

Note on identities inspired by new soft theorems

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ABSTRACT: The new soft theorems, for both gravity and gauge amplitudes, have inspired a number of works, including the discovery of new identities related to amplitudes. In this note, we present the proof and discussion for two sets of identities. The first set includes an identity involving the half-soft function which had been used in the soft theorem for one-loop rational gravity amplitudes, and another simpler identity as its byproduct. The second set includes two identities involving the KLT momentum kernel, as the consistency conditions of the KLT relation plus soft theorems for both gravity and gauge amplitudes. We use the CHY formulation to prove the first identity, and transform the second one into a convenient form for future discussion.

KEYWORDS: Scattering Amplitudes, Gauge Symmetry

ARXIV EPRINT: [1604.00650](https://arxiv.org/abs/1604.00650)

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1 Introduction

Scattering amplitudes often have an universal soft behavior when the momentum of one external leg tends to zero. This soft limit can be traced back to the works [1–6]. Recently, a new soft theorem for tree level gravity amplitudes was studied in [7]. By using the on-shell recursion relation [8, 9] and imposing the holomorphic soft limit, Cachazo and Strominger have proved that

$$\begin{aligned}
 &M_n(\lambda_n \rightarrow \varepsilon\lambda_n) \\
 &= \frac{1}{\varepsilon^3} \sum_{a=1}^{n-2} \frac{\langle n-1, a \rangle^2 [na]}{\langle n-1, n \rangle^2 \langle na \rangle} M_{n-1} \left(\tilde{\lambda}_{n-1} \rightarrow \tilde{\lambda}_{n-1} + \varepsilon \frac{\langle an \rangle}{\langle a, n-1 \rangle} \tilde{\lambda}_n, \tilde{\lambda}_1 \rightarrow \tilde{\lambda}_1 + \varepsilon \frac{\langle n-1, n \rangle}{\langle n-1, a \rangle} \tilde{\lambda}_n \right) + O(\varepsilon^0),
 \end{aligned}
 \tag{1.1}$$

here for M_n and M_{n-1} , the unmentioned external kinematic data are un-deformed and we prefer to suppress them for conciseness. Taylor expansion in ε exhibits three singular terms in orders ε^{-3} , ε^{-2} and ε^{-1} , while higher order terms in ε will be mixed with the less interesting $O(\varepsilon^0)$ parts.

A similar relation for tree level Yang-Mills amplitudes using the on-shell recursion relation, proved by Casali [10], takes the form

$$\begin{aligned}
 &A_n(\lambda_n \rightarrow \varepsilon\lambda_n) \\
 &= \frac{1}{\varepsilon^2} \frac{\langle n-1, 1 \rangle}{\langle n-1, n \rangle \langle n1 \rangle} A_{n-1} \left(\tilde{\lambda}_{n-1} \rightarrow \tilde{\lambda}_{n-1} + \varepsilon \frac{\langle 1n \rangle}{\langle 1, n-1 \rangle} \tilde{\lambda}_n, \tilde{\lambda}_1 \rightarrow \tilde{\lambda}_1 + \varepsilon \frac{\langle n-1, n \rangle}{\langle n-1, 1 \rangle} \tilde{\lambda}_n \right) + O(\varepsilon^0),
 \end{aligned}
 \tag{1.2}$$

where two singular terms in orders ε^{-2} and ε^{-1} appear after Taylor expansion. The mixing between higher order terms from the deformed A_{n-1} and $O(\varepsilon^0)$ parts also persists to this case.

Based on this new discovery, many related studies have been done. In [23–36], the soft theorem has been generalized to arbitrary dimensions and other theories or categories:

string theory, ABJM theory, theories with fermions or massive particles, and form factors. In [37–51, 53–58], the theorem has been understood from various perspectives, especially those of symmetries and invariance. In [11, 28, 59–63], its generalization to loop level has been discussed. In [64–71], the relevant double (or multiple) soft theorem has also been discussed.

Among these studies, we have met two sets of identities which have not been proved so far. We will present the proof in this note.

One identity of the first set was mentioned in [11], which explored loop correction to the soft theorem. It involves the so-called half-soft function h (first defined in [12] and reinterpreted in [13]), which appears naturally for all-plus one-loop gravity amplitude. Its general proof was not given in [11], but explicit checks up to 12 points had been done. The identity reads

$$\sum_{b \neq n} \langle bn \rangle^2 \sum_{M,N} h(b, n, M) h(b, n, N) \langle b | K_M | n \rangle \langle n | K_N | b \rangle^3 = 0, \tag{1.3}$$

where M, N are two nonempty partition sets of the $(n - 2)$ particles other than b and n , and K_M and K_N are the corresponding total momenta. During the proof, we had also discovered another simpler identity, which can serve as its logical preliminary. It reads

$$\frac{[1n]}{\langle 1n \rangle} \frac{|\psi_{N \cup M}|_w^w}{\langle w1 \rangle \langle wn \rangle} + \sum_N \langle 1 | K_N | n \rangle \langle n | K_N | 1 \rangle \frac{|\psi_N|_x^x}{\langle x1 \rangle \langle xn \rangle} \frac{|\psi_M|_y^y}{\langle y1 \rangle \langle yn \rangle} = 0, \tag{1.4}$$

where the ψ matrix is related to h , and other symbols above will be explained shortly.

The second set of identities was conjectured in [14], which is a consequence of consistency conditions between the soft theorems for gravity and gauge amplitudes, under the well-known KLT relation [15]. It involves the KLT momentum kernel [12, 16–18], and the transformation matrices (D and C below) between BCJ basis of gauge amplitudes [19]. These two identities are

$$\sum_{\alpha_{t'}, \beta_{t'} \in \mathcal{S}_{n-3}} D[t, \alpha_t, n - 1, n | t', \alpha_{t'}, n - 1, n] S[\alpha_{t'} | \beta_{t'}]_{p_{n-1}} \cdot C[t', n - 1, \beta_{t'}, n | t, n - 1, \beta_t, n] = S[\alpha_t | \beta_t]_{p_{n-1}}, \tag{1.5}$$

$$\sum_{t'=1}^{n-2} \sum_{\alpha_{t'}, \beta_{t'} \in \mathcal{S}_{n-3}} D[t, \alpha_t, n - 1, n | t', \alpha_{t'}, n - 1, n] S[\alpha_{t'} | \beta_{t'}]_{p_{n-1}} \cdot J_{t'} (C[t', n - 1, \beta_{t'}, n | t, n - 1, \beta_t, n]) = 0, \tag{1.6}$$

where $S[\alpha_t | \beta_t]_{p_{n-1}}$ is the KLT momentum kernel of pivot p_{n-1} , and $J_{t'} \equiv J_{t', \dot{\alpha} \dot{\beta}}$ is the anti-holomorphic angular momentum operator. We will use the CHY formulation [20–22] to prove the first identity and discuss the second one.

This note is organized as follows. In section 2, we prove identity (1.3) of the half-soft function, and also the byproduct identity (1.4). In section 3, we prove identity (1.5) of the KLT momentum kernel by using the CHY formulation, while we transform identity (1.6) into a convenient form for possible future attempts and end with some discussion.

