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BCJ identities and d-dimensional generalized unitarity

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Abstract: We present a set of relations between one-loop integral coefficients for dimensionally regulated QCD amplitudes. Within dimensional regularization, the combined use of color-kinematics duality and integrand reduction yields the existence of relations between the integrand residues of partial amplitudes with different orderings of the external particles. These relations can be established for the cut-constructible contributions as well for the ones responsible for rational terms. Starting from the general parametrization of one-loop residues and applying Laurent expansion in order to extract the coefficients of the amplitude decomposition in terms of master integrals, we show that the full set of relations can be obtained by considering the BCJ identities between d-dimensional tree-level amplitudes. We provide explicit examples for multi-gluon scattering amplitudes at one loop.

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Contents

1 Introduction

Tree-level amplitudes in gauge theories are known to satisfy color-kinematics (C/K) duality, i.e. they admit an expansion in terms of Feynman diagrams where the kinematic parts of the numerators satisfy the same antisymmetry and Lie Algebra identities as their corresponding color factors. This property was first observed by Bern, Carrasco and Johansson for pure gauge amplitudes in [1, 2] and later extended to both massless and massive QCD in [3–5]. One of most striking implications of the C/K duality is the existence of relations between color-ordered tree-level amplitudes [1] which, together with U(1) symmetry and Kleiss-Kuijf relations [6], can be used to further reduce the number of independent partial amplitudes to be considered in tree-level calculations. In [7], by adopting the Four-Dimensional-Formulation (FDF) [8] variant of the Four-Dimensional-Helicity (FDH) [9–11] regularization scheme, we studied the C/K-duality for tree-level amplitudes in d dimensions and we derived a set of BCJ identities, for four- and five-point amplitudes, which take into account the explicit dependence on the regulating parameter, together with a general strategy for the determination of analogous relations between higher-multiplicity amplitudes.

The recent development of on-shell [12, 13] and generalized unitarity techniques [14] for quadruple-[15, 16], triple-[16–18], double-[19, 20] and single-[21–23] cut allowed tremendous simplifications in one-loop calculations, where the knowledge of tree-level amplitudes can be exploited in order to determine the coefficients of the known basis of integrals in which any amplitude can be decomposed [24, 25]. In the framework of four-dimensional generalized unitarity, the BCJ identities for tree-level amplitudes were used in [26] to derive relations between coefficients of one-loop amplitudes in $\mathcal{N} = 4$ super Yang-Mills theory and, more recently, in [27] these relations have been extended to integral coefficients for the cut-constructible part of one-loop QCD amplitudes by showing that tree-level the C/K-duality can significantly decrease the number of independent coefficients needed in one-loop computations. When moving to d-dimensional generalized unitarity, extensions of tree-level identities to one-loop amplitudes are expected to hold also between rational contributions, as it was investigated in [28, 29]. Within a different approach, the BCJ relations have been used in [30] to reconstruct the non-planar two-loop integrand contributions to the all plus five-gluon amplitude from the planar ones.

In this paper, by making use of the BCJ identities for dimensionally regulated trees, we provide a set of coefficient relations for one-loop QCD amplitudes which include the contributions from rational terms. The paper is organized as follows: in section 2 we recall the main results regarding the BCJ identities for tree-level amplitudes in d dimensions, obtained by using the FDF scheme. In section 3 we review the decomposition of one-loop amplitudes via integrand reduction [31–37] and we apply the d-dimensional BCJ identities between four-point amplitudes in order to establish general relations between the coefficients appearing in the decomposition. In section 4 we verify the coefficient identities on a few concrete examples, by showing relations between the analytic expression of the coefficients for scalar loop contributions to multi-gluon amplitudes, up to six points. Finally, in appendix A we extend the results of section 3 by providing the set of coefficient relations that can be derived from the BCJ identities between five-point amplitudes. Both algebraic manipulations and numerical evaluations have been carried out by using the MATHEMATICA package S@M [38].

2 Color-kinematics duality in d dimensions

In this section we briefly review the study the C/K-duality for dimensionally regulated amplitudes presented in [7] in the framework of FDF. FDF is a dimensional regularization scheme, first introduced in [8], which allows a purely four-dimensional representation of the additional degrees of freedom associated to the analytic continuation of the space-time dimension. FDF has been recently applied to the computation of one-loop QCD corrections in [39, 40], where the processes $gg \to gg$, $q\bar{q} \to gg$, $gg \to Hg$, $gg \to Hgg$ (in the heavy top limit) and $gg \to ggg(g)$ were studied. In this formulation, virtual states are associated to massive four-dimensional particles, whose mass acts as regulating parameter. The four-dimensional degrees of freedom of the gauge bosons are carried by *massive vector bosons* (denoted by $g^\bullet$) of mass μ and their $(d-4)$-dimensional ones by *real scalar particles* ($s^\bullet$)

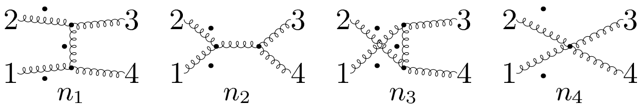

Figure 1. Feynman diagrams for $g^\bullet g^\bullet \to gg$.

of mass μ. At the same time, d-dimensional fermions of mass m are treated as a *tardyonic Dirac fields* ($q^\bullet$) with mass $m + i\mu\gamma^5$.

In order to show how the BCJ identities can be derived taking into account the effects of dimensional regularization, we consider the process $g^\bullet(p_1)g^\bullet(p_2) \to g(p_3)g(p_4)$, where two generalized gluons, i.e. with on-shell momentum $p^2 = \mu^2$, produce a final state with two massless ones. Analogous results can be proven for $s^\bullet s^\bullet \to gg$ as well. The four Feynman diagrams contributing to the amplitude are shown in figure 1, where massive particles are indicated with a dot. The color factors of the first three diagrams, which involve the exchange of a virtual particle, are, respectively,

$$c_1 = \tilde{f}^{a_2 a_3 b}\tilde{f}^{b a_4 a_1}, \qquad c_2 = \tilde{f}^{a_1 a_2 b}\tilde{f}^{b a_3 a_4}, \qquad c_3 = \tilde{f}^{a_1 a_3 b}\tilde{f}^{b a_4 a_2}. \tag{2.1}$$

The four-gluon interaction gives contribution to all of these color structures so that, by labelling with n_i the kinematic parts of Feynman graph numerators, it can be decomposed as

$$c_4 n_4 = c_1 n_{1;4} + c_2 n_{2;4} + c_3 n_{3;4}. \tag{2.2}$$

Therefore, each $n_{i;4}$, conveniently multiplied and divided by the corresponding kinematic pole, can be absorbed into the definition of the numerators of the cubic graphs. As a result, the amplitude is expressed in terms of diagrams involving three-gluon vertices only,

$$\mathcal{A}_4(p_1, p_2, p_3, p_4) = c_1 \frac{n_1}{P_{23}^2 - \mu^2} + c_2 \frac{n_2}{P_{12}^2} + c_3 \frac{n_3}{P_{24}^2 - \mu^2}, \tag{2.3}$$

being $P_{ij}^2 = (p_i + p_j)^2$. We observe that FDF amplitudes receive contributions from both massless and massive virtual states, as it is evident from the pole structure of the r.h.s. of (2.3). The three color factors c_i are related by the Jacobi identity

$$-c_1 + c_2 + c_3 = 0, \tag{2.4}$$

which allows us, for example by eliminating c_2, to rewrite (2.3) in terms of two color-stripped terms only,

$$\mathcal{A}_4(p_1, p_2, p_3, p_4) = c_1 K_1 + c_3 K_3, \tag{2.5}$$

with

$$K_1 = \frac{n_1}{P_{23}^2 - \mu^2} + \frac{n_2}{P_{12}^2}, \qquad K_3 = \frac{n_3}{P_{24}^2 - \mu^2} - \frac{n_2}{P_{12}^2}. \tag{2.6}$$

From the explicit Feynman rules-expression of the numerators n_i's, it can be proven that, when on-shell and transversality conditions ($\epsilon(p_i) \cdot p_i = 0$) are imposed, the amplitude satisfies the C/K-duality, i.e. the kinematic numerators obey the same Jacobi identity as the color factors,

$$-n_1 + n_2 + n_3 = 0. \tag{2.7}$$

The set of equations (2.6) and (2.7) can be conveniently organized into a linear system $\mathbb{A}\,\mathbf{n} = \mathbf{K}$,

$$\begin{pmatrix} \frac{1}{P_{23}^2-\mu^2} & \frac{1}{P_{12}^2} & 0 \\ 0 & -\frac{1}{P_{12}^2} & \frac{1}{P_{24}^2-\mu^2} \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} K_1 \\ K_3 \\ 0 \end{pmatrix}. \tag{2.8}$$

Due to momentum conservation $P_{12}^2 + P_{23}^2 + P_{24}^2 = 2\mu^2$ one can verify that the matrix $\mathbb{A}$ has

$$\text{rank}(\mathbb{A}) = 2 \tag{2.9}$$

or, equivalently, that a linear relation can be established between its rows,

$$(P_{23}^2 - \mu^2)\mathbb{A}_1 - (P_{24}^2 - \mu^2)\mathbb{A}_2 + \mathbb{A}_3 = 0. \tag{2.10}$$

Therefore, because of the consistency condition of the inhomogeneous system (2.8),

$$\text{rank}(\mathbb{A}) = \text{rank}(\mathbb{A}|\mathbf{K}) = 2, \tag{2.11}$$

a constraint analogous to (2.10) must hold between the elements of the vector $\mathbf{K}$,

$$K_3 = \frac{P_{23}^2 - \mu^2}{P_{24}^2 - \mu^2} K_1. \tag{2.12}$$

Starting from the Feynman diagram expansions (2.6), it can be checked that the kinematic factors K_i exactly correspond to two different color-orderings of the amplitude,

$$K_1 = A(1,2,3,4), \qquad\qquad K_3 = A(2,1,3,4), \tag{2.13}$$

so that (2.12) can be rewritten as

$$A(2,1,3,4) = \frac{P_{23}^2 - \mu^2}{P_{24}^2 - \mu^2} A(1,2,3,4). \tag{2.14}$$

With similar considerations one can verify that

$$A(2,4,1,3) = \frac{P_{12}^2}{P_{24}^2 - \mu^2} A(1,2,3,4), \qquad A(2,4,1,3) = \frac{P_{12}^2}{P_{23}^2 - \mu^2} A(2,1,3,4). \tag{2.15}$$

Eqs. (2.14) and (2.15) show that the four-point amplitude involving two gluons and two massive vector bosons, transforming under the adjoint representation of the gauge group, satisfies the same BCJ identities which have been presented in [4, 5] for the scattering of gluons with massive particles in the fundamental representation as a generalization of the BCJ relation for pure gluon amplitudes [1] (here recovered by setting the dimensional

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regulator μ^2 to zero) and for massless QCD amplitudes [3]. It has been verified that all $2 \to 2$ tree-level amplitudes involving FDF particles, including adjoint scalars and tardyonic fermions, obey the same kind of massive BCJ relations.

In general, when moving to higher-point amplitudes,

$$\mathcal{A}_m(p_1, p_2, \ldots, p_m) = \sum_{i=1}^{N} \frac{c_i n_i}{D_i}, \tag{2.16}$$

the kinematic numerators obtained in the standard Feynman rules-approach do not satisfy the C/K-duality, because of the rising of anomalous terms, which have been shown to originate from contact interactions. Nevertheless, starting from the set of Feynman rules numerators n_i, one can build a dual representation of the amplitude by means of a *generalized gauge transformation* [41–47], i.e. a shift of the numerators

$$n_i \to n_i' + \Delta_i, \tag{2.17}$$

which leaves the amplitude unchanged,

$$\delta\mathcal{A}_m^{\text{tree}}(p_1, p_2, \ldots, p_m) \equiv \sum_{i=1}^{N} \frac{c_i \Delta_i}{D_i} = 0, \tag{2.18}$$

and reshuffles the contact terms among numerators, in such a way to restore the C/K-duality. In [7] a diagrammatic approach was proposed to determine the explicit expressions of the shifts to be performed on the numerators, purely based on the algebraic properties of the higher-point generalization of the linear system (2.8) and on a systematic way to generate the anomalous terms through the introduction of off-shell currents. In particular, the computation of the rank of the kinematic matrix $\mathbb{A}$ can be used as constructive criterion in order to detect the existing relations between color-ordered amplitudes.

