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On pentagon identity in Ding-Iohara-Miki algebra

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Abstract: We notice that the famous pentagon identity for quantum dilogarithm functions and the five-term relation for certain operators related to Macdonald polynomials discovered by Garsia and Mellit can both be understood as specific cases of a general "master pentagon identity" for group-like elements in the Ding-Iohara-Miki (or quantum toroidal, or elliptic Hall) algebra. We prove this curious identity and discuss its implications.

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1 Introduction

Pentagon identities are ubiquitous in mathematics and physics. A non-exhaustive list of their appearances includes representation theory (where they describe coherence of associators, in particular quantum $6j$-symbols [1]), topological field theory [2, 3], hyperbolic geometry (Teichmüller theory, both classical and quantum [4]–[6]) and integrable systems [7]–[9]. In general a pentagon identity has the form

$$F \cdot F = F \cdot F \cdot F, \tag{1.1}$$

where the precise meaning of F depends on the context: for example it can be an operator with the dot denoting a contraction or composition, or it can be a simplex of a three-dimensional triangulation in which case the dot implies gluing along a face.

In the present short note we propose a new pentagon identity for the generating series of elements of Ding-Iohara-Miki (DIM) algebra $U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}_1)$ [11, 12]. This algebra has many alternative names because it has been rediscovered several times in different contexts: elliptic Hall algebra [13], spherical double affine Hecke algebra of type $\mathrm{GL}(\infty)$ [14], quantum continuous $\mathfrak{gl}_\infty$ algebra [15], deformed $W_{1+\infty}$-algebra [12], quantum toroidal algebra of type $\mathfrak{gl}_1$ [16]. DIM algebra plays an important role in modern mathematical physics (see e.g. [17]–[21]) and its representation theory is very rich.

Let us state our main result.

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Theorem 1. *Let $\vec{\gamma} \in \mathbb{Z}^2 \backslash \{(0,0)\}$. Consider the formal generating function*

$$T_{\vec{\gamma}}(u) = \exp\left(-\sum_{n\geq 1} \frac{(-u)^n}{n} e_{n\vec{\gamma}}\right). \tag{1.2}$$

where $e_{\vec{\gamma}} \in U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}_1)$ with $\vec{\gamma} \in \mathbb{Z}^2 \backslash \{(0,0)\}$ denote the standard generators of the DIM algebra (see Definition 1 for the definition of the algebra). Then the following identity holds

$$T_{(0,1)}(v)T_{(1,0)}(u) = T_{(1,0)}(u)T_{(1,1)}(uv)T_{(0,1)}(v) \tag{1.3}$$

One can view the pentagon identity (1.3) as a q_3-deformation of the simplest of the motivic Kontsevich-Soibelman wall-crossing formulas [24]. Indeed, DIM algebra is a q_3-deformation of the Lie algebra of functions on a quantum torus (see section 2), so for $q_3 = 1$ the identity (1.3) reduces to a motivic KS wall-crossing formula.

In what follows we first verify some consequences of the identity (1.3) and then prove it. If the pentagon identity (1.3) is valid in the algebra, then it should hold in all of its representations, of which there are plenty. In section 3.1 we prove that in the so-called vector representation the pentagon identity reduces to the famous Faddeev-Kashaev-Volkov pentagon identity [9, 10] for quantum dilogarithm functions. In section 3.2 we evaluate (1.3) in the Fock representation of $U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}_1)$ producing an identity for certain operators related to Macdonald polynomials. This identity has been proven in the recent work of Garsia and Mellit [25] and actually implies Theorem 1 since the Fock representation of DIM algebra is known to be faithful. Finally, in section 3.3 we demonstrate the nontriviality of the relation by deriving several first terms in its u and v expansion directly from the commutation relations of the DIM algebra.

Theorem 1 implies some new identities which we briefly discuss in section 4. For example, there are homomorphisms of DIM algebra to spherical double affine Hecke algebras $\mathbb{H}_n$ for any n, therefore the pentagon identity is also valid in $\mathbb{H}_n$. In more down to earth terms this means an identity involving Ruijsenaars-Schneider Hamiltonians with n particles. Evaluating (1.3) in the MacMahon representation one could get an identity for certain combinatorial operators acting on plane partitions. In principle, one can take any representation of DIM algebra and obtain the corresponding pentagon identity, which usually turns out to be new.

We present our conclusions and list some puzzles related to the DIM pentagon identity in section 5.

