Published for SISSA by Springer

Received: October 30, 2015
Revised: March 21, 2016
Accepted: April 6, 2016
Published: April 15, 2016

On gauge enhancement and singular limits in G_2 compactifications of M-theory

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Abstract: We study the physics of singular limits of G_2 compactifications of M-theory, which are necessary to obtain a compactification with non-abelian gauge symmetry or massless charged particles. This is more difficult than for Calabi-Yau compactifications, due to the absence of calibrated two-cycles that would have allowed for direct control of W-boson masses as a function of moduli. Instead, we study the relationship between gauge enhancement and singular limits in G_2 moduli space where an associative or coassociative submanifold shrinks to zero size; this involves the physics of topological defects and sometimes gives indirect control over particle masses, even though they are not BPS. We show how a lemma of Joyce associates the class of a three-cycle to any U(1) gauge theory in a smooth G_2 compactification. If there is an appropriate associative submanifold in this class then in the limit of nonabelian gauge symmetry it may be interpreted as a gauge theory worldvolume and provides the location of the singularities associated with non-abelian gauge or matter fields. We identify a number of gauge enhancement scenarios related to calibrated submanifolds, including Coulomb branches and non-isolated conifolds, and also study examples that realize them.

Keywords: M-Theory, Superstring Vacua

ArXiv ePrint: 1507.05965

doi:10.1007/JHEP04(2016)100

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1 Introduction

M-theory compactifications to four dimensions are of fundamental physical interest. Since the weakly coupled superstring theories can be obtained as limits of M-theory [1–3], it is reasonable to expect that M-theory compactifications give a broad view of the four-dimensional landscape. However, detailed progress has historically been limited by both our primitive understanding of M-theory and also a lack of control over the relevant seven-manifolds, which are chosen to have G_2 holonomy in order to preserve $\mathcal{N} = 1$ supersymmetry in four dimensions [4].

Seven-manifolds with G_2 holonomy, which we shall simply refer to as G_2 manifolds, are notoriously difficult to construct, in part due to the lack of an analog of Yau's theorem that might provide a topological criterion sufficient for the existence of a G_2 metric. Additionally, they are real manifolds and therefore cannot be described directly using the techniques of complex algebraic geometry. The first compact G_2 manifolds were constructed by Joyce [5, 6] by resolving singularities on orbifolds; another construction was subsequently discovered by Kovalev [7] utilizing "twisted connected sums." In recent years, there has been much progress [8, 9] in using generalizations of Kovalev's construction to produce large classes of compact G_2 manifolds. For example, [10] identified fifty million appropriate "matching pairs" of asymptotically cylindrical Calabi-Yau threefolds that can be used to construct compact G_2 manifolds; perhaps this should be compared to the five hundred million reflexive polytopes that can be used to construct Calabi-Yau threefolds. See [11] for a recent study of M-theory on twisted connected sum G_2 manifolds.

Encouraged by this significant mathematical progress, we will study some interesting open questions concerning G_2 compactifications of M-theory. Our aims are twofold. First, as the number of compact examples has grown we will study the physics of G_2 compactifications utilizing objects that are natural in global compactifications, rather than relying on local descriptions. Second, since M-theory on a smooth G_2 manifold can realize at most an abelian gauge sector without massless charged matter, we will study various limits in moduli space that may give rise to non-abelian gauge enhancement or massless charged matter. In such limits, the G_2 manifold X must develop singularities. The non-abelian gauge group is determined by the structure of singularities in codimension four while the non-chiral and chiral matter spectra are determined by codimension six and seven singularities, respectively. We note that there is an extensive literature on this topic, particularly in the local case, including early non-compact work on singularities and gauge enhancement [12–14], the relationship to anomaly cancellation [15], the study of metrics with G_2 holonomy including singular limits [16, 17], uplifts from chiral type IIa models with intersecting D6-branes [18, 19], brane probes [20] and fibrations [21], the physical structure of singularities ([22] and references therein), a G_2-motivated extension of the MSSM [23] and related phenomenological models [24].

Regarding gauge enhancement, there are critical differences between G_2 and Calabi-Yau compactifications of M-theory; namely, the method typically utilized to study gauge enhancement in Calabi-Yau compactifications cannot be applied directly in the case of G_2 compactifications. That method is to choose a smooth manifold in which the mass of W-bosons is determined by the volume of particular two-cycles and then to take a limit in the moduli space such that the two-cycles shrink to zero volume and gauge enhancement occurs. The method is reliable for Calabi-Yau compactification because the Kähler form calibrates two-cycles, and thus the volumes of holomorphic curves (which determine W-boson masses) can be reliably computed as functions of the Kähler moduli. Physically, this correlates with the fact that Calabi-Yau compactifications of M-theory have BPS particles. In contrast, there is no calibration form for two-cycles in a G_2 manifold and there are no BPS particles in G_2 compactifications of M-theory, and therefore one must find other means to study gauge enhancement. This is the subject of the current paper.[1]

[1]For an earlier analysis of another situation in which BPS particles do not control gauge enhancement, see [25]. In that case, BPS strings provided the confining flux tubes emanating from magnetically charged states. For Calabi-Yau cases in which gauge enhancement occurs when two-cycles collapse due to movement in complex structure moduli, rather than Kähler moduli, see [26–28].

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Earlier studies of gauge enhancement [12–14] used a weak coupling limit which geometrically corresponds to making the volume of the singular locus large; this is difficult to control in G_2 moduli space. Our approach will be to study gauge enhancement via limits in G_2 moduli space in which an associative threefold or coassociative fourfold shrinks to zero size. These limits are closely related to the physics of domain walls, instantons, and strings obtained from wrapped M-branes, but we will also see in some scenarios that they are related to charged matter. We will show that a lemma of Joyce determines the class of a three-cycle for any U(1) gauge theory in a smooth G_2 compactification and that, if this class is realized by a submanifold satisfying some additional assumptions, then in the limit of nonabelian gauge symmetry it is natural to interpret the submanifold as the location of the gauge theory worldvolume. (Instantons and 't Hooft-Polyakov monopoles play an interesting role in this analysis.) In such a case the submanifold provides a useful guide for gauge enhancement, since the appearance of singularities in or along it may correspond to the appearance of massless charged particle states. We will also identify a number of different scenarios in which it is possible to control charged particle masses indirectly via calibrated submanifolds, and also discuss the physics of defects (such as BPS strings) associated with symmetry breaking.

We begin in section 2 by reviewing elements of smooth G_2 compactifications of M-theory. In section 3 we discuss the parameter space of G_2 manifold structures (up to diffeomorphism) on a fixed differentiable manifold and make a natural proposal concerning its boundary. In section 4 we will study gauge enhancement broadly in G_2 compactifications, discussing associated topological defects and singular limits. BPS defects can be studied as functions of moduli since they are obtained by wrapping M2-branes and M5-branes on calibrated submanifolds; a natural singular limit of X is to tune the volume of these cycles to zero. Additionally, there is a moduli-dependent inequality due to Joyce that determines the class of a non-trivial three-cycle that is useful for thinking about non-abelian gauge enhancement. We will use these ideas in section 5 to study the physics of a number of different scenarios, including Coulomb branches (associated to the breaking of a non-abelian group by an adjoint chiral multiplet) and also conifold transitions. Finally, in section 6 we will apply these techniques to concrete examples and see the appearance of some well-known defects that appear in broken gauge theories, for example the 't Hooft-Polyakov monopole [29, 30].

2 Compactifications of M-theory on G_2 manifolds

In this section we will review elementary facts about Kaluza-Klein reduction of eleven dimensional supergravity on seven manifolds with G_2 holonomy.

G_2 manifolds. First, let us review some basic facts about G_2 manifolds. See [9, 31] for further details.

A G_2*-structure* on a seven-manifold X is a principal subbundle of the frame bundle of X which has structure group G_2. Practically, each G_2 structure gives rise to a three-form Φ and a metric g_Φ such that every tangent space of X admits an isomorphism with $\mathbb{R}^7$

which identifies g_Φ with $g_0 \equiv dx_1^2 + \cdots + dx_7^2$ and identifies Φ with

$$\Phi_0 = dx_{123} + dx_{145} + dx_{167} + dx_{246} - dx_{257} - dx_{347} - dx_{356}, \tag{2.1}$$

where $dx_{ijk} \equiv dx_i \wedge dx_j \wedge dx_k$. Note that the subgroup of $\mathrm{GL}(7,\mathbb{R})$ which preserves Φ_0 is the exceptional compact Lie group G_2. The three-form Φ, sometimes called the *G_2-form*, determines an orientation, a Riemannian metric g_Φ, and a Hodge star[2] operator $\star_\Phi$. We will refer to the pair (Φ, g_Φ) as a G_2-structure.

For a seven-manifold X with a G_2-structure (Φ, g_Φ) and associated Levi-Civita connection ∇, the *torsion* of the G_2-structure is $\nabla\Phi$, and when $\nabla\Phi = 0$ the G_2 structure is said to be *torsion-free*. The following are equivalent:

- $Hol(g_\Phi) \subseteq G_2$
- $\nabla\Phi = 0$, and
- $d\Phi = d \star_\Phi \Phi = 0$.

The triple (X, Φ, g_Φ) is called a *G_2-manifold* if (Φ, g_Φ) is a torsion-free G_2-structure on X. Then by the above equivalence, the metric g_Φ has $Hol(g_\Phi) \subseteq G_2$ and g_Φ is Ricci-flat. For a compact G_2-manifold X, $Hol(g_\Phi) = G_2$ if and only if $\pi_1(X)$ is finite [6]. In this case the moduli space of metrics with holonomy G_2 is a smooth manifold of dimension $b_3(X)$.

We assume for simplicity that the cohomology of X is torsion-free. Let us set the notation we will use for classical intersection numbers. We take Poincaré dual integral cohomology bases $\sigma_I \in H^2(X,\mathbb{Z})$ and $\tilde{\sigma}^I \in H^5(X,\mathbb{Z})$ and corresponding dual homology bases $\Sigma^I \in H_2(X,\mathbb{Z})$ and $\tilde{\Sigma}_I \in H_5(X,\mathbb{Z})$. These satisfy

$$\int_X \sigma_I \wedge \tilde{\sigma}^J = \tilde{\Sigma}_I \cdot \Sigma^J = \delta_I^J = \sigma_I(\Sigma^J) = \tilde{\sigma}^J(\tilde{\Sigma}_I). \tag{2.2}$$

Similarly, we take Poincaré dual integral cohomology bases $\Phi_i \in H^3(X,\mathbb{Z})$ and $\tilde{\Phi}^i \in H^4(X,\mathbb{Z})$, along with the corresponding dual homology bases $T^i \in H_3(X,\mathbb{Z})$ and $\tilde{T}_i \in H_4(X,\mathbb{Z})$. These satisfy

$$\int_X \Phi_i \wedge \tilde{\Phi}^j = \tilde{T}_i \cdot T^j = \delta_i^j = \Phi_i(T^j) = \tilde{\Phi}^j(\tilde{T}_i). \tag{2.3}$$

We use capital Latin indices for two-forms, five-forms, two-cycles, and five-cycles; similarly, we use lower case Latin indices for three-forms, four-forms, three-cycles, and four-cycles. The classical intersection numbers in a simply-connected G_2 manifold are

$$C_{IJk} \equiv \int_X \sigma_I \wedge \sigma_J \wedge \Phi_k = \tilde{\Sigma}_I \cdot \tilde{\Sigma}_J \cdot \tilde{T}_k \tag{2.4}$$

and, as one would expect, these numbers play a critical role in determining the physics of the four-dimensional $\mathcal{N} = 1$ compactification.

[2] For the four- and eleven-dimensional Hodge star, we will use $\star_4$ and $\star_{11}$, respectively.

Since X is a manifold with G_2 structure, vectors and differential forms on X fall into G_2-representations. The decomposition of forms into G_2 representations extends to cohomology [32]:

$$\begin{aligned} H^2(X,\mathbb{R}) &= H^2_7 \oplus H^2_{14} \\ H^3(X,\mathbb{R}) &= H^3_1 \oplus H^3_7 \oplus H^3_{27} \end{aligned} \tag{2.5}$$

where the summands are defined as[3]

$$\begin{aligned} H^2_7 &= \{\star(\alpha \wedge \star\Phi) \mid \alpha \in H^1(X,\mathbb{R})\} \\ &= \{\sigma \in H^2(X,\mathbb{R}) \mid \star(\sigma \wedge \phi) = 2\sigma\} \\ H^2_{14} &= \{\sigma \in H^2(X,\mathbb{R}) \mid \star(\sigma \wedge \phi) = -\sigma\} \\ H^3_1 &= \{r\,\Phi \mid r \in \mathbb{R}\} \\ H^3_7 &= \{\star(\alpha \wedge \Phi) \mid \alpha \in H^1(X,\mathbb{R})\} \\ H^3_{27} &= \{\alpha \in H^3(X,\mathbb{R}) \mid \alpha \wedge \Phi = 0 \ \text{ and } \ \alpha \wedge \star\Phi = 0\}. \end{aligned} \tag{2.6}$$

It will be important for us that if X has holonomy exactly G_2, instead of simply being a manifold with G_2 structure, then H^2_7 is empty and therefore $H^2(X,\mathbb{R}) = H^2_{14}$.

This last observation is useful for the following reason. Consider any harmonic two-form σ. Then we have [32]

$$\star_\Phi \sigma = -\sigma \wedge \Phi. \tag{2.7}$$

This is a nice relation; the Hodge star $\star_\Phi$ explicitly depends on the metric, but this dependence is encoded in a simple way in the metric moduli. Therefore, anywhere the Hodge star of a two-form appears in a Kaluza-Klein reduction, the moduli dependence implicit in $\star_\Phi$ can be encoded directly in terms of Φ. For example for harmonic two-forms σ^1 and σ^2,

$$\int \sigma^1 \wedge \star_\Phi \sigma^2 = -\int \sigma^1 \wedge \sigma^2 \wedge \Phi = -\int (s^1_I\,\sigma_I) \wedge (s^2_J \wedge \sigma_J) \wedge (\phi_k\,\Phi_k) = -s^1_I s^2_J \phi_k\, C_{IJk}. \tag{2.8}$$

Expressions of this form determine the structure of abelian gauge coupling functions and kinetic mixings obtained from Kaluza-Klein reduction, for example.

Calibrated geometry for G_2 manifolds. In both the Calabi-Yau and G_2 manifold cases, we lack explicit knowledge of the metric. However, in both cases, the volumes of certain cycles in this metric can be computed via calibrated geometry as developed in the seminal work of Harvey and Lawson [33]. Their fundamental observation is the following. Let X be a Riemannian manifold and α a closed p-form such that $\alpha|_\xi \leq vol_\xi$ for all oriented tangent p-planes ξ on X. Then any compact oriented p-dimensional submanifold T of X with the property that $\alpha|_T = vol_T$ is a minimum volume representative of its homology class, that is

$$vol(T) = \int_T \alpha = \int_{T'} \alpha \leq vol(T') \tag{2.9}$$

[3]Since this section is devoted to descriptions in cohomology rather than specific forms, we suppress the dependence of $\star$ on the G_2 form.

