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Einstein-scalar field solutions in AdS spacetime: clouds, boundary conditions, and scalar multipoles

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Abstract: We consider an Einstein-scalar field model which is a consistent truncation of $\mathcal{N} = 8$ $D = 4$ gauged supergravity, the scalar field possessing a potential which is unbounded from below and a tachyonic mass above the Breitenlohner-Freedman bound. We investigate the spherically symmetric asymptotically anti-de Sitter soliton and black hole solutions, with the aim of clarifying the asymptotics and the possible boundary conditions at infinity. The emerging picture is contrasted with that found for an Einstein-scalar field model with the same scalar mass and a quartic selfinteraction term. We also provide arguments for the existence of solitonic solutions which can be viewed as non-linear continuation of the (probe) scalar multipolar clouds, with emphasis on the dipole case. Apart from numerical results, exact solutions are found for solitons with a monopole and dipole scalar field, as perturbations around the AdS background.

Keywords: Black Holes, Supergravity Models, AdS-CFT Correspondence, Black Holes in String Theory

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1 Introduction

The study of scalar fields in AdS spacetime can be traced back at least to the work [1, 2], where the massive Klein-Gordon equation has been solved in an AdS_4 background. More recently, this subject has been of particular interest mainly due to the AdS/CFT duality [3–5], which asserts that a consistent theory of quantum gravity in D-dimensions has an equivalent formulation in terms of a non-gravitational theory in $(D-1)$-dimensions.

One well understood limit of the duality is when, in the AdS bulk, it is sufficient to consider the low energy limit of the superstring theory, namely, supergravity (for a review, see ref. [6]). The supergravity models usually contain tachyonic scalar fields and, in some cases, there exist consistent truncations such that the matter content consists only of scalar fields. The solutions of these models are particularly interesting due to non-trivial boundary conditions satisfied by the scalar field(s) [7] that are relevant for the dual theory [8]. For example, the scalar fields can break the conformal symmetry on the boundary [9], and exact hairy black hole (BH) solutions with this property were presented in [10]. Also, the lowest energy solitonic solutions can be considered the true ground state of the theory [7, 11], while the study of the thermodynamic properties of AdS BHs offers the possibility to better understand the non-perturbative aspects of certain dual field theories.

Of interest in this context is the $\mathcal{N}=8$ $D=4$ gauged supergravity model that can be obtained as a compactification of $D=11$ supergravity on S^7 [12]. As discussed in [13] (see also [14] and [15]), this model possesses a consistent truncation with three real scalar fields of equal mass, coupled to four U(1) fields, which can be set to zero. The scalar fields can also vanish, a case which results in Einstein gravity with a negative cosmological constant $\Lambda = -3/L^2$ (with L the AdS radius). The main solution of interest is the AdS_4 spacetime in global coordinates

$$ds^2 = -N(r)dt^2 + \frac{dr^2}{N(r)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad \text{where} \quad N(r) = 1 + \frac{r^2}{L^2}, \tag{1.1}$$

with r, t the radial and time coordinate, respectively, while θ, φ are the usual coordinates on S^2. Otherwise, as discussed in appendix A, we can (consistently) take only one scalar field to be nonzero ($n=1$), or two of them ($n=2$) (in which case, they are equal); finally, all scalars can be taken nonzero and equal, $n=3$, the resulting Einstein-(single, real) scalar field model being presented in section 2. For any nonzero n, the scalar field possesses a tachyonic mass $\mu^2 = -2/L^2$.

In this work, we are interested in static and localized solutions of the considered Einstein-scalar field model, which are also regular and have a finite mass. They may possess an event horizon of spherical topology (and then correspond to BHs with scalar hair) or just correspond to solitonic deformations of the globally AdS spacetime (1.1). In both cases, the generic expression of the (static) scalar field as $r \to \infty$ (which is found by considering the linearized Klein-Gordon equation in a fixed AdS background) is the sum of two modes

$$\phi = \frac{\alpha(\theta,\varphi)}{r} + \frac{\beta(\theta,\varphi)}{r^2} + \ldots, \tag{1.2}$$

α and β being two real functions. For a well defined theory, one has to specify a boundary condition on α, β, the natural choice corresponding to either $\alpha = 0$ or $\beta = 0$. However, as shown in [7, 16–18], one may consider a larger class of mixed boundary conditions, with nonzero α and β, for which the conserved global charges are still well defined and finite. The boundary conditions in this case are defined by an essentially arbitrary function W connecting α and β, with

$$\beta = \frac{dW(\alpha)}{d\alpha}\,. \tag{1.3}$$

Since their properties depend significantly on the choice of W, this type of models have been called *designer gravity theories* [7].

The expressions of α and β are not determined a priori, various boundary conditions being possible, that lead to different properties of the solutions. Also, α and β in the large-r expansion (1.2) are not arbitrary, being determined by the imposed data at the origin (for solitons), or at the horizon (for BHs). For example, a priori one expects the existence of solutions with $\alpha = 0$ or $\beta = 0$. However, to our best of knowledge, no systematic study of this aspect has been presented in the literature. For $n = 1$, we mention the study in refs. [18–20], where spherically symmetric solutions with $\beta = f\alpha^2$ are studied (with f a negative constant).

The first goal of the present work is to consider a systematic study of the spherically symmetric solitonic and BH solutions of the considered consistent truncation of the $\mathcal{N} = 8$ $D = 4$ model. The aim is to clarify the dependence of the data at infinity on the data at the origin/horizon,

$$\alpha \equiv \alpha(r_0, \phi(r_0)), \quad \beta \equiv \beta(r_0, \phi(r_0)), \quad M_0 \equiv M_0(r_0, \phi(r_0)), \tag{1.4}$$

(with $r_0 = 0$ or r_h for solitons and BHs, while M_0 is an extra-constant which enters the far field expression of the metric), without imposing any relation between α and β.[1] The main results can be summarized as follows. Firstly, both perturbative and non-perturbative solutions are considered. The perturbative results are found for solitons, the perturbation parameter being $\phi(0)$, the value of the scalar field at the origin. The non-perturbative solutions are found by solving numerically the field equations, in which case we aim for a systematic scan of the parameter space for a large range of $r_h, \phi(r_h)$. Secondly, our results provide strong evidence for the *absence* of (soliton or BH) solutions satisfying the 'standard' conditions $\alpha = 0$ or $\beta = 0$. That is, the $\mathcal{N} = 8$ $D = 4$ model possesses *designer gravity* solutions only.

A natural question which arises in this context concerns the generality of these results. In particular, is it possible to find solutions with $\alpha = 0$ or $\beta = 0$ in (1.2) for a different choice of the scalar field potential? To address this aspect, we study also solutions of a model in which the scalar field still possesses the same mass as in the $\mathcal{N} = 8$ case; however, the selfinteraction is given by a quartic term. As a result, the asymptotic behaviour of the scalar field is less constrained and one finds e.g. spherically symmetric solutions with $\alpha = 0$ or $\beta = 0$ in the expansion (1.2).

Another goal of this work is motivated by the observation that, to our best knowledge, all Einstein-(real) scalar field solutions reported in the literature correspond to spherically

[1] A similar analysis in five dimensions, but with a different goal, was presented in [21].

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symmetric configurations. However, the (static) solution of the linearized Klein-Gordon equation in a fixed AdS background possesses a general solution, the scalar field being a superposition of modes, $\phi = \sum_{\ell m} Y_{\ell m}(\theta,\varphi) R_\ell(r)$ (with $Y_{\ell m}$ the real spherical harmonics and $R_\ell(r)$ the radial amplitude). From this perspective, no value of (ℓ, m) is privileged, with the same asymptotic decay (1.2) for all modes (see, also, [22]). Also, since the spherically symmetric Einstein-scalar field solitons can be viewed as a non-linear continuation of the $\ell = m = 0$ mode, one expects similar results to exist for higher modes. In this work we consider mainly the simplest case $\ell = 1, m = 0$ and we provide evidence that the qualitative picture found in the spherically symmetric case still holds. First, one finds again an exact, perturbative solitonic solution, which is interpreted as a deformation of AdS spacetime with a dipolar scalar field. Nonperturbative solutions with a $1/r^2$-decay at infinity of the scalar field are shown to exist in a model with ϕ^4-selfinteraction

This paper is organized as follows. In the next section we present the general framework, while in section 3 we consider the probe limit of the problem, with a study of scalar clouds in a fixed (Schwarzschild-)AdS background. The case of spherically symmetric configurations is discussed in section 4, while the gravitating scalar dipoles are studied in section 5. We conclude in section 6 with a discussion and some further remarks. The appendix A explains how the considered sugra-action is obtained starting with the general results in ref. [13]. In the appendix B, we provide some details on the perturbative axially symmetric solutions, including the Einstein-scalar field soliton with a quadrupole scalar field.

2 The general framework

2.1 The action, equations of motion and scalar field potentials

We consider the Einstein-(real)scalar field model with a negative cosmological constant

$$I = \int_{\mathcal{M}} d^4x\sqrt{-g}\left[\frac{1}{4\kappa^2}\left(R + \frac{6}{L^2}\right) - \frac{1}{2}g^{ab}\phi,_a\phi,_b - U(\phi)\right] - \frac{1}{2\kappa^2}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}K, \tag{2.1}$$

where $\kappa^2 \equiv 4\pi G$ (with G the Newton's constant) and $U(\phi)$ is the scalar field potential. Also, the last term in (2.1) is the Hawking-Gibbons surface term [23], where K is the trace of the extrinsic curvature for the boundary $\partial\mathcal{M}$ and h is the induced metric of the boundary.

The corresponding Einstein-scalar field equations, as obtained from the variation of the action (2.1) with respect to the metric and scalar field, respectively, read:

$$R_{ab} - \frac{1}{2}g_{ab}R - \frac{3}{L^2}g_{ab} = 2\kappa^2\, T_{ab}, \qquad \nabla^2\phi = \frac{\partial U}{\partial\phi}, \tag{2.2}$$

where T_{ab} is the stress-energy tensor of the scalar field,

$$T_{ab} = \phi_{,a}\phi_{,b} - g_{ab}\left(\frac{1}{2}g^{cd}\phi_{,c}\phi_{,d} + U(\phi)\right). \tag{2.3}$$

In this work we shall consider two different expressions of the scalar field potential $U(\phi)$. The first case is of main interest, occurring in a consistent trucation of the $\mathcal{N} = 8$ $D = 4$ gauged supergravity model [12], with

$$\text{sugra}: \qquad U(\phi) = -\frac{n}{\kappa^2 L^2}\sinh^2\left(\frac{\kappa}{\sqrt{n}}\phi\right), \quad \text{with} \quad n = 1, 2, 3. \tag{2.4}$$

The appendix A presents some details on how the action (2.1) with the above potential can be obtained starting with the general results in ref. [13].

The small-ϕ expansion of the scalar field potential (2.4) is

$$U(\phi) \sim -\frac{\phi^2}{L^2} - \frac{\kappa^2\phi^4}{3nL^2} + O(\phi^6). \tag{2.5}$$

Therefore, for any n, the scalar field possesses a tachyonic mass, with

$$\mu^2 = \left.\frac{d^2U}{d\phi^2}\right|_{\phi=0} = -\frac{2}{L^2}, \tag{2.6}$$

which implies the asymptotic behaviour (1.2).

The second case considered in this work corresponds to a massive scalar field with quartic selfinteraction,

$$\phi^4\text{-model}: \qquad U(\phi) = -\frac{\phi^2}{L^2} + \lambda\phi^4, \tag{2.7}$$

where λ is an arbitrary constant. Note that for the particular value $\lambda = -\kappa^2/(3nL^2)$, the potential (2.7) can be considered as a truncation of the sugra-potential (2.4).

2.2 Mixed boundary conditions and holographic mass

In this section we present a general discussion of possible boundary conditions for a scalar field with the tachyonic mass (2.6), the interpretation within AdS/CFT duality, and a concrete method to obtain the holographic mass. We are going to follow closely the refs. [24] and [25], where counterterms for the scalar fields were used to regularize the action and to obtain the holographic mass. These results can be directly applied to various examples considered in the next sections.

While in theories of gravity coupled to matter the theory is usually fully determined by the action, in the presence of scalar fields with tachyonic mass the situation is quite different. That is, both modes in the scalar field's fall off (1.2) are normalizable and represent physically acceptable fluctuations. Therefore, specifying boundary conditions for the scalar field is equivalent to fixing the boundary data α, β or a specific relation between them. It is common to denote the mixed boundary conditions on the scalar field by $\beta \equiv W'(\alpha)$, where $W(\alpha)$ is an arbitrary differentiable function. This restriction on α and β can be obtained from the vanishing symplectic flux flow through the boundary [26] and it is interpreted as an integrability condition for the mass in the Hamiltonian formalism [7, 27].

The existence of various boundary conditions for the scalar fields fits very well in the context of AdS/CFT duality where they are interpreted as multitrace deformations in the dual field theory [8]. The interpretation is as follows: if α is identified with the source for an operator in the dual field theory, $\mathcal{O}$, the dual field theory action should contain a term $\int \alpha(x)\mathcal{O}(x)d^3x$ and then β is identified with the vacuum expectation value (VEV) of the operator, $\beta = \langle\mathcal{O}\rangle$ (and the other way around with β the source and α the VEV). However, for the current work, the mixed boundary condition, $\beta = W'(\alpha)$, is the relevant one with the following interpretation: these general boundary conditions are multitrace deformations of the boundary CFT, of the form $\int W[\mathcal{O}(x)]d^3x$. A generic deformation can

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break the conformal symmetry in the boundary, but since it is still invariant under global time translations, there exists a conserved total mass/energy. As we are going to explicitly show below, the mixed boundary conditions that preserve the conformal symmetry can be obtained from the vanishing trace of the dual stress tensor and correspond to triple trace deformations.

After this brief review of mixed boundary conditions for the scalar field and their interpretation within the AdS/CFT duality, let us obtain the holographic mass by using the 'counterterm method', that consists in adding suitable additional surface terms to regularize the action (2.1). These counterterms are usually built up with curvature invariants on the boundary $\partial\mathcal{M}$ (which is sent to infinity after the integration); as such, they do not alter the bulk equations of motion. In four spacetime dimensions, the following counterterms are sufficient to cancel divergences for (electro-)vacuum solutions with negative cosmological constant [28–30]

$$I_{\rm ct}^{(0)} = -\frac{1}{2\kappa^2}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\left[\frac{2}{L}+\frac{L}{2}\mathcal{R}\right], \tag{2.8}$$

where $\mathcal{R}$ is the Ricci scalar of the boundary metric h. Within this approach, the mass computation goes as follows. First step consists in constructing a divergence-free boundary stress tensor $\mathrm{T}_{\mu\nu}$ from the total action $I{=}I_{\rm bulk}{+}I_{\rm surf}{+}I_{\rm ct}^{(0)}$ by defining

$$\mathrm{T}_{\mu\nu} = \frac{2}{\sqrt{-h}}\frac{\delta I}{\delta h^{\mu\nu}} = \frac{1}{2\kappa^2}\left(K_{\mu\nu}-Kh_{\mu\nu}-\frac{2}{L}h_{\mu\nu}+LE_{\mu\nu}\right), \tag{2.9}$$

where $E_{\mu\nu}$ is the Einstein tensor of the boundary metric, $K_{\mu\nu}=-1/2(\nabla_\mu n_\nu+\nabla_\mu n_\nu)$ is the extrinsic curvature, n^μ being an outward pointing normal vector to the boundary. Here one supposes that the boundary geometry is foliated by spacelike surfaces Σ with metric σ_{ij}

$$h_{\mu\nu}dx^\mu dx^\nu = -N_\Sigma^2 dt^2+\sigma_{ij}(dx^i+N_\sigma^i dt)(dx^j+N_\sigma^j dt). \tag{2.10}$$

Then, if ξ^μ is a Killing vector generating an isometry of the boundary geometry, there should be an associated conserved charge. In this approach, $\rho = u^a u^b \mathrm{T}_{ab}$ is the proper energy density while u^a is a timelike unit vector normal to Σ. Thus the conserved charge associated with time translation $\partial/\partial t$ is the mass of the spacetime

$$M = \int_\Sigma d^2x\sqrt{\sigma}N_\Sigma\rho. \tag{2.11}$$

The presence of the scalar field in the bulk action (2.1) brings the potential danger of having divergent contributions coming from both, the gravitational and matter actions [31]. This is the case for a scalar field which behaves asymptotically as $O(1/r)$ (i.e. $\alpha\neq 0$ in (1.2)), and then the counterterms (2.8) will not yield a finite mass. However, it is still possible to obtain a finite mass by allowing the boundary counterterms to depend not only on the boundary metric $h_{\mu\nu}$, but also on the scalar field. This means that the quasilocal stress-energy tensor (2.9) also acquires a contribution coming from the matter field. The counterterm that regularizes the action and has a valid variational principle compatible with the boundary condition $\beta\equiv W'(\alpha)$ is [25]

$$I_\phi^{ct} = -\int_{\partial M} d^3x\sqrt{-h}\left[\frac{1}{2L}\phi^2+\frac{1}{L}\frac{W(\alpha)}{\alpha^3}\phi^3\right] \tag{2.12}$$

that yields a supplementary contribution to the boundary stress tensor (2.9),

$$\mathrm{T}_{\mu\nu}^{(\phi)} = -\frac{1}{L} h_{\mu\nu} \left(\phi^2 + \frac{W(\alpha)}{\alpha^3} \phi^3 \right), \tag{2.13}$$

which should be taken into account when obtaining a finite mass, eq. (2.11).