As an example, we consider the scattering of two generalized gluons producing three massless ones in the final state, $g^\bullet(p_1)g^\bullet(p_2) \to g(p_3)g(p_4)g(p_5)$. After absorbing the contributions from four-gluon vertices into the redefinition of the numerators of the cubic graphs, the amplitude can be expressed in terms of 15 diagrams, each of them identified by its pole structure, i.e. its two internal propagators. The color factors associated to these diagrams satisfy a set of 9 independent Jacobi identities of the type

$$-c_i + c_j + c_k = 0, \tag{2.19}$$

which allow us to express the amplitude in terms of six individually gauge invariant terms only,

$$\mathcal{A}_5(p_1, p_2, p_3, p_4, p_5) = \sum_{i=1}^{6} c_i K_i, \tag{2.20}$$

with

$$\begin{aligned}
c_1 &= \tilde{f}^{a_1a_2b}\tilde{f}^{ba_3c}\tilde{f}^{ca_4a_5}, & c_4 &= \tilde{f}^{a_2a_3b}\tilde{f}^{bca_1}\tilde{f}^{ca_4a_5},\\
c_2 &= \tilde{f}^{a_2a_3b}\tilde{f}^{ba_4c}\tilde{f}^{ca_5a_1}, & c_5 &= \tilde{f}^{a_2bc}\tilde{f}^{ba_3a_4}\tilde{f}^{ca_5a_1},\\
c_3 &= \tilde{f}^{a_1a_2b}\tilde{f}^{bca_5}\tilde{f}^{ca_3a_4}, & c_6 &= \tilde{f}^{a_2a_5b}\tilde{f}^{ba_3c}\tilde{f}^{ca_4a_1},
\end{aligned} \tag{2.21}$$

and

$$
\begin{aligned}
K_1 &= \frac{n_1}{P_{12}^2 P_{45}^2} + \frac{n_{12}}{P_{12}^2 P_{35}^2} + \frac{n_{13}}{(P_{24}^2-\mu^2)P_{35}^2} - \frac{n_{10}}{(P_{13}^2-\mu^2)(P_{24}^2-\mu^2)} + \frac{n_{15}}{(P_{13}^2-\mu^2)P_{45}^2}, \\
K_2 &= \frac{n_2}{(P_{23}^2-\mu^2)(P_{15}^2-\mu^2)} + \frac{n_7}{(P_{14}^2-\mu^2)(P_{23}^2-\mu^2)} - \frac{n_{14}}{(P_{14}^2-\mu^2)P_{35}^2} + \frac{n_{13}}{(P_{24}^2-\mu^2)P_{35}^2} \\
&\quad + \frac{n_{11}}{(P_{24}^2-\mu^2)(P_{15}^2-\mu^2)}, \\
K_3 &= \frac{n_3}{P_{12}^2 P_{34}^2} + \frac{n_9}{(P_{13}^2-\mu^2)(P_{25}^2-\mu^2)} - \frac{n_{12}}{P_{12}^2 P_{35}^2} - \frac{n_{13}}{(P_{24}^2-\mu^2)P_{35}^2} + \frac{n_{10}}{(P_{13}^2-\mu^2)(P_{24}^2-\mu^2)} \\
&\quad - \frac{n_8}{(P_{25}^2-\mu^2)P_{34}^2}, \\
K_4 &= \frac{n_4}{(P_{23}^2-\mu^2)P_{45}^2} - \frac{n_7}{(P_{14}^2-\mu^2)(P_{23}^2-\mu^2)} + \frac{n_{14}}{(P_{14}^2-\mu^2)P_{35}^2} - \frac{n_{13}}{(P_{24}^2-\mu^2)P_{35}^2} \\
&\quad + \frac{n_{10}}{(P_{13}^2-\mu^2)(P_{24}^2-\mu^2)} - \frac{n_{15}}{(P_{13}^2-\mu^2)P_{45}^2}, \\
K_5 &= \frac{n_5}{P_{34}^2(P_{15}^2-\mu^2)} - \frac{n_9}{(P_{13}^2-\mu^2)(P_{25}^2-\mu^2)} - \frac{n_{10}}{(P_{13}^2-\mu^2)(P_{24}^2-\mu^2)} + \frac{n_8}{(P_{25}^2-\mu^2)s_{43}} \\
&\quad - \frac{n_{11}}{(P_{24}^2-\mu^2)(P_{15}^2-\mu^2)}, \\
K_6 &= \frac{n_6}{(P_{14}^2-\mu^2)(P_{25}^2-\mu^2)} + \frac{n_9}{(P_{13}^2-\mu^2)(P_{25}^2-\mu^2)} + \frac{n_{14}}{(P_{14}^2-\mu^2)P_{35}^2} - \frac{n_{13}}{(P_{24}^2-\mu^2)P_{35}^2} \\
&\quad + \frac{n_{10}}{(P_{13}^2-\mu^2)(P_{24}^2-\mu^2)}.
\end{aligned} \tag{2.22}
$$

The number of distinct gauge invariant contributions, obtained after Jacobi identities are taken into account, corresponds to the number of independent color-ordered amplitudes one gets after imposing Kleiss-Kuijf identity, [6, 48]. Since, conversely to the four-point case, the numerators n_i's do not satisfy the same Jacobi identity as the color factors, (2.8) is generalized to a system of 15 equations,

$$\mathbb{A}\mathbf{n} = \mathbf{K} + \boldsymbol{\phi}, \tag{2.23}$$

where

$$
\begin{aligned}
\mathbf{n} &= (n_1, n_2, \ldots, n_{15})^T, \\
\mathbf{K} &= (\{K_1, K_2, \ldots, K_6\}, 0, 0, \ldots, 0)^T, \\
\boldsymbol{\phi} &= (0, 0, \ldots, 0, \{\phi_{[i,j,k]}\})^T
\end{aligned} \tag{2.24}
$$

and the elements of the matrix $\mathbb{A}$ take values in

$$(\mathbb{A})_{ij} \in \{0, \pm 1, \pm(P_{ij}^2)^{-1}, \pm(P_{ij}^2-\mu^2)^{-1}\}. \tag{2.25}$$

The anomalous terms $\phi_{[i,j,k]} = -n_i + n_j + n_k$ can be recursively determined starting from four-point off-shell currents. By performing the set of shifts (2.18), one can build an alternative representation of the amplitude, where the n_i's are substituted by a new set of numerators n_i''s satisfying the C/K dual system

$$\mathbb{A}\mathbf{n}' = \mathbf{K}. \tag{2.26}$$

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Because of momentum conservation, the rank of the matrix $\mathbb{A}$ turns out to be non-maximal, $\text{rank}(\mathbb{A}) = 11$, and the consistency condition

$$\text{rank}(\mathbb{A}|\mathbf{K}) = 11 \tag{2.27}$$

implies the existence of four linear relations between the kinematic factors K_i's, which can be simply found by determining a complete set of vanishing linear combinations of the rows of $\mathbb{A}$. In this way, we obtain the set of identities

$$\begin{aligned}
P_{45}^2 K_1 - P_{34}^2 K_3 - (P_{14}^2 - \mu^2) K_6 &= 0, \\
P_{12}^2 K_1 - (P_{23}^2 - \mu^2) K_4 - (P_{25}^2 - \mu^2) K_6 &= 0, \\
(P_{15}^2 - \mu^2) K_2 - P_{45}^2 K_4 - (P_{25}^2 - \mu^2) K_6 &= 0, \\
(P_{23}^2 - \mu^2) K_2 - P_{34}^2 K_5 + (P_{23}^2 + P_{35}^2 - \mu^2) K_6 &= 0,
\end{aligned} \tag{2.28}$$

which reduce to two the numbers of independent K_i's. At higher multiplicities, rather than corresponding to a single partial amplitude, each kinematic factor can be expressed as a linear combination of color-ordered amplitudes. The relations between the K_i's and color-ordered amplitudes can be found either by comparing their expansions in terms of Feynman diagrams or, more conveniently, by first performing the usual color algebra on (2.20), in order to express all $c_i's$ in terms of traces of generators T^{a_i}, and then by identifying the combinations of K_i's that multiply each single trace with the corresponding color-ordered amplitude. In this case, it can be shown that

$$\begin{aligned}
K_1 &= A_5(1,2,3,4,5) + A_5(1,2,4,3,5) + A_5(1,3,2,4,5), \\
K_2 &= -A_5(1,4,2,3,5), \\
K_3 &= A_5(1,3,4,2,5) - A_5(1,2,4,3,5), \\
K_4 &= A_5(1,4,2,3,5) - A_5(1,3,2,4,5), \\
K_5 &= -A_5(1,3,4,2,5), \\
K_6 &= A_5(1,3,4,2,5) + A_5(1,4,2,3,5) + A_5(1,4,3,2,5).
\end{aligned} \tag{2.29}$$

Therefore, by substituting (2.29) in (2.28), one can reduce from six to two the number of independent color-ordered amplitudes and express all the others through the set of relations

$$\begin{aligned}
A_5(1,3,4,2,5) &= \frac{-P_{12}^2 P_{45}^2 A_5(1,2,3,4,5) + (P_{14}^2 - \mu^2)(P_{24}^2 + P_{25}^2 - 2\mu^2) A_5(1,4,3,2,5)}{(P_{13}^2 - \mu^2)(P_{24}^2 - \mu^2)}, \\
A_5(1,2,4,3,5) &= \frac{-(P_{14}^2 - \mu^2)(P_{25}^2 - \mu^2) A_5(1,4,3,2,5) + P_{45}^2 (P_{12}^2 + P_{24}^2 - \mu^2) A_5(1,2,3,4,5)}{P_{35}^2 (P_{24}^2 - \mu^2)}, \\
A_5(1,4,2,3,5) &= \frac{-P_{12}^2 P_{45}^2 A_5(1,2,3,4,5) + (P_{25}^2 - \mu^2)(P_{14}^2 + P_{25}^2 - 2\mu^2) A_5(1,4,3,2,5)}{P_{35}^2 (P_{24}^2 - \mu^2)}, \\
A_5(1,3,2,4,5) &= \frac{-(P_{14}^2 - \mu^2)(P_{25}^2 - \mu^2) A_5(1,4,3,2,5) + P_{12}^2 (P_{24}^2 + P_{45}^2 - \mu^2) A_5(1,2,3,4,5)}{(P_{13}^2 - \mu^2)(P_{24}^2 - \mu^2)}.
\end{aligned} \tag{2.30}$$

Identities involving other color-ordered amplitudes can be obtained by making use of Kleiss-Kuijf identities such as

$$A_5(1,2,3,4,5)+A_5(1,2,3,5,4)+A_5(1,2,4,3,5)+A_5(1,4,2,3,5)=0, \tag{2.31}$$

which, substituted in (2.30), gives

$$A_5(1,2,4,3,5)=\frac{(P_{14}^2+P_{45}^2-\mu^2)A_5(1,2,3,4,5)+(P_{14}^2-\mu^2)A_5(1,2,3,5,4)}{(P_{24}^2-\mu^2)}. \tag{2.32}$$

The structure of the identities (2.30)–(2.32) for the five-point amplitude involving two adjoint massive vectors bosons is analogous to the one of the BCJ identities for QCD amplitudes with massive quarks [5, 49]. The BCJ relations for the five-gluon (massless) amplitudes [1] can be recovered by setting $\mu^2 = 0$. The very same identities are satisfied by the color-ordered amplitudes where the generalized gluons in the initial state are replaced by massive scalars ($s^\bullet s^\bullet \to ggg$) and similar relations have been verified in [7] for a five-point amplitude involving both FDF particles and quarks, namely $g^\bullet g^\bullet(s^\bullet s^\bullet) \to q\bar{q}g$. These diagrammatic construction of dual representations by means of generalized gauge transformations (2.17) can find a straightforward generalization to higher multiplicities.

In the following section, we will show how FDF formulation of the BCJ identities for d-dimensional tree-level amplitudes, such as (2.14) and (2.32), can be used in order to determine coefficient relations for full d-dimensional one-loop amplitudes, including both cut-constructible part and rational terms.

3 Coefficient relations for one-loop amplitudes in d dimensions

Since the introduction of generalized unitarity [14, 15] and complex kinematics for on-shell particles [12, 13], the study of analyticity and factorization properties of scattering amplitudes has turned into an extremely powerful tool for their computation. Relying on the decomposition of any amplitude in terms of a linear combination of master integrals (MI's) [24, 25], the basic idea of unitarity based methods consists in extracting the coefficients of the MI's by matching multiple cuts of the amplitude with the cuts of the MI's themselves. In this framework, the *integrand reduction method*, first introduced for one-loop amplitudes in [31] and [32], in four- and d-dimensions respectively, and more recently extended to multi-loop case [33–37], exploits the knowledge of the algebraic structure of Feynman integrands, which allows to decompose each numerator as a combination of products of denominators with polynomial coefficients, in order to reach the decomposition of scattering amplitudes in terms of MI's.

