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# Holomorphic field realization of SH$^c$ and quantum geometry of quiver gauge theories

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Abstract: In the context of 4D/2D dualities, SH$^c$ algebra, introduced by Schiffmann and Vasserot, provides a systematic method to analyse the instanton partition functions of $\mathcal{N}=2$ supersymmetric gauge theories. In this paper, we rewrite the SH$^c$ algebra in terms of three holomorphic fields $D_0(z)$, $D_{\pm 1}(z)$ with which the algebra and its representations are simplified. The instanton partition functions for arbitrary $\mathcal{N}=2$ super Yang-Mills theories with $A_n$ and $A_n^{(1)}$ type quiver diagrams are compactly expressed as a product of four building blocks, Gaiotto state, dilatation, flavor vertex operator and intertwiner which are written in terms of SH$^c$ and the orthogonal basis introduced by Alba, Fateev, Litvinov and Tarnopolskiy. These building blocks are characterized by new conditions which generalize the known ones on the Gaiotto state and the Carlsson-Okounkov vertex. Consistency conditions of the inner product give algebraic relations for the chiral ring generating functions defined by Nekrasov, Pestun and Shatashvili. In particular we show the polynomiality of the qq-characters which have been introduced as a deformation of the Yangian characters. These relations define a second quantization of the Seiberg-Witten geometry, and, accordingly, reduce to a Baxter TQ-equation in the Nekrasov-Shatashvili limit of the Omega-background.

Keywords: Conformal and W Symmetry, Differential and Algebraic Geometry, Gauge Symmetry, String Duality

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## Contents



# 1 Introduction

$\mathrm{SH}^c$ is an algebra introduced by Shiffmann and Vasserot in [1] (see also [2]) to describe the equivariant cohomology of the instanton moduli space of $\mathcal{N}=2$ gauge theories in four dimensions. It has been defined as a spherical (symmetric) version of the degenerate double affine Hecke algebra (DAHA) which has been developed by Cherednick for many years [3].[1] While DAHA encodes the algebraic (recursive) properties of Macdonald polynomials [4]

[1] In fact, $\mathrm{SH}^c$ is a short notation introduced in [1] for central extension of the Spherical degenerate double affine Hecke algebra. It may be better referred to as Schiffmann-Vasserot algebra, but we will use $\mathrm{SH}^c$ in the text.

with two deformation parameters, degenerate DAHA is obtained by taking a limit $q, t \to 1$ such that one parameter $\beta$ remains, with $t = q^{-\beta}$. In this limit, Macdonald polynomials degenerate into Jack polynomials.

This algebra precisely describes the algebraic structure behind Nekrasov instanton partition functions [5] with the Omega background $\mathbb{R}^2_{\epsilon_1} \times \mathbb{R}^2_{\epsilon_2}$ with the identification $\beta = -\epsilon_1/\epsilon_2$ and has been used to prove the 4D/2D correspondence which generalizes Alday, Gaiotto and Tachikawa's proposal [6] (AGT conjecture) for various types of quiver gauge theories — namely pure super Yang-Mills theory [1], the gauge theories with fundamental [7] and bifundamental hypermultiplets [8] (see also the recent preprint [9]). For pure super Yang-Mills, the theory is characterized by a Gaiotto state [10], a coherent state affiliated to the Whittaker vector appearing in the representation theory of noncompact Lie algebra. To address higher quiver gauge theories, an operator which intertwines different representations is required for the description of bifundamental multiplets. It is defined by a direct product of the Carlsson-Okounkov operator [11] which describes the U(1) part and the vertex operator of Toda field theory. This operator must be properly generalized to describe the gauge theories on arbitrary quiver diagrams. Aside from SUSY gauge theories, $\mathrm{SH}^c$ has also revealed itself particularly useful in the study of vortex dynamics [12].

In such developments, the important role devoted to $\mathrm{SH}^c$ as a "universal" symmetry came principally from the fact that it contains all $W_N$ algebras for arbitrary $N$, together with an additional U(1) factor. The parameter $\beta$ is identified with a deformation parameter that defines the central charge $c = (N-1)(1+Q^2 N(N+1))$ of $W_N$ representations through the combination $Q = \sqrt{\beta} - \sqrt{\beta}^{-1}$. In this sense, $\mathrm{SH}^c$ should be regarded as a one parameter deformation of the $W_{1+\infty}$ algebra. The latter is known to have realizations in terms of $N$ free fermions acting on a space of $N$-tuple Young diagrams. These representations, that we call here rank $N$ representations, are identical to those defined by Fateev and Lukyanov in [13]. The correspondence between the two algebras has also been confirmed in more general cases where the Hilbert space contains singular vectors. The most typical example is the minimal models of $W_N$. There, it has been demonstrated explicitly in [14] that $\mathrm{SH}^c$ reproduces the proper descriptions of the Hilbert space constrained by the so-called N-Burge conditions [15, 16]. This universality is essential when we have to treat a system that contains gauge groups of different rank, as it is the case for quiver theories.

On the other hand, the action of $\mathrm{SH}^c$ on instanton partition functions of quiver $\mathcal{N} = 2$ gauge theories is very different from the representation of $W_N$ algebras. It is better understood after the introduction of an orthonormal basis constructed by Alba, Fateev, Litvinov and Tarnopolsky (AFLT basis) to prove the 4D/2D duality [17, 18]. AFLT basis should be regarded as a generalization of Jack polynomials [19–22], it is the proper basis to describe the action of degenerate DAHA. Instead of the description in terms of chiral primary fields with different spins, it is defined more abstractly through generators $D_{m,n}$ with two indices $m, n$. The first index $m \in \mathbb{Z}$ is identified with the index of the Virasoro generators $L_m$ while the second one $n \in \mathbb{Z}^{\geq 0}$ corresponds to the spin $n + 1$ of the generator. Since it is a nonlinear symmetry with a reasonably complicated structure, we have to be careful how to organize the generators. One of the authors [23] has recently found that holomorphic expansion in terms of the *second* index, $D_{\pm 1}(z), D_0(z)$, gives a compact description of $\mathrm{SH}^c$



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through the study of the Nekrasov-Shatashvili [24] limit of AGT conjecture. This turns out to be very useful and is a main tool of this paper.

The focus of the paper is to provide an $\mathrm{SH}^c$ description of Nekrasov partition functions for general $A_Q$ and $A_Q^{(1)}$-type quiver gauge theories. To do so, the action of $\mathrm{SH}^c$ operators on Gaiotto states, together with the adjoint action on the intertwiner operator describing bifundamental fields, is worked out. These actions are conveniently expressed in terms of the vertex operators $\mathcal{Y}(z)$ associated with the current $D_0(z)$. They extend the work on the covariance of the partition function presented in [8] by giving us the possibility to consider quiver theories with gauge groups of different ranks. As a consequence of our results, several useful identities can be established among correlators of the gauge theories. In particular, we were able to recover the expression of the qq-characters recently introduced by Nekrasov, Pestun and Shatashvili (NPS) [25]. For the simplest $A_1$ case with fundamental multiplets (4.10),

$$\chi(z) = \left\langle \mathcal{Y}(z+\epsilon_+) + q\frac{m(z)}{\mathcal{Y}(z)} \right\rangle .$$

Here $\epsilon_+ := \epsilon_1+\epsilon_2$ and $\langle\cdots\rangle$ denotes an average weighted by the instanton partition function, which is defined in (4.4). The operator $\mathcal{Y}(z)$ is interpreted as an operator version of the chiral ring generating function. These characters, presented as further deformation of the characters of Yangian algebras [26], encode in a compact form a recursion relation among the instanton partition functions [27]. Here we show that $\mathrm{SH}^c$ provides a proper symmetry behind the qq-character formulae, as was already predicted by NPS, and that the polynomiality property naturally follows from our description.

The qq-characters define a double deformation of the Seiberg-Witten geometry in a form of second quantization. In the above example, the Seiberg-Witten curve is expressed as (5.1),

$$y + q\frac{m(z)}{y} = \prod_{\ell=1}^{N}(z-a_\ell).$$

Seiberg-Witten theory is well-known to provide an effective description of the infrared sector of $\mathcal{N}=2$ gauge theories on $\mathbb{R}^4$ [28, 29]. The effective Lagrangian is written in terms of an holomorphic function, the prepotential, obtained from the knowledge of an algebraic curve and a differential form. This formulation is identified with the construction of finite gap solutions for classical integrable hierarchies, the algebraic curve corresponding to the spectral curve of the system [30].[2] When the gauge theory is considered in the Nekrasov-Shatashvili (NS) limit $\epsilon_2\to 0$ of the Omega-background, the associated integrable systems are quantized, with the remaining parameter $\epsilon_1$ playing the role of the Planck constant [24]. The algebraic curve becomes the Baxter TQ-equation of the quantum system [31, 32] (see also [27, 33] for the extension to quivers), it is equivalent to a Schrödinger equation under a quantum change of variables [34], in a form of ODE/IM correspondence [35].[3] In this

[2]These finite gap solutions can also be obtained from Hitchin systems.

[3]In this classical version of AGT correspondence, the Shrödinger equation is obtained as the semiclassical limit of the null vector decoupling equations obeyed by Liouville correlators containing a degenerate operators. It is sometimes referred as the *bispectral duality* [36, 37].



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framework, the two complex variables of the algebraic curve become non-commutative, thus defining a first quantization of the Seiberg-Witten curve [38–40]. In the full Omega-background, the qq-character is an operator acting in a Hilbert space of quantum states. In the NS limit, the expectation value of this operator in the Gaiotto state (which plays the role of a coherent state) becomes the T-polynomial of the TQ-equation, while its defining relation in terms of vertex operators reproduces Baxter's relation. In this sense, the qq-character formula presents a second quantization of the integrable system in which the TQ-relation emerges in the classical $\epsilon_2 \to 0$ limit.

We organize the paper as follows. In section 2, we introduce the holomorphic field description of SH$^c$ algebra and the rank $N$ representation. We also provide useful expressions for the adjoint actions of the vertex operators. In section 3, after a brief review of the general construction of the instanton partition functions, we introduce the building blocks (Gaiotto states, flavor vertex operator, intertwiner) with which the partition functions are written as a product. We show that the Gaiotto state satisfies stronger constraints which are compactly expressed in terms of SH$^c$ fields. A generalized intertwiner which connects different rank gauge groups is also defined. It satisfies similar conditions as the Gaiotto states and indeed it reduces to the Gaiotto state when the gauge group of one side is trivial. The flavor vertex operator is used to include the fundamental hypermultiplets in the gauge theories. These results are used in section 4 to build an infinite number of constraints among the correlation functions of the vertex operator $\mathcal{Y}$. The new characterizations of the Gaiotto states and the intertwiner play an essential role to give a closed and compact expression for these constaints — written in the form of qq-characters. Finally in section 5 we present their interpretation as quantum Seiberg-Witten geometry along the line of [23, 27, 41]. The concluding section proposes some perspectives for future research, and several technical details are gathered in the appendix.

## 2 Reformulation of SH$^c$ algebra

### 2.1 SH$^c$ algebra in terms of holomorphic fields

The SH$^c$ algebra is defined on a set of operators $D_{m,n}$ with the double grading $(m,n) \in \mathbb{Z} \times \mathbb{Z}^{\geq 0}$ [1]. The first index is called the *degree* and the second index the *order*. The algebraic relations involving $D_{\pm 1,n}$ and $D_{0,n}$ are written as

$$[D_{0,n}, D_{\pm 1,m}] = \pm D_{\pm 1,n+m-1}, \quad n \geq 1\,, \tag{2.1}$$

$$[D_{-1,n}, D_{1,m}] = E_{n+m} \quad n,m \geq 0\,, \tag{2.2}$$

$$[D_{0,n}, D_{0,m}] = 0\,,\ \ n,m \geq 0\,. \tag{2.3}$$

where $E_k$ denotes a linear combination of powers of the generators $D_{0,n}$ that will be given shortly. Additional relations can be found in [2], but they will not be used here. The algebra is spanned by the operators of degree 0 and $\pm 1$ upon the recursive use of the following commutation relations,

$$D_{\pm(m+1),0} = \pm\frac{1}{m}[D_{\pm 1,1}, D_{\pm m,0}], \quad D_{\pm m,n} = \pm[D_{0,n+1}, D_{\pm m,0}], \tag{2.4}$$

for $n \geq 0$ and $m > 0$. Rank $N$ representations of $\mathrm{SH}^c$ match with those of a semidirect product of the $W_N$ algebra and a U(1) current. The Heisenberg generators (U(1) currents) are related to $D_{m,0}$, the Virasoro generators to $D_{m,0}, D_{m,1}$, and operators $D_{m,n}$ with $n > 1$ to the currents of spin $n+1$ [1, 7].

It is useful to assemble the generators in the form of holomorphic fields [23],

$$D_{\pm1}(z) = \sum_{n=0}^{\infty} z^{-n-1} D_{\pm1,n}, \quad D_0(z) = \sum_{n=0}^{\infty} z^{-n-1} D_{0,n+1}, \quad E(z) = 1 + \epsilon_+ \sum_{n=0}^{\infty} z^{-n-1} E_n, \tag{2.5}$$

where $\epsilon_+ = \epsilon_1 + \epsilon_2$. We use here the Omega-background equivariant deformation parameters $\epsilon_1, \epsilon_2$ [5, 42] instead of the $\mathrm{SH}^c$ deformation parameter $\beta = -\epsilon_1/\epsilon_2$ in order to simplify the comparison with gauge theories.[4] It is noted that these Laurent series are vanishing at $z = \infty$, in which they are different from usual holomorphic fields in CFT.

We rewrite the defining properties of the generators $D_{\pm1,n}$ and $D_{0,n}$ in terms of the holomorphic fields,

$$[D_0(z), D_{\pm1}(w)] = \pm \frac{D_{\pm1}(w) - D_{\pm1}(z)}{z - w}, \quad [D_{-1}(z), D_1(w)] = \frac{E(w) - E(z)}{z - w} \epsilon_+^{-1}. \tag{2.6}$$

For the definition of $E(z)$ and the vertex operators which will appear later, it is essential to introduce

$$\Phi(z) := \log(z) D_{0,1} - \sum_{n=1}^{\infty} \frac{1}{n z^n} D_{0,n+1} \quad \Rightarrow \quad D_0(z) = \partial_z \Phi(z). \tag{2.7}$$

This definition of the field $\Phi(z)$ resembles the mode expansion of an holomorphic free bosonic field in CFT and the series $D_0(z)$ can be interpreted as the associated current. As we noted, however, the usual U(1) current in CFT is expanded as a sum over the degree as $J(\zeta) = \sum_{n\in\mathbb{Z}} (\text{coeff.}) D_{-n,0} \zeta^{-n-1}$ while (2.7) is expanded with respect to the order. In addition, fields at different points are commuting, $[\Phi(z), \Phi(w)] = 0$, as a consequence of (2.3). In this sense, the interpretation of the complex variable $z$ is different from the holomorphic coordinate of a Riemann surface but it should rather be seen as the spectral parameter of an integrable model. We will come back to this description later.

The following dressed combination of vertex operators will play a central role in our reformulation of the $\mathrm{SH}^c$ algebra and the correspondence with gauge theories,

$$\mathcal{Y}(z) = e^{c(z)} e^{\Phi(z-\epsilon_1)} e^{\Phi(z-\epsilon_2)} e^{-\Phi(z)} e^{-\Phi(z-\epsilon_+)}. \tag{2.8}$$

The function $c(z)$ encodes the dependence in the infinite number of the central charges $c_n$ $(n \geq 0)$ of the algebra. It expands as

$$c(z) = c_0 \log(z) - \sum_{n=1}^{\infty} \frac{c_n}{n z^n}. \tag{2.9}$$

The generating series $E(z)$ can now be expressed using the newly defined vertex operator,

$$E(z) = \mathcal{Y}(z + \epsilon_+) \mathcal{Y}(z)^{-1}. \tag{2.10}$$

[4]The correspondence between the convention of [8] and this paper is summarized in appendix A.



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### 2.2 Rank $N$ representations

Among possible representations of $\mathrm{SH}^c$, the best studied one is the rank $N$ representation where the Hilbert space is spanned by a basis labeled by $N$-tuple Young diagrams $\vec{Y} = (Y_1, \cdots, Y_N)$. The representation is characterized by $N$ complex numbers $a_\ell$ ($\ell = 1, \cdots, N$) that define the central charges $c_n$ through the relation

$$e^{c(z)} = \prod_{\ell=1}^{N} (z - a_\ell). \tag{2.11}$$

To emphasize the dependence of the representation in the parameters $a_\ell$ through the central charges, they will be included in the notation of the vector basis $|\vec{a}, \vec{Y}\rangle$. These vectors form an orthonormal basis of the representation space,

$$\langle \vec{a}, \vec{Y} | \vec{a}, \vec{Y}'\rangle = \delta_{\vec{Y},\vec{Y}'}, \quad 1 = \sum_{\vec{Y}} |\vec{a}, \vec{Y}\rangle\langle \vec{a}, \vec{Y}|. \tag{2.12}$$

For $N = 2$, this basis is actually proportional to the one employed in the proof of AGT conjecture in [17] and is usually referred as the AFLT basis. For a generic $N$, they can be identified with the generalized Jack polynomials introduced in [21] and studied in [22]. The vacuum state is obtained by taking the $N$-tuple of empty Young diagrams denoted $\vec{\emptyset}$, it satisfies

$$D_{0,n} |\vec{a}, \vec{\emptyset}\rangle = D_{-1,n} |\vec{a}, \vec{\emptyset}\rangle = 0, \quad \text{or} \quad D_0(z) |\vec{a}, \vec{\emptyset}\rangle = D_{-1}(z) |\vec{a}, \vec{\emptyset}\rangle = 0. \tag{2.13}$$

The Hilbert space spanned by $|\vec{a}, \vec{Y}\rangle$ will be denoted $\mathcal{V}_{\vec{a}}$. When several Hilbert spaces are considered, an extra label $\vec{a}$ will be inserted on the notation of the operators $D_r^{\vec{a}}(z)$ to specify in which space $\mathcal{V}_{\vec{a}}$ they act. The rank $N$ representations of $\mathrm{SH}^c$ are equivalent to the representations of $\mathcal{W}_N \times \mathrm{U}(1)$ [1, 14].