We also mention that the background metric upon which the dual field theory resides is $\gamma_{\mu\nu} = \lim_{r\to\infty} \frac{L^2}{r^2} h_{\mu\nu}$ (with r the radial coordinate). Then, the expectation value of the dual CFT stress-tensor can be calculated using the relation [40]

$$\sqrt{-\gamma}\gamma^{\mu\lambda} < \tau_{\lambda\nu} >= \lim_{r\to\infty} \sqrt{-h} h^{\mu\lambda} \mathrm{T}_{\lambda\nu}, \tag{2.14}$$

with the trace [25]

$$< \tau_\nu^\nu >= -\frac{3}{L^4} \left(W - \frac{\alpha\beta}{3} \right). \tag{2.15}$$

It follows that the only mixed boundary condition, which preserves the conformal symmetry corresponds to a triple trace deformation in dual field theory, $\beta \sim \alpha^2$. In this case, the trace of the dual stress tensor vanishes.

In this approach, given some data at infinity (α, β), one can assign a well defined mass M only *after* defining the function $W(\alpha)$, cf. eq. (1.3). In principle, this result can be circumvented by using the counterterm in refs. [10, 32–34],

$$I_{ct}^{(\phi)} = \frac{1}{3} \int_{\partial M} d^3x \sqrt{-h} \left(\phi n^\nu \partial_\nu \phi - \frac{1}{2L} \phi^2 \right), \tag{2.16}$$

which does not require to specify a condition for α and β. However, this counterterm is problematic because it is not intrinsic to the boundary and also, for mixed boundary conditions, the variational principle is not satisfied. Moreover, it implies generically a vanishing trace of the boundary stress tensor (although for $\beta \sim \alpha^2$ the mass expression coincides with that found using (2.13)).

2.3 Remarks on numerics

While it was possible to find some partial analytical results, the non-perturbative solutions are constructed numerically, by integrating the system of Einstein-scalar field equations (2.2) subject to suitable boundary conditions. Both solitons and BHs will be considered. However, note that in order to simplify the problem, in the BH case we restrict the study to the region outside the event horizon. For spherically symmetric solutions, we use a standard Runge-Kutta ordinary differential equation solver. All numerical calculations in the axially symmetric case have been performed by using a professional package, which uses a Newton-Raphson finite difference method with an arbitrary grid and arbitrary consistency order [35, 36].

The numerics are done working with a scaled scalar field and a scaled radial coordinate,

$$\phi \to \phi/\kappa, \quad r \to r/L, \tag{2.17}$$

that results in the following Lagrangian of the considered models (note that for the ϕ^4 case, we supplement (2.17) with $\lambda \to \lambda\kappa^2/L^2$)

$$\mathcal{L} = R + 6 - \frac{1}{2} g^{ab}\phi_{,a}\phi_{,b} - U(\phi) \tag{2.18}$$

with

$$\text{sugra}: \quad U(\phi) = -n \sinh^2\left(\frac{\phi}{\sqrt{n}}\right), \qquad \phi^4\text{-model}: \quad U(\phi) = -\phi^2 + \lambda\phi^4. \tag{2.19}$$

However, for the sake of clarity, all equations displayed in what follows are given in terms of dimensionful variables.

3 Probe limit: static scalar clouds in AdS

Before considering the full problem, it is interesting to consider first the probe limit, and to study solutions of the Klein-Gordon equation in a fixed geometry, while neglecting the backreaction of the scalar field. The background can be the AdS spacetime or the Schwarzschild-AdS (SAdS) BH, which possesses a line element of the form (1.1), with[2]

$$N(r) = \left(1 - \frac{r_h}{r}\right)\left(1 + \frac{r^2}{L^2} + \frac{r r_h}{L^2} + \frac{r_h^2}{L^2}\right), \tag{3.1}$$

where r_h is the event horizon radius and M the BH mass.

This approximation greatly simplifies the problem but retains some of the interesting physics. Both *linear* (i.e. with a mass term only in the potential $U(\phi)$) and *non-linear* clouds will be considered. In both cases, the mass-energy density of a configuration, as measured by a static observer with 4-velocity $U^a \sim \delta^a_t$, is $\rho = -T_t^t$. Then one can define a total mass of a cloud,

$$M^{(\text{cloud})} = -\int d^3x T_t^t = -\int_{r_0}^{\infty} dr \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi r^2 \sin\theta\, T_t^t. \tag{3.2}$$

One can see that, in order for $M^{(\text{cloud})}$ to be finite, T_t^t should decay faster than $1/r^3$ as $r \to \infty$. Then, for the scalar field asymptotics (1.2), the term proportional with $1/r$ should be absent, i.e. $\alpha(\theta, \varphi) = 0$ in the large r-limit.

3.1 The linear case

Let us start with the case of a massive scalar field with no selfinteraction (i.e. with $U = \frac{1}{2}\mu^2\phi^2$) in a fixed AdS background (1.1), and consider solutions of the (linear) KG equation

$$\nabla^2\phi = \mu^2\phi. \tag{3.3}$$

The scalar field can be decomposed in a sum of modes

$$\phi = \sum_{\ell m} \phi_{\ell m}(r, \theta, \varphi), \qquad \text{with} \quad \phi_{\ell m} = Y_{\ell m}(\theta, \varphi) R_\ell(r), \tag{3.4}$$

[2]The expression (3.1) results from the usual SAdS expression, $N = 1 - 2M/r + r^2/L^2$ with $M = r_h(1 + r_h^2/L^2)/2$.

where $Y_{\ell m}(\theta,\varphi)$ are the real spherical harmonics (with $\ell = 0, 1, \ldots$ and $-\ell \leq m \leq \ell$), while the radial amplitude $R_\ell(r)$ is a solution of the equation

$$\frac{1}{r^2}(r^2 N R_\ell)' = \left(\mu^2 + \frac{\ell(\ell+1)}{r^2}\right) R_\ell, \tag{3.5}$$

where a prime denotes the derivative w.r.t. the radial coordinate r.

For the case of interest in this work with $\mu^2 = -2/L^2$, the general solution of the above equation reads (with ${}_2F_1$ the hypergeometric function)

$$\begin{aligned} R_\ell(r) = {} & c_1 \left(\frac{r}{L}\right)^{\ell} {}_2F_1\left(\frac{1+\ell}{2}, \frac{2+\ell}{2}; \frac{3}{2}+\ell; -\frac{r^2}{L^2}\right) \\ & + c_2 \left(\frac{L}{r}\right)^{\ell+1} {}_2F_1\left(\frac{1-\ell}{2}, -\frac{\ell}{2}; \frac{1}{2}-\ell; -\frac{r^2}{L^2}\right) \end{aligned} \tag{3.6}$$

being the sum of two modes (with c_1, c_2 arbitrary constants). However, the second term in the above relation diverges as $r \to 0$ and thus we set $c_2 = 0$ (also, in what follows, we take $c_1 = 1$). The explicit form of the solution for the first three values of ℓ reads

$$\begin{aligned} R_0(r) &= \frac{L}{r}\arctan\left(\frac{r}{L}\right), \quad R_1(r) = \frac{3L}{r} - \frac{3L^2}{r^2}\arctan\left(\frac{r}{L}\right), \\ R_2(r) &= \frac{6L^2}{\pi r^2}\left(-1 + \left(1 + \frac{r^2}{3L^2}\right)\arctan\left(\frac{r}{L}\right)\right) \end{aligned} \tag{3.7}$$

the expressions for higher ℓ becoming increasingly complicated.

As $r \to 0$, the (regular) solution has the following form

$$R_\ell(r) = \left(\frac{r}{L}\right)^{\ell} - \frac{(\ell+1)(\ell+2)}{2(2\ell+3)}\left(\frac{r}{L}\right)^{\ell+2} + \ldots. \tag{3.8}$$

As spatial infinity is approached, all multipoles decay according to (1.2), such that the approximate form of a (ℓ, m)-mode reads

$$\phi_{\ell m}(r,\theta,\varphi) = \frac{\alpha(\theta,\varphi)}{r} + \frac{\beta(\theta,\varphi)}{r^2} + \ldots,$$

where

$$\alpha(\theta,\varphi) = \frac{\sqrt{\pi}\,\Gamma(\ell+\frac{3}{2})L}{(\Gamma(\frac{\ell}{2}+1))^2} Y_{\ell m}(\theta,\varphi), \;\; \beta(\theta,\varphi) = \frac{\sqrt{\pi}\,\Gamma(\ell+\frac{3}{2})L^2}{(\Gamma(\frac{\ell+1}{2}))^2} Y_{\ell m}(\theta,\varphi).$$

This behaviour strongly contrasts with that found for a Minkowski spacetime background, where the scalar mode which is regular at $r = 0$ diverges as $r \to \infty$. This feature can be traced back to the "box"-like behaviour of the AdS spacetime, and is present also for a Maxwell field [37–39].

The above solution contains already several features that will also be found in the (self-gravitating) sugra-case. First, one notices that all multipoles share the same far field decay. Second, the radial amplitude $R_\ell(r) =$ is always nodeless. Moreover, both parameters α and β are non-zero. As such, the linear cloud mass, as computed according to (3.2) diverges, although the energy density ρ is finite everywhere.

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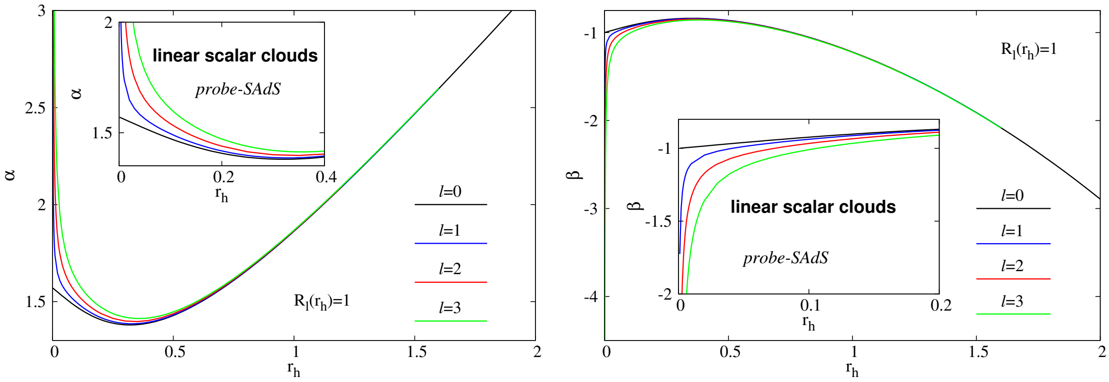

Figure 1. The parameters α and β which enter the far field expansion of the scalar field are shown as a function of the event horizon radius for several values of the harmonic index ℓ. The results are found for linear scalar clouds in a Schwarzschild-AdS background.

One may ask if the situation is different when considering a SAdS BH background. Although no exact solutions of the radial equation appear to exist, the eq. (3.5) can be solved numerically. The approximate expansion of the radial amplitude as $r \to r_h$ reads

$$R_\ell(r) = R_\ell(r_h)\left(1 + \frac{\ell(\ell+1)L^2 - 2r_h^2}{r_h(L^2+3r_h^2)}(r-r_h)\right) + \dots, \tag{3.9}$$

while the asymptotic expansion of a mode is given by (1.2), with

$$\alpha = p_1 Y_{\ell m}(\theta,\varphi), \quad \beta = p_2 Y_{\ell m}(\theta,\varphi), \tag{3.10}$$

where the constants p_1, p_2 depend on the value of r_h.

As seen in figure 1, the presence of a horizon does not change the picture found for solitons. In particular, there are no scalar clouds[3] with $\alpha = 0$ or $\beta = 0$.

3.2 Non-linear clouds in the ϕ^4 model

One may inquire how general are the above results and what are the new results induced by the scalar field selfinteraction. In particular, are there scalar clouds with $\alpha = 0$ (which then would possess a finite (cloud) mass)?

While the picture found for the sugra-potential (2.4) appears to be qualitatively similar to that found for linear clouds (and we could not find solutions with $\alpha = 0$), the situation is different for a model with a quartic selfinteraction. An indication in this direction[4] as provided by the existence of the following exact solution[5] describing a spherically symmetric

[3] The apparent divergence for $\ell > 0$ of α, β as $r_h \to 0$ is an artifact of solving the (linear) equation (3.5) with the boundary condition $R_\ell(r_h) = 1$, while $R_\ell(0) = 0$ in the solitonic limit.

[4] An exact solution with $\alpha = 0$ and a finite mass $M^{(\mathrm{cloud})} = 8\pi^2 L/(27\lambda^2)$ is found in a model with a *cubic*-selfinteraction

$$U(\phi) = -\frac{\phi^2}{L^2} + \lambda\phi^3, \quad \text{and} \quad \phi(r) = -\frac{4}{3\lambda}\frac{1}{1+\frac{r^2}{L^1}}.$$

[5] Note that the solution with real ϕ exists for $\lambda < 0$ only.