2 Two identities of the half-soft function

In this section we will prove (1.3) and (1.4), first let's set up a bit convenient facilitation. For reader's reference, we write (1.3) again below

$$\sum_{b \neq n} \langle bn \rangle^2 \sum_{M, N} h(b, n, M) h(b, n, N) \langle b | K_M | n \rangle \langle n | K_N | b \rangle^3 = 0, \quad (2.1)$$

where M, N are two non-overlapping nonempty sets satisfying $M \cup N = \{1, \dots, n-1\} \setminus b$, and momentum conservation enforces $k_b + k_n + K_M + K_N = 0$. The half-soft function h above is defined as [13]

$$h(b, n, N) = \frac{1}{\prod_i^N \langle ib \rangle^2 \langle in \rangle^2} |\Psi|_r^r = \frac{1}{\prod_i^N \langle ib \rangle^2 \langle in \rangle^2} \|\Psi\|, \quad (2.2)$$

where $|\Psi|_r^r$ denotes the determinant of matrix Ψ after deleting its r -th row and r -th column, and $\|\Psi\|$ indicates this quantity is *independent* of the choice $r \in N$. If there is only one row and one column, the determinant is 1 after deletion. The matrix Ψ is defined as

$$\Psi_{ij}(b, n) = -\frac{[ij]}{\langle ij \rangle} \langle ib \rangle \langle in \rangle \langle jb \rangle \langle jn \rangle \text{ for } i \neq j, \quad \Psi_{ii} = \sum_{j \neq i}^N \Psi_{ij}, \quad (2.3)$$

where b and n serve as auxiliary spinors. The sum of each row is zero, so Ψ is degenerate. Observe that the summand in (1.3) has even power of K_M and K_N , by momentum conservation this sum is symmetric between M and N , then we can replace K_M by $-K_N$ and rewrite (1.3) as

$$\sum_{b \neq n} \langle bn \rangle^2 \sum_N h(b, n, N) h(b, n, M) \langle b | N | n \rangle \langle n | N | b \rangle^3 = 0, \quad (2.4)$$

for brevity N stands for K_N in spinorial products (and later N also represents the number of elements in the set N , depending on the context).

To simplify the proof, we define the matrix ψ as

$$\psi_{ij}(b, n) = -\frac{[ij]}{\langle ij \rangle} \langle jb \rangle \langle jn \rangle \text{ for } i \neq j, \quad \psi_{ii} = \sum_{j \neq i}^N \psi_{ij}, \quad (2.5)$$

where the common factor $\langle ib \rangle \langle in \rangle$ of the i -th row in Ψ has been stripped off. One can easily verify that

$$h(b, n, N) = \frac{1}{\prod_i^N \langle ib \rangle^2 \langle in \rangle^2} \|\Psi\| = \frac{1}{\prod_i^N \langle ib \rangle \langle in \rangle} \frac{|\psi_N|_x^x}{\langle xb \rangle \langle xn \rangle}, \quad (2.6)$$

where N has been added to ψ to label the corresponding set, note that $|\psi_N|_x^x / \langle xb \rangle \langle xn \rangle$ is *independent* of the choice $x \in N$. Then we have

$$\begin{aligned} h(b, n, N) h(b, n, M) &= \frac{1}{\prod_i^{N+M} \langle ib \rangle \langle in \rangle} \frac{|\psi_N|_x^x}{\langle xb \rangle \langle xn \rangle} \frac{|\psi_M|_y^y}{\langle yb \rangle \langle yn \rangle} \\ &= \frac{\langle bn \rangle}{\prod_{i \neq n} \langle in \rangle \prod_i^{N+M} \langle ib \rangle} \frac{|\psi_N|_x^x}{\langle xb \rangle \langle xn \rangle} \frac{|\psi_M|_y^y}{\langle yb \rangle \langle yn \rangle}, \end{aligned} \quad (2.7)$$

where $\prod_{i \neq n} \langle in \rangle$ is a common factor independent of b so it can be dropped, hence (2.4) becomes

$$\sum_{b \neq n} \frac{\langle bn \rangle^3}{\prod_{i \neq b, n} \langle ib \rangle} \sum_N \langle b|N|n \rangle \langle n|N|b \rangle^3 \frac{|\psi_N|_x^x}{\langle xb \rangle \langle xn \rangle} \frac{|\psi_M|_y^y}{\langle yb \rangle \langle yn \rangle} = 0. \quad (2.8)$$

2.1 A simpler byproduct identity

In the proof of (2.8), we happened to discover (1.4). For reader's reference, it is given below

$$\frac{[1n]}{\langle 1n \rangle} \frac{|\psi_{N \cup M}|_w^w}{\langle w1 \rangle \langle wn \rangle} + \sum_N \langle 1|N|n \rangle \langle n|N|1 \rangle \frac{|\psi_N|_x^x}{\langle x1 \rangle \langle xn \rangle} \frac{|\psi_M|_y^y}{\langle y1 \rangle \langle yn \rangle} = 0, \quad (2.9)$$

where N, M are two non-overlapping nonempty sets satisfying $N \cup M = \{2, \dots, n-1\}$, and the auxiliary spinors are 1 and n . Also note that $w \in N \cup M$, $x \in N$, $y \in M$ and it is free to switch the choices w, x, y within each set. Since this is mandatory for (2.8) to hold, we will prove it first as the tricks used here are analogous to those for (2.8).

Now we will adopt the BCFW deformation and reduce it into an identity of the same form, but with one particle removed, in other words, we will perform an inductive proof. Before induction, the identity is confirmed analytically at lower points for $n = 4, 5, 6$. For later convenience, we multiply it by a non-zero factor, yields

$$\frac{1}{\prod_{i \neq 1, n} \langle i1 \rangle} \left(\frac{[1n]}{\langle 1n \rangle} \frac{|\psi_{N \cup M}|_w^w}{\langle w1 \rangle \langle wn \rangle} + \sum_N \langle 1|N|n \rangle \langle n|N|1 \rangle \frac{|\psi_N|_x^x}{\langle x1 \rangle \langle xn \rangle} \frac{|\psi_M|_y^y}{\langle y1 \rangle \langle yn \rangle} \right) = 0, \quad (2.10)$$

which is of course equivalent to (1.4). But now there are two advantages: the large z behavior of its l.h.s. is improved, and it has the desired simple pole for residue evaluation, as we will soon see.

For generic n , consider BCFW deformation $\langle 1|n \rangle$ and a particular pole $\langle 21 \rangle$. Note that particles 1 and n are special while the rest ($n-2$) ones are symmetric, so it is sufficient to consider the residue of $\langle 21 \rangle$ only, as all $\langle i1 \rangle$'s with $i \in \{2, \dots, n-1\}$ behave similarly. At $\langle 2\hat{1} \rangle = 0$, we have

$$|\hat{1} \rangle = |1 \rangle - |n \rangle \frac{\langle 12 \rangle}{\langle n2 \rangle} = |2 \rangle \frac{\langle 1n \rangle}{\langle 2n \rangle}, \quad |\hat{n} \rangle = |n \rangle + |1 \rangle \frac{\langle 12 \rangle}{\langle n2 \rangle}, \quad (2.11)$$

and

$$|\hat{1} \rangle |1 \rangle + |2 \rangle |2 \rangle \equiv |2 \rangle |\hat{2} \rangle, \quad |\hat{2} \rangle = |2 \rangle + |1 \rangle \frac{\langle 1n \rangle}{\langle 2n \rangle}, \quad (2.12)$$

by which we mean to combine the momenta of particle $\hat{1}$ and 2 into that of particle $\hat{2}$, or more physically, particles $\hat{1}$ and 2 merge into particle $\hat{2}$. Including the deformed particle \hat{n} , the set $\{1, 2, \dots, n\}$ now shrinks into $\{\hat{2}, \dots, \hat{n}\}$ while momentum conservation still holds, as what induction requires.

