

Lattice formulation for 2d $\mathcal{N} = (2, 2)$, $(4, 4)$ super Yang-Mills theories without admissibility conditions

So Matsuura^a and Fumihiko Sugino^b

^a*Department of Physics, and Research and Education Center for Natural Science, Keio University, 4-1-1 Hiyoshi, Yokohama, 223-8521, Japan*

^b*Okayama Institute for Quantum Physics, Kyoyama 1-9-1, Kita-ku, Okayama 700-0015, Japan*

E-mail: s.matsu@phys-h.keio.ac.jp, fumihiko_sugino@pref.okayama.lg.jp

ABSTRACT: We present a lattice formulation for two-dimensional $\mathcal{N} = (2, 2)$ and $(4, 4)$ supersymmetric Yang-Mills theories that resolves vacuum degeneracy for gauge fields without imposing admissibility conditions. Cases of $U(N)$ and $SU(N)$ gauge groups are considered, gauge fields are expressed by unitary link variables, and one or two supercharges are preserved on the two-dimensional square lattice. There does not appear fermion doubler, and no fine-tuning is required to obtain the desired continuum theories in a perturbative argument. This formulation is expected to serve as a more convenient basis for numerical simulations. The same approach will also be useful to other two-dimensional supersymmetric lattice gauge theories with unitary link variables constructed so far — for example, $\mathcal{N} = (8, 8)$ supersymmetric Yang-Mills theory and $\mathcal{N} = (2, 2)$ supersymmetric QCD.

KEYWORDS: Lattice Gauge Field Theories, Lattice Quantum Field Theory, Extended Supersymmetry, Topological Field Theories

ARXIV EPRINT: [1402.0952](https://arxiv.org/abs/1402.0952)

Contents

1	Introduction	1
2	Brief review of the lattice formulation for 2d $\mathcal{N} = (2, 2)$ and $(4, 4)$ SYM theories	3
2.1	2d $\mathcal{N} = (2, 2)$ SYM theory on the lattice	3
2.2	2d $\mathcal{N} = (4, 4)$ SYM theory on the lattice	4
3	2d $\mathcal{N} = (2, 2)$ lattice SYM theory without the admissibility condition	6
3.1	$U(N)$ gauge group	7
3.2	$SU(N)$ gauge group	9
3.3	Convenient expressions for numerical simulation	11
4	2d $\mathcal{N} = (4, 4)$ lattice SYM theory without the admissibility condition	11
5	Summary and discussion	13
A	Potential walls of the gauge action for $SU(N)$ gauge group	14

1 Introduction

Lattice formulation is one of the most powerful ways to investigate nonperturbative aspects of quantum field theories, in particular gauge field theories. It gives not only rigorous definitions of quantum field theories, but also yields an environment to carry out numerical “experiments” on the theories. This approach has achieved a great success for lattice QCD. As a step along this line, it is quite natural and significant to head for lattice regularizations of supersymmetric gauge theories. In spite of difficulties of realizing supersymmetry on lattice, several lattice formulations of supersymmetric gauge theories which need no fine-tuning in taking the continuum limit have been developed. In particular, for theories with extended supersymmetries, some of supercharges are exactly preserved on lattice by applying the so-called orbifolding procedure [1–9] or the topological twists [10–19] to the discretization.

For $\mathcal{N} = 1$ pure supersymmetric Yang-Mills (SYM) theories in three and four dimensions, preserving chiral symmetry rather than supersymmetry on the lattice plays a key role to restore the supersymmetry and all the other symmetries in the continuum limit [20, 21]. Except for these cases, however, it will be hard to consider lattice regularizations for three- and four-dimensional supersymmetric theories in such a way that the continuum limit requires no tuning. In fact, the number of symmetries preserved on the lattice is generally too small to forbid relevant operators that prevent the lattice theory from restoring all the symmetries (including supersymmetries) of the target theory in the continuum limit.

As an approach to circumvent this issue, a hybrid regularization has been proposed for four-dimensional $\mathcal{N} = 2, 4$ SYM theories [22–24], where two different discretizations by lattice and matrix [25–27] are combined. Regarding to the planar limit, four-dimensional $\mathcal{N} = 4$ SYM theory can be obtained by using a large- N reduction technique [28, 29].

In [1–9], various theories have been formulated on exotic lattices by the orbifolding procedure from SYM matrix theories. In [10–12, 30–33], essentially the same formulations as the orbifold lattice theories have been independently developed [34–37] from different approaches.¹ In these formulations, the bosonic link variables are not unitary but complex matrices and the lattice spacing is introduced by fixing the trace part of these variables at a specific point in the flat directions. As a result, gauge groups which the formulations allow are $U(N)$ rather than $SU(N)$. Differently from the continuum theory, the overall $U(1)$ modes in $U(N)$ are always coupled with the remaining $SU(N)$ modes in the lattice theory. In particular, there are zero-modes in the $U(1)$ sector of fermions. In numerical simulations, therefore, we must introduce a large mass in the $U(1)$ part of the complex link variables in order to fix the lattice spacing and take care of the fermionic zero-modes in computing the Dirac matrix [39–41]. It is important to check decoupling of the $U(1)$ sector in the continuum limit in order to confirm that the correct continuum theory is obtained.

In [13–18], one of the authors of the present paper (F.S.) discretized topologically twisted gauge theories with preserving one or two supercharges. One characteristic feature of this formulation is that lattice gauge fields are expressed by compact link variables on the hypercubic lattice as in the conventional lattice gauge theories, which will be more convenient for numerical simulations [42–44]. In addition, it is valid for both of the gauge groups $U(N)$ and $SU(N)$. In particular, for the gauge group $SU(N)$, we do not need to be bothered about the $U(1)$ sector mentioned above. On the other hand, we have to take care of vacuum degeneracy of lattice gauge fields. In [14], an admissibility condition is imposed in order to single out the physical vacuum from the other unphysical vacua. Although this prescription works well, it yields complicated simulation codes in practice.

In this paper, we modify the formulation for two-dimensional $\mathcal{N} = (2, 2)$ and $(4, 4)$ SYM theories [13, 14] so that the vacuum degeneracy is resolved without imposing the admissibility condition. It is possible with keeping relevant symmetries to ensure no fine-tuning in the continuum limit for both of the gauge groups $U(N)$ and $SU(N)$. The modified formulation simplifies lattice actions, which will serve a more convenient basis for numerical simulations.

This paper is organized as follows. In the next section, we give a short review of the lattice formulation for two-dimensional $\mathcal{N} = (2, 2)$ and $(4, 4)$ SYM theories in [13, 14]. In section 3, the modification of the $\mathcal{N} = (2, 2)$ theory is discussed for the gauge groups $U(N)$ and $SU(N)$. Convenient expressions for actual numerical simulations are also presented. In section 4, the $\mathcal{N} = (4, 4)$ theory is modified in parallel with the $\mathcal{N} = (2, 2)$ case. The results obtained so far are summarized and some future directions are discussed in section 5. Appendix A is devoted to a proof that potential barriers of the $SU(N)$ lattice gauge action infinitely grow away from the physical vacuum, which makes any unphysical vacuum ineffective.

¹For a review, see [38].

2 Brief review of the lattice formulation for 2d $\mathcal{N} = (2, 2)$ and $(4, 4)$ SYM theories

In this section, we present a brief review of the lattice formulation for two-dimensional $\mathcal{N} = (2, 2)$ and $(4, 4)$ SYM theories constructed in [13, 14]. In what follows, x denotes a site of the two-dimensional square lattice \mathbb{Z}_L^2 , where L is the number of the sites in one direction. Lattice gauge fields are expressed by the group-valued variables $U_\mu(x) = e^{iaA_\mu(x)}$ on the link $(x, x + \hat{\mu})$, while all the other fields are algebra-valued variables put on the sites.