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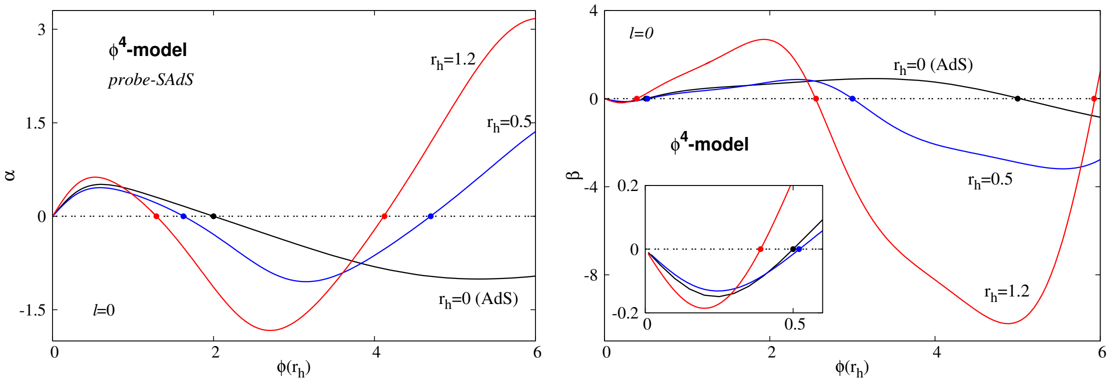

Figure 2. The parameters α and β which enter the far field expansion (1.2) are shown as a function of the scalar field at the horizon for solutions of a ϕ^4-model in a fixed background. Note the existence of configurations with $\alpha = 0$ or $\beta = 0$ (marked with dots).

soliton in the ϕ^4-model:

$$\phi(r) = \frac{1}{2\sqrt{-\lambda}} \frac{1}{\sqrt{1+\frac{r^2}{L^2}}},$$
$$\text{with} \quad \phi(r) \to \frac{1}{2\sqrt{-\lambda}} \frac{L}{r} - \frac{L^3}{4\sqrt{-\lambda} r^3} + \ldots \quad \text{as} \quad r \to \infty \quad \text{i.e.} \quad \beta = 0. \tag{3.11}$$

We have studied generalizations of this exact solution, by solving numerically the scalar field equation for a (S)AdS background and varying $\phi(r_h)$ (the value of the scalar field at the horizon or the origin, $r_h = 0$, for the soliton), the values of the parameters α and β being extracted from the numerical output. In the numerics, we set $\lambda = -1$ without any loss of generality, via a suitable scaling of the scalar field.

Some numerical results are shown in figure 2 where one can observe that the parameters α and β can be zero, for a (presumably infinite) set of discrete values of $\phi(r_h)$. For example, in the solitonic case ($r_h = 0$), the exact solution (3.11) corresponds to $\phi(0) = 1/2$, while the first configuration with $\alpha = 0$ is found for $\phi(0) = 2$ (which strongly suggests the existence of an exact solution also in that case). Moreover, for $\phi(0) > 2$ the solutions possess at least a node.

In figure 3 (left panel) we present a number of relevant quantities as a function of the horizon radius for the subset of (finite mass) non-linear cloud solutions with $\alpha = 0$. As one can see, no restrictions seem to exist on the BH size, while for large enough BHs, both M and β increase linearly with r_h.

Since one has to solve numerically a (nonlinear) partial differential equation (PDE), the case of higher multipoles is technically more complicated. Restricting to axisymmetric configurations, this is a particular case of the problem discussed in section 3, being solved by using a similar numerical approach. Moreover, we have mainly studied the case of finite mass of the dipole clouds.[6] i.e. with $\alpha = 0$ in the scalar far field expansion, the $1/r^2$ decay being imposed by introducing a new function $\psi = r\phi$, and requiring $\psi \to 0$ as $r \to \infty$. Other boundary conditions satisfied by the scalar field are $\partial_\theta \phi = 0$ at $\theta = 0, \pi$ and $\phi = 0$ at $r = 0$,

[6]However, we have confirmed the existence of similar solutions also for $\alpha = 0$ quadrupoles.

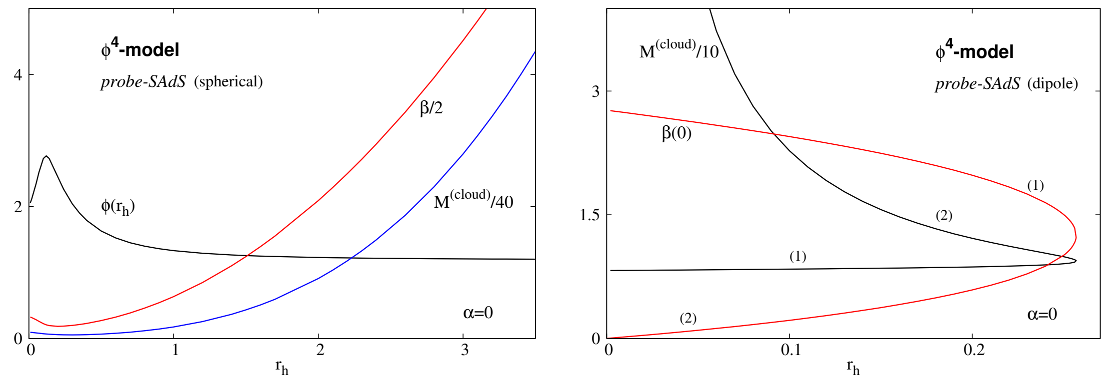

Figure 3. Several quantities of interest are shown as a function of the event horizon radius for $\ell = 0,\ 1$ solutions of the ϕ^4-model in a fixed Schwarzschild-AdS background. The scalar field here decays asymptotically as $1/r^2$.

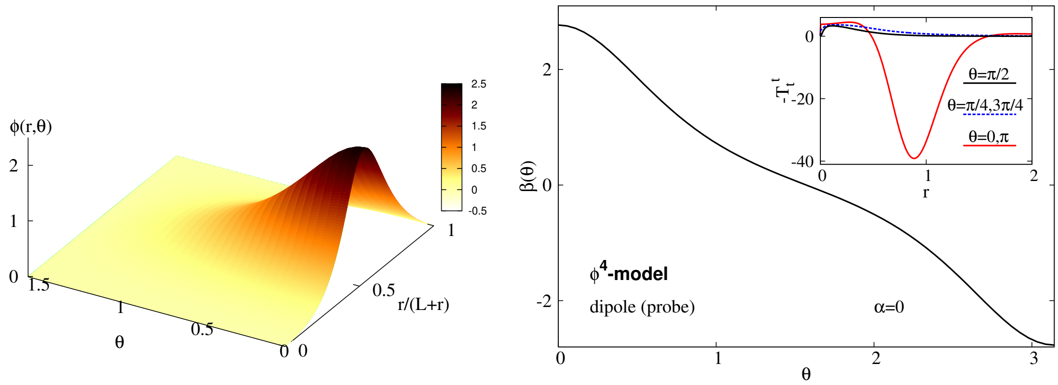

Figure 4. *Left panel:* the scalar field profile is shown for the (fundamental) dipole solution with $1/r^2$ decay in the ϕ^4-model and a fixed AdS background. Only half of the space is shown here, with $\phi(r, \theta) = -\phi(r, \pi - \theta)$. *Right panel:* the function β which enters the large-r asymptotics (1.2) of ϕ is shown for the same solution. The inset shows the energy density $\rho = -T_t^t$ for several angular directions.

while the field is (still) odd-parity, $\phi(\theta) = -\phi(\pi - \theta)$. In figure 4 (left panel) we display the profile of the AdS dipole solution, which possesses a finite cloud mass $M^{(\text{cloud})} \simeq 8.208$.

We note that the extrema of the field are located on the z-axis (with $z = r\cos\theta$) and are symmetric w.r.t. the equatorial plane.

We shall also mention that the ℓ-labeling of the solutions in terms of multipoles is ambiguous in a non-linear setup, since the 'pure-cloud' feature of the linear solutions is lost due to selfinteraction. As such, ℓ stands rather for the number of angular nodes of the scalar profiles (with $\ell = 0$ for spherical solutions, $\ell = 1$ for dipoles (one node at $\theta = \pi/2$), etc). For example, for a dipole solution, one can write the following expansion of the scalar field

$$\phi(r, \theta) = \sum_{k \geq 0} f_k(r) \mathcal{P}_{2k+1}(\cos\theta), \tag{3.12}$$

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with $\mathcal{P}_n(x)$ the Legendre polynomials. Although the $k = 0$ term dominates, the contribution of the higher order terms is also nontrivial, as can be seen already in the profile of the function $\beta(\theta)$ (in the right panel of figure 4). While for a linear cloud $\beta \sim \cos(\theta)$, this is not the case when there exists a nonzero contribution of higher order $\mathcal{P}_{2k+1}$-terms.

As expected, similar solutions are found (numerically) in the presence of a BH horizon, i.e. for a SAdS background. The numerics is done in terms of a new radial coordinate $\bar{r} = \sqrt{r^2 - r_h^2}$ in the line-element (1.1), (3.1), such that the horizon is located at $\bar{r} = 0$, where we impose $\partial_{\bar{r}}\psi = 0$ (with $\phi = r\psi$). The boundary conditions on the z-axis and at infinity are similar to those imposed for $r_h = 0$. In figure 3, we have shown the mass of the non-linear cloud solutions and the value of $\beta(0)$ as a function of the horizon radius. One can see that the picture is very different as compared to that found in the spherically symmetric case. As for $\ell = 0$, a branch of solutions (label (1) in figure 3) smoothly emerges when adding a horizon at the center of a soliton. Along this branch the mass increases, while the maximal value of β decreases. However, for $\ell = 1$, one finds the existence of a maximal value of r_h, with a back bending and the occurrence of a secondary branch of solutions (label (2)), which extends backwards in r_h. As $r_h \to 0$ along this secondary branch, $\beta \to 0$, while $M^{(\mathrm{cloud})}$ of the dipolar cloud solutions diverges.[7]

4 Spherically symmetric Einstein-scalar field solutions

4.1 The Ansatz, equations and asymptotics

The spherically symmetric solutions are constructed by using the following Ansatz for the metric and scalar field

$$ds^2 = -N(r)e^{-2\delta(r)}dt^2 + \frac{dr^2}{N(r)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad \text{and} \quad \phi \equiv \phi(r)\,, \tag{4.1}$$

where it is convenient to take

$$N(r) = 1 + \frac{r^2}{L^2} - \frac{2m(r)}{r}, \tag{4.2}$$

with $m(r)$ a mass function. From (2.2) we find the following equations for the metric functions and the scalar field:

$$m' = \kappa^2 r^2\left(\frac{1}{2}N\phi'^2 + U(\phi)\right), \quad \delta' = -\kappa^2 r\phi'^2, \quad \phi'' + \left(\frac{2}{r} + \frac{N'}{N} - \delta'\right)\phi' - \frac{1}{N}\frac{dU(\phi)}{d\phi} = 0. \tag{4.3}$$

There is also a 2nd order constraint equation

$$\frac{1}{2}N'' - N\delta'' + \frac{N'}{r} - \delta'\left(\frac{N}{r} + \frac{3}{2}N' - N\delta'\right) - \frac{3}{L^2} + \kappa^2(N\phi'^2 + 2U(\phi)) = 0, \tag{4.4}$$

which, however, is a differential consequence of the equations for m, δ in (4.3).

[7]This behaviour can be understood by noticing that, for the (scaled) units (2.17) we employ in numerics, $r_h \to 0$ can also be approached as $L \to \infty$ (and thus a vanishing cosmological constant), while the background metric becomes the Schwarzschild BH. However, no smooth solution exists in this case [41].

The Ricci scalar R and the Kretschmann scalar $K = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$ are given by

$$
\begin{aligned}
R &= 2N\delta'' - N'' + 3\delta' N' - \frac{2N(r\delta' - 1)^2}{r^2} - \frac{4N'}{r} + \frac{2}{r^2}, \\
K &= \frac{4(N-1)^2}{r^4} + \frac{2N'^2}{r^2} + \frac{2(N' - 2N\delta')^2}{r^2} + \Big(2N(\delta'^2 - \delta'') + N'' - 3\delta' N'\Big)^2 .
\end{aligned}
\tag{4.5}
$$

For all solutions reported in this work, both R and K are regular everywhere on the considered domain of integration.[8]

The system of equations (4.3) will be solved first perturbatively and then numerically. In both cases, it is useful to find the approximate form of the solutions at the boundaries of the domain of integration.

4.1.1 The small-r expansion

Starting with the solitonic case, the small-r solution can be written in the form

$$
m(r) = \sum_{k\geq 3} m_{(k)} r^k, \quad \delta(r) = \delta(0) + \sum_{k\geq 1} \delta_{(k)} r^k, \quad \phi(r) = \phi(0) + \sum_{k\geq 1} \phi_{(k)} r^k, \tag{4.6}
$$

the series coefficients $m_{(k)}$, $\delta_{(k)}$, $\phi_{(k)}$, being determined by the values of the functions ϕ and δ at $r = 0$. No general pattern for these coefficients appears to exist, the first terms being

$$
\begin{aligned}
& m_{(3)} = \frac{1}{3}\kappa^2 U_0,\ m_{(4)} = 0,\ m_{(5)} = \frac{2\kappa^2 U'^2 - 0}{45},\ m_{(6)} = 0,\ m_{(7)} = \frac{5\kappa^2 U_0'^2}{252L^2}\left(-1 + L^2\left(\frac{2}{3}\kappa^2 U_0 + \frac{6}{25}U_0''\right)\right), \\
& \delta_{(1)} = \delta_{(2)} = \delta_{(3)} = 0,\ \delta_{(4)} = -\frac{1}{36}\kappa^2 U_0'^2,\ \delta_{(5)} = 0,\ \delta_{(6)} = \frac{1}{27L^2}\kappa^2 U_0'^2\left(1 - L^2\left(\frac{2}{3}\kappa^2 U_0 + \frac{1}{10}U_0''\right)\right), \\
& \phi_{(1)} = 0,\ \phi_{(2)} = \frac{1}{6}U_0',\ \phi_{(3)} = 0,\ \phi_{(4)} = \frac{U_0'}{12L^2}\left(-1 + \frac{2}{3}\kappa^2 L^2 U_0 + \frac{1}{10}L^2 U_0''\right), \phi_{(5)} = 0,
\end{aligned}
$$

with $U_0 = U(\phi(0))$, $U_0' = U'(\phi(0))$, $U_0'' = U''(\phi(0))$.

4.1.2 The near-horizon solution

Apart from solitons, we are also interested in BH solutions. They possess a non-extremal horizon[9] located at $r = r_h > 0$, where $N(r_h) = 0$. Close to the horizon, we assume the existence of a power series expansion of the solution in $r - r_h$, with

$$
m(r) = \sum_{k\geq 0} \bar{m}_{(k)}(r - r_h)^k,\ \delta(r) = \delta(r_h) + \sum_{k\geq 1} \bar{\delta}_{(k)}(r - r_h)^k,\ \phi(r) = \phi(r_h) + \sum_{k\geq 1} \bar{\phi}_{(k)}(r - r_h)^k, \tag{4.7}
$$

[8]Thus, for BHs, this holds on and outside the event horizon.

[9]We did not find any indication for the existence of extremal BH solutions. In fact, their absence is also suggested by the absence of an attractor solution with an $AdS_2 \times S^2$ geometry, which would describe the near horizon of the extremal BHs.

the coefficients being determined by the horizon values of the functions ϕ and δ. One finds e.g.

$$\bar{m}_{(0)}=\frac{r_h(r_h^2+L^2)}{2L^2},\quad \bar{m}_{(1)}=\kappa^2 r_h^2 U_h,\quad \bar{\delta}_{(1)}=-\frac{\kappa^2 r_h^3 U_h'^2}{\left(1+\frac{3r_h^2}{L^2}-2\kappa^2 r_h^2 U_h\right)^2},\quad \bar{\phi}_{(1)}=\frac{r_h U_h'}{1+\frac{3r_h^2}{L^2}-2\kappa^2 r_h^2 U_h},$$

$$\bar{m}_{(2)}=\kappa^2 r_h\left(U_h+\frac{3r_h^2}{4}\frac{U_h'^2}{1+\frac{3r_h^2}{L^2}-2\kappa^2 r_h^2 U_h}\right),$$

$$\bar{\delta}_{(2)}=-\left(\left(1+\frac{3r_h^2}{L^2}\right)\left(1-\frac{3r_h^2}{L^2}-r_h^2 U_h''\right)+2\kappa^2 r_h^4\left(\left(\frac{6}{L^2}-U_h''-6\right)U_h-2\kappa^2 U_h\right)\right)\frac{2\kappa^2 r_h^2 U_h'^2}{4\left(1+\frac{3r_h^2}{L^2}-2\kappa^2 r_h^2 U_h\right)^4},$$

$$\bar{\phi}_{(2)}=-\left(\left(1+\frac{3r_h^2}{L^2}\right)\left(\frac{6}{L^2}-U_h''\right)U_h'-2\kappa^2 U_h'\left(\left(2\left(1+\frac{6r_h^2}{L^2}\right)-r_h^2 U_h''\right)U_h+r_h^2 U_h'^2-4\kappa^2 r_h^2 U_h^2\right)\right)$$
$$\times\frac{r_h^2}{4\left(1+\frac{3r_h^2}{L^2}-2\kappa^2 r_h^2 U_h\right)^3}, \tag{4.8}$$

where we denote $U_h=U(\phi(r_h))$, $U_h'=U'(\phi(r_h))$, $U_h''=U''(\phi(r_h))$. Also, since the model (2.18) is invariant when taking $\phi\to-\phi$, it is enough to consider the case $\phi(r_h)>0$, only (or $\phi(0)>0$ for solitons).