At one loop, if we split the $d = 4-2\epsilon$ dimensional loop momentum $\bar{l}^\alpha$ into its four-dimensional part l^α and a vector μ^α belonging to the -2ϵ-subspace,

$$\bar{l}^\alpha = l^\alpha + \mu^\alpha, \qquad \bar{l}^2 = l^2 - \mu^2, \tag{3.1}$$

we can write an arbitrary one-loop n-point color-ordered amplitude as

$$A_n^{\text{1-loop}} = \int d^d \bar{l} \frac{\mathcal{N}(l,\mu^2)}{D_0 D_1 \dots D_{n-1}}, \tag{3.2}$$

with

$$D_i = (\bar{l} + p_i)^2 - m_i^2 = (l + p_i)^2 - m_i^2 - \mu^2. \tag{3.3}$$

The integrand reduction algorithm allows us to write the numerators $\mathcal{N}(l,\mu^2)$ in terms of denominators and, consequently, to obtain a decomposition of the integrand of the type

$$\begin{aligned} \frac{N(l,\mu^2)}{D_0 D_1 \dots D_{n-1}} = & \sum_{i \ll m}^{n-1} \frac{\Delta_{ijklm}(l,\mu^2)}{D_i D_j D_k D_l D_m} + \sum_{i \ll l}^{n-1} \frac{\Delta_{ijkl}(l,\mu^2)}{D_i D_j D_k D_l} + \sum_{i \ll k}^{n-1} \frac{\Delta_{ijk}(l,\mu^2)}{D_i D_j D_k} \\ & + \sum_{i<j}^{n-1} \frac{\Delta_{ij}(l,\mu^2)}{D_i D_j} + \sum_{i}^{n-1} \frac{\Delta_{i}(l,\mu^2)}{D_i}, \end{aligned} \tag{3.4}$$

where $i \ll m$ indicates lexicographic ordering. The functions $\Delta_{i\cdots k}(l,\mu^2)$, called residues, are polynomials in μ^2 and in the components $\{x_i\}$ of l^α, which, according to the cut $D_i = D_j = D_k = \cdots = 0$ under consideration, is decomposed with respect to a suitable basis $\mathcal{E}^{(i\cdots k)} = \{e_1, e_2, e_3, e_4\}$ of four-dimensional massless vectors defined in terms of spinor variables,

$$l^{(i\dots k)\,\nu} = -p_i + \sum_{j=1}^{4} x_j\, e_j^\nu. \tag{3.5}$$

Given such a decomposition of the loop momentum, the parametric expression of $\Delta_{i\cdots k}(l,\mu^2)$ is process independent and, for renormalizable theories [31, 32, 35], it turns out to be

$$\begin{aligned} \Delta_{ijklm} &= c\mu^2, \\ \Delta_{ijkl} &= c_0 + c_1 x_4 + c_2 \mu^2 + c_3 x_4 \mu^2 + c_4 \mu^4, \\ \Delta_{ijk} &= c_{0,0} + c^+_{1,0} x_4 + c^+_{2,0} x_4^2 + c^+_{3,0} x_4^3 + c^-_{1,0} x_3 + c^-_{2,0} x_3^2 + c^-_{3,0} x_3^3 + c_{0,2}\mu^2 + c^+_{1,2} x_4 \mu^2 + c^-_{1,2} x_3 \mu^2, \\ \Delta_{ij} &= c_{0,0,0} + c_{0,1,0} x_1 + c_{0,2,0} x_1^2 + c^+_{1,0,0} x_4 + c^+_{2,0,0} x_4^2 + c^-_{1,0,0} x_3 + c^-_{2,0,0} x_3^2 + c^+_{1,1,0} x_1 x_4 \\ &\quad + c^-_{1,1,0} x_1 x_3 + c_{0,0,2}\mu^2, \\ \Delta_i &= c_{0,0,0,0} + c_{0,1,0,0} x_1 + c_{0,0,1,0} x_2 + c^-_{1,0,0,0} x_3 + c^+_{1,0,0,0} x_4, \end{aligned} \tag{3.6}$$

where, for each coefficient, a superscript labelling the specific cut is understood, $c_l = c_l^{(i\cdots k)}$.

As a consequence of (3.6), by neglecting all *spurious* terms which vanish upon integration, the amplitude (3.2) can be written in terms of MI's

$$I_{i\cdots k}[\alpha] = \int d^d \bar{l} \frac{\alpha}{D_i \cdots D_k} \tag{3.7}$$

and of the coefficients of the residues as

$$
\begin{aligned}
A_n^{\text{1-loop}} = &\sum_{i\ll l}^{n-1}\left[c_0^{(ijkl)} I_{ijkl}[1] + c_4^{(ijkl)} I_{ijkl}[\mu^4]\right] + \sum_{i\ll k}^{n-1}\left[c_{0,0}^{(ijk)} I_{ijk}[1] + c_{0,2}^{(ijk)} I_{ijk}[\mu^2]\right] \\
&+ \sum_{i\ll j}^{n-1}\left[c_{0,0,0}^{(ij)} I_{ij}[1] + c_{0,1,0}^{(ij)} I_{ij}[(l+p_i)\cdot e_2] + c_{0,2,0}^{(ij)} I_{ij}[((l+p_i)\cdot e_2)^2] + c_{0,0,2}^{(ij)} I_{ij}[\mu^2]\right] \\
&+ \sum_{i}^{n-1} c_{0,0,0,0}^{(i)} I_i[1]. \qquad (3.8)
\end{aligned}
$$

In the original top-down formulation of the algorithm [31, 50, 51], all coefficients of the integrand decomposition (3.4) are computed by sampling the numerator of the integrand, after all non-vanishing contribution to higher-point residues have been subtracted, on a finite set of on-shell solutions of the multiple cuts. Alternatively, starting from the techniques presented in [16, 18], it has been shown in [52] that, by performing a suitable Laurent expansion of the cut integrand with respect to one of the components of the loop momenta which are left unconstrained by the on-shell conditions, one can determine the unknown coefficients of the integrand reduction by comparison with the ones of the Laurent expansion itself.

A full color-dressed amplitude is obtained as a combination of color-ordered amplitudes, multiplied for the corresponding color structure. For instance, in the pure gluon case, we have [53]

$$
\begin{aligned}
\mathcal{A}_n^{\text{1-loop}} &= g^n \sum_{c=1}^{[n/2]+1} \sum_{\sigma\in S_n/S_{n;c}} \mathrm{Gr}_{n;c}(\sigma) A_{n;c}^{\text{1-loop}}(\sigma), \\
\mathrm{Gr}_{n;1}(\sigma) &= N_c \mathrm{Tr}(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(n)}}), \\
\mathrm{Gr}_{n;c}(\sigma) &= N_c \mathrm{Tr}(T^{a_{\sigma(1)}}\cdots T^{a_{\sigma(c-1)}})\mathrm{Tr}(T^{a_{\sigma(c)}}\cdots T^{a_{\sigma(c-1)}}), \quad c>1. \qquad (3.9)
\end{aligned}
$$

Although it is sufficient to consider leading color contributions $A_{n;1}(\sigma) \equiv A_n(\sigma)$, since amplitudes associated to subleading colors can be obtained as a sum over permutations of $A_n(\sigma)$'s [14], one should, in principle, fit the coefficients of the residues (3.6) for each color-ordering. However, the C/K-duality satisfied by tree-level amplitudes, in which the integrand factorizes when evaluated on unitarity cuts, can be used to determine relations between coefficients of residues which differ from the ordering of external particles, and thus to reduce the total number of coefficients to be individually computed.

In the following, we recall the extraction of coefficients via Laurent expansion, for which we refer to [52] and [54], and we make use of the d-dimensional BCJ identities presented in section 2 in order to determine the full set of relations between integral coefficients. As we will explicitly show, these identities holds separately for both independent cut solutions that must be averaged in the extraction of the integral coefficients. For sake of simplicity, we derive relations between integral coefficient that can be obtained starting from the BCJ identities at four points only and we collect in appendix A the set of relations that follow from the C/K-duality for five points amplitudes. We expect similar results to hold even

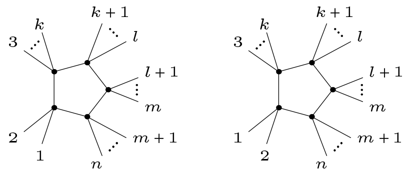

Figure 2. Pentagon topologies for the cuts $C_{12|3\ldots k|(k+1)\ldots l|(l+1)\ldots m|(m+1)\ldots n}$ and $C_{21|3\ldots k|(k+1)\ldots l|(l+1)\ldots m|(m+1)\ldots n}$.

when the BCJ identities for higher multiplicity amplitudes are taken into account but we leave this generalization to future studies. For this reason, we will not discuss relations between tadpoles coefficients, which would at least require the use of the BCJ identities between six points tree-level amplitudes.

3.1 Relations for pentagon coefficients

The solutions of the quintuple cut $D_i = D_j = D_k = D_l = D_m = 0$ can be parametrized as

$$l_+^{(ijklm)\nu} = -p_i^\nu + x_1 e_1^{(ijklm)\nu} + x_2 e_2^{(ijklm)\nu} + x_3 e_3^{(ijklm)\nu} + \frac{x_4 + \mu^2}{x_3} e_4^{(ijklm)\nu}, \tag{3.10}$$

$$l_-^{(ijklm)\nu} = -p_i^\nu + x_1 e_1^{(ijklm)\nu} + x_2 e_2^{(ijklm)\nu} + x_3 e_4^{(ijklm)\nu} + \frac{x_4 + \mu^2}{x_3} e_3^{(ijklm)\nu}, \tag{3.11}$$

where the full set of parameters x_1, x_2, x_3, x_4 and μ^2 is fixed by the cut conditions. The single pentagon coefficient appearing in (3.4) can be computed evaluating the integrand on the two on-shell solutions,

$$C^{\pm}_{i|j|k|l|m} = \frac{N_\pm}{\prod_{h \neq i,j,k,l,m} D_{h,\pm}} = c^{(ijklm)\pm} \mu^2. \tag{3.12}$$

In order to see how the BCJ identities for tree-level amplitudes can be used to relate different pentagon coefficients, let us consider the contributions shown in figure 2, which share the same cut solutions. In addition, since these two pentagons differ in the ordering of the external particles p_1 and p_2 only, they can be obtained as the product of the same tree-level amplitudes, with the only exception of the color-ordering of the four-point amplitude involving p_1 and p_2. More precisely, for the ordering $\{1, 2, \ldots, n\}$ we have

$$\begin{aligned} &C^{\pm}_{12|3\ldots k|(k+1)\ldots l|(l+1)\ldots m|(m+1)\ldots n} \\ &\quad = A_4^{\text{tree}}\left(-l_1^\pm, 1, 2, l_3^\pm\right) A_k^{\text{tree}}\left(-l_3^\pm, P_{3\cdots k}, l_{k+1}^\pm\right) A_{l-k+2}^{\text{tree}}\left(-l_{k+1}^\pm, P_{k+1\ldots,l^\pm}, l_l^\pm\right) \\ &\quad \times A_{m-l+2}^{\text{tree}}\left(-l_{l+1}^\pm, P_{l^\pm+1\ldots,m}, l_m^\pm\right) A_{n-m+2}^{\text{tree}}\left(-l_{m+1}^\pm, P_{m+1\ldots,n}, l_1^\pm\right) \end{aligned} \tag{3.13}$$

and $C^{\pm}_{21|3\ldots k|(k+1)\ldots l|(l+1)\ldots m|(m+1)\ldots n}$ is obtained just by changing $1 \leftrightarrow 2$. The tree-level amplitudes $A_4^{\text{tree}}\left(-l_1^\pm, 1, 2, l_3^\pm\right)$ and $A_4^{\text{tree}}\left(-l_1^\pm, 2, 1, l_3^\pm\right)$ are related by the d-dimensional BCJ identity (2.15),

$$A_4^{\text{tree}}(-l_1^\pm, 2, 1, l_3^\pm) = \frac{P^2_{l_3^\pm 2} - \mu^2}{P^2_{-l_1^\pm 2} - \mu^2} A_4^{\text{tree}}(-l_1^\pm, 1, 2, l_3^\pm), \tag{3.14}$$

which, substituted into the expression of $C^{\pm}_{21|3\ldots k|(k+1)\ldots l|(l+1)\ldots m|(m+1)\ldots n}$, allow us to identify

$$C^{\pm}_{21|3\ldots k|(k+1)\ldots l|(l+1)\ldots m|(m+1)\ldots n} = \frac{P^2_{l_3^{\pm}2} - \mu^2}{P^2_{-l_1^{\pm}2} - \mu^2} C^{\pm}_{12|3\ldots k|(k+1)\ldots l|(l+1)\ldots m|(m+1)\ldots n}. \tag{3.15}$$

The ratio of the two propagators appearing in (3.15) evaluates to same constant value for both cut solutions,

$$\frac{P^2_{l_3^{\pm}2} - \mu^2}{P^2_{-l_1^{\pm}2} - \mu^2} = \alpha, \tag{3.16}$$

so that, by making use of (3.12), (3.15) becomes

$$c^{(21|\ldots)\pm} = \alpha c^{(12|\ldots)\pm}. \tag{3.17}$$

Therefore, as simple byproduct of the BCJ identities at tree-level, the knowledge of a single pentagon coefficient completely determines the other one.