To locate pole $\langle 21 \rangle$ in (2.10), we immediately find one in the overall factor. Naively, there might be another one under $|\psi_N|_x^x$ if we take $x = 2$, for example. However, the expansion of $|\psi_N|_x^x$ in terms of $\langle 21 \rangle$ will cancel this pole. In other words, $|\psi_N|_x^x / \langle x1 \rangle \langle xn \rangle$ is a polynomial of $\langle 21 \rangle$ (one may also choose $x \neq 2$ to invalidate this pole), that's why the overall factor is mandatory.

The next step is to analyze the large z behavior of the l.h.s. in (2.10) before evaluating its residues at finite locations. To clarify the analysis, we further separate the second term in the parenthesis, and from now on we redefine N, M to exclude particle 2 from them while N', M' denote the original sets. Depending on whether N' or M' contains particle 2, the set $\{2, \dots, n-1\}$ has three types of splitting: $\{\{2\} \cup N, M\}$, $\{N, \{2\} \cup M\}$ and $\{\{2\}, N \cup M\}$, where $N \cup M = \{3, \dots, n-1\}$. So the second term becomes

$$\begin{aligned} & \sum_{N'} \langle 1|N'|n \rangle \langle n|N'|1 \rangle \frac{|\psi_{N'}|_x^x}{\langle x1 \rangle \langle xn \rangle} \frac{|\psi_{M'}|_y^y}{\langle y1 \rangle \langle yn \rangle} \\ &= \sum_{\{2\} \cup N, M} (\langle 1|N|n \rangle + \langle 12 \rangle [2n]) (\langle n|N|1 \rangle + \langle n2 \rangle [21]) \frac{|\psi_{N \cup \{2\}}|_2^2}{\langle 21 \rangle \langle 2n \rangle} \frac{|\psi_M|_y^y}{\langle y1 \rangle \langle yn \rangle} \\ & \quad + \sum_{N, \{2\} \cup M} \langle 1|N|n \rangle \langle n|N|1 \rangle \frac{|\psi_N|_x^x}{\langle x1 \rangle \langle xn \rangle} \frac{|\psi_{M \cup \{2\}}|_2^2}{\langle 21 \rangle \langle 2n \rangle} + [n2] [12] \frac{|\psi_{N \cup M}|_w^w}{\langle w1 \rangle \langle wn \rangle}. \end{aligned} \quad (2.13)$$

Also, the first term in (2.10) can be written as

$$\frac{[1n]}{\langle 1n \rangle} \frac{|\psi_{N' \cup M'}|_w^w}{\langle w1 \rangle \langle wn \rangle} = \frac{[1n]}{\langle 1n \rangle} \frac{|\psi_{N \cup M \cup \{2\}}|_2^2}{\langle 21 \rangle \langle 2n \rangle}. \quad (2.14)$$

Since the three ψ 's in (2.14) and the first and second terms of (2.13) contain particle 2, we can choose to delete its corresponding row and column. Large z power counting shows that all four terms in (2.13) and (2.14) behave as $z^{N+M-1} = z^{n-4}$ under $\langle 1|n \rangle$, but the overall factor in the front of (2.10) behaves as $z^{-(n-2)}$, which renders the entire expression as z^{-2} , so there is no boundary contribution. Therefore, via contour integration, the l.h.s. of (2.10) (denoted I below) can be expressed as

$$\oint_{z=0} \frac{dz}{z} I(z) = - \oint_{\langle 2\hat{1} \rangle=0} \frac{dz}{z} I(z) - \dots - \oint_{\langle n-1, \hat{1} \rangle=0} \frac{dz}{z} I(z), \quad (2.15)$$

if the residue at $\langle 2\hat{1} \rangle = 0$ vanishes, by the symmetry among particles $\{2, \dots, n-1\}$ the entire un-deformed expression must also vanish. Note the contribution from the overall factor in (2.10) is universal, so it can be dropped. At $\langle 2\hat{1} \rangle = 0$, after some algebra, the residue evaluation gives

$$\frac{|\psi_{N \cup \{2\}}|_2^2}{\langle 21 \rangle \langle 2n \rangle} \rightarrow \left(\frac{\langle 1n \rangle}{\langle 2n \rangle} \right)^{N-1} (-\langle n|N|2 \rangle) \frac{|\psi_N|_x^x}{\langle x2 \rangle \langle xn \rangle}, \quad (2.16)$$

recall that $\langle 21 \rangle$ above is not a pole, while the real pole comes from the overall factor. Here $|\hat{1} \rangle$ is replaced by $|2 \rangle$ up to a factor, after recalling (2.11). By expanding the determinant to the first order of $\langle 21 \rangle$, then using the independence of choice x to switch the deleted row and column for each term, we can collect a factor $(-\langle n|N|2 \rangle)$ as above. The similar (and simpler) story happens for

$$\frac{|\psi_M|_y^y}{\langle y1 \rangle \langle yn \rangle} \rightarrow \left(\frac{\langle 1n \rangle}{\langle 2n \rangle} \right)^{M-2} \frac{|\psi_M|_y^y}{\langle y2 \rangle \langle yn \rangle}. \quad (2.17)$$

Plugging them back, up to a factor $(\langle 1n \rangle / \langle 2n \rangle)^{N+M-2}$, the sum of (2.13) and (2.14) becomes

$$\begin{aligned}
 & (-\langle n|N+M|2\rangle) \frac{|\psi_{N\cup M}|_w^w [1n]}{\langle w2 \rangle \langle wn \rangle \langle 2n \rangle} \\
 & + \sum_N \langle 2|N|\hat{n}\rangle (\langle n|N|1\rangle + \langle n2|21\rangle) (-\langle n|N|2\rangle) \frac{|\psi_N|_x^x}{\langle x2 \rangle \langle xn \rangle} \frac{|\psi_M|_y^y}{\langle y2 \rangle \langle yn \rangle} \\
 & + \sum_N \langle 2|N|\hat{n}\rangle \langle n|N|1\rangle (-\langle n|M|2\rangle) \frac{|\psi_N|_x^x}{\langle x2 \rangle \langle xn \rangle} \frac{|\psi_M|_y^y}{\langle y2 \rangle \langle yn \rangle} + [\hat{n}2][12] \frac{|\psi_{N\cup M}|_w^w}{\langle w2 \rangle \langle wn \rangle}.
 \end{aligned} \tag{2.18}$$

By momentum conservation, up to a factor [12], it can be simplified into

$$\begin{aligned}
 & \left(\frac{\langle 1n \rangle}{\langle 2n \rangle} [n1] + [\hat{n}2] \right) \frac{|\psi_{N\cup M}|_w^w}{\langle w2 \rangle \langle wn \rangle} \\
 & + \langle n2 \rangle \sum_N \langle 2|N|\hat{n}\rangle \left(\langle n|N|2\rangle + \langle n|N|1\rangle \frac{\langle 1n \rangle}{\langle 2n \rangle} \right) \frac{|\psi_N|_x^x}{\langle x2 \rangle \langle xn \rangle} \frac{|\psi_M|_y^y}{\langle y2 \rangle \langle yn \rangle} \\
 & = \langle n2 \rangle \left(\frac{[\hat{2}\hat{n}]}{\langle 2n \rangle} \frac{|\psi_{N\cup M}|_w^w}{\langle w2 \rangle \langle wn \rangle} + \sum_N \langle 2|N|\hat{n}\rangle \langle n|N|\hat{2}\rangle \frac{|\psi_N|_x^x}{\langle x2 \rangle \langle xn \rangle} \frac{|\psi_M|_y^y}{\langle y2 \rangle \langle yn \rangle} \right) = 0,
 \end{aligned} \tag{2.19}$$

after assuming the identity of $(n-1)$ particles holds. This finishes the inductive proof of (1.4).