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for any T' such that $[T-T']=0$ in $H_p(X,\mathbb{R})$. Note in particular the useful fact that $vol(T)$ is computed precisely by $\int_T \alpha$, even though one may not know the metric on X explicitly.

If X is a Calabi-Yau threefold, the Kähler form J, the holomorphic three-form Ω, and the square of the Kähler form $J\wedge J$ are calibration forms for two-cycles, three-cycles, and four-cycles; they calibrate holomorphic curves, special Lagrangian submanifolds, and holomorphic divisors. Note that, in M-theory compactifications on X the presence of calibrated two-cycles allows for control over massive charged particle states obtained from wrapped M2-branes [34]. This computes particle masses as a function of moduli.

If X is a G_2 manifold, Φ and $\star_\Phi \Phi$ are calibration forms which calibrate so-called associative three-cycles and coassociative four-cycles, respectively. As we will discuss, this allows for control over topological defects obtained from wrapping M2-branes and M5-branes on calibrated three-cycles and four-cycles; these are instantons, domain walls, and strings. Note the absence of calibrated two-cycles, however.

Elements of Kaluza-Klein reduction. While we do not aim to be exhaustive, we would like to briefly review certain aspects of the Kaluza-Klein reduction of eleven dimensional supergravity on X, pioneered in [4]. We ignore the α' corrections, which have been considered recently [35–37].

Consider a compactification of M-theory on a smooth G_2 manifold X at large volume, where the metric of X is determined by a torsion-free G_2 form $\Phi\in H^3(X)$, and $\Psi\equiv\star_\Phi\Phi$ is the dual four-form. This gives rise to a four-dimensional $\mathcal{N}=1$ effective theory, which can be obtained via Kaluza-Klein reduction of 11-dimensional supergravity. In order to perform this reduction, we expand the M-theory three-form C_3 as

$$C_3 = A_I\wedge\sigma_I + \theta_i\wedge\Phi_i, \tag{2.10}$$

where the A_I are four-dimensional abelian vector potentials and the θ_i are pseudo-scalars in four-dimensions. The latter, together with the scalars obtained from reduction of metric moduli via the G_2-form Φ, form complex scalars that sit in chiral multiplets. Similarly, the dual six-form can be expanded as

$$C_6 = \tilde{A}^I\wedge\tilde{\sigma}^I + B^i\wedge\tilde{\Phi}^i, \tag{2.11}$$

where $\tilde{A}^I$ are the dual magnetic vector potentials upon reduction to four dimensions and the B^i are two-forms which are four-dimensional Hodge duals of the θ, i.e. $d\theta=\star_4\, dB$. We call them B^i rather than $\tilde{\theta}^i$ to align conventions with the weakly coupled type II superstring literature, where such two-forms B give rise to the $\int d^4x\, B\wedge F$ type Stückelberg couplings critical for anomaly cancellation via the Green-Schwarz mechanism. In summary, the supergravity sector of M-theory on X gives rise to $b_3(X)$ neutral chiral multiplets and $b_2(X)$ U(1) vector multiplets.

The bosonic part of the eleven-dimensional supergravity action is given by

$$S=\frac{1}{2\kappa_{11}^2}\left(\int d^{11}x\,\sqrt{-g}\,\left(R-\frac{1}{2}|G_4|^2\right)-\frac{1}{6}\int C_3\wedge G_4\wedge G_4\right) \tag{2.12}$$

where G_4 is the field strength of the M-theory three-form. The Chern-Simons term gives rise to abelian θ-terms in the four-dimensional action

$$S_{4d,\text{theta}} = -\frac{1}{12\kappa_{11}^2} C_{iJK} \int \theta_i \, F_J \wedge F_K. \tag{2.13}$$

The kinetic term for the M-theory three-form gives rise to kinetic terms for the four-dimensional gauge fields. Generically in a four-dimensional $\mathcal{N} = 1$ supergravity theory these are of the form

$$S_{4d,\text{kinetic}} = -\frac{1}{4} Re\, f_{JK}(\Phi) \int F_J \wedge \star_4 F_K \tag{2.14}$$

where the prefactor f_{JK} is the gauge coupling function. These kinetic terms arise from the $\int G_4 \wedge \star_{11} G_4$ terms in (2.12) and their dependence on G_2 takes the form $\int_X \sigma_J \wedge \star_\Phi \sigma_K$ which can be rewritten in terms of G_2 moduli using the property $\star_\Phi \sigma = -\sigma \wedge \Phi$ characteristic of $H^2_{14}(X, \mathbb{Z})$. We will use this moduli dependence in the following.

Alternatively, a different Lagrangian for eleven-dimensional supergravity arises if different mass dimensions are chosen for the three-form field in the theory. Following [38], taking $2\kappa_{11}^2 = l_{11}^9/2\pi$, $\tilde{C}_3 = C_3/l_{11}^3 = C_3/(4\pi\kappa_{11}^2)^{1/3}$, and associated field strength $\tilde{G}_4$ the action is

$$S_{11} = 2\pi \left(\frac{1}{l_{11}^9} \int d^{11}x \, \sqrt{-g} R - \frac{1}{2l_{11}^3} \int d^{11}x \, |\tilde{G}_4|^2 - \frac{1}{6} \int d^{11}x \, \tilde{C}_3 \wedge \tilde{G}_4 \wedge \tilde{G}_4 \right). \tag{2.15}$$

Of course, changing the mass dimensions in this way affects the induced mass dimensions of four-dimensional fields upon compactification, and therefore also the mass dimensions of the four-dimension coupling constants. We note this alternate form of the action since it gives rise to canonical mass dimensions for four-dimensional coupling constants, such as a dimensionless gauge coupling.

3 The boundary of the moduli space

In this section we will review what is known about the boundary of the Kähler moduli space of Calabi-Yau threefolds, and point out analogies with the moduli space of G_2 metrics on a fixed differentiable manifold. We identify limits in the moduli space that necessarily lead to singularities, and speculate that all such limits take this general form. In the case of Calabi-Yau threefolds, thanks to the theorems of Calabi [39] and Yau [40], the Kähler moduli space can be identified with a subset of the second cohomology group, which in fact forms a cone.

According to Wilson [41], the Kähler cone of a Calabi-Yau threefold Z is completely determined by three conditions on the Kähler form ω:

$$\text{(1) a topological condition:} \quad \int_Z \omega \wedge \omega \wedge \omega > 0, \tag{3.1}$$

$$\text{(2) a characteristic class condition:} \quad \int_Z p_1(Z) \wedge \omega < 0, \text{ and} \tag{3.2}$$

(3) two calibrated cycle conditions:

$$\text{for every effective algebraic curve } C, \quad \int_C \omega > 0, \text{ and} \tag{3.3}$$

$$\text{for every effective algebraic surface } S, \quad \int_S \star\omega > 0. \tag{3.4}$$

The characteristic class condition [42] is usually stated in terms of $c_2(Z) = -\frac{1}{2}p_1(Z)$ rather than $p_1(Z)$, and the condition on surfaces, which is normally written in the form $\int_S \omega \wedge \omega > 0$ [43, 44], is usually omitted entirely since it follows from the condition on curves [45].

As Wilson further explains, except for the theoretical possibility of a curved part of the boundary of the cone (for which no examples are known), the boundaries of the cone either correspond to curves or divisors that collapse to lower dimension as the boundary is approached, or correspond to fibrations by elliptic curves or K3 surfaces where the fibers collapse at the boundary. When the Kähler cone is complexified, the codimension one boundary components become divisors on the boundary of the moduli space, and the complexified cone itself becomes a neighborhood of an point of intersection of such divisors [46].

By mirror symmetry, a similar structure holds near large complex structure limit points in the complex structure moduli space. Such a point will be the intersection of divisors along which singularities are acquired via variation of complex structure; for such singularities, there are always "vanishing cycles," which are 3-cycles whose volume goes to zero at the divisor. Although we do not have mathematical results which guarantee that those 3-cycles will have special Lagrangian representatives, by mirror symmetry, they must (since they are mirrors of calibrated cycles, i.e., the algebraic cycles on the mirror which collapse). We thus see a very similar structure for complex moduli.

We propose here an analogous structure for the parameter space of G_2 manifold structures up to diffeomorphism-isotopic-to-the-identity on a fixed differentiable manifold X. By a result of Joyce ([5], Theorem C) by associating the cohomology class $[\Phi]$ to each G_2-manifold structure, we can identify this parameter space with an open subset of $H^3(X, \mathbb{R})$.

According to Joyce ([6], Lemma 1.1.2) and Harvey-Lawson [33], there are three conditions which such a the cohomology class of a G_2 form Φ must satisfy:

$$\text{(1) a topological condition: } \int_X \sigma \wedge \sigma \wedge \Phi < 0 \text{ for every } \sigma \in H^2(X, \mathbb{R}), \tag{3.5}$$

(2) a characteristic class condition: $\displaystyle\int_X p_1(X) \wedge \Phi < 0$, or more generally,[4] for any bundle

(3.6)

$$E \text{ admitting a ``}G_2\text{-instanton'' [48, 49]}, \quad \int_X p_1(E) \wedge \Phi < 0, \text{ and} \tag{3.7}$$

[4] We thank Thomas Walpuski for bringing this condition to our attention. The fact that the Levi-Civita connection is a G_2-instanton on the tangent bundle of X is pointed out in Example 3.4 of [47]; this is why $p_1(X) = p_1(TX)$ occurs as a special case.

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(3) two calibrated cycle conditions:[5]

$$\text{for every associative cycle } A, \quad \int_A \Phi > 0, \text{ and} \tag{3.8}$$

$$\text{for every coassociative cycle } C, \quad \int_C \star_\Phi \Phi > 0. \tag{3.9}$$

Unlike the case of the Kähler cone, however, there is no evidence that these conditions determine the parameter space completely.. However, since no other phenomena are known which lead to constraints on the G_2 parameter space, we propose that these known conditions should be a complete list. In particular, in this paper we shall attempt to study limits of smooth G_2 manifolds by studying the approach to boundary components which are defined by the conditions above.

Under appropriate assumptions about the behavior of the volume and diameter of the metric as the boundary of the parameter space is approached, the Gromov-Hausdorff theory of limiting Riemannian metrics [50] can be used to show that the singular limit has the structure of a G_2 manifold away from a subset of real codimension 4 [51], and that the generic singularities in real codimension 4 are orbifold singularities of the type which lead to non-abelian gauge symmetry in the compactification of M-theory. Any further singularities are in codimension 6 or codimension 7, and in fact the physics of isolated codimension 7 singularities has been studied extensively. In particular, for the examples which have been analyzed in detail [13], there is always a vanishing 3-cycle[6] or a vanishing 4-cycle (associative, or coassociative, respectively). This is the phenomenon we are proposing for the general case.

4 Gauge enhancement, defects, and singular limits

In this section we will study gauge enhancement in G_2 compactifications of M-theory and its relationship to topological defects and singular limits of X. We will first present the logic for studying gauge enhancement via singular limits in which calibrated submanifolds vanish. After presenting the logic, we will study topological defects and singular limits in detail, and then will conclude the section with some further comments on gauge enhancement.

Before proceeding, we would like to re-emphasize a main point of our work discussed in the introduction: understanding and controlling gauge enhancement is more difficult in G_2 compactifications than in Calabi-Yau compactifications, since (unlike Calabi-Yau manifolds) there is no calibration form for two-cycles, and therefore it is more difficult to control W-boson masses as a function of moduli. Therefore a controlled description of

[5] There is a potential problem with this formulation: the set of cycles with associative or coassociative representatives may depend on Φ. Thus, some of these conditions may not actually generate boundary walls, since by the time the integral vanishes, the cycle may no longer be associative and the condition may no longer apply. We thank Thomas Walpuski for a discussion on this point.

[6] As pointed out in [13], in the case of vanishing 3-cycles there can be membrane instantons which affect the C-field superpartner of Φ, and cause the quantum moduli space to differ from the classical moduli space.

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non-abelian gauge enhancement requires either gaining some mathematical control over two-cycles or utilizing different physical techniques; we will do both, though two-cycles will only be controlled indirectly.

4.1 Logic behind gauge enhancement in G_2 compactifications

We use the following logic as a guide for obtaining either non-abelian gauge enhancement or the existence of massless charged particles in limits of G_2 compactifications.

Since compactification of M-theory on a smooth G_2 manifold gives rise to an abelian theory with no massless charged fields, the existence of non-abelian gauge symmetry or massless charged fields in a these compactifications requires taking a singular limit in the metric moduli space. Conversely, if the singular limit gives rise to non-abelian gauge symmetry, the process of returning to a smooth G_2 manifold from the singular limit[7] necessarily corresponds to spontaneous symmetry breaking via the Higgs mechanism. The D-flat direction associated to the breaking has a corresponding massless scalar fluctuation from the broken vacuum, represented by a supergravity mode; as such, $b_3(X)$ necessarily increases in the Higgsing process. Since the metric on the G_2 manifold X is determined by the G_2-form Φ, taking a singular limit of X requires performing a deformation $\Phi \mapsto \Phi + \delta\Phi$, which also determines a deformation of the associated four-form $\star_\Phi \Phi \mapsto \star_{\Phi+\delta\Phi}(\Phi + \delta\Phi)$.

How might one detect such a singularity, necessary for the existence of non-abelian gauge symmetry or massless charged matter, after performing a Φ-deformation? Unlike in complex algebraic geometry, G_2 manifolds do not admit the luxury of detecting singularities via the structure of algebraic equations; furthermore, though Φ determines a metric, its form is not known in general compact examples. Given current knowledge, as explained in the previous section, it seems to us that the most natural way to do so is to perform a Φ-deformation of X which shrinks an associative and / or coassociative submanifold to zero size. Certainly this is sufficient to produce a singularity.

We therefore propose studying the physics of gauge enhancement via its relation to vanishing associative or coassociative submanifolds. This includes the physics of M2-branes and M5-branes wrapped on such calibrated submanifolds, which give rise to instantons, strings, and domain walls. These defects can be controlled because they wrap calibrated submanifolds in the geometry: they are BPS defects. We will see the appearance of other topological defects, such as 't Hooft-Polyakov monopoles, but these cannot be controlled by calibrated geometry, as expected since there are no BPS particles in $d = 4$ $\mathcal{N} = 1$ gauge theories. They will still be useful, though, as we will see in section 5.1 and the first example of section 6.2.

4.2 Defects from wrapped membranes and five-branes

In addition to contributions from Kaluza-Klein reduction of eleven-dimensional supergravity, the physics of M-theory compactifications depend also on the physics of wrapped M2-branes and M5-branes. Depending on the details of the wrapped cycles in X, a variety of

[7]It is possible that if singularities of higher codimension are also present, they might obstruct any "return to a smooth G_2 manifold".

