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Supersymmetric Rényi entropy and Anomalies in $6d$ (1,0) SCFTs

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Abstract: A closed formula of the universal part of supersymmetric Rényi entropy S_q for six-dimensional $(1,0)$ superconformal theories is proposed. Within our arguments, S_q across a spherical entangling surface is a cubic polynomial of $\nu = 1/q$, with 4 coefficients expressed as linear combinations of the 't Hooft anomaly coefficients for the R-symmetry and gravitational anomalies. As an application, we establish linear relations between the c-type Weyl anomalies and the 't Hooft anomaly coefficients. We make a conjecture relating the supersymmetric Rényi entropy to an equivariant integral of the anomaly polynomial in even dimensions and check it against known data in $4d$ and $6d$.

Keywords: AdS-CFT Correspondence, Anomalies in Field and String Theories, Conformal Field Theory, Holography and condensed matter physics (AdS/CMT)

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1 Introduction

Six-dimensional superconformal theories provide a framework to understand various features of lower-dimensional supersymmetric dynamics. By themselves, they are difficult to study by traditional quantum field theory techniques. All known examples of interacting CFTs in six dimensions are supersymmetric. The $(2,0)$ theories should be the simplest ones [1–3]. A large class of interacting $(1,0)$ fixed points have been constructed in string theory or brane constructions [4–8]. Recently, F-theory provides a way to classify the known and new $(1,0)$ fixed points [9–11].

Since all the known interacting fixed points are supersymmetric, it is expected that supersymmetry constraints are important in computing their physical characteristic quantities, such as Weyl anomalies. Indeed, the a-anomaly in $(1,0)$ superconformal theories has been recently determined in terms of their 't Hooft anomaly coefficients [12, 13] for the R-symmetry and gravitational anomalies [14] by analyzing supersymmetric RG flows on the tensor branch [15][1]

$$\bar{a} = \frac{a}{a_{\mathfrak{u}(1)}} = \frac{16}{7}\,(\alpha - \beta + \gamma) + \frac{6}{7}\,\delta\,, \tag{1.1}$$

where $\alpha\,,\beta\,,\gamma\,,\delta$ are the coefficients appearing in the anomaly polynomial

$$\mathcal{I}_8 = \frac{1}{4!}\left(\alpha\, c_2^2(R) + \beta\, c_2(R)p_1(T) + \gamma\, p_1^2(T) + \delta\, p_2(T)\right)\ . \tag{1.2}$$

Here $c_2(R)$ is the second Chern class of the R-symmetry bundle and $p_{1,2}$ are the Pontryagin classes of the tangent bundle. The relation (1.1) is analogous to the known relation [18] in four-dimensional $\mathcal{N}=1$ SCFTs, $a_{d=4} = \frac{9}{32}k_{RRR} - \frac{3}{32}k_R$, where k_{RRR} and k_R are the $\mathrm{Tr}\,\mathrm{U}(1)_R^3$ and $\mathrm{Tr}\,\mathrm{U}(1)_R$ 't Hooft anomalies. Although the anomaly multiplet in six dimensions has not yet been constructed, such linear relations are believed to follow from the anomaly supermultiplets which include 't Hooft anomalies as well as the anomalous trace of the stress tensor. The Weyl anomaly coefficients in $6d$ are defined from the latter [19–22]

$$\langle T_\mu^{\ \mu}\rangle \sim a\,E_6 + \sum_{i=1}^{3} c_i\, I_i\,, \tag{1.3}$$

where E_6 is the Euler density and $I_{i=1,2,3}$ are three Weyl invariants. In the presence of $(1,0)$ supersymmetry, $c_{i=1,2,3}$, satisfying a constraint $c_1 - 2c_2 + 6c_3 = 0$ [23–25], are also believed to be linearly related to the 't Hooft anomaly coefficients [26–28]. Assuming that the linear relation indeed exist, one could determine its coefficients by considering the known values of the corresponding Weyl and 't Hooft anomalies in four independent examples. Unfortunately only three are known, i.e. the free hyper multiplet, the free tensor multiplet and supergravity [23, 29]. The naive vector multiplet is not conformal and the conformal version [30] involves higher derivatives. Evaluating the anomalies via the heat kernel method will involve higher powers of the Laplacian operator in curved space and, hence, difficult to compute. We will, therefore, consider another approach.

In even dimensions, it is known that the a-anomaly determines both the universal log divergence of the round-sphere partition function[2] and the universal log divergence in the vacuum state entanglement entropy associated with a ball in flat space [31]. On the other hand, by the conformal Ward identities, the 2-point and 3-point functions of the stress tensor in the vacuum in flat space can be determined up to 3 coefficients [32, 33], which are linearly related to c-type Weyl anomalies $c_{1,2,3}$. In the presence of $(1,0)$ supersymmetry, only two of them are independent as mentioned before.

Because the round sphere is conformally flat, one expects that the nearly-round sphere partition function, which includes the response to a small deviation of the metric from the round sphere, is determined by the flat space stress tensor correlators. Due to these intrinsic relations and supersymmetric constraints, it is therefore tempting to ask whether

[1]The subscript $\mathfrak{u}(1)$ means an Abelian $(2,0)$ tensor multiplet. See [16] for the result in $(2,0)$ theories and [17] for earlier investigation.

[2]The a-anomaly is proportional to the coefficient of the log divergence.

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one can fully determine the partition function on a q-branched sphere,[3] which is directly related to the supersymmetric Rényi entropy S_q.

Supersymmetric Rényi entropy was first introduced in three dimensions [34–36], and later studied in four dimensions [37–39], in five dimensions [40, 41], in six dimensions ((2, 0) theories) [42, 43] and also in two dimensions ((2, 2) SCFTs) [44, 45]. By turning on a certain R-symmetry background field $\mu(q)$, one can calculate the supersymmetric partition function $Z_q[\mu(q)]$ on a q-branched sphere $\mathbb{S}^d_q$,

$$ds^2_{\mathbb{S}^d_q}/\ell^2 = q^2 \sin^2\theta d\tau^2 + d\theta^2 + \cos^2\theta d\Omega^2_{d-2}\,, \tag{1.4}$$

where $\theta \in [0, \pi/2]$ and $\tau \in [0, 2\pi)$. The supersymmetric Rényi entropy is defined as

$$S_q = \frac{qI_1 - I_q}{1-q}\,, \quad I_q := -\log Z_q[\mu(q)]\,. \tag{1.5}$$

The quantities defined in (1.5) are UV divergent in general but one can extract universal parts free of ambiguities. Notice that the ordinary Rényi entropy is not supersymmetric because of the conical singularity.[4]

1.1 Summary of results

The main result of this paper is the exact universal part of the supersymmetric Rényi entropy in 6d (1, 0) SCFTs. We show that, for theories characterized by the anomaly polynomial (1.2), it is given by a cubic polynomial of $\nu = 1/q$

$$S^{(1,0)}_\nu = \sum_{n=0}^{3} s_n(\nu-1)^n\,, \tag{1.6}$$

with four coefficients

$$\begin{aligned} s_0 &= \frac{1}{6}(8\alpha - 8\beta + 8\gamma + 3\delta)\,,\\ s_1 &= \frac{1}{4}(2\alpha - 3\beta + 4\gamma + \delta)\,,\\ s_2 &= \frac{1}{24}(2\alpha - 5\beta + 8\gamma)\,,\\ s_3 &= \frac{1}{192}(\alpha - 4\beta + 16\gamma)\,. \end{aligned} \tag{1.7}$$

where $\alpha\,, \beta\,, \gamma\,, \delta$ are the 't Hooft anomaly coefficients defined in (1.2). The basic ingredients in our arguments are the following:

[3] A q-branched sphere is a sphere with a conical singularity with the deformation parameter $q - 1$, see (1.4).

[4] Consider CFTs in flat space with the metric, $ds^2_{\mathbb{R}^d} = d\tau_E^2 + dr^2 + r^2 d\Omega^2_{d-2}$. The entangling surface Σ is $(\tau_E = 0, r = R)$. In the replica trick approach, the Rényi entropy can be computed from the path integral on the conic space with Σ the fixed sphere. After the transformations $\tau_E = \ell\frac{\sin\tau}{\cosh\eta+\cos\tau}, r = \ell\frac{\sinh\eta}{\cosh\eta+\cos\tau}$, the conic space becomes a hyperbolic space $\mathbb{S}^1_q \times \mathbb{H}^{d-1}$ up to a warp factor $ds^2_{\mathbb{R}^d}/\ell^2 = \Omega^2(d\tau^2 + d\eta^2 + \sinh^2\eta d\Omega^2_{d-2})$, where $\theta \in [0, \pi/2]$, $\tau \in [0, 2\pi]$ and $\Omega = \frac{1}{\cosh\eta+\cos\tau}$. A further Weyl transformation with $\cot\theta = \sinh\eta$ maps $\mathbb{S}^1_q \times \mathbb{H}^{d-1}$ to the branched sphere $\mathbb{S}^d_q$ (1.4), where Σ is mapped to $\theta = 0$. Throughout this work we take the same boundary condition as the "smooth cone" boundary condition in [46], which means that we smooth out the cone.

