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Monopole star products are non-alternative

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Abstract: Non-associative algebras appear in some quantum-mechanical systems, for instance if a charged particle in a distribution of magnetic monopoles is considered. Using methods of deformation quantization it is shown here, that algebras for such systems cannot be alternative, i.e. their associator cannot be completely anti-symmetric.

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1 Introduction

Deformation quantization [1, 2] has been explored much in the associative setting. If one drops the condition that the star product be associative, some of the usual methods are no longer available. The classification of such star products therefore remains open. In this paper, we present one general result in this direction, motivated by a recent resurgence of interest in magnetic-monopole systems [3–8], where standard quantization methods show that associative algebras cannot constitute consistent quantizations of the relevant observables [9, 10].

In the original version of deformation quantization, associativity of the star product represents an important condition on the coefficients in the formal power series of the product. If one works with star products without the condition of associativity, at first sight it may seem easier to find acceptable versions because they may appear to be subject to fewer consistency requirements. However, if one is forced to use a non-associative star product for physical reasons, one is not fully liberated from imposing conditions on the associator

$$[a, b, c] = a * (b * c) - (a * b) * c\,. \tag{1.1}$$

For a specific set of basic observables, the associator, like the usual commutator

$$[a, b] = a * b - b * a\,, \tag{1.2}$$

is prescribed based on physical arguments.

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Formulated for position and momentum components as basic observables, the commutator of an acceptable star product should be $[q_i, p_j] = i\hbar\{q_i, p_j\} = i\hbar\delta_{ij}$, mimicking the Poisson bracket. If these are coordinates of a charged particle (with electric charge e) moving in the magnetic field $B^l(q_i)$ of a magnetic monopole distribution, so that $\mathrm{div}B = \partial_l B^l \neq 0$, the classical brackets are modified: they are twisted Poisson brackets for which the Jacobi identity does not hold [11–13]. An algebra that quantizes the bracket endows phase-space functions with a new product $\star$ and the associated commutator (1.2) and associator (1.1). The Jacobiator of the commutator is proportional to the totally antisymmetric part of the associator and can be non-zero for non-associative $\star$-products. In the present context, one is led to the relations [9, 10]

$$
\begin{aligned}
[q_i, q_j] &= 0 && (1.3)\\
[q_i, p_j] &= i\hbar\delta_{ij} && (1.4)\\
[p_i, p_j] &= i\hbar e\epsilon_{ijk}B^k && (1.5)\\
\left[q_i, x^I, x^J\right] &= 0 && (1.6)\\
[p_i, p_j, p_k] &= -\hbar^2 e\epsilon_{ijk}\partial_l B^l && (1.7)
\end{aligned}
$$

to be realized by a star product. Here $(x^I)_{I=1}^6$ is a collective notation for the Cartesian coordinates $(q_i, p_i)_{i=1}^3$. In the absence of a magnetic charge density, one can introduce a canonical momentum π_i with zero brackets for its components. However, the definition, $\pi_i := p_i + A_i$, makes use of a vector potential A through $B = \mathrm{rot}A$, which does not exist if $\mathrm{div}B$ does not vanish. Instead of a zero associator in standard star products, the specific form of (1.7) imposes restrictions on acceptable star products for magnetic-monopole systems.

Most of the usual properties of quantum mechanics are no longer valid and must be modified when observables cannot be represented as associative operators on a Hilbert space. In some studies, a weaker condition given by an alternative algebra has been found advantageous [14–16] — if it can be realized. An alternative algebra is one where the associator (1.1) is completely antisymmetric, or, equivalently, where the $*$-product obeys

$$
\begin{aligned}
a * (a * b) &= (a * a) * b\\
(a * b) * b &= a * (b * b) \qquad (1.8)
\end{aligned}
$$

for any a, b in the algebra. Many well-known non-associative algebras are of this form, such as the octonionic ones. Requiring an algebra to be alternative, provides a priori a tempting option for the case of a charged particle in the background of magnetic monopoles, in particular in view of the total anti-symmetry of the basic relation (1.7).

However, in this report we demonstrate the impossibility of such an algebra as a set of quantized observables of a charged particle in the presence of magnetic monopole densities, obtained by deformation quantization. While (1.7) implies a totally antisymmetric associator for linear functions of the basic observables, the associator of general algebra elements is not guaranteed to be totally antisymmetric. Different examples for algebras consistent with the relations (1.3)–(1.7) have been constructed using star products [3–8], one of which has explicitly been shown to be non-alternative [17, 18]. In what follows, we will analyze the possibility of alternative monopole star products in general terms, using deformation theory, the basics of which we first recall in the next section.

2 Deformation quantization with non-associativity

The classical theory is described by the commutative algebra of smooth functions on $T^*\mathbb{R}^3$, equipped with the bivector field[1]

$$\Pi = \left(\frac{\partial}{\partial q_i} + \epsilon_{jik} B^k(q) \frac{\partial}{\partial p_j}\right) \wedge \frac{\partial}{\partial p_i} \,, \tag{2.1}$$

in the canonical linear coordinates $(x^I)_{I=1}^6 \equiv (q_1, q_2, q_3, p_1, p_2, p_3)$. For a vector field B with non-vanishing divergence, this is only a twisted Poisson bivector: its Schouten bracket with itself does not vanish but is given by

$$\frac{1}{2}[\Pi, \Pi] = \Pi^\sharp(H) \tag{2.2}$$

where the 3-form H takes the form

$$H = \pi^* \mathrm{d}B \,. \tag{2.3}$$

Here the magnetic field B is considered a 2-form on $\mathbb{R}^3$ by means of $B = \epsilon_{ijk} B^i \mathrm{d}q^j \wedge \mathrm{d}q^k$ and $\pi\colon T^*\mathbb{R}^3 \to \mathbb{R}^3$ is the canonical projection. Maxwell's equations link $\mathrm{d}B$ directly to the magnetic monopole density: $\mathrm{d}B = *\rho_{\text{magnetic}}$.