2.2 Proof of the first identity

Now we move to prove (2.8) by applying the similar pack of tricks: to consider deformation $\langle 1|n\rangle$ acting on its l.h.s., and the pole $\langle 21\rangle$. First, we separate the expression into three parts corresponding to $b=1$, $b=2$ and $b=3, \dots, n-1$, namely

$$\sum_{b \neq n} \frac{\langle bn \rangle^3}{\prod_{i \neq b, n} \langle ib \rangle} \sum_N \langle b|N|n\rangle \langle n|N|b\rangle^3 \frac{|\psi_N|_x^x}{\langle xb \rangle \langle xn \rangle} \frac{|\psi_M|_y^y}{\langle yb \rangle \langle yn \rangle} = I_1 + I_2 + I_{b \neq 1, 2, n}. \tag{2.20}$$

Similarly, we now redefine N and M to exclude particles 2 and 1, with respect to I_1 and I_2 . For I_1 , the set $\{2, \dots, n-1\}$ has three types of splitting: $\{\{2\} \cup N, M\}$, $\{N, \{2\} \cup M\}$ and $\{\{2\}, N \cup M\}$, where $N \cup M = \{3, \dots, n-1\}$. For I_2 , we have $\{\{1\} \cup N, M\}$, $\{N, \{1\} \cup M\}$ and $\{\{1\}, N \cup M\}$. For $I_{b \neq 1, 2, n}$, there are four types: $\{\{1, 2\} \cup N_b, M_b\}$, $\{N_b, \{1, 2\} \cup M_b\}$, $\{\{1\} \cup N_b, \{2\} \cup M_b\}$ and $\{\{2\} \cup N_b, \{1\} \cup M_b\}$, where $N_b \cup M_b = \{3, \dots, n-1\} \setminus b$, but the last two will not contribute to the residue of $\langle 21\rangle$ and hence the corresponding terms are neglected, which will be explained shortly.

According to the splittings above, we can write

$$\begin{aligned}
 \frac{I_1}{\langle 1n \rangle^3} &= \frac{1}{\prod_{i \neq 1, 2, n} \langle i1 \rangle \langle 21 \rangle} \sum_N \left(\langle 1|N+2|n\rangle \langle n|N+2|1\rangle^3 \frac{|\psi_{N \cup \{2\}}|_2^2}{\langle 21 \rangle \langle 2n \rangle} \frac{|\psi_M|_y^y}{\langle y1 \rangle \langle yn \rangle} \right. \\
 & \quad \left. + \langle 1|N|n\rangle \langle n|N|1\rangle^3 \frac{|\psi_N|_x^x}{\langle x1 \rangle \langle xn \rangle} \frac{|\psi_{M \cup \{2\}}|_2^2}{\langle 21 \rangle \langle 2n \rangle} \right) \\
 & + \frac{1}{\prod_{i \neq 1, 2, n} \langle i1 \rangle \langle 21 \rangle} \langle 1|2|n\rangle \langle n|2|1\rangle^3 \frac{1}{\langle 21 \rangle \langle 2n \rangle} \frac{|\psi_{N \cup M}|_w^w}{\langle w1 \rangle \langle wn \rangle},
 \end{aligned} \tag{2.21}$$

$$\begin{aligned}
 \frac{I_2}{\langle 2n \rangle^3} &= \frac{1}{\prod_{i \neq 1,2,n} \langle i2 \rangle \langle 12 \rangle} \sum_N \left(\langle 2|N+1|n \rangle \langle n|N+1|2 \rangle^3 \frac{|\psi_{N \cup \{1\}}|_1^1}{\langle 12 \rangle \langle 1n \rangle} \frac{|\psi_M|_y^y}{\langle y2 \rangle \langle yn \rangle} \right. \\
 &\quad \left. + \langle 2|N|n \rangle \langle n|N|2 \rangle^3 \frac{|\psi_N|_x^x}{\langle x2 \rangle \langle xn \rangle} \frac{|\psi_{M \cup \{1\}}|_1^1}{\langle 12 \rangle \langle 1n \rangle} \right) \\
 &\quad + \frac{1}{\prod_{i \neq 1,2,n} \langle i2 \rangle \langle 12 \rangle} \langle 2|1|n \rangle \langle n|1|2 \rangle^3 \frac{1}{\langle 12 \rangle \langle 1n \rangle} \frac{|\psi_{N \cup M}|_w^w}{\langle w2 \rangle \langle wn \rangle}, \tag{2.22}
 \end{aligned}$$

$$\begin{aligned}
 I_{b \neq 1,2,n} &= \sum_{b \neq 1,2,n} \frac{\langle bn \rangle^3}{\prod_{i \neq 1,b,n} \langle ib \rangle \langle 1b \rangle} \\
 &\quad \times \sum_{N_b} \left(\langle b|N_b+1+2|n \rangle \langle n|N_b+1+2|b \rangle^3 \frac{|\psi_{N_b \cup \{1,2\}}|_1^1}{\langle 1b \rangle \langle 1n \rangle} \frac{|\psi_{M_b}|_y^y}{\langle yb \rangle \langle yn \rangle} \right. \\
 &\quad \left. + \langle b|N_b|n \rangle \langle n|N_b|b \rangle^3 \frac{|\psi_{N_b}|_x^x}{\langle xb \rangle \langle xn \rangle} \frac{|\psi_{M_b \cup \{1,2\}}|_1^1}{\langle 1b \rangle \langle 1n \rangle} \right) \\
 &\quad + (\text{two neglected terms}). \tag{2.23}
 \end{aligned}$$

For $I_{b \neq 1,2,n}$ one can verify that, only terms for which 1 and 2 are in the same splitting set, have pole $\langle 21 \rangle$ and hence contribute to the residue, which explains why we only need the first two terms. Moreover, N_b in $\{\{1,2\} \cup N_b, M_b\}$ can be empty (similarly for M_b). While for I_1 , N in $\{\{2\} \cup N, M\}$ cannot be empty, otherwise such a splitting belongs to type $\{\{2\}, N \cup M\}$ (similarly for I_2).

After the separation, we now analyze the large z behavior. Under $\langle 1|n \rangle$, large z power counting shows that $I_1 \sim z^{-2}$, $I_2 \sim z^{-1}$ and $I_{b \neq 1,2,n} \sim z^{-1}$, so there is no boundary contribution. Then we can repeat the contour integration (2.15). Again, thanks to the symmetry among particles $\{2, \dots, n-1\}$, it is sufficient to consider the residue of $\langle 21 \rangle$ only.