(A) S_ν of $(1,0)$ free hyper multiplet and free tensor multiplet can be computed by the heat kernel method closely following [43]. The results are given by

$$S^h_\nu = \frac{7}{2880}(\nu-1)^3 + \frac{7}{720}(\nu-1)^2 + \frac{1}{40}(\nu-1) + \frac{11}{360}\,, \tag{1.8}$$
$$S^t_\nu = \frac{1}{360}(\nu-1)^3 + \frac{1}{90}(\nu-1)^2 + \frac{1}{10}(\nu-1) + \frac{199}{360}\,. \tag{1.9}$$

These are the main results of section 2.

(B) S_ν of A_{N-1} type $(2,0)$ theories (which are of course $(1,0)$ conformal theories) in the large N has been computed in [43]. The result is given by

$$\frac{S_\nu[A_{N\to\infty}]}{N^3} = \frac{1}{192}(\nu-1)^3 + \frac{1}{12}(\nu-1)^2 + \frac{1}{2}(\nu-1) + \frac{4}{3}\,. \tag{1.10}$$

(C) Based on (A)(B) and (F) below, a reasonable *assumption* is that the general form of S_ν for $(1,0)$ SCFTs is a cubic polynomial in $\nu-1$. However, so far we do not have a sharp argument for this assumption.[5] Furthermore, based on (D)(E)(F) below, the four coefficients of the cubic polynomial are linear combinations of $\alpha,\beta,\gamma,\delta$.

(D) The value of S_ν at $\nu=1$ is the entanglement entropy associated with a spherical entangling surface, which is proportional to the a-anomaly (1.1).

(E) The first and second derivatives of S_ν at $\nu=1$ can be written as linear combinations of integrated two- and three-point functions of operators in supersymmetric stress tensor multiplet. Because of this, one can relate the first and second derivatives at $\nu=1$ to c_1 and c_2,

$$\partial_\nu S_\nu\big|_{\nu=1} = \frac{3}{2}c_2 - \frac{3}{4}c_1\,, \quad \partial^2_\nu S_\nu\big|_{\nu=1} = c_2 - \frac{5}{16}c_1\,, \tag{1.11}$$

where c_1 and c_2 are believed to be given by linear combinations of 't Hooft anomaly coefficients $\alpha,\beta,\gamma,\delta$.

(F) The large ν behavior of S_ν is controlled by the "supersymmetric Casimir energy" [47]. This gives

$$\lim_{\nu\to\infty}\frac{S_\nu}{\nu^3} = \frac{1}{192}(\alpha-4\beta+16\gamma)\,. \tag{1.12}$$

(G) In the large ν expansion, the second Pontryagin class (with coefficient δ) will not contribute to the ν^3 term (as we see from (F)) and the ν^2 term. Because of the latter, one has

$$\partial_\delta\left(\partial^2_\nu S_\nu\big|_{\nu=1}\right) = 0\,. \tag{1.13}$$

[5] We are interested only in the universal part, i.e. the coefficient of the UV log divergent part. This part should be given by a finite number of counter-terms, each of them an integral of local functions of the supersymmetric background including the metric (squashed sphere). Unfortunately the supersymmetric smooth squashed sphere in $6d$ has not yet been constructed.

(H) For the conformal non-unitary $(1,0)$ vector multiplet, a constraint for the c-type Weyl anomalies, $c_1+4c_2=\frac{62}{45}$, can be obtained by studying the higher-derivative operators on the Ricci flat background [26].[6] Together with (E), one has

$$16\left(\partial_\nu^2 S_\nu^{\text{“Vector”}}\big|_{\nu=1}\right)-8\left(\partial_\nu S_\nu^{\text{“Vector”}}\big|_{\nu=1}\right)=(c_1+4c_2)\big|_{\text{“Vector”}}=\frac{62}{45}\,. \tag{1.14}$$

From (A)(B)(C)(D)(E)(F)(G)(H), one can uniquely find the general expression of the supersymmetric Rényi entropy given in (1.6)(1.7). We emphasize that among all these ingredients (C) is an assumption, all the rest are derived results. The results (A),(D),(E),(F),(G) are new as far as we know. The precise agreement between (F) and (A)(B) can be considered as a nontrivial test of (F). Independently, we conjecture a relation between the supersymmetric Rényi entropy and the anomaly polynomial in any even dimension, which perfectly agrees with (A)-(H). We consider this precise agreement as a strong support of our result (1.6), (1.7). Note that (E) and (1.7) also establish the linear relations between c-type Weyl anomalies and the 't Hooft anomaly coefficients,[7]

$$\begin{aligned} c_1 &= -\frac{2}{9}(6\alpha-7\beta+8\gamma+4\delta)\,,\\ c_2 &= -\frac{1}{18}(6\alpha-5\beta+4\gamma+5\delta)\,,\\ c_3 &= \frac{1}{18}(2\alpha-3\beta+4\gamma+\delta)\,. \end{aligned} \tag{1.15}$$

This paper is organized as follows. In section 2 we employ heat kernel method to study the supersymmetric Rényi entropy of free $(1,0)$ multiplets. In section 3 we propose a form of the universal supersymmetric Rényi entropy with four non-trivial coefficients, which works for general $6d$ $(1,0)$ SCFTs. We determine the coefficients one by one. We study the relation between the supersymmetric Rényi entropy and the supersymmetric Casimir energy in section 4, which is used to determine one of the coefficients in the previous section. In section 5 we conjecture a relation between the supersymmetric Rényi entropy and the anomaly polynomial for SCFTs in even dimensions and test this conjecture in $6d$ and $4d$. In section 6, we discuss some open questions, further applications of our results and some future directions of research.

2 Free $6d$ (1,0) multiplets

We begin by studying the supersymmetric Rényi entropy of free $(1,0)$ multiplets, following [42]. For free fields, the Rényi entropy associated with a spherical entangling surface in flat space can be computed by conformally mapping the conic space to a hyperbolic

[6] We thank Matteo Beccaria for explaining us this result first presented in [26].

[7] The numerical coefficients for the c-anomalies here are different from those presented in [26], where an assumption concerning the structure of the linear combinations was made. We thank Matteo Beccaria for discussion on this issue. After our paper appeared on the arXiv the authors of [26] clarified to us that the data they used did not allow them to fix $c_{1,2,3}$ unambiguously. There was still a 1-parameter freedom consistent with our result. They fixed this freedom by another assumption/conjecture relating anomalies in $4d$ and $6d$.

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space $\mathbb{S}^1_\beta \times \mathbb{H}^5$ and using the heat kernel method.[8] A six-dimensional $(1,0)$ hyper multiplet includes 4 real scalars, 1 Weyl fermion and a tensor multiplet includes 1 real scalar, 1 Weyl fermion and a 2-form field with self-dual strength. The 2-form field has a self-duality constraint which reduces the number of degrees of freedom by half.