The bivector field Π then induces the following bracket on the functions $f, g \in C^\infty(T^*\mathbb{R}^3)$,

$$\{f, g\} = \frac{1}{2} \Pi^{IJ}(x) \frac{\partial f}{\partial x^I} \frac{\partial g}{\partial x^J} \,. \tag{2.4}$$

This bracket is an antisymmetric bi-derivation, but no longer a Lie bracket and thus not a Poisson bracket: the r.h.s. of (2.2) provides precisely the non-zero Jacobiator.

2.1 Star product

Deformation quantization turns the classical commutative algebra $(C^\infty(T^*\mathbb{R}^3), \cdot)$ into the quantum algebra $\mathcal{A} := (C^\infty(T^*\mathbb{R}^3)[[\lambda]], \star)$, where $\lambda = \frac{1}{2} i\hbar$ is considered as a formal deformation or expansion parameter:

$$f \star g = \sum_{j=0}^{\infty} \lambda^j B_j(f, g) \,. \tag{2.5}$$

Here $B_j\colon \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ are bilinear maps on $\mathcal{A}$.[2] To zeroth order in λ, we have the classical product given by pointwise multiplication, $B_0(f, g) = f \cdot g \equiv fg$. Following [1], we will

[1] We set the electric charge to $e = 1$ from now on.

[2] Using the same letter for these bilinear maps and the magnetic field should not cause confusion.

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assume that B_j is a bi-differential operator of maximum degree j which is zero on constants for strictly positive j:

$$B_j(f,g) = \sum_{k,l=1}^{j} B_j^{k,l}(f,g) \qquad \text{for} \quad j \geq 1 \tag{2.6}$$

$$B_j^{k,l}(f,g) = \sum_{I_1,\dots,I_k,J_1,\dots,J_l=1}^{6} B^{k,l}_{j;I_1,\dots,I_k,J_1,\dots,J_l}(q) \frac{\partial^k f}{\partial x^{I_1}\cdots\partial x^{I_k}} \frac{\partial^l g}{\partial x^{J_1}\cdots\partial x^{J_l}} \tag{2.7}$$

The property implies in particular that the star product defines a unital algebra, with the unit function as unit.

Let us for a moment assume that $\star$ would be associative. In this case, we would have that the commutator (1.2) evidently satisfies the Jacobi identity and also that $[f, g\star h] = [f,g]\star h + g\star[f,h]$. Both equations together, evaluated at lowest non-vanishing order in λ, imply that the antisymmetric part $B_1^-(f,g) = \frac{1}{2}(B_1(f,g) - B_1(g,f))$ of $B_1(f,g)$ is a Poisson bivector. On the other hand, for physical reasons, we want that the antisymmetric part of the first order deformation is determined by the classical bracket:

$$B_1^-(f,g) = \{f,g\}\,. \tag{2.8}$$

This then shows that the $\star$-product cannot be associative for the deformation quantization of the above classical system, cf., in particular, eq. (2.2) — as anticipated already in the Introduction.

In fact, in the present article, we want to strengthen eq. (2.8) in a two-fold way: first, we require in addition that B_1 is antisymmetric itself already, so that

$$B_1(f,g) = \{f,g\}\,. \tag{2.9}$$

This, in fact, is not really a restriction: it can be shown that every star product either satisfies this condition or has an equivalent deformation for which (2.9) is fulfilled. We will come back to this below and assume it for now in any case. Second, we want that for linear coordinate functions on $T^*\mathbb{R}^3$ the bracket determines the commutator even to next-to-leading order, i.e. we require

$$\frac{x^I\star x^J - x^J\star x^I}{i\hbar} = \{x^I, x^J\} + O(\hbar^2)\,. \tag{2.10}$$

The first condition is equivalent to requiring $B_1^+(f,g) = 0$ for all functions f, g, the second one to demanding

$$B_2^-(x^I, x^J) = 0\,. \tag{2.11}$$

We remark in parenthesis that the equation (2.10) is implied if the x^I are implemented as distinguished observables in the sense of [2].

2.2 Monopole star products

Since we found above that the associator of the monopole star product cannot be zero, we also expand it into a formal power series in λ:

$$A(f,g,h) = f\star(g\star h) - (f\star g)\star h := \sum_{j=0}^{\infty} \lambda^j A_j(f,g,h)\,. \tag{2.12}$$

The maps B_i and A_j are not independent; in fact, A_j is determined by the B_i with $i \leq j$. It is easy to evaluate the low orders: we always have $A_0 = 0$, because the point-wise multiplication of phase-space functions is associative. At first order, we have

$$A_1(f,g,h) = f B_1(g,h) - B_1(f,g)h + B_1(f,gh) - B_1(fg,h) = 0 \tag{2.13}$$

simply since B_1 is bi-differential of order $(1,1)$.

At second order, one finds

$$\begin{aligned} A_2(f,g,h) \,=\, & f B_2(g,h) - B_2(f,g)h + B_2(f,gh) - B_2(fg,h) \\ & + B_1(f,B_1(g,h)) - B_1(B_1(f,g),h)\,. \end{aligned} \tag{2.14}$$

For a non-associative star product, the coefficient A_2, as the first non-zero one in the expansion (2.12), plays a role similar to the coefficient B_1 in specifying conditions on the star product as a quantization of the classical bracket. The totally antisymmetric contribution

$$\begin{aligned} A_2^-(f,g,h) := \frac{1}{6}\,\big(& A_2(f,g,h) + A_2(h,f,g) + A_2(g,h,f) \\ & - A_2(f,h,g) - A_2(g,f,h) - A_2(h,g,f)\big) \end{aligned}$$

to A_2, in view of (2.11), only depends on B_1 *if* it is evaluated on linear functions of the basic variables x^I: we have

$$A_2^-(x^I,x^J,x^K) = \frac{1}{2} J(x^I,x^J,x^K) \tag{2.15}$$

where $J(f,g,h)$ is the Jacobiator of B_1, i.e. of the classical bracket $\{\cdot,\cdot\}$. In particular, $A_2^-(p_1,p_2,p_3) = 4\pi^*\mathrm{d}B$ for a star product that quantizes a twisted Poisson bivector obeying (2.2). It is then consistent to assume that $A_2(p_1,p_2,p_3) = A_2^-(p_1,p_2,p_3)$ is totally antisymmetric, as written in the basic relation (1.7). The basic relations do not give us direct statements about A_2 evaluated on functions not linear in the global coordinates x^I. We will assume that $A_2(f,g,h)$ can be chosen totally antisymmetric even in this case — since our aim is to prove that monopole star products cannot be alternative, there would be nothing to show if this assumption were violated. However, this condition does not already imply that the star product is alternative, since non-linear functions generically lead to contributions to $A(f,g,h)$ of higher order in λ, which do not directly follow from simple combinations of the basic relations (1.7).