The action of $\mathrm{SH}^c$ generators of degrees $\pm 1$ on the state $|\vec{a}, \vec{Y}\rangle$ involves the $N$-tuple Young diagram with a box added/removed. As such, they can be seen as an analog of creation/annihilation operators while the total number of boxes in $\vec{Y}$ represents the number of particles (later identified with the instanton charge). Following [23], $N$-tuple Young diagrams $\vec{Y}$ with a box $x$ added/removed will be denoted $\vec{Y} \pm x$ (respectively). We further introduce the sets $A(\vec{Y})$ and $R(\vec{Y})$ containing all the boxes that can be added to/removed from the Young diagrams composing $\vec{Y}$. In figure 1, we illustrate the locations of the boxes in the sets $A(Y)$ and $R(Y)$ with the example of a single Young diagram $Y$.

The boxes $x \in \vec{Y}$ are characterized by a triplet of indices $(\ell, i, j)$ where $\ell = 1 \cdots N$ and $(i, j) \in Y_\ell$ gives the position of the box in the $\ell$th Young diagram. To each box $x$ is associated a complex number $\phi_x$ depending on the central charges using the map

$$x = (\ell, i, j) \in \vec{Y} \quad \longrightarrow \quad \phi_x = a_\ell + (i-1)\epsilon_1 + (j-1)\epsilon_2 \in \mathbb{C}. \tag{2.14}$$

With these definitions, the action of the spanning subalgebra takes the simple form [8, 23]

$$\begin{aligned}
D_{+1}(z)|\vec{a}, \vec{Y}\rangle &= \sum_{x \in A(\vec{Y})} \frac{\Lambda_x(\vec{Y})}{z - \phi_x} |\vec{a}, \vec{Y} + x\rangle, \qquad D_{-1}(z)|\vec{a}, \vec{Y}\rangle = \sum_{x \in R(\vec{Y})} \frac{\Lambda_x(\vec{Y})}{z - \phi_x} |\vec{a}, \vec{Y} - x\rangle, \\
D_0(z)|\vec{a}, \vec{Y}\rangle &= \sum_{x \in \vec{Y}} \frac{1}{z - \phi_x} |\vec{a}, \vec{Y}\rangle,
\end{aligned} \tag{2.15}$$

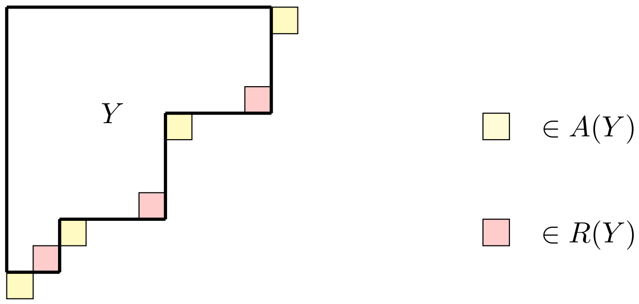


**Figure 1**. $A(Y)$ and $R(Y)$.

which are equivalent to their component form ($n \geq 0$):

$$D_{+1,n}|\vec{a},\vec{Y}\rangle = \sum_{x\in A(\vec{Y})} (\phi_x)^n \Lambda_x(\vec{Y})|\vec{a},\vec{Y}+x\rangle, \quad D_{-1,n}|\vec{a},\vec{Y}\rangle = \sum_{x\in R(\vec{Y})} (\phi_x)^n \Lambda_x(\vec{Y})|\vec{a},\vec{Y}-x\rangle,$$
$$D_{0,n+1}|\vec{a},\vec{Y}\rangle = \sum_{x\in\vec{Y}} (\phi_x)^n |\vec{a},\vec{Y}\rangle. \tag{2.16}$$

We note that the second relation in (2.16) implies that the moments of $\phi_{x\in\vec{Y}}$ are the eigenvalues of the commuting charges $D_{0,n}$. In the (generalized) Calogero-Sutherland system, $D_{0,n}$ plays the role of infinite commuting charges and $\phi_x$ is interpreted as the momentum of each particle. The interpretation of $z$ as the spectral parameter is natural in this sense. The left action of $\mathrm{SH}^c$ generators on bra $\langle\vec{a},\vec{Y}|$ is identical for the diagonal operators $D_0(z)$, $E(z)$, $e^{\Phi(z)}$. However it is reversed for the operators $D_{\pm1}(z)$,

$$\langle\vec{a},\vec{Y}|D_{+1}(z) = \sum_{x\in R(\vec{Y})} \frac{\Lambda_x(\vec{Y})}{z-\phi_x}\langle\vec{a},\vec{Y}-x|, \quad \langle\vec{a},\vec{Y}|D_{-1}(z) = \sum_{x\in A(\vec{Y})} \frac{\Lambda_x(\vec{Y})}{z-\phi_x}\langle\vec{a},\vec{Y}+x|. \tag{2.17}$$

The series $E(z)$ is also diagonal on the states $|\vec{a},\vec{Y}\rangle$, with eigenvalues given by the function

$$\Lambda(z)^2 = \prod_{x\in A(\vec{Y})} \frac{z-\phi_x+\epsilon_+}{z-\phi_x} \prod_{x\in R(\vec{Y})} \frac{z-\phi_x-\epsilon_+}{z-\phi_x}. \tag{2.18}$$

The coefficients $\Lambda_x(\vec{Y})$ in the action (2.15) of $D_\eta(z)$ correspond to the residues of this function $\Lambda(z)^2$ at $z=\phi_x$ with $x \in A(\vec{Y})$ or $R(\vec{Y})$:

$$\Lambda(z)^2 = 1 + \epsilon_+ \sum_{x\in A(\vec{Y})} \frac{\Lambda_x(\vec{Y})^2}{z-\phi_x} - \epsilon_+ \sum_{x\in R(\vec{Y})} \frac{\Lambda_x(\vec{Y})^2}{z-\phi_x},$$
$$\Lambda_x(\vec{Y})^2 = \prod_{\substack{y\in A(\vec{Y})\\ y\neq x}} \frac{\phi_x-\phi_y+\epsilon_+}{\phi_x-\phi_y} \prod_{\substack{y\in R(\vec{Y})\\ y\neq x}} \frac{\phi_x-\phi_y-\epsilon_+}{\phi_x-\phi_y}. \tag{2.19}$$

Eventually, the action of the vertex operator is expressed in terms of a product over the boxes of $\vec{Y}$,

$$e^{\Phi(z)}|\vec{a},\vec{Y}\rangle = Q_{\vec{Y}}(z)|\vec{a},\vec{Y}\rangle, \quad \text{with} \quad Q_{\vec{Y}}(z) = \prod_{x\in\vec{Y}} (z-\phi_x). \tag{2.20}$$

The specific combination of vertex operators entering in the definition (2.8) of $\mathcal{Y}(z)$ leads to a remarkable simplification of its eigenvalues

$$\mathcal{Y}(z)|\vec{a},\vec{Y}\rangle=\prod_{\ell=1}^{N}(z-a_\ell)\prod_{x\in\vec{Y}}\frac{(z-\phi_x-\epsilon_1)(z-\phi_x-\epsilon_2)}{(z-\phi_x)(z-\phi_x-\epsilon_+)}|\vec{a},\vec{Y}\rangle=\frac{\prod_{x\in A(\vec{Y})}(z-\phi_x)}{\prod_{x\in R(\vec{Y})}(z-\epsilon_+-\phi_x)}|\vec{a},\vec{Y}\rangle. \tag{2.21}$$

We note that there is a cancellation of factors between the numerators and the denominators in the middle term, and the resulting expression in the r.h.s. bears contributions only from the edges of the Young diagrams. Taking the ratio (2.10) defining the operator $E(z)$, we recover the expression (2.18) for the function $\Lambda(z)^2$:

$$E(z)|\vec{a},\vec{Y}\rangle=\Lambda(z)^2|\vec{a},\vec{Y}\rangle\,. \tag{2.22}$$

In appendix C, we provide an explicit computation of the commutation relations of $D_0(z)$, $D_{\pm1}(z)$ in the rank $N$ representation.

Finally we would like to mention the existence of an automorphism of representation. Under the shift of $\vec{a}$, $a_i\to\vec{a}'=\vec{a}+\mu\vec{e}$ where $\vec{e}=(1,1,\cdots,1)$, the representation (2.16) implies that

$$D_{+1}^{\vec{a}+\mu\vec{e}}(z)|\vec{a}+\mu\vec{e},\vec{Y}\rangle=\sum_{x\in A(\vec{Y})}\frac{\Lambda_x(\vec{Y})}{z-\mu-\phi_x}|\vec{a}+\mu\vec{e},\vec{Y}+x\rangle, \tag{2.23}$$

$$D_{-1}^{\vec{a}+\mu\vec{e}}(z)|\vec{a}+\mu\vec{e},\vec{Y}\rangle=\sum_{x\in R(\vec{Y})}\frac{\Lambda_x(\vec{Y})}{z-\mu-\phi_x}|\vec{a}+\mu\vec{e},\vec{Y}-x\rangle\,. \tag{2.24}$$

The coefficients appearing here may be identified with the representation of $D_{\pm1}^{\vec{a}}(z-\mu)$. It implies that there is an automorphism of the algebra by shifting the variable $z$: $D_r^{\vec{a}+\mu\vec{e}}(z)\sim D_r^{\vec{a}}(z-\mu)$ for $r=0,\pm1$. This shift symmetry of the representations is referred as the spectral flow in the context of $W_{1+\infty}$-algebra [43].

### 2.3 Adjoint action of the vertex operators

In order to prepare for the computations necessary in the next sections, we would like to evaluate the commutation relations between the vertex operators $e^{\pm\Phi(z)}$ and the elements spanning the $\mathrm{SH}^c$ algebra. The generators of degree zero form a commutative subalgebra, as a consequence the field $\Phi(z)$ commute with the series $D_0(z)$.

The evaluation of the adjoint action on $D_{\pm1}$ is slightly more involved. We introduce a vertex operator depending on two finite sets of points $z_i$ and $w_j$ with $i\in I$, $j\in J$,

$$U(\{z_i\},\{w_j\}):=\exp\left(\sum_{i\in I}\Phi(z_i)-\sum_{j\in J}\Phi(w_j)\right). \tag{2.25}$$

We claim the following identities:

$$\begin{aligned}U(\{z_i\},\{w_j\})^{-1}D_1(u)U(\{z_i\},\{w_j\})&=\mathrm{P}^-_{u=\infty,z_{i\in I}}\left[\frac{\prod_{j\in J}(w_j-u)}{\prod_{i\in I}(z_i-u)}D_1(u)\right],\\ U(\{z_i\},\{w_j\})^{-1}D_{-1}(u)U(\{z_i\},\{w_j\})&=\mathrm{P}^-_{u=\infty,w_{j\in J}}\left[\frac{\prod_{i\in I}(z_j-u)}{\prod_{j\in J}(w_j-u)}D_{-1}(u)\right],\end{aligned} \tag{2.26}$$

with the projector $\mathrm{P}^-_{u=\infty,z_{i\in I}}$ acting on functions of the variable $u$ as

$$\mathrm{P}^-_{u=\infty,z_{i\in I}} f(u) := f(u) - \sum_{i\in I} \frac{\mathrm{Res}_{\zeta=z_i} f(\zeta)}{u-z_i} - \mathrm{P}^+_u f(u). \tag{2.27}$$

Here $\mathrm{P}^+_z$ picks up the positive powers of a Laurent series in $z$, namely for a function $f(z)=\sum_{n=-m}^{\infty} a_n z^{-n}$, it operates as $\mathrm{P}^+_z f(z) = \sum_{n=-m}^{0} a_n z^{-n}$. Later we will use a similar notation for the orthogonal projector $\mathrm{P}^-_z = 1-\mathrm{P}^+_z = \mathrm{P}^-_{z=\infty}$ which picks up the negative powers of $f(z)$. The second term in (2.27) also plays the role to remove singularities at $u=z_i$. One may use a contour integration to write these projections in a compact form, for example,

$$\mathrm{P}^-_{u=\infty,z_{i\in I}} f(u) = \oint_C \frac{f(w)}{u-w}\frac{dw}{2\pi i}, \tag{2.28}$$

where the contour $C$ is defined by $|w| = R$ with $R < \mathrm{Min}_i(|z_i|)$. The formula (2.26) formally resembles an OPE in CFT, up to the existence of the projection operator which is necessary here since there is no singularity except for $u=0$ on the left hand side.

Before proving (2.26), it may be instructive to give some specific examples which will be used later. The first one is when one of the sets $I, J$ is null:

$$\begin{aligned} e^{-\eta\sum_{i=1}^M \Phi(z_i)} D_\eta(z_0) e^{\eta\sum_{i=1}^M \Phi(z_i)} &= \sum_{i=0}^M D_\eta(z_i) \prod_{j(\neq i)}^M \frac{1}{z_j - z_i}, \\ e^{\eta\sum_{i=1}^M \Phi(z_i)} D_\eta(w) e^{-\eta\sum_{i=1}^M \Phi(z_i)} &= \mathrm{P}^-_w\left[D_\eta(w)\prod_{i=1}^M (z_i - w)\right]. \end{aligned} \tag{2.29}$$

The other one is the adjoint action of $\mathcal{Y}(z)$ which is defined as a product of vertex operators with shifted arguments:

$$\begin{aligned} \frac{1}{\mathcal{Y}(z)} D_{-1}(w)\mathcal{Y}(z) &= S(w-z)D_{-1}(w) + \frac{\epsilon_1\epsilon_2}{\epsilon_+}\left(\frac{D_{-1}(z)}{z-w} - \frac{D_{-1}(z-\epsilon_+)}{z-w-\epsilon_+}\right), \\ \mathcal{Y}(z+\epsilon_+) D_1(w)\frac{1}{\mathcal{Y}(z+\epsilon_+)} &= S(z-w)D_1(w) - \frac{\epsilon_1\epsilon_2}{\epsilon_+}\left(\frac{D_1(z)}{z-w} - \frac{D_1(z+\epsilon_+)}{z-w+\epsilon_+}\right), \end{aligned} \tag{2.30}$$

where $S(z)$ denotes a scattering factor $S(z)$ defined as

$$S(z) = \frac{(z+\epsilon_1)(z+\epsilon_2)}{z(z+\epsilon_+)}. \tag{2.31}$$

**Proof of the formula (2.26).** To end up this section, we would like to give a short derivation of the identity (2.26). Rather than working with the commutator (2.6) directly, it is easier to evaluate the action on the states $|\vec{a},\vec{Y}\rangle$ which form a faithful representation of the $\mathrm{SH}^c$ algebra. We use the property

$$\frac{Q_{\vec{Y}\pm x}(z)}{Q_{\vec{Y}}(z)} = (z-\phi_x)^{\pm 1}, \tag{2.32}$$

which is a direct consequence of (2.20), and the action (2.15) of $D_\eta(z)$ on the states $|\vec{a},\vec{Y}\rangle$. It follows that

$$U(\{z_i\},\{w_j\})^{-1}D_1(u)U(\{z_i\},\{w_j\})|\vec{a},\vec{Y}\rangle = \sum_{x\in A(\vec{Y})}\frac{\prod_{j\in J}(w_j-\phi_x)}{\prod_{i\in I}(z_i-\phi_x)}\frac{\Lambda_x(\vec{Y})}{u-\phi_x}|\vec{a},\vec{Y}+x\rangle. \tag{2.33}$$

The product in the r.h.s. can be rewritten as a sum over single poles in $\phi_x$, with an extra polynomial term, using the algebraic identity,

$$\begin{aligned}\frac{\prod_{j\in J}(w_j-\phi)}{(u-\phi)\prod_{i\in I}(z_i-\phi)} &= \sum_{n=0}^{|J|-|I|-1} a_n(u|z,w)\phi^n + \frac{\prod_{j\in J}(w_j-u)}{(u-\phi)\prod_{i\in I}(z_i-u)}\\ &\quad+\sum_{i\in I}\frac{\prod_{j\in J}(w_j-z_i)}{(z_i-\phi)(u-z_i)\prod_{j\in I\setminus\{i\}}(z_j-z_i)}\,.\end{aligned} \tag{2.34}$$

Here $a_n$ are the coefficients appearing in Laurent expansion of the l.h.s. in $\phi$, they depend on the parameter $z_i$, $w_j$ and $u$:

$$\mathrm{P}_\phi^+\frac{\prod_{j\in J}(w_j-\phi)}{(u-\phi)\prod_{i\in I}(z_i-\phi)} = \sum_{n=0}^{|J|-|I|-1} a_n(u|z,w)\phi^n. \tag{2.35}$$

The sum over single poles in $\phi$ can be used to reform $D_1(z)$, while the polynomial part in $\phi_x$ gives the transformations $D_{1,n}$, and (2.33) becomes

$$\left(\frac{\prod_{j\in J}(w_j-u)}{\prod_{i\in I}(z_i-u)}D_1(u)+\sum_{n=0}^{|J|-|I|-1} a_n(u|z,w)D_{1,n}+\sum_{i\in I}\frac{\prod_{j\in J}(w_j-z_i)}{(u-z_i)\prod_{j\in I\setminus\{i\}}(z_j-z_i)}D_1(z_i)\right)|\vec{a},\vec{Y}\rangle\,. \tag{2.36}$$

