lambda}_{p\dot{\alpha}} = \frac{1}{\sqrt{2p^1}} \begin{pmatrix} -p_\perp \\ \bar{p}_\perp \end{pmatrix}. \quad (\text{A.4})$$

Note that these definitions give $(\lambda_{p\alpha})^* = \tilde{\lambda}_{p\dot{\alpha}}$ for a real-valued momentum p^μ . Moreover, for the principal square roots they are also consistent with the momentum inversion rule

$$\lambda_{-p} = \text{sgn}(p^0) i \lambda_p, \quad \tilde{\lambda}_{-p} = \text{sgn}(p^0) i \tilde{\lambda}_p. \quad (\text{A.5})$$

B Massive spinor parametrizations

Here we give the massive spinor-helicity variables that are consistent with the parametrizations (A.2) through (A.4) in the massless limit. The spinors (2.20) can be rewritten as

$$\lambda_{p\alpha}^a = \sqrt{\frac{E+P}{2P}} \begin{pmatrix} -\bar{p}_\perp/\sqrt{p_+} \\ \sqrt{p_+} \end{pmatrix}_\alpha \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}^a + \sqrt{\frac{E-P}{2P}} \begin{pmatrix} \sqrt{p_+} \\ p_\perp/\sqrt{p_+} \end{pmatrix}_\alpha \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}^a, \quad (\text{B.1a})$$

$$\tilde{\lambda}_{p\dot{\alpha}}^a = \sqrt{\frac{E+P}{2P}} \begin{pmatrix} -p_\perp/\sqrt{p_+} \\ \sqrt{p_+} \end{pmatrix}_{\dot{\alpha}} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}^a - \sqrt{\frac{E-P}{2P}} \begin{pmatrix} \sqrt{p_+} \\ \bar{p}_\perp/\sqrt{p_+} \end{pmatrix}_{\dot{\alpha}} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}^a, \quad (\text{B.1b})$$

where now we take $p_\pm = P \pm p^3$, $p_\perp = p^1 + ip^2$. Moreover, as mentioned in section 2, we introduce sign functions in the definitions

$$P = \text{sgn}(E) \sqrt{\vec{p}^2}, \quad m = \text{sgn}(E) \sqrt{E^2 - P^2}. \quad (\text{B.2})$$

This allows the massless limit to keep $(E+P)$ finite and send $(E-P) \rightarrow \text{sgn}(E)m^2/(2P)$ for all real-valued energies $E = p^0$, as well as preserve $\det\{\lambda_{p\alpha}^a\} = \det\{\tilde{\lambda}_{p\dot{\alpha}}^a\} = m$ and the

form (2.17) of the Dirac equation for antiparticles. If p_+ happens to vanish, an equivalent solution may be used:⁸

$$\lambda_{p\dot{\alpha}}^a = \sqrt{\frac{E+P}{2P}} \begin{pmatrix} -\sqrt{p^-} \\ p_\perp/\sqrt{p^-} \end{pmatrix}_\alpha \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}^a + \sqrt{\frac{E-P}{2P}} \begin{pmatrix} \bar{p}_\perp/\sqrt{p^-} \\ \sqrt{p^-} \end{pmatrix}_\alpha \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}^a, \quad (\text{B.3a})$$

$$\tilde{\lambda}_{p\dot{\alpha}}^a = \sqrt{\frac{E+P}{2P}} \begin{pmatrix} -\sqrt{p^-} \\ \bar{p}_\perp/\sqrt{p^-} \end{pmatrix}_{\dot{\alpha}} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}^a - \sqrt{\frac{E-P}{2P}} \begin{pmatrix} p_\perp/\sqrt{p^-} \\ \sqrt{p^-} \end{pmatrix}_{\dot{\alpha}} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}^a. \quad (\text{B.3b})$$