2.1 2d $\mathcal{N} = (2, 2)$ SYM theory on the lattice

Field contents of $\mathcal{N} = (2, 2)$ SYM theory on the two-dimensional lattice [13, 14] are as follows. Bosonic variables are the unitary link variables $U_\mu(x)$, scalar variables² $\phi(x)$ and $\bar{\phi}(x)$ and a hermitian auxiliary field $H(x)$. Fermionic variables are denoted by $\{\psi_\mu(x), \chi(x), \eta(x)\}$. Note that, when the gauge group is $SU(N)$, all the fields excluding $U_\mu(x)$ are traceless. These lattice variables are connected by the supersymmetry transformation:

$$\begin{aligned} QU_\mu(x) &= i\psi_\mu(x)U_\mu(x), & Q\psi_\mu(x) &= iD_\mu\phi(x) + i\psi_\mu(x)\psi_\mu(x), \\ Q\bar{\phi}(x) &= \eta(x), & Q\eta(x) &= [\phi(x), \bar{\phi}(x)], \\ Q\chi(x) &= H(x), & QH(x) &= [\phi(x), \chi(x)], & Q\phi(x) &= 0, \end{aligned} \quad (2.1)$$

where D_μ represents the covariant forward difference given by $D_\mu\varphi(x) \equiv U_\mu(x)\varphi(x + \hat{\mu})U_\mu(x)^\dagger - \varphi(x)$ for any adjoint field $\varphi(x)$ on the site. Note that Q is nilpotent up to infinitesimal gauge transformation with the parameter $\phi(x)$. The lattice action can be expressed as a Q -exact form:

$$\begin{aligned} S_{\text{lat}}^{(2,2)} &= Q \frac{1}{2g_0^2} \sum_x \text{Tr} \left[\frac{1}{4} \eta(x) [\phi(x), \bar{\phi}(x)] - i\chi(x) (\Phi(x) + iH(x)) - i \sum_{\mu=1}^2 \psi_\mu(x) D_\mu \bar{\phi}(x) \right] \\ &= \frac{1}{2g_0^2} \sum_x \text{Tr} \left[\frac{1}{4} [\phi(x), \bar{\phi}(x)]^2 + H(x)^2 - iH(x)\Phi(x) + \sum_{\mu=1}^2 D_\mu \phi(x) D_\mu \bar{\phi}(x) \right. \\ &\quad - \frac{1}{4} \eta(x) [\phi(x), \eta(x)] - \chi(x) [\phi(x), \chi(x)] \\ &\quad - \sum_{\mu=1}^2 \psi_\mu(x) \psi_\mu(x) \left(\bar{\phi}(x) + U_\mu(x) \bar{\phi}(x + \hat{\mu}) U_\mu(x)^\dagger \right) \\ &\quad \left. + i\chi(x) Q\Phi(x) + i \sum_{\mu=1}^2 \psi_\mu(x) D_\mu \eta(x) \right], \end{aligned} \quad (2.2)$$

where $\Phi(x)$ is a hermitian matrix depending on the plaquette variables,

$$U_{12}(x) \equiv U_1(x)U_2(x + \hat{1})U_1(x + \hat{2})^\dagger U_2(x)^\dagger \quad \text{and} \quad U_{21}(x) = U_{12}(x)^\dagger. \quad (2.3)$$

² $\phi(x)$ and $\bar{\phi}(x)$ can be treated as independent hermitian matrices in path-integrals of the theory.

After the auxiliary field is integrated out, the action of the gauge fields is given by

$$S_G^{U(N)} = \frac{1}{8g_0^2} \sum_x \text{Tr} (\Phi(x)^2) \quad \text{for } U(N), \quad (2.4)$$

$$S_G^{SU(N)} = \frac{1}{8g_0^2} \sum_x \text{Tr} \left(\left\{ \Phi(x) - \left(\frac{1}{N} \text{Tr} \Phi(x) \right) \mathbf{1}_N \right\}^2 \right) \quad \text{for } SU(N). \quad (2.5)$$

For the gauge group $SU(N)$, since the auxiliary field $H(x)$ is traceless, only the traceless part of $\Phi(x)$ contributes to the action. In order to obtain the correct continuum action, the expansion by the lattice spacing a of $\Phi(x)$ around $U_{12}(x) = \mathbf{1}_N$ must be

$$\Phi(x) = 2a^2 F_{12}(x) + \mathcal{O}(a^3), \quad (2.6)$$

where $F_{12}(x) \equiv \partial_1 A_2(x) - \partial_2 A_1(x) + i[A_1(x), A_2(x)]$. The simplest choice satisfying (2.6) is

$$\Phi(x) = -i (U_{12}(x) - U_{21}(x)). \quad (2.7)$$

However, this causes a problem: there are a number of unphysical degenerate vacua [13]. In the diagonal gauge,

$$U_{12}(x) = \text{diag} (e^{i\theta_1(x)}, \dots, e^{i\theta_N(x)}), \quad (2.8)$$

(2.7) becomes $\Phi(x) = \text{diag} (2 \sin \theta_1(x), \dots, 2 \sin \theta_N(x))$ which has zeros at $\theta_i(x) = 0$ and π . There remain $(N+1)^{L^2}$ degenerate vacua for the gauge group $U(N)$ after a subgroup of the gauge symmetry (permutations of the eigenvalues) is taken into account. See [14, 45] for vacuum degeneracy in the case of $SU(N)$. In [14], an admissibility condition is introduced in order to single out the trivial vacuum $U_{12} = \mathbf{1}_N$ as follows. When $\|1 - U_{12}(x)\| < \epsilon$ for $\forall x$, $S_{\text{lat}}^{(2,2)}$ is defined by (2.2) with the choice of $\Phi(x)$ as

$$\Phi(x) = \frac{-i (U_{12}(x) - U_{21}(x))}{1 - \frac{1}{\epsilon^2} \|1 - U_{12}(x)\|^2}, \quad (2.9)$$

otherwise $S_{\text{lat}}^{(2,2)} = +\infty$. Here, $\|\cdot\|$ is a norm of a matrix defined by $\|A\| \equiv \sqrt{\text{Tr} (AA^\dagger)}$, and ϵ is a positive number chosen in the range $0 < \epsilon < 2$ for $U(N)$, $0 < \epsilon < 2\sqrt{2}$ for $SU(N)$ with $N = 2, 3, 4$ and $0 < \epsilon < 2\sqrt{N} \sin(\frac{\pi}{N})$ for $SU(N)$ with $N \geq 5$. Thanks to this admissibility condition, we can restrict the value of the plaquette variables to the range $\|1 - U_{12}(x)\| < \epsilon$ without breaking the Q supersymmetry, and as a result we can single out the trivial vacuum $U_{12} = \mathbf{1}_N$ from the other unphysical vacua.

2.2 2d $\mathcal{N} = (4, 4)$ SYM theory on the lattice

In the lattice model for two-dimensional $\mathcal{N} = (4, 4)$ SYM theory [13, 14], there are scalar fields $B(x)$, $C(x)$, $\phi_\pm(x)$,³ and auxiliary fields $\tilde{H}_\mu(x)$ and $H(x)$ in addition to the link

³Here, we write the scalar fields $\phi(x)$ and $\bar{\phi}(x)$ in [13, 14] as $\phi_+(x)$ and $\phi_-(x)$, respectively.

variables $U_\mu(x)$. Fermionic fields are $\psi_{\pm\mu}(x)$, $\chi_\pm(x)$ and $\eta_\pm(x)$. The supersymmetry transformations preserved on the lattice

$$\begin{aligned}
Q_\pm U_\mu(x) &= i\psi_{\pm\mu}(x)U_\mu(x), \\
Q_\pm \psi_{\pm\mu}(x) &= i\psi_{\pm\mu}(x)\psi_{\pm\mu}(x) \pm iD_\mu\phi_\pm(x), \\
Q_\mp \psi_{\pm\mu}(x) &= \frac{i}{2}\{\psi_{+\mu}(x), \psi_{-\mu}(x)\} + \frac{i}{2}D_\mu C(x) \mp \tilde{H}_\mu(x), \\
Q_\pm \tilde{H}_\mu(x) &= -\frac{1}{2}\left[\psi_{\mp\mu}(x), \phi_\pm(x) + U_\mu(x)\phi_\pm(x + \hat{\mu})U_\mu(x)^\dagger\right] \\
&\quad \pm \frac{1}{4}\left[\psi_{\pm\mu}(x), C(x) + U_\mu(x)C(x + \hat{\mu})U_\mu(x)^\dagger\right] \\
&\quad \mp \frac{i}{2}D_\mu\eta_\pm(x) + \frac{i}{2}\left[\psi_{\pm\mu}(x), \tilde{H}_\mu(x)\right] \pm \frac{1}{4}\left[\psi_{\pm\mu}(x)\psi_{\pm\mu}(x), \psi_{\mp\mu}(x)\right], \\
Q_\pm B(x) &= \chi_\pm(x), \\
Q_\pm \chi_\pm(x) &= \pm[\phi_\pm(x), B(x)], \\
Q_\mp \chi_\pm(x) &= \frac{1}{2}[C(x), B(x)] \mp H(x), \\
Q_\pm H(x) &= [\phi_\pm(x), \chi_\mp(x)] \pm \frac{1}{2}[B(x), \eta_\pm(x)] \mp \frac{1}{2}[C(x), \chi_\pm(x)], \\
Q_\pm C(x) &= \eta_\pm(x), \\
Q_\pm \eta_\pm(x) &= \pm[\phi_\pm(x), C(x)], \\
Q_\mp \eta_\pm(x) &= \mp[\phi_+(x), \phi_-(x)], \\
Q_\pm \phi_\pm(x) &= 0, \\
Q_\mp \phi_\pm(x) &= \mp\eta_\pm(x)
\end{aligned} \tag{2.10}$$