Since the states $|\vec{a},\vec{Y}\rangle$ generate a faithful representation, the equality of vectors can be lifted at the level of operators

$$\begin{aligned}U(z,w)^{-1}D_1(u)U(z,w) &= \frac{\prod_{j\in J}(w_j-u)}{\prod_{i\in I}(z_i-u)}D_1(u)+\sum_{n=0}^{|J|-|I|-1} a_n(u|z,w)D_{1,n}\\ &\quad+\sum_{i\in I}\frac{\prod_{j\in J}(w_j-z_i)}{(u-z_i)\prod_{j\in I\setminus\{i\}}(z_j-z_i)}D_1(z_i)\end{aligned} \tag{2.37}$$

The expression for $U(z,w)^{-1}D_{-1}(u)U(z,w)$ is similarly obtained and is written as (2.37) with the substitution of the variables $z_i \leftrightarrow w_j$. The right hand side of (2.37) can be simplified by analyzing the first term: the second term cancels the poles (and the constant part) at $u=\infty$ of the first term, while the third term cancels the simple poles at $u=z_i$. The existence of such terms is natural since the left hand side of (2.37) is not singular at these points. The procedure of removing the unwanted poles is performed by the projector $\mathrm{P}^-_{u=\infty,z_{i\in I}}$ defined in (2.27), and (2.37) produces (2.26). It is noted that to analyze the pole at infinity, the following property should be employed,

$$\mathrm{P}_\phi^+\frac{r(\phi)}{u-\phi} = \frac{\mathrm{P}_\phi^+ r(\phi)-\mathrm{P}_u^+ r(u)}{u-\phi} = -\mathrm{P}_u^+\frac{r(u)}{u-\phi}, \tag{2.38}$$

for any meromorphic function $r(z)$. It implies in particular

$$\sum_{n=0}^{|J|-|I|-1} a_n(u|z,w)\phi^n = -\sum_{n=0}^{|J|-|I|-1} a_n(\phi|z,w)u^n. \tag{2.39}$$

# 3 Instanton partition function and SH$^c$ algebra

## 3.1 Nekrasov instanton partition function

Class $\mathcal{S}$ gauge theories with $\mathcal{N}=2$ supersymmetry are obtained by compactification of the six dimensional $\mathcal{N}=(2,0)$ theory on a Riemann surface. They are classified by a quiver diagram where each node $i$ is in correspondence with the simple group component $\mathrm{SU}(N_i)$ of the total gauge group $G=\otimes_i \mathrm{SU}(N_i)$. Thus, to each node corresponds a gauge multiplet containing a vector, two fermions and a scalar field in the adjoint representation. The arrows $i\to j$ of the quiver represents bifundamental matter fields, i.e. a chiral multiplet containing a fermion and a scalar field, with mass $m_{ij}$, and transforming in the fundamental representation of $\mathrm{SU}(N_i)\times\mathrm{SU}(N_j)$. In addition, a number $\tilde{N}_i$ of fundamental (or anti-fundamental) matter fields can be attached to each node $i$. They consist in chiral multiplets of masses $m_i^{(f)}$ with $f=1\cdots\tilde{N}_i$, encoded in the $\tilde{N}_i$-vector $\vec{m}_i$ (see figure 2).

The instanton partition functions of class $\mathcal{S}$ theories have been evaluated using localization in the Omega-background [5]. The theory is considered on the Coulomb branch where the adjoint scalar fields take non-zero vacuum expectation values. These complex parameters will be denoted $a_\ell^{(i)}$ with $\ell=1\cdots N_i$, they form the $N_i$-vector $\vec{a}_i$ attached to the node $i$. Localization provides a sum over nested integrals that can be computed by residues. The residues are in one-to-one correspondence with the boxes of the $N_i$-tuple Young diagrams for each node $i$ of the quiver. The resulting formula is a sum over realizations of these diagrams weighted by the multiplets contributions [42, 44–47]:

$$\mathcal{Z}_{\text{inst.}} = \sum_{\vec{Y}_1,\cdots\vec{Y}_Q}\prod_{i=1}^{Q} q_i^{|\vec{Y}_i|}\mathcal{Z}_{\text{vect.}}(\vec{a}_i,\vec{Y}_i)\mathcal{Z}_{\text{fund.}}(\vec{a}_i,\vec{Y}_i;\vec{m}_i)\prod_{i\to j\in E_Q}\mathcal{Z}_{\text{bfd.}}(\vec{a}_i,\vec{Y}_i;\vec{a}_j,\vec{Y}_j|m_{ij}), \tag{3.1}$$

where $Q$ is the number of nodes in the quiver, $E_Q$ its set of links, and $|\vec{Y}|$ denotes the total number of boxes in the N-tuple Young diagram $\vec{Y}$. The instanton counting parameter $q_i$ corresponds to the exponentiated gauge coupling at the node $i$, suitably renormalized in asymptotically free theories .

It is known that the contribution from each representation can be systematically derived from that for the bifundamental representation. Taking a bifundamental field of mass $m_{12}$ coupled to the two gauge groups $\mathrm{SU}(N_1)$ and $\mathrm{SU}(N_2)$, the contribution reads:

$$\mathcal{Z}_{\text{bfd.}}(\vec{a},\vec{Y};\vec{b},\vec{W}|m_{12}) = \prod_{\ell=1}^{N_1}\prod_{\ell'=1}^{N_2} g_{Y_\ell,W_{\ell'}}(a_\ell-b_{\ell'}-m_{12}) \tag{3.2}$$

$$\begin{aligned} g_{\lambda,\mu}(x) = &\prod_{(i,j)\in\lambda}(x+\epsilon_1(\lambda'_j-i+1)-\epsilon_2(\mu_i-j)) \\ &\cdot\prod_{(i,j)\in\mu}(-x+\epsilon_1(\mu'_j-i)-\epsilon_2(\lambda_i-j+1))\,. \end{aligned} \tag{3.3}$$



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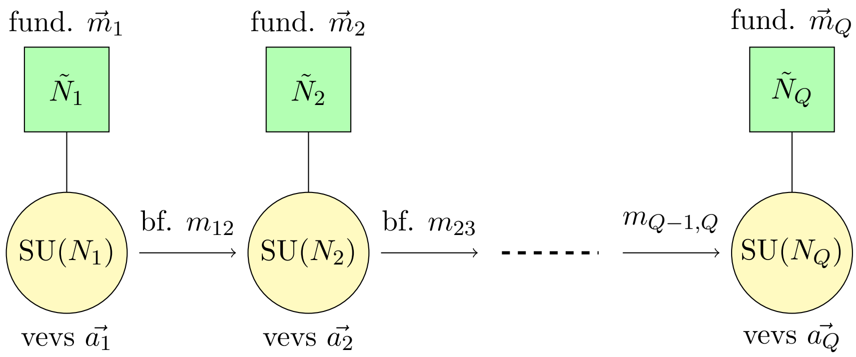


**Figure 2**. $A_Q$ linear quiver.

Here $\lambda_i$ is the height of $i^{\text{th}}$ column and $\lambda'_i$ is the length of $i^{\text{th}}$ row of Young diagram $\lambda$ (see figure 3).

The other building blocks can be written from (3.2) as follows.

- Fundamental hypermultiplets transforming under the gauge group $\mathrm{SU}(N)$ and the flavor group $\mathrm{SU}(\tilde{N})$, with masses $m_1, \cdots, m_{\tilde{N}}$: we take a vanishing bifundamental mass $m_{12} = 0$, for the first node $N_1 = \tilde{N}$, $\vec{a}_1 = \vec{m} := (m^{(1)}, \cdots, m^{(\tilde{N})})$ and $\vec{Y}_1 = \vec{\emptyset}$, and for the second node $N_2 = N$, $\vec{a}_2 = \vec{a}$ and $\vec{Y}_2 = \vec{Y}$ arbitrary:[5]

$$\mathcal{Z}_{\text{fund.}}(\vec{m}; \vec{a}, \vec{Y}) = \mathcal{Z}_{\text{bfd.}}(\vec{m}, \vec{\emptyset}; \vec{a}, \vec{Y}|0). \tag{3.4}$$

- Antifundamental hypermultiplet: in a symmetric way, we take $m_{12} = 0$, for the first node $N_1 = N$, $\vec{a}_1 = \vec{a}$ and $\vec{Y}_1 = \vec{Y}$ and for the second one $N_2 = \tilde{N}$, $\vec{a}_2 = -\vec{m}$ and $\vec{Y}_2 = \vec{\emptyset}$,

$$\mathcal{Z}_{\text{af.}}(\vec{m}; \vec{a}, \vec{Y}) = \mathcal{Z}_{\text{bfd.}}(\vec{a}, \vec{Y}; -\vec{m}, \vec{\emptyset}|0) = \mathcal{Z}_{\text{fund.}}(-\epsilon_+ - \vec{m}; \vec{a}, \vec{Y}). \tag{3.5}$$

- Adjoint hypermultiplet: we take $N_1 = N_2 = N$, $\vec{Y}_1 = \vec{Y}_2 = \vec{Y}$ and $\vec{a}_1 = \vec{a}_2 = \vec{a}$,

$$\mathcal{Z}_{\text{adj.}}(\vec{a}, \vec{Y}|m) := \mathcal{Z}_{\text{bfd.}}(\vec{a}, \vec{Y}; \vec{a}, \vec{Y}|m). \tag{3.6}$$

- Vector multiplet: inverse of the adjoint hypermultiplet with zero mass,

$$\mathcal{Z}_{\text{vect.}}(\vec{a}, \vec{Y}) := \mathcal{Z}_{\text{bfd.}}(\vec{a}, \vec{Y}; \vec{a}, \vec{Y}|0)^{-1}. \tag{3.7}$$

We note that the fundamental matter can be seen as bifundamental matter where one of the two gauge groups is taken in the weak coupling limit, effectively becoming a flavor group. In this limit, the corresponding exponentiated gauge coupling $q$ is sent to zero, and due to the presence of the factor $q^{|\vec{Y}|}$, only empty Young diagrams contribute in the

[5]We have chosen to shift the definition of the fundamental masses by $\epsilon_+$ in order to simplify formulas: $\vec{m}_{\text{fund.}} = \vec{m} + \epsilon_+$, $\vec{m}_{\text{af.}} = \vec{m} + \epsilon_+$. Note also that antifundamental contributions will be not discussed here as they are equivalent to fundamental contributions with shifted masses.

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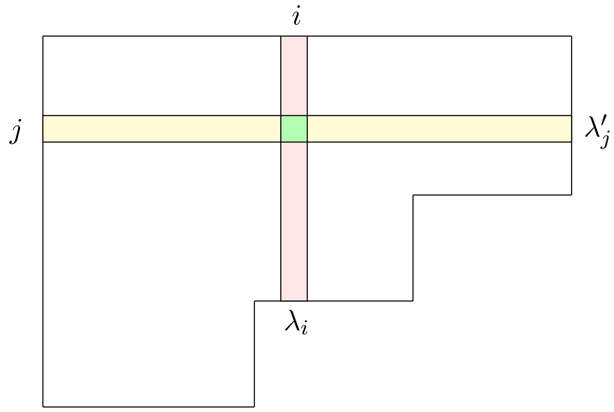


**Figure 3**. Young diagram.

summations. As a result, the contribution of a fundamental matter multiplet is derived from the bifundamental contribution by attaching to each set of $\tilde{N}_i$ fundamental flavors an $\tilde{N}_i$-tuple of empty Young diagrams.

The fact that the various contributions to the partition function of the different multiplets are derived from the bifundamental contribution implies important consequences for their SH$^c$ realization that will be presented in the next section.

### 3.2 Action of SH$^c$ operators on instanton partition functions

In this paper we focus on the linear quiver $A_Q$ and its affine version $A_Q^{(1)}$, they are characterized by the set of arrows $E_Q = \{i \to i+1, i = 1 \cdots Q-1\}$ and $E_Q = \{i \to i+1, i = 1 \cdots Q\}$ respectively, with the identification of indices modulo $Q$. One of the goal of this paper is to formulate the action of SH$^c$ operators on the (affine) linear quiver instanton partition function. For this purpose, we need to rewrite the partition function in terms of elements of the representation theory of SH$^c$: Gaiotto states, intertwiner and the vertex operator (figure 4).

**Gaiotto state.** To each node $i$ of the quiver diagram is associated a vector space of representation $\mathcal{V}_{\vec{a}_i}$ spanned by the vectors $|\vec{a}_i, \vec{Y}_i\rangle$ where $\vec{Y}_i$ takes values in all the possible realization of $N_i$-tuple Young diagrams. The set of complex parameters $\vec{a}_i$ is fixed in each $\mathcal{V}_{\vec{a}_i}$ and define the central charges of the representation of SH$^c$. The Gaiotto state has been introduced in [10] as a specific Whittaker vector of the Virasoro algebra with respect to the maximal nilpotent subalgebra $\{L_n, n > 0\}$. This algebra is spanned by the two elements $L_1$ and $L_2$, and the Gaiotto state is defined up to a normalization by the conditions,[6]

$$L_1|G\rangle = \alpha|G\rangle, \quad L_2|G\rangle = 0, \tag{3.8}$$

where $\alpha$ is a constant. This definition has been generalized to the case with fundamental flavors [48], and to higher rank [49, 50], and eventually implemented in the space of

[6]As a consequence of the Virasoro commutation relations, the second condition implies $L_n|G\rangle = 0$ for $n > 2$.

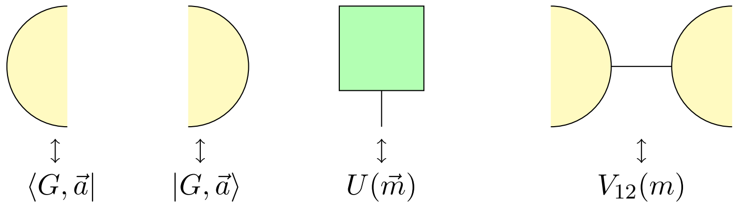


**Figure 4**. Correspondence between quiver diagram and Gaiotto states/vertex operator/intertwiner.

representation of $\mathrm{SH}^c$ (which contains a Virasoro sub-algebra) [1],

$$|G, \vec{a}\rangle = \sum_{\vec{Y}} \sqrt{\mathcal{Z}_{\text{vect}}(\vec{a}, \vec{Y})}|\vec{a}, \vec{Y}\rangle, \quad |G, \vec{a}\rangle \in \mathcal{V}_{\vec{a}}. \tag{3.9}$$

This state is known to provide the instanton partition function of pure $\mathcal{N} = 2$ SYM ($A_1$ quiver, $\tilde{N} = 0$),

$$\mathcal{Z}_{\text{inst}} = \langle G, \vec{a}|q^D|G, \vec{a}\rangle = \sum_{\vec{Y}} q^{|\vec{Y}|}\mathcal{Z}_{\text{vect.}}(\vec{a}, \vec{Y}), \tag{3.10}$$

where the operator $D = D_{0,1}$ counts the number of boxes in $\vec{Y}$, $D|\vec{a}, \vec{Y}\rangle = |\vec{Y}||\vec{a}, \vec{Y}\rangle$. It is identified with $L_0$ in Virasoro algebra up to the zero mode and $q^D$ may be regarded as the propagator in string theory. In the following, we refer to the operator of the form $q^D$ as the dilatation operator. It satisfies,

$$q^D D_{\pm 1}(z) = D_{\pm 1}(z) q^{D \pm 1}\,. \tag{3.11}$$

In terms of $\mathrm{SH}^c$ operators written in the form of holomorphic fields, the Gaiotto state has a new characterization. This is one of the main results of the paper:

$$D_{-1}(z)|G, \vec{a}\rangle = \frac{1}{\sqrt{-\epsilon_1\epsilon_2}}\frac{1}{\mathcal{Y}(z)}|G, \vec{a}\rangle, \tag{3.12}$$

$$D_1(z)|G, \vec{a}\rangle = \frac{-1}{\sqrt{-\epsilon_1\epsilon_2}}\mathrm{P}_z^-\mathcal{Y}(z + \epsilon_+)|G, \vec{a}\rangle, \tag{3.13}$$

$$\langle G, \vec{a}|D_{-1}(z) = \frac{-1}{\sqrt{-\epsilon_1\epsilon_2}}\langle G, \vec{a}|\mathrm{P}_z^-\mathcal{Y}(z + \epsilon_+), \tag{3.14}$$

$$\langle G, \vec{a}|D_1(z) = \frac{1}{\sqrt{-\epsilon_1\epsilon_2}}\langle G, \vec{a}|\frac{1}{\mathcal{Y}(z)}\,. \tag{3.15}$$

These formulae are a consequence of a more general result, presented in (3.22) and (3.23) below, and proven in appendix B.

These new expressions contain more information than the previously known relations given in (3.16). They reveal themselves powerful enough to derive several useful relations, presented in [27], among instanton partition function for arbitrary (A-type) quiver diagrams. The asymptotic of the operators $\mathcal{Y}(z)$ at infinity is deduced from (2.21): $\mathcal{Y}(z) \sim z^N$



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since $|A(\vec{Y})| - |R(\vec{Y})| = N$ for any $N$-tuple $\vec{Y}$. Expanding the first relation at infinite spectral parameter $z$ allows to recover the characterization of the Gaiotto states in [7, 8],

$$D_{-1,n}|G,\vec{a}\rangle = 0, \quad D_{-1,N-1}|G,\vec{a}\rangle = \frac{1}{\sqrt{-\epsilon_1\epsilon_2}}|G,\vec{a}\rangle, \quad D_{-1,N}|G,\vec{a}\rangle = \frac{1}{\sqrt{-\epsilon_1\epsilon_2}}\left(\sum_{\ell=1}^{N} a_\ell\right)|G,\vec{a}\rangle, \tag{3.16}$$

where $n = 1 \cdots N-2$, and the last property has been obtained using the formula (A.3) in [23]. These identities suggest to see the Gaiotto state as a (partial) coherent state in the physical sense of eigenstate of the annihilation operators $D_{-1,n}$.