Recalling (2.16) and (2.17), at $\langle 2\hat{1} \rangle = 0$ the residue evaluation gives

$$\begin{aligned}
 \frac{\langle 21 \rangle}{\langle 1n \rangle^3} I_1 &\rightarrow \frac{\langle 2n \rangle^3}{\langle 1n \rangle^3} \frac{1}{\prod_{i \neq 1,2,n} \langle i2 \rangle} \\
 &\quad \times \sum_N \left[\langle \hat{1}|N+2|\hat{n} \rangle \langle n|N+2|1 \rangle^3 (-\langle n|N|2 \rangle) \right. \\
 &\quad \left. + \langle \hat{1}|N|\hat{n} \rangle \langle n|N|1 \rangle^3 (-\langle n|M|2 \rangle) \right] \frac{|\psi_N|_x^x}{\langle x2 \rangle \langle xn \rangle} \frac{|\psi_M|_y^y}{\langle y2 \rangle \langle yn \rangle} \tag{2.24} \\
 &\quad - \frac{\langle 2n \rangle^4}{\langle 1n \rangle^2} \frac{[12]^3 [2\hat{n}]}{\prod_{i \neq 1,2,n} \langle i2 \rangle} \frac{|\psi_{N \cup M}|_w^w}{\langle w2 \rangle \langle wn \rangle},
 \end{aligned}$$

or after a bit simplification,

$$\begin{aligned}
 \frac{\langle 21 \rangle \langle 1n \rangle}{\langle 2n \rangle^2} I_1 &\rightarrow \frac{\langle 1n \rangle^2}{\prod_{i \neq 1,2,n} \langle i2 \rangle} \sum_N \langle 2|N|\hat{n} \rangle \left(-\langle n|N|2 \rangle \langle n|N+2|1 \rangle^3 \right. \\
 &\quad \left. + \langle n|N+1|2 \rangle \langle n|N|1 \rangle^3 \right) \frac{|\psi_N|_x^x}{\langle x2 \rangle \langle xn \rangle} \frac{|\psi_M|_y^y}{\langle y2 \rangle \langle yn \rangle} \tag{2.25} \\
 &\quad - \langle 1n \rangle^2 \langle 2n \rangle^2 \frac{[12]^3 [2\hat{n}]}{\prod_{i \neq 1,2,n} \langle i2 \rangle} \frac{|\psi_{N \cup M}|_w^w}{\langle w2 \rangle \langle wn \rangle}.
 \end{aligned}$$

Similarly for I_2 ,

$$\begin{aligned} \frac{\langle 21 \rangle \langle 1n \rangle}{\langle 2n \rangle^2} I_2 \rightarrow & \frac{\langle 2n \rangle^2}{\prod_{i \neq 1,2,n} \langle i2 \rangle} \sum_N \langle 2|N|\hat{n} \rangle (\langle n|N|1 \rangle \langle n|N+1|2 \rangle^3 \\ & - \langle n|N+2|1 \rangle \langle n|N|2 \rangle^3) \frac{|\psi_N|_x^x}{\langle x2 \rangle \langle xn \rangle} \frac{|\psi_M|_y^y}{\langle y2 \rangle \langle yn \rangle} \\ & - \langle 1n \rangle^3 \langle 2n \rangle \frac{[12]^3 [1n]}{\prod_{i \neq 1,2,n} \langle i2 \rangle} \frac{|\psi_{N \cup M}|_w^w}{\langle w2 \rangle \langle wn \rangle}. \end{aligned} \quad (2.26)$$

Combining I_1 and I_2 , we find

$$\begin{aligned} \frac{\langle 21 \rangle \langle 1n \rangle}{\langle 2n \rangle^2} (I_1 + I_2) \rightarrow & - \frac{[12] \langle 2n \rangle^3}{\prod_{i \neq 1,2,n} \langle i2 \rangle} \sum_N \langle 2|N|\hat{n} \rangle \langle n|N|\hat{2} \rangle^3 \frac{|\psi_N|_x^x}{\langle x2 \rangle \langle xn \rangle} \frac{|\psi_M|_y^y}{\langle y2 \rangle \langle yn \rangle} \\ & - \frac{[12]^3 \langle 1n \rangle^2 \langle 2n \rangle^3}{\prod_{i \neq 1,2,n} \langle i2 \rangle} \sum_N \langle 2|N|\hat{n} \rangle \langle n|N|\hat{2} \rangle \frac{|\psi_N|_x^x}{\langle x2 \rangle \langle xn \rangle} \frac{|\psi_M|_y^y}{\langle y2 \rangle \langle yn \rangle} \\ & - \langle 1n \rangle^2 \langle 2n \rangle^2 \frac{[12]^3 [\hat{2}\hat{n}]}{\prod_{i \neq 1,2,n} \langle i2 \rangle} \frac{|\psi_{N \cup M}|_w^w}{\langle w2 \rangle \langle wn \rangle}, \end{aligned} \quad (2.27)$$

after using the following identity

$$\begin{aligned} & \langle 1n \rangle^2 (-\langle n|N|2 \rangle \langle n|N+2|1 \rangle^3 + \langle n|N+1|2 \rangle \langle n|N|1 \rangle^3) \\ & + \langle 2n \rangle^2 (\langle n|N|1 \rangle \langle n|N+1|2 \rangle^3 - \langle n|N+2|1 \rangle \langle n|N|2 \rangle^3) \\ & = -[12] (\langle 1n \rangle \langle n|N|1 \rangle + \langle 2n \rangle \langle n|N|2 \rangle)^3 - [12]^3 \langle 1n \rangle^2 \langle 2n \rangle^2 (\langle 1n \rangle \langle n|N|1 \rangle + \langle 2n \rangle \langle n|N|2 \rangle) \\ & = -[12] \langle 2n \rangle^3 \langle n|N|\hat{2} \rangle^3 - [12]^3 \langle 1n \rangle^2 \langle 2n \rangle^3 \langle n|N|\hat{2} \rangle. \end{aligned} \quad (2.28)$$

Now note the second and third terms in (2.27) can be regrouped as

$$- \frac{[12]^3 \langle 1n \rangle^2 \langle 2n \rangle^3}{\prod_{i \neq 1,2,n} \langle i2 \rangle} \left(\sum_N \langle 2|N|\hat{n} \rangle \langle n|N|\hat{2} \rangle \frac{|\psi_N|_x^x}{\langle x2 \rangle \langle xn \rangle} \frac{|\psi_M|_y^y}{\langle y2 \rangle \langle yn \rangle} + \frac{[\hat{2}\hat{n}]}{\langle 2n \rangle} \frac{|\psi_{N \cup M}|_w^w}{\langle w2 \rangle \langle wn \rangle} \right) = 0, \quad (2.29)$$

which is exactly identity (1.4) for the set $\{\hat{2}, \dots, \hat{n}\}$! Therefore we are left with

$$\frac{\langle 21 \rangle \langle 1n \rangle}{\langle 2n \rangle^2} (I_1 + I_2) \rightarrow - \frac{[12] \langle 2n \rangle^3}{\prod_{i \neq 1,2,n} \langle i2 \rangle} \sum_N \langle 2|N|\hat{n} \rangle \langle n|N|\hat{2} \rangle^3 \frac{|\psi_N|_x^x}{\langle x2 \rangle \langle xn \rangle} \frac{|\psi_M|_y^y}{\langle y2 \rangle \langle yn \rangle}. \quad (2.30)$$