are nilpotent in the sense that

$$\begin{aligned}
Q_+^2 &= (\text{infinitesimal gauge transformation with the parameter } \phi_+(x)), \\
Q_-^2 &= (\text{infinitesimal gauge transformation with the parameter } -\phi_-(x)), \\
\{Q_+, Q_-\} &= (\text{infinitesimal gauge transformation with the parameter } C(x)).
\end{aligned} \tag{2.11}$$

The lattice action can be expressed as the Q_+ and Q_- transformations of gauge invariant terms:

$$\begin{aligned}
S_{\text{lat}}^{(4,4)} &= Q_+ Q_- \frac{1}{2g_0^2} \sum_x \text{Tr} \left[-iB(x)\Phi(x) - \sum_{\mu=1}^2 \psi_{+\mu}(x)\psi_{-\mu}(x) \right. \\
&\quad \left. - \chi_+(x)\chi_-(x) - \frac{1}{4}\eta_+(x)\eta_-(x) \right],
\end{aligned} \tag{2.12}$$

which is written down more explicitly as

$$\begin{aligned}
S_{\text{lat}}^{(4,4)} = & \frac{1}{2g_0^2} \sum_x \text{Tr} \left[-i \left(\frac{1}{2} [C(x), B(x)] + H(x) \right) \Phi(x) + H(x)^2 \right. \\
& + i\chi_-(x)Q_+\Phi(x) - i\chi_+(x)Q_-\Phi(x) - iB(x)Q_+Q_-\Phi(x) \\
& - [\phi_+(x), B(x)][\phi_-(x), B(x)] - \frac{1}{4} [C(x), B(x)]^2 \\
& + \chi_+(x)[\phi_-(x), \chi_+(x)] - \chi_-(x)[\phi_+(x), \chi_-(x)] + \chi_+(x)[C(x), \chi_-(x)] \\
& - \chi_-(x)[B(x), \eta_+(x)] - \chi_+(x)[B(x), \eta_-(x)] \\
& + \frac{1}{4} [\phi_+(x), \phi_-(x)]^2 - \frac{1}{4} [\phi_+(x), C(x)][\phi_-(x), C(x)] \\
& \left. - \frac{1}{4} \eta_-(x)[\phi_+(x), \eta_-(x)] + \frac{1}{4} \eta_+(x)[\phi_-(x), \eta_+(x)] - \frac{1}{4} \eta_+(x)[C(x), \eta_-(x)] \right] \\
& + \frac{1}{2g_0^2} \sum_x \sum_{\mu=1}^2 \text{Tr} \left[\tilde{H}_\mu(x)^2 + \frac{1}{2} \psi_{+\mu}(x) \psi_{+\mu}(x) \psi_{-\mu}(x) \psi_{-\mu}(x) + D_\mu \phi_+(x) D_\mu \phi_-(x) \right. \\
& + \frac{1}{4} (D_\mu C(x))^2 + i\psi_{+\mu}(x) D_\mu \eta_-(x) + i\psi_{-\mu}(x) D_\mu \eta_+(x) \\
& - \psi_{+\mu}(x) \psi_{+\mu}(x) \left(\phi_-(x) + U_\mu(x) \phi_-(x + \hat{\mu}) U_\mu(x)^\dagger \right) \\
& + \psi_{-\mu}(x) \psi_{-\mu}(x) \left(\phi_+(x) + U_\mu(x) \phi_+(x + \hat{\mu}) U_\mu(x)^\dagger \right) \\
& \left. - \frac{1}{2} \{ \psi_{+\mu}(x), \psi_{-\mu}(x) \} \left(C(x) + U_\mu(x) C(x + \hat{\mu}) U_\mu(x)^\dagger \right) \right]. \quad (2.13)
\end{aligned}$$

Here, $\Phi(x)$ has the same property as that in the $\mathcal{N} = (2, 2)$ case, which provides the gauge field action (2.4) or (2.5) after $H(x)$ is integrated out. The problem of vacuum degeneracy mentioned in the $\mathcal{N} = (2, 2)$ case commonly arises in the $\mathcal{N} = (4, 4)$ case. Thus, imposing the same admissibility condition as in (2.9) resolves the problem [14].

3 2d $\mathcal{N} = (2, 2)$ lattice SYM theory without the admissibility condition

As mentioned in the introduction, it is better to find an expression of $\Phi(x)$ that has a unique vacuum at $U_{12} = \mathbf{1}_N$ without imposing the admissibility condition. At the first sight, however, it seems impossible because of the condition (2.6) and the periodicity of the plaquette variable. In fact, in the diagonal gauge (2.8), we can express $\Phi(x)$ as $\Phi(x) = \text{diag}(f(\theta_1(x)), \dots, f(\theta_N(x)))$, where $f(\theta)$ is a periodic function with the period 2π . The condition (2.6) means that $f(\theta)$ is linear around the origin: $f(\theta) = 2\theta + \mathcal{O}(\theta^2)$, and the periodicity of the plaquette variable suggests $f(-\pi) = f(\pi)$. There must be at least one zero in $\theta \in (-\pi, \pi]$ except for the origin as long as $f(\theta)$ is a regular function. However, if the regularity of $f(\theta)$ is relaxed, one can construct a desirable expression of $\Phi(x)$. In this section, we provide such an expression in two-dimensional $\mathcal{N} = (2, 2)$ lattice SYM theory for both of the gauge groups $U(N)$ and $SU(N)$.

3.1 $U(N)$ gauge group

For the gauge group $U(N)$, we show that the expression

$$\begin{aligned} \Phi^{(1)}(x) &= 2i(2 - U_{12}(x) - U_{21}(x))(U_{12}(x) - U_{21}(x))^{-1} \\ &\quad + 2i(U_{12}(x) - U_{21}(x))^{-1}(2 - U_{12}(x) - U_{21}(x)) \\ &\equiv \frac{4(2 - U_{12}(x) - U_{21}(x))}{-i(U_{12}(x) - U_{21}(x))} \end{aligned} \quad (3.1)$$

has desirable properties for $\Phi(x)$. In the diagonal gauge (2.8), $\theta_i(x)$ takes values in the range:

$$-\pi < \theta_i(x) \leq \pi, \quad (3.2)$$

and (3.1) can be expressed as

$$\Phi^{(1)}(x) = \text{diag}(f_1(\theta_1(x)), \dots, f_1(\theta_N(x))) \quad \text{with} \quad f_1(\theta) = 4 \tan \frac{\theta}{2}. \quad (3.3)$$

This clearly satisfies (2.6), and $f_1(\theta)$ is zero only at the origin. The point we could avoid the discussion above is that $f_1(\theta)$ is a periodic but not regular function; it diverges at the boundary of the region (3.2). Another important point we should notice is that the interactions in (3.1) are local, similarly to the case of the admissibility condition (2.9) discussed in [14, 46]. This guarantees a local field theory to be obtained from the lattice model in the continuum limit. Actually, (3.1) connects link variables separated by at most a few lattice sites.⁴

Let us next check the absence of fermion doublers. By plugging $U_\mu(x) = e^{iaA_\mu(x)}$ into a part of the action concerning fermion kinetic terms, i.e. the last line of the r.h.s. in (2.2) with (3.1) used for $\Phi(x)$, we read off the kinetic terms of the fermions in the limit $U_\mu(x) \rightarrow 1$. Then, some care is needed in taking this limit for

$$\begin{aligned} Q\Phi^{(1)}(x) &= -2iQ(U_{12}(x) + U_{21}(x))(U_{12}(x) - U_{21}(x))^{-1} \\ &\quad - 2i(2 - U_{12}(x) - U_{21}(x))(U_{12}(x) - U_{21}(x))^{-1} \\ &\quad \times Q(U_{12}(x) - U_{21}(x))(U_{12}(x) - U_{21}(x))^{-1} + (\text{h.c.}). \end{aligned} \quad (3.4)$$