2.1 Heat kernel and Rényi entropy

The partition function of free fields on $\mathbb{S}^1_\beta \times \mathbb{H}^5$ can be computed by the heat kernel[9]

$$\log Z(\beta) = \frac{1}{2}\int_0^\infty \frac{dt}{t} K_{\mathbb{S}^1_\beta \times \mathbb{H}^5}(t)\,, \tag{2.1}$$

where $K_{\mathbb{S}^1_\beta \times \mathbb{H}^5}(t)$ is the heat kernel of the associated conformal Laplacian and β is the length of the unit circle $2\pi q$. The kernel factorizes because the spacetime is a direct product,

$$K_{\mathbb{S}^1_\beta \times \mathbb{H}^5}(t) = K_{\mathbb{S}^1_\beta}(t)\, K_{\mathbb{H}^5}(t)\ . \tag{2.2}$$

On a circle, the kernel is given by

$$K_{\mathbb{S}^1_\beta}(t) = \frac{\beta}{\sqrt{4\pi t}} \sum_{n\neq 0,\in\mathbb{Z}} e^{\frac{-\beta^2 n^2}{4t}}\ . \tag{2.3}$$

In the presence of a chemical potential μ, it will be twisted [48]

$$\widetilde{K}_{\mathbb{S}^1_\beta}(t) = \frac{\beta}{\sqrt{4\pi t}} \sum_{n\neq 0,\in\mathbb{Z}} e^{\frac{-\beta^2 n^2}{4t}+i2\pi n\mu + i\pi n f}\,, \tag{2.4}$$

where f controls the periodic/anti-periodic boundary conditions, namely $f=0$ for bosons and $f=1$ for fermions. The volume factor can be factorized in the kernels on the hyperbolic space, because $\mathbb{H}^5$ is homogeneous. Thus $K_{\mathbb{H}^5}(t)$ can be written in terms of the equal-point kernel,

$$K_{\mathbb{H}^5}(t) = \int d^5x\sqrt{g}\ K_{\mathbb{H}^5}(x,x,t) = V_5\, K_{\mathbb{H}^5}(0,t)\ . \tag{2.5}$$

The regularized volume is given by $V_5 = \pi^2\log(\ell/\epsilon)$, where ϵ is actually the UV cutoff in the original flat space before the conformal mapping[10] and ℓ is the curvature radius of $\mathbb{H}^5$. For the $K_{\mathbb{H}^5}(0,t)$ of free fields with different spins we refer to [42] and references there in.

The Rényi entropy of a hyper multiplet can be obtained by summing up the contributions of 4 real scalars, 1 Weyl fermion and the Rényi entropy of a tensor multiplet can be obtained by summing up the contributions of 1 real scalar, 1 Weyl fermion and a self-dual 2-form,

$$S_q^{\text{hyper}} = 4\times\frac{S_q^s}{2} + S_q^f\,, \tag{2.6}$$

$$S_q^{\text{tensor}} = \frac{S_q^s}{2} + S_q^f + \frac{S_q^v}{2}\ . \tag{2.7}$$

[8] In this section we use $\beta = 1/T$ as the inverse temperature and hopefully this will not be confusing with the anomaly coefficient β.

[9] For Rényi entropy of free fields in other dimensions less than six, see for instance [49–52].

[10] In flat space with entangling region bounded by Σ, there is a UV divergence at Σ and so we need to introduce a short distance cut-off, which eventually becomes an IR cut-off by the conformal mapping between the two spaces. See footnote 4 for more details.

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where the Rényi entropy for free fields with different spins can be computed by using the corresponding heat kernels.[11] The final results for the Rényi entropy of a $6d$ complex scalar, a $6d$ Weyl fermion and a $6d$ 2-form field are

$$S_q^s = \frac{(q+1)\left(3q^2+1\right)\left(3q^2+2\right)}{15120q^5}\frac{V_5}{\pi^2}\,, \tag{2.8}$$

$$S_q^f = \frac{(q+1)\left(1221q^4+276q^2+31\right)}{120960q^5}\frac{V_5}{\pi^2}\,, \tag{2.9}$$

$$S_q^v = \frac{(q+1)\left(37q^2+2\right)+877q^4+4349q^5}{5040q^5}\frac{V_5}{\pi^2}\,, \tag{2.10}$$

respectively. Note that, to obtain the correct Rényi entropy for the two form field, we have taken a q-independent constant shift which is associated with possible boundary contributions [42]. Before moving on, let us represent S_q in terms of

$$S_\nu = \frac{\pi^2}{V_5} S_q\,, \text{ with } \ \nu = 1/q\ .$$

The Rényi entropy of free $(1,0)$ multiplets are given by

$$S_\nu^{\text{hyper}} = \frac{(\nu-1)^5}{1920} + \frac{(\nu-1)^4}{320} + \frac{31(\nu-1)^3}{2880} + \frac{(\nu-1)^2}{45} + \frac{\nu-1}{30} + \frac{11}{360}\,, \tag{2.11}$$

$$S_\nu^{\text{tensor}} = \frac{(\nu-1)^5}{1920} + \frac{(\nu-1)^4}{320} + \frac{13(\nu-1)^3}{960} + \frac{(\nu-1)^2}{30} + \frac{2(\nu-1)}{15} + \frac{199}{360}\ . \tag{2.12}$$

The reason why S_ν is convenient is obvious, the series expansion near $\nu = 1$ has finite terms while the expansion of S_q near $q = 1$ has infinite number of terms. We will use S_ν instead of S_q to express Rényi entropy and supersymmetric Rényi entropy from now on. It is worth to remember the relations between the derivatives with respect to q and the derivatives with respect to ν at $q = 1/\nu = 1$,

$$\partial_\nu S_\nu\big|_{\nu=1} = -\partial_q S_q\big|_{q=1}\cdot\frac{\pi^2}{V_5}\,,\quad \partial_\nu^2 S_\nu\big|_{\nu=1} = \left(2\partial_q S_q + \partial_q^2 S_q\right)\big|_{q=1}\cdot\frac{\pi^2}{V_5}\,, \tag{2.13}$$

which will be useful later. Finally, one can check that $\partial^0_{q=1}$, $\partial^1_{q=1}$ and $\partial^2_{q=1}$ of both S_q^{hyper} and S_q^{tensor} are consistent with the previous results about the free $(1,0)$ multiplets [23, 29]. By "consistent", we refer to the relations between the first and the second derivatives of the Rényi entropy at $q = 1$ and the two- and three-point functions of the stress tensor derived in [53, 54].

2.2 Supersymmetric Rényi entropy

The supersymmetric Rényi entropy of free multiplets can be computed by the twisted kernel (2.4) on the supersymmetric background. The R-symmetry group of $6d$ $(1,0)$ theories is $\mathrm{SU}(2)_R$, which has a single $\mathrm{U}(1)$ Cartan subgroup. Therefore one can turn on a single R-symmetry background gauge field (chemical potential) to twist the boundary conditions

[11] For some relevant details of this computation we refer to [42].

for scalars and fermions along the replica circle $\mathbb{S}^1_\beta$ [48]. The R-symmetry chemical potential can be solved by studying the Killing spinor equation on the conic space ($\mathbb{S}^6_q$ or $\mathbb{S}^1_{\beta=2\pi q}\times\mathbb{H}^5$),[12]

$$\mu(q) := k\, A_\tau = \frac{q-1}{2}\,, \tag{2.14}$$

with k being the R-charge of the Killing spinor under the Cartan U(1). We choose $k=1/2$ and the background field turns out to be

$$A_\tau = (q-1)\ . \tag{2.15}$$

For each component field in the free multiplets, one has to first figure out the associated Cartan charge k_i and then compute the chemical potential by $k_i\,A_\tau$. After that one can compute the free energy on $\mathbb{S}^1_\beta\times\mathbb{H}^5$ using the twisted heat kernel with the chemical potential $\mu = k_i\,A_\tau$ and obtain the supersymmetric Rényi entropy.

After summing up the component fields, the supersymmetric Rényi entropy of a free $(1,0)$ hyper multiplet and a free $(1,0)$ tensor multiplet are

$$S^h_\nu = \frac{7}{2880}(\nu-1)^3+\frac{7}{720}(\nu-1)^2+\frac{1}{40}(\nu-1)+\frac{11}{360}\,, \tag{2.16}$$
$$S^t_\nu = \frac{1}{360}(\nu-1)^3+\frac{1}{90}(\nu-1)^2+\frac{1}{10}(\nu-1)+\frac{199}{360}\,, \tag{2.17}$$

respectively.

3 Interacting 6d (1,0) SCFTs

Having obtained the free multiplet results (2.16), (2.17), we will use them to rewrite $S^{(1,0)}_\nu$ in a general form which, we hope, works for general interacting 6d $(1,0)$ SCFTs,

$$S^{(1,0)}_\nu = A\,(\nu-1)^3+B\,(\nu-1)^2+C\,(\nu-1)+D\,, \tag{3.1}$$

where the coefficients $A\,,B\,,C\,,D$ will depend on the specific theories.[13]

Before determining $A\,,B\,,C\,,D$ for general $(1,0)$ fixed points, let us summarize what we have learned so far for the existing examples. These are free $(1,0)$ hyper multiplet, free $(1,0)$ tensor multiplet, A_{N-1} type $(2,0)$ theories in the large N limit and non-unitary conformal $(1,0)$ vector multiplet [26, 30]. We list $A\,,B\,,C\,,D$ and the relevant anomaly data for them in table 1.[14] The anomaly data are from [14, 21, 23].