To settle this leftover, we look back to $I_{b \neq 1,2,n}$ in (2.23) and find

$$\frac{|\psi_{N_b \cup \{1,2\}}|_1^1}{\langle 1b \rangle \langle 1n \rangle} \rightarrow \frac{[21]}{\langle 21 \rangle} |\psi_{N_b \cup \{\hat{2}\}}|_2^{\hat{2}} = - \frac{[12]}{\langle 21 \rangle} \langle 2b \rangle \langle 2n \rangle \frac{|\psi_{N_b \cup \{\hat{2}\}}|_x^x}{\langle xb \rangle \langle xn \rangle}, \quad (2.31)$$

where again we have used the independence of choice x to switch the deleted row and column. Now

$$\begin{aligned} \frac{\langle 21 \rangle \langle 1n \rangle}{\langle 2n \rangle^2} I_{b \neq 1,2,n} \rightarrow & - [12] \sum_{b \neq 1,2,n} \frac{\langle bn \rangle^3}{\prod_{i \neq 1,b,n} \langle ib \rangle} \\ & \times \sum_{N_b} \left(\langle b|N_b + \hat{2}|\hat{n} \rangle \langle n|N_b + \hat{2}|b \rangle^3 \frac{|\psi_{N_b \cup \{\hat{2}\}}|_x^x}{\langle xb \rangle \langle xn \rangle} \frac{|\psi_{M_b}|_y^y}{\langle yb \rangle \langle yn \rangle} \right. \\ & \left. + \langle b|N_b|\hat{n} \rangle \langle n|N_b|b \rangle^3 \frac{|\psi_{N_b}|_x^x}{\langle xb \rangle \langle xn \rangle} \frac{|\psi_{M_b \cup \{\hat{2}\}}|_y^y}{\langle yb \rangle \langle yn \rangle} \right). \end{aligned} \quad (2.32)$$

Summing (2.30) and (2.32), we get

$$\begin{aligned}
 & -\frac{\langle 21 \rangle \langle 1n \rangle}{\langle 2n \rangle^2 [12]} (I_1 + I_2 + I_{b \neq 1, 2, n}) \\
 & \rightarrow \frac{\langle 2n \rangle^3}{\prod_{i \neq 1, 2, n} \langle i2 \rangle} \sum_N \langle 2|N|\hat{n}\rangle \langle n|N|\hat{2}\rangle^3 \frac{|\psi_N|_x^x}{\langle x2 \rangle \langle xn \rangle} \frac{|\psi_M|_y^y}{\langle y2 \rangle \langle yn \rangle} \\
 & \quad + \sum_{b \neq 1, 2, n} \frac{\langle bn \rangle^3}{\prod_{i \neq 1, b, n} \langle ib \rangle} \sum_{N'_b} \langle b|N'_b|\hat{n}\rangle \langle n|N'_b|b\rangle^3 \frac{|\psi_{N'_b}|_x^x}{\langle xb \rangle \langle xn \rangle} \frac{|\psi_{M'_b}|_y^y}{\langle yb \rangle \langle yn \rangle} \\
 & = \sum_{b \neq 1, n} \frac{\langle bn \rangle^3}{\prod_{i \neq 1, b, n} \langle ib \rangle} \sum_{N'_b} \langle b|N'_b|\hat{n}\rangle \langle n|N'_b|b\rangle^3 \frac{|\psi_{N'_b}|_x^x}{\langle xb \rangle \langle xn \rangle} \frac{|\psi_{M'_b}|_y^y}{\langle yb \rangle \langle yn \rangle} = 0,
 \end{aligned} \tag{2.33}$$

which returns to the form of (2.8) for the set $\{\hat{2}, \dots, \hat{n}\}$! It vanishes after assuming the identity of $(n-1)$ particles (without particle 1) holds. Similar to N', M' , here N'_b, M'_b denote the sets including $\hat{2}$ but not b . This finishes the inductive proof of (1.3).

3 Two identities of the KLT momentum kernel

In this section we will prove (1.5) and (1.6) as conjectured in [14]. To understand these relations, we must first define the transformation matrices D and C between BCJ basis of gauge amplitudes via

$$A_n(t, \alpha_t, n-1, n) = \sum_{\alpha_{t'} \in S_{n-3}} A_n(t', \alpha_{t'}, n-1, n) D[t', \alpha_{t'}, n-1, n | t, \alpha_t, n-1, n], \tag{3.1}$$

$$\tilde{A}_n(t, n-1, \beta_t, n) = \sum_{\beta_{t'} \in S_{n-3}} C[t, n-1, \beta_t, n | t', n-1, \beta_{t'}, n] \tilde{A}_n(t', n-1, \beta_{t'}, n), \tag{3.2}$$

where $\alpha_{t'}$ and $\beta_{t'}$ denote the permutations of $(n-3)$ particles other than t' , $(n-1)$ and n . In a tensorial sense, D and C are the transformation matrices with respect to the summation of all $(n-3)!$ permutations, which is defined as the inner product. For reader's reference, we write (1.5) and (1.6) again below

$$\begin{aligned}
 & \sum_{\alpha_{t'}, \beta_{t'} \in S_{n-3}} D[t, \alpha_t, n-1, n | t', \alpha_{t'}, n-1, n] S[\alpha_{t'} | \beta_{t'}]_{p_{n-1}} \\
 & \quad \cdot C[t', n-1, \beta_{t'}, n | t, n-1, \beta_t, n] = S[\alpha_t | \beta_t]_{p_{n-1}},
 \end{aligned} \tag{3.3}$$

$$\begin{aligned}
 & \sum_{t'=1}^{n-2} \sum_{\alpha_{t'}, \beta_{t'} \in S_{n-3}} D[t, \alpha_t, n-1, n | t', \alpha_{t'}, n-1, n] S[\alpha_{t'} | \beta_{t'}]_{p_{n-1}} \\
 & \quad \cdot J_{t'} (C[t', n-1, \beta_{t'}, n | t, n-1, \beta_t, n]) = 0,
 \end{aligned} \tag{3.4}$$

where $S[\alpha_t | \beta_t]_{p_{n-1}}$ is the KLT momentum kernel of pivot p_{n-1} , and $J_{t'} \equiv J_{t', \hat{\alpha}\hat{\beta}}$ is the anti-holomorphic angular momentum operator. Here we follow the convention of S in [16–18], namely

$$S[\alpha_1, \dots, \alpha_k | \beta_1, \dots, \beta_k]_{p_{n-1}} = \prod_{i=1}^k \left(s_{\alpha_i, n-1} + \sum_{j=i+1}^k \theta(\alpha_i, \alpha_j) s_{\alpha_i, \alpha_j} \right), \tag{3.5}$$

where s_{ij} is each Mandelstam variable, and $\theta(\alpha_i, \alpha_j)$ is zero when the pair (α_i, α_j) has the same ordering at both sets $\{\alpha_1, \dots, \alpha_k\}$ and $\{\beta_1, \dots, \beta_k\}$, and unity otherwise.

For the first identity, its physical interpretation is straightforward: if we regard the KLT momentum kernel S as the metric, it is simply the tensorial transformation rule for metric. In fact, such a tensorial formulation had been established in [20, 22] (known as the KLT orthogonality or the CHY formulation) and we will use it to formally prove the first identity shortly. The second identity is however more intricate, as it roughly represents angular momentum conservation in an entangled way. The CHY formulation can help transform it into a relation that may reveal very nontrivial properties of scattering process, while to prove it directly is yet beyond our understanding.