**Flavor vertex operator.** Due to the presence of the empty Young diagram in the definition (3.4), the fundamental matter contribution can be written in a simpler form,

$$\mathcal{Z}_{\text{fund.}}(\vec{m};\vec{a},\vec{Y}) = \prod_{x\in\vec{Y}}\prod_{f=1}^{\tilde{N}}(\phi_x - m^{(f)}) = \prod_{f=1}^{\tilde{N}}(-1)^{|\vec{Y}|}Q_{\vec{Y}}(m^{(f)}), \tag{3.17}$$

where $Q_{\vec{Y}}(z)$ denotes the eigenvalue of the vertex operator defined in (2.20). This expression implies that the vertex operator can be used to insert fundamental multiplets in the quiver gauge theories.

$$U(\vec{m}) = (-1)^{\tilde{N}D}\exp\left(\sum_{f=1}^{\tilde{N}}\Phi(m^{(f)})\right) \quad \Rightarrow \quad U(\vec{m})|\vec{a},\vec{Y}\rangle = \mathcal{Z}_{\text{fund.}}(\vec{m};\vec{a},\vec{Y})|\vec{a},\vec{Y}\rangle. \tag{3.18}$$

This operator generates the modified Gaiotto states in the presence of fundamental multiplets, as studied in [7]. Since it plays the role to add the contribution of fundamental hypermultiplets with flavor group SU($\tilde{N}$), it will sometimes be referred to as the flavor vertex operator. The instanton partition function for this theory can be written

$$\mathcal{Z}_{\text{inst}} = \langle G,\vec{a}|q^D U(\vec{m})|G,\vec{a}\rangle = \sum_{\vec{Y}} q^{|\vec{Y}|}\mathcal{Z}_{\text{vect.}}(\vec{a},\vec{Y})\mathcal{Z}_{\text{fund.}}(\vec{m};\vec{a},\vec{Y}). \tag{3.19}$$

It is noted that the vertex operator $U(\vec{m})$ commutes with the dilatation operator $q^D$.

**Intertwiner.** Up to now, only partition functions of $\mathcal{N}=2$ theories with a single gauge group have been reproduced. To address the case of bifundamental matter coupled with multiple gauge groups, the construction of a new operator $V_{12}(\vec{a}_1,\vec{a}_2|m_{12}) : \mathcal{V}_{\vec{a}_2} \to \mathcal{V}_{\vec{a}_1}$ is required. This operator intertwines two SH$^c$ representations specified by $\vec{a}_1,\vec{a}_2$, with a different rank $N_1$ for $\mathcal{V}_{\vec{a}_1}$ and $N_2$ for $\mathcal{V}_{\vec{a}_2}$,

$$V_{12}(\vec{a}_1,\vec{a}_2|m_{12}) = \sum_{\vec{Y}_1,\vec{Y}_2}\bar{\mathcal{Z}}_{\text{bfd.}}(\vec{a}_1,\vec{Y}_1;\vec{a}_2,\vec{Y}_2|m_{12})\ |\vec{a}_1,\vec{Y}_1\rangle\langle\vec{a}_2,\vec{Y}_2|, \tag{3.20}$$

where a renormalized version of the bifundamental contribution has been used,

$$\bar{\mathcal{Z}}_{\text{bfd.}}(\vec{a}_1,\vec{Y}_1;\vec{a}_2,\vec{Y}_2|m_{12}) = \sqrt{\mathcal{Z}_{\text{vect.}}(\vec{a}_1,\vec{Y}_1)\mathcal{Z}_{\text{vect.}}(\vec{a}_2,\vec{Y}_2)}\mathcal{Z}_{\text{bfd.}}(\vec{a}_1,\vec{Y}_1;\vec{a}_2,\vec{Y}_2|m_{12}). \tag{3.21}$$



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Several algebraic properties of the intertwiner operator were studied from the viewpoint of $\mathrm{SH}^c$ in [8], in relation with a recursion formula satisfied by $\bar{\mathcal{Z}}_{\mathrm{bfd.}}$.[7]

The intertwiner satisfies a set of identities which resemble the conditions (3.12)–(3.15) for the Gaiotto states:

$$\begin{aligned} & D_{-1}^{\vec{a}_1}(z) V_{12}(\vec{a}_1,\vec{a}_2|m_{12}) - V_{12}(\vec{a}_1,\vec{a}_2|m_{12}) D_{-1}^{\vec{a}_2}(z-m_{12}) \\ & \qquad = \frac{1}{\sqrt{-\epsilon_1\epsilon_2}} \mathrm{P}_z^- \left( \frac{1}{\mathcal{Y}^{(1)}(z)} V_{12}(\vec{a}_1,\vec{a}_2|m_{12}) \mathcal{Y}^{(2)}(z+\epsilon_+ - m_{12}) \right), \end{aligned} \tag{3.22}$$

$$\begin{aligned} & D_{+1}^{\vec{a}_1}(z) V_{12}(\vec{a}_1,\vec{a}_2|m_{12}) - V_{12}(\vec{a}_1,\vec{a}_2|m_{12}) D_{+1}^{\vec{a}_2}(z+\epsilon_+-m_{12}) \\ & \qquad = -\frac{1}{\sqrt{-\epsilon_1\epsilon_2}} \mathrm{P}_z^- \left( \mathcal{Y}^{(1)}(z+\epsilon_+) V_{12}(\vec{a}_1,\vec{a}_2|m_{12}) \frac{1}{\mathcal{Y}^{(2)}(z+\epsilon_+ - m_{12})} \right). \end{aligned} \tag{3.23}$$

Here the notation $\mathcal{Y}^{(i)}$ $(i=1,2)$ represents the action of the vertex operor $\mathcal{Y}$ in the space $\mathcal{V}_{\vec{a}_i}$. These formulas characterize the transformation of the bifundamental contribution under the action of $\mathrm{SH}^c$. The proof of the formulae is summarized in appendix B.

In the 4D/2D correspondence, the intertwiner is described as a vertex operator of the form, $V = V^{\mathrm{CO}} V^{\mathrm{Toda}}$ where $V^{\mathrm{CO}}$ is the Carlsson-Okounkov vertex [11] for the U(1) factor and $V^{\mathrm{Toda}}$ is the vertex operator of Toda field theory associated with the $W_N$ algebra. This construction, however, has some limitations. One issue is the technical difficulty to define the transformation properties of $V^{\mathrm{Toda}}$ for higher spin generators. We have to face a nonlinear expression in terms of $W$ generators or Toda fields which is usually not manageable. A more serious issue is the impossibility to define an intertwiner between $W_N$ and $W_M$ Toda systems with $N \neq M$ since there is no obvious correspondence between the generators in $W_N$ algebras with a different $N$ (see, for example [51, 52], for attempts to explore such a setup). At the level of $\mathrm{SH}^c$, the correspondence between the generators for representations of a different rank becomes obvious and the transformation properties (3.22) and (3.23) are compact and tractable. Furthermore, it was confirmed in [7, 8] that these conditions contain the modified Ward identities for the U(1) current and the Virasoro operator for $V = V^{\mathrm{CO}} V^{\mathrm{Toda}}$ when $N_1 = N_2$. In this sense, our characterizations of the intertwiner is a natural generalization of the conventional vertex operators in Toda field theories to study the 4D/2D correspondence.

**Gaiotto state from intertwiner.** As briefly recalled in the previous subsection, the study of the instanton partition functions for miscellaneous field content can be reduced to the analysis of the bifundamental hypermultiplet. This fact has an important consequence for the $\mathrm{SH}^c$ realization that we explain here. We first consider the special case $N_1 = N$, $N_2 = 0$ and $m_{12} = 0$ for the intertwiner. Here the rank 0 representation means a trivial representation which consists of one state — the vacuum $|,\rangle$ (empty slots means that we have no Fock space). Since $\mathcal{Z}_{\mathrm{bfd.}}(\vec{a},\vec{Y};,|0) = \mathcal{Z}_{\mathrm{vect.}}(,) = 1$, we find after omitting the trivial bra vacuum,

$$V_{12}(\vec{a},|0) = |G,\vec{a}\rangle. \tag{3.24}$$

[7] In [8] it was assumed that the ranks of the two representations are the same. However, the computation performed there can be straightforwardly generalized to the case $N_1 \neq N_2$.

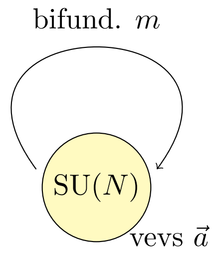


**Figure 5**. $A_1^{(0)}$ quiver.

Taking the opposite case of a rank zero representation for the first node produces the bra Gaiotto state in a similar way. In this sense, the recursion properties of the Gaiotto states (3.12)–(3.15) are straightforward consequences of (3.22) and (3.23). We note that we can take $D_\eta^{(2)}(z) = 0$ for the trivial representation and the action operator $\mathcal{Y}$ on $\mathcal{V}_2$ is replaced by 1.

Inclusion of (anti-)fundamental hypermultiplet is also straightforward. From (3.4) and (3.5), after fixing $N_1 = N$, $N_2 = \tilde{N}$, $m_{12} = 0$ and $\vec{a}_2 = \vec{m} + \epsilon_+$, we obtain that the action of the intertwiner on the vacuum produces the Gaiotto state with a flavor vertex operator inserted:

$$V_{12}(\vec{a}, \vec{m} + \epsilon_+|0)|\vec{m} + \epsilon_+, \vec{\emptyset}\rangle = U(\vec{m})|G, \vec{a}\rangle\,. \tag{3.25}$$

**Full partition function.** We now have all the elements to write down the instanton partition function of any linear quiver as a product of operators,

$$\mathcal{Z}_{\text{inst}} = \begin{cases} \langle G, \vec{a}_1|q_1^D U(\vec{m}_1)V_{12}(\vec{a}_1, \vec{a}_2|m_{12})q_2^D U(\vec{m}_2)V_{23}(\vec{a}_2, \vec{a}_3|m_{23})\cdots|G, \vec{a}_Q\rangle & \text{for } A_Q\,, \\ \mathrm{Tr}_{\mathcal{V}_{\vec{a}_1}}\left[q_1^D U(\vec{m}_1)V_{12}(\vec{a}_1, \vec{a}_2|m_{12})q_2^D U(\vec{m}_2)V_{23}(\vec{a}_2, \vec{a}_3|m_{23})\cdots V_{Q1}(\vec{a}_Q, \vec{a}_1|m_{Q1})\right] & \text{for } A_Q^{(1)}. \end{cases} \tag{3.26}$$

To each arrow $i \to j$ of the quiver is associated the intertwiner $V_{ij}(\vec{a}_i, \vec{a}_j|m_{ij})$, and to each node $i$ an operator $q_i^D U(\vec{m}_i)$. For the linear quivers, the resulting operator is sandwiched between the Gaiotto states attached to the first and the last node. On the contrary, a trace is directly obtained for the affine quiver from the intertwiners, it is defined as

$$\mathrm{Tr}_{\mathcal{V}_{\vec{a}}} \cdots = \sum_{\vec{Y}} \langle \vec{a}, \vec{Y}| \cdots |\vec{a}, \vec{Y}\rangle. \tag{3.27}$$

As an example, the partition function of $\mathcal{N} = 2^*$ theory represented in figure 5 with bifundamental fields of mass $m$ reads

$$\begin{aligned}\mathcal{Z}_{\text{inst}} = \mathrm{Tr}_{\mathcal{V}_{\vec{a}}}\left[q^D V_{11}(\vec{a}, \vec{a}|m)\right] &= \sum_{\vec{Y}} q^{|\vec{Y}|}\mathcal{Z}_{\text{vect.}}(\vec{a}, \vec{Y})\mathcal{Z}_{\text{bfd.}}(\vec{a}, \vec{Y}; \vec{a}, \vec{Y}|m) \\ &= \sum_{\vec{Y}} q^{|\vec{Y}|}\mathcal{Z}_{\text{vect.}}(\vec{a}, \vec{Y})\mathcal{Z}_{\text{adj}}(\vec{a}, \vec{Y}|m),\end{aligned} \tag{3.28}$$

thanks to the orthonormality property of the states $|\vec{a}, \vec{Y}\rangle$.



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**Alternative expressions.** Although it will not be used in this paper, we would like to provide, as a side remark, a new expression for the bifundamental contribution (3.2) involving the vertex operator $\mathcal{Y}(z)$. This expression is a consequence of the property (B.1) expressing the variation of $\mathcal{Z}_{\text{bfd.}}(\vec{a},\vec{Y};\vec{b},\vec{W}|m)$ under the addition of a box in the Young diagrams $\vec{Y}$. It turns out that the right hand side of (B.1) is independent of the actual content of boxes in the Young diagrams $\vec{Y}$. As a result, this formula can be used to build recursively $\mathcal{Z}_{\text{bfd.}}(\vec{a},\vec{Y};\vec{b},\vec{W}|m)$ from $\mathcal{Z}_{\text{bfd.}}(\vec{a},\vec{\emptyset};\vec{b},\vec{W}|m)$, adding boxes one by one. Since, $\mathcal{Z}_{\text{bfd.}}(\vec{a},\vec{\emptyset};\vec{b},\vec{W}|m)$ can be further identified with a fundamental contribution of mass $\vec{a}-m$, it is shown that

$$\begin{aligned}\mathcal{Z}_{\text{bfd.}}(\vec{a},\vec{Y};\vec{b},\vec{W}|m) &= \langle\vec{b},\vec{W}|U(\vec{a}-m)|\vec{b},\vec{W}\rangle\prod_{x\in\vec{Y}}\left\langle\vec{b},\vec{W}|\mathcal{Y}(\phi_x-m+\epsilon_+)|\vec{b},\vec{W}\right\rangle\\ &= \langle\vec{a},\vec{Y}|U(\vec{b}+m-\epsilon_+)|\vec{a},\vec{Y}\rangle\prod_{x\in\vec{W}}\left\langle\vec{a},\vec{Y}|\mathcal{Y}(\phi_x+m)|\vec{a},\vec{Y}\right\rangle\end{aligned}\tag{3.29}$$

where the second equality has been obtained by exploiting the symmetry under the exchange of $(\vec{a},\vec{Y})\leftrightarrow(\vec{b},\vec{W})$ and $m\leftrightarrow\epsilon_+-m$. As a special case of this expression, new formulae for the vector contribution and the Gaiotto states can also be deduced,

$$\begin{aligned}\mathcal{Z}_{\text{vect.}}(\vec{a},\vec{Y}) &= \left\langle\vec{a},\vec{Y}\middle|U(\vec{a}-\epsilon_+)^{-1}\prod_{x\in\vec{Y}}\mathcal{Y}(\phi_x)^{-1}\middle|\vec{a},\vec{Y}\right\rangle,\\ |G,\vec{a}\rangle &= \sum_{\vec{Y}}\frac{1}{\sqrt{U(\vec{a}-\epsilon_+)}}\prod_{x\in Y}\frac{1}{\sqrt{\mathcal{Y}(\phi_x)}}|\vec{a},\vec{Y}\rangle.\end{aligned}\tag{3.30}$$

## 4 Ward identities of SH$^c$ and qq-character

In the previous section, we have seen that the instanton partition function for any $A_Q$ type quiver gauge theories can be written by combining Gaiotto states, dilatation operators, flavor vertex operators and intertwiners as in (3.26). The behavior of these states/operators under the action of SH$^c$ generators has been characterized through the set of relations (3.12)–(3.15), (3.11), (2.26) and (3.22)–(3.23). As a side result, one obtains a series of consistency conditions by inserting $D_{\pm1}(z)$ in the correlator and evaluating the inner product in two different ways,

$$(\langle G,\vec{a}_1|\mathcal{O}_1D_{\pm1}(z))\,\mathcal{O}_2|G,\vec{a}_Q\rangle = \langle G,\vec{a}_1|\mathcal{O}_1\,(D_{\pm1}(z)\mathcal{O}_2|G,\vec{a}_Q\rangle)\,,\tag{4.1}$$

where $\mathcal{O}_i$ denotes a combination of flavor vertex operator, dilatation operators and intertwiners. These conditions may be regarded as the Ward identities for the correlation functions of SH$^c$. Since it is written as a generating function with parameter $z$, it gives an infinite number of constraints. In the following, we evaluate the explicit form of these identities. We observe that their structure takes the form of a double quantum deformation of the character formulae for $A_Q$, the so-called *qq-character* proposed by Nekrasov, Pestun and Shatashvili [25, 41]. In the next section, we will discuss another interpretation of these formulae as a quantum deformation of the Seiberg-Witten curve.