We have

$$\Delta_1\psi_2(x) - \Delta_2\psi_1(x) + F_{12}(x)(\Delta_1\psi_2(x) - \Delta_2\psi_1(x))F_{12}(x)^{-1} + \mathcal{O}(a) \quad (3.5)$$

from the first term in the r.h.s. of (3.4), and

$$-F_{12}(x)(\Delta_1\psi_2(x) - \Delta_2\psi_1(x))F_{12}(x)^{-1} + \mathcal{O}(a) \quad (3.6)$$

from the second term, where Δ_μ is the forward difference given by $\Delta_\mu\varphi(x) \equiv \varphi(x+\hat{\mu}) - \varphi(x)$. Consequently, terms containing $F_{12}(x)^{-1}$ cancel with each other, and the regular limit

$$Q\Phi^{(1)}(x) \Big|_{U_\mu \rightarrow 1} = 2(\Delta_1\psi_2(x) - \Delta_2\psi_1(x)) \quad (3.7)$$

⁴Note that $(U_{12}(x) - U_{21}(x))^{-1}$ is local on the lattice, since the inverse is taken with respect not to lattice sites but to internal gauge indices.

is obtained. By using the backward difference Δ_μ^* ($\Delta_\mu^* \varphi(x) \equiv \varphi(x) - \varphi(x - \hat{\mu})$), the fermion kinetic terms are expressed as

$$S_F^{\text{kin}} = \frac{1}{2g_0^2} \sum_x \text{Tr} \left[-\Psi(x)^T \frac{1}{2} \gamma_\mu (\Delta_\mu + \Delta_\mu^*) \Psi(x) - \Psi(x)^T \frac{1}{2} P_\mu \Delta_\mu \Delta_\mu^* \Psi(x) \right], \quad (3.8)$$

where $\Psi(x) = (\psi_1(x), \psi_2(x), \chi(x), \frac{1}{2}\eta(x))^T$, and matrices

$$\gamma_1 = -i\sigma_1 \otimes \sigma_1, \quad \gamma_2 = i\sigma_1 \otimes \sigma_3, \quad P_1 = \sigma_1 \otimes \sigma_2, \quad P_2 = \sigma_2 \otimes \mathbf{1}_2 \quad (3.9)$$

satisfy the algebra:

$$\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}, \quad \{P_\mu, P_\nu\} = 2\delta_{\mu\nu}, \quad \{\gamma_\mu, P_\nu\} = 0. \quad (3.10)$$

This is the same kinetic action obtained in [13], and thus no fermion doubler appears.⁵

Notice that the lattice action has symmetries under

- lattice translation
- gauge transformation
- Q -supersymmetry transformation (2.1)
- global $U(1)_R$ rotation:

$$\begin{aligned} U_\mu(x) &\rightarrow U_\mu(x), & \psi_\mu(x) &\rightarrow e^{i\alpha} \psi_\mu(x), \\ \phi(x) &\rightarrow e^{2i\alpha} \phi(x), & \bar{\phi}(x) &\rightarrow e^{-2i\alpha} \bar{\phi}(x), \\ H(x) &\rightarrow H(x), & \chi(x) &\rightarrow e^{-i\alpha} \chi(x), & \eta(x) &\rightarrow e^{-i\alpha} \eta(x) \end{aligned} \quad (3.11)$$

- reflection: $x \equiv (x_1, x_2) \rightarrow \tilde{x} \equiv (x_2, x_1)$ with

$$\begin{aligned} (U_1(x), U_2(x)) &\rightarrow (U_2(\tilde{x}), U_1(\tilde{x})), \\ (\psi_1(x), \psi_2(x)) &\rightarrow (\psi_2(\tilde{x}), \psi_1(\tilde{x})), \\ (H(x), \chi(x)) &\rightarrow (-H(\tilde{x}), -\chi(\tilde{x})), \\ (\phi(x), \bar{\phi}(x), \eta(x)) &\rightarrow (\phi(\tilde{x}), \bar{\phi}(\tilde{x}), \eta(\tilde{x})) \end{aligned} \quad (3.12)$$

which are the same as the symmetries discussed for renormalization in [14].⁶ Hence we can repeat the renormalization argument in [14], and it is shown that no fine-tuning is required in taking the continuum limit to all orders in the perturbation theory. We conclude that the choice (3.1) provides a nonperturbative definition of the two-dimensional $U(N)$ $\mathcal{N} = (2, 2)$ SYM theory.

⁵The second term of (3.8) can be regarded as a kind of the Wilson term twisted by the matrices P_μ . Lattice fermion actions with such twisted terms are discussed in [47].

⁶Path-integral measures are also invariant under these transformations [16].

3.2 SU(N) gauge group

For the gauge group SU(N), (2.5) means that the vacua for the gauge fields are determined by

$$\Phi(x) - \left(\frac{1}{N} \text{Tr} (\Phi(x)) \right) \mathbf{1}_N = 0. \quad (3.13)$$

Unfortunately, it turns out that the expression (3.1) for $\Phi(x)$ cannot be applied here except for the SU(2) case. In fact, plugging (2.8) with

$$-\pi < \theta_i(x) \leq \pi \quad (i = 1, \dots, N-1), \quad \theta_N(x) = -\theta_1(x) - \dots - \theta_{N-1}(x) \quad (3.14)$$

into the equation (3.13), we find

$$U_{12}(x) = \exp \left(\frac{2\pi ni}{N} \right) \mathbf{1}_N \quad (n = 0, \dots, N-1) \quad (3.15)$$

which are nothing but the \mathbb{Z}_N center of SU(N). There are still N degenerate vacua for each plaquette.

Here, we should note that in the SU(2) case the degeneracy is harmless and (3.1) remains valid. Since the minimum at $\theta = \pi$ coincides with the singular point of the function $\tan \frac{\theta}{2}$ and the gauge field action around this solution diverges, it is decoupled from the theory.

Motivated by this observation, in order to avoid the degeneracy for general N , we propose the following expression:

$$\begin{aligned} \Phi^{(M)}(x) &= \frac{2i}{M} \left((2 - U_{12}(x)^M - U_{21}(x)^M) (U_{12}(x)^M - U_{21}(x)^M)^{-1} \right. \\ &\quad \left. + (U_{12}(x)^M - U_{21}(x)^M)^{-1} (2 - U_{12}(x)^M - U_{21}(x)^M) \right) \\ &\equiv \frac{4}{M} \frac{2 - U_{12}(x)^M - U_{21}(x)^M}{-i(U_{12}(x)^M - U_{21}(x)^M)} \end{aligned} \quad (3.16)$$

with $M = 1, 2, \dots$. This is applicable for any N satisfying $N \leq 2M$. In the diagonal gauge (2.8),

$$\Phi^{(M)}(x) = \text{diag} (f_M(\theta_1(x)), \dots, f_M(\theta_N(x))) \quad \text{with} \quad f_M(\theta) = \frac{4}{M} \tan \frac{M\theta}{2}. \quad (3.17)$$

There are potential walls of infinite height at $\theta_i(x) = \pm \frac{\pi}{M}, \pm \frac{3\pi}{M}, \dots$ in the gauge action (2.5) with (3.16) used for $\Phi(x)$:

$$\begin{aligned} S_G^{\text{SU}(N)} &= \frac{1}{8g_0^2} \sum_x \text{Tr} \left(\left\{ \Phi^{(M)}(x) - \left(\frac{1}{N} \text{Tr} \Phi^{(M)}(x) \right) \mathbf{1}_N \right\}^2 \right) \\ &= \frac{2}{M^2 N^2 g_0^2} \sum_x \mathcal{L}_G^{\text{SU}(N)}(x), \\ \mathcal{L}_G^{\text{SU}(N)}(x) &\equiv \left\{ (N-1) \tan \left(\frac{M\theta_1(x)}{2} \right) - \sum_{i=2}^N \tan \left(\frac{M\theta_i(x)}{2} \right) \right\}^2 \\ &\quad + (\text{cyclic permutations of } \theta_1(x), \dots, \theta_N(x)). \end{aligned} \quad (3.18)$$