The coefficient D in (3.1) can be determined by using the fact that, the entanglement entropy associated with a spherical entangling surface, which is nothing but $S_{\nu=1}$, is proportional to the Weyl anomaly a. This is true for general CFTs in even dimensions as shown in [31]. Therefore

$$\frac{S^{(1,0)}_{\nu=1}}{S^{(2,0)}_{\nu=1}} = \frac{a}{a_{\mathrm{u}(1)}} =: \bar{a}\ . \tag{3.2}$$

[12]See the appendix in [42].

[13]This structure is not true for the ordinary (non-supersymmetric) Rényi entropy [55].

[14]We denote the conformal non-unitary vector multiplet by "Vector".

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	A	B	C	D	α	β	γ	δ	c_1	c_2	c_3
Hyper	$\frac{7}{2880}$	$\frac{7}{720}$	$\frac{1}{40}$	$\frac{11}{360}$	0	0	$\frac{7}{240}$	$-\frac{1}{60}$	$-\frac{1}{27}$	$-\frac{1}{540}$	$\frac{1}{180}$
Tensor	$\frac{1}{360}$	$\frac{1}{90}$	$\frac{1}{10}$	$\frac{199}{360}$	1	$\frac{1}{2}$	$\frac{23}{240}$	$-\frac{29}{60}$	$-\frac{8}{27}$	$-\frac{11}{135}$	$\frac{1}{45}$
$A_{N-1}\frac{1}{N^3}$	$\frac{1}{192}$	$\frac{1}{12}$	$\frac{1}{2}$	$\frac{4}{3}$	1	0	0	0	$-\frac{4}{3}$	$-\frac{1}{3}$	$\frac{1}{9}$
"Vector"	–	–	–	–	-1	$-\frac{1}{2}$	$-\frac{7}{240}$	$\frac{1}{60}$	–	–	–

Table 1. Supersymmetric Rényi entropy and anomalies of known $(1,0)$ fixed points.

By studying supersymmetric RG flows on the tensor branch, $a/a_{\mathrm{u}(1)}$ has been computed in [15], see (1.1). This allows us to fix

$$D = S^{(1,0)}_{\nu=1} = \frac{7}{12}\left(\frac{16}{7}(\alpha-\beta+\gamma)+\frac{6}{7}\delta\right) = \frac{4}{3}(\alpha-\beta+\gamma)+\frac{\delta}{2}\,. \tag{3.3}$$

The coefficients C and B in (3.1) are the first and the second ν-derivatives of $S^{(1,0)}_\nu$ at $\nu=1$, respectively. The transformations between the ν-derivatives and the q-derivatives are given by (2.13). The relations between the q-derivatives and the integrated correlators are given in appendix A. Namely, the first q-derivative at $q=1$ is given by a linear combinations of integrated $\langle TT\rangle$ and integrated $\langle JJ\rangle$ in (A.23),

$$S'_{q=1} = -V_{d-1}\left(\frac{\pi^{\frac{d}{2}+1}\Gamma(\frac{d}{2})(d-1)}{(d+1)!}C_T - g^2\frac{\pi^{\frac{d+3}{2}}}{2^{d-3}(d-1)\Gamma(\frac{d-1}{2})}C_J\right)\,. \tag{3.4}$$

This relation holds for general SCFTs with conserved R-symmetry in d-dimensions. Similarly the second q-derivative at $q=1$ is given by a linear combination of the integrated stress tensor 3-point function, the integrated R-current 3-point function and some mixed 3-point functions. This is given explicitly in (A.27)

$$S''_{q=1} = \frac{1}{6}I'''_{q=1} = \frac{4\pi^3}{3}\left[\langle\hat{E}\hat{E}\hat{E}\rangle^c - g^3\langle\hat{Q}\hat{Q}\hat{Q}\rangle^c - 3g\langle\hat{E}\hat{E}\hat{Q}\rangle^c + 3g^2\langle\hat{E}\hat{Q}\hat{Q}\rangle^c\right]_{\mathbb{S}^1_{q=1}\times\mathbb{H}^{d-1}}\,. \tag{3.5}$$

In $6d$ $(1,0)$ SCFTs, by the conformal Ward identities, the two- and three-point functions of the stress tensor multiplet (including R-current) may be determined in terms of two independent coefficients, which are linearly related to c_1 and c_2. Because of this, C and B in (3.1) are also linear combinations of c_1 and c_2. These relations can be obtained by fitting to the free hyper multiplet and the free tensor multiplet in table 1,

$$B = c_2 - \frac{5}{16}c_1\,,\quad C = \frac{3}{2}c_2 - \frac{3}{4}c_1\,. \tag{3.6}$$

Assuming B and C are linear combinations of $\alpha\,,\beta\,,\gamma\,,\delta$, we shall establish the explicit relations. Because the second Pontryagin class $p_2(T)$ does not contribute to the ν^2 term, we get

$$\partial_\delta B = 0\,. \tag{3.7}$$

To see that the ν^2 term is independent of $p_2(T)$, let us consider the free energy on $\mathbb{S}^5_q\times\mathbb{H}^1$, which can be used to compute S_q because $\mathbb{S}^5_q\times\mathbb{H}^1$ is conformally equivalent to $\mathbb{S}^6_q$ or

$\mathbb{S}^1_q \times \mathbb{H}^5$. $\mathbb{S}^5_q \times \mathbb{H}^1$ is similar to $\mathbb{S}^5_q \times \mathbb{S}^1_{\beta\to\infty}$, but they are not the same due to different boundary conditions on $\mathbb{H}^1$ and $\mathbb{S}^1_\beta$. The latter background preserving supersymmetry is used to compute the supersymmetric Casimir energy in $6d$. One can formally define a supersymmetric Rényi entropy on $\mathbb{S}^5_q \times \mathbb{S}^1_{\beta\to\infty}$ with the Rényi parameter q by using the free energy $\beta E_c[\mathbb{S}^5_q]$. As we will see in the next section, $p_2(T)$ will not contribute to the $1/q^2$ term in this supersymmetric Rényi entropy, because $p_2(T)$ contributes to E_c in the following way (4.7)

$$\frac{p_2(T)}{\omega_1\omega_2\omega_3} \to \frac{1}{\omega_1\omega_2\omega_3}\sum_{i<j}^{3}\omega_i^2\omega_j^2\,, \quad \omega_1=\omega_2=1\,, \omega_3=1/q\ . \tag{3.8}$$

The different boundary conditions on $\mathbb{S}^5_q \times \mathbb{H}^1$ will not change the property that the $1/q^2$ term is independent of δ. We further confirm this fact by establishing a concrete relation between S_q and the anomaly polynomial in section 5.

Since B depends only on $\alpha\,,\beta\,,\gamma\,,$ it can be fixed by fitting to the three independent examples, the free hyper multiplet, the free tensor multiplet and the A_{N-1} type theories in the large N,

$$B=\frac{1}{24}(2\alpha-5\beta+8\gamma)\ . \tag{3.9}$$

The same fitting method can be used to determine the $\alpha\,,\beta\,,\gamma\,,\delta$ dependence of C, but since C depends on all four of them, one free parameter is left. We fix the remaining free parameter by making use of the result of c_1+4c_2 for the conformal non-unitary $(1,0)$ vector multiplet in [26] (obtained by the heat kernel computation on the Ricci flat background)

$$(c_1+4c_2)\big|_{\text{“Vector”}}=\frac{62}{45}\ . \tag{3.10}$$

Thus, the coefficient C as a linear combination of $\alpha\,,\beta\,,\gamma\,,\delta$ is determined

$$C=\frac{1}{4}(2\alpha-3\beta+4\gamma+\delta)\ . \tag{3.11}$$

Eqs. (3.6), (3.9), (3.11) also establish the linear relations between $c_{1,2,3}$ and $\alpha\,,\beta\,,\gamma\,,\delta$

$$\begin{aligned} c_1 &= -\frac{2}{9}(6\alpha-7\beta+8\gamma+4\delta)\,,\\ c_2 &= -\frac{1}{18}(6\alpha-5\beta+4\gamma+5\delta)\,,\\ c_3 &= -\frac{1}{6}(c_1-2c_2)=\frac{1}{18}(2\alpha-3\beta+4\gamma+\delta)\ . \end{aligned} \tag{3.12}$$

The remaining coefficient A will be fixed as

$$A=\frac{1}{192}(\alpha-4\beta+16\gamma)\ . \tag{3.13}$$

in the next section by studying the large ν behavior of the supersymmetric Rényi entropy. Obviously, the leading contribution in the limit $\nu\to\infty$ is determined only by A.