### 4.1 $A_1$ quiver

We start from the expression (3.10) of the instanton partition function for pure SYM with SU($N$) gauge group, and consider the insertion of the operator of $D_{-1}(z)$, $\langle G,\vec{a}|D_{-1}(z)q^D|G,\vec{a}\rangle$, evaluated in two different ways as in (4.1). After the use of the identities (3.11), (3.12), (3.14), we arrive at,

$$\langle G,\vec{a}|\mathrm{P}_z^-\left(\mathcal{Y}(z+\epsilon_+)+\frac{q}{\mathcal{Y}(z)}\right)q^D|G,\vec{a}\rangle=0\,. \tag{4.2}$$

The insertion of $\mathcal{Y}$ has the effect of adding extra factors to the partition function. For example, from (2.21),

$$\langle G,\vec{a}|\mathcal{Y}(z+\epsilon_+)q^D|G,\vec{a}\rangle=\sum_{\vec{Y}}q^{|\vec{Y}|}\left(\frac{\prod_{x\in A(\vec{Y})}(z+\epsilon_+-\phi_x)}{\prod_{x\in R(\vec{Y})}(z-\phi_x)}\right)\mathcal{Z}_{\text{vect.}}(\vec{a},\vec{Y}). \tag{4.3}$$

We will use the following notation for the expectation value of the Gaiotto state:

$$\langle\cdots\rangle=\frac{1}{\mathcal{Z}_{\text{inst}}}\sum_{\vec{Y}}q^{|\vec{Y}|}\mathcal{Z}_{\text{vect.}}(\vec{a},\vec{Y})\langle\vec{a},\vec{Y}|\cdots|\vec{a},\vec{Y}\rangle. \tag{4.4}$$

This defines an average of operators acting on states $|\vec{a},\vec{Y}\rangle$ that is normalized to $\langle 1\rangle=1$. The relation (4.2) is rewritten in the form:

$$\mathrm{P}_z^-\left\langle\mathcal{Y}(z+\epsilon_+)+\frac{q}{\mathcal{Y}(z)}\right\rangle=0\,. \tag{4.5}$$

This condition is the generating function of an infinite number of constraints on the instanton partition function. At the same time, this formula implies that

$$\chi(z):=\langle\mathcal{Y}(z+\epsilon_+)\rangle+\left\langle\frac{q}{\mathcal{Y}(z)}\right\rangle=\left\langle\mathrm{P}_z^+(\mathcal{Y}(z+\epsilon_+))\right\rangle \tag{4.6}$$

has no negative powers of $z$ in the Laurent expansion at $z=\infty$. We note that $\mathcal{Y}(z)$ behaves as $\mathcal{Y}(z)\sim z^N$ as $z\to\infty$. It implies that $\chi(z)$ thus defined is a polynomial in $z$ of degree $N$. The expression $\chi\sim y+1/y$ is reminiscent of the character of sl(2) for the fundamental representation. The formula (4.6) is deformed by two parameters $\epsilon_{1,2}$ and was referred as a fundamental qq-character in [25, 27, 41] for the quantum deformed Yangian $Y_\epsilon(\mathrm{sl}(2))$.

The inclusion of fundamental hypermultiplets with $\tilde{N}$ flavor is a straightforward generalization. The only necessary modification is to insert a flavor vertex operator $U(\vec{m})$ in front of $|G,\vec{a}\rangle$. The commutator with $D_{-1}(z)$ is obtained from (2.29):

$$D_{-1}(z)U(\vec{m})=U(\vec{m})\mathrm{P}_z^-\left[D_{-1}(z)m(z)\right],\quad m(z)=\prod_{f=1}^{\tilde{N}}(z-m^{(f)})\,. \tag{4.7}$$

Inserting this relation between two Gaiotto states (with an operator $q^D$), and then evaluating the action of $D_{-1}(z)$ through (3.12) and (3.14) leads to

$$\mathrm{P}_z^-\left\langle\mathcal{Y}(z+\epsilon_+)+q\frac{m(z)}{\mathcal{Y}(z)}\right\rangle=0, \tag{4.8}$$

where the average acquired an extra factor $\mathcal{Z}_{\text{fund.}}(\vec{m};\vec{a},\vec{Y})$,

$$\langle\cdots\rangle = \frac{1}{\mathcal{Z}_{\text{inst}}}\sum_{\vec{Y}} q^{|\vec{Y}|}\mathcal{Z}_{\text{fund.}}(\vec{m};\vec{a},\vec{Y})\mathcal{Z}_{\text{vect.}}(\vec{a},\vec{Y})\langle\vec{a},\vec{Y}|\cdots|\vec{a},\vec{Y}\rangle. \tag{4.9}$$

After including the fundamental hypermultiplets, the qq-character is modified to

$$\chi(z) = \left\langle \mathcal{Y}(z+\epsilon_+) + \frac{qm(z)}{\mathcal{Y}(z)} \right\rangle. \tag{4.10}$$

As a consequence of (4.8), $\mathrm{P}_z^-\chi(z) = 0$ and the qq-character is again a polynomial of degree $N$ in $z$. A more detailed discussion of the qq-character in the presence of fundamental hypermultiplets is presented in appendix D.

In the case $\tilde{N} < N$, the ratio $m(z)/\mathcal{Y}(z)$ has no polynomial part and the character $\chi(z)$ equals the average of the operator $\mathrm{P}_z^+\mathcal{Y}(z+\epsilon_+)$. As a result, explicit expressions for the qq-character can be obtained by expansion of $\mathcal{Y}(z+\epsilon_+)$ at infinity using the properties (2.8), (2.21):

$$\begin{aligned}\mathcal{Y}(z+\epsilon_+) &= \prod_{\ell=1}^{N}(z+\epsilon_+-a_\ell)\left(1-\epsilon_1\epsilon_2\frac{d}{dz}D_0(z)+\text{higher terms in }\epsilon\right)\\ &= \prod_{\ell=1}^{N}(z+\epsilon_+-a_\ell)+\epsilon_1\epsilon_2 z^{N-2}\ D+O(z^{N-3}).\end{aligned} \tag{4.11}$$

In the average (4.4) the operator $D$ with eigenvalue $|\vec{Y}|$ can be replaced by a logarithmic $q$-derivative,

$$\chi(z) = \prod_{\ell=1}^{N}(z+\epsilon_+-a_\ell)+\epsilon_1\epsilon_2 z^{N-2}q\partial_q\log\mathcal{Z}_{\text{inst}}+O(z^{N-3}). \tag{4.12}$$

Specializing to $N=1$ and to $N=2$ with $a_1=-a_2=a$, we deduce the following expressions

$$\begin{aligned}\mathrm{U}(1):&\quad \chi(z) = z+\epsilon_+-a,\\ \mathrm{SU}(2):&\quad \chi(z) = (z+\epsilon_+)^2-a^2+\epsilon_1\epsilon_2 q\partial_q\log\mathcal{Z}_{\text{inst}}.\end{aligned} \tag{4.13}$$

### 4.2 qq-characters of higher representations for the $A_1$ quiver

In a series of recent lectures, Nekrasov proposed a generalization of the qq-character for higher representations of $Y_\epsilon(\mathrm{sl}(2))$ [25]. Higher qq-characters involve a set of complex parameters $\nu_1,\cdots,\nu_r\in\mathbb{C}$, and they are defined as

$$\chi_r(z|\nu_1,\cdots\nu_r) = \sum_{I\sqcup J=\{1,\cdots,r\}} q^{|J|}\prod_{\substack{i\in I\\ j\in J}}S(\nu_i-\nu_j)\left\langle\prod_{i\in I}\mathcal{Y}(z+\epsilon_++\nu_i)\prod_{j\in J}\frac{m(z+\nu_j)}{\mathcal{Y}(z+\nu_j)}\right\rangle. \tag{4.14}$$

Here $S(z)$ is the scattering factor (2.31). It is claimed in [25] that the expectation value of these operators is again a polynomial in $z$. This proposal has been verified using our formalism in appendix D for the second character of pure SU($N$) SYM in the restricted cases $N = 1$ and $N = 2$. The second character can be rewritten using the shifted spectral variables $z_1 = z + \nu_1$, $z_2 = z + \nu_2$,

$$\begin{aligned}\chi_2(z_1, z_2) = &\left\langle \left(\mathcal{Y}(z_1 + \epsilon_+) + \frac{qm(z_1)}{\mathcal{Y}(z_1)}\right)\left(\mathcal{Y}(z_2 + \epsilon_+) + \frac{qm(z_2)}{\mathcal{Y}(z_2)}\right)\right\rangle \\ &+ q\frac{\epsilon_1\epsilon_2}{z_{12}}\left\langle \frac{\mathcal{Y}(z_1 + \epsilon_+)}{z_{12} + \epsilon_+}\frac{m(z_2)}{\mathcal{Y}(z_2)} + \frac{\mathcal{Y}(z_2 + \epsilon_+)}{z_{12} - \epsilon_+}\frac{m(z_1)}{\mathcal{Y}(z_1)}\right\rangle,\end{aligned} \tag{4.15}$$

with $z_{12} = z_1 - z_2$. This condition is equivalent to (4.14). From the insertion of two operators $D_1(z_1)D_{-1}(z_2)$ within two Gaiotto states, it is possible to show that

$$\mathrm{P}^-_{z_1}\mathrm{P}^-_{z_2}\chi_2(z_1, z_2) = 0. \tag{4.16}$$

The polynomiality is further obtained in the cases $N = 1$ and $N = 2$ by employing the explicit expression (4.11) of the operator $\mathcal{Y}(z + \epsilon_+)$. In both cases, it was found that

$$\chi_2(z_1, z_2) = \mathrm{P}^+_{z_1}\mathrm{P}^+_{z_2}\left\langle \mathcal{Y}(z_1 + \epsilon_+)\mathcal{Y}(z_2 + \epsilon_+)\right\rangle + \frac{2q\epsilon_1\epsilon_2}{z_{12}^2 - \epsilon_+^2}. \tag{4.17}$$

### 4.3 Generalization to the $A_Q$-type quiver

For simplicity here we will only treat explicitly the case of the $A_2$ quiver without fundamental matter fields. For any operator $\mathcal{O}$ we introduce the index $\alpha = 1, 2$ labeling the space $\mathcal{V}_{\vec{a}_\alpha}$ in which the operator acts, and we associate the expectation value

$$\begin{aligned}\left\langle \mathcal{O}^{(\alpha)}(z)\right\rangle = \frac{1}{\mathcal{Z}_{\text{inst}}}\sum_{\vec{Y}_1,\vec{Y}_2} q_1^{|\vec{Y}_1|}q_2^{|\vec{Y}_2|}\mathcal{Z}_{\text{vect.}}(\vec{a}_1, \vec{Y}_1)\mathcal{Z}_{\text{vect.}}(\vec{a}_2, \vec{Y}_2)\mathcal{Z}_{\text{bfd.}}(\vec{a}_1, \vec{Y}_1; \vec{a}_2, \vec{Y}_2|m_{12}) \\ \langle\vec{a}_\alpha, \vec{Y}_\alpha|\mathcal{O}_\alpha(z)|\vec{a}_\alpha, \vec{Y}_\alpha\rangle.\end{aligned} \tag{4.18}$$

To derive the qq-character relations, we consider the commutation relation (3.22) between the SH$^c$ generating series $D_\eta(z)$ and the intertwiner operator. We consider the operator insertion of the following type,

$$\langle G, \vec{a}_1|D^{\vec{a}_1}_{-1}(z)q_1^DV_{12}(\vec{a}_1, \vec{a}_2|m_{12})q_2^D|G, \vec{a}_2\rangle, \quad \langle G, \vec{a}_1|q_1^DV_{12}(\vec{a}_1, \vec{a}_2|m_{12})q_2^DD^{\vec{a}_2}_{+1}(z)|G, \vec{a}_2\rangle, \tag{4.19}$$

and then using the action of the SH$^c$ modes on Gaiotto states, it is possible to derive the following identity, obtained respectively from the former and latter expressions:

$$\begin{aligned}\mathrm{P}^-_z\left\langle \mathcal{Y}^{(1)}(z + \epsilon_+) + q_1\frac{\mathcal{Y}^{(2)}(z + \epsilon_+ - m_{12})}{\mathcal{Y}^{(1)}(z)} + q_1q_2\frac{1}{\mathcal{Y}^{(2)}(z - m_{12})}\right\rangle = 0, \\ \mathrm{P}^-_z\left\langle \mathcal{Y}^{(2)}(z + \epsilon_+) + q_2\frac{\mathcal{Y}^{(1)}(z + m_{12})}{\mathcal{Y}^{(2)}(z)} + q_1q_2\frac{1}{\mathcal{Y}^{(1)}(z + m_{12} - \epsilon_+)}\right\rangle = 0.\end{aligned} \tag{4.20}$$

These identities imply that the two following qq-characters are polynomials in $z$:

$$\begin{aligned}\chi^{(1)}(z) = \left\langle \mathcal{Y}^{(1)}(z + \epsilon_+) + q_1\frac{\mathcal{Y}^{(2)}(z + \epsilon_+ - m_{12})}{\mathcal{Y}^{(1)}(z)} + q_1q_2\frac{1}{\mathcal{Y}^{(2)}(z - m_{12})}\right\rangle, \\ \chi^{(2)}(z) = \left\langle \mathcal{Y}^{(2)}(z + \epsilon_+) + q_2\frac{\mathcal{Y}^{(1)}(z + m_{12})}{\mathcal{Y}^{(2)}(z)} + q_1q_2\frac{1}{\mathcal{Y}^{(1)}(z + m_{12} - \epsilon_+)}\right\rangle.\end{aligned} \tag{4.21}$$

Generalization of these formulae to the $A_Q$ quiver with fundamental multiplets is straightforward. For example, the first one is generalized to

$$\chi^{(1)}(z) = \sum_{i=1}^{Q+1} \left[ \prod_{j=1}^{i-1} q_j m_j(z-\zeta_j) \right] \left\langle \frac{\mathcal{Y}_i(z+\epsilon_+-\zeta_i)}{\mathcal{Y}_{i-1}(z-\zeta_{i-1})} \right\rangle, \quad \zeta_j = \sum_{k=1}^{j-1} m_{k,k+1}, \tag{4.22}$$

with $\zeta_0 = \zeta_1 = 0$, $\mathcal{Y}_0 = \mathcal{Y}_{Q+1} := 1$ and the mass polynomials $m_i(z) = \prod_{f=1}^{\tilde{N}_i}(z - m_f^{(i)})$ associated to the fundamental multiplet of the node $i$, with flavor group $\mathrm{SU}(\tilde{N}_i)$. We note that for a linear quiver with $N = N_1 \geq N_2 \geq \cdots \geq N_Q$, the rank of the flavor group need to satisfy $\tilde{N}_i \leq 2N_i - N_{i+1} - N_{i-1}$ with $N_0 = N_{Q+1} = 0$. In this set-up, the character $\chi^{(1)}(z)$ is a polynomial of degree at most $N$.

As already stated, in the weak coupling limit $q_2 \to 0$ the second node of the quiver diagram acts as a set of $N_2$ fundamental flavors of mass $a_\ell^{(2)}$ coupled to the first node. This relation can also be observed at the level of characters. As can be seen from (4.18) in this limit only the empty $N_2$-tuple $\vec{Y}_2 = \vec{\emptyset}$ contribute to the sum, $\mathcal{Z}_{\text{vect.}}(\vec{a}_2, \vec{\emptyset}) = 1$ and $\mathcal{Z}_{\text{bfd.}} \to \mathcal{Z}_{\text{fund.}}$. We further notice that the operator $\mathcal{Y}^{(2)}(z+\epsilon_+)$ becomes polynomial and, as such, can be identify with $\chi^{(2)}(z)$, it reproduces a mass polynomial with masses $a_\ell^{(2)} - \epsilon_+$,

$$\chi^{(2)}(z) \to \prod_{\ell=1}^{N_2}(z - a_\ell^{(2)} + \epsilon_+) =: m(z). \tag{4.23}$$

In this weak coupling limit, the first equation in (4.21) becomes the equation (4.10) for the massive qq-character, with an extra shift of the fundamental masses by $m_{12}$.

## 5 Quantum Seiberg-Witten geometry

In the limit $\epsilon_1, \epsilon_2 \to 0$, the Omega-background reduces to $\mathbb{R}^4$ and the infrared theory is characterized by a complex algebraic curve. This curve, together with a differential form, determines the prepotential of the theory through the Seiberg-Witten relations. It is also associated to the spectral curve of a classical integrable system in the Bethe/gauge correspondence (see for instance [30] and references inside). For simplicity here, we focus our discussion on the case of a single node with gauge group $\mathrm{SU}(N)$ and a number $\tilde{N}$ of fundamental multiplets. In this case, the algebraic curve can be written in the form

$$y + q\frac{m(z)}{y} = \prod_{\ell=1}^{N}(z - a_\ell). \tag{5.1}$$

This expression should be compared with the definition (4.6) of the qq-character. It is then appealing to interpret the qq-character as a double deformation of the Seiberg-Witten geometry, where the expectation value of the operator $\mathcal{Y}(z)$ reduces to the complex parameter $y$ of the curve $E(y,z) = 0$, while the qq-character $\chi(z)$ reproduces the gauge polynomial in the r.h.s. of (5.1). This is indeed the case, as we will demonstrate shortly.

The discussion becomes even more illuminating if we introduce the intermediate background $\mathbb{R}^2_{\epsilon_1} \times \mathbb{R}^2$ obtained in the Nekrasov-Shatashvili limit $\epsilon_2 \to 0$ of the Omega-background.

This $\epsilon_1$-deformation of the Euclidean background is known to be responsible for the quantization of the classical integrable system associated to the $\mathcal{N}=2$ gauge theory [42]. In this background, the Seiberg-Witten curve is replaced by a Baxter TQ-equation that has been derived in [31, 32] (the derivation was later extended to quivers in [27, 33]),

$$T(z)Q(z) = Q(z+\epsilon_1) + qm(z)Q(z-\epsilon_1), \quad Q(z) = \prod_r (z-u_r), \tag{5.2}$$

where $T(z)$ and $Q(z)$ denote respectively the Baxter T- and Q-polynomials. The TQ-equation can be recast in a form more similar to the original Seiberg-Witten curve (5.1) by the introduction of the ratio $Y(z) = Q(z)/Q(z-\epsilon_1)$:[8]

$$T(z) = Y(z+\epsilon_1) + q\frac{m(z)}{Y(z)}. \tag{5.4}$$

In this form, it readily reproduces (5.1) in the limit $\epsilon_1 \to 0$. In order to show that the qq-character defines a sort of second quantization of the Seiberg-Witten geometry, we will take the NS limit and reproduce the TQ-relation (5.4), the operator $\mathcal{Y}(z)$ being reduced to the rational function $Y(z)$, and the qq-character to the T-polynomial.