4.1.3 The large-r approximate solution

Finally, the large-r approximate expression of the solutions holds for both solitons and BHs. For a generic scalar potential with

$$U|_{\phi=0}=0,\qquad \frac{\partial U}{\partial\phi}|_{\phi=0}=0,\qquad \frac{\partial^2 U}{\partial\phi^2}|_{\phi=0}=-\frac{2}{L^2},\qquad \frac{\partial^3 U}{\partial\phi^3}|_{\phi=0}=0, \tag{4.9}$$

an approximate form of the solutions[10] can be written as series in $1/r$, with

$$m(r)=M_0-\frac{\alpha^2\kappa^2}{2L^2}r+\sum_{k\geq1}\frac{\tilde{m}_{(k)}}{r^k},\quad \delta(r)=\sum_{k\geq1}\frac{\tilde{\delta}_{(k)}}{r^k},\quad \phi(r)=\frac{\alpha}{r}+\frac{\beta}{r^2}+\sum_{k\geq3}\frac{\tilde{\phi}_{(k)}}{r^k}, \tag{4.10}$$

with the coefficients depending on the free parameters $\{M_0,\alpha,\beta\}$. One finds, e.g.,

$$\tilde{m}_{(1)}=-\frac{\kappa^2}{2}\left(\alpha^2+\frac{2\beta^2}{L^2}+\alpha^4\left(\frac{2\kappa^2}{L^2}+\frac{1}{4}U^{(4)}\right)\right),$$
$$\tilde{m}_{(2)}=-\frac{\alpha\kappa^2}{6}\left(\alpha\left(-M_0+2\alpha\beta\left(\frac{8\kappa^2}{L^2}+U^{(4)}\right)\right)+\frac{1}{15}\alpha^3 U^{(5)}+4\beta\right), \tag{4.11}$$
$$\tilde{\delta}_{(1)}=0,\quad \tilde{\delta}_{(2)}=\frac{1}{2}\alpha^2\kappa^2,\ \tilde{\delta}_{(3)}=\frac{4}{3}\alpha\beta\kappa^2,\ \tilde{\delta}_{(4)}=\frac{\kappa^2}{8}\left(8\beta^2+\alpha^2L^2\left(\frac{6\kappa^2}{L^2}+U^{(4)}\right)\right),$$
$$\tilde{\phi}_{(3)}=\frac{\alpha^2L^2}{12}\left(\frac{6\kappa^2}{L^2}+U^{(4)}\right),\ \tilde{\phi}_{(4)}=\frac{L^2}{12}\left(\alpha\left(4M+\alpha\beta\left(\frac{8\kappa^2}{L^2}+U^{(4)}\right)\right)+\frac{1}{12}\alpha^3U^{(5)}-4\beta\right),$$

where $U^{(n)}$ denotes $\frac{\partial^n U}{\partial\phi^n}|_{\phi=0}$.

[10] Let us remark that the equations of the model are invariant when taking $\delta\to\delta+const.$, a symmetry which is lost when imposing $\delta(\infty)=0$.

Also, the leading order expression of the metric functions g_{rr} and g_{tt} reads

$$g_{rr} = \frac{1}{N(r)} = \left(\frac{L}{r}\right)^2 - \left(1 + \frac{\alpha^2\kappa^2}{L^2}\right)\left(\frac{L}{r}\right)^4 + \frac{2M_0}{L}\left(\frac{L}{r}\right)^5 + O(1/r^6),$$
$$-g_{tt} = N(r)e^{-2\delta(r)} = 1 + \frac{r^2}{L^2} - \frac{2M_0 + \frac{8\alpha\beta\kappa^2}{3L^2}}{r} + O(1/r^2),$$

such that the spacetime is still asymptotically (locally) AdS.

4.1.4 Quantities of interest

The Hawking temperature and horizon area of the BH solutions are fixed by the horizon data, with

$$T_H = \frac{1}{4\pi}N'(r_h)e^{-\delta(r_h)}, \quad A_H = 4\pi r_h^2. \tag{4.12}$$

The mass computation is a straightforward application of the general formalism in section 2.2. For a given design function W (as given by (1.3)), the non-vanishing components of the resulting boundary stress-tensor are (here we choose $\partial\mathcal{M}$ to be a three surface of fixed r, while $n_\nu = \sqrt{g_{rr}}\delta_{r\nu} = \delta_{r\nu}/\sqrt{N}$):

$$\mathrm{T}_\theta^\theta = \mathrm{T}_\varphi^\varphi = \left(\frac{M_0 L}{2\kappa^2} - \frac{1}{L}(W - \alpha\beta)\right)\frac{1}{r^3} + O\left(\frac{1}{r^4}\right), \quad \mathrm{T}_t^t = \left(-\frac{M_0 L}{\kappa^2} - \frac{1}{L}(W + \alpha\beta) -\right)\frac{1}{r^3} + O\left(\frac{1}{r^4}\right).$$

Then the mass of these solutions, as computed from section 2.2 is

$$M = 4\pi\left(\frac{M_0}{\kappa^2} + \frac{\alpha\beta + W}{L^2}\right), \tag{4.13}$$

with W the function (1.3) imposing a condition between α and β.

4.2 Solutions in the $\mathcal{N} = 8$ $D = 4$ model

4.2.1 Perturbative solitons

In the solitonic case, a simple enough exact solution can be found perturbatively in terms of the scalar amplitude $\phi(0) = \epsilon$. The Ansatz for a perturbative approach is:

$$m(r) = \sum_{k\geq 2} \epsilon^k m_k(r), \quad \delta(r) = \sum_{k\geq 2} \epsilon^k \delta_k(r), \quad \phi(r) = \sum_{k\geq 1} \epsilon^k \phi_k(r). \tag{4.14}$$

The solution for the lowest order in ϵ is valid for any scalar selfinteraction, since only the mass term is relevant here. One finds

$$\phi_1(r) = \frac{L}{r}\mathcal{X}(r), \; \phi_2(r) = 0, \; m_2(r) = \frac{\kappa^2 L}{2}\mathcal{X}(r)\left(1 - \frac{L}{r}\mathcal{X}(r)\right), \; m_3(r) = 0, \tag{4.15}$$
$$\delta_2(r) = -\frac{\kappa^2}{2}\left(\frac{1}{N_0(r)} + \mathcal{X}(r)\left(\frac{2L}{r} + \left(1 - \frac{L^2}{r^2}\right)\mathcal{X}(r)\right) - \frac{\pi^2}{4}\right), \; \delta_3(r) = 0,$$

where we define the auxiliary functions

$$N_0(r) = 1 + \frac{r^2}{L^2}, \quad \mathcal{X}(r) = \arctan\left(\frac{r}{L}\right). \tag{4.16}$$

Although the equations can be solved to the next order in ϵ, the solution with a generic parameter n in the potential (2.4) is exceedingly complicated. Thus in what follows, we shall restrict our study to the special case $n=1$, where the solution still possesses a simple enough form, with

$$\begin{aligned}
\phi_3(r) &= -\frac{\kappa^2}{2N_0(r)}\left(1-\frac{L}{3r}\mathcal{X}(r)\left(3+N_0(r)-\frac{L^2}{r^2}N_0^2(r)\mathcal{X}^2(r)\right)\right), \quad \phi_4(r)=0,\\
m_4(r) &= \frac{\kappa^4 L}{24N_0(r)}\left(-\frac{9r}{L}+\left(\frac{19r^2}{L^2}+25\right)\mathcal{X}(r)-\frac{2r}{L}\left(13+\frac{2r^2}{L^2}+\frac{11L^2}{r^2}\right)\mathcal{X}^2(r)\right.\\
&\quad \left.+6\left(\frac{L^2}{r^2}-\frac{r^2}{L^2}\right)\mathcal{X}^3(r)+\frac{4L}{r}N_0^2(r)\mathcal{X}^4(r)\right),\\
\delta_4(r) &= \frac{\kappa^4}{192}\left[\frac{8}{N_0(r)^2}\left(14+\frac{5r^2}{L^2}-\frac{10r}{L}\left(1-\frac{r^2}{L^2}+\frac{16L^2}{5r^2}\right)\mathcal{X}(r)\right)+\left(\frac{5r^2}{L^2}+\frac{16L^2}{5r^2}-3\right)N_0(r)\mathcal{X}(r)^2\right.\\
&\quad \left.-\frac{8r}{L}\left(1-\frac{L^4}{r^4}\right)N_0(r)\mathcal{X}(r)^3-2\left(1+\frac{2L^2}{r^2}+\frac{3L^4}{r^4}\right)N_0(r)^2\mathcal{X}(r)^4+\pi^2(\pi^2-10)\right].
\end{aligned} \tag{4.17}$$

While $\phi_4(r)=0$, the function $\phi_5(r)$ is more complicated, with the presence of the polylogarithm function $Li_n(x)$,

$$\phi_5(r) = \kappa^4\sum_{k=0}^{5} f_k(r)\mathcal{X}(r)^k, \tag{4.18}$$

where

$$\begin{aligned}
f_0(r) &= -\frac{1}{4N_0(r)}\left(1+\frac{2i\pi^4 N_0(r)}{15}\frac{L}{r}\left(1-\frac{90}{\pi^4}Li_4\left(-\frac{r+iL}{r-iL}\right)\right)\right),\\
f_1(r) &= \frac{7}{40N_0(r)^2}\frac{L}{r}\left(1+\frac{2r^4}{7L^4}+\frac{19r^2}{7L^2}-\frac{40}{7}N_0(r)^2\left(5Li_3\left(-\frac{r+iL}{r-iL}\right)+\zeta(3)\right)\right),\\
f_2(r) &= \left(1+\frac{r^2}{L^2}+\frac{2r^4}{3L^4}\right)\frac{45L}{r}-480iN_0(r)^2 Li_2\left(-\frac{r+iL}{r-iL}\right),\\
f_3(r) &= \frac{L}{24r}\left(1-24i\pi-\frac{8L^2}{r^2}+48\log\left(\frac{L+ir}{2r}\right)\right),\\
f_4(r) &= \frac{L}{24r}\left(-8i+\frac{5L}{r}-\frac{L^3}{r^3}\right), \qquad f_5(r)=\frac{N_0(r)}{120}\frac{L^3}{r^3}\left(1+\frac{9L^2}{r^2}\right),
\end{aligned}$$

where $\zeta(x)$ is the Riemann zeta function.[11]

The above expressions allow for a discussion of some basic properties of the solitonic solution. For example, the small-r expansion of the scalar field reads

$$\phi(r) = \epsilon-\left(\epsilon+\frac{2\kappa^2}{3}\epsilon^3+\frac{2\kappa^4}{15}\epsilon^5\right)\frac{r^2}{3L^2}+O(r^4), \quad \text{thus} \quad \phi(0)=\epsilon. \tag{4.19}$$

[11]Note that $\phi_5(r)$ is a real function, despite the presence of i in its expression.

In the context of this work, the coefficients α, β, M_0 which enter the far field asymptotics are of special interest, with

$$\begin{aligned}
\alpha &= \frac{\pi L}{2}\epsilon + \frac{\pi L}{36}\kappa^2\left(1 - \frac{1}{16}(12+\pi^2)\right)\epsilon^3 + \frac{\pi L}{3840}\kappa^4 a_5\epsilon^5 + \dots, \\
\beta &= -L^2\epsilon + \frac{\pi^2 L^2\kappa^2}{8}\left(1 - \frac{16}{3\pi^2}\right)\epsilon^3 + \frac{L^2\kappa^4}{480}b_5\epsilon^5 + \dots, \\
M_0 &= \frac{3\pi L}{4}\epsilon^2\kappa^2\left(1 + \frac{1}{72}\epsilon^2\kappa^2(54 - 11\pi^2)\right) + \dots,
\end{aligned} \tag{4.20}$$

(where we denote $a_5 = 96 + \pi^4 + \pi^2(20 - 960\log 2) + 5280\zeta(3)$, $b_5 = -144 + 5\pi^2(3 + \pi^2 - 48\log 2) + 840\zeta(3))$). One notices that α is positive and β negative to order $\mathcal{O}(\epsilon)^5$, which suggests the absence of solutions with $\alpha \leq 0$ and $\beta \geq 0$ to all orders, a conjecture which is confirmed by the nonperturbative results in the next subsection. Also, one should remark that (α, β, M_0) are not independent, being parameterized by $\phi(0)$. The choice of the function W (cf. eq. (1.3)) fixes this parameter; for example, $\beta = -\alpha^2$ for $\epsilon = \phi(0) \simeq 0.4$, while $\beta = -\alpha^3$ for $\epsilon = \phi(0) \simeq 0.519$).

The expression of the metric potential at the origin (which provides a measure on how strong are the gravity effects) is also of interest, with

$$-g_{tt}(0) = 1 - \frac{1}{4}\kappa^2(\pi^2 - 8)\epsilon^2 + \frac{1}{48}\kappa^4(96 - 19\pi^2 + \pi^4)\epsilon^4 + \dots. \tag{4.21}$$

Finally, we mention that given a design function W (which would fix the parameter $\phi(0)$), the mass M of the solitons results directly from the eqs. (4.13), (4.20).

4.2.2 Nonperturbative results

The nonperturbative solutions are found by integrating numerically the equations (4.3). In our approach, suitable initial conditions resulting from (4.6), (4.7), are imposed at $r = r_0 + 10^{-6}$ (with $r_0 = (0, r_h)$ for solitons and BHs respectively), for global tolerance 10^{-15}, the equations being integrated towards $r \to \infty$.

In principle, the full set of solutions can be scanned in this way by varying the boundary data at $r = r_0$ (which is provided by $\phi(r_0)$) and extracting from the numerical output the parameters (α, β, M_0) in the far field, together with $\delta(r_0)$.

The profile of a typical soliton solution is shown in figure 5 for $n = 1$ (left panel) and $n = 3$ (right panel). Both configurations have the value of the scalar field at the origin, $\phi(0) = 0.65$; however, some features of the solutions depend on the value of n (one finds e.g. $\alpha = 0.9294$, $\beta = -0.5226$ for $n = 1$ and $\alpha = 1.1263$, $\beta = -1.0134$ for $n = 3$). A similar picture is found for BHs, as shown in figure 6 for $n = 1, 3$ solutions with $\phi(r_h) = 1.4$ and $r_h = 2.3$ (in which case e.g. $\alpha = 5.234$, $\beta = -2.919$ for $n = 1$ and $\alpha = 11.3536$, $\beta = -74.9321$ for $n = 3$). Also, for $n = 1$ we have included the profile of the perturbative solution with $\epsilon = 0.65$; as one can see, this provides a good approximation of the non-perturbative result. Moreover, the displayed profiles are typical and so far we could not find any indication for the existence of solutions with the function $\phi(r)$ changing sign (i.e. with the existence of nodes).