3.1 A closed formula

As a summary, we can completely determine a closed formula for the universal part of supersymmetric Rényi entropy for $6d$ $(1,0)$ SCFTs,

$$S_\nu^{(1,0)} = \frac{1}{192}(\alpha - 4\beta + 16\gamma)(\nu-1)^3 + \frac{1}{24}(2\alpha - 5\beta + 8\gamma)(\nu-1)^2 + \frac{1}{4}(2\alpha - 3\beta + 4\gamma + \delta)(\nu - 1) + \frac{1}{6}(8\alpha - 8\beta + 8\gamma + 3\delta)\,. \tag{3.14}$$

Given that 't Hooft anomalies for general $6d$ $(1,0)$ SCFTs can be computed [13], the above formula tells us the universal supersymmetric Rényi entropy for any $(1,0)$ SCFT.

For $(2,0)$ theories labeled by a simply-laced Lie algebra $\mathfrak{g}$, (3.14) reduces to [43]

$$S_\nu^{(2,0)} = (\bar{c} - \bar{a})\,\frac{7}{12}\,H_\nu + (7\bar{a} - 4\bar{c})\,\frac{1}{3}\,T_\nu\,, \tag{3.15}$$

where $\bar{a}$ and $\bar{c}$ are determined by the rank, dimension and dual Coxeter number of $\mathfrak{g}$,

$$\bar{a} = \frac{16}{7}\,d_\mathfrak{g} h_\mathfrak{g}^\vee + r_\mathfrak{g}\,, \quad \bar{c} = 4\,d_\mathfrak{g} h_\mathfrak{g}^\vee + r_\mathfrak{g}\,. \tag{3.16}$$

T_ν and H_ν are the supersymmetric Rényi entropy of the $(2,0)$ tensor multiplet and that of the $(2,0)$ supergravity (large N), respectively

$$T_\nu = \frac{1}{192}(\nu-1)^3 + \frac{1}{48}(\nu-1)^2 + \frac{1}{8}(\nu-1) + \frac{7}{12}\,, \tag{3.17}$$

$$H_\nu = \frac{1}{192}(\nu-1)^3 + \frac{1}{12}(\nu-1)^2 + \frac{1}{2}(\nu-1) + \frac{4}{3}\,. \tag{3.18}$$

4 Relation with supersymmetric Casimir energy

In this section we clarify the relation between the supersymmetric Rényi entropy and the supersymmetric Casimir energy in $6d$. Similar relation in $4d$ has been obtained in [38]. Recall that the partition function Z on $\mathcal{M}^{D-1} \times \mathbb{S}^1_{\widetilde{\beta}}$ is determined by the Casimir energy on the compact space $\mathcal{M}^{D-1}$ in the limit $\widetilde{\beta} \to \infty$

$$E_c := -\lim_{\widetilde{\beta}\to\infty} \partial_{\widetilde{\beta}} \log Z(\widetilde{\beta})\,, \tag{4.1}$$

which is equivalent to the statement[15]

$$\lim_{\widetilde{\beta}\to\infty} \log Z(\widetilde{\beta}) = -\widetilde{\beta} E_c\,. \tag{4.2}$$

We consider the cases with supersymmetry. In even-dimensional superconformal theories, the supersymmetric Casimir energy on $\mathbb{S}^1 \times \mathbb{S}^{D-1}$ has been conjectured to be equal to the equivariant integral of the anomaly polynomial in [47], where the authors provided strong supports for this conjecture by examining a number of SCFTs in two, four and six

[15] In this section we use $\widetilde{\beta} = 1/T$ for the inverse temperature in order to distinguish it from the 't Hooft anomaly β.

dimensions.[16] The equivariant integration is defined with respect to the Cartan subalgebra of the global symmetries (that commute with a given supercharge) and one can write this as

$$E_D(\mu_j) = \int_{\mu_j} I_{D+2}\,, \tag{4.3}$$

where the equivariant parameters μ_j are the chemical potentials corresponding to the Cartan generators. In equivariant cohomology, doing the integration (4.3) in $6d$ is equivalent to the replacement rules (4.7).[17]

Let us consider $6d$ $(1,0)$ SCFTs on $\mathbb{S}^1_{\tilde{\beta}} \times \mathbb{S}^5_{\vec{\omega}}$ with squashing parameters $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$. The squashing parameters are defined by coefficients appearing in the Killing vector

$$K = \omega_1 \frac{\partial}{\partial \phi_1} + \omega_2 \frac{\partial}{\partial \phi_2} + \omega_3 \frac{\partial}{\partial \phi_3}\,, \tag{4.4}$$

where ϕ_1, ϕ_2, ϕ_3 are three circles representing the $\mathrm{U}(1)^3$ isometries of the 5-sphere. The supersymmetric Casimir energy of superconformal $(1,0)$ theories is given by the equivariant integral (4.3)

$$E_6^{(1,0)}(\mu_j) = -\int_{\mu_j} \mathcal{I}_8\,, \tag{4.5}$$

where the 8-form anomaly polynomial is[18]

$$\mathcal{I}_8 = \frac{1}{4!}\left(\alpha\, c_2^2(R) + \beta\, c_2(R) p_1(T) + \gamma\, p_1^2(T) + \delta\, p_2(T)\right) \tag{4.6}$$

as introduced in the introduction. The integration (4.5) is equivalent to the following replacement rules [47]

$$c_2(R) \to -\sigma^2\,, \quad p_1(T) \to \sum_{i=1}^{3} \omega_i^2\,, \quad p_2(T) \to \sum_{i<j}^{3} \omega_i^2 \omega_j^2\,, \tag{4.7}$$

where σ is the chemical potential for the R-symmetry Cartan and $\omega_{1,2,3}$ are the chemical potentials for the rotation generators (commuting with the supercharge). After the replacement, the result should be divided by the equivariant Euler class,

$$e(T) = \omega_1 \omega_2 \omega_3\ . \tag{4.8}$$

In the particular background of $\mathbb{S}^5_q \times \mathbb{S}^1_{\tilde{\beta}}$, where $\mathbb{S}^5_q$ is a q-deformed 5-sphere with the metric

$$\mathrm{d}s^2 = (\sin^2\theta + q^2\cos^2\theta)\mathrm{d}\theta^2 + q^2 \sin^2\theta\, \mathrm{d}\tau^2 + \cos^2\theta\, \mathrm{d}\Omega_3^2\,, \tag{4.9}$$

one should identify the shape parameters as

$$\omega_1 = \omega_2 = 1\,, \quad \omega_3 = \frac{1}{q}\ . \tag{4.10}$$

[16] For $6d$ superconformal index, see [56–58].
[17] See the appendix in [47] for details on the equivalence.
[18] We consider the minimal set of global symmetries without extra flavor symmetries.

Note that there is a supersymmetric constraint for the chemical potentials, $\sigma = \frac{1}{2}\sum_j \omega_j$. Evaluating (4.5) one obtains

$$E_6^{(1,0)} = -\frac{1}{24\omega_1\omega_2\omega_3}\left(\alpha\,\sigma^4 - \beta\,\sigma^2\sum_{j=1}^{3}\omega_j^2 + \gamma\left(\sum_{j=1}^{3}\omega_j^2\right)^2 + \delta\left(\sum_{i<j}^{3}\omega_i^2\omega_j^2\right)\right) . \tag{4.11}$$

Therefore the free energy in the $q \to 0$ limit[19]

$$f[\mathbb{S}^5_{q\to 0}\times\mathbb{S}^1_{\widetilde{\beta}\to\infty}] = \frac{1}{\widetilde{\beta}\pi^2/2}\widetilde{\beta}E_c\bigg|_{q\to 0} = -\frac{1}{192\pi^2}\frac{\alpha - 4\beta + 16\gamma}{q^3}\,, \tag{4.12}$$

where we have divided by a q-independent volume factor Vol $[\mathbb{D}^4\times\mathbb{S}^1_{\widetilde{\beta}}] = \widetilde{\beta}\pi^2/2$. Because of the conformal equivalence between $\mathbb{S}^5_q\times\mathbb{H}^1$ and $\mathbb{S}^1_q\times\mathbb{H}^5$, we have

$$f[\mathbb{S}^5_{q\to 0}\times\mathbb{S}^1_{\widetilde{\beta}\to\infty}] = f[\mathbb{S}^5_{q\to 0}\times\mathbb{H}^1] = f[\mathbb{S}^1_{q\to 0}\times\mathbb{H}^5]\,, \tag{4.13}$$

where the first equality follows from the background coincidence and the second one follows from the conformal invariance of (supersymmetric) Rényi entropy and

$$S_{q\to 0} = -I_{q\to 0}\,,\quad I_q := -\log Z_q\ . \tag{4.14}$$

From (4.13) we obtain the asymptotic supersymmetric Rényi entropy on $\mathbb{S}^1_q\times\mathbb{H}^5$

$$S_{q\to 0} = -I_{q\to 0} = \frac{1}{192}\frac{\alpha - 4\beta + 16\gamma}{q^3}\ . \tag{4.15}$$

This fixes the undetermined coefficient A in (3.1) as

$$A = \frac{1}{192}(\alpha - 4\beta + 16\gamma)\ . \tag{4.16}$$

Notice that this result perfectly agrees with the supersymmetric Rényi entropy of the known $(1,0)$ fixed points listed in table 1.