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objects can appear in the non-compact spacetime. These include not only ordinary particles, but also a variety of topological defects such as instantons, monopoles, strings, and domain walls. In some cases these objects that arise in M-theory compactifications can be identified with objects known to exist in spontaneously broken gauge theories, and in some cases we will see their parametric dependence on the expectation values of scalar fields via the structure of the geometry.

Charged particles and monopoles from wrapped branes and topological protection. Let us begin by considering particle states arising from wrapped branes. Suppose that X has[8] non-trivial positive two-cycles Σ_I, $I \in \{1, \dots, b_2(X)\}$; then since X is a smooth manifold, it also has dual five-cycles $\tilde{\Sigma}_I$, and if these cycles are submanifolds then particles in four-dimensions arise from wrapping M2-branes and M5-branes on Σ_I and $\tilde{\Sigma}_I$, respectively.

There are two important physical consequences of the fact that these particles arise from wrapping branes on non-trivial cycles Σ_I and $\tilde{\Sigma}_I$. The first is related to intersection theory. To see this, recall that the Kaluza-Klein ansätz for C_3 and also C_6 takes the form

$$C_3 = A_I \wedge \sigma_I + \cdots \qquad \text{and} \qquad C_6 = \tilde{A}^I \wedge \tilde{\sigma}^I + \cdots\,, \tag{4.1}$$

where A and $\tilde{A}$ are electric and magnetic vector potentials in four dimensions. An M2-brane (for example) on $L \times \Sigma$ where $\Sigma \equiv n^I \Sigma^I$ and L is the worldline of the spacetime particle couples as

$$\int_{L\times\Sigma} C_3 = \sum_I n^I \int_{L\times\Sigma^I} A_J \wedge \sigma_J = n^I \int_L A_I, \tag{4.2}$$

by virtue of (2.2). This is the coupling of a particle with charge n^I under the I^{th} U(1) associated to the vector potential A_I. These massive particles could be electrons or W-bosons; it would be interesting to derive their spacetime quantum numbers explicitly.[9] The topologically non-trivial two-cycles are related to the existence of geometric[10] U(1) symmetries, and the topology determines an observable in the four-dimensional theory, namely, the charge of the particle.

The second physical point is related to stability: an M2-brane wrapped on a non-trivial volume-minimizing two-cycle Σ could be stable against decay due to charge conservation in the low-energy theory; this charge conservation would arise from topological protection. This should be contrasted with a vacuum where the gauge symmetry is completely broken. There, the W-bosons are not charged, therefore not protected, and have significantly different physics; this correlates with the disappearance of non-trivial two-forms after the abelian theory is Higgsed. If the corresponding two-cycles are still present in the geometry, they must be trivial in homology.

[8] Recall that unlike for Calabi-Yau manifolds, a G_2 manifold may have $b_2(X) = 0$.

[9] In M-theory on Calabi-Yau manifolds, the corresponding particles are BPS and this fact is used in deriving the spacetime quantum numbers [34, 52].

[10] Here we study U(1)'s at the level of geometry, and currently have nothing to say about whether other couplings in the theory may give them a mass.

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Submanifold	Dimension	M2	M5
Σ	2	charged particle	spacetime filling brane
T	3	spacetime instanton	domain wall
$\tilde{T}$	4	—	axionic string
$\tilde{\Sigma}$	5	—	magnetic monopole

Table 1. G_2 manifolds can have non-trivial two-cycles, three-cycles, four-cycles, and five-cycles. By wrapping M2-branes and M5-branes on cycles that are submanifolds, a variety of defects can arise in four-dimensions; only those associated to calibrated three-cycle and four-cycles are BPS defects.

Wrapping M5-branes on 5-cycles can also give rise to particles; doing so on $L \times \tilde{\Sigma}$, where $\tilde{\Sigma} = n_I \tilde{\Sigma}_I$ and L is its worldline, we see that it couples as

$$\int_{L\times\tilde{\Sigma}} C_6 = \int_{L\times n_I\tilde{\Sigma}_I} \tilde{A}^J \wedge \tilde{\sigma}^J = n_I \int_L \tilde{A}^I \tag{4.3}$$

by virtue of (2.2), where n_I is the charge of the particle under the I^{th} *magnetic* U(1) vector potential $\tilde{A}^I$. This is a magnetic monopole. If the U(1)'s of the vacuum state obtained from M-theory on X arise from the spontaneous breaking of a non-abelian gauge theory, and new non-trivial five-manifolds appear in X as a result of that breaking, then these may be the standard monopoles of 't Hooft [29] and Polyakov [30]. The monopole mass is given by

$$m^2 = 16\pi^2 \frac{|v|^2}{g^2} \tag{4.4}$$

where v is the Higgs field expectation value and g is the gauge coupling. As we will see in sections 5 and 6, this $|v|^2/g^2$ dependence can be understood geometrically.

BPS defects: instantons, strings, and domain walls. While the charged particles (both electrons and monopoles) discussed in the previous section are difficult to control geometrically, calibration of three-cycles and four-cycles in G_2 manifolds allows for control over the associated topological defects, which *are* BPS. These are instantons, strings, and domain walls from M2-branes on three-cycles, M5-branes on four-cycles, and M5-branes on three-cycles, respectively. See table 1 for a listing of the possibilities.

Let us begin by discussing a spacetime instanton obtained by wrapping an M2-brane on an associative submanifold $T \equiv n^i T^i$. The classical instanton action depends on the volume

$$vol(T) = \int_T \Phi = \int_{n^i T^i} \phi_j \Phi_j = n^i \phi_i \tag{4.5}$$

and also its coupling to the M-theory three-form

$$\int_T C_3 = \int_{n^i T^i} \theta_j \Phi_j = n^i \theta_i. \tag{4.6}$$

That is, the instanton couples to the chiral supermultiplet in four-dimensions with associated complex scalar field $n^j(\phi_j + i\theta_j)$. In the instanton background, the four-dimensional effective action receives corrections

$$\Delta S_{4d} = \int [dZ]\, A\, e^{-S_{\rm inst}} = \int [dZ] A\, e^{-n^i\,(\phi_i + i\theta_i) + \dots} \tag{4.7}$$

where $[dZ]$ represents integration over the instanton zero modes and we note the exponential dependence on the four-dimensional complex scalars $\phi_i + i\theta_i$. The instanton prefactor A has been studied in [53], though currently more can be said in Calabi-Yau compactifications of M-theory and also in F-theory; see e.g. [54].

The precise nature of the instanton correction depends critically on its zero modes; for example if there is an identification $[dZ] = d^4x\, d^2\theta$ of the instanton zero modes with the spacetime coordinates x^μ and superspace spinor θ_α of the $d = 4$ $\mathcal{N} = 1$ theory, then the instanton correction is a superpotential correction. These (x, θ) zero modes are the Goldstone bosons and Goldstinos of the spacetime translations and supersymmetries broken by the instanton. Note that, while one might assert the irrelevance of so-called $\overline{\tau}$ instanton zero modes in M-theory compactifications, since (unlike in type II compactifications) the geometric background only preserves four supercharges, it is known that in passing from type IIb to F-theory these modes are repackaged [55, 56] and there is a condition that must be checked to ensure their absence. It would be interesting to understand whether a similar repackaging occurs in the IIa to M-theory on G_2 limit.

In addition to the zero modes associated to the super-Poincaré invariance broken by the instanton, there might also be deformation zero modes. These were some of the zero modes studied by Harvey and Moore [53], who showed that an M2-brane instanton on T corrects the superpotential if T is rigid and supersymmetric, i.e. rigid and associative. Together these ensure the absence of deformation modes and the presence of θ modes, respectively. More specifically, they studied the case with $b_1(T) = 0$, and if $b_1(T) \neq 0$ the associated Wilson line modulini must also be carefully studied. The first known examples of compact rigid associative submanifolds of compact G_2 manifolds were given in [9], the form of their superpotential corrections (modulo potential complications from Wilson line modulini) were studied in [11].

Finally, note that the instanton prefactor A may not have any dependence on G_2 moduli that violates the axion shift symmetries of M-theory, and therefore it seems that any possible dependence must be exponential in the moduli. It is often stated that the prefactor must be a moduli independent constant due to the shift symmetries, but we believe that this claim cannot be consistent with M-theory lifts of IIa configurations with chiral matter prefactors [57–59] for Euclidean D2-brane instantons. In that context the prefactors are chiral supermultiplets, not constants, and we see no reason that such configurations should be absent in M-theory. Furthermore, the complex scalars in these chiral multiplets have vacuum expectation values that depend on open string moduli that are lifted in M-theory to G_2 moduli; these could be turned into G_2-moduli dependent constant prefactors by Higgsing the gauge group that charges the supermultiplets. We see no reason that such IIa configurations should be absent in M-theory lifts, and conclude only that any moduli dependence must be consistent with axionic shift symmetries, as in the case of exponential dependence.

A four-dimensional string arises from wrapping an M5-brane on a four-cycle $\tilde{T} = n_i \tilde{T}_i$, and its worldsheet in $\mathbb{R}^{3,1}$ couples to a two-form B which is the four-dimensional Hodge dual of an axion. This is straightforward to understand since the string coupling is

$$\int_{\mathrm{WS} \times n_i \tilde{T}_i} C_6 = n_i \int_{\mathrm{WS}} B^i \tag{4.8}$$

where WS is the worldsheet of the string. If T is coassociative, then the string is BPS.

On the other hand, considering just simple Lagrangian quantum field theories rather than compactifications of string theory or M-theory, Abrikosov-Nielsen-Olesen (ANO) vortex strings exist in vacua where some number of U(1) symmetries are spontaneously broken. The simplest case in field theory is given by the abelian Higgs model. This theory has $G = \mathrm{U}(1)$, but the vacuum spontaneously breaks the symmetry to a subgroup $H = \emptyset$; it is completely broken. The defects in the theory are determined by the homotopy groups $\pi_i(G/H) = \pi_i(\mathrm{U}(1))$. Thus, since $\pi_1(\mathrm{U}(1)) = \mathbb{Z}$ the theory exhibits ANO vortex strings. The tension of the critical or BPS ANO vortex string is

$$T_{\mathrm{ANO}} = 2\pi |v|^2 n, \tag{4.9}$$

where v is the vacuum expectation value of the charged scalar field which performs the breaking and n is the winding number of the vortex. There are also the semi-local strings [60] associated with U(1) symmetry breaking in theories with flat directions, which also has a tension that scales as $T \sim |v|^2$.

Finally, a domain wall arises on four-dimensions from wrapping an M5-brane on a three-cycle $T = n^i T^i$. We do not have much to say about these at this point, except that BPS domain walls may exist in four-dimensional theories. See for example [61].

4.3 Two physical singular limits

Having prepared some preliminaries, let us study two types of singular limits.

One singular limit we will study, which we have already mentioned, is when an associative T or coassociative $\tilde{T}$ submanifold goes to zero volume. Since

$$vol(T) = \int_T \Phi = \phi_i \int_T \Phi_i > 0 \tag{4.10}$$

this gives a moduli-dependent inequality that is violated in the singular limit where T vanishes, and is therefore similar to a Kähler cone condition in Calabi-Yau manifolds. A similar statement holds for coassociative submanifolds. The other singular limit we will study derives from an inequality due to Joyce that holds for any non-zero two-form class on a G_2 manifold; this makes it particularly relevant for vacua on Coulomb branches. We will see that, although the limit itself is not physically natural, this study suggests the existence of a calibrated submanifold which would be physically important; in fact, submanifolds in this class may be "gauge theory worldvolumes" in certain cases.

Boundaries from a lemma of Joyce and gauge theory worldvolumes. We begin with the singular limit associated to the Joyce lemma. Since non-trivial two-forms and two-cycles determine the structure of abelian gauge fields and charged particle states in four-dimensions, it would be useful for understanding gauge enhancement to have a G_2-moduli dependent condition which involves $H^2(X)$ or $H_2(X)$ in some way.

We recall from section 3 a lemma of Joyce (which we called there the "topological condition" on Φ): let X be a G_2 manifold with G_2-form Φ and associated class $[\Phi] \in H^3(X,\mathbb{R})$. Then for any non-zero class $[\sigma] \in H^2(X,\mathbb{R})$ we have $[\sigma] \cup [\sigma] \cup [\Phi] < 0$. For simplicity we will write the condition as

$$\int_X \sigma \wedge \sigma \wedge \Phi < 0 \tag{4.11}$$

for a representative σ of $[\sigma]$. Of course, we can restrict to integral cohomology, taking $[\sigma] \in H^2(X,\mathbb{Z})$, and there is a dual five-cycle $[\tilde{\Sigma}] \in H_5(X,\mathbb{Z})$, i.e.,

$$\eta(\tilde{\Sigma}) = \int \sigma \wedge \eta \tag{4.12}$$

for all $\eta \in H^5(X,\mathbb{Z})$. However, (4.11) also implies that $[-\sigma\wedge\sigma]$ is non-trivial[11] in $H^4(X,\mathbb{Z})$, and therefore by Poincaré duality we also have[12] a three-cycle $[D_\Sigma]$ dual to $[-\sigma\wedge\sigma]$, i.e.,

$$\xi(D_\Sigma) = -\int \sigma \wedge \sigma \wedge \xi \tag{4.13}$$

for all $\xi \in H^3(X,\mathbb{Z})$. Thus, to any two-form $[\sigma] \in H^2(X,\mathbb{Z})$ we have associated five-cycle *and* three-cycle classes

$$[\tilde{\Sigma}] \in H_5(X,\mathbb{Z}) \qquad \text{and} \qquad [D_\Sigma] \in H_3(X,\mathbb{Z}). \tag{4.14}$$

In the following we will try to consistently use bracketed expressions to denote homology classes, while those expressions without the brackets will denote submanifold (perhaps calibrated) representatives of those classes; e.g. $[D_\Sigma]$ will be a three-cycle class, but D_Σ will be an associative representative of that class.

This is a remarkable condition! It says that for *any* U(1) symmetry determined by some $\sigma \in H^2(X,\mathbb{Z})$ in a G_2 compactification of M-theory, we have a canonical map not only to the expected non-trivial five-cycle $\tilde{\Sigma}$ via duality with homology, but also a map to a non-trivial three-cycle D_Σ via duality with homology and the Joyce lemma. In general, however, there is no requirement that either of these classes have representatives that are submanifolds, let alone calibrated ones.

[11]We introduce the sign so that the inequality may in some cases be a positivity condition on an associative cycle.

[12]Since σ is non-trivial then so is $\star_\Phi\sigma$ and $\star_\Phi(-\sigma\wedge\sigma)$, and accordingly by dimension counting there must also exist an integral two-cycle Σ and four-cycle $\tilde{D}_\Sigma$. However, unlike for $\tilde{\Sigma}$ and D_Σ, there does not exist a canonical map $\sigma \to \Sigma$ or $\sigma \to \tilde{D}_\Sigma$; this is because $\star_\Phi$ introduces the metric and does not have a canonical action on integral (co)homology. Therefore, we focus on $\tilde{\Sigma}$ and D_Σ.