We summarize our conditions on A_2 in

Definition 1. *A* monopole star product *is a non-associative star product $\star$ on $C^\infty(T^*\mathbb{R}^3)[[\lambda]]$ such that (2.10) holds, its associator to second order in λ is totally antisymmetric and further obeys the following conditions:*

1. $A_2(p_1,p_2,p_3) \neq 0$,

2. $A_2(q_i,x^I,x^J) = 0$ *for all* $i = 1,2,3$ *and* $I,J = 1,\dots,6$, *and*

3. $B_1(q_i, A_2(p_1,p_2,p_3)) = 0$ *for* $i = 1,2,3$.

where $(x^I)_{I=1}^6 = (q_1,q_2,q_2,p_1,p_2,p_3)$ *are the canonical linear coordinates on* $T^*\mathbb{R}^3$.

2.3 Hochschild cohomology

For an associative algebra $\mathcal{A}$, the space of multilinear maps from $\mathcal{A}$ to itself can be equipped with a coboundary operator d, used in Hochschild cohomology. For a multilinear map $\phi\colon \mathcal{A}^{\otimes n} \to \mathcal{A}$ of n arguments, $\mathrm{d}\phi$ is a multilinear function of $n+1$ arguments given by

$$\begin{aligned} \mathrm{d}\phi(a_0,a_1,\dots,a_n) = a_0\cdot\phi(a_1,\dots,a_n) + \sum_{j=0}^{n-1}(-1)^j\phi(a_0,\dots,a_{j-1},a_j\cdot a_{j+1},a_{j+2},\dots,a_n) \\ +(-1)^n\phi(a_0,\dots,a_{n-1})\cdot a_n\,. \end{aligned} \tag{2.16}$$

Hochschild cohomology plays an important role in classifying equivalent star products with respect to a redefinition of higher orders in a λ-expansion: if

$$D(f) = \sum_{j=0}^{\infty} D_j(f)\lambda^j \tag{2.17}$$

with linear differential operators D_j starting with $D_0 = \mathrm{id}$, for any given star product $\star$ a new product $\star'$ can be defined by means of

$$D(f)\star' D(g) = D(f\star g)\,. \tag{2.18}$$

The condition on D_0 ensures that D is invertible as a map on formal power series. If functions in $C^\infty(M)$ are written as symbols of operators, for instance by a Weyl correspondence, a non-trivial map D changes the factor-ordering choice in the correspondence. To first order, $B_1' = B_1 - \mathrm{d}D_1$ while $\mathrm{d}B_1 = 0$; see (2.13). The first Hochschild cohomology therefore classifies inequivalent choices of B_1 which cannot be related by a different choice of factor ordering. For a given bracket $\{\cdot,\cdot\}$, all star products quantizing it respect the condition (2.8), but not necessarily (2.9).

If $\mathcal{A}$ is not associative, $\not{\mathrm{d}}$, defined just like d for an associative algebra, is not a coboundary operator: for a linear function $\phi\colon \mathcal{A}\to\mathcal{A}$, we have

$$\not{\mathrm{d}}\phi(a_0,a_1) = a_0\star\phi(a_1) - \phi(a_0\star a_1) + \phi(a_0)\star a_1 \tag{2.19}$$

and

$$\not{\mathrm{d}}^2\phi(a_0,a_1,a_2) = A(a_0,a_1,\phi(a_2)) + A(a_0,\phi_0(a_1),a_2) + A(\phi(a_0),a_1,a_2) - \phi_0(A(a_0,a_1,a_2)) \tag{2.20}$$

with the associator A. Therefore, Hochschild cohomology is not available for non-associative algebras. However, the coboundary operator d of the classical associative commutative algebra of smooth functions may still be used in constructing non-associative deformations, as we will do below. For instance, the product in (2.13) refers to $\cdot$, not to $\star$. Moreover, we can refer to the standard argument [19] for changing the star product within its equivalence class to show that the symmetric part in B_1 can always be set to zero and (2.9) be achieved. Thus, up to operator ordering, we can always assume that B_1 is given by the classical bracket, even if it is not Poisson, but for example twisted Poisson as here.

3 The main result

Our main result is

Theorem 1. *Let $\star$ be a monopole star product as defined above, cf. Definition 1. Then the associator $A(f,g,h) \equiv f \star (g \star h) - (f \star g) \star h$ cannot be totally antisymmetric in its arguments.*

We will prove this result by making use of three lemmas:

Lemma 1. *Let $\star$ be a star product obeying (2.10). If $\star$ is flexible at second order, that is $A_2(f,g,h) = -A_2(h,g,f)$, then B_2 is symmetric.*

Proof: We evaluate A_2 in (2.14) on functions with $f = h$, writing the result as

$$\begin{aligned} A_2(f,g,f) &= fB_2(g,f) - B_2(f,g)f + B_2(f,gf) - B_2(fg,f) \\ &= -2fB_2^-(f,g) + 2B_2^-(f,fg) \end{aligned} \tag{3.1}$$

using the antisymmetric part $B_2^-(f,g) := \frac{1}{2}(B_2(f,g) - B_2(g,f))$ of B_2. If $A_2(f,g,h) = -A_2(h,g,f)$ holds, $A_2(f,g,f) = 0$, and we obtain

$$B_2^-(f,fg) = fB_2^-(f,g)\,. \tag{3.2}$$

For an antisymmetric bi-differential form, this equation can hold only if the degree is $(1,1)$. However, if B_2^- has a contribution of degree $(1,1)$, (2.11) cannot hold. Therefore, $B_2^- = 0$ and B_2 is symmetric. □

In particular, the conclusion holds for a monopole star product (2.5). All explicit star products that have been constructed for monopole systems indeed have a symmetric B_2. For associative star products, Kontsevich's formula [20] has the same property. If symmetry of B_j holds at all even orders j, the star product gives rise to a formal deformation of the twisted Poisson bracket by powers of λ^2, or a Vey deformation as defined in [1].