2 Definition and properties of the DIM algebra

Definition 1. *Let q_1, q_2 and q_3 be formal parameters such that $q_1q_2q_3 = 1$. The algebra $U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}_1)$ is multiplicatively generated by the central elements c_1, c_2 and the elements $e_{\vec{\gamma}}$, with $\vec{\gamma} \in \mathbb{Z}^2 \backslash \{(0,0)\}$, satisfying the following relations (we follow the conventions of* [22, 23]):

1. *For* $\vec{\gamma} \in \mathbb{Z}^2 \backslash \{(0,0)\}$ *we define*[1] $\gcd(\vec{\gamma}) = \gcd(\gamma_1, \gamma_2)$. *We call* $\vec{\gamma} \in \mathbb{Z}^2 \backslash \{(0,0)\}$ *coprime if* $\gcd(\vec{\gamma}) = 1$. *Then for any nonzero coprime* $\vec{\gamma}$

$$[e_{n\vec{\gamma}}, e_{m\vec{\gamma}}] = \frac{n}{\kappa_n} \delta_{n+m,0} (c_{n\vec{\gamma}} - c_{-n\vec{\gamma}}), \tag{2.1}$$

where we abbreviate $c_{\vec{\gamma}} = c_1^{\gamma_1} c_2^{\gamma_2}$ *and*

$$\kappa_n = (1 - q_1^n)(1 - q_2^n)(1 - q_3^n). \tag{2.2}$$

2. *For* $\vec{\alpha}, \vec{\beta} \in \mathbb{Z}^2 \backslash \{(0,0)\}$, $\vec{\alpha} \neq \vec{\beta}$ *we set* $\vec{\alpha} > \vec{\beta}$ *if* (α_1, α_2) *and* (β_1, β_2) *are ordered lexicographically, i.e.* $\alpha_1 > \alpha_2$ *or* $\alpha_1 = \alpha_2$ *and then* $\beta_1 > \beta_2$. *Let* $\vec{\alpha} \times \vec{\beta} = \alpha_1 \beta_2 - \alpha_2 \beta_1$ *be the two-dimensional cross-product. We call a pair* $(\vec{\alpha}, \vec{\beta})$ *admissible if the triangle* $T_{\vec{\alpha},\vec{\beta}} = (-\vec{\beta}, 0, \vec{\alpha})$ *does not contain any* $\mathbb{Z}^2$ *points in the interior and on at least two of its edges. We call the middle* $M(T_{\vec{\alpha},\vec{\beta}})$ *of the triangle* $T_{\vec{\alpha},\vec{\beta}}$ *the vertex, which is in the middle according to the lexicographic ordering described above. Then for any* $\vec{\alpha}, \vec{\beta} \in \mathbb{Z}^2 \backslash \{(0,0)\}$ *such that* $T_{\vec{\alpha},\vec{\beta}}$ *is admissible we have*

$$[e_{\vec{\alpha}}, e_{\vec{\beta}}] = \begin{cases} \frac{1}{\kappa_1} c_{M(T_{\vec{\alpha},\vec{\beta}})} h_{\vec{\alpha}+\vec{\beta}}, & \vec{\alpha} \times \vec{\beta} > 0, \\ -\frac{1}{\kappa_1} c_{M(T_{\vec{\beta},\vec{\alpha}})} h_{\vec{\alpha}+\vec{\beta}}, & \vec{\alpha} \times \vec{\beta} < 0, \end{cases} \tag{2.3}$$

where $h_{\vec{\gamma}}$ *is defined using a generating series as follows. Let* $\vec{\gamma}$ *be coprime, then*

$$1 + \sum_{n \geq 1} h_{n\vec{\gamma}} z^{-n} = \exp\left(- \sum_{k \geq 1} \frac{\kappa_k}{k} e_{k\vec{\gamma}} z^{-k} \right), \tag{2.4}$$

or more explicitly

$$h_{n\vec{\gamma}} = (-1)^n E_n(\kappa_k \vec{e}_{k\vec{\gamma}}), \tag{2.5}$$

where E_n *are elementary symmetric functions,*[2] *so that*

$$h_{\vec{\gamma}} = -\kappa_1 e_{\vec{\gamma}}, \tag{2.6}$$

$$h_{2\vec{\gamma}} = \frac{\kappa_1^2}{2} e_{\vec{\gamma}}^2 - \frac{\kappa_2}{2} e_{2\vec{\gamma}}, \tag{2.7}$$

$$h_{3\vec{\gamma}} = -\frac{\kappa_1^3}{6} e_{\vec{\gamma}}^3 + \frac{\kappa_1 \kappa_2}{2} e_{2\vec{\gamma}} e_{\vec{\gamma}} - \frac{\kappa_3}{3} e_{3\vec{\gamma}}, \tag{2.8}$$

$$\cdots \tag{2.9}$$

Commutation relations for non-admissible triangles are defined implicitly, i.e. they can be obtained by the successive application of the relations for admissible triangles. In fact the whole algebra can be generated from just six elements: $e_{(\pm 1,0)}$, $e_{(0,\pm 1)}$ and c_1, c_2. We list some examples of commutation relations to make Definition 1 more accessible and to demonstrate where the nontrivial dependence on q_i and c_1, c_2 appears.