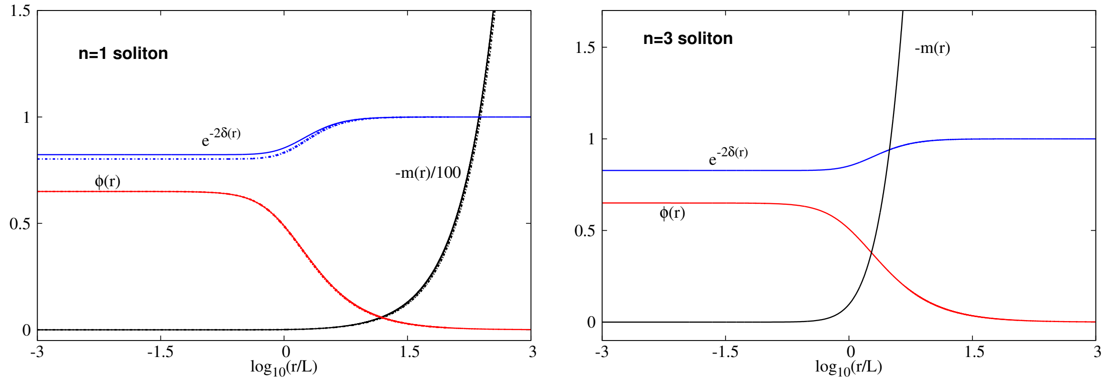

Figure 5. Typical profiles of $n = 1, 3$ soliton solutions with the same value of the scalar field at the origin $\phi(0) = 0.65$ are shown as a function of the radial coordinate. The corresponding perturbative solution is also shown for the $n = 1$ case (dotted curves).

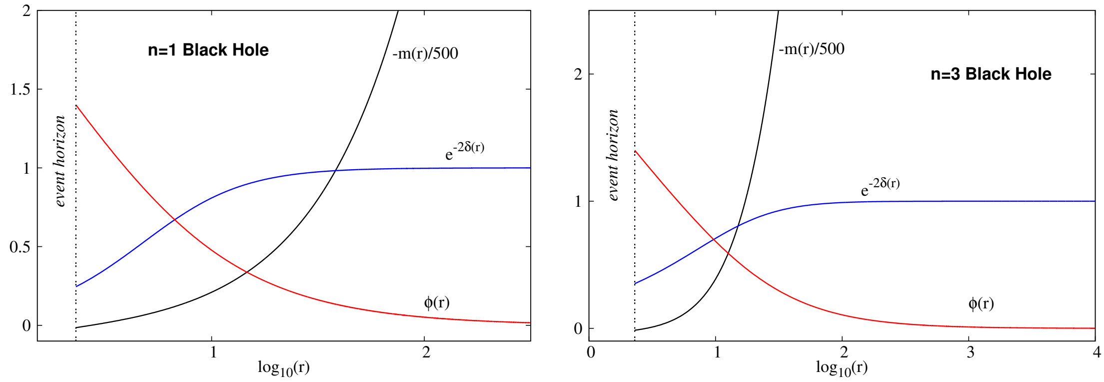

Figure 6. Typical profiles of $n = 1, 3$ black hole solutions with the same value of the scalar field at the horizon $\phi(r_h) = 1.4$ are shown as a function of the radial coordinate.

In figure 7 we show how the parameters α, β, M_0 and $g_{tt}(0)$ vary with $\phi(0)$ for soliton solutions with[12] $n = 1, 2, 3$. In particular, we remark that α is always strictly positive (and increasing with $\phi(0)$) while $\beta < 0$. For $n = 1$, we have included also the corresponding perturbative results. As one can see, they stop to be reliable when $\phi(0)$ becomes around one.

Turning now to BH solutions, we have considered a systematic scan of $n = 1, 2, 3$ configurations by varying both r_h and $\phi(r_h)$ in steps of 10^{-3}. The emerging picture is displayed in figures 8, 9 and can be summarized as follows. First, no configurations with $\alpha \leq 0$ or $\beta \geq 0$ exist, at least for the considered range of $(r_h, \phi(r_h))$ (note, however, the existence of local extrema of these quantities). Second, for any horizon size, both the parameter M_0 and the Hawking temperature increase with $\phi(r_h)$. Also, we mention that the function $e^{-2\delta(r_h)}$ decreases monotonically with $\phi(r_h)$ which makes the study of solutions with large values of the scalar field at the horizon difficult.

[12]We have considered as well solutions with $n = 4,\ 5$ and have found that they follow the $n = 2, 3$ pattern.

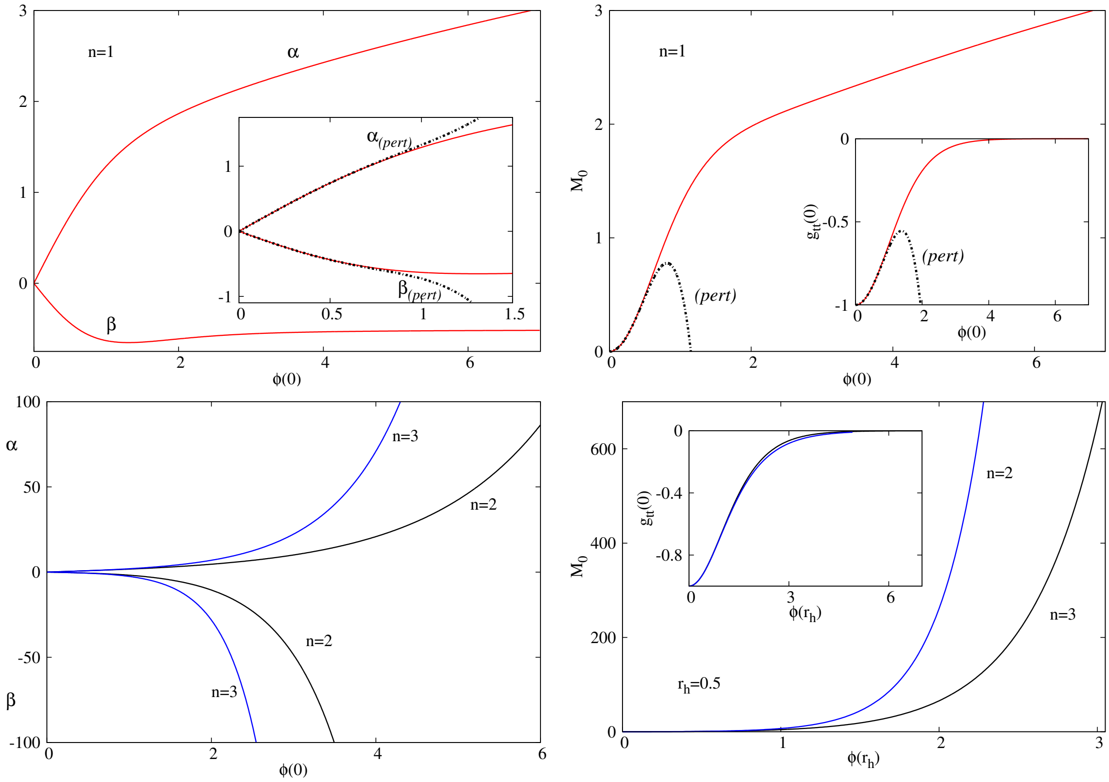

Figure 7. Several quantities of interest are shown as a function of the scalar field at the origin for $n = 1$ and $n = 2, 3$ solitonic solutions. The perturbative results are also shown for $n = 1$ (dotted curves).

Finally, let us remark that the results in figure 8 imply only rather weak restrictions on the function $\mathcal{W}$ which connects α and β in *designer gravity theories*, since all positive (negative) values of α (β) are realized.[13] In figure 10 we show the mass of $n = 1$ BHs as a function of $(r_h, \phi(r_h))$ for several different functions W (which corresponds to consider specific slices in the general plots above). As one can see, the minimal value of M is achieved in the solitonic limit, with the existence of two solitons for the same value of the scalar field at the horizon. A similar picture has been found for $n = 2, 3$ solutions.

4.3 Solutions in the ϕ^4-model

Some of the features above are shared by the gravitating solutions in the ϕ^4-model. For example, a continuum of solutions is found again when varying the values of r_h and $\phi(r_h)$ (or $\phi(0)$), without any indication for the existence of an upper bound for these parameters. As with the sugra case, the generic ϕ^4-solutions have nonzero parameters α, β. In figure 10 (right panel) we show the mass of $\lambda = -3$ BH solutions with three different choices of the condition $\beta(\alpha)$. One can notice a (qualitatively) similar picture to that found for solutions with the $n = 1$ potential (2.4).

[13]For example, the choice $\beta = f\alpha^k$ imposes only $f < 0$.

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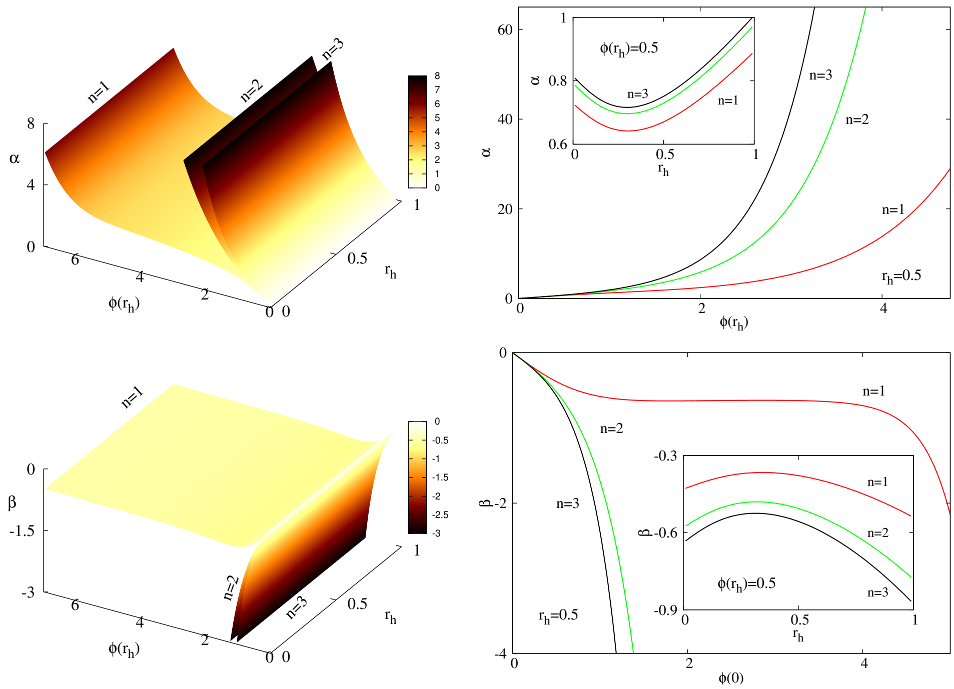

Figure 8. The parameters α and β which enter the far field expansion of the scalar field are shown as a function of the value of the horizon radius and of the scalar field at the horizon for families of $n = 1, 2, 3$ black hole solutions. The right panel shows different slices of these plots.

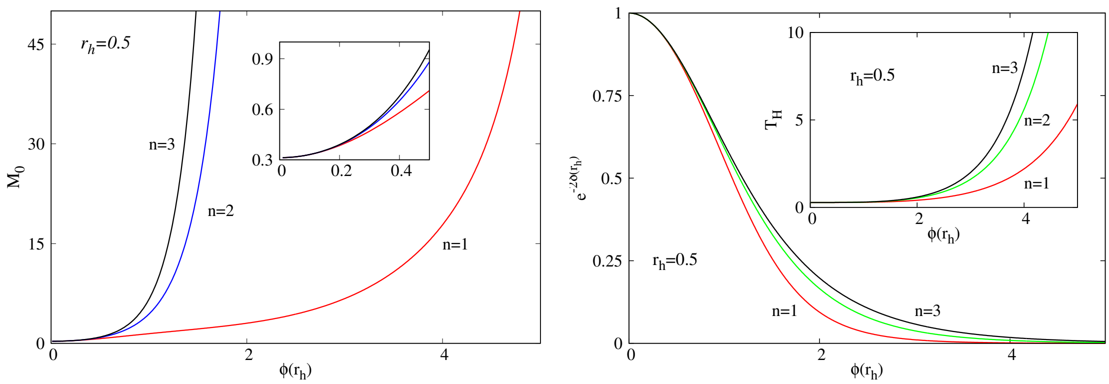

Figure 9. The mass-parameter M_0, the value of the function $e^{-2\delta}$ at the horizon and the Hawking temperature are shown as a function of $\phi(r_h)$ for families of $n = 1, 2, 3$ black hole solutions with a fixed value of the horizon radius.

There are also a number of new specific properties, the most interesting one being that a scalar potential with quartic selfinteraction allows for solutions with $\alpha = 0$ or $\beta = 0$ in the asymptotic expansion (1.2).

In what follows we shall restrict our study to configurations with $\alpha = 0$, in which case the mass function $m(r)$ approaches a constant value at infinity. The profile of two typical solutions are shown in figure 11 (note that $m'(r) < 0$ for some range of r, such that solutions violate the weak energy conditions). In figure 12 (left panel) we show how several

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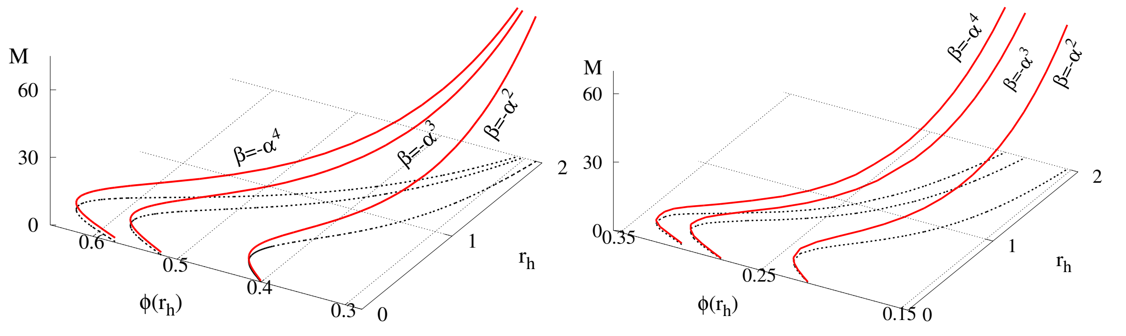

Figure 10. The mass of the solutions is shown as a function of the values of the event horizon radius and the scalar field at the horizon for sugra solutions with $n = 1$ (left panel) and for $\lambda = -3$ solutions of the ϕ^4 model (right panel), with three different choices of the boundary condition $\beta(\alpha)$.

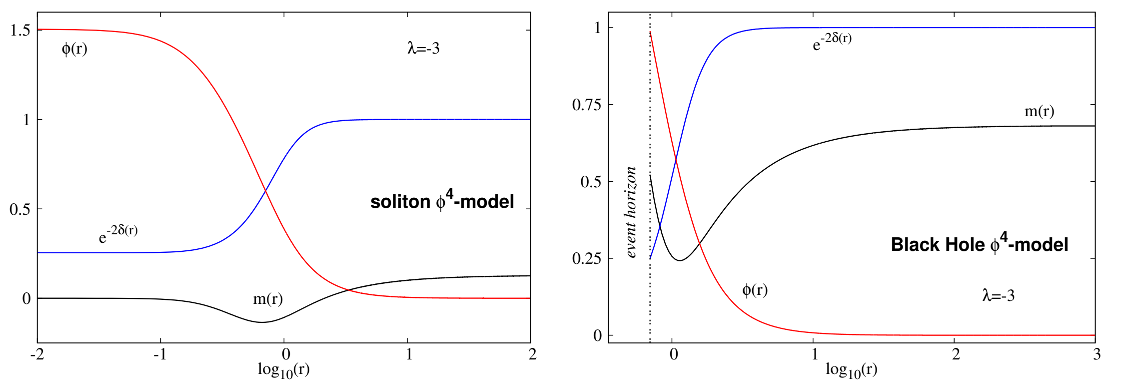

Figure 11. The profile of the nodeless soliton of the ϕ^4-model is shown together with a typical black hole solution. In both cases, the scalar field decays as $1/r^2$.