Lemma 2. *If (2.5) is a star product with symmetric B_2, then the totally anti-symmetric part of A_3 is equal to zero.*

Proof: Using the definition of the associator and the star product, we derive

$$\begin{aligned} A_3(f,g,h) &= \mathrm{d}B_3(f,g,h) + B_2(f,B_1(g,h)) \\ &\quad - B_2(B_1(f,g),h) + B_1(f,B_2(g,h)) - B_1(B_2(f,g),h)\,, \end{aligned} \tag{3.3}$$

where d is the coboundary operator of Hochschild cohomology, cf. eq. (2.16). In particular, $\mathrm{d}B_3(f,g,h) \equiv fB_3(g,h) + B_3(f,gh) - hB_3(f,g) - B_3(fg,h)$. The totally anti-symmetric part A_3^- of A_3, defined as in (2.15), is given by

$$
\begin{aligned}
3A_3^-(f,g,h) = {} & B_2^-(f, 2B_1^-(g,h)) + B_2^-(h, 2B_1^-(f,g)) + B_2^-(g, 2B_1^-(h,f)) \\
& + B_1^-(f, 2B_2^-(g,h)) + B_1^-(f, 2B_2^-(g,h)) + B_1^-(f, 2B_2^-(g,h))
\end{aligned}
\tag{3.4}
$$

where, as before, $B_j^-(f,g) = \frac{1}{2}(B_j(f,g) - B_j(g,f))$ is the antisymmetric part of B_j.[3] Since all terms on the right-hand side of (3.4) contain a B_2^-, $B_2^- = 0$ implies $A_3^- = 0$. □

We remark that for the last conclusion it is important that the antisymmetric part of A_3, unlike the full A_3, does not depend on B_3.

Lemma 3. *Let $\star$ be a star product such that*

$$
\begin{aligned}
O(f,g,h,k) := {} & A_2(f,g,B_1(h,k)) - A_2(f,B_1(g,h),k) + A_2(B_1(f,g),h,k) \\
& + B_1(A_2(g,h,k),f) - B_1(A_2(f,g,h),k)
\end{aligned}
\tag{3.5}
$$

is not identically zero. Then the third-order contribution A_3 to the associator is non-zero.

Proof: Again, we use the Hochschild coboundary operator and consider

$$
\mathrm{d}A_3(f,g,h,k) = fA_3(g,h,k) - A_3(fg,h,k) + A_3(f,gh,k) - A_3(f,g,hk) + kA_3(f,g,h)\,. \tag{3.6}
$$

Our goal is to show that $\mathrm{d}A_3$ is non-zero for algebras with non-zero O, which implies immediately also that $A_3 \neq 0$. The Pentagon identity

$$
f \star A(g,h,k) + A(f,g,h) \star k = A(f \star g,h,k) - A(f,g \star h,k) + A(f,g,h \star k) \tag{3.7}
$$

for non-associative algebras can be used for a compact proof of this statement. Expanding it to third order in λ, we obtain

$$
\begin{aligned}
& fA_3(g,h,k) + B_1(f, A_2(g,h,k)) + kA_3(f,g,h) + B_1(A_2(f,g,h),k) \\
& \quad = A_3(fg,h,k) - A_3(f,gh,k) + A_3(f,g,hk) \\
& \qquad + A_2(B_1(f,g),h,k) - A_2(f,B_1(g,h),k) + A_2(f,g,B_1(h,k))
\end{aligned}
\tag{3.8}
$$

where we used $A_1 = 0$, cf. eq. (2.13). These terms can be organized to obtain

$$
\begin{aligned}
\mathrm{d}A_3(f,g,h,k) = {} & A_2(f,g,B_1(h,k)) - A_2(f,B_1(g,h),k) + A_2(B_1(f,g),h,k) \\
& + B_1(A_2(g,h,k),f) - B_1(A_2(f,g,h),k)\,.
\end{aligned}
\tag{3.9}
$$

Alternatively, one can prove directly that $\mathrm{d}A_3$ is of this form without invoking the Pentagon identity, as shown in appendix B. The right-hand side of this equation is equal to $O(f,g,h,k)$. If it is not identically zero, A_3 is non-zero. □

We are now ready to prove our main result:

[3] See appendix A for a detailed derivation of (3.4).

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Proof (of Theorem 1)**:** by Lemmas 1 and 2, a monopole star product has an A_3 with zero totally antisymmetric part. If the star product is alternative, we must then have $A_3 = 0$. If the obstruction O provided by Lemma 3 is not identically zero, however, it is not possible that $A_3 = 0$. We now show that $O \neq 0$ for a monopole star product, discussing two cases separately depending on whether the associator (the monopole density) is constant or a function of the position.

For a constant associator, we may choose $f = p_1$, $g = p_2$, $h = p_3$ and $k = q_3 p_3$. Using the twisted Poisson bracket for B_1, all but the first term in $O(f, g, h, k)$ are zero, while $A_2(f, g, B_1(h, k))$ is proportional to the monopole density and therefore non-zero.