Let us consider the interval $(-\frac{\pi}{M}, \frac{\pi}{M})$ between the two walls nearest from the origin for each $\theta_i(x)$. Then, the point $(\theta_1(x), \dots, \theta_N(x))$ can move inside the N -dimensional hypercube

$$(\theta_1(x), \dots, \theta_N(x)) \in \left(-\frac{\pi}{M}, \frac{\pi}{M}\right)^N \tag{3.19}$$

satisfying the unimodular condition

$$\theta_1(x) + \dots + \theta_N(x) = 0. \tag{3.20}$$

It is easy to see that the gauge action (3.18) has a unique minimum at the origin in the region defined by (3.19) and (3.20). The region does not include any nontrivial \mathbb{Z}_N vacuum because of $\frac{2\pi}{N} \geq \frac{\pi}{M}$ for $N \leq 2M$. In appendix A, we show that whenever the point $(\theta_1(x), \dots, \theta_N(x))$ approaches the boundary of the region, (3.18) diverges as the inverse square of distance from the boundary. Although there are several vacua outside the region we are considering, the growth of the potential near the boundary implies that the tunneling probability from the trivial vacuum to any other vacuum is zero. Namely, the trivial vacuum is effectively singled out as long as initial field configurations (in a numerical simulation) are around $U_{12}(x) = \mathbf{1}_N$ for $\forall x$. Similarly to the $U(N)$ case (3.1), the interactions in (3.16) are local on the lattice.

Furthermore, it is straightforward to show

$$Q\Phi^{(M)}(x) \Big|_{U_\mu \rightarrow 1} = 2(\Delta_1\psi_2(x) - \Delta_2\psi_1(x)) \tag{3.21}$$

by noting

$$\begin{aligned} U_{12}(x)^M + U_{21}(x)^M &= 2 - a^4 M^2 F_{12}(x)^2 + \mathcal{O}(a^5), \\ U_{12}(x)^M - U_{21}(x)^M &= i2a^2 M F_{12}(x) + \mathcal{O}(a^3). \end{aligned} \tag{3.22}$$

We see the absence of fermion doublers by repeating the same discussion as in the $U(N)$ case. The fact that the lattice action enjoys the same symmetries as mentioned in the $U(N)$ case is sufficient to obtain the desired continuum theory without fine-tuning at the level of perturbative expansions to all orders [14]. Therefore, we can conclude that the choice (3.16) provides a nonperturbative definition of the two-dimensional $SU(N)$ $\mathcal{N} = (2, 2)$ SYM theory.

Before closing this subsection, it should be noted that (3.16) with general M can also be applied to the $U(N)$ case.⁷ There is a unique vacuum at the origin in the region (3.19). (Note that (3.20) is not imposed in the $U(N)$ case.) Since the gauge action (2.4) with (3.16) used for $\Phi(x)$ in the diagonal gauge

$$S_G^{U(N)} = \frac{1}{8g_0^2} \sum_x \text{Tr} \left(\Phi^{(M)}(x)^2 \right) = \frac{2}{M^2 g_0^2} \sum_x \sum_{i=1}^N \tan^2 \left(\frac{M\theta_i(x)}{2} \right) \tag{3.23}$$

infinitely grows near the boundary of the N -dimensional hypercube, configurations are always confined inside the hypercube once we start with initial configurations satisfying (3.19).

⁷(3.1) is included in (3.16) as a special case of $M = 1$.

3.3 Convenient expressions for numerical simulation

Although the expressions (3.1) and (3.16) are well-defined from the mathematical point of view, they would not be appropriate for numerical simulations. In fact, since importance sampling is expected to mainly pick up configurations near the vacuum $U_{12}(x) = \mathbf{1}_N$, we would encounter the loss of significance in both of the numerator and the denominator due to $U_{12}(x) \sim U_{21}(x) \sim \mathbf{1}_N$. In addition, it would take much machine time to compute the inverse of $U_{12}(x) - U_{21}(x)$ which becomes almost zero-matrix. For actual numerical simulations, it is better to rewrite them in a form which is apparently regular at the origin.

In the $SU(N)$ gauge group, (3.16) with M even can be recast as

$$\begin{aligned} \Phi^{(2m)}(x) &= \frac{-i}{m} \left((U_{12}(x)^m - U_{21}(x)^m)(U_{12}(x)^m + U_{21}(x)^m)^{-1} \right. \\ &\quad \left. + (U_{12}(x)^m + U_{21}(x)^m)^{-1}(U_{12}(x)^m - U_{21}(x)^m) \right) \\ &\equiv \frac{2}{m} \frac{-i(U_{12}(x)^m - U_{21}(x)^m)}{U_{12}(x)^m + U_{21}(x)^m} \end{aligned} \tag{3.24}$$

with $m = 1, 2, \dots$. This is apparently regular at $U_{12}(x) = \mathbf{1}_N$ and applicable to $SU(N)$ with $N \leq 4m$. Interestingly, (3.24) also does the job in the $U(N)$ case for any m .⁸ Upon using (3.24) for numerical simulations, initial configurations should be chosen in the range $\theta_i(x) \in (-\frac{\pi}{2m}, \frac{\pi}{2m})$ for $\forall i, x$ in the diagonal gauge (2.8). Then, the expression (3.24) will give a convenient numerical means for both of the gauge groups $U(N)$ and $SU(N)$.

We here add another comment on an actual numerical simulation. In a hybrid Monte Carlo simulation on a computer, a field configuration develops along the Monte Carlo time not continuously but stepwise by an appropriate molecular dynamics. Due to the effect of a finite time step, even if there are potential walls of infinite height, the configuration might “jump” over the wall from the physical vacuum to an unphysical vacuum. Although such a phenomenon is harder to occur as the time step becomes finer, it is safe to check if the configuration always stays around the physical vacuum by measuring the distance of the link variables from the physical vacuum (the unit matrix) during the simulation. When the “jump” is detected, we are to reject the corresponding configuration by hand. In case that the “jump” occurs frequently, the result of the simulation is no longer reliable because the detailed balance condition will be seriously broken by the rejection. In a simulation, therefore, it would be important to tune the time step of the molecular dynamics keeping the configuration staying inside the walls.

4 2d $\mathcal{N} = (4, 4)$ lattice SYM theory without the admissibility condition

We can repeat almost all discussions in section 3 in two-dimensional $\mathcal{N} = (4, 4)$ lattice SYM theory for the gauge groups $U(N)$ and $SU(N)$. The only differences are arguments for the absence of fermion doublers and renormalization.

⁸In practice, $m = 1$ will be convenient.

Absence of fermion doublers. Similarly to (3.21) in the $\mathcal{N} = (2, 2)$ case, we obtain

$$Q_{\pm}\Phi^{(M)}(x)\Big|_{U_{\mu}\rightarrow 1} = 2(\Delta_1\psi_{\pm 2}(x) - \Delta_2\psi_{\pm 1}(x)), \quad (4.1)$$

which leads to the fermion kinetic terms of the form (3.8) with

$$\Psi(x) = \left(\psi_{+1}(x), \psi_{+2}(x), \chi_{+}(x), \frac{1}{2}\eta_{+}(x), \psi_{-1}(x), \psi_{-2}(x), \chi_{-}(x), \frac{1}{2}\eta_{-}(x)\right)^T$$

and

$$\begin{aligned} \gamma_1 &= \begin{pmatrix} & & -i\sigma_1 \\ & -\sigma_2 & \\ \sigma_2 & & \\ -i\sigma_1 & & \end{pmatrix}, & \gamma_2 &= \begin{pmatrix} & & i\sigma_3 \\ & -i\mathbf{1}_2 & \\ i\sigma_3 & & \\ & & \end{pmatrix}, \\ P_1 &= \begin{pmatrix} & & \sigma_2 \\ & i\sigma_1 & \\ -i\sigma_1 & & \\ \sigma_2 & & \end{pmatrix}, & P_2 &= \begin{pmatrix} & & -i\mathbf{1}_2 \\ & -i\sigma_3 & \\ i\sigma_3 & & \\ i\mathbf{1}_2 & & \end{pmatrix}. \end{aligned} \quad (4.2)$$

This is valid for both of the gauge groups $U(N)$ and $SU(N)$. The matrices have the same form as discussed in [14] and satisfy (3.10), which guarantees the absence of fermion doublers.