To perform the NS limit, we will follow the procedure described in [23] (see also [27]) and first re-derive the Bethe equations. In the NS limit, the sum over Young diagrams entering the expression (3.19) of the partition function is dominated by a Young diagram $\vec{Y}^*$ with infinitely many boxes.[9] This critical Young diagram minimizes the summation and its profile is obtained by solving the discrete saddle point equations:

$$\frac{q^{|\vec{Y}^*+x|}\mathcal{Z}_{\text{vect.}}(\vec{a},\vec{Y}^*+x)\mathcal{Z}_{\text{fund.}}(\vec{m};\vec{a},\vec{Y}^*+x)}{q^{|\vec{Y}^*|}\mathcal{Z}_{\text{vect.}}(\vec{a},\vec{Y}^*)\mathcal{Z}_{\text{fund.}}(\vec{m};\vec{a},\vec{Y}^*)} = 1, \quad \forall x \in A(\vec{Y}^*). \tag{5.5}$$

Taking into account the variation of the vector and fundamental contributions, we find

$$-\frac{q}{\epsilon_1\epsilon_2}m(\phi_x)\frac{\prod_{y\in R(\vec{Y}^*)}(\phi_x-\phi_y)(\phi_x-\phi_y-\epsilon_+)}{\prod_{\substack{y\in A(\vec{Y}^*)\\ y\neq x}}(\phi_x-\phi_y)(\phi_x-\phi_y+\epsilon_+)} = 1, \quad \forall x \in A(\vec{Y}^*). \tag{5.6}$$

Following [23], we now consider only Young diagrams with infinitely high columns, and such that a box can be added to (or removed from) each column. Up to $\epsilon_2$-corrections, the images under $\phi_x$ of a box $x \in A(\vec{Y}^*)$ and the box immediately below $x' \in R(\vec{Y}^*)$ are

[8] The TQ-equation can also be written in an operatorial form,

$$\left(\hat{y} + qm(z)\hat{y}^{-1}\right) Q(z) = T(z)Q(z), \quad \hat{y} = e^{\epsilon_1\partial_z} \tag{5.3}$$

where $\hat{y}$ is a shift operator. Here the non-commutativity of the variables $\hat{y}$ and $z$ becomes manifest and the previous relation defines a quantum curve. This difference equation is actually equivalent to a Schrödinger equation under a quantum change of variables [34]. This correspondence goes under the name of *bispectral duality* [38–40] and can be seen as a degenerate version of the AGT correspondence relating the gauge theory in the NS background with the semiclassical Liouville/Toda theory.

[9] This argument is similar to the one employed by Nekrasov and Okounkov in [42] to perform the limit $\epsilon_1, \epsilon_2 \to 0$. The main difference here is that the critical Young diagram doesn't have a continuous profile but is instead described by a step-function where the plateaux are given by the Bethe roots.

equal, they define the set of Bethe roots $u_r = \phi_x$ for $x \in R(\vec{Y}^*)$.[10] This is true for all boxes $x \in A(\vec{Y}^*)$, except for $N$ extra boxes (one for each diagram) that lie on the top right of the diagrams, and for which $\phi_x = \xi_\ell$ with $\xi_l = a_\ell + n_\ell\epsilon_1$ and $n_\ell$ the number of columns for the Young diagram $Y_\ell^*$.It is emphasized that these extra boxes are necessary to fulfill the relation $|A(\vec{Y}^*)| = |R(\vec{Y}^*)| + N$ between the cardinal of the two sets. The number of columns $n_\ell$ in each diagram will play the role of a cut-off sent to infinity at the end of the computation. Under this identification, and taking into account the factor $-\epsilon_1\epsilon_2$ from the box $y$ of coordinate $\phi_y = \phi_x - \epsilon_2$ just below $x$, we find in the limit $\epsilon_2 \to 0$:

$$1 = q\frac{m(u_r)}{\Xi(u_r)\Xi(u_r+\epsilon_1)}\prod_{\substack{s=1\\ s\neq r}}^{M}\frac{u_r-u_s-\epsilon_1}{u_r-u_s+\epsilon_1}, \quad \Xi(z) = \prod_{\ell=1}^{N}(z-\xi_\ell), \tag{5.7}$$

and the number of Bethe roots is $M = \sum_\ell n_\ell$. These equations resemble the Bethe equations of an inhomogeneous sl(2) XXX spin chain with a twist parameter $q$.[11] The TQ-equation associated to this system of Bethe roots reads

$$T(z)Q(z) = \Xi(z)\Xi(z+\epsilon_1)Q(z+\epsilon_1) + qm(z)Q(z-\epsilon_1), \tag{5.8}$$

Introducing the Q-polynomial as in (5.2), it is indeed possible to show that the r.h.s. is a polynomial of degree $M + 2N$, with $M$ zeros at $z = u_r$ as a consequence of the Bethe equations (5.7). The TQ-equation (5.2) is reproduced by further sending the number of Bethe roots $M$ to infinity, together with the cut-offs $\xi_\ell$ after a proper rescaling of the T and Q polynomials. More details on this limit will be provided in the work [53] to appear.

The NS limit of the expectation value (4.9) of operators is also dominated by the single state $|\vec{a}, \vec{Y}^*\rangle$, and diagonal operators in the basis $|\vec{a}, \vec{Y}\rangle$ can be identified with their eigenvalues:[12]

$$\langle\mathcal{O}\rangle \sim \left\langle \vec{a}, \vec{Y}^*|\mathcal{O}|\vec{a}, \vec{Y}^*\right\rangle \tag{5.9}$$

due to the simplification $\mathcal{Z}_{\text{inst}} \sim \mathcal{Z}_{\text{vect.}}(\vec{a}, \vec{Y}^*)\mathcal{Z}_{\text{fund.}}(\vec{m}; \vec{a}, \vec{Y}^*)$. From the action (2.21) of the operator $\mathcal{Y}(z)$ on states $|\vec{a}, \vec{Y}\rangle$, replacing the coordinate $\phi_x$ of boxes that can be added to /removed from $\vec{Y}^*$ by Bethe roots, we find

$$\langle\mathcal{Y}(z)\rangle \sim \left\langle \vec{a}, \vec{Y}^*|\mathcal{Y}(z)|\vec{a}, \vec{Y}^*\right\rangle \sim \frac{Q(z)\Xi(z)}{Q(z-\epsilon_1)}, \quad \left\langle\frac{1}{\mathcal{Y}(z)}\right\rangle \sim \left\langle \vec{a}, \vec{Y}^*|\frac{1}{\mathcal{Y}(z)}|\vec{a}, \vec{Y}^*\right\rangle \sim \frac{Q(z-\epsilon_1)}{Q(z)\Xi(z)}. \tag{5.10}$$

Denoting $\chi_{\text{NS}}(z)$ the limit of the qq-character $\chi(z)$, the identity (4.6) reproduces the TQ-equation (5.8) with the T-polynomial $T(z) = \chi_{\text{NS}}(z)\Xi(z)$ (the presence of the extra cut-off factor $\Xi(z)$ will be explained in [53]).

Our observation can be easily generalized to apply to linear quiver gauge theories. The NS limit for the $A_2$ quiver has been performed in [23]. Using the same procedure, the qq-character identity (4.21) reproduces the TQ-relation for an inhomogeneous sl(3) XXX spin chain characterized by two sets of Bethe roots (equation (7.15) of [27]).

[10] In a Young diagram $\lambda$, the image $\phi_x$ of $x = (i, \lambda_i) \in R(\lambda)$ is given explicitly by $\phi_x = a + (i-1)\epsilon_1 + (\lambda_i - 1)\epsilon_2$, it is finite in the limit $\epsilon_2 \to 0$ since $\lambda_i$ tends to infinity such that $\epsilon_2\lambda_i$ remains finite.

[11] In fact, in the superconformal case $\tilde{N} = 2N$, it exactly reproduces the inhomogeneous XXX spin chain for an appropriate choice of masses $m_f$.

[12] This is true for well-behaved operators for which the insertion does not modify significantly the saddle point equations.

## 6 Summary and concluding remarks

In this paper, we developed a holomorphic field representation of $\mathrm{SH}^c$ algebra. It has the merit to express the commutation relations and the finite rank representations of the operators in a compact form. Instanton partitions for $A_Q$ and $A_Q^{(1)}$-type quiver gauge theories can be expressed concisely in terms of Gaiotto state, intertwiner operator, and a newly introduced flavor vertex operator that insert the contribution of fundamental hypermultiplets. A new characterization of the Gaiotto state and intertwiner has been established using the adjoint action of $\mathrm{SH}^c$ holomorphic fields. It provided the infinite set of constraint on the instanton partition function from the chiral ring generating function proposed in [27]. These constraints are summarized in simple algebraic relations which were referred as the qq-character.

The qq-characters describe a quantum version of the Seiberg-Witten geometry [41] in the NS limit $\epsilon_2 \to 0$. In this setup, the $\epsilon_2$-deformation introduces a form of second quantization of the TQ-system in which the T-polynomial is replaced by an operator acting in the Hilbert space of the rank $N$ representation. This should be compared with the recent results obtained in [54, 55], where the subleading corrections to the NS limit have been derived. These corrections are compatible (up to a quantum correction) with the second quantization of the NS action that is interpreted as the Yang-Yang functional of the underlying quantum integrable system. By comparing these two different approaches, a unified interpretation for the $\epsilon_2$-deformation of quantum integrable systems should emerge.

There are some obvious generalizations of the current study for future work. In [27], similar algebraic relations for the chiral generating functional were proposed for ADE type quiver gauge theories. In order to describe the bifurcation in the quiver diagram, we need to find a $\mathrm{SH}^c$ description of trivalent vertex which takes of the form:

$$|\vec{b}\rangle\!\rangle_{\alpha\beta\gamma} = \sum_{\vec{W}} \mathcal{Z}_{\text{vect.}}(\vec{b},\vec{W})^{-1/2} |\vec{b},\vec{W}\rangle_\alpha \otimes |\vec{b},\vec{W}\rangle_\beta \otimes |\vec{b},\vec{W}\rangle_\gamma . \tag{6.1}$$

Here we added extra labels $\alpha, \cdots$ to specify the Hilbert spaces. With the help of such operator, one may give the partition for the $D_4$ quiver partition function, for example, as

$$\begin{aligned} &\otimes_{i=1}^3 \left( {}_{\alpha_i}\langle G, \vec{a}_i | q_i^{D_{0,1}^{(i)}} V_{\alpha_i\beta_i}(\vec{a}_i,\vec{b}|m_i) \right) \cdot \; q_4^{D_{0,1}^{(1)}} \; |\vec{b}\rangle\!\rangle_{\beta_1\beta_2\beta_3} \\ &\quad = \sum_{\vec{Y}_1,\vec{Y}_2,\vec{Y}_3,\vec{W}} q_4^{|\vec{W}|} \mathcal{Z}_{\text{vect.}}(\vec{b},\vec{W}) \prod_{i=1}^{3} q_i^{|\vec{Y}_i|} \mathcal{Z}_{\text{vect.}}(\vec{a}_i,\vec{Y}_i) \mathcal{Z}_{\text{bfd.}}(\vec{a}_i,\vec{Y}_i;\vec{b},\vec{W}|m_i) \,. \end{aligned} \tag{6.2}$$

In order to derive the qq-character for such extended cases, we need to find an analog of (3.22), (3.23) for the trivalent vertex. At this moment, however, this seems not so simple and we would like to leave it for future study. Another possible direction proposed in [27] is the 5D version of the current analysis. It corresponds to the algebra studied by many authors in [56–59] and has implications in 4D [60, 61]. Since the building blocks are already known (for example, [62, 63]), it would not be so difficult to perform a similar analysis in such set-up. In addition, our formulation of $\mathrm{SH}^c$ seems particularly suited to the generalization to the six-dimensional $\Omega$-background $\mathbb{R}^2_{\epsilon_1} \times \mathbb{R}^2_{\epsilon_2} \times \mathbb{R}^2_{\epsilon_3}$ in which the instanton partition function of $\mathcal{N}=2$ theories are expressed as a sum over plane partitions [64].



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In a different perspective, it is important to clarify the relations with integrable models. On one hand, a proposal of NPS in [27] suggests a connection between quantum geometry and the representation of the Yangian associated to the quiver Dynkin diagram. On the other hand, Maulik and Okounkov have proposed in [65] the expression of the Yangian of $\widehat{\mathrm{gl}}(1)$. In their formalism, the Dynkin diagram is regarded as the finite lattice of a spin system, where the spin degree of freedom is actually described by a free boson Fock space. In [66], a short summary of [65] and a possible supersymmetric generalization were presented. While the two approaches are very different, the coproduct defined by the authors of [65] coincides with the one employed in [1]. Our analysis which relates the NPS qq-character [27] to the SH$^c$ algebra [1] could provide an interesting link between the two Yangians.

**Note added.** After the first version of this paper was submitted to arXiv, N. Nekrasov published a paper [67] where he studied Dyson-Schwinger equations for the instanton partition functions. He evaluated the effect of adding point-like instantons, and express this effect by an operator $\mathcal{Y}$ which is identical to ours. There seems to be a direct relation with our analysis and we hope to provide a more detailed comparison in the near future.

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## A Comments on the notations

In this paper, in order to ease the comparison with the gauge theory, we use the omega background parameters $\epsilon_1, \epsilon_2$ instead of the CFT parameter $\beta = -\epsilon_1/\epsilon_2$ in [7, 8]. Since some results of these papers are used here, we summarize the correspondence between the notations in this appendix. As we use two parameters instead of one, we need some rescaling and shift of parameters to compare with the results there. Adding a tilde to the notations in [7, 8], the comparison goes as follows:

$$D_{0,n+1} = (\epsilon_2)^n \tilde{D}_{0,n+1}, \qquad D_{\pm 1,n} = (\epsilon_2)^n \tilde{D}_{\pm 1,n}, \qquad E_n = (\epsilon_2)^n \tilde{E}_n\,, \tag{A.1}$$

$$a_\ell = -\epsilon_2 \tilde{a}_\ell + \epsilon_+, \qquad z = \epsilon_2/\tilde{\zeta}, \qquad c_n = (-\epsilon_2)^n \tilde{c}_n, \tag{A.2}$$

$$\phi(x)|_{x\in A(Y_\ell)} = -\epsilon_2(\tilde{a}_\ell + \tilde{A}_t(Y_\ell)), \quad \phi(x)|_{x\in R(Y_\ell)} = -\epsilon_2(\tilde{a}_\ell + \tilde{B}_t(Y_\ell)). \tag{A.3}$$

We note that under the rescaling (A.1), the algebra (2.1)–(2.3) remains the same.

## B Proof of the recursion formulae for Gaiotto states and intertwiner

Since the Gaiotto state can be derived from the intertwiner (3.25), it will be sufficient to prove (3.22), (3.23). We need a few formulae to characterize the behavior of the instanton partition function building blocks (here bifundamental and the vector contributions) under the variations of the number of boxes in the $N$-tuple $\vec{Y}$.

$$\frac{\mathcal{Z}_{\text{bfd.}}(\vec{a},\vec{Y}+x;\vec{b},\vec{W}|m)}{\mathcal{Z}_{\text{bfd.}}(\vec{a},\vec{Y};\vec{b},\vec{W}|m)}=\frac{\prod_{y\in A(\vec{W})}(\phi_x-\phi_y+\epsilon_+-m)}{\prod_{y\in R(\vec{W})}(\phi_x-\phi_y-m)},\tag{B.1}$$

$$\frac{\mathcal{Z}_{\text{bfd.}}(\vec{a},\vec{Y}-x;\vec{b},\vec{W}|m)}{\mathcal{Z}_{\text{bfd.}}(\vec{a},\vec{Y};\vec{b},\vec{W}|m)}=\frac{\prod_{y\in R(\vec{W})}(\phi_x-\phi_y-m)}{\prod_{y\in A(\vec{W})}(\phi_x-\phi_y+\epsilon_+-m)},\tag{B.2}$$

$$\frac{\mathcal{Z}_{\text{bfd.}}(\vec{a},\vec{Y};\vec{b},\vec{W}+x|m)}{\mathcal{Z}_{\text{bfd.}}(\vec{a},\vec{Y};\vec{b},\vec{W}|m)}=\frac{\prod_{y\in A(\vec{Y})}(\phi_x-\phi_y+m)}{\prod_{y\in R(\vec{Y})}(\phi_x-\phi_y+m-\epsilon_+)},\tag{B.3}$$

$$\frac{\mathcal{Z}_{\text{bfd.}}(\vec{a},\vec{Y};\vec{b},\vec{W}-x|m)}{\mathcal{Z}_{\text{bfd.}}(\vec{a},\vec{Y};\vec{b},\vec{W}|m)}=\frac{\prod_{y\in R(\vec{Y})}(\phi_x-\phi_y+m-\epsilon_+)}{\prod_{y\in A(\vec{Y})}(\phi_x-\phi_y+m)},\tag{B.4}$$

$$\frac{\mathcal{Z}_{\text{vect.}}(\vec{a},\vec{Y}+x)}{\mathcal{Z}_{\text{vect.}}(\vec{a},\vec{Y})}=-\frac{1}{\epsilon_1\epsilon_2}\frac{\prod_{y\in R(\vec{Y})}(\phi_x-\phi_y)(\phi_x-\phi_y-\epsilon_+)}{\prod_{\substack{y\in A(\vec{Y})\\ y\neq x}}(\phi_x-\phi_y)(\phi_x-\phi_y+\epsilon_+)},\tag{B.5}$$

$$\frac{\mathcal{Z}_{\text{vect.}}(\vec{a},\vec{Y}-x)}{\mathcal{Z}_{\text{vect.}}(\vec{a},\vec{Y})}=-\frac{1}{\epsilon_1\epsilon_2}\frac{\prod_{y\in A(\vec{Y})}(\phi_x-\phi_y)(\phi_x-\phi_y+\epsilon_+)}{\prod_{\substack{y\in R(\vec{Y})\\ y\neq x}}(\phi_x-\phi_y)(\phi_x-\phi_y-\epsilon_+)}.\tag{B.6}$$