If the monopole density is not constant, we specialize $O(f, g, h, k)$ to

$$O(f, g, h, g) = A_2(B_1(f, g), h, g) - B_1(A_2(f, g, h), g)\,. \tag{3.10}$$

Since the associator is not constant, it depends on at least one position coordinate, say q_1 without loss of generality. If we then choose $f = p_2$, $g = p_1$ and $h = p_3$ we have $B_1(A_2(f, g, h), g) \neq 0$ while $A_2(B_1(f, g), h, g) = 0$. □

The conclusion is independent of the choice of the star product within an equivalence class, with [4] or [8] as concrete examples, because alternativity is independent of the choice of the ordering (the "gauge") [21].

More generally, Lemma 3 gives us an obstruction to alternativity which only depends on B_1 and A_2, and therefore can be tested for general non-associative star products more easily than the full associator.

4 Monopole Weyl star product

Two different star products have been proposed recently for the magnetic-monopole system, one by using the Kontsevich formula [3–7], and one from Weyl products [8]. The former is known to be non-alternative [17, 18]. Since it satisfies our assumptions, it provides an explicit example for our general result. We now discuss the star product of [8] in more detail.

Example (Weyl star product): The star product of [8] has the first coefficient $B_1(f, g) = \frac{1}{2}\{f, g\}$ with an atisymmetric bracket $\{f, g\} = \frac{1}{2}\Pi^{IJ}\partial_I f \partial_J g$ given by an arbitrary bivector Π^{IJ}. It can therefore be applied to monopole star products. The second coefficient is

$$B_2(f, g) = -\frac{1}{2}\Pi^{IJ}\Pi^{KL}(\partial_I\partial_K f)(\partial_J\partial_L g) - \frac{1}{3}\Pi^{IJ}\partial_J\Pi^{KL}\left((\partial_I\partial_K f)(\partial_L g) - (\partial_K f)(\partial_I\partial_L g)\right)\,, \tag{4.1}$$

transferred to our notation. It obeys our assumptions. In particular, B_2 has no contribution of bi-differential degree $(1, 1)$, and it is symmetric thanks to the antisymmetry of the twisted Poisson tensor Π^{IJ}. Therefore, our conditions on monopole star products are satisfied and the algebra cannot be alternative. □

In [21, 22], an explicit expression for B_3 is given as well. It is therefore possible to compute A_3 in specific examples and show that it is not totally antisymmetric. In particular, for monopole star products, it is not difficult to find functions $f(p_1, p_2, p_3)$ such that $A_3(f, f, f) \neq 0$.

Lemma 4. *Let $\star$ be a Weyl star product on $C^\infty(T^*\mathbb{R}^3)[[\lambda]]$ according to [8] which quantizes a twisted Poisson tensor (2.1), and let $f(p_1,p_2,p_3)$ be a function of the fiber coordinates of $T^*\mathbb{R}^3$ such that $\partial_{p_i}\partial_{p_j}f=0$ whenever $i\neq j$. The third coefficient of the associator of $\star$ then obeys*

$$A_3(f,f,f)=\frac{4}{3}i\left(\partial_{q_1}\Pi^{p_2p_3}+\partial_{q_2}\Pi^{p_3p_1}+\partial_{q_3}\Pi^{p_1p_2}\right)\sum_{\sigma\in Z_3}\Pi^{p_{\sigma(1)}p_{\sigma(2)}}\partial_{p_{\sigma(3)}}f\,\partial^2_{p_{\sigma(1)}}f\,\partial^2_{p_{\sigma(2)}}f\,, \tag{4.2}$$

summing over elements of the alternating group $A_3=Z_3$ of cyclic permutations.

Proof: We have explicitly computed $A_3(f,f,f)$ for arbitrary f using Cadabra software [23, 24]:

$$\begin{aligned}A_3(f,f,f)=\frac{2i}{3}\Big(&\Pi^{LM}\partial_L\Pi^{NO}\,\partial_N\Pi^{PQ}\,\partial_Mf\,\partial_Pf\,\partial_O\partial_Qf\\&-\Pi^{LM}\partial_L\Pi^{NO}\,\partial_N\Pi^{PQ}\,\partial_Of\,\partial_Pf\,\partial_M\partial_Qf\\&-2\,\Pi^{LM}\Pi^{NO}\partial_L\Pi^{PQ}\,\partial_Pf\,\partial_M\partial_Nf\,\partial_O\partial_Qf\\&+\Pi^{LM}\Pi^{NO}\partial_L\Pi^{PQ}\,\partial_Mf\,\partial_{NP}f\,\partial_O\partial_Qf\Big)\,.\end{aligned} \tag{4.3}$$

For a monopole star product, the bivector Π is a function only of the position coordinates q_i via the magnetic field. Therefore, L and N must be position indices for non-zero contributions in the first two terms of (4.3). These terms are then identically zero because each contains a factor of $\partial_L\Pi^{NO}$, which is zero for a bivector of the form (2.1).

In the third term, only L is required to be a position index, while M, N, O, P, and Q are momentum indices if f depends only on momenta. The components Π^{LM} then equal δ^{LM} since they contain one position and one momentum index. The remaining terms in (4.3) yield

$$\begin{aligned}\frac{3}{2i}A_3(f,f,f)=-2\,\Pi^{NO}\big(&\partial_{q_1}\Pi^{UQ}\,\partial_Uf\,\partial_{p_1}\partial_Nf\,\partial_O\partial_Qf\;+\partial_{q_2}\Pi^{UQ}\,\partial_Uf\,\partial_{p_2}\partial_Nf\,\partial_O\partial_Qf\\&+\partial_{q_3}\Pi^{UQ}\,\partial_Uf\,\partial_{p_3}\partial_Nf\,\partial_O\partial_Qf\,\big)\\+\Pi^{NO}\big(&\partial_{q_1}\Pi^{UQ}\,\partial_{p_1}f\,\partial_N\partial_Uf\,\partial_O\partial_Qf+\partial_{q_2}\Pi^{UQ}\,\partial_{p_2}f\,\partial_N\partial_Uf\,\partial_O\partial_Qf\\&+\partial_{q_3}\Pi^{UQ}\,\partial_{p_3}f\,\partial_N\partial_Uf\,\partial_O\partial_Qf\big)\end{aligned}$$