Though in section 5 we will present physical arguments that this is the case under some circumstances, we simply assume it for now. Then M2-branes and M5-branes can be wrapped on the representatives $\tilde{\Sigma}$ and D_Σ, giving defects in four dimensions, which are a

$$\begin{aligned} \text{Magnetic Monopole:} &\quad \text{from an M5-brane on } \tilde{\Sigma} \\ \text{Spacetime Instanton:} &\quad \text{from an M2-brane on } D_\Sigma \\ \text{Domain Wall:} &\quad \text{from an M5-brane on } D_\Sigma, \end{aligned} \tag{4.15}$$

where the Joyce lemma ensures the existence of the class $[D_\Sigma]$, but the assumed positivity is necessary for the existence of the instanton and the domain wall.

In addition to its role in the existence of these defects, there are other physical consequences of the lemma (4.11). Again assuming (for now) that $[D_\Sigma]$ has an associative representative D_Σ, note that (4.11) can be rewritten

$$0 < \int_{D_\Sigma} \Phi = vol(D_\Sigma) \tag{4.16}$$

in which case the lemma can also be interpreted as a condition on calibrated cycles; when it is violated, the volume of an associative submanifold goes to zero. As such, the generic physical lessons from conditions of the second type (which we will discuss shortly) also apply here.

What is the physical meaning of $vol(D_\Sigma)$? Since D_Σ is related to a U(1) gauge theory, it is natural that $vol(D_\Sigma)$ might determine a parameter in the U(1) theory, and it is reasonable to guess that it is related to the gauge coupling. If this is true, the limit in which (4.11) is violated should be a limit in the gauge coupling. In fact this can be seen directly: we can write

$$Vol(D_\Sigma) = \int_X \sigma \wedge \sigma \wedge \Phi = -\int_X \sigma \wedge \star_\Phi \sigma \tag{4.17}$$

using the identity $\star_\Phi \sigma = -\sigma \wedge \Phi$, and from section 2 we recall that this expression appears in the four-dimensional gauge coupling $\frac{1}{g^2_{\text{YM}}}$ via Kaluza-Klein reduction of eleven-dimensional supergravity. Specifically, in the case $b_2(X) = 1$ we have

$$\frac{1}{4g^2_{\text{YM}}} = -\frac{\pi}{l^3_{11}} \int_X \sigma \wedge \star_\Phi \sigma = \frac{\pi}{l^3_{11}} \int_X \sigma \wedge \sigma \wedge \Phi = \pi \frac{vol(D_\Sigma)}{l^3_{11}}. \tag{4.18}$$

and so we see $g^2_{\text{YM}} = \frac{l^3_{11}}{4\pi vol(D_\Sigma)}$. This is also a result expected from considering M-theory lifts of IIa compactifications with spacetime filling D6-branes: there the four-dimensional /gauge coupling is determined by the volume of the supersymmetric three-cycle wrapped by the D6-brane, which lifts to a supersymmetric three-cycle — an associative submanifold — in M-theory. From this and (more importantly) the general argument we conclude

violation of the Joyce lemma is the limit $g^2_{\text{YM}} \to \infty$,

which is not the physical limit we'd like to consider to control gauge enhancement. We emphasize that this conclusion holds regardless of whether the class $[D_\Sigma]$ has an associative representative.

Cycle	Cycle Dimension	M2	M5
D_Σ	3	spacetime instanton	domain wall
$\tilde{\Sigma}$	5	—	magnetic monopole

Table 2. Any non-trivial two-form σ in a G_2 manifold determines not only a dual five-cycle, but also a related three-cycle. If these have representatives that are submanifolds, a variety of topological defects in four-dimensions, listed above, can be obtained by wrapping M2-branes and M5-branes.

Instead, the physical lesson we would like to draw is that for every U(1) in a G_2 compactification, the Joyce lemma implies the existence of a non-trivial three-cycle $[D_\Sigma]$. If $[D_\Sigma]$ has an appropriate associative representative D_Σ, then the volume of this associative determines the G_2 moduli dependence of the four-dimensional gauge coupling. This result matches the expectation from dimensional reduction of a seven-dimensional gauge theory on $\mathbb{R}^{3,1} \times D_\Sigma$. Moreover, an associative submanifold is expected to play the role of gauge theory worldvolume in the singular limit, as we will discuss.

When an associative submanifold D_Σ exists, wrapping an M2-brane on it gives a spacetime instanton with associated classical suppression factor

$$e^{-Vol(D_\Sigma)} \sim e^{-\frac{1}{g_{\rm YM}^2}}. \tag{4.19}$$

This instanton may correct the superpotential in some models depending on the structure of instanton zero modes. It is initially confusing that such an effect would appear, since it has the dependence of a gauge instanton but U(1) theories on their own do not exhibit such effects. One possible resolution of this puzzle is that such effects can appear in abelian theories if they arise as broken phases of non-abelian theories; see section 5.1 for further discussion.

Singularities from calibrated submanifolds and vanishing defects. When the metric moduli of a manifold are varied such that a non-trivial cycle goes to zero volume, a singularity develops, and this can be done in G_2 moduli for associative and coassociative submanifolds since they are calibrated. That is, since

$$Vol(T) = \int_T \Phi \qquad \text{and} \qquad Vol(\tilde{T}) = \int_{\tilde{T}} \star_\Phi \Phi \tag{4.20}$$

for T or $\tilde{T}$ any associative or coassociative submanifold, one simple tunes the moduli such that one of the cycles vanishes and a singularity develops. Note that in many examples the vanishing of a three-cycle is accompanied by the vanishing of a two-cycle that it contains; we will discuss the physics of this phenomenon in sections 5 and 6.

M2- and M5-branes wrapped on associative or coassociative submanifolds can give rise to BPS instantons, BPS domain walls, and BPS strings, and the energetics of these defects can be controlled by the calibration forms Φ and $\star_\Phi \Phi$. Since in gauge theories the existence of topological defects is often determined by the properties of spontaneous symmetry breaking, it is natural to wonder whether the vanishing limit of these instantons, domain walls, and strings may be related to gauge enhancement. For example, the vanishing tension limit of an Abrikosov-Nielsen-Olesen vortex string or semi-local string is a limit in which the abelian Higgs model is unbroken.

5 Scenarios for gauge enhancement

In this section we would like to identify some scenarios in which these ideas may be put to use. That is, if M-theory on a particular G_2 manifold X describes a phase of a broken gauge theory coupled to gravity, how might one extract some data of the unbroken gauge theories in the singular limit?

A simple case that we begin with is to understand G_2 compactifications which describe the Coulomb branch of a non-abelian theory obtained when an adjoint chiral multiplet receives an expectation value. We will then describe the physics associated with a circle full of conifolds, including the Higgs and Coulomb branches associated with the M-theory lift of the deformation and small resolution. We speculate that a deformation of that setup would produce a theory with electrons and positron localized at different codimension seven singularities, but we leave the detailed explanation of such a deformation to future work.

5.1 Non-Abelian gauge theories with adjoint chiral multiplets

Consider a four-dimensional $\mathcal{N} = 1$ non-abelian gauge theory with adjoint chiral multiplets, and suppose that there are flat directions in the scalar potential along which one (or more) of the adjoint chiral multiplets can receive an expectation value. If the expectation values for components of the adjoint chiral are generic, the non-abelian theory is broken from G to $\mathrm{U}(1)^{rk(G)}$. The broken theory exhibits charged massive W-bosons and massless photons, which give rise to long range forces. This is a Coulomb branch.

In the $d = 4$ $\mathcal{N} = 1$ super-Higgs mechanism a massless vector multiplet eats a massless chiral multiplet to produce a massive vector multiplet, and only $dim(G)$-$rk(G)$ chiral multiplets may be eaten in the breaking $G \mapsto \mathrm{U}(1)^{rk(G)}$ since this is the number of vector multiplets that receive a mass. In the case of a single adjoint chiral multiplet all but $rk(G)$ of its components are eaten in the breaking. The $rk(G)$ leftover components must be uncharged under $\mathrm{U}(1)^{rk(G)}$; these are the $rk(G)$ Cartan elements of the adjoint. More massless chiral multiplets remain (on the Coulomb branch) if there are more adjoint chirals in the non-abelian theory, or other matter fields.

Such a theory exhibits 't Hooft-Polyakov monopoles. These were first shown to exist [29, 30] in a model of Georgi and Glashow [62] where $G = \mathrm{SU}(2)$ is broken to $H = \mathrm{U}(1)$ by an adjoint scalar field. The existence of the monopole solution is guaranteed by the non-trivial homotopy $\pi_2(G/H) = \mathbb{Z}$, which in this case can be seen since G/H is topologically an S^2, or alternatively since $\pi_2(G/H) = \pi_1(H) = \mathbb{Z}$. The first equality holds by virtue of a long exact sequence

$$\ldots \longrightarrow \pi_2(G) \longrightarrow \pi_2(G/H) \longrightarrow \pi_1(H) \longrightarrow \pi_1(G) \longrightarrow \ldots \tag{5.1}$$

and the facts that $\pi_2(G) = \pi_1(G) = 0$ for $G = \mathrm{SU}(2)$. This situation generalizes easily: for any breaking of $G \mapsto H = \mathrm{U}(1)^{rk(G)}$ for a Lie group G with $\pi_2(G) = \pi_1(G) = 0$ we have

$$\pi_2(G/H) = \pi_1(H) = \mathbb{Z}^{rk(G)} \tag{5.2}$$

and the broken theory exhibits magnetic monopoles; these include the cases where $G = \mathrm{SU}(N), \mathrm{Sp}(N)$, or G an exceptional simple Lie group, but not $G = \mathrm{SO}(N > 2)$ since $\pi_1(\mathrm{SO}(N > 2)) = \mathbb{Z}_2$. Even in this case $\pi_2(G/H)$ must be non-trivial. This can be seen using the exact sequence with $G = \mathrm{SO}(N > 2)$ and $H = \mathrm{U}(1)^{rk(G)}$

$$0 \to \pi_2(G/H) \xrightarrow{f} \mathbb{Z}^{rk(G)} \xrightarrow{g} \mathbb{Z}_2 \to \dots \tag{5.3}$$

where if $\pi_2(G/H)$ were trivial, then g would be injective, which cannot be true. Thus $\pi(G/H)$ is non-trivial for G a simple Lie group and $H = \mathrm{U}(1)^{rk(G)}$, so these theories contain monopoles.

For G a simple Lie group $\pi_3(G) \neq 0$ and the theory exhibits Yang-Mills (henceforth, gauge) instantons. It is common to think of these topologically non-trivial gauge field configurations in unbroken non-abelian gauge theories, but it is also natural to consider what happens to them if a charged scalar obtains an expectation value that breaks G. This is an important question in the context of M-theory, since gauge instantons exist in singular G_2 compactifications with non-abelian gauge symmetry, but if they also exist in the Higgs regime then they may arise geometrically from the structure of a smooth G_2 manifold or a submanifold thereof.

In gauge theory this question was answered by 't Hooft [63], who showed that gauge instantons associated to a gauge group G contribute to the effective action even if G is spontaneously broken to an abelian or trivial group H; see [64] for a general treatment, including for a variety of unbroken subgroups H. The usual BPST instanton solution [65] can be used to construct SU(2) instantons for SU(2) subgroups of G. Suppose a Higgs field h receives an expectation value $\langle h \rangle = v$. In the case $v \neq 0$ the instanton size ρ is no longer a modulus and therefore the finite size instanton does not solve the equations of motion in the Higgs regime. Nevertheless their contributions to the effective action may be computed (for example, using the formalism of constrained instantons [66]) and the standard suppression factor $e^{-1/g^2_{\mathrm{YM}}}$ still appears. In the original example of [63] the extra term in the instanton action induced by $v \neq 0$ is

$$\Delta S = 2\pi^2 |v|^2 \rho^2 \tag{5.4}$$

where the numerical prefactor is model dependent. The $|v|^2\rho^2$ dependence is more general, however, as it follows directly from the Higgs kinetic term $|D_\mu h|^2 \supset |A_\mu h|^2$, as clearly presented in [64], for example. It will be important for us that the small ($\rho = 0$) instanton of the four-dimensional gauge theory solves the equations of motion even in the Higgs regime!

Before addressing this issue in M-theory, it is useful to consider the same question in a type IIa compactification with a stack of N D6-branes wrapped on a special Lagrangian submanifold $\mathcal{M}_3$. Reducing to four dimensions gives a four-dimensional U(N) gauge theory with gauge coupling depending on $vol(\mathcal{M}_3)$. The small instanton of this four-dimensional gauge theory is a brane within a brane [67], which in this case is a Euclidean D2-brane that is wrapped on $\mathcal{M}_3$. The contribution to the effective action of this ED2 arises as $e^{-1/g^2_{\rm YM}}$ by the relationship between the gauge coupling and $vol(\mathcal{M}_3)$. Now, if $\mathcal{M}_3$ is in a family of special Lagrangians $\mathcal{M}_{3,t}$ then there is an adjoint chiral multiplet that breaks the $\mathrm{U}(N)$ theory to $\mathrm{U}(1)^N$ by spreading out the N D6-branes so that they wrap $\mathcal{M}_{3,t_i}$ where $i = 1, \ldots, N$. The non-abelian theory has been broken, but nevertheless an ED2 on any member of this family is a solution and still contributes to the effective action as $e^{-1/g^2_{\rm YM}}$ by virtue of the fact that every member of the family has the same volume. Whether this instanton contributes to the superpotential or some other aspect of the effective $\mathcal{N} = 1$ action depends on details of zero modes, but it is a solution in any case, as predicted by 't Hooft's gauge theory calculation [63]. Lifting this configuration to M-theory should preserve the volume dependence of the ED2-instanton in the broken theory as the volume of an associative submanifold.

Now we turn to the more general question of interest: if M-theory on a smooth G_2 manifold X were to realize such a phase of a gauge theory, accounting for many or all of these interesting features, how would one tell from the topology of X, and how might one take a singular limit in which the non-abelian gauge symmetry is restored? One would like to identify the presence of charged massive W-bosons, adjoint chiral multiplets, 't Hooft-Polyakov monopoles,[13] and gauge instantons. For simplicity we assume that $rk(G) = b_2(X)$; one could also apply the argument below to a $rk(G)$-dimensional subset of the two-forms, five-cycles, and other associated topological data.

The $\mathcal{N} = 1$ supersymmetric Georgi-Glashow model is, itself, a good starting point: suppose M-theory on a G_2 manifold X realized a phase of a gauge theory with $G = \mathrm{SU}(2)$ broken to $H = \mathrm{U}(1)$ by the expectation value of single adjoint chiral multiplet X_a, $a \in \{1, 2, 3\}$, which is the only charged chiral multiplet in the unbroken theory. A D-flat direction exists for the adjoint chiral by virtue of the fact that $X_a X_b \delta_{ab}$ is a gauge invariant holomorphic function. The associated branch of moduli space is a Coulomb branch, which exhibits massive charged W-bosons, 't Hooft-Polyakov monopoles, gauge instantons and a finite gauge coupling. This theory is an $\mathcal{N} = 1$ avatar of pure Seiberg-Witten theory, which also exhibits the above physical effects on its Coulomb branch.