These formulae were used in [8] to prove the recursive properties of the quiver gauge theories. Essentially the same computation shows up here. We evaluate the action of $D_\eta(z)$ on the intertwiner:

$$V(\vec{a},\vec{b}|m)=\sum_{\vec{Y},\vec{W}}\bar{\mathcal{Z}}_{\text{bfd.}}(\vec{a},\vec{Y};\vec{b},\vec{W}|m)|\vec{a},\vec{Y}\rangle\langle\vec{b},\vec{W}|\,,\tag{B.7}$$

with $\bar{\mathcal{Z}}_{\text{bfd.}}(\vec{a},\vec{Y};\vec{b},\vec{W}|m):=\sqrt{\mathcal{Z}_{\text{vect.}}(\vec{a},\vec{Y})\mathcal{Z}_{\text{vect.}}(\vec{b},\vec{W})}\mathcal{Z}_{\text{bfd.}}(\vec{a},\vec{Y}:\vec{b},\vec{W}|m)$. We first evaluate the action of $D_{-1}(z)$ on the intertwiner from the left. It is easily deduced from its action on states $|\vec{a},\vec{Y}\rangle$,

$$D_{-1}^{\vec{a}}(z)V(\vec{a},\vec{b}|m)=\sum_{\vec{Y},\vec{W}}\bar{\mathcal{Z}}_{\text{bfd.}}(\vec{a},\vec{Y};\vec{b},\vec{W}|m)\sum_{x\in R(\vec{Y})}\frac{\Lambda_x(\vec{Y})}{z-\phi_x}|\vec{a},\vec{Y}-x\rangle\langle\vec{b},\vec{W}|.\tag{B.8}$$

An alternative expression can be obtained after noticing that the inverse images of a state $|\vec{a},\vec{Y}\rangle$ under the mapping $D_{-1}(z)$ are the states $|\vec{a},\vec{Y}+x\rangle$ for $x\in A(\vec{Y})$ because the action of the operator removes one box. Since the states $|\vec{a},\vec{Y}\rangle$ form a basis of the vector space $\mathcal{V}_{\vec{a}}$, it is possible to write

$$D_{-1}^{\vec{a}}(z)V(\vec{a},\vec{b}|m)=\sum_{\vec{Y},\vec{W}}\sum_{x\in A(\vec{Y})}\frac{\Lambda_x(\vec{Y}+x)}{z-\phi_x}\bar{\mathcal{Z}}_{\text{bfd.}}(\vec{a},\vec{Y}+x;\vec{b},\vec{W}|m)|\vec{a},\vec{Y}\rangle\langle\vec{b},\vec{W}|.\tag{B.9}$$



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It is convenient to rewrite this expression as follows, using the fact that $\Lambda_x(\vec{Y}+x)^2 = \Lambda_x(\vec{Y})^2$, $\forall x \in A(\vec{Y})$:[13]

$$D^{\vec{a}}_{-1}(z)V(\vec{a},\vec{b}|m)=\sum_{\vec{Y},\vec{W}}\sum_{x\in A(\vec{Y})}\frac{\Lambda_x(\vec{Y})}{z-\phi_x}\frac{\bar{\mathcal{Z}}_{\text{bfd.}}(\vec{a},\vec{Y}+x;\vec{b},\vec{W}|m)}{\bar{\mathcal{Z}}_{\text{bfd.}}(\vec{a},\vec{Y};\vec{b},\vec{W}|m)}\bar{\mathcal{Z}}_{\text{bfd.}}(\vec{a},\vec{Y};\vec{b},\vec{W}|m)|\vec{a},\vec{Y}\rangle\langle\vec{b},\vec{W}|. \tag{B.10}$$

We evaluate the ratio for $\bar{\mathcal{Z}}_{\text{bfd.}}$ from (B.1) and (B.5) and put the explicit form of $\Lambda_x(\vec{Y})$ (2.19):

$$\Lambda_x(\vec{Y})\frac{\bar{\mathcal{Z}}_{\text{bfd.}}(\vec{a},\vec{Y}+x;\vec{b},\vec{W}|m)}{\bar{\mathcal{Z}}_{\text{bfd.}}(\vec{a},\vec{Y};\vec{b},\vec{W}|m)}=\frac{1}{\sqrt{-\epsilon_1\epsilon_2}}\frac{\prod_{y\in R(\vec{Y})}(\phi_x-\phi_y-\epsilon_+)}{\prod_{\substack{y\in A(\vec{Y})\\ y\neq x}}(\phi_x-\phi_y)}\frac{\prod_{y\in A(\vec{W})}(\phi_x-\phi_y-m+\epsilon_+)}{\prod_{y\in R(\vec{W})}(\phi_x-\phi_y-m)}. \tag{B.11}$$

The action of $D_{-1}$ from the right can be evaluated similarly,

$$-V(\vec{a},\vec{b}|m)D^{\vec{b}}_{-1}(z')=\sum_{\vec{Y},\vec{W}}\sum_{x\in R(\vec{W})}\frac{\Lambda_x(\vec{W})}{z'-\phi_x}\frac{\bar{\mathcal{Z}}_{\text{bfd.}}(\vec{a},\vec{Y};\vec{b},\vec{W}-x|m)}{\bar{\mathcal{Z}}_{\text{bfd.}}(\vec{a},\vec{Y};\vec{b},\vec{W}|m)}\bar{\mathcal{Z}}_{\text{bfd.}}(\vec{a},\vec{Y};\vec{b},\vec{W}|m)|\vec{a},\vec{Y}\rangle\langle\vec{b},\vec{W}|. \tag{B.12}$$

From (B.4) and (B.6), the factor in the middle takes the form:

$$\Lambda_x(\vec{W})\frac{\bar{\mathcal{Z}}_{\text{bfd.}}(\vec{a},\vec{Y};\vec{b},\vec{W}-x|m)}{\bar{\mathcal{Z}}_{\text{bfd.}}(\vec{a},\vec{Y};\vec{b},\vec{W}|m)}=\frac{1}{\sqrt{-\epsilon_1\epsilon_2}}\frac{\prod_{y\in A(\vec{W})}(\phi_x-\phi_y+\epsilon_+)}{\prod_{\substack{y\in R(\vec{W})\\ y\neq x}}(\phi_x-\phi_y)}\frac{\prod_{y\in R(\vec{Y})}(\phi_x-\phi_y+m-\epsilon_+)}{\prod_{y\in A(\vec{Y})}(\phi_x-\phi_y+m)}. \tag{B.13}$$

To add (B.10) and (B.12), we use the following identity to simplify the formula (we put $z'=z-m$)

$$\begin{aligned}&\sum_{x\in A(\vec{Y})}\frac{1}{z-\phi_x}\frac{\prod_{y\in R(\vec{Y})}(\phi_x-\phi_y-\epsilon_+)}{\prod_{\substack{y\in A(\vec{Y})\\ y\neq x}}(\phi_x-\phi_y)}\frac{\prod_{y\in A(\vec{W})}(\phi_x-\phi_y-m+\epsilon_+)}{\prod_{y\in R(\vec{W})}(\phi_x-\phi_y-m)}\\ &\quad+\sum_{x\in R(\vec{W})}\frac{1}{z-m-\phi_x}\frac{\prod_{y\in A(\vec{W})}(\phi_x-\phi_y+\epsilon_+)}{\prod_{\substack{y\in R(\vec{W})\\ y\neq x}}(\phi_x-\phi_y)}\frac{\prod_{y\in R(\vec{Y})}(\phi_x-\phi_y+m-\epsilon_+)}{\prod_{y\in A(\vec{Y})}(\phi_x-\phi_y+m)}\\ &\qquad=\mathrm{P}^-_z\left(\frac{\prod_{y\in R(\vec{Y})}(z-\phi_y-\epsilon_+)}{\prod_{y\in A(\vec{Y})}(z-\phi_y)}\frac{\prod_{y\in A(\vec{W})}(z-\phi_y-m+\epsilon_+)}{\prod_{y\in R(\vec{W})}(z-\phi_y-m)}\right).\end{aligned} \tag{B.14}$$

This formula is obtained by comparing the residue of the poles on both sides. Finally, we note that by using (2.21), we obtain

$$\begin{aligned}&\mathrm{P}^-_z\left(\frac{\prod_{y\in R(\vec{Y})}(z-\phi_y-\epsilon_+)}{\prod_{y\in A(\vec{Y})}(z-\phi_y)}\frac{\prod_{y\in A(\vec{W})}(z-\phi_y-m+\epsilon_+)}{\prod_{y\in R(\vec{W})}(z-\phi_y-m)}\right)|\vec{a},\vec{Y}\rangle\langle\vec{b},\vec{W}|\\ &\qquad=\mathrm{P}^-_z\left(\frac{1}{\mathcal{Y}^{(1)}(z)}|\vec{a},\vec{Y}\rangle\langle\vec{b},\vec{W}|\mathcal{Y}^{(2)}(z-m+\epsilon_+)\right).\end{aligned} \tag{B.15}$$

After combining everything, we prove (3.22). The proof of (3.23) is completely parallel and we omit it here.

[13] In this expression, and the analysis hereafter, the correct choice of sign is verified by comparing with the direct action of $\mathrm{SH}^c$ generators on states with a small number of boxes.

## C Some calculations of commutation relations in the rank $N$ representation

For completeness, we present here some explicit computations for the commutators of holomorphic generators.

**$[D_0(z), D_1(w)]$.** From (2.15),

$$D_1(w)D_0(z)|\vec{a},\vec{Y}\rangle = \sum_{x\in\vec{Y}}\frac{1}{z-\phi_x}\sum_{y\in A(\vec{Y})}\frac{\Lambda_y(\vec{Y})}{w-\phi_y}|\vec{a},\vec{Y}+y\rangle, \tag{C.1}$$

$$D_0(z)D_1(w)|\vec{a},\vec{Y}\rangle = \sum_{y\in A(\vec{Y})}\sum_{x\in\vec{Y}+y}\frac{1}{z-\phi_x}\frac{\Lambda_y(\vec{Y})}{w-\phi_y}|\vec{a},\vec{Y}+y\rangle. \tag{C.2}$$

The difference is the inclusion of the box $y$ in $\vec{Y}+y$. So we obtain the first relation in (2.6).

$$\begin{aligned}[D_0(z), D_1(w)]\,|\vec{a},\vec{Y}\rangle &= \sum_{y\in A(\vec{Y})}\frac{\Lambda_y(\vec{Y})}{(z-\phi_y)(w-\phi_y)}|\vec{a},\vec{Y}\rangle = \sum_{y\in A(\vec{Y})}\frac{1}{z-w}\left(\frac{\Lambda_y(\vec{Y})}{w-\phi_y}-\frac{\Lambda_y(\vec{Y})}{z-\phi_y}\right)|\vec{a},\vec{Y}\rangle\\ &= \frac{D_1(w)-D_1(z)}{z-w}|\vec{a},\vec{Y}\rangle.\end{aligned} \tag{C.3}$$

**$[D_1(z), D_{-1}(w)]$.**

$$\begin{aligned}D_1(z)D_{-1}(w)|\vec{a},\vec{Y}\rangle &= \sum_{x\in R(\vec{Y})}\sum_{y\in A(\vec{Y}-x)}\frac{\Lambda_x(\vec{Y})}{w-\phi_x}\frac{\Lambda_y(\vec{Y}-x)}{z-\phi_y}|\vec{a},\vec{Y}-x+y\rangle\\ &= \sum_{x\in R(\vec{Y})}\sum_{y\in A(\vec{Y})}\frac{\Lambda_x(\vec{Y})}{w-\phi_x}\frac{\Lambda_y(\vec{Y}-x)}{z-\phi_y}|\vec{a},\vec{Y}-x+y\rangle\\ &\quad+\sum_{x\in R(\vec{Y})}\frac{\Lambda_x(\vec{Y})}{w-\phi_x}\frac{\Lambda_x(\vec{Y}-x)}{z-\phi_x}|\vec{a},\vec{Y}\rangle,\end{aligned} \tag{C.4}$$

where we have assumed for $\vec{Y}$ a generic form such that $A(\vec{Y}-x) = A(\vec{Y})\cup\{x\}$. From the explicit form of $\Lambda_x(\vec{Y})$, one may prove for $y\in A(\vec{Y})$:

$$\Lambda_y(\vec{Y}-x)^2 = \Lambda_y(\vec{Y})^2\frac{S(\phi_x-\phi_y)}{S(\phi_y-\phi_x)}, \quad \Lambda_x(\vec{Y}-x)^2 = \Lambda_x(\vec{Y})^2\,, \tag{C.5}$$

where $S(z)$ is the scattering factor which appeared in (2.31). After combining them, $D_1(z)D_{-1}(w)|\vec{a},\vec{Y}\rangle$ becomes,

$$\sum_{x\in R(\vec{Y})}\sum_{y\in A(\vec{Y})}\frac{\Lambda_x(\vec{Y})}{w-\phi_x}\frac{\Lambda_y(\vec{Y})}{z-\phi_y}\left(\frac{S(\phi_x-\phi_y)}{S(\phi_y-\phi_x)}\right)^{1/2}|\vec{a},\vec{Y}-x+y\rangle+\sum_{x\in R(\vec{Y})}\frac{(\Lambda_x(\vec{Y}))^2}{(w-\phi_x)(z-\phi_x)}|\vec{a},\vec{Y}\rangle\,. \tag{C.6}$$



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$D_{-1}(w)D_1(z)|\vec{a},\vec{Y}\rangle$ is evaluated similarly, assuming that $R(\vec{Y}+x)=R(\vec{Y})$,

$$\sum_{y\in A(\vec{Y})}\sum_{x\in R(\vec{Y})}\frac{\Lambda_x(\vec{Y})}{w-\phi_x}\frac{\Lambda_y(\vec{Y})}{z-\phi_y}\left(\frac{S(\phi_x-\phi_y)}{S(\phi_y-\phi_x)}\right)^{1/2}|\vec{a},\vec{Y}-x+y\rangle+\sum_{x\in A(\vec{Y})}\frac{(\Lambda_x(\vec{Y}))^2}{(w-\phi_x)(z-\phi_x)}|\vec{a},\vec{Y}\rangle\,. \tag{C.7}$$

In $[D_1(z),D_{-1}(w)]$, the first term cancels and the second term gives $\frac{E(z)-E(w)}{\epsilon_+(z-w)}|\vec{a},\vec{Y}\rangle$ after the use of the relations (2.18), (2.19) and (2.22). Degenerate situations should be evaluated case by case, but the general conclusion remains unchanged.