Renormalization. The lattice theories with the gauge groups $U(N)$ and $SU(N)$ have symmetries under

- lattice translation
- gauge transformation
- Q_{\pm} -supersymmetry transformations (2.10)
- global $SU(2)_R$ rotation generated by

$$\begin{aligned} J_{\pm\pm} &= \sum_{x,\alpha} \left[\sum_{\mu=1}^2 \psi_{\pm\mu}^{\alpha}(x) \frac{\partial}{\partial\psi_{\mp\mu}^{\alpha}(x)} + \chi_{\pm}^{\alpha}(x) \frac{\partial}{\partial\chi_{\mp}^{\alpha}(x)} - \eta_{\pm}^{\alpha}(x) \frac{\partial}{\partial\eta_{\mp}^{\alpha}(x)} \right. \\ &\quad \left. \pm 2\phi_{\pm}^{\alpha}(x) \frac{\partial}{\partial C^{\alpha}(x)} \mp C^{\alpha}(x) \frac{\partial}{\partial\phi_{\mp}^{\alpha}(x)} \right], \\ J_0 &= \sum_{x,\alpha} \left[\sum_{\mu=1}^2 \left(\psi_{+\mu}^{\alpha}(x) \frac{\partial}{\partial\psi_{+\mu}^{\alpha}(x)} - \psi_{-\mu}^{\alpha}(x) \frac{\partial}{\partial\psi_{-\mu}^{\alpha}(x)} \right) + \chi_{+}^{\alpha}(x) \frac{\partial}{\partial\chi_{+}^{\alpha}(x)} \right. \\ &\quad - \chi_{-}^{\alpha}(x) \frac{\partial}{\partial\chi_{-}^{\alpha}(x)} + \eta_{+}^{\alpha}(x) \frac{\partial}{\partial\eta_{+}^{\alpha}(x)} - \eta_{-}^{\alpha}(x) \frac{\partial}{\partial\eta_{-}^{\alpha}(x)} \\ &\quad \left. + 2\phi_{+}^{\alpha}(x) \frac{\partial}{\partial\phi_{+}^{\alpha}(x)} - 2\phi_{-}^{\alpha}(x) \frac{\partial}{\partial\phi_{-}^{\alpha}(x)} \right] \end{aligned} \quad (4.3)$$

(α labels a basis of gauge group generators) which satisfy

$$[J_0, J_{\pm\pm}] = \pm 2J_{\pm\pm}, \quad [J_{++}, J_{--}] = J_0 \quad (4.4)$$

- $Q_+ \leftrightarrow Q_-$ with

$$\begin{aligned} \phi_{\pm} &\rightarrow -\phi_{\mp}, & B &\rightarrow -B, & \tilde{H}_{\mu} &\rightarrow -\tilde{H}_{\mu}, \\ \chi_{\pm} &\rightarrow -\chi_{\mp}, & \psi_{\pm\mu} &\rightarrow \psi_{\mp\mu}, & \eta_{\pm} &\rightarrow \eta_{\mp} \end{aligned} \quad (4.5)$$

- reflection: $x \equiv (x_1, x_2) \rightarrow \tilde{x} \equiv (x_2, x_1)$ with

$$\begin{aligned} (U_1(x), U_2(x)) &\rightarrow (U_2(\tilde{x}), U_1(\tilde{x})), \\ (\psi_{\pm 1}(x), \psi_{\pm 2}(x)) &\rightarrow (\psi_{\pm 2}(\tilde{x}), \psi_{\pm 1}(\tilde{x})), \\ (\tilde{H}_1(x), \tilde{H}_2(x)) &\rightarrow (\tilde{H}_2(\tilde{x}), \tilde{H}_1(\tilde{x})), \\ (H(x), B(x), \chi_{\pm}(x)) &\rightarrow (-H(\tilde{x}), -B(\tilde{x}), -\chi_{\pm}(\tilde{x})), \\ (\phi_{\pm}(x), C(x), \eta_{\pm}(x)) &\rightarrow (\phi_{\pm}(\tilde{x}), C(\tilde{x}), \eta_{\pm}(\tilde{x})). \end{aligned} \quad (4.6)$$

As discussed in [14], these symmetries are sufficient to show that the lattice theories do not need fine-tuning in taking the continuum limit to all orders in perturbative expansions.

As a conclusion, the $\mathcal{N} = (4, 4)$ lattice model (2.12) with (3.16) used for $\Phi(x)$ non-perturbatively defines two-dimensional $\mathcal{N} = (4, 4)$ SYM theories with gauge groups $U(N)$ and $SU(N)$. In actual numerical simulations for the $\mathcal{N} = (4, 4)$ SYM theories, the lattice action with (3.24) will be convenient for both of the gauge groups $U(N)$ and $SU(N)$.

5 Summary and discussion

In this paper, we have modified the lattice formulation of two-dimensional $\mathcal{N} = (2, 2)$ and $(4, 4)$ SYM theories discussed in [13, 14] in such a way that lattice actions resolve vacuum degeneracy for gauge fields without imposing admissibility conditions for both of the gauge groups $U(N)$ and $SU(N)$. The modification yields simpler lattice actions, which will make numerical simulations more feasible.

This method will also be useful to other two-dimensional supersymmetric lattice gauge theories with unitary link variables constructed so far — for example, $\mathcal{N} = (8, 8)$ SYM theory with mass deformations [22] and $\mathcal{N} = (2, 2)$ supersymmetric QCD [17, 18]. Since the former case is expected to give a nonperturbative construction of four-dimensional $\mathcal{N} = 4$ SYM theory [22], it would lead to some simplification for the construction of the four-dimensional theory. For the latter case, especially the model constructed in [18], chiral flavor symmetry of matter supermultiplets is preserved on the lattice by using the Ginsparg-Wilson formulation [46, 48]. The admissibility condition is used there for resolving the vacuum degeneracy for gauge fields as well as for locality of the overlap Dirac operator [49–51]. Interestingly, it suggests that our method can also be applied to formulate chiral gauge theories without imposing admissibility conditions.

Acknowledgments

The authors would like to thank T. Misumi and K. Ohta for useful discussion. The work of S. M. is supported in part by Grant-in-Aid for Young Scientists (B), 23740197 and Keio Gijuku Academic Development Funds. The work of F. S. is supported in part by Grant-in-Aid for Scientific Research (C), 25400289.

A Potential walls of the gauge action for $SU(N)$ gauge group

In this appendix, we show that near the boundary of the region defined by (3.19) and (3.20) the gauge action (3.18) increases as the inverse square of distance from the boundary.

Let us consider $\mathcal{L}_G^{\text{SU}(N)}(x)$ in (3.18) for an arbitrary fixed x . In $N = 2$ case, $|\theta_1(x)| = \frac{\pi}{M} - \epsilon(x)$ ($0 < \epsilon(x) \ll 1$) and $\theta_2(x) = -\theta_1(x)$ near the boundary. Then, $\mathcal{L}_G^{\text{SU}(2)}(x)$ behaves as

$$\mathcal{L}_G^{\text{SU}(2)}(x) = 8 \tan^2 \left(\frac{M\theta_1(x)}{2} \right) = 8 \cot^2 \left(\frac{M\epsilon(x)}{2} \right) = \frac{32}{M^2} \frac{1}{\epsilon(x)^2} + (\text{finite}), \quad (\text{A.1})$$

which diverges as the inverse square of the distance from the boundary $\epsilon(x)$.

For $N \geq 3$, when the point $(\theta_1(x), \dots, \theta_N(x))$ approaches the boundary, all the possibilities are exhausted by the following cases (I) and (II):

- (I) Among the coordinates $\{\theta_1(x), \dots, \theta_N(x)\}$, some coordinate (say $\theta_k(x)$) approaches $\frac{\pi}{M}$ or $-\frac{\pi}{M}$, while some other coordinate ($\theta_\ell(x)$) does not.
- (II) Every coordinate approaches $\frac{\pi}{M}$ or $-\frac{\pi}{M}$.