If M-theory on X gives a vacuum on the Coulomb branch of this model, the existence of only this $\mathrm{U}(1)$ requires $b_2(X) = 1$, and we have associated homology classes $[\tilde{\Sigma}] \in H_5(X)$ and $[\Sigma] \in H_2(X)$. In a general abelian theory $[\tilde{\Sigma}]$ and $[\Sigma]$ need not satisfy any other conditions. However, the Georgi-Glashow model is not a general abelian theory; it is obtained by spontaneous symmetry breaking from a non-abelian theory, and it exhibits both massive W-bosons and monopoles. Kaluza-Klein reduction of $d = 11$ supergravity on X does not account for this physics and therefore it should arise from another source. Wrapped M2-branes and M5-branes provide such a source, as discussed in section 4.2. Doing so, however, requires that the classes $[\tilde{\Sigma}]$ and $[\Sigma]$ have representatives that are

[13] More specifically, we will look for monopoles that are associated with symmetry breaking in some way. We will have nothing to say about the field profiles characteristic of 't Hooft-Polyakov monopoles, but will focus instead on the dependence of M-theory monopoles on gauge theory parameters, finding that they have a dependence characteristic of 't Hooft-Polyakov monopoles.

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positive volume submanifolds; we call these $\tilde{\Sigma}$ and Σ. Thus, the physics of the Coulomb branch requires that X satisfy certain properties if the massive W-bosons and monopoles are to arise from wrapped M2-branes and M5-branes.

What about the gauge instantons? They are naturally present if the three-cycle ensured by the Joyce lemma has a representative that is an associative submanifold. Recall that this lemma gives

$$[D_\Sigma] \equiv -[\tilde{\Sigma}] \cap [\tilde{\Sigma}] \in H_3(X, \mathbb{Z}), \tag{5.5}$$

a non-trivial three-cycle associated to the U(1), and also that the gauge coupling is

$$g^2_{\rm YM} = \frac{l_{11}^3}{-4\pi \int_X \sigma \wedge \sigma \wedge \Phi} = \frac{l_{11}^3}{4\pi\, vol(D_\Sigma)} \tag{5.6}$$

where σ is the harmonic two-form that defines the U(1) and the latter equality holds only if there is an associative submanifold D_Σ in the class $[D_\Sigma]$. If D_Σ does exist, then M2-branes wrapped on D_Σ are spacetime instantons with $e^{-1/g^2_{\rm YM}}$ type dependence characteristic of gauge instantons. Since (for example) the small instanton[14] does solve the equations of motion in the Higgs regime, these effects should be visible on the Coulomb branch, that is in M-theory on X! Furthermore, since they do not arise from supergravity reduction, an M2-brane instanton on D_Σ would give one of the required $e^{-1/g^2_{\rm YM}}$ suppressed contributions to the effective action, and the non-triviality of the class $[D_\Sigma]$ is ensured on general grounds by the Joyce lemma, it seems reasonable that an associative representative D_Σ exists. Henceforth we assume the existence of such a cycle.

So far, the existence of massive W-bosons, 't Hooft-Polyakov monopoles, and gauge instantons have given us submanifold representatives of the classes $[\Sigma]$, $[\tilde{\Sigma}]$, and $[D_\Sigma] = -[\tilde{\Sigma}] \cdot [\tilde{\Sigma}]$ that are submanifolds. This additional structure on X is not required in general, but arises from physics. It is interesting to ask whether physics suggests a relation between the physical parameters in the Higgs vacuum and the geometry of these submanifolds, in particular their volumes. Using the standard dependences of W-boson and monopole masses on the Higgs field expectation value $|v|$ and the gauge coupling $g_{\rm YM}$ and relating this scaling to the expected scaling of these the masses with the volumes of Σ and $\tilde{\Sigma}$, we have

$$M_W \propto g_{\rm YM}|v| \propto vol(\Sigma) \qquad \text{and} \qquad M_M \propto \frac{|v|}{g_{\rm YM}} \propto vol(\tilde{\Sigma}) \tag{5.7}$$

and therefore the non-abelian limit with $|v| \mapsto 0$ sends both Σ and $\tilde{\Sigma}$ to zero volume, but for the gauge coupling to remain finite $vol(D_\Sigma)$ must also remain finite.

The reader may have already noticed that (in a certain sense) we have an overconstrained system: the volumes of the three submanifolds Σ, $\tilde{\Sigma}$ and D_Σ scale with two physical parameters v and $g_{\rm YM}$. Since $\frac{1}{g^2_{\rm YM}} \propto vol(D_\Sigma)$ we can write another scaling for the monopole mass

$$M_M \propto \frac{|v|}{g_{\rm YM}} \propto \frac{M_W}{g^2_{\rm YM}} \propto vol(\tilde{\Sigma}) \propto vol(\Sigma) vol(D_\Sigma) \tag{5.8}$$

[14]The M-theory lift of a D6-ED2 system would yield such an configuration, where the ED2 can be interpreted as the small instanton of the D6 worldvolume theory [67].

so that the volume of the five-manifold $\tilde{\Sigma}$ depends on the volumes of the two-manifold Σ and the three-manifold D_Σ.

While perhaps not absolutely necessary, this dependence is *suggestive* of a fibration! Specifically, a fibration $\tilde{\Sigma} \to D_\Sigma$ with generic fibers of class $[\Sigma]$. If it were true that $\tilde{\Sigma} = \Sigma \times D_\Sigma$ then we would have $vol(\tilde{\Sigma}) = vol(\Sigma)\, vol(D_\Sigma)$, but this would also be true if equivalent volume two-manifolds of class $[\Sigma]$ were fibered over D_Σ.[15] Though the possibility of a fibration has arisen in our analysis from rather general properties of G_2 manifolds and Coulomb branches, this structure matches other known ideas in the literature. In non-compact models a singular ALE space (with ADE singularity) is fibered over a three-manifold that we might as well call D_Σ; alternatively one could get an ADE singularity fibered over D_Σ from models with heterotic duals, in which X fibered over D_Σ by coassociative K3 manifolds. In either case, the resolution of the singularity would produce five-manifolds that are fibered over three-manifolds by two-spheres, which in our language would be $\tilde{\Sigma}$, D_Σ and fibers of class $[\Sigma]$. If there was such a fibration, then in the gauge enhanced singular limit the fiber must collapse and therefore $\tilde{\Sigma}$ collapses to the three-manifold D_Σ. The enhanced non-abelian gauge symmetry is localized on the three-manifold D_Σ over which the singularity has developed.

Now, D_Σ may have its own topology. If $b_1(D_\Sigma) \neq 0$ it has one-cycles, and the S^2-fibration over the one-cycles give three-manifolds that are just S^2-fibrations over circles. In the gauge enhanced singular limit where the W-boson mass goes to zero the three-manifold would collapse to a circle and we would have the seven-dimensional gauge theory on D_Σ. What does the topology $b_1(D_\Sigma)$ correspond to physically? Reduction of the seven-dimensional (on $\mathbb{R}^{3,1} \times D_\Sigma$) gauge fields on the $b_1(D_\Sigma)$ one-cycles in D_Σ give rise to adjoint chiral supermultiplets in the $d = 4$ $\mathcal{N} = 1$ effective theory, where in the case of the Georgi-Glashow model we must have $b_1(D_\Sigma) = 1$.

To this point we have given physical arguments for what must occur in the singular limit, but we have not discussed how one gets there. In a Calabi-Yau compactification (again cf. appendix A) one would could simply calibrate the fibers to zero, since they are a family of holomorphic curves. Since this cannot be done in G_2 manifolds, let us use our proposal to use associative or coassociative submanifolds to develop singularities. Recall that the D_Σ ensured by the Joyce lemma is not the one to calibrate to zero, though, since this would be the infinite coupling limit.

However, physical considerations lead to another three-cycle, namely the S^2 fibration over the one-cycle in D_Σ corresponding to the adjoint chiral. Since in the gauge enhanced singular limit the S^2 must go to zero volume, this seems to be the natural cycle to try to send to zero. Doing so via calibration requires an associative submanifold T in that class, so that our proposal is to take a limit in G_2 moduli such that

$$vol(T) = \int_T \Phi \qquad \mapsto \qquad 0, \tag{5.9}$$

[15] The latter situation is precisely what occurs (with appropriate dimension changes for $\tilde{\Sigma}$ and D_Σ) in the M-theory Coulomb branch obtained from compactification on a resolved Calabi-Yau elliptic fibration, which is often used as a tool to study F-theory; see appendix A for a discussion.

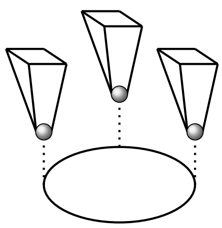
where the collapse of T to an S^1 seems most natural in order to retain a finite gauge coupling.

The proposal put forth here in the Georgi-Glashow model extends to Coulomb branches associated to other non-abelian groups G, as well, with some small modifications. For example, in cases with $rk(G) > 1$ there are a wider variety of monopoles and therefore there must be additional five-manifolds on which to wrap M5-branes; this, too, matches the Calabi-Yau case discussed in appendix A, since higher rank groups give rise to additional Cartan divisors in the resolution. For gauge coupling unification in the singular limit, it seems that distinct five-cycles $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ associated the monopoles of the Coulomb branch must have associated associative submanifold D_{Σ_1} and D_{Σ_2} in the same class, i.e. $[D_{\Sigma_1}] = [D_{\Sigma_2}]$. The relationship between the geometry and the gauge coupling, W-boson mass, and monopole mass also generalizes.

5.2 Massless charged matter from circles of conifolds

In this section we will utilize a similar idea — where two-cycles are controlled since they sit in calibrated three-cycles — in theories that realize conifold transitions, following [11]. Recall first the physics of an M-theory compactification on a Calabi-Yau threefold with conifold singularities, focusing for now on one of the conifolds. It gives rise to a U(1) gauge theory with massless charged particles, as can be understood by taking a limit of the resolved conifold. In the resolved conifold, the singular tip has been replaced by a two-sphere, and an M2-brane wrapped on the two-sphere gives rise to a massive charged particle. The Kähler form calibrates holomorphic curves in Calabi-Yau manifolds, and thus upon taking a limit in Kähler moduli space the curve shrinks, giving a massless charged hypermultiplet localized at a codimension six singularity. When the conifold is deformed this charged hypermultiplet receives an expectation value, the two-form which gave rise to the U(1) factor no longer exists and the theory is spontaneously broken.

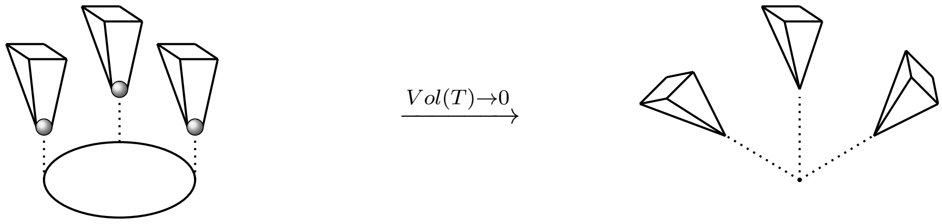
Similar singularities may exist at codimension six in G_2 compactifications of M-theory, and though there is no calibration for two-cycles, the proposal from section 4 is to control them indirectly by other calibrations. We begin from the resolution and will address the deformation in the next section. Consider a smooth G_2 manifold X where one can identify a circle full of resolved conifolds. Compact G_2 manifolds with such a feature are known to exist, as studied for example in [11], and such a geometry locally appears as



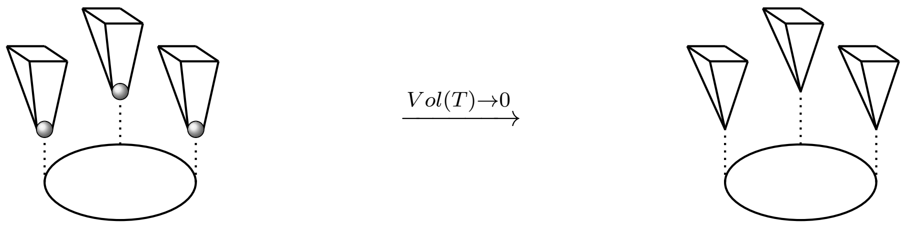
where the cone object is the six-dimensional resolved conifold, a two-sphere has replaced the singular tip as usual and in fact the conifold is fibered over a circle. Though we cannot control the two-spheres directly, note that the two-spheres fibered over the circle give a three-cycle $[T] \in H_3(X,\mathbb{Z})$ which contains the two-spheres, and we consider an associative representative T of this class. If we move in moduli space such that

$$Vol(T) = \int_T \Phi \qquad \mapsto \qquad 0 \tag{5.10}$$

there are at least two natural ways in which the three-manifold T might collapse, analogous to how a divisor may collapse to a curve or a point in an algebraic variety. First, T may collapse to a point



or alternatively T may collapse to a circle



which is the case that we will focus on. In such a limit X exhibits a circle full of conifolds, and there are massless charged particles localized at these codimension six singularities. In summary, given a circle full of resolved conifolds in X, the three-cycle T which is a two-sphere fibration over a circle can be calibrated to zero volume to control the mass of the charged particles associated with the two-spheres; this matter is necessarily non-chiral.

Furthermore, in this case (where the associative threefold collapses to a circle) one can be quite explicit about the local geometry. To do so, recall that the conifold can be represented by a gauged linear sigma model (GLSM). Let (x_1, x_2, x_3, x_4) be fields (homogeneous coordinates) with U(1) charges $(1, 1, -1, -1)$ respectively. The vacuum moduli space of this GLSM solves the D-term constraint

$$|x_1|^2 + |x_2|^2 - |x_3|^2 - |x_4|^2 = \xi, \tag{5.11}$$

where ξ is the Fayet-Iliopoulos (FI) parameter of the GLSM; this is not to be confused with the FI parameter of the U(1) theory on spacetime. The vacuum moduli space is a toric Calabi-Yau manifold, which is a conifold for $\xi = 0$. There are two different small resolutions of the conifold; here they are given by $\xi \neq 0$, where the sign of ξ determines which of the two small resolutions is realized. Tuning ξ corresponds to movement in the Kähler moduli space of the Calabi-Yau manifold, and knowing this the G_2 interpretation is immediate. In the case of a Calabi-Yau product with a circle, movement in Kähler moduli space induces movement in G_2 moduli space, and thus either of the small resolutions not only moves in the Kähler moduli space but also deforms the G_2-form Φ. An associative submanifold may therefore collapse in the $\xi \mapsto 0$ limit, which we see explicitly here: the cycle T above collapse to a circle. This phenomenon also occurs in the compact model studied in [11], which we will review in section 6.3.

It may be possible to deform these theories by letting ξ vary along the circle, leaving isolated singularities at the points where ξ vanishes. We leave the investigation of such a deformation to future work.