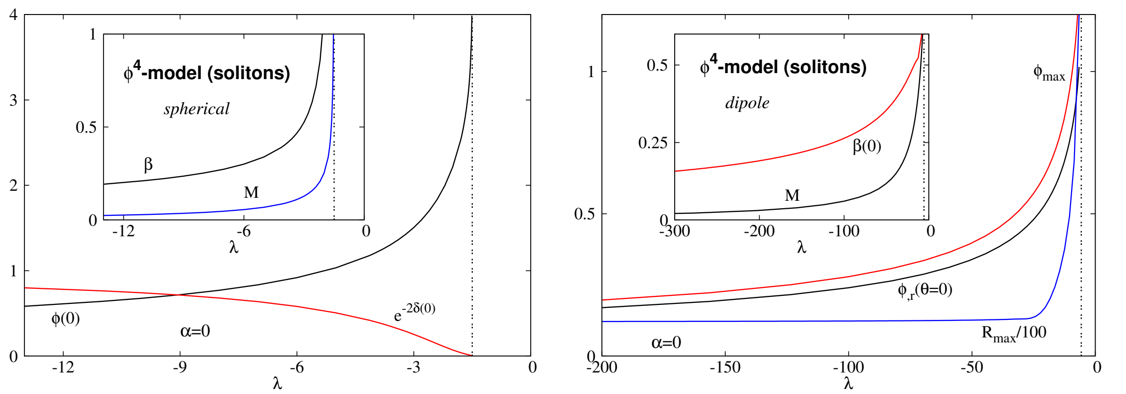

Figure 12. Several quantities of interest are shown for spherically symmetric and axially symmetric (dipole) solitons of the ϕ^4-model as a function of the coupling constant λ. The scalar field decays for these solutions as $1/r^2$.

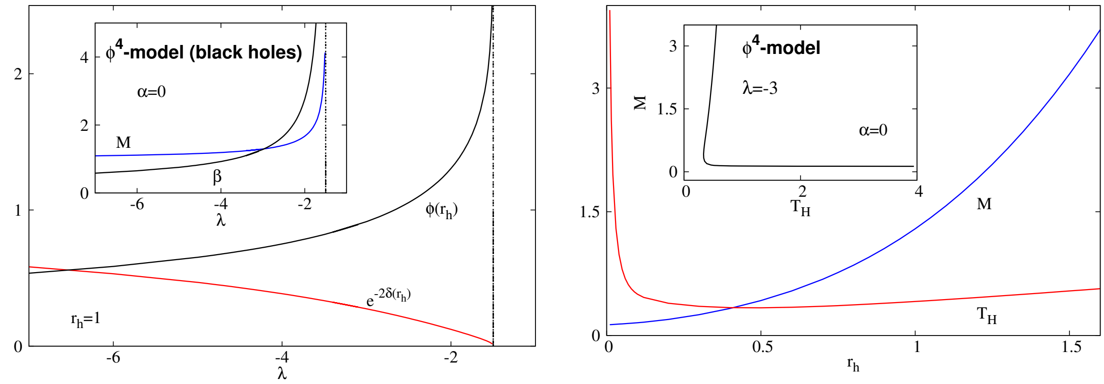

Figure 13. Several quantities of interest are shown for spherically symmetric black hole solutions with $\alpha = 0$ of the ϕ^4-model as a function of the coupling constant λ (left panel) and as a function of event horizon radius (right panel).

quantities of interest vary as a function of λ for ϕ^4-solitons. No solutions with $\alpha = 0$ were found for $\lambda > 0$, while our results suggest[14] the existence of a maximal value of λ, with $|\lambda| \geq 1.5$. As $|\lambda|_{\rm min}$ is approached, the function $e^{-2\delta(0)}$ takes very small values close to zero, and the Ricci scalar appears to diverge.

As seen in the left panel of figure 13, a similar picture is found in the presence of a BH horizon, without the existence of an upper bound on the horizon size (we mention that the same picture was found for other values of r_h). In figure 13 (right panel) we show the result for solutions with a fixed value of $\lambda = -3$ and a varying horizon size. One can see that the familiar SAdS thermodynamics is recovered for $\alpha = 0$ BHs with scalar hair, with the existence of two branches of solutions which join for a minimal value of the Hawking temperature.

5 Beyond spherical symmetry: gravitating scalar dipoles

On general grounds, one expects that each (linear) AdS scalar cloud with given numbers (ℓ, m) would possess nonlinear continuations in the full Einstein-scalar field model (and thus, the spherically symmetric ($\ell = 0$ mode) case discussed above is not special). In what follows we present results for the simplest case of (axially symmetric) scalar dipoles (note, however the perturbative construction of the quadrupole solution in the appendix B). Such configurations are first constructed within a perturbative approach, by considering the backreacting version of the $\ell = 1$ (linear) mode in section 3.1. Non-perturbative solutions of the Einstein-scalar field equations with a $1/r^2$ far field decay of the scalar field are constructed in the ϕ^4-model.

[14]Note that this is smaller than the value $\lambda = -1/(3n)$ found for the truncation of the sugra-potential (2.4); the absence of configurations with $\alpha = 0$ or $\beta = 0$ in the sugra-case can presumably be attributed to the fact that the quartic term in the potential (2.4) never becomes dominant.

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5.1 Perturbative results

In constructing perturbatively axially symmetric solutions it is convenient to consider a generalization of the pure AdS line element (1.1) with three unknown functions F_i,

$$ds^2 = -F_1(r,\theta)N_0(r)dt^2 + F_2(r,\theta)\frac{dr^2}{N_0(r)} + F_3(r,\theta)r^2\Big(d\theta^2 + \sin^2\theta d\varphi^2\Big), \tag{5.1}$$

with

$$N_0(r) = 1 + \frac{r^2}{L^2}. \tag{5.2}$$

The scalar field only depends on r, θ, with the following perturbative ansatz up to order $\mathcal{O}(\epsilon^3)$:

$$\phi(r,\theta) = \epsilon\phi^{(1)}(r,\theta) + \epsilon^3\phi^{(3)}(r,\theta) + \dots, \tag{5.3}$$

where $\phi^{(1)}(r,\theta)$ is a linear scalar on AdS studied in section 3.1 and ϵ is an infinitesimally small parameter. The backreaction of the scalar field on the geometry is taken into account by defining (with $i = 1, 2, 3$)

$$F_i(r,\theta) = 1 + \epsilon^2 F_{i2}(r,\theta) + \dots. \tag{5.4}$$

Then the coupled Einstein-scalar field equations are solved order by order in ϵ, the constants which enter the solution being fixed by imposing regularity at $r = 0$ and AdS asymptotics.

To illustrate this procedure, let us consider the backreaction on the geometry of a scalar dipole cloud (similar results for the $\ell = 2, m = 0$ case are given in the appendix B.3). Thus the lowest order data is

$$\phi^{(1)}(r,\theta) = R_1(r)\cos\theta \quad \text{with} \quad R_1(r) = \frac{L}{r}\left(1 - \frac{L}{r}\arctan\left(\frac{r}{L}\right)\right). \tag{5.5}$$

Then, the perturbed metric solution is constructed by considering an angular expansion of F_i in terms of Legendre functions, with coefficients given by radial functions. To lowest order one takes the consistent ansatz

$$F_{i2}(r,\theta) = \kappa^2(a_i(r) + \mathcal{P}_2(\cos\theta)b_i(r)).$$

In solving the Einstein equations, one uses a residual gauge freedom to set the radial function $a_3 = 0$, the expressions of the other functions being (we recall $\mathcal{X} = \arctan(r/L)$)

$$\begin{aligned}
a_1(r) &= \frac{\pi^2}{6} - \frac{2}{3N_0(r)} - \frac{4r}{LN_0(r)}\left(1 + \frac{3L^2}{4r^2}\right)\mathcal{X}(r) - \frac{2}{3}(\mathcal{X}(r))^2,\\
a_2(r) &= -\frac{L^2}{3r^2}\left(1 + \frac{1}{N_0(r)}\right) + \frac{4L^3}{3r^3}\mathcal{X}(r)\left(\left(1 + \frac{3r^2}{4L^2}\right)\frac{1}{N_0(r)} - \frac{L}{2r}\mathcal{X}(r)\right),\\
b_1(r) &= \frac{2L^2}{9L^2}\left(\frac{1}{N_0(r)} - 7 + 3\mathcal{X}(r)\left(\mathcal{X}(r) + \frac{2L}{r}\right)\right),\\
b_2(r) &= \frac{2}{9N_0(r)} + \frac{4L^3}{3r^3}\mathcal{X}(r)\left(1 - \frac{L}{2r}(1 + N_0(r))\mathcal{X}(r)\right),\\
b_3(r) &= -\frac{10}{9} + \frac{L^2}{r^2} + \frac{2L}{3r}\left(1 - \frac{L^2}{r^2}\right)\mathcal{X}(r) + \frac{1}{3}\left(1 - \frac{L^4}{r^4}\right)(\mathcal{X}(r))^2.
\end{aligned} \tag{5.6}$$

Moving now to the next order in ϵ, we shall restrict again to the case $n = 1$ in the potential (2.4). The correction induced by the metric corrections to the scalar field are found by taking a (consistent) ansatz for $\phi^{(3)}$ with two unknown functions,

$$\phi^{(3)}(r,\theta) = \phi_{31}(r)\mathcal{P}_1(\cos\theta) + \phi_{33}(r)\mathcal{P}_3(\cos\theta). \tag{5.7}$$

The explicit form of the functions $\phi_{31}(r)$ and $\phi_{33}(r)$ is given in the appendix B.2. In deriving it, we impose them to be regular at $r = 0$ and to decay as $1/r^2$ in $r \to \infty$. As such, the expansion parameter ϵ can be identified with the function α (evaluated at $\theta = 0$) that enters the far field expansion (1.2) of the scalar field, and thus

$$\alpha(\theta) = \epsilon L \cos\theta. \tag{5.8}$$

The corresponding expression of β is

$$\beta(\theta) = \left(\frac{L^2\pi}{2}\epsilon + \bar{\beta}_1\epsilon^3\right)\cos\theta + \bar{\beta}_3\epsilon^3 L_3(\cos\theta), \tag{5.9}$$

where we denote

$$\bar{\beta}_1 = \frac{8L^2\pi}{25}\left(1+\frac{\pi^2}{32}(-4+\log(256))-\frac{9}{8}\zeta(3)+\frac{6403\kappa^2}{2016}\left(-1+\frac{\pi^2}{6403}(801-1272\log(2))+\frac{5724}{6403}\zeta(3)\right)\right),$$

$$\bar{\beta}_3 = \frac{L^2\pi}{7000}\left(61+72\zeta(3)-\frac{29\pi^2}{12}\left(1+\frac{192}{29}\log(2)\right)+\kappa^2\left(-\frac{94}{3}+\frac{\pi^2}{8}(-47+384\log(2)-216\zeta(3))\right)\right).$$

The presence of the term proportional with $\mathcal{P}_3(\cos\theta)$ in the far field expansion of β above indicates that, to order $1/r^2$, the asymptotic behaviour of the scalar field deviates from that of a dipole.

Different from the spherical case, we were not able to solve the equations to higher order in ϵ. However, likely the above solution already captures same basic features of the general configurations. One finds, e.g.,

$$-g_{tt}(0) = 1 - \frac{1}{6}(10-\pi^2)\kappa^2\epsilon^2, \tag{5.10}$$

while the leading order terms in the large-r expressions of the metric potentials are

$$\begin{aligned}
g_{rr} &= \frac{L^2}{r^2} - \left(1+\frac{1}{24}\left(\pi^2+\frac{20}{3}+(3\pi^2-4)\cos 2\theta\right)\kappa^2\epsilon^2\right)\frac{L^4}{r^4}+\ldots,\\
g_{\varphi\varphi} &= \sin^2\theta g_{\theta\theta} = \left(1+\frac{\kappa^2\epsilon^2}{24}\left(\frac{40}{3}-\pi^2\right)(1-3\cos^2\theta)\right)r^2+\frac{1}{6}L^2(1+3\cos 2\theta)\kappa^2\epsilon^2+\ldots,\\
g_{tt} &= -\frac{r^2}{L^2} - \left(1+\frac{1}{72}(3\pi^2-28)(1+3\cos 2\theta)\kappa^2\epsilon^2\right)+\frac{\kappa^2\epsilon^2\pi L}{18r}+\ldots.
\end{aligned} \tag{5.11}$$

Also, the non-vanishing components of the boundary stress tensor, as computed by using the prescription in section 2.2 are

$$\mathrm{T}_\theta^\theta = \mathrm{T}_\varphi^\varphi = -\frac{\pi L^2(2+3\cos 2\theta)}{24}\frac{\epsilon^2}{r^3}+\ldots, \qquad \mathrm{T}_t^t = -\frac{\pi L^2}{12}\frac{\epsilon^2}{r^3}+\ldots. \tag{5.12}$$

Then, to order ϵ^2, the (cubic) term (which is multiplied with the function W) in the scalar counterterm (2.13) does not show up, and one finds the following simple expression for the mass of the gravitating dipole solution

$$M = \frac{L\pi^2}{3}\epsilon^2, \tag{5.13}$$

(where we choose ∂M to be a surface at constant r, while $n_\nu = \delta_{\nu r}\sqrt{F_2/N_0}$).

5.2 Non-perturbative solitons in the ϕ^4-model

5.2.1 The framework

The (axially symmetric) non-perturbative solutions are constructed by employing the Einstein-De Turck approach [42, 43]. Therefore, instead of the Einstein equations, we solve the so called Einstein-DeTurck (EDT) equations

$$R_{ab} - \nabla_{(a}\xi_{b)} = -\frac{3}{L^2}g_{ab} + 2\kappa^2\left(T_{ab} - \frac{1}{2}Tg_{ab}\right), \quad \text{with} \quad \xi^a = g^{bc}(\Gamma^a_{bc} - \bar{\Gamma}^a_{bc}), \tag{5.14}$$

Γ^a_{bc} being the Levi-Civita connection associated to the spacetime metric g that one wants to determine. Also, a reference metric $\bar{g}$ is introduced, with $\bar{\Gamma}^a_{bc}$ the corresponding Levi-Civita connection. Solutions to (5.14) solve the Einstein equations iff $\xi^a \equiv 0$ everywhere on $\mathcal{M}$. To achieve this, we shall impose boundary conditions which are compatible with $\xi^a = 0$ on the boundary of the domain of integration.

Within this approach, the (static, axially symmetric) metric Ansatz is more complicated than the perturbative one, eq. (5.1), with five metric functions

$$ds^2 = -f_0(r,\theta)N(r)dt^2 + f_1(r,\theta)\frac{dr^2}{N(r)} + S_1(r,\theta)(rd\theta + S_2(r,\theta)dr)^2 + f_2(r,\theta)r^2\sin^2\theta d\varphi^2. \tag{5.15}$$

For solitons with a $1/r^2$ decay of the scalar field (the only considered case), the obvious reference metric is AdS spacetime, while the numerics is done with a scalar field Ansatz

$$\phi = \frac{\psi(r,\theta)}{r}, \tag{5.16}$$

such that a vanishing ψ as $r \to \infty$ corresponds to $\alpha = 0$ in (1.2).