[1] Notice that $\gcd(\vec{\gamma})$ is always positive.

[2] We use the upper-case letter to avoid confusion with the generators $e_{\vec{\gamma}}$.

Example 1. *If* $\vec{\alpha} > (0,0)$*,* $\vec{\beta} > (0,0)$*, then* $M(T_{\vec{\alpha},\vec{\beta}}) = (0,0)$*, so the factor* $c_{M(T_{\vec{\alpha},\vec{\beta}})} = 1$ *does not appear in eq. (2.3). If in addition* $\gcd(\vec{\alpha}+\vec{\beta}) = 1$*, then*

$$[e_{\vec{\alpha}}, e_{\vec{\beta}}] = -e_{\vec{\alpha}+\vec{\beta}} \tag{2.10}$$

Example 2. *We give a sample of the simplest commutation relations:*

$$[e_{(1,0)}, e_{(0,1)}] = -e_{(1,1)}, \tag{2.11}$$
$$[e_{(0,1)}, e_{(-1,0)}] = -c_{(0,1)}e_{(-1,1)}, \tag{2.12}$$
$$[e_{(-1,0)}, e_{(0,-1)}] = -e_{(-1,-1)}, \tag{2.13}$$
$$[e_{(2,1)}, e_{(0,1)}] = \frac{\kappa_1}{2}e_{(1,1)}^2 - \frac{\kappa_2}{2\kappa_1}e_{(2,2)}. \tag{2.14}$$

DIM algebra so defined is obviously invariant under any permutation of the parameters q_1, q_2 and q_3. It is doubly graded, with $e_{\vec{\gamma}}$ having grades (γ_1, γ_2). The algebra also has a large group of automorphisms which contains the universal cover $\widetilde{SL}(2,\mathbb{Z})$ of $\mathrm{SL}(2,\mathbb{Z})$ [23]. This universal cover retains the knowledge of the "winding" of a given $\mathbb{Z}^2$ vector around the origin and fits into the short exact sequence

$$0 \to \mathbb{Z} \to \widetilde{SL}(2,\mathbb{Z}) \to \mathrm{SL}(2,\mathbb{Z}) \to 0. \tag{2.15}$$

More concretely, the generator[3] $S = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ of $\mathrm{SL}(2,\mathbb{Z})$, which rotates $\mathbb{Z}^2$ by $-\frac{\pi}{2}$ is lifted to a generator of $\widetilde{SL}(2,\mathbb{Z})$ which fourth power is no longer trivial, but changes winding number by one. The $T = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ generator remains unchanged. The action of automorphisms on the DIM generators is

$$\mathcal{S}(e_{\vec{\alpha}}) = \begin{cases} e_{S(\vec{\alpha})}, & \alpha_1 \geq 0, \alpha_2 > 0, \text{ or } \alpha_1 \leq 0, \alpha_2 < 0, \\ c_{-S(\vec{\alpha})}e_{S(\vec{\alpha})}, & \alpha_1 \geq 0, \alpha_2 > 0, \text{ or } \alpha_1 \leq 0, \alpha_2 < 0, \end{cases} \tag{2.16}$$
$$\mathcal{S}(c_{\vec{\alpha}}) = c_{S(\vec{\alpha})} \tag{2.17}$$
$$\mathcal{T}(e_{\vec{\alpha}}) = e_{T(\vec{\alpha})}, \tag{2.18}$$
$$\mathcal{T}(c_{\vec{\alpha}}) = c_{T(\vec{\alpha})}. \tag{2.19}$$

Notice in particular that

$$\mathcal{S}^4(e_{\vec{\alpha}}) = c_{-2\vec{\alpha}}e_{\vec{\alpha}} \tag{2.20}$$

for any $\vec{\alpha}$, which is precisely the element of winding number one in $\widetilde{SL}(2,\mathbb{Z})$. The complicated picture with universal cover arises because of the lexicographic ordering entering the definition of $c_{M(T_{\vec{\alpha},\vec{\beta}})}$. In the absence of central charges the automorphism group is just $\mathrm{SL}(2,\mathbb{Z})$. In fact since the pentagon identity involves only the generators from the first quadrant, the central charges will never appear in the commutation process.