3.2 Relations for box coefficients

Next we consider the quadrupole cut $D_i = D_j = D_k = D_l = 0$, whose solutions can be parametrized as

$$\begin{aligned} l_{+}^{(ijkl)\nu} &= -p_i^{\nu} + x_1 e_1^{(ijkl)\nu} + x_2 e_2^{(ijkl)\nu} + x_3\, e_3^{(ijkl)\nu} + \frac{x_4 + \mu^2}{x_3} e_4^{(ijkl)\nu}, \\ l_{-}^{(ijkl)\nu} &= -p_i^{\nu} + x_1 e_1^{(ijk)\nu} + x_2 e_2^{(ijkl)\nu} + \frac{x_4 + \mu^2}{x_3} e_3^{(ijkl)\nu} + x_3\, e_4^{(ijkl)\nu}, \end{aligned} \tag{3.18}$$

being x_1, x_2, x_3 and x_4 coefficients fixed by the cut conditions. The two non-spurious coefficients $c_0^{(ijkl)}$ and $c_4^{(ijkl)}$ can be extracted in the $\mu^2 \to 0$ and $\mu^2 \to \infty$ limits,

$$C^{\pm}_{i|j|k|l} = \left.\frac{N_{\pm}}{\prod_{h\neq i,j,k,l} D_{h,\pm}}\right|_{\mu^2\to 0} = c_0^{(ijkl)\pm}, \tag{3.19a}$$

$$C^{\pm}_{i|j|k|l} = \left.\frac{N_{\pm}}{\prod_{h\neq i,j,k,l} D_{h,\pm}}\right|_{\mu^2\to \infty} = c_4^{(ijkl)\pm}\mu^4 + \mathcal{O}\left(\mu^3\right), \tag{3.19b}$$

and the box contribution to the amplitude (3.8) is obtained by averaging over the two cut solutions,

$$A_n^{\text{1-loop}}\Big|_{\text{box}} = \frac{1}{2}\left(c_0^{(ijkl)+} + c_0^{(ijkl)-}\right) I_{ijkl}\,[1] + c_4^{(ijkl)}\, I_{(ijkl)}\,[\mu^4]\,, \tag{3.20}$$

where we used $c_4^{(ijkl)} \equiv c_4^{(ijkl)+} = c_4^{(ijkl)-}$. Analogously to the pentagon case, we consider two box topologies differing just from the ordering of the external particles p_1 and p_2, as depicted in figure 3. When the integrand associated to the ordering $\{1, 2, \ldots, n\}$ is evaluated on the on-shell solutions it factorizes into

$$\begin{aligned} C^{\pm}_{12|3\ldots k|(k+1)\ldots l|(l+1)\ldots n} = {} & A_4^{\text{tree}}\left(-l_1^{\pm}, 1, 2, l_3^{\pm}\right) A_k^{\text{tree}}\left(-l_3^{\pm}, P_{3\ldots k}, l_{k+1}^{\pm}\right) \\ & \times A_{l-k+2}^{\text{tree}}\left(-l_{k+1}^{\pm}, P_{k+1\ldots,l}, l_{l+1}^{\pm}\right) A_{n-l+2}^{\text{tree}}\left(-l_{l+1}^{\pm}, P_{l+1\ldots,n}, l_1^{\pm}\right) \end{aligned} \tag{3.21}$$

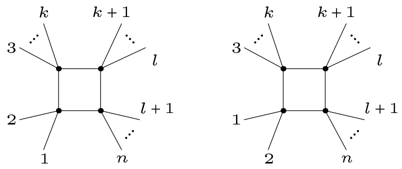

Figure 3. Box topologies for the cuts $C_{12|3\dots k|k+1\dots l|l+1\dots n}$ and $C_{21|3\dots k|k+1\dots l|l+1\dots n}$.

and the expression of $C^{\pm}_{21|3\dots k|(k+1)\dots l|(l+1)\dots n}$ in terms of tree-level amplitudes can be obtained by exchanging $1 \leftrightarrow 2$. Therefore, thanks to the BCJ identity between tree-level amplitudes (3.14), we can write

$$C^{\pm}_{21|3\dots k|(k+1)\dots l|(l+1)\dots n} = \frac{P^2_{l_3^{\pm}2} - \mu^2}{P^2_{-l_1^{\pm}2} - \mu^2} C^{\pm}_{12|3\dots k|(k+1)\dots l|(l+1)\dots n}. \tag{3.22}$$

It can be verified that the ratio of propagators sampled on the cut solutions converges to a constant both for $\mu^2 \to 0$ and $\mu^2 \to \infty$ limits,

$$\left. \frac{P^2_{l_3^{\pm}2} - \mu^2}{P^2_{-l_1^{\pm}2} - \mu^2} \right|_{\mu^2 \to 0} = \alpha_0^{\pm}, \tag{3.23}$$

$$\left. \frac{P^2_{l_3^{\pm}2} - \mu^2}{P^2_{-l_1^{\pm}2} - \mu^2} \right|_{\mu^2 \to \infty} = \alpha_4^{\pm} + \mathcal{O}\left(\frac{1}{\mu}\right), \tag{3.24}$$

so that, by evaluating both sides of (3.22) in the two limits, we can trivially obtain the contributions from $C_{21|3\dots k|(k+1)\dots l|(l+1)\dots n}$, once $C_{12|3\dots k|(k+1)\dots l|(l+1)\dots n}$ has been calculated,

$$c_i^{(21|\dots)\pm} = \alpha_i^{\pm} c_i^{(12|\dots)\pm}, \qquad i = 0, 4. \tag{3.25}$$

3.3 Relations for triangle coefficients

The solutions of the triple cut $D_i = D_j = D_k = 0$ can be parametrized in terms of μ^2 and one free parameter t as

$$\begin{aligned} l_+^{(ijk)\nu} &= -p_i^{\nu} + x_1 e_1^{(ijk)\nu} + x_2 e_2^{(ijk)\nu} + t\, e_3^{(ijk)\nu} + \frac{x_3 + \mu^2}{t} e_4^{(ijk)\nu}, \\ l_-^{(ijk)\nu} &= -p_i^{\nu} + x_1 e_1^{(ijk)\nu} + x_2 e_2^{(ijk)\nu} + \frac{x_3 + \mu^2}{t} e_3^{(ijk)\nu} + t\, e_4^{(ijk)\nu}, \end{aligned} \tag{3.26}$$

where the coefficients x_1, x_2 and x_3 are fixed by the cut conditions. By considering the expansion of the integrand in the large-t limit,

$$C^{\pm}_{i|j|k}\left(t, \mu^2\right) = \left. \frac{N_{\pm}}{\prod_{h \neq i,j,k} D_{h,\pm}} \right|_{t \to \infty} = \sum_{m=0}^{3} c_{m,0}^{(ijk)\pm} t^m + \mu^2 \sum_{m=0}^{1} c_{m,2}^{(ijk)\pm} t^m, \tag{3.27}$$

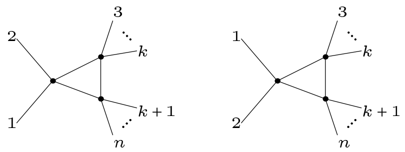

Figure 4. Triangle topologies for the cuts $C_{12|3\ldots k|(k+1)\ldots n}$ and $C_{21|3\ldots k|(k+1)\ldots n}$.

the triangle contribution to the one-loop amplitude (3.8) can be obtained by averaging on the two solutions (3.26).

$$A_n^{\text{1-loop}}\big|_{\text{triangle}} = \frac{1}{2}\left(c_{0,0}^{+} + c_{0,0}^{-}\right) I_3\left[1\right] + \frac{1}{2}\left(c_{0,2}^{+} + c_{0,2}^{-}\right) I_3\left[\mu^2\right]. \tag{3.28}$$

The C/K-duality for tree-level amplitudes can be used to relate all coefficients of the expansions (3.27) for different triangles. As an example, we consider the two triangle contributions depicted in figure 4. When evaluated on the on-shell solutions, the triangle with external ordering $\{1, 2, \ldots, n\}$ factorizes into

$$C^{\pm}_{12|3\ldots k|(k+1)\ldots n} = A_4^{\text{tree}}\left(-l_1^{\pm}, 1, 2, l_3^{\pm}\right) A_k^{\text{tree}}\left(-l_3^{\pm}, P_{3\cdots k}, l_{k+1}^{\pm}\right) A_{n-k+2}^{\text{tree}}\left(-l_{k+1}^{\pm}, P_{k+1\ldots,n}, l_1^{\pm}\right) \tag{3.29}$$

and the analogous expression for $C^{\pm}_{21|3\ldots k|(k+1)\ldots n}$ is obtained by changing $1 \leftrightarrow 2$. As for the previous cases, we can make use of the BCJ identity (3.14) in order to establish a relation between $C^{\pm}_{21|3\ldots k|(k+1)\ldots n}$ and $C^{\pm}_{12|3\ldots k|(k+1)\ldots n}$,

$$C^{\pm}_{21|3\ldots k|(k+1)\ldots n} = \frac{P^2_{l_3^{\pm}2} - \mu^2}{P^2_{-l_1^{\pm}2} - \mu^2} C_{12|3\ldots k|(k+1)\ldots n}. \tag{3.30}$$

According to the expansion (3.27), both $C^{\pm}_{21|3\ldots k|(k+1)\ldots n}$ and $C^{\pm}_{12|3\ldots k|(k+1)\ldots n}$ can be parametrized as

$$\begin{aligned} C^{\pm}_{12|3\ldots k|(k+1)\ldots n} &= \sum_{m=0}^{3} c^{(12|\ldots)\pm}_{m,0} t^m + \mu^2 \sum_{m=0}^{1} c^{(12|\ldots)\pm}_{m,2} t^m, \\ C^{\pm}_{21|3\ldots k|(k+1)\ldots n} &= \sum_{m=0}^{3} c^{(21|\ldots)\pm}_{m,0} t^m + \mu^2 \sum_{m=0}^{1} c^{(21|\ldots)\pm}_{m,2} t^m. \end{aligned} \tag{3.31}$$

There, if we consider the large-t limit of the ratio of the two propagators evaluated on the cut solution, which is found in the form

$$\left.\frac{P^2_{l_3^{\pm}2} - \mu^2}{P^2_{-l_1^{\pm}2} - \mu^2}\right|_{t\to\infty} = \sum_{m=-3}^{0} \alpha^{\pm}_{m,0} t^m + \mu^2 \sum_{m=-3}^{-2} \alpha^{\pm}_{m,2} t^m + \mathcal{O}\left(\frac{1}{t^4}\right), \tag{3.32}$$

we can insert the expansions (3.31) and (3.32) into (3.30) and, by matching each monomial between the two sides, obtain the set of relations

$$c^{(21|\ldots)\pm}_{m,0} = \sum_{l=0}^{3-m} \alpha^{\pm}_{-l,0}\, c^{(12|\ldots)\pm}_{l+m,0}, \quad c^{(21|\ldots)\pm}_{m,2} = \sum_{l=0}^{1-m} \left(\alpha^{\pm}_{-l-2,2}\, c^{(12|\ldots)\pm}_{l+m+2,0} + \alpha^{\pm}_{-l0}\, c^{(12|\ldots)\pm}_{l+m,2}\right). \tag{3.33}$$

Eqs. (3.33) show that $C^{\pm}_{21|3\ldots k|(k+1)\ldots n}$ can be fully reconstructed from the knowledge of $C^{\pm}_{12|3\ldots k|(k+1)\ldots n}$.

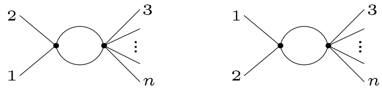

Figure 5. Bubble topologies for the cuts $C_{12|3\ldots n}$ and $C_{21|3\ldots n}$.