5 Relation with anomaly polynomial

Inspired by the relation between the supersymmetric Casimir energy and the anomaly polynomial [47], we conjecture in this section a relation between the supersymmetric Rényi entropy and the anomaly polynomial. Following this relation, the supersymmetric Rényi entropy in even dimensions can be extracted directly from the anomaly polynomial of the theory. We conjecture that S_q is determined by an equivariant integral of the anomaly polynomial $\mathcal{I}_{D+2}$ with respect to the subalgebra formed by generators $(r, h_{j=1,\dots D/2}, h_{[\frac{D}{2}+1]})$, where r is the R-symmetry Cartan generator and h_j is the j-th orthogonal rotation generator in $\mathbb{R}^D$, while $h_{[\frac{D}{2}+1]}$ generates an additional U(1) rotation. We emphasize that we do not have yet a physical understanding of the extra U(1), but just employ it in the same way as the other rotational U(1)'s. We will check our conjecture against existing data in

[19] $f := \frac{I}{V}$, $I := -\log Z$.

6d and 4d. To simplify the notation, we will use $\widetilde{h} = h_{[\frac{D}{2}+1]}$ from now on. The Cartan generators commuting with a given supercharge Q have the corresponding chemical potentials denoted by $\sigma, \vec{\omega}, \widetilde{\omega}$. Define an equivariant integral[20]

$$F(\sigma, \vec{\omega}, \widetilde{\omega}) = \int_{(\sigma,\vec{\omega},\widetilde{\omega})} \mathcal{I}_{D+2} \tag{5.1}$$

with the corresponding chemical potentials as the equivariant parameters. The supersymmetric Rényi entropy can be determined as follow

$$S_q = V_{\mathbb{H}^1} \frac{qF_1 - F_q}{1-q}\,, \quad F_q = F(\sigma, \vec{\omega}, \widetilde{\omega})\Big|_{\vec{\omega}=\vec{1},\widetilde{\omega}=1/q}\,. \tag{5.2}$$

Note that in the second equation in (5.2), the supersymmetric constraint for the chemical potentials was implicitly assumed. A volume $V_{\mathbb{H}^1} = 2\log(\ell/\epsilon)$ was factorized in S_q because we work effectively on $\mathbb{S}^{D-1}_q \times \mathbb{H}^1$. We will test this conjecture for SCFTs in 4d and 6d in the following subsections. We have not been able, so far, to prove this conjecture. The fact that an equivariant integral appears in this conjecture may hint towards some localization.

5.1 Six dimensions

In $\mathbb{R}^6$, there is a $\mathrm{U}(1)^3$ subalgebra in the rotation symmetries. The generators commuting with the supercharge have the corresponding chemical potentials, $\omega_{1,2,3}$. The additional chemical potential is $\widetilde{\omega} = \omega_4$. Consider superconformal theories with $\mathrm{SU}(2)_R$ R-symmetry. For the 8-form anomaly polynomial given in (4.6), the replacement rule in carrying out the equivariant integration (5.1) should be

$$c_2(R) \to -\sigma^2\,, \quad p_1(T) \to \sum_{i=1}^{4} \omega_i^2\,, \quad p_2(T) \to \sum_{i<j}^{4} \omega_i^2 \omega_j^2\,. \tag{5.3}$$

After these replacements in the anomaly polynomial, we divide it by $\widetilde{e}(T) = \omega_1\omega_2\omega_3\omega_4$. The result is given by

$$F(\sigma, \omega_{1,2,3,4}) = -\frac{1}{24\omega_1\omega_2\omega_3\omega_4}\left(\alpha\,\sigma^4 - \beta\,\sigma^2 \sum_{j=1}^{4}\omega_j^2 + \gamma\left(\sum_{j=1}^{4}\omega_j^2\right)^2 + \delta\left(\sum_{i<j}^{4}\omega_i^2\omega_j^2\right)\right)\,. \tag{5.4}$$

Upon plugging in

$$\sigma = \frac{1}{2}\sum_{i=1}^{4}\omega_i\,, \quad \omega_1 = \omega_2 = \omega_3 = 1\,, \omega_4 = 1/q\,, \tag{5.5}$$

one obtains

$$\frac{S_q}{V_{\mathbb{H}^1}} = \frac{qF_1 - F_q}{1-q} = \frac{\alpha - 4\beta + 16\gamma}{384q^3} + \frac{13\alpha - 28\beta + 16\gamma}{384q^2} + \frac{67\alpha - 76\beta + 112\gamma + 48\delta}{384q} + \frac{1}{384}(175\alpha - 148\beta + 112\gamma + 48\delta). \tag{5.6}$$

[20] One can come up, for now, with some loose arguments that this equivariant integral gives the coefficient of the universal log divergence in the free energy on a general D-dimensional squashed sphere.

The above result can be rewritten as S_ν,

$$S_\nu = \frac{1}{192}(\nu-1)^3(\alpha-4\beta+16\gamma)+\frac{1}{24}(\nu-1)^2(2\alpha-5\beta+8\gamma)$$
$$+\frac{1}{4}(\nu-1)(2\alpha-3\beta+4\gamma+\delta)+\frac{1}{6}(8\alpha-8\beta+8\gamma+3\delta)\ . \tag{5.7}$$

This agrees precisely with (3.14). Remarkably, a single conjectured formula by the equivariant integral (5.2) can give the a-anomaly, $c_{1,2,3}$-anomalies and also a certain part of the supersymmetric Casimir energy simultaneously and precisely. We consider these agreements as a strong support of both our results (1.6) and the conjecture itself.

5.2 Four dimensions

In $\mathbb{R}^4$, there is a $\mathrm{U}(1)^2$ subalgebra in the rotation symmetries. The generators commuting with the supercharge have the corresponding chemical potentials, $\omega_{1,2}$. The additional chemical potential is $\widetilde{\omega}=\omega_3$. Consider superconformal theories with $\mathrm{U}(1)_R$ R-symmetry. The 6-form anomaly polynomial is

$$\mathcal{I}_6=\frac{1}{3!}(k_{RRR}c_1(R)^3+k_Rc_1(R)p_1(T))\ . \tag{5.8}$$

The supersymmetric Casimir energy is given by the equivariant integral of $\mathcal{I}_6$ [47]

$$E_4=\int\mathcal{I}_6=\frac{k_{RRR}}{6\omega_1\omega_2}\sigma^3-\frac{k_R}{24\omega_1\omega_2}(\omega_1^2+\omega_2^2)\sigma\,, \tag{5.9}$$

where the chemical potentials satisfy a supersymmetric constraint $\sigma=\frac{1}{2}(\omega_1+\omega_2)$. Note that the relation between the conformal and the 't Hooft anomalies in a $4d$ $\mathcal{N}=1$ theory is

$$k_{RRR}=\frac{16}{9}(5a-3c)\,,\quad k_R=16\,(a-c)\ . \tag{5.10}$$

Plugging this in (5.9), one reproduces the familiar result [59, 60]

$$E_4=\frac{2}{3}(a-c)(\omega_1+\omega_2)+\frac{2}{27}(3c-2a)\frac{(\omega_1+\omega_2)^3}{\omega_1\omega_2}\ . \tag{5.11}$$

For our purpose, the equivariant parameters have been generalized to $\sigma,\omega_1,\omega_2,\omega_3$. The equivariant integration (5.9) now becomes

$$F(\sigma,\omega_{1,2,3})=\frac{k_{RRR}}{6\omega_1\omega_2\omega_3}\sigma^3-\frac{k_R}{24\omega_1\omega_2\omega_3}(\omega_1^2+\omega_2^2+\omega_3^2)\sigma\,, \tag{5.12}$$

with a constraint $\sigma=\frac{1}{2}(\omega_1+\omega_2+\omega_3)$. Evaluating the supersymmetric Rényi entropy (5.2), one obtains

$$S_\nu=\frac{3}{8}(k_R-3k_{RRR})+\left(\frac{5}{24}k_R-\frac{3}{8}k_{RRR}\right)(\nu-1)+\frac{1}{24}(k_R-k_{RRR})(\nu-1)^2\,, \tag{5.13}$$
$$=-4\,a-\frac{4}{3}\,c\,(\nu-1)-\frac{4}{27}\,(3c-2a)\,(\nu-1)^2\ . \tag{5.14}$$

This is precisely the universal supersymmetric Rényi entropy in $4d$ $\mathcal{N}=1$. A few remarks are in order. The leading coefficient in large ν, $-4(3c-2a)/27$, precisely agrees with the result in [38]. The first ν-derivative at $\nu=1$, $-4c/3$, agrees with (A.23).[21] The constant term, $-4a$, agrees with the linear relation between the a-anomaly and the entanglement entropy in [31]. From (5.13) to (5.14), we have used the relations (5.10). Demanding the equivalence between (5.13) and (5.14), one can reproduce the famous known relations between the conformal and the 't Hooft anomalies.