In the limit when one of the parameters q_i goes to one, the algebra $U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}_1)$ becomes the Lie algebra of functions on a quantum torus, also known as $qW_{1+\infty}$-algebra. Since q_i

[3] The standard identification of $\mathrm{SL}(2,\mathbb{Z})$ generators with matrices is $S_{\text{standard}} = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $T_{\text{standard}} = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, so our generators are $S = S_{\text{standard}}^{-1}$, $T = S_{\text{standard}}T_{\text{standard}}^{-1}S_{\text{standard}}^{-1}$.

enter in the definition symmetrically, we can consider $q_3 \to 1$ without loss of generality. We introduce the rescaled generators and central charges in the limit as follows:

$$w_{\vec{\alpha}} = \left(q_1^{\frac{\gcd(\vec{\alpha})}{2}} - q_1^{-\frac{\gcd(\vec{\alpha})}{2}}\right) e_{\vec{\alpha}}, \qquad c_i \to q_3^{\tilde{c}_i} \quad i = 1, 2. \tag{2.21}$$

Then the commutation relations turn into Lie algebraic form:

$$[w_{\vec{\alpha}}, w_{\vec{\beta}}] = \left(q_1^{\frac{\vec{\alpha}\times\vec{\beta}}{2}} - q_1^{-\frac{\vec{\alpha}\times\vec{\beta}}{2}}\right) w_{\vec{\alpha}+\vec{\beta}} + \delta_{\vec{\alpha}+\vec{\beta},0}(\alpha_1\tilde{c}_1 + \alpha_2\tilde{c}_2) \tag{2.22}$$

Notice that the identities (2.22) with $\tilde{c}_i = 0$ would follow from the multiplication rule

$$w_{\vec{\alpha}} w_{\vec{\beta}} = q_1^{\frac{\vec{\alpha}\times\vec{\beta}}{2}} w_{\vec{\alpha}+\vec{\beta}}, \tag{2.23}$$

but in a general representation of the Lie algebra (2.22) the rule (2.23) does not hold.

3 Checks and proof of the DIM pentagon relation

In this section we check the pentagon relation (1.3) by evaluating it in a vector representation of DIM algebra (section 3.1), and prove using a Fock representation (section 3.2). We also do some brute force calculations using the commutation relations of the DIM algebra (section 3.3).

3.1 Reineke's character, vector representations and quantum dilogarithm

Theorem 2. *Let $\mathcal{O}_{q_i}$ be the associative algebra of functions on a two-dimensional quantum torus $(\mathbb{C}^\times \times \mathbb{C}^\times)_{q_i}$ with non-commutativity parameter q_i. The algebra $\mathcal{O}_{q_i}$ is multiplicatively generated by $\mathbf{x}$ and $\mathbf{y}$, satisfying the q_i-commutation relations*

$$\mathbf{y}\,\mathbf{x} = q_i \mathbf{x}\,\mathbf{y}. \tag{3.1}$$

Then the map ρ_i (for $i = 1, 2, 3$) called Reineke's character [27]

$$\rho_i : U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}_1) \to \mathcal{O}_{q_i}, \tag{3.2}$$

described by the formulas

$$\rho_i(e_{\vec{\gamma}}) = \frac{q_i^{\frac{\gamma_1\gamma_2}{2}}}{q_i^{\frac{\gcd(\vec{\gamma})}{2}} - q_i^{-\frac{\gcd(\vec{\gamma})}{2}}} \mathbf{x}^{\gamma_1}\mathbf{y}^{\gamma_2}, \tag{3.3}$$

$$\rho_i(c_{\vec{\gamma}}) = 1 \tag{3.4}$$

for any $\vec{\gamma} \neq 0$ is a homomorphism.

Proof. By explicit verification of the commutation relations. □

Under the character map ρ_i the elements $T_{\vec{\gamma}}(u)$ for coprime $\vec{\gamma}$ become

$$\rho_i(T_{\vec{\gamma}}(u)) = \left(-q_i^{\frac{1+\gamma_1\gamma_2}{2}} u\, \mathbf{x}^{\gamma_1}\mathbf{y}^{\gamma_2}; q_i\right)_\infty^{-1}, \tag{3.5}$$

where

$$(x;q)_\infty = \prod_{n\geq 0}(1-q^i x). \tag{3.6}$$

The pentagon identity (1.3) then becomes the well-known identity for quantum dilogarithms (we get rid of the $q_i^{\frac{1}{2}}$ factors by rescaling the generating function parameters $u \mapsto q_i^{\frac{1}{2}} u$, $v \mapsto q_i^{\frac{1}{2}} v$):

$$(-v\,\mathbf{y};q_i)_\infty^{-1}(-u\,\mathbf{x};q_i)_\infty^{-1} = (-u\,\mathbf{x};q_i)_\infty^{-1}(-uv\,\mathbf{x}\,\mathbf{y};q_i)_\infty^{-1}(-v\,\mathbf{y};q_i)_\infty^{-1} \tag{3.7}$$

Thus, under Reineke's map DIM pentagon identity (1.3) maps to a true statement. It is curious that we can obtain three "different" quantum dilogarithm identities, one for each q_i, from a single "master" identity (1.3).