3.4 Relations for bubble coefficients

Finally, we consider the double cut $D_i = D_j = 0$, whose solutions are parametrized as

$$\begin{aligned} l_{+}^{(ij)\nu} &= -p_i^\nu + y\, e_1^{(ij)\nu} + (a_0 + y\, a_1)\, e_2^{(ij)\nu} + t\, e_3^{(ij)\nu} + \frac{\mu^2 + b_0 + b_1 y + b_2 y^2}{t} e_4^{(ij)\nu}, \\ l_{-}^{(ij)\nu} &= -p_i^\nu + y\, e_1^{(ij)\nu} + (a_0 + y\, a_1)\, e_2^{(ij)\nu} + \frac{\mu^2 + b_0 + b_1 y + b_2 y^2}{t} e_3^{(ij)\nu} + t\, e_4^{(ij)\nu}, \end{aligned} \tag{3.34}$$

where a_i and b_i are kinematic factors fixed by the cut conditions, whereas t and y are free parameters. The bubble coefficients are extracted from the large-t expansion,

$$C_{i|j}^{\pm}\left(t, y, \mu^2\right) = \frac{N_\pm}{\prod_{h \neq i,j} D_{h,\pm}} - \sum_{k \neq i,j}^{n-1} \frac{\Delta_{ijk,\pm}^R}{D_{k,+}}\Bigg|_{t\to\infty} = \sum_{l=0}^{2}\sum_{m=0}^{2-l} c_{l,m,0}^{(ij)\pm}\, t^l\, y^m + \mu^2 c_{0,0,2}^{(ij)\pm}. \tag{3.35}$$

Here the reduced residues $\Delta_{ijk,\pm}^R$, defined in [52], are needed in order to subtract *spurious* contributions originating from triangle coefficients. The bubble contribution to the amplitude (3.8) is

$$A_n^{\text{1-loop}}\Big|_{\text{bubble}} = c_{0,0,0}^{(ij)} I_{ij}\,[1] + c_{0,1,0} I_{ij}\left[(q+p_i)\cdot e_2\right] + c_{0,2,0} I_{ij}\left[((q+p_i)\cdot e_2)^2\right] + c_{0,0,2}^{(ij)} I_{ij}\left[\mu^2\right], \tag{3.36}$$

where we dropped the "$\pm$" label, since the coefficients appearing in the r.h.s. turn out to be the identical for the two solutions. As usual, in order to show the role of the C/K-duality in the reduction of the number of coefficients to be actually computed, we consider two bubble contributions differing by the ordering of the external particles p_1 and p_2, as illustrated in figure 5. The two coefficients are given by

$$\begin{aligned} C_{12|3\ldots n}^{\pm} &= A_4^{\text{tree}}\left(-l_1^\pm, 1, 2, l_3^\pm\right) A_n^{\text{tree}}\left(-l_3^\pm, P_{3\cdots n}, l_1^\pm\right), \\ C_{21|3\ldots n}^{\pm} &= A_4^{\text{tree}}\left(-l_1^\pm, 2, 1, l_3^\pm\right) A_n^{\text{tree}}\left(-l_3^\pm, P_{3\cdots n}, l_1^\pm\right) \end{aligned} \tag{3.37}$$

and, using (3.14) to relate $A_4^{\text{tree}}\left(-l_1^\pm, 1, 2, l_3^\pm\right)$ and $A_4^{\text{tree}}\left(-l_1^\pm, 2, 1, l_3^\pm\right)$, we obtain

$$C_{21|3\ldots n}^{\pm} = \frac{P_{l_3^\pm 2}^2 - \mu^2}{P_{-l_1^\pm 2}^2 - \mu^2} C_{12|3\ldots n}^{\pm}. \tag{3.38}$$

The ratio of the two propagators in the large-t limit is parametrized as

$$\frac{P_{l_3^\pm 2}^2 - \mu^2}{P_{-l_1^\pm 2}^2 - \mu^2}\Bigg|_{t\to\infty} = \sum_{l=-2}^{0}\sum_{m=0}^{-l} \alpha_{l,m,0}\, t^l\, y^m + \frac{\mu^2}{t^2}\alpha_{-2,0,2} + \mathcal{O}\left(\frac{1}{t^3}\right), \tag{3.39}$$

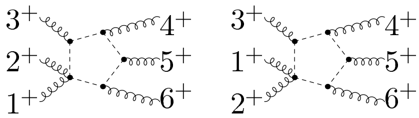

Figure 6. Pentagon topologies for the cuts $C_{12|3|4|5|6}$ and $C_{21|3|4|5|6}$.

so that, by plugging in (3.38) the expansions

$$\begin{aligned} C^{\pm}_{12|3\ldots n} &= \sum_{l=0}^{2}\sum_{m=0}^{2-l} c^{(12|\ldots)\pm}_{l,m,0}\, t^l\, y^m + \mu^2 c^{(12|\ldots)\pm}_{0,0,2}, \\ C^{\pm}_{21|3\ldots n} &= \sum_{l=0}^{2}\sum_{m=0}^{2-l} c^{(21|\ldots)\pm}_{l,m,0}\, t^l\, y^m + \mu^2 c^{(21|\ldots)\pm}_{0,0,2}, \end{aligned} \tag{3.40}$$

one can verify that the coefficients of $C^{\pm}_{21|3\ldots n}$ are completely determined by

$$\begin{aligned} c^{(21|\ldots)\pm}_{l,m,0} &= \sum_{r=l}^{2}\left(\sum_{s=\max[0,l+m-r]}^{\min[m,2-r]} \alpha^{\pm}_{l-r,m-s,0}\, c^{(12|\ldots)\pm}_{r,s,0}\right), \\ c^{(21|\ldots)\pm}_{0,0,2} &= \alpha^{\pm}_{-2,0,2}\, c^{(12|\ldots)\pm}_{2,0,0} + \alpha^{\pm}_{0,0,0}\, c^{(12|\ldots)\pm}_{0,0,2}. \end{aligned} \tag{3.41}$$

4 Examples

We hereby verify on some explicit examples the coefficient relations we have derived in the previous section. In order to obtain compact expressions and keep the discussion as simple possible, we consider scalar loop contributions to gluon amplitudes only and we present analytic results for convenient helicity configurations. Nevertheless, numerical checks of the coefficient relations have been performed for all helicity configurations and gluon loop contributions have been included as well. All results presented in this section have been numerically validated against the ones provided by the C++ library NJet [55].

In addition, we would like to mention that, besides constituting one of the FDF ingredients needed for the computation of the full amplitude, the scalar contributions presented in this section can been thought as the generators of rational terms in alternative frameworks, such as supersymmetric decomposition [56, 57].

4.1 Pentagons

To begin with, we consider the six-gluon helicity amplitude $\mathcal{A}_6^{\text{1-loop}}(1^+,2^+,3^+,4^+,5^+,6^+)$ and we compute the quintuple cuts $C^{\pm}_{12|3|4|5|6}$ and $C^{\pm}_{21|3|4|5|6}$ of figure 6, with the use the basis $\mathcal{E}^{(45012)} = \{e_1, e_2, e_3, e_4\}$, where

$$e_1^\nu = p_4^\nu, \qquad e_2^\nu = p_5^\nu, \qquad e_3^\nu = \frac{1}{2}\langle 4\,|\gamma^\mu|\,5], \qquad e_4^\nu = \frac{1}{2}\langle 5\,|\gamma^\mu|\,4], \tag{4.1}$$

The solutions of the quintuple cut are

$$l_5^{+\nu} = c\, e_3^\nu - \frac{\mu^2}{s_{45}c}\, e_4^\nu, \qquad l_5^{-\nu} = c\, e_4^\nu - \frac{\mu^2}{s_{45}c}\, e_3^\nu, \tag{4.2}$$

where the parameters c and μ^2 are fixed by the on-shell conditions. From the product of tree-level amplitudes we obtain

$$\begin{aligned}C^{\pm}_{12|3|4|5} &= A_4^{\text{tree}}\left(-l_1^{\pm},1^+,2^+,l_3^{\pm}\right)A_3^{\text{tree}}\left(-l_3^{\pm},3^+,l_4^{\pm}\right)A_3^{\text{tree}}\left(-l_4^{\pm},4^+,l_5^{\pm}\right)\\ &\quad\times A_3^{\text{tree}}\left(-l_5^{\pm},5^+,l_6^{\pm}\right)A_3^{\text{tree}}\left(-l_6^{\pm},6^+,l_1^{\pm}\right)\\ &= \frac{i\mu^2[2|1]\langle 3|l_5|4]\langle 4|l_4|3]\langle 5|l_1|6]\langle 6|l_6|5]}{\langle 1|2\rangle\langle 3|4\rangle^2\langle 5|6\rangle^2\langle 1|l_1|1]}\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}C_{21|3|4|5} &= A_4^{\text{tree}}\left(-l_1^{\pm},2^+,1^+,l_3^{\pm}\right)A_3^{\text{tree}}\left(-l_3^{\pm},3^+,l_4^{\pm}\right)A_3^{\text{tree}}\left(-l_4^{\pm},4^+,l_5^{\pm}\right)\\ &\quad\times A_3^{\text{tree}}\left(-l_5^{\pm},5^+,l_6^{\pm}\right)A_3^{\text{tree}}\left(-l_6^{\pm},6^+,l_1^{\pm}\right),\\ &= \frac{i\mu^2[2|1]\langle 3|l_5|4]\langle 4|l_4|3]\langle 5|l_1|6]\langle 6|l_6|5]}{\langle 1|2\rangle\langle 3|4\rangle^2\langle 5|6\rangle^2\langle 2|l_1|2]}.\end{aligned} \tag{4.4}$$

The two cuts are related by the BCJ identity (3.15),

$$C^{\pm}_{21|3|4|5} = \frac{\left(l_3^{\pm}+p_2\right)^2-\mu^2}{\left(l_1^{\pm}-p_2\right)^2-\mu^2}C\pm_{12|3|4|5|6}. \tag{4.5}$$

By using momentum conservation to express $l_5^{\pm}$ in terms of $l_1^{\pm}, l_3^{\pm}, l_4^{\pm}$,

$$l_1^{\pm} = l_5^{\pm} - p_5 - p_6, \qquad l_3^{\pm} = l_5^{\pm} + p_3 + p_4, \qquad l_4^{\pm} = l_5^{\pm} + p_4, \qquad l_6^{\pm} = l_5^{\pm} - p_5, \tag{4.6}$$

one can verify that $C^{\pm}_{12|3|4|5|6}$ takes the form

$$\begin{aligned}&C^{\pm}_{12|3|4|5|6}\\ &= \frac{i\mu^2 {s_{34}}^2 {s_{45}}^2 {s_{56}}^2 [2|1][4|3][6|5]\langle 3|1+2|6]^2\langle 6|1+2|3]^2}{\text{tr}_5(6,3,5,4)^3\langle 1|2\rangle\langle 3|4\rangle\langle 5|6\rangle(s_{45}\text{tr}_5(1,5,2,6)+s_{345}\text{tr}_5(1,5,4,6)-s_{16}\text{tr}_5(3,4,5,6))},\end{aligned} \tag{4.7}$$

where $s_{ij} = \langle ij\rangle[ji]$ and $\text{tr}_5(1,2,3,4) = \langle 1\,|234|\,1] - \langle 1\,|432|\,1]$.

In a similar way, according to (3.16), we find

$$\frac{\left(l_3^{\pm}+p_2\right)^2-\mu^2}{\left(l_1^{\pm}-p_2\right)^2-\mu^2} = \frac{s_{45}\text{tr}_5(1,5,2,6)+s_{345}\text{tr}_5(1,5,4,6)-s_{16}\text{tr}_5(3,4,5,6)}{s_{45}\text{tr}_5(2,5,1,6)+s_{345}\text{tr}_5(2,5,4,6)-s_{26}\text{tr}_5(3,4,5,6)}. \tag{4.8}$$

Hence, substituting (4.7) and (4.8) in (4.5), we obtain

$$\begin{aligned}&C^{\pm}_{21|3|4|5|6}\\ &= \frac{i\mu^2 {s_{34}}^2 {s_{45}}^2 {s_{56}}^2 [2|1][4|3][6|5]\langle 3|1+2|6]^2\langle 6|1+2|3]^2}{\text{tr}_5(6,3,5,4)^3\langle 1|2\rangle\langle 3|4\rangle\langle 5|6\rangle(s_{45}\text{tr5}(2,5,1,6)+s_{345}\text{tr5}(2,5,4,6)-s_{26}\text{tr5}(3,4,5,6))},\end{aligned} \tag{4.9}$$

which reproduces the same result one could obtain from similar algebraic manipulations on (4.4). The analytic expressions for the two cuts find numerical agreement with NJet.

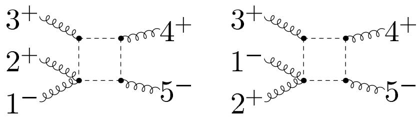

Figure 7. Box topologies for the cuts $C_{12|3|4|5}$ and $C_{21|3|4|5}$.

4.2 Boxes

As an example of identities between box coefficients, we consider the quadruple cuts $C^{\pm}_{12|3|4|5}$ and $C^{\pm}_{21|3|4|5}$ for the helicity amplitude $\mathcal{A}_5^{\text{1-loop}}(1^-, 2^+, 3^+, 4^+, 5^-)$, depicted in figure 7. For this configuration we use the basis $\mathcal{E}^{(40123)}$ of eq. (4.1), where the cut solutions can be parametrized as

$$l_5^{+\nu} = c_+ \, e_3^\nu - \frac{\mu^2}{s_{45} c_+} \, e_4^\nu, \qquad\qquad l_5^{-\nu} = c_- \, e_4^\nu - \frac{\mu^2}{s_{45} c_-} \, e_3^\nu, \tag{4.10}$$

being c_+ and c_- coefficients determined by behavior of the on-shell solutions for $\mu^2 \to 0$ and $\mu^2 \to \infty$. By combining tree-level amplitudes we can write

$$\begin{aligned} C^{\pm}_{12|3|4|5} &= A_4^{\text{tree}}\left(-l_1^{\pm}, 1^-, 2^+, l_3^{\pm}\right) A_3^{\text{tree}}\left(-l_3^{\pm}, 3^+, l_4^{\pm}\right) A_3^{\text{tree}}\left(-l_4^{\pm}, 4^+, l_5^{\pm}\right) A_3^{\text{tree}}\left(-l_5^{\pm}, 5^-, l_1^{\pm}\right) \\ &= \frac{\langle 1|l_1|2]^2 \langle 3|l_5|4] \langle 4|l_4|3] \langle 5|l_1|1]}{s_{12} [5|1] \langle 3|4\rangle^2 \langle 1|l_1|1]} \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} C^{\pm}_{21|3|4|5} &= A_4^{\text{tree}}\left(-l_1^{\pm}, 2^+, 1^-, l_3^{\pm}\right) A_3^{\text{tree}}\left(-l_3^{\pm}, 3^+, l_4^{\pm}\right) A_3^{\text{tree}}\left(-l_4^{\pm}, 4^+, l_5^{\pm}\right) A_3^{\text{tree}}\left(-l_5^{\pm}, 5^-, l_1^{\pm}\right) \\ &= \frac{\langle 1|l_1|2]^2 \langle 3|l_5|4] \langle 4|l_4|3] \langle 5|l_1|1]}{s_{12} [5|1] \langle 3|4\rangle^2 \langle 2|l_1|2]} \end{aligned} \tag{4.12}$$

and then relate two cuts through (3.22),

$$C^{\pm}_{21|3|4|5} = \frac{\left(l_3^{\pm} + p_2\right)^2 - \mu^2}{\left(l_1^{\pm} - p_2\right)^2 - \mu^2} C^{\pm}_{21|3|4|5}. \tag{4.13}$$