Then the EDT equations (5.14) together with the scalar field equation result in a set of six elliptic partial differential equations, which are solved numerically as a boundary value problem. Following the standard approach [44], the boundary conditions are found by constructing an approximate form of the solutions on the boundary of the domain of integration compatible with the requirement $\xi^a = 0$. They read

$$\begin{gathered}
\partial_r f_1\big|_{r=0} = \partial_r f_2\big|_{r=0} = \partial_r f_0\big|_{r=0} = \partial_r S_1\big|_{r=0} = \partial_r S_2\big|_{r=0} = 0, \quad \psi|_{r=0} = 0, \\
\partial_\theta f_1\big|_{\theta=0,\pi} = \partial_\theta f_2\big|_{\theta=0,\pi} = \partial_\theta f_0\big|_{\theta=0,\pi} = \partial_\theta S_1\big|_{\theta=0,\pi} = S_2\big|_{\theta=0,\pi} = 0, \quad \partial_\theta \psi\big|_{\theta=0,\pi}, \\
f_1\big|_{r=\infty} = f_2\big|_{r=\infty} = f_0\big|_{r=\infty} = 1, \quad S_1\big|_{r=\infty} == S_2\big|_{r=\infty} = 0, \quad \psi\big|_{r=\infty} = 0.
\end{gathered}$$

Moreover, we shall assume again that the solutions are symmetric w.r.t. a reflection in the equatorial plane, which implies that the functions f_1, f_2, f_0, S_1 satisfy Neumann boundary conditions at $\theta = \pi/2$ while S_2 and ψ vanish there. It is also of interest to display the far field behaviour of the solution

$$\begin{aligned}
&\phi = \frac{\beta(\theta)}{r^2} + O(1/r^4), \qquad f_0 = 1 + \frac{f_{03}(\theta)}{r^3} + O(1/r^4), \qquad f_1 = 1 + \frac{24\pi G\beta(\theta)^2}{r^4} + O(1/r^5),\\
&f_2 = 1 + \frac{f_{23}(\theta)}{r^3} + O(1/r^4), \quad S_1 = 1 - \frac{f_{03}(\theta) + f_{23}(\theta)}{r^3} + O(1/r^4), \quad S_2 = O(1/r^5),
\end{aligned} \tag{5.17}$$

the functions $\beta(\theta)$ and $f_{03}(\theta)$, $f_{23}(\theta)$, $s_{13}(\theta)$ being determined from the numerics.

One finds in this way the following large-r expressions of the non-vanishing components of the boundary stress tensor (note the absence of a contribution from the scalar counterterm (2.13)):

$$T_\theta^\theta = -\frac{3}{4\kappa^2 L}\frac{(f_{03}(\theta) + f_{23}(\theta))}{r^3} + \dots, \quad T_\varphi^\varphi = \frac{3}{4\kappa^2 L}\frac{f_{23}(\theta)}{r^3} + \dots, \quad T_t^t = \frac{3}{4\kappa^2 L}\frac{f_{03}(\theta)}{r^3} + \dots, \tag{5.18}$$

which is traceless, as expected. Then a straightforward computation leads to the following expression for the mass:

$$M = \frac{3\pi}{2\kappa^2 L^2}\int_0^\pi d\theta \sin\theta f_{03}(\theta).$$

5.2.2 Numerical results

In this approach, the only input parameter is λ, the constant of the quartic selfinteraction. Instead of r, the numerics is done by using a compactified radial coordinate $x = r/(1+r)$, the equations being discretized on a $(x,\ \theta)$ grid with around 250×50 points. Then the resulting system is solved iteratively until convergence is achieved. The typical numerical error for the solutions reported in this work is estimated to be of the order of 10^{-4} (also, the order of the difference formulae was 6).

The profile of the typical scalar field, the function $\beta(\theta)$ and the energy density $\rho = -T_t^t$ are (qualitatively) similar to those displayed in figure 4 for solutions in the probe limit. As expected, the ϕ^4-(AdS probe) solution with $\alpha = 0$ found in section 3.2 possesses gravitating generalizations. The resulting picture shares the basic features found for spherically symmetric solitons, see figure 12. The solutions with a $1/r^2$ decay exist up to a minimal value of $|\lambda|$ (while again no such solutions are found for $\lambda > 0$). As the minimal value of $|\lambda|$ is approached, both the mass and $\beta(0)$ increase, while the numerics become increasingly challenging, with large numerical errors.

Also, the solutions appear to exist for arbitrary large values of $|\lambda|$. To understand this limit, one notes that these Einstein-scalar field solutions can also be constructed by using an alternative scaling, with $\lambda \to \lambda c^2$, $\phi \to \phi/c$, and $\kappa^2 \to \kappa^2 c^2$, with c an arbitrary nonzero constant. This can be used to set $\lambda = -1$, and work instead with the following form of the EDT equations $R_{ab} - \nabla_{(a}\xi_{b)} = 3g_{ab}/L^2 + 2\bar{\kappa}^2(T_{ab} - \frac{1}{2}Tg_{ab})$, with $\bar{\kappa}^2 = \kappa^2/|\lambda|$. As such, $\bar{\kappa}^2 \to 0$ corresponds to solutions in the probe limit (being approached for large values of $|\lambda|$).

We mention that the preliminary results indicate the existence of BH generalizations of these solutions, with the presence, as in the probe limit in section 3.2, of two branches of solutions which merge for a maximal value for the horizon area. However, their study is more involved, being beyond the purposes of this work.

Returning to the solitonic case, such solutions should exist as well for a pure $1/r$ asymptotic decay of the scalar field, or, more generally, with nonzero α and β in eq. (1.2). However, so far we did not manage to adapt our numerical scheme to these cases. The obstacle is that the EDT approach requires the choice of a suitable background metric $\bar{g}$, which is not obvious for a $1/r$ decay of the scalar field. For example, when choosing AdS for $\bar{g}$, we could not find a consistent far field expression of the solutions which is compatible with the requirement $\xi^a = 0$. This obstacle has also prevented us to find non-perturbative solutions in the $\mathcal{N} = 8$ model.

We also mention that no results were found when modifying the code used in the ϕ^4-model for a scalar potential given by (2.4), while keeping the same set of boundary conditions, which strongly suggests the absence of solutions with $\alpha = 0$ in that case.

6 Discussion

The Einstein-scalar field system with mass $\mu^2 = -2/L^2$ in AdS_4 spacetime provides an interesting toy model to investigate the issues of asymptotics and possible boundary conditions, together with the existence of scalar multipolar solutions. Moreover, for a suitable scalar potential, this model is a consistent truncation of $\mathcal{N} = 8$ $D = 4$ gauged supergravity [13], this being the main case studied in this work. Apart from this case, we have considered also a model with a quartic selfinteraction of the scalar field.

The main results can be summarized as follow. First, both the perturbative and the numerical results for the $\mathcal{N} = 8$ model strongly suggest that no (soliton or BH) solutions can be found subject to the 'standard' boundary conditions $\alpha = 0$ or $\beta = 0$ (with α and β the parameters which enter the asymptotic scalar field expansion (1.2)). As such, all solutions of (the considered truncation) of the $\mathcal{N} = 8$ model belong to *designer gravity* theories [7]. Then the existence of the relation between α and β of the form (1.3) is essential, from a physical point of view, for obtaining an integrable mass for the solutions. The fact that the scalar selfinteraction potential in the $\mathcal{N} = 8$ gauged supergravity supports only mixed boundary conditions implies that the bulk solution is consistent with RG flows generated in the dual field theory by multi-trace deformations. In particular, the (marginal) triple-trace deformation is consistent with mixed boundary conditions that preserve the conformal symmetry, in which case $\beta \sim \alpha^2$.

However, this result depends on the precise form of the scalar field selfinteraction. As shown in this work, a different picture is found for a scalar field with quartic selfinteraction, with the existence as well of spherically symmetric solitons and BHs with a $1/r$ or $1/r^2$ asymptotic decay of the scalar field.

In a different direction, our results suggest that the spherically symmetric Einstein-scalar field solitons are only the first member of a discrete family of solutions, which can be viewed as non-linear continuations of the linear scalar clouds in a fixed AdS background. Moreover, similar configurations should exist when adding a BH horizon at the center of these solitons. The main case studied in our work was that of (axially symmetric) dipoles, where we have found both perturbative and non-perturbative results. However,

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we emphasize that the multipole structure in AdS is quite different than in flat spacetimes because all the multipoles come at the same order in AdS.

As avenue for future research, we mention first the possible existence of configurations *without* isometries, which would be the backreacting version of the $m \neq 0$ scalar multipoles. Moreover, already in the dipole case, similar solutions were shown to exist in a model with U(1) fields. Also, it would be interesting to consider a similar study for other AdS parametrizations (here we mention the existence in the $n = 3$ sugra-model of an exact solution describing a BH with scalar hair,[15] whose event horizon is a surface of negative constant curvature [46]). Finally, we conjecture the existence of (qualitatively) similar results for any value of the scalar field mass above the Breitenlohner-Freedman bound [45].

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A The $\mathcal{N} = 8$ $D = 4$ gauged supergravity action: the Einstein-scalar field(s) truncation

Among other results, ref. [13] shows the existence of a consistent truncation of the bosonic sector of the gauged $N = 8$ supergravity, which, apart from the Einstein term, contains three scalar fields $\phi^{(12)}$, $\phi^{(13)}$, $\phi^{(14)}$ and four U(1) gauge fields $F^{(C)}_{\mu\nu}$ ($C = 1, \ldots, 4$). Its (bulk) action reads (eq. (2.11) in ref. [13]):

$$\begin{aligned} I = \frac{1}{4\kappa^2}\int d^4x\sqrt{-g}\Big[R - \frac{1}{e}\Big((\partial_\mu\phi^{(12)})^2 + (\partial_\mu\phi^{(13)})^2 + (\partial_\mu\phi^{(14)})^2\Big) - U(\phi) \\ - 2\Big(e^{-\lambda_1}(F^{(1)}_{\mu\nu})^2 + e^{-\lambda_2}(F^{(2)}_{\mu\nu})^2 + e^{-\lambda_3}(F^{(3)}_{\mu\nu})^2 + e^{-\lambda_4}(F^{(4)}_{\mu\nu})^2\Big)\Big], \end{aligned} \tag{A.1}$$

with the scalar potential

$$U = -4g^2\Big(\cosh\phi^{(12)} + \cosh\phi^{(13)} + \cosh\phi^{(14)}\Big). \tag{A.2}$$

[15] See also [47] for exact solutions in an extended supergravity model.

(with $4\pi G = \kappa^2$ and $2g^2 = 1/L^2$ for the notation in this work) while λ_i are linear combination of the scalar fields, as given by

$$\lambda_1 = -\phi^{(12)} - \phi^{(13)} - \phi^{(14)}, \quad \lambda_2 = -\phi^{(12)} + \phi^{(13)} + \phi^{(14)},$$
$$\lambda_3 = \phi^{(12)} - \phi^{(13)} + \phi^{(14)}, \qquad \lambda_4 = \phi^{(12)} + \phi^{(13)} - \phi^{(14)}.$$

Let us remark that one can take consistently $F^{(C)}_{\mu\nu} = 0$ and thus we are left with a model with three gravitating scalar fields

$$\phi^{(12)} \equiv 2\kappa\phi^{(1)}, \;\; \phi^{(13)} \equiv 2\kappa\phi^{(2)}, \;\; \phi^{(14)} \equiv 2\kappa\phi^{(3)}. \tag{A.3}$$

The case of only one nonzero scalar field $\phi^{(a)}$ results in the action (2.1) with $n = 1$ in the potential (2.4). Let us assume now that two scalar fields (for example $a = 1, 2$) are equal, while the third one vanishes. Then the redefinition

$$\phi^{(1)} = \phi^{(2)} = \frac{\phi}{\sqrt{2}}\,, \tag{A.4}$$

leads to the $n = 2$ case in (2.1), (2.4). Finally, when all scalars are equal, the sugra-model in section 2 with $n = 3$ is recovered via the redefinition

$$\phi^{(1)} = \phi^{(2)} = \phi^{(3)} = \frac{\phi}{\sqrt{3}}. \tag{A.5}$$

Also, it was pointed out in [46] that, for a scalar field potential (2.4) with $n = 3$, the model (2.1) can be obtained via the field redefinition from the action of a scalar field conformally coupled to Einstein gravity with a negative cosmological constant.

To clarify if this result holds for the general n-case, we consider the following transformation in (2.1)

$$\hat{g}_{\mu\nu} = (1 - \frac{\kappa^2}{n}\psi^2)^{-1} g_{\mu\nu}, \qquad \psi = \sqrt{\frac{n}{\kappa^2}}\tanh\left(\sqrt{\frac{\kappa^2}{n}}\phi\right). \tag{A.6}$$

Then the original action (2.1) becomes

$$S = \int\sqrt{-\hat{g}}\left(\frac{1}{4\kappa^2}\left(\hat{R}+\frac{6}{L^2}\right)+\frac{3\kappa^2\psi^2-n^2}{2n(n-\kappa^2\psi^2)}\hat{\nabla}^a\psi\hat{\nabla}_a\psi-\frac{\hat{R}}{4n}\psi^2+\frac{(n-3)n}{n^2L^2}\psi^2+\frac{(3-2n)\kappa^2}{2n^2L^2}\psi^4\right). \tag{A.7}$$

It is obvious that the case $n = 3$ is special, with a simple form of the above expression

$$S = \int\sqrt{-\hat{g}}\left(\frac{1}{4\kappa^2}\left(\hat{R} + \frac{6}{L^2}\right) - \frac{1}{2}\hat{g}^{\mu\nu}\nabla_a\psi\nabla_a\psi - \frac{1}{12}\hat{R}\psi^2 - \frac{\kappa^2}{6L^2}\psi^4\right). \tag{A.8}$$

Also, this is the only case where the matter part in (A.7) (which includes also the $\hat{R}\psi^2$ term) is conformally invariant.

It is worth to mention that (A.1) corresponds to a particular truncation of the gauged $N = 8$ supergravity, with a specific choice for the 28 vectors and the 70 real scalar degrees of freedom [13], leading to a model without direct interaction terms between the scalar fields and

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the gauge potentials. However, other consistent truncations are possible, resulting in more complicated theories. In the context of this work, it is of interest to mention the truncation in ref. [48] (see also ref. [49]) that keeps the SU(3) invariant sector of the maximal SO(8) gauged supergravity, and also the truncation in ref. [50], which maintains $\mathrm{U}(1)^4 \subset \mathrm{SO}(8)$ (see also refs. [48, 51, 52]). Solutions of a general Einstein-Maxwell-scalar field model which encompasses the two aforementioned truncations have been discussed in ref. [53]. In addition to the kinetic terms for the U(1) gauge field $F = dA$ and the scalar field ϕ (which has a nontrivial potential and the same far field decay as in eq. (1.2)), the matter action there also contains a direct coupling term between ϕ and A, akin to the Abelian-Higgs theory.[16] This leads to a very rich set of configurations, with an intricate pattern of branching and some unexpected results. For example, we mention the existence of solutions which are *not* perturbatively connected to the (globally) AdS vacuum and possess a planar ((Poincaré)) AdS limit. Hairy, charged BHs providing holographic superconductors do also exist in this model. However, the (charged) solutions in ref. [53] are spherically symmetric. As a direct continuation of one of the directions in our work, it would be interesting to consider their generalizations for a set of boundary conditions at infinity corresponding to scalar multipoles.