The Reineke's character can be used to obtain a representation of the DIM algebra from a representation of $\mathcal{O}_{q_i}$ e.g. using difference operators on functions of a single variable w. One can take

$$\mathbf{x} = q_i^{-w\partial_w}, \tag{3.8}$$
$$\mathbf{y} = w. \tag{3.9}$$

These are known as vector representations $\mathcal{V}_{q_i}$ of $U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}_1)$.

Another way to view DIM pentagon identity (1.3) is as a *refinement*, or *categorification* of the pentagon identity in $\mathcal{O}_{q_i}$, because it splits the terms with the same powers of $\mathbf{x}$ and $\mathbf{y}$ into separate entities, e.g. a term $\mathbf{x}^2\mathbf{y}^2$ in the identity for quantum dilogarithms can be either $e_{(2,2)}$ or $e_{(1,1)}^2$ in the DIM algebra (see section 3.3 for precisely this computation). The Reineke's character, as its name suggests, plays the role of taking the trace, or the grading, of a refined formula.

3.2 The proof of pentagon theorem using the results of Garsia-Mellit

Theorem 3. *Let $\mathcal{F}^{(1,0)}_{q_1,q_2} = \mathbb{C}[p_1,p_2,\ldots]$ be the vector space of polynomials in time variables p_n. Denote $q = q_1$, $t = q_2^{-1}$ and let $M_\lambda^{(q,t)}(p_n)$ be Macdonald polynomial in the standard normalization labelled by a Young diagram $\lambda = \{\lambda_1,\lambda_2,\ldots,\lambda_{l(\lambda)}\}$. Then the map*

$$f : U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}_1) \to \mathrm{End}(\mathcal{F}^{(1,0)}_{q,t^{-1}}) \tag{3.10}$$

defined by relations below gives a representation of $U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}_1)$ *on* $\mathcal{F}^{(1,0)}_{q,t^{-1}}$*:*

$$f(e_{(0,n)}) = \frac{1}{1-t^{-n}} n \frac{\partial}{\partial p_n}, \qquad n>0, \tag{3.11}$$

$$f(e_{(0,-n)}) = -\left(\frac{q}{t}\right)^{\frac{n}{2}} \frac{1}{1-q^n} p_n, \qquad n>0, \tag{3.12}$$

$$f(e_{(1,n)}) = -\frac{1}{(1-q)(1-t^{-1})} \times \oint_{\mathcal{C}_0} \frac{dz}{z} z^n e^{\sum_{k\geq 1}\frac{z^k}{k}(1-t^{-k})p_k} e^{-\sum_{k\geq 1} z^{-k}(1-q^k)\frac{\partial}{\partial p_k}}, \quad n\in\mathbb{Z}, \tag{3.13}$$

$$f(e_{(-1,n)}) = \frac{1}{(1-q^{-1})(1-t)} \times \oint_{\mathcal{C}_0} \frac{dz}{z} z^n e^{-\sum_{k\geq 1}\frac{z^k}{k}(1-t^{-k})\left(\frac{t}{q}\right)^{\frac{k}{2}}p_k} e^{\sum_{k\geq 1} z^{-k}(1-q^k)\left(\frac{t}{q}\right)^{\frac{k}{2}}\frac{\partial}{\partial p_k}}, \quad n\in\mathbb{Z}, \tag{3.14}$$

$$f(e_{\pm n,0})M^{(q,t)}_\lambda(p_m) = \pm\left(-\frac{1}{(1-q^{\pm n})(1-t^{\mp n})} + \sum_{(i,j)\in\lambda} q^{\pm n(j-1)}t^{\pm n(1-i)}\right) M^{(q,t)}_\lambda(p_m), \quad n>0, \tag{3.15}$$

$$f(e_{n,1})M^{(q,t)}_\lambda(p_m) = \frac{\left(\frac{t}{q}\right)^{\frac{1}{2}(1+\delta_{n\geq 0})}}{1-q^{-1}} \times \oint_{\mathcal{C}_0} \frac{dz}{z} z^n \sum_{i=1}^{l(\lambda)} \frac{1-q\frac{z}{t^{1-i}}}{1-t\frac{z}{t^{1-i}}} \prod_{j=i+1}^{l(\lambda)} \frac{\psi\left(\frac{z}{q^{\lambda_j}t^{1-j}}\right)}{\psi\left(\frac{z}{t^{1-j}}\right)} \delta\left(\frac{z}{q^{\lambda_i-1}t^{1-i}}\right) M^{(q,t)}_{\lambda-1_i}(p_m), \tag{3.16}$$