5.3 Strings from deformed circles of conifolds

To this point we have not yet used a BPS defect to understand symmetry breaking: the monopoles of section 5.1, which played a critical role, were not BPS. In this section we would like to understand symmetry breaking of U(1) theories in terms of the appearance of BPS strings, and their relation to wrapped branes on the new cycles of the deformation of conifolds.

Consider the circle of conifolds. From wrapping M2-branes and anti M2-branes on the two-sphere of the resolved circle of conifolds, the conifold limit has two chiral multiplets of massless charged particles, with opposite charge. Consider the field theoretic description of this setup. The D-term contribution to the scalar potential is

$$V_D = \frac{g^2}{2}\left(\xi + \overline{\phi}_+\phi_+ - \overline{\phi}_-\phi_-\right)^2 \tag{5.12}$$

where ϕ_+ and ϕ_- are the complex scalars in the charged chiral multiplets and ξ is the Fayet-Iliopoulos parameter of the four-dimensional U(1) theory, which must be zero at the conifold point since the U(1) is unbroken there. In the super-Higgs mechanism for $\mathcal{N} = 1$ theories in four dimensions a massless vector multiplet eats a massless chiral multiplet to become a massive vector multiplet. Since $\xi = 0$ at the conifold point the expectation values

$$\langle \phi_+ \rangle = \langle \phi_- \rangle = v \tag{5.13}$$

determine a D-flat direction in the scalar potential that Higgses the U(1), which are a subset of the full space of D-flat directions $\xi + \overline{\phi}_+\phi_+ - \overline{\phi}_-\phi_-$, where ξ is field-dependent.

D-flat directions correspond to holomorphic gauge invariant combinations of the fields, and this one in particular corresponds to

$$\phi_+\phi_-, \tag{5.14}$$

Rewriting the scalar fields in polar coordinates $\phi_i = \rho_i e^{i\theta_i}$, the phase of $\phi_+\phi_-$ is $e^{i(\theta_+ + \theta_-)}$, and $\theta_+ + \theta_-$ is uncharged under the U(1). The combination $\theta_+ - \theta_-$, on the other hand, is the eaten by the photon in the Higgsing process. This field theoretic description is natural in M-theory: two chiral multiplets became massless in taking the singular limit of the resolved circle of conifolds, and in Higgsing the theory by deformation one combination of these chiral multiplets is eaten, while the one corresponding to the flat direction is left massless. The existence of the new chiral multiplet of the flat direction on the Higgs branch corresponds to the new contribution to b_3 from the $S^3 \times S^1$ of the deformed circle of conifolds.

Now wrap an M5-brane on the $S^3 \times S^1$. This gives a string in four dimensions that exists in the deformation, but not the resolution. Classically, its tension depends on

$$T \sim R\, vol(S^3) \tag{5.15}$$

where R is the radius of the S^1, and since the volume of the S^3 is an order parameter for the symmetry breaking the tension is $T \sim R\, f(v)$ for some function f of the Higgs expectation value. This string is charged under the two-form that is the four-dimensional Hodge dual of the linear combination of axions $\theta_+ + \theta_-$ associated to the D-flat direction. If the $S^3 \times S^1$ is a coassociative submanifold then this string is BPS.

We see that we have a topological defect — a BPS string — that appears in connection with the breaking of a U(1) gauge symmetry in a four dimensional theory, and its tension depends on the Higgs expectation value. This basic phenomenon is known in the breaking of U(1) theories, of course. Famously, the non-supersymmetric abelian Higgs model supports a stable, Abrikosov-Nielsen-Olesen (ANO) vortex string with tension

$$T_{\text{ANO}} = 2\pi |v|^2 \tag{5.16}$$

for the critical vortex that satisfies the Bogolmo'nyi bound, which with slight modifications [68, 69] can be extended to $d = 4$ $\mathcal{N} = 1$ theories. There are also the semi-local strings [60] associated with U(1) symmetry breaking in theories with flat directions. In fact, the simplest model with semi-local strings has two charged scalars, just like this model, and furthermore the semi-local string tension $T_{\text{SL}} \sim |v|^2$.

We leave the detailed study of the type of string we obtained from a wrapped M5-brane to future work, but since its energetics match the discussed field theoretic expectations relating string tensions to order parameters for U(1) symmetry breaking and furthermore it is charged (via C_6 reduction) under the two-form dual to the axion $\theta_+ + \theta_-$ of the flat direction, it seems a promising harbinger of U(1) symmetry breaking in G_2 compactifications of M-theory. Calibrating it to zero via calibrating a coassociative to obtain an (abelian) gauge enhanced singular limit fits with the general proposal we put forth in section 4.

6 Examples

In this section we would like to study a number of examples, in particular circle products with Calabi-Yau threefolds, some of the G_2 manifolds of Joyce, and also twisted connected sums.

6.1 Circle products with Calabi-Yau threefolds

Gauge enhancement is relatively easy to understand if $X = Z \times S^1$ for Z a Calabi-Yau three-fold. Some of the physics of gauge enhancement follows directly from the Calabi-Yau geometry in the usual way, but the circle factor also introduces additional topological structure that will be important. But let us first review the relevant mathematical facts.

Consider $X = Z \times S^1$, where Z is a Calabi-Yau threefold. Then X is a compact seven-manifold with SU(3) holonomy. It is a manifold with G_2-structure, and therefore it has a G_2-form Φ. Letting θ be an angle coordinate on S^1, Φ is determined by the Kähler form J and holomorphic three-form Ω on Z as

$$\Phi = Re(\Omega) + d\theta \wedge J. \tag{6.1}$$

and similarly, $\star_\Phi \Phi$ also depends on Ω and J.

Though volumes of two-manifolds cannot be easily computed in a general compact G_2 manifold, in this case we can make use of the product structure of the metric and also the fact that any $[\Sigma] \in H^2(X, \mathbb{Z})$ is also an element of $[\Sigma] \in H^2(Z, \mathbb{Z})$. In particular a holomorphic curve Σ in Z is also a volume minimizing two-cycle in X. Therefore,

$$vol(\Sigma) = \int_\Sigma J = \int_\Sigma \frac{\partial}{\partial \theta} \lrcorner \Phi, \tag{6.2}$$

where we have written the last equality to emphasize the connection between the two-cycle volume and the G_2-form Φ. Note also that to any $\Sigma \in H^2(Z, \mathbb{Z}) = H^2(X, \mathbb{Z})$ we have a three-cycle $T_\Sigma \in H^3(X, \mathbb{Z})$ given by $\Sigma \times S^1$, and furthermore that any three-cycle in Z is also a three-cycle in X.

Gauge enhancement upon movement in the moduli space of X can occur in a few different ways. We will study those cases where gauge enhancement can be seen via the existence of new massless particles in the limit where the volume of some two-cycles goes to zero, which is under control in the case of a circle product for the reasons just discussed. Since X has SU(3) holonomy, the four-dimensional theory actually has $\mathcal{N} = 2$ supersymmetry. This theory exhibits BPS particles and accordingly there is a calibration form for two-manifolds, which is given by

$$\frac{\partial}{\partial \theta} \lrcorner \Phi = J. \tag{6.3}$$

One can use this fact to calibrate two-cycles to zero in the usual way. However, given the relationship to Φ this may also collapse three-manifolds if they contain non-trivial two-manifolds, as suggested in the more general proposal we have put forth in section 4.

Let us consider two natural singular limits of the Calabi-Yau case, in the language of X and Φ. Let Σ be a holomorphic curve in Z that comes in a real two-parameter family, parameterized by a curve C of genus g_C. If we have

$$R^6_{S^1} \gg vol(Z). \tag{6.4}$$

then for energy scales above the Kaluza-Klein scale of the S^1 but below that of Z we have an effective five-dimensional theory, and M2-branes on Σ give [34, 70] a $d = 5$ $\mathcal{N} = 1$ massive vector multiplet and g_C adjoint hypermultiplets. These can be reduced to $d = 4$ $\mathcal{N} = 2$ multiplets by compactification on the S^1. From the point of view of X, Σ comes in a three real parameter family parameterized by $S^1 \times C$ and this family gives a massive $\mathcal{N} = 2$ vector multiplet in four dimensions. Taking a limit in the moduli space of Φ such that $vol(\Sigma) \to 0$, the massive vector multiplet becomes massless and the theory exhibits an SU(2) gauge enhancement; if multiple curves go to zero volume, generic ADE gauge groups are possible. In a second scenario, Σ could instead be rigid, in which case M2-branes on it give rise to charged hypermultiplets in four dimensions; then if Σ goes to zero volume there is no gauge enhancement, but the charged hypermultiplet becomes massless. If this gives a conifold in Z, then we have a circle of conifolds in X. In both of these two cases, from the point of view of X gauge enhancement occurs due to the collapse of two-cycles inside associative threefolds, exemplifying our proposal from section 4.

To close this section we would like to remind the reader why two-cycles were controllable in this case, and speculate as to how this might generalize. On general grounds one might hope that, since the metric is determined by Φ for any G_2 manifold, integrating $\int_\Sigma v_\Sigma \lrcorner \Phi$ for some distinguished vector field v_Σ computes the volume of two-manifold Σ. This was, in fact, what worked for $X = Z \times S^1$, where $v_\Sigma = \partial/\partial\theta$ and we note that v_Σ is independent of Σ. Therefore in this case $v_\Sigma \lrcorner \Phi$ is the same two-form for any Σ. The vector $\partial/\partial\theta$ generates the direction normal to Σ in $\Sigma \times S^1$.

It would be interesting in future work to study this idea in more general setups, where v_Σ may differ for various Σ. This is somewhat natural from our proposal to collapse two-manifolds by collapsing associatives or coassociatives. In the case of an associative T containing a two-manifold Σ, there is a natural vector field v_Σ that generates the normal direction to Σ in T; the idea would be to study a potential relationship between $\int_\Sigma v_\Sigma \lrcorner \Phi$ and the volume of Σ.

6.2 Joyce's examples

We now turn to study M-theory on compact seven-manifolds with holonomy precisely G_2. The first examples of these manifolds were due to Joyce. The idea in this case is to resolve a toroidal orbifold in a way that allows one to prove that there exists a G_2 metric on a small deformation of the resolved space. Therefore we will denote this space

$$X = Res(T^7/\Gamma). \tag{6.5}$$

Here Γ is some discrete group, which typically acts on the T^7 coordinates in simple ways. For early work on the physics of M-theory on Joyce orbifolds, see [71].

We will study two specific examples. The two examples differ qualitatively: in first, there is only one topologically distinct smoothing of the singular space, whereas there are two in the second example. The basic idea is that the singularities can locally be written as a three-manifold times a real codimension four ADE singularity, and these singularities can be deformed or resolved into a smooth four-manifold. Though these four-manifolds are diffeomorphic, the actions of orbifold elements on them are not necessarily topologically equivalent. In the first example we study they are, whereas in the second example they are not.

Before proceeding to the examples, let us discuss some details of the two desingularizations which are common to both examples; there the singular sets are locally modeled by $T^3 \times \mathbb{C}^2/\{\pm 1\}$ and we focus on one. Define (x_1, x_2, x_3) to be the T^3 coordinates and $z_1 \equiv x_4 + ix_5$ and $z_2 \equiv x_6 + ix_7$ to be the coordinates on $\mathbb{C}^2$, with the orbifold action locally giving the quotient of $\mathbb{C}^2$. There are two possible choices for resolving $\mathbb{C}^2/\{\pm 1\}$ into a smooth space Y_i:

A) Let Y_1 be the blowup of $\mathbb{C}^2/\{\pm 1\}$ at the origin. The exceptional divisor in Y_1 is a $\Sigma_1 \cong \mathbb{P}^1$ and its homology class generates $H_2(Y_1, \mathbb{R})$.

B) Define the map $\sigma : \mathbb{C}^2/\{\pm 1\} \to \mathbb{C}^3$ such that $\sigma : \pm(z_1, z_2) \mapsto (z_1^2 - z_2^2, iz_1^2 + iz_2^2, 2z_1 z_2)$. Then $\mathbb{C}^2/\{\pm 1\}$ is $\{(w_1, w_2, w_3) \in \mathbb{C}^3 : w_1^2 + w_2^2 + w_3^2 = 0\}$ and the *smoothing* of the singularity is defined by

$$Y_2 \equiv \{(w_1, w_2, w_3) \in \mathbb{C}^2 : w_1^2 + w_2^2 + w_3^2 = \epsilon\} \tag{6.6}$$

for $\epsilon \in \mathbb{C}$. Defining $\epsilon = re^{2i\theta}$ we have

$$\Sigma_2 \equiv \{(e^{i\theta}x_1, e^{i\theta}x_2, e^{i\theta}x_3) : x_j \in \mathbb{R}, x_1^2 + x_2^2 + x_3^2 = r\} \tag{6.7}$$

an S^2 in Y_2 whose homology class generates $H_2(Y_2, \mathbb{R})$.

In the respective cases the singular set is now smoothed into $T^3 \times Y_i$, and Y_1 and Y_2 are diffeomorphic. The question in examples will be to study how the orbifold action on each Y_i could be topologically distinct, and the associated physical implications.

To fully control this behavior, we would like certain cycles within the blowup locus to have associative or coassociative representatives, but this is hard to guarantee. However, at least in some cases [47], there are G_2-instantons whose degeneracy locus approaches the singular locus as nonabelian gauge symmetry is restored, and we obtain a condition of the form (3.7).

Example one: a Higgsed SU(2) theory with a single branch of moduli space. In this section we will analyze a geometry that determines a moduli space that spontaneously breaks an $\mathrm{SU}(2)^{12}$ gauge symmetry to $\mathrm{U}(1)^{12}$ with three adjoint Higgs fields for each SU(2) factor. We will see that the geometry exhibits topological defects of the type expected from gauge theoretic arguments, in particular 't Hooft-Polyakov monopoles, including the correct dependence of monopole masses on the parameters of the spontaneously broken gauge theory. This closely matches the discussion of section 5.1.