Momentum conservation allows us to write

$$l_1^{\pm} = l_5^{\pm} - p_5, \qquad\qquad l_4^{\pm} = l_5^{\pm} + p_4, \qquad\qquad l_3 = l_5^{\pm} + p_3 + p_4 \tag{4.14}$$

and, consequently, to express $C^{\pm}_{12|3|4|5}$ as

$$C^{\pm}_{12|3|4|5} = -\frac{i\mu^4 [4|3] \mathrm{tr}_5(\eta_{1,2}, 4, 3, 5)^2}{s_{12} \mathrm{tr}_5(3, 4, 1, 5) [5|3] [5|4] \langle 3|4\rangle^2}, \tag{4.15}$$

where we have introduced the complex momenta $\eta^\nu_{i,j} = \frac{1}{2} \langle i | \gamma^\nu | j]$. We have verified that, for this particular helicity configuration, the box coefficient is given by the $\sim \mu^4$ term only. Therefore, we just need to compute the ratio of the propagators in the large-μ^2 limit,

$$\left.\frac{\left(l_3^{\pm} + p_2\right)^2 - \mu^2}{\left(l_1^{\pm} - p_2\right)^2 - \mu^2}\right|_{\mu^2 \to \infty} = -1 + \mathcal{O}\left(\frac{1}{\mu}\right). \tag{4.16}$$

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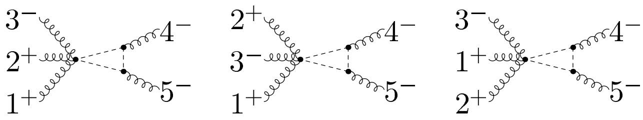

Figure 8. Triangle topologies for the cuts $C_{123|4|5}, C_{132|4|5}$ and $C_{213|4|5}$.

Thanks to this result, the expression for $C_{21|3|4|5}$ obtained from (4.15) is

$$C^{\pm}_{21|3|4|5} = \frac{i\mu^4[4|3]\mathrm{tr}_5(\eta_{1,2},4,3,5)^2}{s_{12}\mathrm{tr}_5(3,4,1,5)[5|3][5|4]\langle 3|4\rangle^2}, \tag{4.17}$$

which finds again agreement with NJet.

4.3 Triangles

For triple cuts we give an example of the coefficient relations obtained through identities between five-point tree-level amplitudes, which are discussed in appendix A. Let us consider $\mathcal{A}_5^{\text{1-loop}}(1^+,2^+,3^-,4^-,5^-)$ and the three cuts of figure 8, $C^{\pm}_{213|4|5}$, $C^{\pm}_{123|4|5}$ and $C^{\pm}_{132|4|5}$, respectively. We use the basis $\mathcal{E}^{(40123)}$ of eq. (4.1), where the cut solutions are given by

$$l_5^{+\nu} = t\, e_3^\nu - \frac{\mu^2}{s_{45}t}\, e_4^\nu, \qquad\qquad l_5^{-\nu} = t\, e_4^\nu - \frac{\mu^2}{s_{45}t}\, e_3^\nu, \tag{4.18}$$

and from the product of tree-level amplitudes we obtain

$$\begin{aligned}
C^{\pm}_{123|4|5} &= A_5^{\text{tree}}\left(-l_1^{\pm},1^+,2^+,3^-,l_4^{\pm}\right) A_3^{\text{tree}}\left(-l_4^{\pm},4^-,l_5^{\pm}\right) A_3^{\text{tree}}\left(-l_5^{\pm},5^-,l_1^{\pm}\right) \\
&= \frac{i\langle 5|l_1^{\pm}|4|l_5^{\pm}|5]\langle 3|1+2|l_1^{\pm}|3\rangle^2}{s_{123}[5|4]^2\langle 1|2\rangle\langle 2|3\rangle\langle 1|l_1^{\pm}|1+2|3\rangle} - \frac{i\mu^2[2|1]\langle 3|l_4^{\pm}|2]^2\langle 5|l_1^{\pm}|4|l_5|5]}{[5|4]^2[3|l_4^{\pm}|3|2]\langle 1|l_1^{\pm}|1]\langle 1|2+3|l_4^{\pm}|3\rangle}, \qquad (4.19)\\
C^{\pm}_{132|4|5} &= A_5^{\text{tree}}\left(-l_1^{\pm},1^+,3^-,2^+,l_4^{\pm}\right) A_3^{\text{tree}}\left(-l_4^{\pm},4^-,l_5^{\pm}\right) A_3^{\text{tree}}\left(-l_5^{\pm},5^-,l_1^{\pm}\right) \\
&= -\frac{i\langle 3|l_1^{\pm}|1]^2\langle 3|l_4^{\pm}|2]^2\langle 5|l_1^{\pm}|4|l_5^{\pm}|5]}{[5|4]^2[2|l_4^{\pm}|2+3|1]\langle 1|l_1^{\pm}|1|3\rangle\langle 2|l_4^{\pm}|2|3\rangle} - \frac{i\mu^2[2|1]^4\langle 5|l_1^{\pm}|4|l_5^{\pm}|5]}{s_{123}[3|1][3|2][5|4]^2[2|l_4|2+3|1]}, \qquad (4.20)\\
C^{\pm}_{213|4|5} &= A_5^{\text{tree}}\left(-l_1^{\pm},2^+,1^+,3^-,l_4^{\pm}\right) A_3^{\text{tree}}\left(-l_4^{\pm},4^-,l_5^{\pm}\right) A_3^{\text{tree}}\left(-l_5^{\pm},5^-,l_1^{\pm}\right) \\
&= -\frac{i\langle 5|l_1^{\pm}|4|l_5^{\pm}|5]\langle 3|1+2|l_1^{\pm}|3\rangle^2}{s_{123}[5|4]^2\langle 1|2\rangle\langle 1|3\rangle\langle 2|l_1^{\pm}|1+2|3\rangle} + \frac{i\mu^2[2|1]\langle 3|l_4^{\pm}|1]^2\langle 5|l_1^{\pm}|4|l_5^{\pm}|5]}{[5|4]^2[3|l_4^{\pm}|3|1]\langle 2|l_1^{\pm}|2]\langle 2|1+3|l_4^{\pm}|3\rangle}. \qquad (4.21)
\end{aligned}$$

The three cuts are related by the BCJ identity (2.32),

$$C^{\pm}_{213|4|5} = \frac{\left(P^2_{l_4^{\pm}2} - \mu^2 + P^2_{23}\right)}{\left(P^2_{-l_1^{\pm}2} - \mu^2\right)} C^{\pm}_{123|4|5} + \frac{\left(P^2_{l_4^{\pm}2} - \mu^2\right)}{\left(P^2_{-l_1^{\pm}2} - \mu^2\right)} C^{\pm}_{132|4|5}. \tag{4.22}$$

By using momentum conservation,

$$l_1^{\pm} = l_5^{\pm} - p_5, \qquad\qquad l_4^{\pm} = l_5^{\pm} + p_4, \tag{4.23}$$

and expanding $C^{\pm}_{123|4|5}$ and $C^{\pm}_{132|4|5}$ for $t \to \infty$, we obtain

$$C^{+}_{123|4|5}\left(t,\mu^2\right) = \frac{i\mu^2\langle 3|4\rangle^2(\langle 1|4\rangle\langle 3|5\rangle + \langle 1|3\rangle\langle 4|5\rangle)}{[5|4]\langle 1|2\rangle\langle 1|4\rangle^2\langle 2|3\rangle} - \frac{i\mu^2\langle 3|4\rangle^3}{[5|4]\langle 1|2\rangle\langle 1|4\rangle\langle 2|3\rangle}t, \tag{4.24a}$$

$$C^{-}_{123|4|5}\left(t,\mu^2\right) = \frac{i\mu^2\langle 3|4\rangle\langle 3|5\rangle^2}{[5|4]\langle 1|2\rangle\langle 1|5\rangle\langle 2|3\rangle} + \frac{i\mu^2\langle 3|5\rangle^3}{[5|4]\langle 1|2\rangle\langle 1|5\rangle\langle 2|3\rangle}t, \tag{4.24b}$$

$$C^{+}_{132|4|5}\left(t,\mu^2\right) = -\frac{i\mu^2\langle 3|4\rangle^3(\langle 1|4\rangle\langle 3|5\rangle + \langle 1|3\rangle\langle 4|5\rangle)}{[5|4]\langle 1|3\rangle\langle 1|4\rangle^2\langle 2|3\rangle\langle 2|4\rangle} + \frac{i\mu^2\langle 3|4\rangle^4}{[5|4]\langle 1|3\rangle\langle 1|4\rangle\langle 2|3\rangle\langle 2|4\rangle}t, \tag{4.24c}$$

$$C^{-}_{132|4|5}\left(t,\mu^2\right) = -\frac{i\mu^2(\langle 2|5\rangle\langle 3|4\rangle - \langle 2|3\rangle\langle 4|5\rangle)\langle 3|5\rangle^3}{[5|4]\langle 1|3\rangle\langle 1|5\rangle\langle 2|3\rangle\langle 2|5\rangle^2} - \frac{i\mu^2\langle 3|5\rangle^4}{[5|4]\langle 1|3\rangle\langle 1|5\rangle\langle 2|3\rangle\langle 2|5\rangle}t. \tag{4.24d}$$

In a similar way, the expansion for large-t of the ratio of propagators returns

$$\left.\frac{\left(P^2_{l_4^+2} - \mu^2 + P^2_{23}\right)}{\left(P^2_{-l_1^+2} - \mu^2\right)}\right|_{t\to\infty} = \frac{\mu^2 s_{12}s_{24}s_{25}}{s_{45}t^3\langle 4|2|5]^3} + \frac{s_{12}{s_{25}}^2}{t^3\langle 4|2|5]^3} + \frac{s_{12}s_{25}}{t^2\langle 4|2|5]^2} + \frac{s_{12}}{t\langle 4|2|5]} - 1 + \mathcal{O}\left(\frac{1}{t^4}\right), \tag{4.25a}$$

$$\left.\frac{\left(P^2_{l_4^-2} - \mu^2 + P^2_{23}\right)}{\left(P^2_{-l_1^-2} - \mu^2\right)}\right|_{t\to\infty} = \frac{\mu^2 s_{12}s_{24}s_{25}}{s_{45}t^3\langle 5|2|4]^3} + \frac{s_{12}{s_{25}}^2}{t^3\langle 5|2|4]^3} + \frac{s_{12}s_{25}}{t^2\langle 5|2|4]^2} + \frac{s_{12}}{t\langle 5|2|4]} - 1 + \mathcal{O}\left(\frac{1}{t^4}\right), \tag{4.25b}$$

$$\begin{aligned}\left.\frac{\left(P^2_{l_4^+2} - \mu^2\right)}{\left(P^2_{-l_1^+2} - \mu^2\right)}\right|_{t\to\infty} &= -\frac{\mu^2 s_{24}s_{25}\left(s_{24}+s_{25}\right)}{s_{45}t^3\langle 4|2|5]^3} - \frac{{s_{25}}^2\left(s_{24}+s_{25}\right)}{t^3\langle 4|2|5]^3} - \frac{s_{25}\left(s_{24}+s_{25}\right)}{t^2\langle 4|2|5]^2} \\ &\quad - \frac{s_{24}+s_{25}}{t\langle 4|2|5]} - 1 + \mathcal{O}\left(\frac{1}{t^4}\right),\end{aligned} \tag{4.25c}$$

$$\begin{aligned}\left.\frac{\left(P^2_{l_4^-2} - \mu^2\right)}{\left(P^2_{-l_1^-2} - \mu^2\right)}\right|_{t\to\infty} &= -\frac{\mu^2 s_{24}s_{25}\left(s_{24}+s_{25}\right)}{s_{45}t^3\langle 5|2|4]^3} - \frac{{s_{25}}^2\left(s_{24}+s_{25}\right)}{t^3\langle 5|2|4]^3} - \frac{s_{25}\left(s_{24}+s_{25}\right)}{t^2\langle 5|2|4]^2} \\ &\quad - \frac{s_{24}+s_{25}}{t\langle 5|2|4]} - 1 + \mathcal{O}\left(\frac{1}{t^4}\right).\end{aligned} \tag{4.25d}$$

Therefore, by inserting these results into (4.22) we obtain

$$C^{+}_{213|4|5}\left(t,\mu^2\right) = -\frac{i\mu^2\langle 3|4\rangle^2(\langle 2|4\rangle\langle 3|5\rangle + \langle 2|3\rangle\langle 4|5\rangle)}{[5|4]\langle 1|2\rangle\langle 1|3\rangle\langle 2|4\rangle^2} + \frac{i\mu^2\langle 3|4\rangle^3}{[5|4]\langle 1|2\rangle\langle 1|3\rangle\langle 2|4\rangle}t, \tag{4.26}$$

$$C^{-}_{213|4|5}\left(t,\mu^2\right) = -\frac{i\mu^2\langle 3|4\rangle\langle 3|5\rangle^2}{[5|4]\langle 1|2\rangle\langle 1|3\rangle\langle 2|5\rangle} - \frac{i\mu^2\langle 3|5\rangle^3}{[5|4]\langle 1|2\rangle\langle 1|3\rangle\langle 2|5\rangle}t, \tag{4.27}$$

which agrees with the $t \to \infty$ expansion of (4.24d). The resulting contributions of the three cuts to $\mathcal{A}_5^{\text{1-loop}}\left(1^+, 2^+, 3^-, 4^-, 5^-\right)$ have been numerically checked with NJet.