$$f(e_{n,-1})M^{(q,t)}_\lambda(p_m) = -\frac{\left(\frac{q}{t}\right)^{\frac{1}{2}\delta_{n\leq 0}}}{1-q} \oint_{\mathcal{C}_0} \frac{dz}{z} z^n \sum_{i=1}^{l(\lambda)+1} \prod_{j=1}^{i-1} \psi\left(\frac{z}{q^{\lambda_j}t^{1-j}}\right) \delta\left(\frac{z}{q^{\lambda_i}t^{1-i}}\right) M^{(q,t)}_{\lambda+1_i}(p_m), \quad n\in\mathbb{Z}, \tag{3.17}$$

$$f(c_1) = 1, \tag{3.18}$$

$$f(c_2) = \left(\frac{t}{q}\right)^{\frac{1}{2}}. \tag{3.19}$$

where $\mathcal{C}_0$ *is a small contour around the origin,* $\lambda\pm 1_i = \{\lambda_1,\dots,\lambda_{i-1},\lambda_i\pm 1,\lambda_{i+1},\cdots\}$ *and*

$$\delta_{n\geq 0} = \begin{cases} 1, & n\geq 0 \\ 0, & n<0 \end{cases}, \qquad \delta(x) = \sum_{n\in\mathbb{Z}} x^n, \qquad \psi(x) = \frac{(1-tx)\left(1-\frac{q}{t}x\right)}{(1-x)(1-qx)}. \tag{3.20}$$

Proof. One combines the "horizontal" and "vertical" Fock representations of the DIM algebra given in [17] to obtain the expressions for $e_{(0,n)}$, $e_{(\pm 1,n)}$ and $e_{(n,0)}$, $e_{(n,\pm 1)}$ respectively. □

Several remarks are in order:

1. Some expressions in eqs. (3.11)–(3.17) define the same operators in two different ways, e.g. $e_{(1,0)}$. In these cases both expressions are valid due to the properties of Macdonald polynomials.

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2. The the strange factors like $\left(\frac{t}{q}\right)^{\frac{1}{2}\delta_{n\geq 0}}$ in eqs. (3.16), (3.17) come from the application of the $\mathcal{S}$ automorphism to the "vertical" Fock representation taken from[4] [17].

Let us now prove Theorem 1 by reducing it to an identity proved in [25].

Proof of Theorem 1. We can directly evaluate the generating series $T_{(1,0)}$ and $T_{(0,1)}$ in the Fock representation:

$$f(T_{(1,0)}(u))M_\lambda^{(q,t)}(p_n) = \exp\left(\sum_{n\geq 1}\frac{(-u)^n}{(1-q^n)(1-t^{-n})}\right)\prod_{(i,j)\in\lambda}(1+uq^{j-1}t^{1-i})M_\lambda^{(q,t)}(p_n), \tag{3.21}$$

$$f(T_{(0,1)}(u)) = \exp\left(-\sum_{n\geq 1}\frac{(-v)^n}{1-t^{-n}}\frac{\partial}{\partial p_n}\right). \tag{3.22}$$

It is left to find $T_{(1,1)}(uv)$. To do this we notice that the element $\mathcal{S}^{-1}\mathcal{T}\mathcal{S}\in\widetilde{SL}(2,\mathbb{Z})$ of the automorphism group of the algebra preserves the generators $e_{(n,0)}$, which are diagonal in the basis of Macdonald operators. It also preserves the central charges of the Fock representation. One can then guess that $\mathcal{S}^{-1}\mathcal{T}\mathcal{S}$ can be realized as an operator τ on the Fock space, diagonal in the basis of Macdonald polynomials:

$$f(\mathcal{S}\mathcal{T}^{-1}\mathcal{S}^{-1}(e_{\vec{\gamma}})) = \tau^{-1}f(e_{\vec{\gamma}})\tau. \tag{3.23}$$

In fact it is enough to verify its action on the generators $e_{(0,\pm 1)}$, $e_{(\pm 1,0)}$ to find

$$\tau M_\lambda^{(q,t)}(p_n) = \left(\prod_{(i,j)\in\lambda}q^{j-1}t^{1-i}\right)M_\lambda^{(q,t)}(p_n). \tag{3.24}$$