6 Discussion

In this paper we proposed a closed formula for the universal log term of the six-dimensional supersymmetric Rényi entropy and made a conjecture that the supersymmetric Rényi entropy in even dimensions is equal to an equivariant integral of the anomaly polynomial. It remains a challenging problem to understand the extra U(1) and to prove this conjecture. We leave it for future work.

Let us mention a few other open question and further directions of research that are related to this work.

1. Proving our assumption that the expansion of the supersymmetric Rényi entropy in $1/q$ terminates (it is just a polynomial of $1/q$ with degree 3 in $6d$). For this we need the dependence of possible counter-terms on $1/q$. Hence, we have to construct the six-dimensional supersymmetric curved background and in particular the smooth squashed six-sphere. The super-Weyl anomalies constructed on this background will give the universal part of the supersymmetric Rényi entropy. This approach will, hopefully, allow us to prove our assumption (C) in the introduction.

2. A generalization of the discussion in appendix A implies that the third derivative of the supersymmetric Rényi entropy is related to a specific linear combination of 4-point functions of the stress tensor and other operators in its multiplet. On the other hand, according to our result (1.6), (1.7) it is related to s_3 and hence via (1.1) and (1.15) to the Weyl anomalies. In $6d$, this is indeed consistent with a long time expectation that the a-anomaly should determine some specific term in the 4-point function of the stress tensor. This consistency becomes manifest for $(2,0)$ theories (3.15). It would be nice to demonstrate the relation between $S_\nu'''|_{\nu=1}$ and the integrated 4-point functions of operators in the stress tensor multiplet in a straight forward way.

3. The supersymmetric Rényi entropy has been proven to satisfy the four inequalities [61],
$$\partial_q S_q \le 0\,, \quad \partial_q\left(\frac{q-1}{q}S_q\right)\ge 0\,, \quad \partial_q((q-1)S_q)\ge 0\,, \quad \partial_q^2((q-1)S_q)\le 0\,. \tag{6.1}$$
Imposing these information theory inequalities for the supersymmetric Rényi entropy, one can get bounds on the 't Hooft anomaly coefficients. For $4d$ $\mathcal{N}=1$ superconformal theories, plugging (5.14) into (6.1) one obtains $\frac{3}{7}\le\frac{a}{c}\le\frac{3}{2}$. Notice that the

[21] In the sense that $\partial_\nu S_{\nu=1}$ is a particular linear combination of C_T and C_J, therefore proportional to c.

lower bound is not as tight as the $4d$ $\mathcal{N}=1$ Hofman-Maldacena bounds. For $6d$ $(1,0)$ superconformal theories, plugging (3.14) into (6.1) one obtains[22]

$$P_1 := \alpha - 4(\beta - 4\gamma) \geq 0\,, \tag{6.2}$$
$$P_2 := 3\alpha - 2\beta \geq 0\,, \tag{6.3}$$
$$67\alpha - 76\beta + 16(7\gamma + 3\delta) \geq 0\,, \tag{6.4}$$
$$9\alpha - 8(\beta - 2\gamma - \delta) \geq 0\,. \tag{6.5}$$

It is interesting to clarify the relations among different bounds in $6d$: the information theory bounds shown above, the unitary bound $C_T \propto c_3 \geq 0$ which reads

$$2\alpha - 3\beta + 4\gamma + \delta \geq 0\,, \tag{6.6}$$

and the $6d$ supersymmetric Hofman-Maldacena bounds (obtained by free-multiplet estimaiton) in terms of $\alpha, \beta, \gamma, \delta$,[23]

$$P_3 := 8\alpha - 6\beta + 4\gamma + 7\delta \geq 0 \tag{6.7}$$
$$P_4 := 2\alpha - 9\beta + 16\gamma - 2\delta \geq 0\,. \tag{6.8}$$

It is interesting to notice that, P_1 is equal to s_3 in (1.7) and, from $P_3 \geq 0$ and $P_4 \geq 0$ one can derive both $C_T \propto c_3 \propto s_1 \geq 0$ and $s_2 \geq 0$. This indicates that the inequalities $P_1 \geq 0, P_2 \geq 0, P_3 \geq 0, P_4 \geq 0$ are more fundamental than the others. Moreover, combining the information theory bounds (6.2), (6.3) and the Hofman-Maldacena bounds (6.7), (6.8), one obtains

$$s_0 = \frac{7}{12}\bar{a} = \frac{1}{6}(8\alpha - 8\beta + 8\gamma + 3\delta) = \frac{1}{48}(P_1 + 9P_2 + 4P_3 + 2P_4) \geq 0\,, \tag{6.9}$$

which gives a proof of the positivity of the a-anomaly. We leave further investigation on different bounds for future work.

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[22] It would be interesting to understand whether these bounds (or some of them) can be saturated by some specific theories.

[23] In terms of c_1 and c_2, the Hofman-Maldacena bound reads from table 1, $20c_2 \leq c_1 \leq \frac{40}{11}c_2$. We thank Clay Cordova for telling us about the free-multiplet estimation approach.

A Perturbative expansion around $q = 1$

We review the perturbative expansion of supersymmetric Rényi entropy (associated with spherical entangling surface) around $q = 1$. The great details have been given in [43] and we will be brief. Although our main concern will be $6d$ $(1, 0)$ SCFTs, we keep the discussions valid for any SCFT with conserved R-symmetry in d-dimensions.

Consider the supersymmetric partition function on $\mathbb{S}^1_{\beta=2\pi q} \times \mathbb{H}^{d-1}$ with R-symmetry background fields (chemical potentials),

$$Z[\beta, \mu] = \mathrm{Tr}\left(e^{-\beta(\hat{E}-\mu\hat{Q})}\right) . \tag{A.1}$$

which can be used to compute the supersymmetric Rényi entropy associated with a spherical entangling surface in flat space. We work with the grand canonical ensemble. The state variables can be computed as follows

$$E = \left(\frac{\partial I}{\partial \beta}\right)_\mu - \frac{\mu}{\beta}\left(\frac{\partial I}{\partial \mu}\right)_\beta , \tag{A.2}$$

$$S = \beta\left(\frac{\partial I}{\partial \beta}\right)_\mu - I , \tag{A.3}$$

$$Q = -\frac{1}{\beta}\left(\frac{\partial I}{\partial \mu}\right)_\beta , \tag{A.4}$$

where $I := -\log Z$. The energy expectation value is given by (A.2)

$$E = \frac{\mathrm{Tr}(\rho\hat{E})}{\mathrm{Tr}(\rho)} , \qquad \rho = e^{-\beta(\hat{E}-\mu\hat{Q})} , \tag{A.5}$$

and the charge expectation value is given by (A.4)

$$Q = \frac{\mathrm{Tr}(\rho\hat{Q})}{\mathrm{Tr}(\rho)} . \tag{A.6}$$

In the presence of supersymmetry, both β and μ are functions of a single variable q therefore I is considered as

$$I_q := I[\beta(q), \mu(q)] . \tag{A.7}$$

The supersymmetric Rényi entropy is defined as

$$S_q = \frac{qI_1 - I_q}{1-q} . \tag{A.8}$$

Consider the Taylor expansion around $q = 1$, with $\delta q := q - 1$,

$$S_q = S_{\mathrm{EE}} + \sum_{n=2}^{\infty} \frac{1}{n!}\frac{\partial^n I_q}{\partial q^n}\bigg|_{q=1} \delta q^{n-1} . \tag{A.9}$$

The first q-derivative of I_q is given by

$$I'_q = \left(\frac{\partial I}{\partial \beta}\right)_\mu \beta'(q) + \left(\frac{\partial I}{\partial \mu}\right)_\beta \mu'(q) . \tag{A.10}$$

Using (A.2) and (A.4), one can rewrite it as

$$I'_q = (E - \mu Q)\,\beta'(q) - \beta Q\,\mu'(q)\ . \tag{A.11}$$

Plugging in the supersymmetric background,

$$\beta(q) = 2\pi q\,, \quad \mu(q) = g\frac{q-1}{q}\,, \tag{A.12}$$

one finally has

$$I'_q = 2\pi(E - gQ)\ . \tag{A.13}$$

Notice that $\mu(q)$ is solved from the Killing spinor equation. g is some number depending on the R-charge of the preserved Killing spinor. In general both E and Q are functions of q. Moreover, E and Q here are expectation values rather than operators.