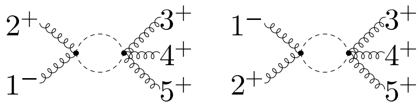

Figure 9. Bubble topologies for the cuts $C_{12|345}$ and $C_{21|345}$.

4.4 Bubbles

As a final example, we compute the double cuts $C_{12|345}$ and $C_{21|345}$ of the helicity amplitude $\mathcal{A}_5^{\text{1-loop}}(1^-, 2^+, 3^+, 4^+, 5^+)$, which are depicted in figure 9. For sake of simplicity, we will consider only pure bubble contributions but we remark that spurious terms originating from triangles, which should subtracted in order to recover the full integral coefficient, can be related through the BCJ identities in the same way as discussed in section 3.4. For this cut we use the basis $\mathcal{E}^{(02)} = \{e_1, e_2, e_3, e_4\}$, where

$$e_1^\nu = (p_1 + p_2)^\nu - \frac{s_{12}}{s_{14} + s_{24}} p_4^\nu, \quad e_2^\nu = p_4^\nu, \quad e_3^\nu = \frac{1}{2}\langle e_1|\gamma^\nu|e_2], \quad e_4^\nu = \frac{1}{2}\langle e_2|\gamma^\nu|e_1], \quad (4.28)$$

and the cut solutions are parametrized by

$$l_1^{+\nu} = y\, e_1^\nu + \frac{(1-y)\, s_{12}}{s_{14} + s_{24}}\, e_2^\nu + t\, e_3^\nu + \frac{(1-y)\, y s_{12} - \mu^2}{(s_{14} + s_{24})\, t}\, e_4^\nu, \quad (4.29)$$

$$l_1^{-\nu} = y\, e_1^\nu + \frac{(1-y)\, s_{12}}{s_{14} + s_{24}}\, e_2^\nu + t\, e_4^\nu + \frac{(1-y)\, y s_{12} - \mu^2}{(s_{14} + s_{24})\, t}\, e_3^\nu. \quad (4.30)$$

By combining the tree amplitudes in which bubble factorizes, we can write the two cuts as

$$\begin{aligned} C_{12|345}^{\pm} &= A_4^{\text{tree}}\left(-l_1^{\pm}, 1^-, 2^+, l_3^{\pm}\right) A_5^{\text{tree}}\left(-l_3^{\pm}, 3^+, 4^+, 5^+, l_1^{\pm}\right) \\ &= \frac{\mu^2 [5|3+4|l_3^{\pm}|3]\langle 1|l_1^{\pm}|2]^2}{s_{12}\langle 3|4\rangle\langle 4|5\rangle\langle 1|l_1^{\pm}|1]\langle 3|l_3^{\pm}|3]\langle 5|l_1^{\pm}|5]}, \end{aligned} \quad (4.31)$$

$$\begin{aligned} C_{21|345}^{\pm} &= A_4^{\text{tree}}\left(-l_1^{\pm}, 2^+, 1^-, l_3^{\pm}\right) A_5^{\text{tree}}\left(-l_3^{\pm}, 3^+, 4^+, 5^+, l_1^{\pm}\right) \\ &= \frac{\mu^2 [5|3+4|l_3|3]\langle 1|l_1^{\pm}|2]^2}{s_{12}\langle 3|4\rangle\langle 4|5\rangle\langle 2|l_1^{\pm}|2]\langle 3|l_3^{\pm}|3]\langle 5|l_1^{\pm}|5]} \end{aligned} \quad (4.32)$$

and, according to (3.38), we can related them through

$$C_{21|345}^{\pm} = \frac{\left(l_3^{\pm} + p_2\right)^2 - \mu^2}{\left(l_1^{\pm} - p_2\right)^2 - \mu^2}\, C_{12|345}^{\pm}. \quad (4.33)$$

If we make use of $l_3^{\pm} = l_1^{\pm} - p_1 - p_2$ and we expand (4.31) in the large-t limit, we get

$$C_{12|345}^{+} = \mu^2 \frac{i[4|2]^3\langle 1|2\rangle}{s_{34}s_{45}[4|1]\langle 3|5\rangle^2}, \quad (4.34)$$

$$C_{12|345}^{-} = \mu^2 \frac{i\langle 1|4\rangle^3}{\langle 1|2\rangle\langle 2|4\rangle\langle 3|4\rangle^2\langle 4|5\rangle^2}, \quad (4.35)$$

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whereas the expansion of the ratios of propagators reads

$$\left.\frac{\left(l_3^+ + p_2\right)^2 - \mu^2}{\left(l_1^+ - p_2\right)^2 - \mu^2}\right|_{t\to\infty} = -\frac{(s_{14}-s_{24})\,[2|1]^2[4|e_1]^2}{(s_{14}+s_{24})\,[4|1]^2[4|2]^2}\frac{y}{t^2} - \frac{[2|1]^2\langle 2|4\rangle[4|e_1]^2}{(s_{14}+s_{24})\,t^2[4|1]^2[4|2]}\frac{1}{t^2} - \frac{[2|1][4|e_1]}{[4|1][4|2]}\frac{1}{t} - 1 + \mathcal{O}\left(\frac{1}{t^3}\right), \tag{4.36}$$

$$\left.\frac{\left(l_3^- + p_2\right)^2 - \mu^2}{\left(l_1^- - p_2\right)^2 - \mu^2}\right|_{t\to\infty} = -\frac{(s_{14}-s_{24})\,\langle 1|2\rangle^2\langle e_1|4\rangle^2}{(s_{14}+s_{24})\,\langle 1|4\rangle^2\langle 2|4\rangle^2}\frac{y}{t^2} - \frac{[4|2]\langle 1|2\rangle^2\langle e_1|4\rangle^2}{(s_{14}+s_{24})\,\langle 1|4\rangle^2\langle 2|4\rangle}\frac{1}{t^2} - \frac{\langle 1|2\rangle\langle e_1|4\rangle}{\langle 1|4\rangle\langle 2|4\rangle}\frac{1}{t} - 1 + \mathcal{O}\left(\frac{1}{t^3}\right). \tag{4.37}$$

These expansions allow us to obtain the analytic expression of $C^{\pm}_{21|345}$ from (4.33),

$$C^{+}_{21|345} = -\mu^2\frac{i[4|2]^3\langle 1|2\rangle}{s_{34}s_{45}[4|1]\langle 3|5\rangle^2} = -C^{+}_{12|345}, \tag{4.38a}$$

$$C^{-}_{21|345} = -\mu^2\frac{i\langle 1|4\rangle^3}{\langle 1|2\rangle\langle 2|4\rangle\langle 3|4\rangle^2\langle 4|5\rangle^2} = -C^{-}_{12|345}, \tag{4.38b}$$

which agree with what we would obtain by considering the large-t expansion of (4.32).

5 Conclusions

In this paper we have presented a set of relations between the coefficients appearing in the decomposition of one-loop QCD amplitudes in terms of master integrals, which have been derived as a byproduct of the color-kinematics duality satisfied by tree-level amplitudes. These relations reduce the number of independent integral coefficients to be individually computed and, being valid for contributions to both cut-constructible part and rational terms, they could play an important role in the optimization of numerical calculations. The complete decomposition of a general one-loop amplitude can be obtained via the d-dimensional integrand reduction algorithm, which can be used to express the amplitude in terms of a known basis of loop integrals, whose coefficient can be extracted through suitable Laurent expansions of the integrand evaluated on the on-shell solutions. Since the on-shell integrand factorizes into a product of tree-level amplitudes, the BCJ identities at tree-level have been exploited in order to establish relations between the integral coefficients themselves. In order to be consistent with d-dimensional unitarity, hence to obtain a set of identities valid for both cut-constructible part and rational terms, we have made use of the BCJ identities for dimensionally regulated tree-level amplitudes, which have been derived by working in the Four-Dimensional-Formulation scheme (FDF), where the effects of dimensional regularization are carried by massive degrees of freedom. The coefficients identities derived in this paper have been verified on a number of contributions to multi-gluon scattering amplitudes, for which we have provided analytic expressions of the integral coefficients. A natural extension of this work would be the study of similar relations for higher-point one-loop amplitudes, which would require the use of the d-dimensional BCJ identities between tree-level amplitudes with more than five external particles. Moreover,

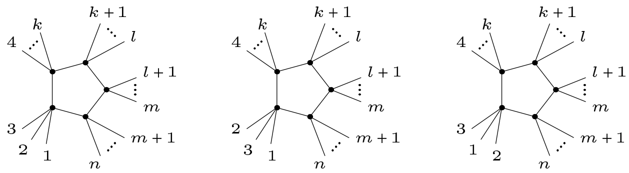

Figure 10. Pentagon topologies for the cuts $C_{123|4\ldots k|(k+1)\ldots l|(l+1)\ldots m|(m+1)\ldots n}$, $C_{132|4\ldots k|(k+1)\ldots l|(l+1)\ldots m|(m+1)\ldots n}$ and $C_{231|4\ldots k|(k+1)\ldots l|(l+1)\ldots m|(m+1)\ldots n}$.

it would be interesting to investigate higher-loop coefficient relations that are expected to descend from the BCJ identities at tree-level. To this end, future work will require, besides a general parametrization of the residues of multi-loop integrands, the derivation of the BCJ identities between FDF amplitudes involving more than two external generalized particles.

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A Coefficient relations from 5-point BCJ identities

In this appendix we collect the set of identities, obtained through the use of the d-dimensional BCJ relations for five-point amplitudes of the type (2.32), that can be used to relate integral coefficients associated to multiple cuts which, besides sharing the same on-shell solutions, differ from the ordering of three external particles.

A.1 Relations for pentagon coefficients

We consider the three quintuple-cuts shown in figure 10, which differ from the ordering of the particles p_1, p_2, p_3. The contribution from the ordering $\{1,2,3\}$ is given by

$$\begin{aligned} &C^{\pm}_{123|4\ldots r|(r+1)\ldots s|(s+1)\ldots t|(t+1)\ldots n} \\ &\quad= A_5^{\text{tree}}\left(-l_1^{\pm},1,2,3,l_4^{\pm}\right) A_{r-1}^{\text{tree}}\left(-l_4^{\pm},P_{4\cdots r},l_{r+1}^{\pm}\right) A_{s-r+2}^{\text{tree}}\left(-l_{r+1}^{\pm},P_{r+1\ldots,s},l_{s+1}^{\pm}\right) \\ &\qquad\times A_{s-t+2}^{\text{tree}}\left(-l_{s+1}^{\pm},P_{s+1\cdots t},l_{t+1}^{\pm}\right) A_{n-t+2}^{\text{tree}}\left(-l_{t+1}^{\pm},P_{t+1\cdots n},l_1^{\pm}\right) \end{aligned} \tag{A.1}$$

and the other two cuts are obtained from the corresponding permutations of $\{1,2,3\}$. Eq. (2.32) can be used in order to relate the amplitudes $A_5^{\text{tree}}\left(-l_1^{\pm},1,2,3,l_4^{\pm}\right)$, $A_5^{\text{tree}}\left(-l_1^{\pm},1,3,2,l_4^{\pm}\right)$ and $A_5^{\text{tree}}\left(-l_1^{\pm},2,1,3,l_4^{\pm}\right)$ and, thus, to identify

$$\begin{aligned} &C^{\pm}_{213|4\ldots r|(r+1)\ldots s|(s+1)\ldots t|(t+1)\ldots n} \\ &= \frac{\left(P^2_{l_4^{\pm}2}+P^2_{23}-\mu^2\right) C^{\pm}_{123|4\ldots r|(r+1)\ldots s|(s+1)\ldots t|(t+1)\ldots n} + \left(P^2_{l_4^{\pm}2}-\mu^2\right) C^{\pm}_{132|4\ldots r|(r+1)\ldots s|(s+1)\ldots t|(t+1)\ldots n}}{\left(P^2_{-l_1^{\pm}2}-\mu^2\right)}. \end{aligned} \tag{A.2}$$

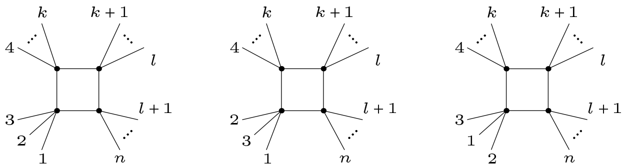

Figure 11. Box topologies for the cuts $C_{123|4\ldots k|k+1\ldots l|l+1\ldots n}$, $C_{132|4\ldots k|k+1\ldots l|l+1\ldots n}$ and $C_{231|4\ldots k|k+1\ldots l|l+1\ldots n}$.