Thus we have

$$f(T_{(1,1)}(uv)) = f(\mathcal{S}\mathcal{T}^{-1}\mathcal{S}^{-1}(T_{(1,0)}(uv))) = \tau^{-1}f(T_{(1,0)}(uv))\tau. \tag{3.25}$$

To make contact with the work of Garsia and Mellit [25] we notice that their parameters q_{GM}, t_{GM} and the power sums p_n^{GM} are related to ours as follows:

$$q_{\mathrm{GM}} = q, \tag{3.26}$$
$$t_{\mathrm{GM}} = t^{-1}, \tag{3.27}$$
$$p_n^{\mathrm{GM}} = (1-t^n)p_n. \tag{3.28}$$

They also define operators τ_v^{GM}, $\Delta_u'^{\mathrm{GM}}$ and ∇_{GM}, which are related to our $T_{(1,0)}$, $T_{(0,1)}$ and τ respectively:

$$\tau_v^{\mathrm{GM}} = f(T_{(0,1)}(-v/t)), \tag{3.29}$$
$$\Delta_u'^{\mathrm{GM}} = f(T_{(1,0)}(-u)), \tag{3.30}$$
$$\nabla_{\mathrm{GM}} = (-1)^d\tau, \tag{3.31}$$

[4]We also rescale the generators of [17] to conform with our commutation relations, which make the symmetry between q_1, q_2, q_3 explicit.

where d counts the total degree of polynomial in $\mathcal{F}^{(1,0)}_{q_1,q_2}$. We also notice that

$$(-1)^d f(T_{(1,1)}(uv))(-1)^d = f(T_{(1,1)}(-uv)). \tag{3.32}$$

Combining all these changes of variables and shifting $v \to tv$ in $T_{(0,1)}$ we find that the DIM pentagon identity under the action of f reduces to

$$(\Delta'^{\text{GM}}_u)^{-1}\tau^{\text{GM}}_v\Delta'^{\text{GM}}_u(\tau^{\text{GM}}_v)^{-1} = \nabla^{-1}\tau^{\text{GM}}_{uv}\nabla, \tag{3.33}$$

which is precisely what is proven in [25].

Furthermore, it has been proven in [26] (Corollary 1.5) that the Fock representation of the DIM algebra is faithful and thus Theorem 1 follows from eq. (3.33).

□

3.3 Direct checks

In this section we show that a direct check of the pentagon relation (1.3) using the commutation relations of the DIM algebra involves some non-obvious cancellations even in the first orders of expansion in u and v.

At first orders u^1v^0 and u^0v^1 the identity holds trivially. Then one gets some identities which can be derived from the commutation relations without deformation parameters q_i making any appearance:

$$u^1v^1: \qquad e_{(0,1)}e_{(1,0)} = e_{(1,0)}e_{(0,1)} + e_{(1,1)}, \tag{3.34}$$

$$u^2v^1: \qquad \frac{1}{2}e_{(0,1)}\left(e^2_{(1,0)} - e_{(2,0)}\right) = \frac{1}{2}(e^2_{(1,0)} - e_{(2,0)})e_{(0,1)} + e_{(1,0)}e_{(0,1)}, \tag{3.35}$$

However, the check at the order u^2v^2 requires the evaluation of the commutator $[e_{(2,0)}, e_{(0,2)}]$ which corresponds to a non-admissible triangle. After sequential application of the commutation relations for admissible triangles we find[5]

$$[e_{(2,0)}, e_{(0,2)}] = \frac{\kappa_1}{2}e^2_{(1,1)} + \left(2 - \frac{\kappa_2}{2\kappa_1}\right)e_{(2,2)}, \tag{3.36}$$

which indeed leads to a complete cancellation with other terms, so that the pentagon identity holds at order u^2v^2. The computations for higher orders quickly become very involved. Probably they can be computerized.

4 Applications and implications

One can evaluate the identity (1.3) in other representation of the DIM algebra, producing some interesting identities for operators in representation spaces. Here we list a couple of preliminary applications of the DIM pentagon identity along these lines.

[5]There is an error in the first coefficient in the r.h.s. in this calculation on p. 30 of [23].