# D Detailed analysis of the qq-characters

## D.1 Matter case

There are different possibilities for the insertion position of the SH$^c$ operator $D_{-1}(z)$, the simplest option is to insert it on the left of the mass operator as in $\langle G,\vec{a}|q^D D_{-1}(z)U(\vec{m})|G,\vec{a}\rangle$, and then use the braiding relation (4.7) to move it to the right. The two formulas (3.12) and (3.14) for the action of $D_{-1}(z)$ provides the identity (4.8). However, this is not the unique choice for the insertion of the $D_{-1}(z)$, a second possibility is to insert it on the right of the mass operator,

$$\langle G,\vec{a}|q^D U(\vec{m})D_{-1}(z)|G,\vec{a}\rangle. \tag{D.1}$$

A new braiding relation is needed in order to move the SH$^c$ operator to the left, which is deduced from (2.29):

$$U(\vec{m})D_{-1}(z)=\left(\frac{D_{-1}(z)}{m(z)}-\sum_{f=1}^{\tilde{N}}\frac{D_{-1}(m^{(f)})}{z-m^{(f)}}\prod_{\substack{f'=1\\f'\neq f}}^{\tilde{N}}\frac{1}{m^{(f)}-m^{(f')}}\right)U(\vec{m}). \tag{D.2}$$

The action of $D_{-1}(z)$ on the Gaiotto state is then computed from (3.12) and (3.13), leading to the identity

$$\frac{1}{m(z)}\mathrm{P}_z^-\left\langle\mathcal{Y}(z+\epsilon_+)\right\rangle-\sum_{f=1}^{\tilde{N}}\frac{1}{z-m^{(f)}}\prod_{\substack{f'=1\\f'\neq f}}^{\tilde{N}}\frac{1}{m^{(f)}-m^{(f')}}\left\langle\mathcal{Y}(m^{(f)}+\epsilon_+)-\Pi(m^{(f)})\right\rangle=-q\left\langle\frac{1}{\mathcal{Y}(z)}\right\rangle, \tag{D.3}$$

where the operator $\Pi(z)=\mathrm{P}_z^+\mathcal{Y}(z+\epsilon_+)$ has been introduced to simplify the expression. As a result, the fundamental qq-character defined in (4.6) obeys

$$\chi(z)=\langle\Pi(z)\rangle+m(z)\sum_{f=1}^{\tilde{N}}\frac{1}{z-m^{(f)}}\prod_{\substack{f'=1\\f'\neq f}}^{\tilde{N}}\frac{1}{m^{(f)}-m^{(f')}}\left\langle\mathcal{Y}(m^{(f)}+\epsilon_+)-\Pi(m^{(f)})\right\rangle\,. \tag{D.4}$$

In the last term, the apparent poles are cancelled by the zeros of $m(z)$ and the r.h.s. is a polynomial. The compatibility with the identity (4.8) implies the following equality,

$$m(z)\sum_{f=1}^{\tilde{N}}\frac{1}{z-m^{(f)}}\prod_{\substack{f'=1\\f'\neq f}}^{\tilde{N}}\frac{1}{m^{(f)}-m^{(f')}}\left\langle\mathcal{Y}(m^{(f)}+\epsilon_+)-\Pi(m^{(f)})\right\rangle=\mathrm{P}_z^+q\left\langle\frac{m(z)}{\mathcal{Y}(z)}\right\rangle. \tag{D.5}$$

### D.2 Second qq-character of the $A_1$ quiver

#### D.2.1 Double action of SH$^c$ operators $D_\eta(z)$

As a preliminary step it is necessary to derive the action of two operators $D_{-1}(z_1)$ and $D_1(z_2)$ with different arguments on a Gaiotto state $|G,\vec{a}\rangle$. The method is the same as in the case of a single operator, and upon using the property

$$\langle\vec{a},\vec{Y}+x|\mathcal{Y}(z+\epsilon_+)|\vec{a},\vec{Y}+x\rangle = S(z-\phi_x)\langle\vec{a},\vec{Y}|\mathcal{Y}(z+\epsilon_+)|\vec{a},\vec{Y}\rangle, \tag{D.6}$$

with $S(z)$ the scattering factor defined in (2.31), it is possible to write down

$$\begin{aligned}&D_{-1}(z_1)D_1(z_2)|G,\vec{a}\rangle\\ &\quad=\frac{1}{\epsilon_1\epsilon_2}\sum_{\vec{Y}}\sqrt{\mathcal{Z}_{\text{vect.}}(\vec{a},\vec{Y})}\sum_{x\in A(\vec{Y})}\frac{1}{z_1-\phi_x}\frac{\prod_{y\in R(\vec{Y})}(\phi_{xy}-\epsilon_+)}{\prod_{\substack{y\in A(\vec{Y})\\ y\neq x}}\phi_{xy}}\mathrm{P}^-_{z_2}S(z_2-\phi_x)\mathcal{Y}(z_2+\epsilon_+)|\vec{a},\vec{Y}\rangle,\end{aligned} \tag{D.7}$$

where the sign ambiguity has been fixed by comparing the coefficient of the vacuum state with the direct action of the SH$^c$ generators. Inserting the pole decomposition of

$$\frac{S(z_2-\phi_x)}{z_1-\phi_x}=\frac{S(z_{21})}{z_1-\phi_x}+\frac{\epsilon_1\epsilon_2}{\epsilon_+z_{12}}\frac{1}{z_2-\phi_x}-\frac{\epsilon_1\epsilon_2}{\epsilon_+(z_{12}-\epsilon_+)}\frac{1}{z_2-\phi_x+\epsilon_+}, \tag{D.8}$$

with the shortcut notation $z_{21}=z_2-z_1$, it is possible to perform the summation over $x\in A(\vec{Y})$:

$$\begin{aligned}&D_{-1}(z_1)D_1(z_2)|G,\vec{a}\rangle\\ &\quad=\frac{1}{\epsilon_1\epsilon_2}\sum_{\vec{Y}}\sqrt{\mathcal{Z}_{\text{vect.}}(\vec{a},\vec{Y})}\left[\mathrm{P}^-_{z_2}S(z_{21})\frac{\mathcal{Y}(z_2+\epsilon_+)}{\mathcal{Y}(z_1)}+\frac{\epsilon_1\epsilon_2}{\epsilon_+z_{12}}\frac{\mathcal{Y}(z_2+\epsilon_+)}{\mathcal{Y}(z_2)}-\frac{\epsilon_1\epsilon_2}{\epsilon_+(z_{12}-\epsilon_+)}\right]|\vec{a},\vec{Y}\rangle,\end{aligned} \tag{D.9}$$

where it has been used that the last two terms in the brackets have no polynomial part at infinity, and consequently in the last term the two factors $\mathcal{Y}(z_2+\epsilon_+)$ have cancelled each other. This result can be written in the compact form

$$D_{-1}(z_1)D_1(z_2)|G,\vec{a}\rangle=\left[\frac{1}{\epsilon_1\epsilon_2}\mathrm{P}^-_{z_2}S(z_{21})\frac{\mathcal{Y}(z_2+\epsilon_+)}{\mathcal{Y}(z_1)}+\frac{1}{\epsilon_+z_{12}}\frac{\mathcal{Y}(z_2+\epsilon_+)}{\mathcal{Y}(z_2)}-\frac{1}{\epsilon_+(z_{12}-\epsilon_+)}\right]|G,\vec{a}\rangle. \tag{D.10}$$

The action of the commuted operators $D_1(z_2)D_{-1}(z_1)$ is derived from the same method,

$$D_1(z_2)D_{-1}(z_1)|G,\vec{a}\rangle=\frac{1}{\mathcal{Y}(z_1)}\left[\frac{S(z_{21})}{\epsilon_1\epsilon_2}\mathrm{P}^-_{z_2}\mathcal{Y}(z_2+\epsilon_+)-\frac{\mathrm{P}^-_{z_1}\mathcal{Y}(z_1+\epsilon_+)}{\epsilon_+z_{21}}+\frac{\mathrm{P}^-_{z_1}\mathcal{Y}(z_1)}{\epsilon_+(z_{21}+\epsilon_+)}\right]|G,\vec{a}\rangle. \tag{D.11}$$

This expression is simplified employing the following identity,

$$\mathrm{P}^-_{z_2}\left[S(z_{21})\mathcal{Y}(z_2+\epsilon_+)\right]=S(z_{21})\mathrm{P}^-_{z_2}\mathcal{Y}(z_2+\epsilon_+)+\frac{\epsilon_1\epsilon_2}{\epsilon_+z_{21}}\mathrm{P}^+_{z_1}\mathcal{Y}(z_1+\epsilon_+)-\frac{\epsilon_1\epsilon_2}{\epsilon_+(z_{21}+\epsilon_+)}\mathrm{P}^+_{z_1}\mathcal{Y}(z_1), \tag{D.12}$$

obtained by decomposition of $S(z_{21})$ as a sum over single poles and of $\mathcal{Y}(z_2+\epsilon_+)$ into positive and negative powers. As a result, we find

$$D_1(z_2)D_{-1}(z_1)|G,\vec{a}\rangle = \left[\frac{1}{\epsilon_1\epsilon_2}\mathrm{P}_{z_2}^- S(z_{21})\frac{\mathcal{Y}(z_2+\epsilon_+)}{\mathcal{Y}(z_1)} + \frac{1}{\epsilon_+ z_{12}}\frac{\mathcal{Y}(z_1+\epsilon_+)}{\mathcal{Y}(z_1)} - \frac{1}{\epsilon_+(z_{12}-\epsilon_+)}\right]|G,\vec{a}\rangle. \tag{D.13}$$

Taking the difference between (D.10) and (D.13), the commutation relation (2.6) between $D_{-1}(z_1)$ and $D_1(z_2)$ is recovered, with the action of $E(z_\alpha)$ on Gaiotto states given in (2.10) by the ratio of $\mathcal{Y}$ operators with shifted arguments.

### D.2.2 Derivation of the second qq-character

The expression of the second qq-character follows from the consideration of the symmetrized action (D.10) of two SH$^c$ operators inside two Gaiotto states,

$$\langle\vec{G},\vec{a}|q^D\left[D_{-1}(z_1)\,D_1(z_2)+D_{-1}(z_2)\,D_1(z_1)\right]|G,\vec{a}\rangle. \tag{D.14}$$

This quantity can be computed either using the right action (D.10) of two operators on the Gaiotto state, or from the right (3.12), (3.13) and left (3.14) actions of a single SH$^c$ operator. These two possible ways of calculation furnish the following identity:

$$\begin{aligned}
&-\frac{2q^{-1}}{\epsilon_1\epsilon_2}\mathrm{P}_{z_1}^-\mathrm{P}_{z_2}^-\left\langle\mathcal{Y}(z_1+\epsilon_+)\mathcal{Y}(z_2+\epsilon_+)\right\rangle \\
&\quad=\frac{1}{\epsilon_+ z_{12}}\left\langle\frac{\mathcal{Y}(z_2+\epsilon_+)}{\mathcal{Y}(z_2)}-\frac{\mathcal{Y}(z_1+\epsilon_+)}{\mathcal{Y}(z_1)}\right\rangle+\frac{1}{\epsilon_1\epsilon_2}\mathrm{P}_{z_2}^-\left[S(z_{21})\left\langle\frac{\mathcal{Y}(z_2+\epsilon_+)}{\mathcal{Y}(z_1)}\right\rangle\right] \\
&\qquad+\frac{1}{\epsilon_1\epsilon_2}\mathrm{P}_{z_1}^-\left[S(z_{12})\left\langle\frac{\mathcal{Y}(z_1+\epsilon_+)}{\mathcal{Y}(z_2)}\right\rangle\right]-\frac{2}{z_{12}^2-\epsilon_+^2}.
\end{aligned} \tag{D.15}$$

The second line involves the commutator of $D_{-1}(z_1)$ with $D_1(z_2)$ evaluated in the Gaiotto states average. The same quantity can also be computed by direct right (3.12), (3.13) and left (3.14) actions on Gaiotto states, leading to a second identity:

$$\begin{aligned}
\frac{1}{\epsilon_+ z_{12}}\left\langle\frac{\mathcal{Y}(z_2+\epsilon_+)}{\mathcal{Y}(z_2)}-\frac{\mathcal{Y}(z_1+\epsilon_+)}{\mathcal{Y}(z_1)}\right\rangle &= -\frac{q^{-1}}{\epsilon_1\epsilon_2}\mathrm{P}_{z_1}^-\mathrm{P}_{z_2}^-\left\langle\mathcal{Y}(z_1+\epsilon_+)\mathcal{Y}(z_2+\epsilon_+)\right\rangle \\
&\quad+\frac{q}{\epsilon_1\epsilon_2}\left\langle\frac{1}{\mathcal{Y}(z_1)\mathcal{Y}(z_2)}\right\rangle.
\end{aligned} \tag{D.16}$$

Replacing the average of the commutator in the first identity, and introducing the positive part $\Pi(z)=\mathrm{P}_z^+\mathcal{Y}(z+\epsilon_+)$ produces

$$\begin{aligned}
0=q^{-1}\left\langle(\mathcal{Y}(z_1+\epsilon_+)-\Pi(z_1))(\mathcal{Y}(z_2+\epsilon_+)-\Pi(z_2))\right\rangle+\mathrm{P}_{z_1}^-\left[S(z_{12})\left\langle\frac{\mathcal{Y}(z_1+\epsilon_+)}{\mathcal{Y}(z_2)}\right\rangle\right] \\
+\mathrm{P}_{z_2}^-\left[S(z_{21})\left\langle\frac{\mathcal{Y}(z_2+\epsilon_+)}{\mathcal{Y}(z_1)}\right\rangle\right]+q\left\langle\frac{1}{\mathcal{Y}(z_1)\mathcal{Y}(z_2)}\right\rangle-\frac{2\epsilon_1\epsilon_2}{z_{12}^2-\epsilon_+^2}.
\end{aligned} \tag{D.17}$$

This relation implies for the second qq-character (4.15),

$$\mathrm{P}_{z_1}^-\mathrm{P}_{z_2}^-\chi_2(z_1,z_2)=\frac{2q\epsilon_1\epsilon_2}{z_{12}^2-\epsilon_+^2}. \tag{D.18}$$

This is not enough to conclude on the polynomiality of the second qq-character, because of the presence of the cross-terms

$$\mathrm{P}^-_{z_1}\mathrm{P}^+_{z_2}\chi_2(z_1,z_2) = \left\langle \Pi(z_2)\left(\mathcal{Y}(z_1+\epsilon_+) - \Pi(z_1) + q\frac{S(z_{21})}{\mathcal{Y}(z_1)}\right)\right\rangle \\ - q\frac{\epsilon_1\epsilon_2}{\epsilon_+}\left\langle \frac{\Pi(z_1)}{z_{21}\mathcal{Y}(z_1)} - \frac{\Pi(z_1+\epsilon_+)}{(z_{21}+\epsilon_+)\mathcal{Y}(z_1)}\right\rangle. \tag{D.19}$$

On the other hand, we know the explicit expression of the second qq-character for small $N$, and can use it to deduce the expression of $\chi_2(z_1,z_2)$. First, it is noted that after the introduction of the orthogonal projector in (D.17), $\chi_2$ can be rewritten as the average of an operator $\Pi_2(z_1,z_2)$

$$\chi_2(z_1,z_2) = \langle \Pi_2(z_1,z_2)\rangle\,, \tag{D.20}$$

defined as

$$\Pi_2(z_1,z_2) = -\Pi(z_1)\Pi(z_2) + \mathrm{P}^+_{z_1}\left[\Pi(z_1)\left(\mathcal{Y}(z_2+\epsilon_+) + q\frac{S(z_{12})}{\mathcal{Y}(z_2)}\right)\right] \\ + \mathrm{P}^+_{z_2}\left[\Pi(z_2)\left(\mathcal{Y}(z_1+\epsilon_+) + q\frac{S(z_{21})}{\mathcal{Y}(z_1)}\right)\right] + \frac{2q\epsilon_1\epsilon_2}{z_{12}^2-\epsilon_+^2}. \tag{D.21}$$

At first sight, it is not clear whether this quantity is a polynomial and we had to check it case by case using the explicit expression of the polynomial operator $\Pi(z)$ given in (4.13) for $N=1,2$.

**Case $N=1$:** in this case $\Pi(z)$ is a scalar and can be taken out of the vacuum expectation values, i.e. $\langle \Pi(z)\cdots\rangle = \Pi(z)\langle\cdots\rangle$. Since it is a polynomial of degree one, it satisfies $\mathrm{P}^+_{z_1}S(z_{12})\Pi(z_1) = \Pi(z_1)$ and as a result

$$\langle \Pi_2(z_1,z_2)\rangle = -\Pi(z_1)\Pi(z_2) + \Pi(z_1)\chi(z_2) + \Pi(z_2)\chi(z_1) + q\frac{2\epsilon_1\epsilon_2}{z_{12}^2-\epsilon_+^2} \tag{D.22}$$

where $\chi(z)$ is the fundamental qq-character given in (4.6). In this simple case, $\chi(z)=\Pi(z)$ which provides the final result

$$\langle \Pi_2(z_1,z_2)\rangle = \chi(z_1)\chi(z_2) + q\frac{2\epsilon_1\epsilon_2}{z_{12}^2-\epsilon_+^2}. \tag{D.23}$$

**Case $N=2$:** in this case, $\Pi(z)$ is a polynomial of degree two that satisfies $\mathrm{P}^+_{z_1}S(z_{12})\Pi(z_1) = \Pi(z_1)+\epsilon_1\epsilon_2$. It follows that

$$\Pi_2(z_1,z_2) = -\Pi(z_1)\Pi(z_2) + \Pi(z_1)\left(\mathcal{Y}(z_2+\epsilon_+) + \frac{q}{\mathcal{Y}(z_2)}\right) + \epsilon_1\epsilon_2\frac{q}{\mathcal{Y}(z_2)} \\ + \Pi(z_2)\left(\mathcal{Y}(z_1+\epsilon_+) + \frac{q}{\mathcal{Y}(z_1)}\right) + \epsilon_1\epsilon_2\frac{q}{\mathcal{Y}(z_1)} + q\frac{2\epsilon_1\epsilon_2}{z_{12}^2-\epsilon_+^2}. \tag{D.24}$$

The explicit expression of $\Pi(z)$ deduced from (4.11) allows to show that

$$\left\langle \Pi(z_1)\left(\mathcal{Y}(z_2+\epsilon_+) + \frac{q}{\mathcal{Y}(z_2)}\right) + \epsilon_1\epsilon_2\frac{q}{\mathcal{Y}(z_2)}\right\rangle = \frac{1}{\mathcal{Z}_{\mathrm{inst}}}D_{z_1}\left(\mathcal{Z}_{\mathrm{inst}}\chi(z_2)\right) = \frac{1}{\mathcal{Z}_{\mathrm{inst}}}D_{z_1}D_{z_2}\mathcal{Z}_{\mathrm{inst}}. \tag{D.25}$$

with the shifted derivative

$$D_{z_\alpha} = (z_\alpha + \epsilon_+)^2 - a^2 + \epsilon_1\epsilon_2 q\partial_q, \quad \chi(z) = \langle \Pi(z) \rangle = \frac{D_z \mathcal{Z}_{\text{inst}}}{\mathcal{Z}_{\text{inst}}}. \tag{D.26}$$

Using this expression we arrive at

$$\chi_2(z_1,z_2) = \langle \Pi_2(z_1,z_2) \rangle = \frac{1}{\mathcal{Z}_{\text{inst}}}\left[D_{z_1}D_{z_2} + 2q\frac{\epsilon_1\epsilon_2}{z_{12}^2 - \epsilon_+^2}\right]\mathcal{Z}_{\text{inst}}, \quad \langle \Pi(z_1)\Pi(z_2) \rangle = \frac{1}{\mathcal{Z}_{\text{inst}}} D_{z_1}D_{z_2}\mathcal{Z}_{\text{inst}}, \tag{D.27}$$

replacing $z_1 = z + \nu_1$ and $z_2 = z + \nu_2$, this quantity is obviously a polynomial in $z$.



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