3.1 Proof of the first identity

Before the proof, we must first rewrite gauge amplitudes in the CHY formulation [22] which is based on the scattering equations [21]. It tells that

$$A_n(t, \alpha_t, n-1, n) = \sum_{i=1}^{(n-3)!} \frac{1}{\det'(\Phi)(\sigma^{(i)})} \Sigma^{(i)}(t, \alpha_t, n-1, n) \text{Pf}'\Psi(\sigma^{(i)}), \quad (3.6)$$

$$\tilde{A}_n(t, n-1, \beta_t, n) = \sum_{i=1}^{(n-3)!} \frac{1}{\det'(\Phi)(\sigma^{(i)})} \Sigma^{(i)}(t, n-1, \beta_t, n) \text{Pf}'\Psi(\sigma^{(i)}), \quad (3.7)$$

where $\sigma^{(i)}$ denotes the i -th solution to the scattering equations

$$\sum_{b \neq a} \frac{s_{ab}}{\sigma_{ab}} = 0, \quad (3.8)$$

with $\sigma_{ab} = \sigma_a - \sigma_b$, and there are $(n-3)!$ solutions in total. The definitions of $\det'(\Phi)$ and $\text{Pf}'\Psi$, namely the reduced determinant of Jacobian Φ and the reduced Pffafian of antisymmetric matrix Ψ , can be found in [22]. The object mainly concerns us is

$$\Sigma^{(i)}(\alpha) \equiv \frac{1}{\sigma_{\alpha(1), \alpha(2)}^{(i)} \cdots \sigma_{\alpha(n-1), \alpha(n)}^{(i)} \sigma_{\alpha(n), \alpha(1)}^{(i)}}. \quad (3.9)$$

On the other hand, the KLT relation gives

$$\begin{aligned} (-)^{n+1} M_n(1, \dots, n) &= \sum_{\alpha_t, \beta_t \in S_{n-3}} A_n(t, \alpha_t, n-1, n) S[\alpha_t | \beta_t]_{p_{n-1}} \tilde{A}_n(t, n-1, \beta_t, n) \\ &= \sum_{i=1}^{(n-3)!} \frac{\text{Pf}'\Psi(\sigma^{(i)}) \text{Pf}'\Psi(\sigma^{(i)})}{\det'(\Phi)(\sigma^{(i)})}, \end{aligned} \quad (3.10)$$

where the second line results from the CHY formulation. There is a subtle issue of the sign above, due to the different conventions $M_n = -M_n^{\text{CHY}}$ and $S[\alpha_t | \beta_t] = S^{\text{CHY}}[\beta_t | \alpha_t^T]$. Plugging (3.6) and (3.7) into this relation, yields

$$\sum_{\alpha_t, \beta_t \in S_{n-3}} \Sigma^{(i)}(t, \alpha_t, n-1, n) S[\alpha_t | \beta_t]_{p_{n-1}} \Sigma^{(j)}(t, n-1, \beta_t, n) = \det'(\Phi)(\sigma^{(i)}) \delta_{ij}, \quad (3.11)$$

or more compactly,

$$G_{i\alpha_t} S[\alpha_t|\beta_t] (H_{j\beta_t})^T = I_{(n-3)! \times (n-3)!}, \quad (3.12)$$

which is the KLT orthogonality, if we define matrices

$$G_{i\alpha_t} \equiv \frac{\Sigma^{(i)}(t, \alpha_t, n-1, n)}{\sqrt{\det'(\Phi)(\sigma^{(i)})}}, \quad H_{j\beta_t} \equiv \frac{\Sigma^{(j)}(t, n-1, \beta_t, n)}{\sqrt{\det'(\Phi)(\sigma^{(j)})}}. \quad (3.13)$$

From this matrix relation we immediately get

$$S[\alpha_t|\beta_t] = (G_{i\alpha_t})^{-1} \left((H_{i\beta_t})^T \right)^{-1}. \quad (3.14)$$

Back to (3.6) and (3.7), if we further define the row vector

$$\Theta_i \equiv \frac{\text{Pf}'\Psi(\sigma^{(i)})}{\sqrt{\det'(\Phi)(\sigma^{(i)})}}, \quad (3.15)$$

then

$$A_n(t, \alpha_t, n-1, n) = \Theta_i G_{i\alpha_t}, \quad \tilde{A}_n(t, n-1, \beta_t, n) = \Theta_i H_{i\beta_t}. \quad (3.16)$$

Plugging them back into (3.1) and (3.2), and assuming their independence of basis Θ_i , we get

$$\begin{aligned} G_{i\alpha_t} &= G_{i\alpha_{t'}} D[t', \alpha_{t'}, n-1, n|t, \alpha_t, n-1, n], \\ H_{i\beta_t} &= H_{i\beta_{t'}} (C[t, n-1, \beta_t, n|t', n-1, \beta_{t'}, n])^T, \end{aligned} \quad (3.17)$$

or equivalently,

$$\begin{aligned} D[t', \alpha_{t'}, n-1, n|t, \alpha_t, n-1, n] &= (G_{i\alpha_{t'}})^{-1} G_{i\alpha_t}, \\ C[t, n-1, \beta_t, n|t', n-1, \beta_{t'}, n] &= (H_{i\beta_t})^T \left((H_{i\beta_{t'}})^{-1} \right)^T. \end{aligned} \quad (3.18)$$

Finally we plug them back into the l.h.s. of (1.5) and interchange t and t' , together with (3.14) we get

$$\begin{aligned} &\sum_{\alpha_{t'}, \beta_{t'} \in S_{n-3}} D[t, \alpha_t, n-1, n|t', \alpha_{t'}, n-1, n] S[\alpha_{t'}|\beta_{t'}]_{p_{n-1}} C[t', n-1, \beta_{t'}, n|t, n-1, \beta_t, n] \\ &= (G_{i\alpha_t})^{-1} G_{i\alpha_{t'}} (G_{j\alpha_{t'}})^{-1} \left((H_{j\beta_{t'}})^T \right)^{-1} (H_{k\beta_{t'}})^T \left((H_{k\beta_t})^{-1} \right)^T \\ &= (G_{i\alpha_t})^{-1} \left((H_{i\beta_t})^T \right)^{-1} = S[\alpha_t|\beta_t]_{p_{n-1}}, \end{aligned} \quad (3.19)$$

which is exactly the r.h.s. of (1.5), hence the proof is finished.

3.2 Discussion of the second identity

Now we move to prove (1.6). Equipped with the matrices defined in the previous subsection, the l.h.s. of (1.6) can be simplified as

$$\begin{aligned}
 & \sum_{t'=1}^{n-2} (G_{i\alpha_t})^{-1} G_{i\alpha_{t'}} (G_{j\alpha_{t'}})^{-1} \left((H_{j\beta_{t'}})^{\text{T}} \right)^{-1} \cdot J_{t'} \left[(H_{k\beta_{t'}})^{\text{T}} \left((H_{k\beta_t})^{-1} \right)^{\text{T}} \right] \\
 &= \sum_{t'=1}^{n-2} (G_{i\alpha_t})^{-1} \left((H_{i\beta_{t'}})^{\text{T}} \right)^{-1} \cdot J_{t'} \left[(H_{j\beta_{t'}})^{\text{T}} \left((H_{j\beta_t})^{-1} \right)^{\text{T}} \right] \\
 &= \sum_{t'=1}^{n-2} (G_{i\alpha_t})^{-1} \left[\left((H_{i\beta_{t'}})^{\text{T}} \right)^{-1} (H_{j\beta_{t'}})^{\text{T}} \cdot J_{t'} \left((H_{j\beta_t})^{-1} \right)^{\text{T}} \right. \\
 &\quad \left. + \left((H_{i\beta_{t'}})^{\text{T}} \right)^{-1} \cdot J_{t'} (H_{j\beta_{t'}})^{\text{T}} \cdot \left((H_{j\beta_t})^{-1} \right)^{\text{T}} \right] \\
 &= \sum_{t'=1}^{n-2} (G_{i\alpha_t})^{-1} \left[J_{t'} \left((H_{i\beta_t})^{-1} \right)^{\text{T}} \cdot (H_{j\beta_t})^{\text{T}} + \left((H_{i\beta_{t'}})^{\text{T}} \right)^{-1} \cdot J_{t'} (H_{j\beta_{t'}})^{\text{T}} \right] \left((H_{j\beta_t})^{-1} \right)^{\text{T}}, \tag{3.20}
 \end{aligned}$$