Analogously to the case discussed in section 3.1, the constant ratios of propagators

$$\frac{P^2_{l_4^\pm 2} - \mu^2}{P^2_{-l_1^\pm 2} - \mu^2} = \alpha^\pm, \qquad \frac{P^2_{l_4^\pm 2} + P^2_{23} - \mu^2}{P^2_{-l_1^\pm 2} - \mu^2} = \beta^\pm, \tag{A.3}$$

allow us to translate (A.2) into a simple identity between the coefficients of the expansion (3.12) for the three cuts,

$$c^{(213|\ldots)\pm} = \beta^\pm c^{(123|\ldots)\pm} + \alpha^\pm c^{(132|\ldots)\pm}. \tag{A.4}$$

A.2 Relations for box coefficients

Similarly to the previous case, we can use the BCJ identities to relate the quadruple cuts depicted in figure 11, given by

$$\begin{aligned} C^\pm_{123|4\ldots r|(r+1)\ldots s|(s+1)\ldots n} = {} & A_5^{\text{tree}}\left(-l_1^\pm, 1, 2, 3, l_4^\pm\right) A_{r-1}^{\text{tree}}\left(-l_4^\pm, P_{4\cdots r}, l_{r+1}^\pm\right) \\ & \times A_{s-r+2}^{\text{tree}}\left(-l_{r+1}^\pm, P_{r+1\ldots,s}, l_{s+1}^\pm\right) A_{n-s+2}^{\text{tree}}\left(-l_{s+1}^\pm, P_{s+1\cdots n}, l_1^\pm\right) \end{aligned} \tag{A.5}$$

and suitable permutations of {1,2,3} for $C^\pm_{132|4\ldots r|(r+1)\ldots s|(s+1)\ldots n}$ and $C^\pm_{213|4\ldots r|(r+1)\ldots s|(s+1)\ldots n}$.

If we make use of (2.32) on the amplitudes involving the particles p_1, p_2 and p_3, we obtain

$$\begin{aligned} & C^\pm_{213|4\ldots r|(r+1)\ldots s|(s+1)\ldots n} \\ & = \frac{\left(P^2_{l_4^\pm 2} + P^2_{23} - \mu^2\right) C^\pm_{123|4\ldots r|(r+1)\ldots s|(s+1)\ldots n} + \left(P^2_{l_4^\pm 2} - \mu^2\right) C^\pm_{132|4\ldots r|(r+1)\ldots s|(s+1)\ldots n}}{\left(P^2_{-l_1^\pm 2} - \mu^2\right)}. \end{aligned} \tag{A.6}$$

As shown in section 3.2, the two box coefficients contributing to the amplitude can be extracted by taking the $\mu^2 \to 0$ and $\mu^2 \to \infty$ limits, where the ratios of propagators behave like

$$\begin{aligned} \left.\frac{P^2_{l_4^\pm 2} - \mu^2}{P^2_{-l_1^\pm 2} - \mu^2}\right|_{\mu^2\to 0} &= \alpha_0^\pm, & \left.\frac{P^2_{l_4^\pm 2} - \mu^2}{P^2_{-l_1^\pm 2} - \mu^2}\right|_{\mu^2\to \infty} &= \alpha_4^\pm + \mathcal{O}\left(\frac{1}{\mu}\right), \\ \left.\frac{P^2_{l_4^\pm 2} + P^2_{23} - \mu^2}{P^2_{-l_1^\pm 2} - \mu^2}\right|_{\mu^2\to 0} &= \beta_0^\pm, & \left.\frac{P^2_{l_4^\pm 2} + P^2_{23} - \mu^2}{P^2_{-l_1^\pm 2} - \mu^2}\right|_{\mu^2\to \infty} &= \beta_4^\pm + \mathcal{O}\left(\frac{1}{\mu}\right). \end{aligned} \tag{A.7}$$

Thus, starting from (A.6) we can relate the coefficients of the expansions (3.19a)–(3.19b) of the three quadruple cuts trough the identities

$$c_i^{(213|\ldots)\pm} = \beta_i^\pm c_i^{(123|\ldots)\pm} + \alpha_i^\pm c_i^{(132|\ldots)\pm}, \qquad i = 0, 4. \tag{A.8}$$

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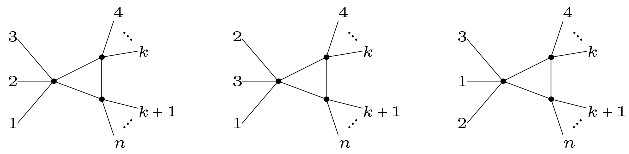

Figure 12. Triangle topologies for the cuts $C_{123|4\ldots k|(k+1)\ldots n}$, $C_{132|4\ldots k|(k+1)\ldots n}$ and $C_{213|4\ldots k|(k+1)\ldots n}$.

A.3 Relations for triangle coefficients

Now we turn our attention to the triangle topologies shown in figure 12. The expression of the cut with external ordering $\{1,2,3,\ldots n\}$ in terms of tree-level amplitudes is given by

$$\begin{aligned} &C^{\pm}_{123|4\ldots k|(k+1)\ldots n} \\ &\quad = A_5^{\text{tree}}\left(-l_1^{\pm},1,2,3,l_4^{\pm}\right) A_{k-1}^{\text{tree}}\left(-l_4^{\pm},P_{4\cdots k},l_{k+1}^{\pm}\right) A_{n-k+2}^{\text{tree}}\left(-l_{k+1}^{\pm},P_{k+1\ldots,n},l_1^{\pm}\right) \end{aligned} \tag{A.9}$$

and, as usual, $C^{\pm}_{132|4\ldots k|(k+1)\ldots n}$ and $C^{\pm}_{213|4\ldots k|(k+1)\ldots n}$ are obtained from the corresponding permutations of $\{1,2,3\}$. Eq. (2.32) allow us to identify

$$C^{\pm}_{213|4\ldots k|(k+1)\ldots n} = \frac{\left(P^2_{l_4^{\pm}2}+P^2_{23}-\mu^2\right)C^{\pm}_{123|4\ldots k|(k+1)\ldots n}+\left(P^2_{l_4^{\pm}2}-\mu^2\right)C^{\pm}_{132|4\ldots k|(k+1)\ldots n}}{\left(P^2_{-l_1^{\pm}2}-\mu^2\right)} \tag{A.10}$$

and, following the procedure of section 3.3, we can take the large-t limit of the two ratios of propagators,

$$\begin{aligned} \left.\frac{P^2_{l_4^{\pm}2}-\mu^2}{P^2_{-l_1^{\pm}2}-\mu^2}\right|_{t\to\infty} &= \sum_{m=-3}^{0}\alpha^{\pm}_{m,0}t^m+\mu^2\sum_{m=-3}^{-2}\alpha^{\pm}_{m,2}t^m+\mathcal{O}\left(\frac{1}{t^4}\right), \\ \left.\frac{P^2_{l_4^{\pm}2}+P^2_{23}-\mu^2}{P^2_{-l_1^{\pm}2}-\mu^2}\right|_{t\to\infty} &= \sum_{m=-3}^{0}\beta^{\pm}_{m,0}t^m+\mu^2\sum_{m=-3}^{-2}\beta^{\pm}_{m,2}t^m+\mathcal{O}\left(\frac{1}{t^4}\right), \end{aligned} \tag{A.11}$$

and use it in (A.10) in order to express the coefficients of the expansion (3.27) of $C^{\pm}_{213|4\ldots k|(k+1)\ldots n}$ in terms of the ones of $C_{123|4\ldots k|(k+1)\ldots n}$ and $C_{132|4\ldots k|(k+1)\ldots n}$,

$$c^{(213|\ldots)\pm}_{m,0} = \sum_{l=0}^{3-m}\left[\beta^{\pm}_{-l,0}\,c^{(123|\ldots)\pm}_{l+m,0}+\alpha^{\pm}_{-l,0}\,c^{(132|\ldots)\pm}_{l+m,0}\right], \tag{A.12}$$

$$c^{(213|\ldots)\pm}_{m,2} = \sum_{l=0}^{1-m}\left[\beta^{\pm}_{-l-2,2}\,c^{(123|\ldots)\pm}_{l+m+2,0}+\beta^{\pm}_{-l,0}\,c^{(123|\ldots)\pm}_{l+m,2}+\alpha^{\pm}_{-l-2,2}\,c^{(132|\ldots)\pm}_{l+m+2,0}+\alpha^{\pm}_{-l,0}\,c^{(132|\ldots)\pm}_{l+m,2}\right]. \tag{A.13}$$

A.4 Relations for bubble coefficients

Finally, we use the BCJ identities in order to determine relations between the coefficients of the bubble contributions shown in figure 13. The double cut with external ordering $\{1,2,3\ldots,n\}$ is given by

$$C^{\pm}_{123|4\ldots n} = A_5^{\text{tree}}\left(-l_1^{\pm},1,2,3,l_4^{\pm}\right) A_{n-1}^{\text{tree}}\left(-l_4^{\pm},P_{4\cdots n},l_1^{\pm}\right), \tag{A.14}$$

whereas $C^{\pm}_{132|4\ldots n}$ and $C^{\pm}_{213|4\ldots n}$ are obtained from the corresponding permutations of {1,2,3}.

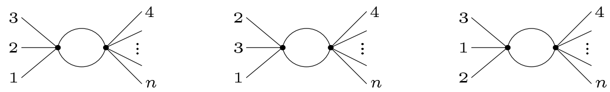

Figure 13. Bubble topologies for the cuts $C_{123|4\ldots n}$, $C_{132|4\ldots n}$ and $C_{213|4\ldots n}$.

Hence, thanks to (2.32), we can identify

$$C^{\pm}_{213|4\ldots n} = \frac{\left(P^2_{l_4^{\pm}2} + P^2_{23} - \mu^2\right) C^{\pm}_{123|4\ldots n} + \left(P^2_{l_4^{\pm}2} - \mu^2\right) C^{\pm}_{132|4\ldots n}}{\left(P^2_{-l_1^{\pm}2} - \mu^2\right)}. \tag{A.15}$$

As we did in section 3.4, after taking the $t \to \infty$ limit of the two ratios of propagators,

$$\begin{aligned} \left.\frac{P^2_{l_4^{\pm}2} - \mu^2}{P^2_{-l_1^{\pm}2} - \mu^2}\right|_{t\to\infty} &= \sum_{l=-2}^{0} \sum_{m=0}^{-l} \alpha^{\pm}_{l,m,0}\, t^l\, y^m + \frac{\mu^2}{t^2} \alpha^{\pm}_{-2,0,2} + \mathcal{O}\left(\frac{1}{t^3}\right), \\ \left.\frac{P^2_{l_4^{\pm}2} + P^2_{23} - \mu^2}{P^2_{-l_1^{\pm}2} - \mu^2}\right|_{t\to\infty} &= \sum_{l=-2}^{0} \sum_{m=0}^{-l} \beta^{\pm}_{l,m,0}\, t^l\, y^m + \frac{\mu^2}{t^2} \beta^{\pm}_{-2,0,2} + \mathcal{O}\left(\frac{1}{t^3}\right), \end{aligned} \tag{A.16}$$

we can substitute the expansion (3.35) for the three cuts in (A.15) and determine the coefficients of $C^{\pm}_{213|4\ldots n}$ from the knowledge of the ones of $C^{\pm}_{123|4\ldots n}$ and $C^{\pm}_{132|4\ldots n}$,

$$c^{(213|\ldots)\pm}_{l,m,0} = \sum_{r=l}^{2} \left(\sum_{s=\max[0,l+m-r]}^{\min[m,2-r]} \left(\alpha^{\pm}_{l-r,m-s,0}\, c^{(132|\ldots)\pm}_{r,s,0} + \beta^{\pm}_{l-r,m-s,0}\, c^{(123|\ldots)\pm}_{r,s,0} \right) \right), \tag{A.17a}$$

$$c^{(213|\ldots)\pm}_{0,0,2} = \alpha^{\pm}_{-2,0,2}\, c^{(132|\ldots)\pm}_{2,0,0} + \alpha^{\pm}_{0,0,0}\, c^{(132|\ldots)\pm}_{0,0,2} + \beta^{\pm}_{-2,0,2}\, c^{(123|\ldots)\pm}_{2,0,0} + \beta^{\pm}_{0,0,0}\, c^{(123|\ldots)\pm}_{0,0,2}. \tag{A.17b}$$

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