We collect terms with the same factor of $\partial_{q_i}\Pi^{IJ}$ from derivatives of the bivector. Such a contribution with $\partial_{q_1}\Pi^{IJ}$ is of the form

$$\begin{aligned}&\Pi^{NO}\left(-2\partial_{q_1}\Pi^{UQ}\,\partial_Uf\,\partial_{p_1}\partial_Nf\,\partial_O\partial_Qf+\partial_{q_1}\Pi^{UQ}\,\partial_{p_1}f\,\partial_N\partial_Uf\,\partial_O\partial_Qf\right)\\&\quad=\Pi^{NO}\big(\partial_{p_1}f\left(-\partial_{q_1}\Pi^{p_1Q}\partial_{p_1}\partial_Nf+\partial_{q_1}\Pi^{p_2Q}\,\partial_N\partial_{p_2}f+\partial_{q_1}\Pi^{p_3Q}\,\partial_N\partial_{p_3}f\right)\\&\qquad-2\partial_{p_2}f\partial_{q_1}\Pi^{p_2Q}\,\partial_{p_1}\partial_Nf-2\partial_{p_3}f\partial_{q_1}\Pi^{p_3Q}\,\partial_{p_1}\partial_Nf\big)\partial_O\partial_Qf\,,\end{aligned}$$

arranging by factors of first-order derivatives $\partial_{p_i}f$. By our assumptions on f, the index N

is determined in all terms for non-zero contributions and we obtain

$$\Big(\partial_{p_1} f\,(-\Pi^{p_1 O}\partial_{q_1}\Pi^{p_1 Q}\partial^2_{p_1} f + \Pi^{p_2 O}\partial_{q_1}\Pi^{p_2 Q}\,\partial^2_{p_2} f + \Pi^{p_3 O}\partial_{q_1}\Pi^{p_3 Q}\,\partial^2_{p_3} f)$$
$$-2\partial_{p_2} f\Pi^{p_1 O}\partial_{q_1}\Pi^{p_2 Q}\,\partial^2_{p_1} f - 2\partial_{p_3} f\Pi^{p_1 O}\partial_{q_1}\Pi^{p_3 Q}\,\partial^2_{p_1} f)\Big)\partial_O\partial_Q f$$
$$= \sum_O \big(\partial_{p_1} f\,(-\Pi^{p_1 O}\partial_{q_1}\Pi^{p_1 O}\partial^2_{p_1} f + \Pi^{p_2 O}\partial_{q_1}\Pi^{p_2 O}\,\partial^2_{p_2} f + \Pi^{p_3 O}\partial_{q_1}\Pi^{p_3 O}\,\partial^2_{p_3} f)$$
$$-2\partial_{p_2} f\Pi^{p_1 O}\partial_{q_1}\Pi^{p_2 O}\,\partial^2_{p_1} f - 2\partial_{p_3} f\Pi^{p_1 O}\partial_{q_1}\Pi^{p_3 O}\,\partial^2_{p_1} f)\partial^2_O f)$$

setting $O = Q$ in the last step, again by our assumptions on f. We now go through all remaining choices of the only free index O. All contributions to terms containing $\partial_{q_1}\Pi^{p_1 O}$ cancel out. We arrive at

$$2\partial_{p_1} f\Pi^{p_2 p_3}\partial_{q_1}\Pi^{p_2 p_3}\partial^2_{p_2} f\partial^2_{p_3} f - 2\partial_{p_2} f\Pi^{p_1 p_3}\partial_{q_1}\Pi^{p_2 p_3}\partial^2_{p_1} f\partial^2_{p_3} f - 2\partial_{p_3} f\Pi^{p_1 p_2}\partial_{q_1}\Pi^{p_3 p_2}\partial^2_{p_1} f\partial^2_{p_2} f$$
$$= 2\partial_{q_1}\Pi^{p_2 p_3}\sum_{\sigma\in Z_3}\Pi^{p_{\sigma(1)}p_{\sigma(2)}}\partial_{p_{\sigma(3)}} f\partial^2_{p_{\sigma(1)}} f\partial^2_{p_{\sigma(2)}} f\,.$$

Bringing back contributions with the remaining $\partial_{q_i}\Pi^{IJ}$, we have (4.2). □

For specific choices of f obeying the condition stated in the Lemma, we can compute $A_3(f, f, f)$ more explicitly. The first parenthesis in (4.2) is half the Jacobiator of the bivector, which is non-zero for a monopole star product. The sum over cyclic permutations depends on the specific f.

Example: Let Π be a bivector as stated in the conditions on a monopole star product.

1. Let $f = |p|^2 = p_1^2 + p_2^2 + p_3^2$. We have

   $$\sum_{\sigma\in Z_3}\Pi^{p_{\sigma(1)}p_{\sigma(2)}}\partial_{p_{\sigma(3)}} f\partial^2_{p_{\sigma(1)}} f\partial^2_{p_{\sigma(2)}} f = 8\sum_{\sigma\in Z_3}\Pi^{p_{\sigma(1)}p_{\sigma(2)}} p_{\sigma(3)}\,.$$

   With a bivector as implied by (1.5),

   $$A_3(|p|^2, |p|^2, |p|^2) = \frac{32}{3} i(p\cdot B)\,\mathrm{div}B\,. \tag{4.4}$$

   For a monopole star product, $\mathrm{div}B \neq 0$, and $p \cdot B$ is generically non-zero for a charged particle with momentum p moving in the magnetic field B. Therefore, a monopole star product obtained from a Weyl star product cannot be alternative to third order in λ.