By introducing an N -dimensional real vector $\vec{A}(x) = (A_1(x), \dots, A_N(x))^T$ with

$$A_i(x) \equiv (N-1) \tan \left(\frac{M\theta_i(x)}{2} \right) - \sum_{j(\neq i)} \tan \left(\frac{M\theta_j(x)}{2} \right) \quad (i = 1, \dots, N) \quad (\text{A.2})$$

and an arbitrary N -dimensional unit vector $\vec{n} = (n_1, \dots, n_N)^T$, we put a lower bound to $\mathcal{L}_G^{\text{SU}(N)}(x)$:

$$\begin{aligned} \mathcal{L}_G^{\text{SU}(N)}(x) &= |\vec{A}(x)|^2 = |\vec{A}(x)|^2 |\vec{n}|^2 \geq (\vec{A}(x) \cdot \vec{n})^2 \\ &= \left[\sum_{i=1}^N \left\{ n_i(N-1) - \sum_{j(\neq i)} n_j \right\} \tan \left(\frac{M\theta_i(x)}{2} \right) \right]^2. \end{aligned} \quad (\text{A.3})$$

Case (I). $\theta_k(x)$ satisfies $|\theta_k(x)| = \frac{\pi}{M} - \epsilon_k(x)$ with $\epsilon_k(x) \rightarrow +0$, and $\theta_\ell(x)$ remains to give a finite value of $\tan \left(\frac{M\theta_\ell(x)}{2} \right)$. Use of the bound (A.3) with the choice of \vec{n} :

$$n_k = -n_\ell = \frac{1}{\sqrt{2}}, \quad \text{the other components are zero} \quad (\text{A.4})$$

leads to

$$\begin{aligned} \mathcal{L}_G^{\text{SU}(N)}(x) &\geq \frac{N^2}{2} \left\{ \tan \left(\frac{M\theta_k(x)}{2} \right) - \tan \left(\frac{M\theta_\ell(x)}{2} \right) \right\}^2 \\ &= \frac{N^2}{2} \left\{ \cot \left(\frac{M\epsilon_k(x)}{2} \right) - (\text{finite}) \right\}^2 = \frac{2N^2}{M^2} \frac{1}{\epsilon_k(x)^2} + \mathcal{O}(\epsilon_k(x)^{-1}). \end{aligned} \quad (\text{A.5})$$

Note that the expression (3.18) becomes no more singular than inverse squared with respect to distance from the boundary. Together with the bound (A.5), we can conclude that $\mathcal{L}_G^{\text{SU}(N)}(x)$ diverges as $\epsilon_k(x)^{-2}$.

Case (II). Clearly from (3.20), there are two coordinates $\theta_k(x)$ and $\theta_\ell(x)$ such that $\theta_k(x) = \frac{\pi}{M} - \epsilon_k(x)$ ($\epsilon_k(x) \rightarrow +0$) and $\theta_\ell(x) = -\frac{\pi}{M} + \epsilon_\ell(x)$ ($\epsilon_\ell(x) \rightarrow +0$). Then, applying the choice (A.4) to the bound (A.3), we find

$$\begin{aligned} \mathcal{L}_G^{\text{SU}(N)}(x) &\geq \frac{N^2}{2} \left\{ \cot\left(\frac{M\epsilon_k(x)}{2}\right) + \cot\left(\frac{M\epsilon_\ell(x)}{2}\right) \right\}^2 \\ &= \frac{2N^2}{M^2} \left(\frac{1}{\epsilon_k(x)} + \frac{1}{\epsilon_\ell(x)} \right)^2 + (\text{finite}). \end{aligned} \tag{A.6}$$

Similarly to the previous case, it is shown that $\mathcal{L}_G^{\text{SU}(N)}(x)$ diverges as the inverse square of distance from the boundary.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] D.B. Kaplan, E. Katz and M. Ünsal, *Supersymmetry on a spatial lattice*, *JHEP* **05** (2003) 037 [[hep-lat/0206019](https://arxiv.org/abs/hep-lat/0206019)] [[INSPIRE](https://inspirehep.net/literature/59471)].
- [2] A.G. Cohen, D.B. Kaplan, E. Katz and M. Ünsal, *Supersymmetry on a Euclidean spacetime lattice. 1. A target theory with four supercharges*, *JHEP* **08** (2003) 024 [[hep-lat/0302017](https://arxiv.org/abs/hep-lat/0302017)] [[INSPIRE](https://inspirehep.net/literature/59471)].
- [3] A.G. Cohen, D.B. Kaplan, E. Katz and M. Ünsal, *Supersymmetry on a Euclidean spacetime lattice. 2. Target theories with eight supercharges*, *JHEP* **12** (2003) 031 [[hep-lat/0307012](https://arxiv.org/abs/hep-lat/0307012)] [[INSPIRE](https://inspirehep.net/literature/59471)].
- [4] D.B. Kaplan and M. Ünsal, *A Euclidean lattice construction of supersymmetric Yang-Mills theories with sixteen supercharges*, *JHEP* **09** (2005) 042 [[hep-lat/0503039](https://arxiv.org/abs/hep-lat/0503039)] [[INSPIRE](https://inspirehep.net/literature/59471)].
- [5] M.G. Endres and D.B. Kaplan, *Lattice formulation of (2, 2) supersymmetric gauge theories with matter fields*, *JHEP* **10** (2006) 076 [[hep-lat/0604012](https://arxiv.org/abs/hep-lat/0604012)] [[INSPIRE](https://inspirehep.net/literature/59471)].
- [6] J. Giedt, *A deconstruction lattice description of the D1/D5 brane world-volume gauge theory*, *Adv. High Energy Phys.* **2011** (2011) 241419 [[hep-lat/0605004](https://arxiv.org/abs/hep-lat/0605004)] [[INSPIRE](https://inspirehep.net/literature/59471)].
- [7] S. Matsuura, *Two-dimensional $N = (2, 2)$ supersymmetric lattice gauge theory with matter fields in the fundamental representation*, *JHEP* **07** (2008) 127 [[arXiv:0805.4491](https://arxiv.org/abs/0805.4491)] [[INSPIRE](https://inspirehep.net/literature/59471)].
- [8] A. Joseph, *Lattice formulation of three-dimensional $\mathcal{N} = 4$ gauge theory with fundamental matter fields*, *JHEP* **09** (2013) 046 [[arXiv:1307.3281](https://arxiv.org/abs/1307.3281)] [[INSPIRE](https://inspirehep.net/literature/59471)].
- [9] A. Joseph, *Supersymmetric quiver gauge theories on the lattice*, *JHEP* **01** (2014) 093 [[arXiv:1311.5111](https://arxiv.org/abs/1311.5111)] [[INSPIRE](https://inspirehep.net/literature/59471)].
- [10] S. Catterall, *Lattice supersymmetry and topological field theory*, *JHEP* **05** (2003) 038 [[hep-lat/0301028](https://arxiv.org/abs/hep-lat/0301028)] [[INSPIRE](https://inspirehep.net/literature/59471)].