A.1 $S'_{q=1}$ and $I''_{q=1}$

From (A.9) we see that

$$S'_{q=1} = \frac{1}{2}I''_{q=1}\ . \tag{A.14}$$

Let us take one more q-derivative of (A.13) and make use of (A.5) and (A.6)

$$I''_q = -4\pi^2\left(\frac{\mathrm{Tr}\left(\rho(\hat{E}-g\hat{Q})^2\right)}{\mathrm{Tr}(\rho)} - \frac{\left[\mathrm{Tr}\left(\rho(\hat{E}-g\hat{Q})\right)\right]^2}{[\mathrm{Tr}(\rho)]^2}\right)\,, \tag{A.15}$$

which can be simplified by using $\rho_0 = \rho(\mu = 0)$ at $q = 1$

$$S'_{q=1} = -2\pi^2\left(\frac{\mathrm{Tr}\left(\rho_0(\hat{E}-g\hat{Q})^2\right)}{\mathrm{Tr}(\rho_0)} - \frac{\left[\mathrm{Tr}\left(\rho_0(\hat{E}-g\hat{Q})\right)\right]^2}{[\mathrm{Tr}(\rho_0)]^2}\right)_{q=1}\ . \tag{A.16}$$

Eq. (A.16) can be written as connected correlators

$$S'_{q=1} = -2\pi^2\left[\langle\hat{E}\hat{E}\rangle^c + g^2\langle\hat{Q}\hat{Q}\rangle^c - 2g\langle\hat{E}\hat{Q}\rangle^c\right]_{\mathbb{S}^1_{q=1}\times\mathbb{H}^{d-1}}\,, \tag{A.17}$$

where we have used the fact that, $\hat{Q}$ is a conserved charge, $[\hat{E},\hat{Q}] = 0$, to flip the order of $\hat{E}$ and $\hat{Q}$. Given that $\langle\hat{E}\hat{Q}\rangle^c = 0$ and $\langle\hat{E}\hat{E}\rangle^c$ has been computed in [53], we get

$$S'_{q=1} = -V_{d-1}\frac{\pi^{d/2+1}\Gamma(d/2)(d-1)}{(d+1)!}C_T - 2\pi^2 g^2\int_{\mathbb{H}^{d-1}}\int_{\mathbb{H}^{d-1}}\langle J_\tau(x)J_\tau(y)\rangle^c_{q=1}\ . \tag{A.18}$$

C_T is defined through the flat space correlator

$$\langle T_{ab}(x)T_{cd}(0)\rangle = \frac{C_T}{x^{2d}}I_{ab,cd}(x)\,, \tag{A.19}$$

where

$$\begin{aligned} I_{ab,cd}(x) &= \frac{1}{2}\left(I_{ac}(x)I_{bd}(x) + I_{ad}(x)I_{bc}(x)\right) - \frac{1}{d}\delta_{ab}\delta_{cd}\,, \\ I_{ab}(x) &= \delta_{ab} - 2\frac{x_a x_b}{x^2}\ . \end{aligned} \tag{A.20}$$

Now the task is to compute the second term in (A.18). Following the way of computing $\langle TT\rangle$ on the hyperbolic space $\mathbb{S}^1_{q=1}\times\mathbb{H}^{d-1}$, one can make use of the flat space correlators in the CFT vacuum,

$$\langle\hat{Q}\hat{Q}\rangle^c = -\frac{\pi^{\frac{d-1}{2}}V_{d-1}}{2^{d-2}(d-1)\Gamma(\frac{d-1}{2})}C_J\,, \tag{A.21}$$

where C_J is defined through the R-current correlator in flat space

$$\langle J_a(x)J_b(0)\rangle = \frac{C_J}{x^{2(d-1)}}I_{ab}(x)\ . \tag{A.22}$$

Our final result of $S'_{q=1}$ becomes

$$S'_{q=1} = -V_{d-1}\left(\frac{\pi^{\frac{d}{2}+1}\Gamma(\frac{d}{2})(d-1)}{(d+1)!}C_T - g^2\frac{\pi^{\frac{d+3}{2}}}{2^{d-3}(d-1)\Gamma(\frac{d-1}{2})}C_J\right)\,, \tag{A.23}$$

which shows that the first q-derivative of S_q at $q=1$ is given by a linear combination of C_T and C_J. This is intuitively expected because in the presence of supersymmetry, taking the derivative with respect to q is equivalent to taking the derivative with respect to $g_{\tau\tau}$ and A_τ at the same time.

In the particular case of $6d$ $(1,0)$ SCFTs, the 2-point function of the stress tensor is determined by the central charge c_3. Therefore the integrated 2-point function is proportional to c_3. Moreover, $S'_{q=1}$ is also proportional to c_3, because the stress tensor and the R-current on the right hand side of (A.23) live in the same multiplet.

A.2 $S''_{q=1}$ and $I'''_{q=1}$

From (A.9) we see that

$$S''_{q=1} = \frac{1}{6}I'''_{q=1}\ . \tag{A.24}$$

It is straightforward to compute I'''_q by taking one more derivative on (A.15)

$$\begin{aligned}\frac{I'''_q}{8\pi^3} = {} & \frac{\mathrm{Tr}\left(\rho(\hat{E}-g\hat{Q})^3\right)}{\mathrm{Tr}(\rho)} - 3\frac{\mathrm{Tr}\left(\rho(\hat{E}-g\hat{Q})^2\right)\mathrm{Tr}\left(\rho(\hat{E}-g\hat{Q})\right)}{[\mathrm{Tr}(\rho)]^2} \\ & +2\frac{\left[\mathrm{Tr}\left(\rho(\hat{E}-g\hat{Q})\right)\right]^3}{[\mathrm{Tr}(\rho)]^3}\,,\end{aligned} \tag{A.25}$$

which may be simplified at $q=1$ where $\mu=0$

$$\begin{aligned}\frac{I'''_{q=1}}{8\pi^3} = {} & \left(\frac{\mathrm{Tr}\left(\rho_0(\hat{E}-g\hat{Q})^3\right)}{\mathrm{Tr}(\rho_0)} - 3\frac{\mathrm{Tr}\left(\rho_0(\hat{E}-g\hat{Q})^2\right)\mathrm{Tr}\left(\rho_0(\hat{E}-g\hat{Q})\right)}{[\mathrm{Tr}(\rho_0)]^2}\right. \\ & \left. +2\frac{\left[\mathrm{Tr}(\rho_0(\hat{E}-g\hat{Q}))\right]^3}{[\mathrm{Tr}(\rho_0)]^3}\right)_{q=1}\ .\end{aligned} \tag{A.26}$$

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This can be further written in terms of connected correlation functions,

$$S''_{q=1}=\frac{1}{6}I'''_{q=1}=\frac{4\pi^3}{3}\left[\langle\hat{E}\hat{E}\hat{E}\rangle^c-g^3\langle\hat{Q}\hat{Q}\hat{Q}\rangle^c-3g\langle\hat{E}\hat{E}\hat{Q}\rangle^c+3g^2\langle\hat{E}\hat{Q}\hat{Q}\rangle^c\right]_{\mathbb{S}^1_{q=1}\times\mathbb{H}^{d-1}}, \tag{A.27}$$

where we have used $[\hat{E},\hat{Q}]=0$. The integrated correlators in (A.27) can be computed by transforming the corresponding flat space correlators, $\langle TTT\rangle,\langle JJJ\rangle,\langle TTJ\rangle,\langle TJJ\rangle$ in the CFT vacuum. These correlators in flat space can be determined up to some coefficients for d-dimensional CFTs by conformal Wald identities [32, 33]. In the case of 6d (1, 0) SCFTs, both the 2- and 3-point functions of the stress tensor multiplet can be determined in terms of three coefficients $c_{1,2,3}$.[24] Therefore the right hand side of (A.27) should be proportional to some linear combinations of $c_{1,2,3}$, because the stress tensor and the R-current belong to the same multiplet.

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