2. Another example in which (4.2) can be used is $f = e^{i\alpha_1 p_1} + e^{i\alpha_2 p_2} + e^{i\alpha^3 p_3}$ for $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$, a family of bounded functions. The sum over cyclic permutations then equals

   $$\sum_{\sigma\in Z_3}\Pi^{p_{\sigma(1)}p_{\sigma(2)}}\partial_{p_{\sigma(3)}} f\partial^2_{p_{\sigma(1)}} f\partial^2_{p_{\sigma(2)}} f = i\alpha_1^2\alpha_2^2\alpha_3^2\left(\frac{\Pi^{p_1 p_2}}{\alpha_3} + \frac{\Pi^{p_2 p_3}}{\alpha_1} + \frac{\Pi^{p_3 p_1}}{\alpha_2}\right)e^{i(p_1+p_2+p_3)}\,.$$

For a bivector as in (1.5), we have

$$A_3(e^{ip_1}+e^{ip_2}+e^{ip_3}, e^{ip_1}+e^{ip_2}+e^{ip_3}, e^{ip_1}+e^{ip_2}+e^{ip_3}) \\ = -\frac{4}{3}\alpha_1^2\alpha_2^2\alpha_3^2 e^{i(\alpha_1p_1+\alpha_2p_2+\alpha_3p_3)}\left(\frac{B^1}{\alpha_1}+\frac{B^2}{\alpha_2}+\frac{B^3}{\alpha_3}\right)\mathrm{div}B\,. \tag{4.5}$$

For any non-zero B, there is a triple $(\alpha_1,\alpha_2,\alpha_3)$ such that $B^1/\alpha_1+B^2/\alpha_2+B^3/\alpha_3$ is not identically zero. Therefore, every magnetic field with non-zero divergence gives rise to an f with $A_3(f,f,f)\neq 0$.

□

The Lemma implies non-alternativity of monopole star products obtained from a Weyl star product quantizing (2.1), but this already follows from Theorem 1. Having explicit examples with $A_3(f,f,f)\neq 0$ implies further results.

A property weaker than alternativity is *flexibility*, for which, by definition, only anti-symmetry with respect to the first and third entry is required:

$$A(f,g,h) = -A(h,g,f)\,. \tag{4.6}$$

Flexibility is important for quantum mechanics because it is a necessary and sufficient condition [25] for the commutator

$$[f,g] = f\star g - g\star f \tag{4.7}$$

to be a derivation of the Jordan product

$$f\circ g := \frac{1}{2}(f\star g - g\star f)\,. \tag{4.8}$$

Heisenberg equations of motion

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{[f,H]}{i\hbar} \tag{4.9}$$

with a Hamiltonian H then obey a product rule of the form

$$\frac{\mathrm{d}(f\circ g)}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}t}\circ g + f\circ\frac{\mathrm{d}g}{\mathrm{d}t}\,. \tag{4.10}$$

To second order in λ, flexibility of the associator follows from (2.14) for any star product with symmetric B_2. However, as with alternativity, this fact does not guarantee that flexibility is realized at higher orders.

Another condition weaker than alternativity is *power-associativity:* a power-associative algebra is defined as an algebra $\mathcal{A}$ such that the subalgebra generated by any single element $a\in\mathcal{A}$ is associative. For any positive integer n, the n-th power a^n is then uniquely defined even though the algebra product may be non-associative. For Weyl star products of monopole systems, we have

Theorem 2. *A Weyl star product which quantizes (2.1) with* $\mathrm{div}B\neq 0$ *cannot be flexible or power associative.*

Proof: Since there is an f such that $A_3(f,f,f) \neq 0$, the associator cannot be antisymmetric in its first and last arguments. Moreover, we have $f \star (f \star f) - (f \star f) \star f = A_3(f,f,f)\lambda^3 + O(\lambda^4)$ and the subalgebra generated by f cannot be associative. □

5 Conclusions

We have shown that, under rather weak conditions, star products that quantize the phase space of a charged particle in the presence of a magnetic monopole density cannot be alternative. More generally, we have provided obstructions for a non-associative star product with symmetric B_2 being alternative. By the non-associative Gelfand-Naimark theorem [26], this result, together with the fact that the algebra is unital, implies that there is no norm that would turn the quantum algebra into a C^*-algebra, even if the algebra can be restricted to bounded functions; see (4.5). This version of our result strengthens the usual statement that non-associative systems cannot be quantized in the standard way by representing observables on a Hilbert space. One way to circumvent the use of Hilbert spaces in associative systems is to take an algebraic view point and define quantum states as positive linear functionals on the C^*-algebra of bounded observables; see for instance [27]. For non-associative systems of the kind studied here, this route must be generalized because the star-product algebra cannot be turned into a C^*-algebra. One can still use positive linear functionals, but only on a $*$-algebra.

Non-alternativity rules out the use of octonions as realizations of observable algebras of the relevant physical systems. Recently, in [28], octonions have been used to realize the relations (1.5) and (1.7) for linear functions of the momentum components. An extension to non-linear functions would encounter the same obstructions found here for star products, and a purely octonionic construction would no longer suffice.

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A Details on a derivation

Starting from (3.3), and using all its cyclic permutations, we can write the fully antisymmetric part of A_3 as

$$\begin{aligned} 6A_3(f,g,h)^- = {} & B_2(f,B_1(g,h)) - B_2(B_1(f,g),h) + B_2(h,B_1(f,g)) \\ & - B_2(B_1(h,f)g) + B_2(g,B_1(h,f)) - B_2(B_1(g,h),f) \\ & - B_2(f,B_1(h,g)) + B_2(B_1(f,h),g) - B_2(g,B_1(f,h)) \\ & + B_2(B_1(g,f),h) - B_2(h,B_1(g,f)) + B_2(B_1(h,g),f) + (B_1 \leftrightarrow B_2)\,. \end{aligned} \tag{A.1}$$

Using the definition of the anti-symmetric parts of the B_i, we have

$$6A_3(f,g,h)^- = 2B_2^-(f,B_1(g,h)) + 2B_2^-(h,B_1(f,g)) + 2B_2^-(g,B_1(h,f)) \\ -2B_2^-(f,B_1(h,g)) - 2B_2(g,B_1(f,h)) - 2B_2^-(h,B_1(g,f)) + (B_1 \leftrightarrow B_2)\,. \tag{A.2}$$

Finally, using the fact that the B_i are linear in their arguments, we obtain the required form for the fully anti-symmetric part of A_3 as in (3.4).