Another singular region is where $P = 0$, and thus $p^0 = m$ and $p_\pm = \pm p^3$. Then we choose

$$\lambda_{p\dot{\alpha}}^a = \sqrt{\bar{p}_\perp} \begin{pmatrix} \sqrt{(m-p_+)/p_\perp} \\ -\sqrt{p_\perp/(m-p_+)} \end{pmatrix}_\alpha \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}^a + \frac{m}{\sqrt{m-p_+}} \begin{pmatrix} 0 \\ -\sqrt{p_\perp/\bar{p}_\perp} \end{pmatrix}_\alpha \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}^a, \quad (\text{B.4a})$$

$$\tilde{\lambda}_{p\dot{\alpha}}^a = \sqrt{p_\perp} \begin{pmatrix} \sqrt{(m-p_+)/\bar{p}_\perp} \\ -\sqrt{\bar{p}_\perp/(m-p_+)} \end{pmatrix}_{\dot{\alpha}} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}^a - \frac{m}{\sqrt{m-p_+}} \begin{pmatrix} 0 \\ -\sqrt{\bar{p}_\perp/p_\perp} \end{pmatrix}_{\dot{\alpha}} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}^a, \quad (\text{B.4b})$$

the massless limit of which is consistent with that of eq. (B.1) (evaluated at $P = 0$). These spinors do not, however, correspond to definite helicities, as the spin vector defined by eq. (2.19) turns out to be $\vec{s}_a = (-1)^{a-1} \{\bar{p}/2m - (0, 0, m/2(m-p_+))\}$ for a complex p^μ . Similarly, in the case where $p_\pm = 0$ and the momentum equals $(m, p^1, \pm ip^1, 0)$, a valid choice is

$$\lambda_{p\dot{\alpha}}^a = \frac{1}{\sqrt{2p^1}} \begin{pmatrix} -\bar{p}_\perp + m \\ p_\perp - m \end{pmatrix}_\alpha \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}^a + \frac{1}{\sqrt{2p^1}} \begin{pmatrix} \mp m \\ \pm m \end{pmatrix}_\alpha \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}^a, \quad (\text{B.5a})$$

$$\tilde{\lambda}_{p\dot{\alpha}}^a = \frac{1}{\sqrt{2p^1}} \begin{pmatrix} -p_\perp \mp m \\ \bar{p}_\perp \mp m \end{pmatrix}_{\dot{\alpha}} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}^a - \frac{1}{\sqrt{2p^1}} \begin{pmatrix} 2p_\perp - m \\ 2\bar{p}_\perp - m \end{pmatrix}_{\dot{\alpha}} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}^a, \quad (\text{B.5b})$$

yielding the spin vectors $\vec{s}_a = (-1)^{a-1} \{\bar{p}/2m - (1, \pm i(1 - m/2p^1), \pm(1 - m/2p^1))\}$. For a real massive momentum at rest $p^\mu = (m, 0, 0, 0)$ we choose

$$\lambda_{p\dot{\alpha}}^a = \begin{pmatrix} 0 \\ \sqrt{m} \end{pmatrix}_\alpha \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}^a + \begin{pmatrix} \sqrt{m} \\ 0 \end{pmatrix}_\alpha \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}^a, \quad \tilde{\lambda}_{p\dot{\alpha}}^a = \begin{pmatrix} 0 \\ \sqrt{m} \end{pmatrix}_{\dot{\alpha}} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}^a - \begin{pmatrix} \sqrt{m} \\ 0 \end{pmatrix}_{\dot{\alpha}} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}^a. \quad (\text{B.6})$$

This choice naturally aligns the spin vector $\vec{s}^a = (-1)^{a-1}(0, 0, 1/2)$ with the z -axis. Finally, we note that for real-valued momenta p^μ the above definitions satisfy $(\lambda_{p\alpha})^* = \tilde{\lambda}_{p\dot{\alpha}}$ and $(\eta_{p\alpha})^* = \tilde{\eta}_{p\dot{\alpha}}$ and are consistent with the momentum inversion rule

$$\lambda_{-p} = \text{sgn}(p^0)i\lambda_p, \quad \tilde{\lambda}_{-p} = \text{sgn}(p^0)i\tilde{\lambda}_p. \quad (\text{B.7})$$

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⁸Using that $\sqrt{p_+} = \sqrt{2P} \cos(\theta/2)$, $\sqrt{p_-} = \sqrt{2P} \sin(\theta/2)$ and $p_\perp = 2Pe^{i\varphi} \sin(\theta/2) \cos(\theta/2)$, one can relate the parametrizations (B.1) and (B.3) by the spin-preserving little-group rotation $\omega_b^a = \begin{pmatrix} e^{-i\varphi} & 0 \\ 0 & e^{i\varphi} \end{pmatrix}$.

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