We begin by reviewing a singular T^7 orbifold, discussed for example in [31], along with the details that will be relevant for our analysis. Let x_i be coordinates on T^7 such that $x_i \sim x_i + 1$ and define the actions

$$\begin{aligned}
\alpha : &\quad (x_1, \ldots, x_7) \mapsto (x_1, x_2, x_3, -x_4, -x_5, -x_6, -x_7) \\
\beta : &\quad (x_1, \ldots, x_7) \mapsto \left(x_1, -x_2, -x_3, x_4, x_5, \frac{1}{2} - x_6, -x_7\right) \\
\gamma : &\quad (x_1, \ldots, x_7) \mapsto \left(-x_1, x_2, -x_3, x_4, \frac{1}{2} - x_5, x_6, \frac{1}{2} - x_7\right).
\end{aligned} \tag{6.8}$$

Then $\Gamma = \langle \alpha, \beta, \gamma \rangle$ is the discrete group that determines the toroidal orbifold. It can be shown that $\alpha^2 = \beta^2 = \gamma^2 = 1$ and furthermore that α, β, and γ commute; as such, $\Gamma \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. The elements $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$, and $\alpha\beta\gamma$ do not have fixed points on T^7. The fixed points of α in T^7 are 16 copies of T^3, and the group $\langle \beta, \gamma \rangle$ acts freely on these 16 T^3's. Similarly, there are another 16 three-tori fixed by β which $\langle \alpha, \gamma \rangle$ acts freely on, and yet another 16 three-tori fixed by γ which $\langle \alpha, \beta \rangle$ acts freely on. The singular set of T^7/Γ is S, which is a disjoint union of 12 three-tori, and the singularity at each three-torus is locally modeled by $T^3 \times \mathbb{C}^2/\{\pm 1\}$. Note that the counting 12, rather than 48, can be seen by noting that (for example) the 16 T^3's fixed by α in T^7 are permuted by $\langle\beta,\gamma\rangle$, and thus these are only 4 distinct T^3's in the quotient. Similar results hold for the other fixed T^3's in T^7, yielding overall 12 copies of T^3 in the orbifold. There are no singularities in codimension seven.

For simplicity, focus one of these singular sets $T^3 \times \mathbb{C}^2/\{\pm 1\}$. Desingularizing this space via method A) above, it is easy to see that that $\Delta b^2(X) = 1$ while $\Delta b^3(X) = 3$; the resolution of the A_1 singularity $\mathbb{C}^2/\{\pm 1\}$ gives the exceptional curve Σ_1 that now contributes to b^2, while defining S^1_i with $i \in \{1,2,3\}$ to be the three S^1's in T^3 we see that $S^1_i \times \Sigma_1$ contributes to $b^3(X)$. To match the notation of section 5, let's instead use Σ for Σ_1, and we will abuse notation by letting Σ denote either a homology class or a representative, with meaning determined by context. Let $\tilde{\Sigma}$ be the five-cycle that is fibered by two-spheres of class Σ over T^3, and we may refer to T^3 as D_Σ. Then M2-branes and M5-branes on Σ and $\tilde{\Sigma}$ give electrically and magnetically charged particles in four dimensions, respectively.

The physics of this example, focusing on this particular resolved singularity, is as follows. Massive W-bosons arise from wrapping M2-branes and anti M2-branes on curves of class Σ where SU(2) has been broken to U(1); the latter follows since $\tilde{\Sigma}$ has a dual two-form σ and therefore gives rise to a massless Z-boson via $C_3 = A \wedge \sigma + \cdots$. The theory on $\mathbb{R}^{3,1} \times D_\Sigma$ has a gauge field, and therefore adjoint chiral multiplets can be obtained by Kaluza-Klein reduction of the gauge field on one-cycles in D_Σ. In addition to the W-bosons, we can obtain other charged particles by wrapping an M5-brane on $\tilde{\Sigma}$, which are magnetically charged. Note from the generic analysis of section 2 and also direct dimensional reduction of the gauge kinetic term in the $d = 7$ theory on $\mathbb{R}^{3,1} \times D_\Sigma$, we have $Vol(D_\Sigma) \sim 1/g^2$, where g is the four-dimensional gauge coupling. Noting also that $Vol(\tilde{\Sigma}) = Vol(\Sigma) \times Vol(D_\Sigma)$ and the fact that $M_W \sim g|v| \sim Vol(\Sigma)$ with v a Higgs expectation value, we see that the mass M_m of an M5-brane on $\tilde{\Sigma}$ is

$$M_m^2 \sim vol(\tilde{\Sigma}) \sim vol(\Sigma) vol(T^3) \sim g|v| \times \frac{1}{g^2} \sim \frac{|v|}{g} \tag{6.9}$$

which is precisely the dependence on v and g expected for the classical mass of a critical 't Hooft-Polyakov monopole. The physical content in this example closely matches our more general discussion (see section 5.1) of Coulomb branches.

Gauge enhancement occurs in this example as discussed in general in section 5.1: associated to each S^1_i in D_Σ is an adjoint chiral multiplet and a vertical three-cycle T_i which is a two-sphere fibration over the S^1_i. The singular limit is the Joyce orbifold itself, in which the volume of T_i is approximated by[16]

$$vol(T_i) \sim \int_{T_i} \Phi \mapsto 0 \tag{6.10}$$

and since D_Σ remains of finite volume in the orbifold limit we know that the two-cycle Σ also vanishes in that limit. Joyce's work shows that in fact all three T_i vanish, and therefore we have a vanishing two-cycle Σ over every point in D_Σ. These give rise to massless W-bosons and the theory in the singular limit exhibits SU(2) gauge symmetry for each D_Σ.

[16] We thank Thomas Walpuski for comments on this point. Finding a way to control this approximation will be an interesting topic for future work.

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Example two: two topologically distinct resolutions. Let's study an explicit example; further mathematical details can be found in section 12.3 of [31]. Let x_i be coordinates on T^7 such that $x_i \sim x_i + 1$. Define the actions

$$\begin{aligned}
\alpha : &\qquad (x_1, \ldots, x_7) \mapsto (x_1, x_2, x_3, -x_4, -x_5, -x_6, -x_7) \\
\beta : &\qquad (x_1, \ldots, x_7) \mapsto \left(x_1, -x_2, -x_3, x_4, x_5, \frac{1}{2} - x_6, -x_7\right) \\
\gamma : &\qquad (x_1, \ldots, x_7) \mapsto \left(-x_1, x_2, -x_3, x_4, -x_5, x_6, \frac{1}{2} - x_7\right).
\end{aligned} \tag{6.11}$$

Defining $\Gamma \equiv \langle \alpha, \beta, \gamma \rangle \cong \mathbb{Z}_2^3$, we are interested in the orbifold T^7/Γ. Important facts are that α, β, and γ commute, that $\alpha^2 = \beta^2 = \gamma^2 = 1$, and that $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$, and $\alpha\beta\gamma$ have no fixed point in T^7. The fixed points of α, β, and γ are each 16 copies of T^3. The set of T^3 in T^7 fixed by α and β are acted on freely by the groups $\langle \beta, \gamma \rangle$ and $\langle \alpha, \gamma \rangle$, respectively; the set of T^3 in T^7 fixed by γ are not acted on freely by $\langle \alpha, \beta \rangle$, however, since $\alpha\beta$ will map some of the T^3 to themselves, albeit without fixed points. The singular set S of T^7/Γ is a disjoint union of 8 copies of T^3 and 8 copies of $T^3/\mathbb{Z}_2$; the former arise from the quotient action on the 32 T^3 in T^7 fixed by α and β, whereas the latter arise from the quotient action on the 16 T^3 fixed by γ.

Codimension four singularities arising along the 8 $T^3/\mathbb{Z}_2$ singular sets are more exotic. In each of these cases the singularity is locally modeled by $(T^3 \times \mathbb{C}^2/\{\pm 1\})/\langle \alpha\beta \rangle$, where the action of $\alpha\beta$ is given by:

$$\alpha\beta : [(x_2, x_4, x_6), \pm(z_1, z_2)] \mapsto \left[\left(-x_2, -x_4, x_6 + \frac{1}{2}\right), \pm(z_1, -z_2)\right] \tag{6.12}$$

where . This corrects a small typo in equation (12.7) of [31]. As in the previous case, the singular sets $T^3 \times \mathbb{C}^2/\{\pm 1\}$ can be resolved in two diffeomorphic ways as $T^3 \times Y_i$, but now we must also take into account the action of $\alpha\beta$ on the resolution. By the above action, one can consider the induced action on the homogeneous coordinates of the exceptional divisor Σ_1 in Y_1; it is such that the orientation of Σ_1 is preserved. Alternatively, in the second description $\alpha\beta$ acts as $(w_1, w_2, w_3) \mapsto (w_1, w_2, w_3)$. The induced action on Σ_2 is $(x_1, x_2, x_3) \mapsto (x_1, x_2, -x_3)$ which fixes Σ_2 but is orientation reversing. Thus, $(\alpha\beta)_*[\Sigma_1] = [\Sigma_1]$ in $H_2(Y_1, \mathbb{R})$ and $(\alpha\beta)_*[\Sigma_2] = -[\Sigma_2]$ in $H_2(Y_2, \mathbb{R})$. Thus, for codimension four singularities of this type, resolutions of type A still contribute a new two-cycle, but resolutions of type B do not, due to the orientation reversal. Let C_j be new three-cycles, with S^1 coordinate x_i, associated with the resolution of $T^3 \times \mathbb{C}^2/\{\pm 1\}$. The action of $\alpha\beta$ on the x_j determines whether the three-cycle C_j is also a three-cycle of the quotient space, and thus the G_2 manifold. For resolutions of type A, C_6 is a three-cycle in the quotient space, whereas C_2 and C_4 are three-cycles of the quotient for resolutions of type B. In summary, after quotienting by $\alpha\beta$, resolutions of type A contribute one new two-cycle and one new three-cycle, and resolutions of type B contribute zero new two-cycles and two new three-cycles.

We see that for codimension four singular loci of this second type, the two different types of smoothings give topologically distinct G_2 manifolds! Due to the existence and non-existence of new two-cycles for cases A and B, respectively, one of the smoothings has a U(1) gauge symmetry and the other does not. We expect that the singular model will have non-abelian gauge symmetry and sit at the intersection of a Higgs branch and Coulomb branch. We leave more detailed study of this model to future work.

6.3 Twisted connected sum G_2 manifolds

In recent years there has been much progress in constructing compact G_2 manifolds using the so-called twisted connected sum (TCS) construction of Kovalev [7]. The compactification of M-theory on TCS G_2 manifolds was the subject [11], to which we refer the reader for further background on the TCS construction. In this section we would like to review the twisted connected sum construction and discuss how the concrete example of [11] exemplifies our proposal from section 4. Though it would require further mathematical developments to realize examples, we would also like to put forward a new proposal for taking singular limits of TCS G_2 manifolds that realize non-abelian gauge symmetry.

Let us begin by briefly reviewing the twisted connected sum construction, giving the reader a qualitative understanding and referring the summary in section 2 of [11] for more details. In two sentences, the construction glues together two non-compact seven-manifolds with G_2 structure to obtain a compact seven-manifold X. If the non-compact seven-manifolds and gluing satisfy certain criteria, then X admits a natural G_2 structure with associated G_2 form $\tilde{\Phi}$, and there is a torsion-free deformation Φ of $\tilde{\Phi}$ (within its cohomology class), so that the space X together with the metric g_Φ associated to Φ form a compact seven-manifold with holonomy G_2.

The critical details lie in the criteria that the gluing map and the non-compact seven-manifolds must satisfy. Let $V_\pm$ be an *asymptotically cylindrical* (ACyl) Calabi-Yau threefold, which is a non-compact Calabi-Yau manifold that asymptotes to a Calabi-Yau cylinder $V_\pm^\infty$ at infinity. A Calabi-Yau cylinder has the form $\Sigma \times \mathbb{C}^*$ where Σ is a smooth K3 surface and the $\mathbb{C}^*$ is often thought of as an interval times S^1. The point is that the Calabi-Yau cylinder, which is a threefold, has a Kähler form and holomorphic three-form inherited from the Kähler and complex structure of Σ together with data from the $\mathbb{C}^*$. This subsequently induces a natural G_2 structure at the asymptotic ends of the non-compact seven-manifolds $M_\pm := V_\pm \times S^1_\pm$, and the TCS construction works by gluing the asymptotic ends of $M_\pm$ in a way such that the compact seven-manifold X obtained by the gluing has a natural G_2 structure and associated G_2 form $\tilde{\Phi}$. To perform such a gluing requires finding a Donaldson matching, which is a diffeomorphism r that glues the smooth K3 surfaces $\Sigma_\pm$ in a way that leaves the G_2 structure on the asymptotic ends of $M_\pm$ invariant. Then, Kovalev's theorem showed that there is a torsion free deformation Φ of $\tilde{\Phi}$ in its cohomology class so that (X, g_Φ) is a compact G_2 manifold.

The construction of compact G_2 manifolds using the TCS construction is then reduced to the construction of ACyl Calabi-Yau threefolds $V_\pm$ and the study of gluing maps. In some cases the gluing map can be chosen somewhat trivially, so that the problem is reduced to constructing the "building blocks" $V_\pm$. Practically, this is often done by constructing associated compact Kähler threefolds $Z_\pm$ with a smooth K3 surface $\Sigma_\pm$ in the anticanonical class, which is then cut out to form $V_\pm := Z_\pm \setminus S_\pm$. In [10] it was shown that such $V_\pm$

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can be constructed from weak Fano threefolds $Z_\pm$, which greatly increased the number of TCS G_2 manifolds to over fifty million [9]. The integral cohomology of TCS G_2 manifolds is known, as are some construction theorems for compact associative submanifolds useful for physics [9].

Our proposal and a U(1)^3 example. In [11] we studied a four-dimensional M-theory compactification on a TCS G_2 manifold X that exhibited $\mathrm{U}(1)^3$ gauge symmetry, a rich spectrum of massive charged particles, and (modulo issues of potential Wilson line moduli) instanton corrections to the superpotential. Furthermore, we studied a non-isolated conifold transition that broke one of the U(1)'s, and Kovalev's theorem ensures that the this deformation, too, is a compact G_2 manifold.

We would like to review this example in just enough detail to make it clear that it exemplifies the proposal we have outlined in section 4 for taking singular limits of compact G_2 manifolds. For the elements of the physics that we studied, it sufficed to focus on "one half" of X, that is, on one of the seven-manifolds $M = V \times S^1$. The ACyl Calabi-Yau threefold, originally studied in [10] but extended with new results in [11], is obtained as follows. Consider a one-parameter linear system of divisors — a pencil — in $\mathbb{P}^3$ generated by K3 surfaces S_0 and S_∞ in the anticanonical class, where S_∞ is a generic quartic and $S_0 = \{x_1x_2x_3x_4 = 0\}$ with x_i the standard homogeneous coordinates for $\mathbb{P}^3$. The base locus of this pencil is the union of curves C_i where each C_i is where $x_i = 0$ intersects S_∞. The threefold Z is obtained by blowing up sequentially along these curves, and then V is obtained from Z by cutting (the inverse image of) a generic member of the pencil $|S_0, S_\infty|$. There are 24 rigid holomorphic curves in Z that arise from the blow-up, and these sit away from the locus where S was cut out, so these are non-trivial in V and also in any G_2 manifold X formed from this building block.

M2-branes wrapped on those 24 rigid two-cycles give rise to massive charged particles in four dimensions, and in [11] we studied a limit in which some of these particles become massless, so that a conifold transition can be performed. Since two-cycles are not calibrated in a G_2 manifold, we took the singular limit by using the proposal of section 4. Namely, by a theorem of [9], to any rigid holomorphic curve C in V there is a compact rigid associative in X that is diffeomorphic to $S^2 \times S^1$, and the conifold limit can be taken in G_2 moduli by calibrating this associative to zero volume. Essentially it occurs by continuously following a family of ACyl Calabi-Yau manifolds V_t to V_0 which has conifold singularities, and Kovalev's theorem can be applied for any $t \neq 0$ to obtain a G_2 manifold. As in the Calabi-Yau case, we assume that there is a singular Ricci-flat metric even for $t = 0$. In summary, we obtained a massless particle limit by calibrating a compact rigid associative submanifold to zero volume, as proposed in section 4.