B Proof of lemma without Pentagon identity

To begin with, let us write the third-order associator as before:

$$A_3(f,g,h) = \mathrm{d}B_3(f,g,h) + B_2(f,B_1(g,h)) \\ -B_2(B_1(f,g),h) + B_1(f,B_2(g,h)) - B_1(B_2(f,g),h)\,, \tag{B.1}$$

where $\mathrm{d}B_n = fB_n(g,h) + B_n(f,gh) - hB_n(f,g) - B_n(fg,h)$. If we apply the Hochschild coboundary operator to A_3, the first term in (B.1) should give zero because $\mathrm{d}^2 = 0$. (Again, when applied to coefficients in an λ-expansion of a non-assocative star product, only the associative multiplication of smooth functions is used in the definition of d.) However, for completeness we will explicitly show this. The part in $\mathrm{d}A_3(f,g,h,k)$ involving contributions only from the B_3 terms has the form

$$f\,\mathrm{d}B_3(g,h,k) - \mathrm{d}B_3(fg,h,k) + \mathrm{d}B_3(f,gh,k) - \mathrm{d}B_3(f,g,hk) + k\,\mathrm{d}B_3(f,g,h)\,. \tag{B.2}$$

Using the definition of $\mathrm{d}B_n$ for $n=3$ gives

$$\begin{aligned} & f\Big(g\,B_3(h,k) + B_3(g,hk) - k\,B_3(g,h) - B_3(gh,k)\Big) \\ & -\Big(f\,g\,B_3(h,k) + B_3(fg,hk) - k\,B_3(fg,h) - B_3(fgh,k)\Big) \\ & +\Big(f\,B_3(gh,k) + B_3(f,ghk) - k\,B_3(f,gh) - B_3(fgh,k)\Big) \\ & -\Big(f\,B_3(g,hk) + B_3(f,ghk) - h\,k\,B_3(f,g) - B_3(fg,hk)\Big) \\ & +k\Big(f\,B_3(g,h) + B_3(f,gh) - h\,B_3(f,g) - B_3(fg,h)\Big)\,. \end{aligned}$$

Upon a close inspection of this expression, we see that there is a counterterm for each term, and thus it is zero. We are left with the action of the coboundary operator on the last four terms in (B.1). Concentrating, for now, on its action on the B_2 terms, using the generic definition of $\mathrm{d}B_n$ for $n=2$, we obtain a part in $\mathrm{d}A_3(f,g,h,k)$ that is of the form:

$$\begin{aligned} & -f\Big(B_2(g,B_1(h,k)) - B_2(B_1(g,h),k)\Big) \\ & -B_2(fg,B_1(h,k)) + B_2(B_1(fg,h),k) \\ & +B_2(f,B_1(gh,k)) - B_2(B_1(f,gh),k) \\ & -B_2(f,B_1(g,hk)) + B_2(B_1(f,g),hk) \\ & +k\Big(B_2(f,B_1(g,h)) - B_2(B_1(f,g),h)\Big)\,. \end{aligned} \tag{B.3}$$

Using the Leibniz property of B_1, and removing terms that identically cancel out, we are left with

$$\begin{aligned}&-f\,B_2(g,B_1(h,k))-f\,B_2(B_1(g,h),k)-B_2(fg,B_1(h,k))\\&+B_2(fB_1(g,h),k)+B_2(f,gB_1(h,k))-B_2(hB_1(f,g),k)\\&-B_2(f,kB_1(g,h))+B_2(B_1(f,g),hk)+k\,B_2(f,B_1(g,h))-k\,B_2(B_1(f,g),h)\,.\end{aligned}$$

This expression can be cast into a more succinct form in terms of $\mathrm{d}A_2$, by adding and subracting a few terms as follows:

$$\begin{aligned}&\mathrm{d}B_2(f,g,B_1(h,k))-\mathrm{d}B_2(f,B_1(g,h),k)+\mathrm{d}B_2(B_1(f,g),h,k)\\&+B_1(h,k)B_2(f,g)-B_2(h,k)B_1(f,g)\,.\end{aligned}\tag{B.4}$$

The action of the differential on the B_1 terms in (B.1) gives an expression similar to (B.3), with the roles of B_1 and B_2 exchanged. Again upon using the Leibniz property of B_1 and cancelling terms, we have the contribution to $\mathrm{d}A_3$ as

$$\begin{aligned}&-f\,B_1(B_2(g,h),k)-g\,B_1(f,B_2(h,k))+B_1(B_2(fg,h),k)+B_1(f,B_2(gh,k))\\&-B_1(B_2(f,gh),k)-B_1(f,B_2(g,hk))+h\,B_1(B_2(f,g),k)+k\,B_1(f,B_2(g,h))\,.\end{aligned}$$

Using anti-symmetry and linearity in either of the arguments of B_1, and again adding and subtracting a few terms, we introduce $\mathrm{d}B_2$ as

$$B_1(\mathrm{d}B_2(g,h,k),f)-B_1(\mathrm{d}B_2(f,g,h),k)-B_2(f,g)B_1(h,k)+B_2(h,k)B_1(f,g)\,.\tag{B.5}$$

As the final result, (B.4) and (B.5) give

$$\begin{aligned}\mathrm{d}A_3(f,g,h,k)=\;&\mathrm{d}B_2(f,g,B_1(h,k))-\mathrm{d}B_2(f,B_1(g,h),k)+\mathrm{d}B_2(B_1(f,g),h,k)\\&+B_1(\mathrm{d}B_2(g,h,k),f)-B_1(\mathrm{d}B_2(f,g,h),k)\,.\end{aligned}\tag{B.6}$$

To get the same result as in (3.9), which was obtained using the Pentagon identity, we just use the definition of $\mathrm{d}B_2$ in terms of the second-order associator as $\mathrm{d}B_2(f,g,h)=A_2(f,g,h)-B_1(f,B_1(g,h))+B_1(B_1(f,g),h)$, and use the linearity of B_1 in its first argument in the last two terms.

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