assuming the two matrices in the front and end of the last line are non-degenerate, we should prove

$$\sum_{t'=1}^{n-2} \left[J_{t'} \left((H_{i\beta_t})^{-1} \right)^{\text{T}} \cdot (H_{j\beta_t})^{\text{T}} + \left((H_{i\beta_{t'}})^{\text{T}} \right)^{-1} \cdot J_{t'} (H_{j\beta_{t'}})^{\text{T}} \right] = 0. \tag{3.21}$$

For the first term above, the summation over t' is trivial since the matrix product involves t only, so it is in fact

$$\left(\sum_{t'=1}^{n-2} J_{t'} \right) \left((H_{i\beta_t})^{-1} \right)^{\text{T}} \cdot (H_{j\beta_t})^{\text{T}} = 0, \tag{3.22}$$

due to angular momentum conservation, as the absence of J_{n-1} and J_n does not matter since $\tilde{\lambda}_{n-1}$ and $\tilde{\lambda}_n$ have been solved by momentum conservation (see [14] for more details). Therefore we are left with

$$\sum_{t=1}^{n-2} \left((H_{i\beta_t})^{\text{T}} \right)^{-1} \cdot J_t (H_{j\beta_t})^{\text{T}} = 0, \tag{3.23}$$

where the dummy variable t' has been replaced by t . We can continue to transform it into a convenient form for further attempts to prove, by isolating its real matrix content. Let's define

$$\Sigma_{j\beta_t} \equiv \Sigma^{(j)}(t, n-1, \beta_t, n), \quad W_{ij} \equiv \frac{1}{\sqrt{\det'(\Phi)(\sigma^{(i)})}} \delta_{ij}, \tag{3.24}$$

then it is clear that $H_{i\beta_t} = W_{ij}\Sigma_{j\beta_t}$. While W_{ij} is a trivial diagonal matrix, $\Sigma_{j\beta_t}$ encodes the real matrix content. Now we can write the l.h.s. of (3.23) as

$$\begin{aligned}
 & \sum_{t=1}^{n-2} W^{-1} \left((\Sigma_{i\beta_t})^T \right)^{-1} \cdot J_t \left((\Sigma_{j\beta_t})^T W \right) \\
 &= \sum_{t=1}^{n-2} W^{-1} \left((\Sigma_{i\beta_t})^T \right)^{-1} \cdot J_t (\Sigma_{j\beta_t})^T \cdot W + \sum_{t=1}^{n-2} W^{-1} \cdot J_t W \\
 &= \sum_{t=1}^{n-2} \left((\Sigma_{i\beta_t})^T \right)^{-1} \cdot J_t (\Sigma_{j\beta_t})^T + W^{-1} \cdot \left(\sum_{t=1}^{n-2} J_t \right) W \\
 &= \sum_{t=1}^{n-2} \left((\Sigma_{i\beta_t})^T \right)^{-1} \cdot J_t (\Sigma_{j\beta_t})^T,
 \end{aligned} \tag{3.25}$$

where in the third line, the second term vanishes again due to angular momentum conservation. Finally, we are left with

$$\sum_{t=1}^{n-2} \left((\Sigma_{i\beta_t})^T \right)^{-1} \cdot J_t (\Sigma_{j\beta_t})^T = 0, \tag{3.26}$$

which can no longer be further simplified.

To get some sense of this very nontrivial identity, it is helpful to see the first nontrivial case $n = 4$, which corresponds to the first nonempty β_t . Recall that

$$\Sigma^{(i)}(\alpha) = \frac{1}{\sigma_{\alpha(1),\alpha(2)}^{(i)} \cdots \sigma_{\alpha(n-1),\alpha(n)}^{(i)} \sigma_{\alpha(n),\alpha(1)}^{(i)}}, \tag{3.27}$$

we have

$$\begin{aligned}
 & \left(\Sigma^{(i)}(1, 3, 2, 4) \right)^{-1} J_1 \Sigma^{(j)}(1, 3, 2, 4) + \left(\Sigma^{(i)}(2, 3, 1, 4) \right)^{-1} J_2 \Sigma^{(j)}(2, 3, 1, 4) \\
 &= \sigma_{13}^{(i)} \sigma_{32}^{(i)} \sigma_{24}^{(i)} \sigma_{41}^{(i)} J_1 \left(\frac{1}{\sigma_{13}^{(j)} \sigma_{32}^{(j)} \sigma_{24}^{(j)} \sigma_{41}^{(j)}} \right) + \sigma_{23}^{(i)} \sigma_{31}^{(i)} \sigma_{14}^{(i)} \sigma_{42}^{(i)} J_2 \left(\frac{1}{\sigma_{23}^{(j)} \sigma_{31}^{(j)} \sigma_{14}^{(j)} \sigma_{42}^{(j)}} \right) \\
 &= \sigma_{13}^{(i)} \sigma_{32}^{(i)} \sigma_{24}^{(i)} \sigma_{41}^{(i)} (J_1 + J_2) \left(\frac{1}{\sigma_{13}^{(j)} \sigma_{32}^{(j)} \sigma_{24}^{(j)} \sigma_{41}^{(j)}} \right) = 0,
 \end{aligned} \tag{3.28}$$

which trivially holds by the antisymmetry of σ_{ab} ! But as n increases, even for $n = 5$ this identity will be much more entangled and simple antisymmetry is insufficient for its proof. The potential toolkit for this purpose includes: (1) relations of spinor derivatives on scattering equations; (2) KK and BCJ relations of $\Sigma_{j\beta_t}$; (3) induction, which may involve contour integration. We will come back to this point in the future after better understanding the scattering equations and their solutions.

A last comment is that in (1.6), the anti-holomorphic angular momentum operator $J_{t',\dot{\alpha}\dot{\beta}}$ should be generalized to $J_{t',\mu\nu}$ in arbitrary dimensions. Since in 4-dimension $J_{\mu\nu} \sim \varepsilon_{\alpha\beta} J_{\dot{\alpha}\dot{\beta}} + \varepsilon_{\dot{\alpha}\dot{\beta}} J_{\alpha\beta}$, and the soft theorem must hold for both holomorphic and anti-holomorphic soft limits, it is more natural to use $J_{\mu\nu}$ as all other quantities are already defined for arbitrary dimensions.

Acknowledgments

The authors would like to thank Qingjun Jin for reading the manuscript. This work is supported by Qiu-Shi Funding and Chinese NSF funding under contracts No.11135006, No.11125523 and No.11575156.

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