Towards non-Abelian gauge sectors from K3 two-cycles. How might one take a singular limit of a TCS G_2 manifold that exhibits non-abelian gauge symmetry? Though the following proposal would require some addition mathematical progress in order to work, let us discuss it since it may be a promising direction for future research. The basic idea is that, since TCS G_2 manifolds are fibered by (not necessarily coassociative) K3 surfaces, one may try to shrink curves in the K3 surfaces such that a singularity develops in the TCS G_2 manifold X.

Recall that one of the ways that we obtained control over particle masses in the case of the resolved circle of conifolds was that there was an associative three-cycle that contained a non-trivial two-cycle, and the limit of vanishing associative led to the collapse of the two-cycle. One of the reasons that this worked was the fact that

$$H^2(X,\mathbb{Z}) = K_+ \oplus K_- \oplus \cdots \qquad\qquad H^3(X,\mathbb{Z}) = K_+ \oplus K_- \oplus \cdots \tag{6.13}$$

where $K_\pm = ker(\rho_\pm)$ with $\rho_\pm : H^2(V_\pm) \to H^2(S_\pm)$ the natural restriction map. This implies that $K_\pm$ contributes both two-cycles and three-cycles, and it is clear how they arise: the former are curves $C_\pm \subset V_\pm$ that are non-trivial in X, but taking the product with $S^1_\pm$ in $M_\pm$ also gives three-cycles that are non-trivial in X. Moreover, these three-cycles have an associative representative, and we used that associative to indirectly control the mass of the M2-branes wrapped on $C_\pm$. However, $C_\pm$ were rigid, so those M2-branes gave rise to chiral multiplets, not vector multiplets.

What other non-trivial two-cycles are there in X that might instead give rise to non-abelian gauge enhancement if we could shrink them via shrinking associatives? The full second cohomology is

$$H^2(X,\mathbb{Z}) = K_+ \oplus K_- \oplus N_+ \cap N_- \tag{6.14}$$

where $N_\pm = Im(\rho_\pm)$ and dualizing to homology there may be two-dimensional submanifold representatives associated to $N_+ \cap N_-$; these would be divisors in $V_\pm$ that restrict to non-trivial curves on both of the K3 surfaces $S_\pm$. So, there can be non-trivial curves in X that are curves in the K3 fibers. M2-branes on these submanifolds give charged particles and one may like to study a massless limit of those particles for the sake of gauge enhancement. However, since (unlike $K_\pm$) $N_+ \cap N_-$ does not also contribute to $H^3(X,\mathbb{Z})$ or $H^4(X,\mathbb{Z})$ it seems more difficult to shrink those curves via a shrinking associative or coassociative. If a singular limit with non-abelian gauge symmetry exists associated to non-trivial $N_+ \cap N_-$, these would be Coulomb branches.

Alternatively, one could try to use associatives or coassociatives to control Higgs branches; that is, to use collapsing associatives or coassociatives to collapse two-cycles in them that are non-trivial in the K3 surfaces $S_\pm$ but are trivial in X. Doing so would require that there are curve classes in $S_\pm$ that contribute to the third or fourth (co) homology but not to the second. But in fact this is the case! Keeping the pieces of the third and fourth cohomology that are related to the second cohomology of the K3 surfaces, we have [9]

$$\begin{aligned} H^3(X,\mathbb{Z}) &= L/(N_+ \oplus N-) \oplus (N_- \cap T_+) \oplus (N_- \cap T_-) \oplus \cdots \\ H^4(X,\mathbb{Z}) &= (T_+ \cap T_-) \oplus L/(N_- \oplus T_+) \oplus L/(N_+ \oplus T_-) \oplus \cdots \end{aligned} \tag{6.15}$$

where $H^2(S_\pm) = L$ and

$$T_\pm = N_\pm^\perp = \{l \in H^2(S_\pm,\mathbb{Z}) | \langle l, n\rangle = 0 \;\forall\; n \in N_\pm\}. \tag{6.16}$$

To our knowledge, there are currently no theorems related to associative or coassociative representatives of these classes. However, if such calibrated submanifolds existed, it is natural to expect that they contain two-cycles that are non-trivial in $S_\pm$, but trivial in X. If those calibrated submanifolds were to collapse in a way that the two-cycles collapsed (as in our conifold idea and examples) then it could develop codimension four singularities in the $K3$ fibers of X, which could be a gauge enhanced singular limit.

Without construction theorems for associative or coassociative submanifolds related to two-cycles in $S_\pm$ it is difficult to move forward, but we find this a promising idea that should be pursued in future work.

7 Conclusions

In this paper we have studied gauge enhancement and singular limits in the compactification of M-theory on a seven-manifold X with holonomy G_2. Such singular limits are necessary for obtaining a realistic compactification, since M-theory on X gives rise to at most abelian gauge symmetry, and therefore cannot realize the standard model of particle physics. In fact, M-theory on X also cannot give rise to massless charged particle states.

Our paper develops a proposal and techniques for identifying when M-theory on a singular limit of X gives rise to massless charged matter or non-abelian gauge sectors. The proposal and techniques are motivated by an important fact about G_2 manifolds: they do not have calibrated two-cycles, and therefore it is difficult to track the volumes of two-cycles as a function of metric moduli. This should be contrasted to the case of M-theory on a Calabi-Yau manifold, where calibrated two-cycles are holomorphic curves and their volumes can be reliably computed as a function of moduli (specifically, Kähler moduli), even though the metric on the Calabi-Yau manifold is not known. This is physically important for the following reason. Since M2-branes on two-cycles in a smooth manifold give rise to massive charged particles, the existence of holomorphic curves in Calabi-Yau manifolds means the W-boson masses can be tracked reliably as a function of moduli, whereas they cannot in G_2 manifolds since the two-cycles are not calibrated. This is related to the fact that there are no BPS particles in the G_2 compactifications, since it has $\mathcal{N}=1$ supersymmetry in four dimensions, whereas compactification on a Calabi-Yau threefold gives an $\mathcal{N}=1$ theory in *five* dimensions, which does support BPS particles. Thus, the method of studying non-abelian gauge enhancement in Calabi-Yau compactifications by sending W-boson masses to zero via calibrating two-cycles to zero volume cannot be utilized in a G_2 compactification.

In section 4 we made a proposal for studying gauge enhancement and / or singular limits with massless charged particles in G_2 compactifications of M-theory, and the proposal may be stated both mathematically and physically. Physically, the proposal is to control harbingers of symmetry breaking other than the massive W-bosons used in Calabi-Yau compactifications. Depending on the example, this could include[17] BPS instantons, strings, or domain walls. Mathematically, this is possible since these objects arise from

[17]Another signal of symmetry breaking are the 't Hooft-Polyakov monopoles, but since these are particles in $d=4$ $\mathcal{N}=1$ theories they are not BPS and they are no more useful than massive W-bosons; this correlates with the fact that the monopoles arise from five-cycles in X, which are not calibrated.

M2-branes or M5-branes on calibrated submanifolds of X, which are so-called associative three-manifolds and coassociative four-manifolds. Therefore, mathematically the proposal is to study singular limits of M-theory on X by collapsing associative or coassociative submanifolds. In some cases this may collapse a two-cycle within the associative or coassociative, in which case one has gained indirect (in the sense that the two-cycle itself is not calibrated) control over a particle mass.

In the remainder of the paper we proceeded forward using the proposal, beginning with general aspects and proceeding toward increasingly concrete scenarios, finishing with concrete examples. We studied defects obtained from wrapped M2-branes and M5-branes in section in 4.2 and studied two types of singular limits in section 4.3. One of those limits collapsed a natural cycle that is explicitly related to symmetry. Namely, for any $[\sigma] \in H^2(X, \mathbb{Z})$ a lemma of Joyce shows that

$$[\sigma] \cup [\sigma] \cup [\Phi] < 0 \tag{7.1}$$

and therefore $-[\sigma] \cup [\sigma]$ is non-trivial in $H^4(X, \mathbb{Z})$ and there is a corresponding three cycle $[D_\Sigma]$ in $H_3(X, \mathbb{Z})$. This is related to a U(1) symmetry via Kaluza-Klein reduction of the M-theory three-form. However, if D_Σ is a calibrated submanifold representative of class $[D_\Sigma]$, then collapsing D_Σ is not a limit with gauge enhancement, since $vol(D_\Sigma)$ computes the gauge coupling; instead it is a strongly coupled limit. We mention it, though, since this class was useful in other ways.

In section 5 we studied some more specific scenarios for gauge enhancement, in particular Coulomb branches of a non-abelian gauge theory in section 5.1 and issues related to massless charged matter, string defects, and issues related to conifold transitions in the latter subsections of section 5. The Coulomb branch scenario involved the breaking the $G \to \mathrm{U}(1)^{rk(G)}$ by the expectation values of scalar fields in adjoint chiral multiplets. Accounting for the 't Hooft-Polyakov monopoles of this theory with M5-branes wrapped on non-trivial five-manifolds and considering the relationship between the (classical) monopole masses and W-boson masses, we were lead to a picture where the five-manifolds are fibered by two-spheres over a parameter space of class $[D_\Sigma]$. A three-cycle class related to the adjoint chiral multiplets appears naturally and is used for gauge enhancement. This general picture was exemplified in the first of Joyce's examples studied in section 6.2. We also studied conifold transitions and saw that the blowdown of the resolution of the circle of conifolds had a collapsing three-cycle, while the deformation to the Higgs branch has an emerging four-cycle; these more general phenomena were exemplified the $\mathrm{U}(1)^3$ TCS example discussed in section 6.3 and studied in [11].

It would be interesting to use the proposal and techniques developed in this paper for finding new singular limits of G_2 compactifications that exhibit non-abelian gauge symmetry. One possibility is to use our proposal for twisted connected sum G_2 manifolds, perhaps along the lines discussed in section 6.3. We leave this to future work.

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Acknowledgments

We thank Bobby Acharya, Allan Adams, Csaba Csáki, Mirjam Cvetič, Keshav Dasgupta, Simon Donaldson, Antonella Grassi, Tom Hartman, Gordy Kane, Nabil Iqbal, Albion Lawrence, André Lukas, Joe Polchinski, Radu Roiban, Julius Shaneson, Washington Taylor, and Scott Watson for useful conversations. We are particularly grateful to Thomas Walpuski for his detailed comments on the first version of this paper. J.H. is grateful to Antonella Grassi and Julius Shaneson for collaboration on related topics in F-theory, and thanks J.L. Halverson for her support and encouragement. We thank the Simons Center for Geometry and Physics for hospitality and the participants of the 2014 Simons Summer workshop for interesting discussions. We also thank the Aspen Center for Physics for hospitality and support under NSF grant PHY-1066293. This work is supported by NSF grants PHY-1307513 and PHY-1125915.

A Monopoles from elliptic Calabi-Yau Coulomb branches

Since M-theory Coulomb branches arising from Calabi-Yau compactification have been studied extensively in recent years, in particular as a tool to study F-theory, we would like to see how that story parallels Coulomb branches of G_2 compactifications.

Consider M-theory compactified on a smooth Calabi-Yau elliptic fibration $X \xrightarrow{\pi} B$ that is the resolution of a singular model $\tilde{X}$ that has ADE singularities along a divisor Z in B. Let us fix $dim_{\mathbb{C}}(B) = 3$ for concreteness, so that this is a $\mathcal{N} = 2$ M-theory compactification in $2+1$ dimensions. M-theory on $\tilde{X}$ has a non-abelian gauge theory on Z with some gauge group G, and the resolution breaks G to $\mathrm{U}(1)^{rk(G)}$ by giving an expectation value to the adjoint scalar field in the non-abelian vector multiplet. Though this discussion generalizes, for simplicity consider again $G = \mathrm{SU}(2)$. This is an M-theory Coulomb branch from Calabi-Yau compactification, and though it parallels the G_2 case, it also has some properties that the G_2 case doesn't have. First, geometrically, W-bosons and monopoles of this theory arise from M2-branes and M5-branes wrapped on curves and divisors, respectively, which (unlike the two-manifolds and the manifolds in real codimension two in the G_2 case) are calibrated submanifolds. Second, field theoretically, there are BPS particles in $d = 3$ $\mathcal{N} = 2$ theories.

The resolved geometry has a so-called Cartan divisor D, which is a $\mathbb{P}^1$ fibration over Z, and since Z is a divisor in a threefold D is itself a threefold. Let the fiber class be Σ. An M5-brane wrapped on Z gives rise to a spacetime instanton, which in $2+1$ dimensions can be a magnetic monopole. This is the correct interpretation in this case since the M5-brane is a magnetic source for C_3, which gives rise to the abelian vector multiplets of the Coulomb branch. M2-branes wrapped on the family of curves of class Σ give rise to the massive W-bosons of the Coulomb branch, though at codimension one subloci in Z it is possible that the fiber becomes a reducible variety, splitting according to $\Sigma = \Sigma_1 + \Sigma_2$. Note, though, that the total fiber volume remains the same even for the split fiber since

$$vol(\Sigma) = \int_\Sigma J = \int_{\Sigma_1} J + \int_{\Sigma_2} J = vol(\Sigma_1) + vol(\Sigma_2). \tag{A.1}$$

The volume of D is

$$vol(D) = \frac{1}{6}\int_D J \wedge J \wedge J = \frac{1}{2} vol(\Sigma)\, vol(Z) \tag{A.2}$$

where the last equality holds due to the fibration. These volumes determine the W-boson and monopole mass, respectively, but we haven't said anything about the gauge coupling yet. In fact, the gauge coupling depends on $vol(Z) \sim \frac{1}{g^2}$ which here plays a role analogous to $vol(D_\Sigma)$ in the G_2 Coulomb branch. So we see that the monopole mass, which depends on the volume of the Cartan divisor D, therefore depends on the parameters of the gauge theory as $|v|/g$ due to $vol(D) \sim vol(\Sigma) vol(Z)$, where $vol(\Sigma) \sim g|v|$ since it determines the W-boson mass.

In summary, this closely matches the G_2 Coulomb branch. In both cases (the $d = 4$ G_2 Coulomb branch and the $d = 3$ Calabi-Yau Coulomb branch of resolution) there is a real codimension two submanifold that is a non-trivial cycle. It is fibered by two-spheres over a real codimension four parameter space, and by wrapping an M5-brane on this real codimension two submanifold we obtain a magnetic monopole. Since in both cases the codimension two submanifold arises due to the smoothing / Higgsing process, the associated monopole is related to symmetry breaking. This is the behavior expected of 't Hooft-Polyakov monopoles, and in both cases the monopole mass dependence $|v|/g$ is recovered due to the relationship between geometric volumes and physical parameters in the broken gauge theory.

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