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4.1 Pentagon identity for spherical double affine Hecke algebra from tensor product of vector representations

There is a homomorphism from $U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}_1)$ to spherical double affine Hecke algebra (sDAHA) $\mathbb{H}_n$ with $n = 1, 2, \dots$ The case $n = 1$ is just the Reineke's character, or vector representation $\mathcal{V}_{q_i}$ discussed in section 3.1. For higher n the homomorphism can be described by noting that DIM algebra carries a coproduct structure (we have not discussed it, since we did not use it here). This allows one to take a tensor product of n copies of vector representation $\mathcal{V}_{q_i}$. This produces an action of $U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}_1)$ on functions of n coordinates w_i, involving Ruijsenaars-Schneider difference operators. For example, in this representation we have

$$\rho_i^{\otimes n}(E_k(e_{(m,0)})) \sim H_k^{\mathrm{RS}}, \tag{4.1}$$

where H_k^{RS} are Ruijsenaars-Schneider Hamiltonians

$$H_k^{\mathrm{RS}} = \sum_{I\subset\{1,\dots,n\},|I|=k} \prod_{i\in I} \prod_{j\in\{1,\dots,n\}\setminus I} \frac{tw_i - w_j}{w_i - w_j} q^{\sum_{i\in I} w_i \partial_{w_i}} \tag{4.2}$$

and E_k are complete symmetric functions.

Taking the limit $n \to \infty$ one recovers the Fock representation and the identity of Garsia-Mellit with

$$p_n = \sum_{i=1}^{\infty} w_i^n \tag{4.3}$$

being power sum variables.

4.2 Pentagon identity for MacMahon operators

There is an interesting representation m of $U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}_1)$ on a vector space $\mathcal{M}$ spanned by plane partitions (or $3d$ Young diagrams), which is called the MacMahon representation. Most of the generators, which we have described for Fock representation in section 3.2 can also be written for MacMahon representation. For example:

$$m(e_{(n,0)})|\pi\rangle = \left(-\frac{1}{\kappa_n} + \sum_{(i,j,k)\in\pi} q_1^{n(j-1)} q_2^{n(i-1)} q_3^{n(k-1)}\right)|\pi\rangle, \qquad n \geq 1 \tag{4.4}$$

where π is a plane partition. The expressions for $m(e_{(0,n)})$ are more involved, but can also be found using certain analogues of Pieri rules for plane partitions. It would be interesting to write down the pentagon relation explicitly in this case.

5 Conclusions and comments

Let us recapitulate. We have introduced an analogue of pentagon relation between certain generating series in the DIM algebra. This identity is very nontrivial, since the commutation relations of the algebra are complicated with some of them defined implicitly by recursion. We have proven the new pentagon identity and have found that in certain representations of DIM algebra it reduces to previously known identities. We have also verified lower orders of the expansion in the generating parameters directly using commutation relations.

There is one puzzling fact about the identity (1.3). The generating series $T_{\vec{\gamma}}$ look like elements of a group and therefore it would be logical to expect that they would have some group-like property, i.e.

$$\Delta(T_{\vec{\gamma}}) \overset{?}{=} T_{\vec{\gamma}} \otimes T_{\vec{\gamma}} \tag{5.1}$$

where Δ is the coproduct on a DIM algebra. However, a problem arises immediately: exactly which coproduct features in eq. (5.1)? In fact the algebra $U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}_1)$ has an infinite number of coproduct structures labelled by an irrational slope in $\mathbb{Z}^2$ lattice. It turns out that eq. (5.1) holds only for the coproduct associated to the slope $\vec{\gamma}+0$, where $+0$ means the "closest irrational slope" to the rational vector $\vec{\gamma}$. For other choice of coproduct eq. (5.1) is manifestly wrong. How then would the identity involving $T_{\vec{\gamma}}$ hold if some of the terms are group-like, while others are not? For $q_3 \to 1$ this problem does not arise, since the coproducts are all equivalent and trivial. One hope might be that coproducts for different slopes s and s' are related by nontrivial Drinfeld twists $F_{s,s'} \in U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}_1) \otimes U_{q_1,q_2,q_3}(\widehat{\widehat{\mathfrak{gl}}}_1)$:

$$\Delta_s = F_{s,s'} \Delta_{s'} F_{s,s'}^{-1}. \tag{5.2}$$

The form of the Drinfeld twist $F_{s,s'}$ is known more or less explicitly:

$$F_{s,s'} \sim \prod_{s<\vec{\gamma}<s',\, \gcd(\vec{\gamma})=1} \exp\left[\sum_{n\geq 1} \kappa_n e_{n\vec{\gamma}} \otimes e_{-n\vec{\gamma}}\right], \tag{5.3}$$

however, the commutation relations of $F_{(1,0),(0,1)}$ with $T_{(1,0)}$ and $T_{(0,1)}$ seem to be complicated. Thus, the mystery remains: how it happens that the coproduct preserves the pentagon relation (1.3)?

Finally, it would be interesting to find if similar identities involving $T_{\vec{\gamma}}$ with non-coprime $\vec{\gamma}$ hold. In the limit $q_3 \to 1$ there are plenty of such identities each corresponding to a motivic wall-crossing formula [24]. One could wonder if any of them survives q_3-deformation.

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