- [11] S. Catterall, *A geometrical approach to $N = 2$ super Yang-Mills theory on the two dimensional lattice*, *JHEP* **11** (2004) 006 [[hep-lat/0410052](#)] [[INSPIRE](#)].
- [12] S. Catterall, *Lattice formulation of $N = 4$ super Yang-Mills theory*, *JHEP* **06** (2005) 027 [[hep-lat/0503036](#)] [[INSPIRE](#)].
- [13] F. Sugino, *A lattice formulation of super Yang-Mills theories with exact supersymmetry*, *JHEP* **01** (2004) 015 [[hep-lat/0311021](#)] [[INSPIRE](#)].
- [14] F. Sugino, *Super Yang-Mills theories on the two-dimensional lattice with exact supersymmetry*, *JHEP* **03** (2004) 067 [[hep-lat/0401017](#)] [[INSPIRE](#)].
- [15] F. Sugino, *Various super Yang-Mills theories with exact supersymmetry on the lattice*, *JHEP* **01** (2005) 016 [[hep-lat/0410035](#)] [[INSPIRE](#)].
- [16] F. Sugino, *Two-dimensional compact $N = (2, 2)$ lattice super Yang-Mills theory with exact supersymmetry*, *Phys. Lett. B* **635** (2006) 218 [[hep-lat/0601024](#)] [[INSPIRE](#)].
- [17] F. Sugino, *Lattice formulation of two-dimensional $N = (2, 2)$ SQCD with exact supersymmetry*, *Nucl. Phys. B* **808** (2009) 292 [[arXiv:0807.2683](#)] [[INSPIRE](#)].
- [18] Y. Kikukawa and F. Sugino, *Ginsparg-Wilson formulation of 2D $N = (2, 2)$ SQCD with exact lattice supersymmetry*, *Nucl. Phys. B* **819** (2009) 76 [[arXiv:0811.0916](#)] [[INSPIRE](#)].
- [19] D. Kadoh, F. Sugino and H. Suzuki, *Lattice formulation of 2D $N = (2, 2)$ SQCD based on the B model twist*, *Nucl. Phys. B* **820** (2009) 99 [[arXiv:0903.5398](#)] [[INSPIRE](#)].
- [20] N. Maru and J. Nishimura, *Lattice formulation of supersymmetric Yang-Mills theories without fine tuning*, *Int. J. Mod. Phys. A* **13** (1998) 2841 [[hep-th/9705152](#)] [[INSPIRE](#)].
- [21] J. Giedt, *Progress in four-dimensional lattice supersymmetry*, *Int. J. Mod. Phys. A* **24** (2009) 4045 [[arXiv:0903.2443](#)] [[INSPIRE](#)].
- [22] M. Hanada, S. Matsuura and F. Sugino, *Two-dimensional lattice for four-dimensional $N = 4$ supersymmetric Yang-Mills*, *Prog. Theor. Phys.* **126** (2011) 597 [[arXiv:1004.5513](#)] [[INSPIRE](#)].
- [23] M. Hanada, *A proposal of a fine tuning free formulation of 4d $N = 4$ super Yang-Mills*, *JHEP* **11** (2010) 112 [[arXiv:1009.0901](#)] [[INSPIRE](#)].
- [24] M. Hanada, S. Matsuura and F. Sugino, *Non-perturbative construction of 2D and 4D supersymmetric Yang-Mills theories with 8 supercharges*, *Nucl. Phys. B* **857** (2012) 335 [[arXiv:1109.6807](#)] [[INSPIRE](#)].
- [25] D.E. Berenstein, J.M. Maldacena and H.S. Nastase, *Strings in flat space and pp waves from $N = 4$ super Yang-Mills*, *JHEP* **04** (2002) 013 [[hep-th/0202021](#)] [[INSPIRE](#)].
- [26] S.R. Das, J. Michelson and A.D. Shapere, *Fuzzy spheres in pp wave matrix string theory*, *Phys. Rev. D* **70** (2004) 026004 [[hep-th/0306270](#)] [[INSPIRE](#)].
- [27] R.C. Myers, *Dielectric branes*, *JHEP* **12** (1999) 022 [[hep-th/9910053](#)] [[INSPIRE](#)].
- [28] T. Ishii, G. Ishiki, S. Shimasaki and A. Tsuchiya, *$N = 4$ super Yang-Mills from the plane wave matrix model*, *Phys. Rev. D* **78** (2008) 106001 [[arXiv:0807.2352](#)] [[INSPIRE](#)].
- [29] G. Ishiki, S. Shimasaki and A. Tsuchiya, *Perturbative tests for a large- N reduced model of super Yang-Mills theory*, *JHEP* **11** (2011) 036 [[arXiv:1106.5590](#)] [[INSPIRE](#)].
- [30] A. D’Adda, I. Kanamori, N. Kawamoto and K. Nagata, *Twisted superspace on a lattice*, *Nucl. Phys. B* **707** (2005) 100 [[hep-lat/0406029](#)] [[INSPIRE](#)].
- [31] A. D’Adda, I. Kanamori, N. Kawamoto and K. Nagata, *Exact extended supersymmetry on a lattice: Twisted $N = 2$ super Yang-Mills in two dimensions*, *Phys. Lett. B* **633** (2006) 645 [[hep-lat/0507029](#)] [[INSPIRE](#)].

- [32] A. D’Adda, I. Kanamori, N. Kawamoto and K. Nagata, *Exact extended supersymmetry on a lattice: twisted $N = 4$ super Yang-Mills in three dimensions*, *Nucl. Phys. B* **798** (2008) 168 [[arXiv:0707.3533](#)] [[INSPIRE](#)].
- [33] K. Nagata and Y.-S. Wu, *Twisted SUSY invariant formulation of Chern-Simons gauge theory on a lattice*, *Phys. Rev. D* **78** (2008) 065002 [[arXiv:0803.4339](#)] [[INSPIRE](#)].
- [34] M. Ünsal, *Twisted supersymmetric gauge theories and orbifold lattices*, *JHEP* **10** (2006) 089 [[hep-th/0603046](#)] [[INSPIRE](#)].
- [35] P.H. Damgaard and S. Matsuura, *Lattice supersymmetry: equivalence between the link approach and orbifolding*, *JHEP* **09** (2007) 097 [[arXiv:0708.4129](#)] [[INSPIRE](#)].
- [36] S. Catterall, *From twisted supersymmetry to orbifold lattices*, *JHEP* **01** (2008) 048 [[arXiv:0712.2532](#)] [[INSPIRE](#)].
- [37] P.H. Damgaard and S. Matsuura, *Geometry of orbifolded supersymmetric lattice gauge theories*, *Phys. Lett. B* **661** (2008) 52 [[arXiv:0801.2936](#)] [[INSPIRE](#)].
- [38] S. Catterall, D.B. Kaplan and M. Ünsal, *Exact lattice supersymmetry*, *Phys. Rept.* **484** (2009) 71 [[arXiv:0903.4881](#)] [[INSPIRE](#)].
- [39] I. Kanamori and H. Suzuki, *Restoration of supersymmetry on the lattice: two-dimensional $N = (2, 2)$ supersymmetric Yang-Mills theory*, *Nucl. Phys. B* **811** (2009) 420 [[arXiv:0809.2856](#)] [[INSPIRE](#)].
- [40] M. Hanada and I. Kanamori, *Lattice study of two-dimensional $N = (2, 2)$ super Yang-Mills at large- N* , *Phys. Rev. D* **80** (2009) 065014 [[arXiv:0907.4966](#)] [[INSPIRE](#)].
- [41] S. Catterall, A. Joseph and T. Wiseman, *Thermal phases of D1-branes on a circle from lattice super Yang-Mills*, *JHEP* **12** (2010) 022 [[arXiv:1008.4964](#)] [[INSPIRE](#)].
- [42] H. Suzuki, *Two-dimensional $N = (2, 2)$ super Yang-Mills theory on computer*, *JHEP* **09** (2007) 052 [[arXiv:0706.1392](#)] [[INSPIRE](#)].
- [43] I. Kanamori, H. Suzuki and F. Sugino, *Euclidean lattice simulation for dynamical supersymmetry breaking*, *Phys. Rev. D* **77** (2008) 091502 [[arXiv:0711.2099](#)] [[INSPIRE](#)].
- [44] I. Kanamori, F. Sugino and H. Suzuki, *Observing dynamical supersymmetry breaking with euclidean lattice simulations*, *Prog. Theor. Phys.* **119** (2008) 797 [[arXiv:0711.2132](#)] [[INSPIRE](#)].
- [45] I. Kanamori, *Lattice formulation of two-dimensional $N = (2, 2)$ super Yang-Mills with $SU(N)$ gauge group*, *JHEP* **07** (2012) 021 [[arXiv:1202.2101](#)] [[INSPIRE](#)].
- [46] M. Lüscher, *Abelian chiral gauge theories on the lattice with exact gauge invariance*, *Nucl. Phys. B* **549** (1999) 295 [[hep-lat/9811032](#)] [[INSPIRE](#)].
- [47] T. Misumi, *Fermion actions extracted from lattice super Yang-Mills theories*, *JHEP* **12** (2013) 063 [[arXiv:1311.4365](#)] [[INSPIRE](#)].
- [48] P.H. Ginsparg and K.G. Wilson, *A remnant of chiral symmetry on the lattice*, *Phys. Rev. D* **25** (1982) 2649 [[INSPIRE](#)].
- [49] H. Neuberger, *Exactly massless quarks on the lattice*, *Phys. Lett. B* **417** (1998) 141 [[hep-lat/9707022](#)] [[INSPIRE](#)].
- [50] H. Neuberger, *More about exactly massless quarks on the lattice*, *Phys. Lett. B* **427** (1998) 353 [[hep-lat/9801031](#)] [[INSPIRE](#)].
- [51] P. Hernández, K. Jansen and M. Lüscher, *Locality properties of Neuberger’s lattice Dirac operator*, *Nucl. Phys. B* **552** (1999) 363 [[hep-lat/9808010](#)] [[INSPIRE](#)].