B Details on the perturbative axially symmetric solutions

B.1 The general equations

For the metric Ansatz (5.1), and $\phi \equiv \phi(r,\theta)$, the Einstein-scalar field equations (2.2) reduce to the following equations:

$$
\begin{aligned}
&-\frac{3}{L^2}-\frac{2\kappa^2\sinh^2(\phi)}{L^2}+\frac{N+rN'-\kappa^2r^2N\phi'^2}{r^2F_2}+\frac{F_3(\cot(\theta)\dot F_1+\ddot F_1)+F_1(\cot(\theta)\dot F_3+\ddot F_3)}{2r^2F_1F_3^2}+\frac{r^2NF_1^2F_3'^2-F_2F_3\dot F_1{}^2}{4r^2F_1^2F_2F_3^2}\\
&\quad+\frac{rF_1N'F_3'+N(2F_3F_1'+(2F_1+rF_1')F_3')}{2rF_1F_2F_3}=0,\\
&\frac{2NF_2\dot F_1+rN'(F_1\dot F_2-F_2\dot F_1)}{4r^3NF_1F_2F_3}+\frac{\dot F_2-4\kappa^2rF_2\dot\phi\phi'}{2r^3F_2F_3}+\frac{(F_2\dot F_1+F_1\dot F_2)(F_3F_1'+F_1F_3')}{4r^2F_1^2F_2F_3^2}+\frac{-F_3^2\dot F_1{}'+F_1(\dot F_3F_3'-F_3\dot F_3{}')}{2r^2F_1F_3^3}=0,\\
&-\frac{3}{L^2}-\frac{2\kappa^2\sinh^2(\phi)}{L^2}+\frac{\dot F_1(2\cot(\theta)F_2+\dot F_2)+F_1(-4\kappa^2F_2\dot\phi^2+2\cot(\theta)\dot F_2)}{4r^2F_1F_2F_3}+\frac{(F_2\dot F_1+F_1\dot F_2)\dot F_3}{4r^2F_1F_2F_3^2}+\frac{2N'+rN''}{2rF_2}\\
&\quad+\frac{3rN'F_1'+2N(2\kappa^2rF_1\phi'^2+F_1')}{4rF_1F_2}+\frac{N(F_1^2F_3F_2'F_3'+F_2(F_3^2F_1'^2+F_1^2F_3'^2))}{4F_1^2F_2^2F_3^2}-\frac{(rF_1N'+N(2F_1+rF_1'))F_2'}{4rF_1F_2^2}\\
&\quad+\frac{(2rF_1N'+N(4F_1+rF_1'))F_3'}{4rF_1F_2F_3}+\frac{N(F_3F_1''+F_1F_3'')}{2F_1F_2F_3}=0,\\
&-\frac{3}{L^2}-\frac{2\kappa^2\sinh^2(\phi)}{L^2}+\frac{2N'+rN''+2\kappa^2rN\phi'^2}{2rF_2}+\frac{\dot F_1(F_3\dot F_1+F_1\dot F_3)}{4r^2F_1^2F_3^2}+\frac{\dot F_1\dot F_2+2F_2\ddot F_1+2F_1\ddot F_2}{4r^2F_1F_2F_3}-\frac{\dot F_2(F_3\dot F_2+F_2\dot F_3)}{4r^2F_2F_3^2}\\
&\quad+\frac{\kappa^2\dot\phi^2}{r^2F_3}-\frac{NF_1'(F_2F_1'+F_1F_2')}{4F_1^2F_2^2}+\frac{(2N+rN')(-F_3F_2'+2F_2F_3')}{4rF_2^2F_3}-\frac{NF_3'(F_3F_2'+F_2F_3')}{4F_2^2F_3^2}+\frac{N(F_1'F_3'+2F_3F_1''+2F_1F_3'')}{4F_1F_2F_3}=0,\\
&-\frac{3}{L^2}-\frac{2\kappa^2\sinh^2(\phi)}{L^2}+\frac{2NF_3+2rF_3N'+2F_2(\kappa^2\dot\phi^2-1)+\cot(\theta)\dot F_2}{2r^2F_2F_3}-\frac{r^2NF_2F_3'^2+F_3(\dot F_2{}^2+2r^2NF_2'F_3')}{4r^2F_2^2F_3^2}\\
&\quad+\frac{F_3(\cot(\theta)\dot F_3+\ddot F_3)-\dot F_3}{2r62F_3^3}+\frac{\ddot F_2+r(rN'F_3'+2N(3F_3'+rF_3''))}{2r^2F_2F_3}+\frac{N(\kappa^2rF_2\phi'^2-F_2')}{rF_2^2}=0,
\end{aligned}
\tag{B.1}
$$

[16] It is interesting to remark that $A = 0$ is a consistent truncation of the general model in ref. [53], which, after taking $\phi \to \phi/\sqrt{2}$, results in a model of the form (2.18). For the $\mathrm{U}(1)^4$-case in ref. [53], the scalar potential is the $n = 4$ sugra-one in eq. (2.19).

where a prime denotes the derivative with respect to r and a dot denotes the derivative with respect to θ.

B.2 The dipole solution: the $\mathcal{O}(\epsilon)^2$ term for the scalar field

The expression of the functions $\phi_{21}(r)$, $\phi_{23}(r)$ which enter the n.l.o. expression (5.7) of the scalar field reads (with $N_0 = 1 + r^2/L^2$, $\mathcal{X} = \arctan(r/L)$):

$$\phi_{31} = \frac{1}{31500 r^6 L N_0} \times \Big(-\kappa^2 r^2 (r(5700L^4 + 424iL^3\pi^4 r N_0 + 5(3520+\gamma_1)L^2 r^2 + 5c_1 r^4)$$
$$-5L(3420L^4 + (2536+\gamma_1)L^2r^2 + (\gamma_1 - 1934)r^4\mathcal{X}) + 6L^2N_0(-120i\gamma_2 L r^4 \mathrm{Li}_4(x)$$
$$+7\pi^2 r^4(4iL\pi^2 - 45r(\ln(4)-1)) + 60\gamma_2 r^4 \mathrm{Li}_3(x)(r+3L\mathcal{X}) + 5((-570\kappa^2L^4 r + 28(-9+22\kappa^2)L^2r^3$$
$$+6(-42+151\kappa^2+2i\gamma_2\pi)r^5 + 24\gamma_2 r^4(r\mathcal{Y} + iL\mathrm{Li}_2(x) - r\ln(2)))\mathcal{X}^2 + 2(95\kappa^2L^5 + 6(7-6\kappa^2)L^3r^2$$
$$+L(\kappa^2(81-106i\pi) + 42i(i+\pi))r^4 + 4i\gamma_2 r^5 + 4\gamma_2 L r^4(-\mathcal{Y}+\ln(2)))\mathcal{X}^3 + 9\gamma_2 r^5\zeta(2)$$
$$+4r^4\mathcal{X}(8i\gamma_2 r\mathrm{Li}_2(x) + 7L(8+\pi^2(-3+\ln(63))) - 3\gamma_2 L\zeta(3)))) \Big),$$

$$\phi_{33} = \frac{L}{126000 r^6} \times \Big(\frac{8400\kappa^2 L^2 r^3}{N_0} - 240\mathcal{X}^2(315\kappa^2L^4 r + L^2r^3(34+3\kappa^2(71-60i\pi)+60i\pi)$$
$$+(34+\kappa^2(-67-48i\pi)+16i\pi)r^5 + 8\gamma_3 r^3(15L^2+4r^2)(\mathcal{Y}-\ln(2))) + 80\mathcal{X}^3(\kappa^2(315L^5 - 507L^3r^2)$$
$$-8\gamma_3\gamma_6 r^2\mathcal{Y} + 2r^2(60i\gamma_3 L^2 r + \gamma_3(\gamma_7+16ir)r^2 + L^3(137+30\gamma_3(i\pi+\ln(4)))))$$
$$+15r^2\mathcal{X}(-15L^3(-88+208\kappa^2+\Gamma_2) + L(376+1392\kappa^2-9\Gamma_2)r^2 + 32\gamma_3(4i\Gamma_1\mathrm{Li}_2(x) + 4\gamma_6\mathrm{Li}_3(x)$$
$$-3\gamma_6\zeta(2))) + 3r^2(6i\gamma_3\gamma_6\pi^4 + 25L^2(1184\kappa^2 + 3(-88+\Gamma_2))r + 20\Gamma_2 r^3 + 80\gamma_3(4\Gamma_1\mathrm{Li}_3(x)$$
$$-8i\gamma_6\mathrm{Li}_4(x) + 3\Gamma_1\zeta(3))) \Big),$$

where we define

$$\gamma_1 = 9\pi^2(151 - 212\ln(2)), \quad \gamma_2 = 21 - 53\kappa^2, \quad \gamma_3 = 3\kappa^2 - 1, \quad \gamma_4 = (17 + 16\ln(2))\pi^2,$$
$$\gamma_5 = (67 + 96\ln(2))\pi^2, \quad \gamma_6 = 3(5L^2 + 3r^2)L, \quad \gamma_7 = 3L(6i\pi - 31 + \ln(4096)),$$
$$\Gamma_1 = 15L^2 r + 4r^3 + \gamma_6\mathcal{X}, \quad \Gamma_2 = 2\gamma_4 - \gamma_5\kappa^2, \quad \mathcal{Y} = \ln(L + ir) - \ln(r),$$

where $Li_n(x)$ is the poly-logarithm function and $\zeta(x)$ is the Riemann zeta function.[17]

B.3 The perturbative quadrupolar solution

In principle, the computation presented in section 5.1 can be repeated starting with any (axisymmetric) scalar ℓ-mode. Here we present some results for the $\ell = 2$, i.e. a scalar quadrupole.

The general equations (5.1)–(5.4) are still valid; however, for $\ell = 2$ the expression of the perturbed metric functions is more complicated, with

$$F_{i2}(r,\theta) = \kappa^2(a_i(r) + \mathcal{P}_2(\cos\theta)b_i(r) + \mathcal{P}_4(\cos\theta)c_i(r)).$$

[17] Both ϕ_{31} and ϕ_{33}, are real functions, although i appears in their expressions.

Then a straightforward but cumbersome computation leads to the following expressions of the functions a_i, b_i, c_i:

$$a_1(r)=\frac{4}{5\pi^2}\left(-13\pi^2+\frac{12L^2}{r^2}\left(1+\frac{10r^2}{3L^2N_0(r)}\right)+\frac{40L^3}{r^3}\mathcal{X}(r)\left(-1+\frac{13r^2}{5L^2}+\frac{2}{5N_0(r)}+\frac{3L}{10r}\left(1+\frac{13r^4}{3L^4}\right)\mathcal{X}(r)\right)\right),$$
$$a_2(r)=-\frac{432L^4}{5\pi^2r^4N_0(r)}\left(1-\frac{L\mathcal{X}(r)}{r}\left(1+\frac{r^2}{3L^2}\right)\right)\left(1+\frac{7r^2}{9L^2}-\frac{LN_0(r)\mathcal{X}(r)}{r}\left(1+\frac{r^2}{9L^2}\right)\right),$$
$$a_3(r)=0,$$
$$b_1(r)=\frac{64L^2}{21\pi^2r^2N_0(r)}\left(-1+34N_0(r)-\frac{42L}{r}N_0(r)\mathcal{X}(r)+9\mathcal{X}^2(r)\left(\frac{L^2}{r^2}-\frac{r^2}{L^2}\right)\right),$$
$$b_2(r)=-\frac{1728L^4}{7\pi^2r^4N_0(r)}\left(1+\frac{38r^2}{27L^2}+\frac{34r^4}{81L^4}-\frac{2L}{r}N_0(r)\mathcal{X}(r)\left(1+\frac{16r^2}{27L^2}-\frac{L\mathcal{X}(r)}{2r}\left(1+\frac{7r^2}{9L^2}\right)\right)\right),$$
$$b_3(r)=-\frac{176L^2}{7\pi^2r^2}\left(1-\frac{68r^2}{33L^2}+\frac{10L\mathcal{X}(r)}{11r}\left(1+\frac{11r^2}{5L^2}\right)+\left(1-\frac{21L^2}{11r^2}\right)N_0(r)\mathcal{X}^2(r)\right),$$
$$c_1(r)=-\frac{768L^4}{5\pi^2r^4N_0(r)}\left(1+\frac{125r^2}{84L^2}+\frac{17r^4}{35L^4}-\frac{LN_0(r)\mathcal{X}(r)}{2r}\left(1+N_0(r)+\frac{2r\mathcal{X}(r)}{7L}\left(1+\frac{5}{4}N_0(r)\right)\right)\right),$$
$$c_2(r)=\frac{384L^4}{7\pi^2r^4N_0(r)}\left(1+\frac{71r^2}{30L^2}+\frac{34r^4}{25L^4}-\frac{4L}{5r}N_0(r)\mathcal{X}(r)\left(-1+\frac{r^2}{4L^2}+\frac{9L\mathcal{X}(r)}{4r}\left(1+\frac{7r^2}{6L^2}+\frac{7r^4}{18L^4}\right)\right)\right),$$
$$c_3(r)=\frac{96L^4}{\pi^2r^4}\left(1+\frac{r^2}{35L^2}-\frac{136r^4}{1575L^4}+\frac{2L\mathcal{X}(r)}{5r}\left(-1+\frac{2r^2}{21L^2}+\frac{r^4}{7L^4}+\frac{r^3N_0(r)\mathcal{X}(r)}{14L^3}\left(1-\frac{21L^4}{r^4}\right)\right)\right),$$

which are found subject to the assumption of regularity at $r=0$ and AdS asymptotics.

To find the ϵ^3-corrections to the scalar field, we consider an expansion similar to (5.4), with

$$\phi^{(3)}(r,\theta)=\phi_{32}(r)\mathcal{P}_2(\cos\theta)+\phi_{34}(r)\mathcal{P}_4(\cos\theta)+\phi_{35}(r)\mathcal{P}_6(\cos\theta). \tag{B.2}$$

Although an exact solution for $\phi_{32}(r)$, $\phi_{34}(r)$, $\phi_{36}(r)$ can be found, its expression is too complicated to include here. As with the dipole case, we impose these functions to be regular at $r=0$ and to decay as $1/r^2$ in $r\to\infty$. Then the expansion parameter ϵ can be identified with the function α (evaluated at $\theta=0$) in eq. (1.2),

$$\alpha=\epsilon L_2(\cos\theta), \tag{B.3}$$

while the expression for β is

$$\beta=\left(-\frac{8L^2}{\pi}\epsilon+\bar\beta_{32}\epsilon^3\right)\mathcal{P}_2(\cos\theta)+(\bar\beta_{34}\mathcal{P}_4(\cos\theta)+\bar\beta_{36}\mathcal{P}_6(\cos\theta))\epsilon^3, \tag{B.4}$$

where we denote

$$\begin{aligned}
\bar\beta_{32}&=\frac{256L^2}{245\pi^3}\left(132-\pi^2(19-48\ln(2))-216\zeta(3)-\frac{\kappa^2}{11}\left(\frac{340742}{15}-\pi^2\left(\frac{11973}{4}-5008\ln(2)\right)+22536\zeta(3)\right)\right),\\
\bar\beta_{34}&=-\frac{3477504L^2}{148225\pi^2}\left(-\frac{864\zeta(3)}{283}+\frac{1676482\kappa^2}{99333}\left(-1+3\pi^2\left(\frac{349357}{6705928}-\frac{90528\ln 2}{838241}\right)+\frac{1222128\zeta(3)}{838241}\right)\right.\\
&\qquad\left.+1+\pi^2\left(\frac{192\ln 2}{283}-\frac{1579}{6792}\right)\right),\\
\bar\beta_{36}&=-\frac{192L^2}{1926925\pi^3}\left(-5\pi^2(6389+18432\ln 2)+1296(303+320\zeta(3))+\kappa^2(\pi^2(382237+153600\ln 2)\right.\\
&\qquad\left.-\left(\frac{59050384}{15}+691200\zeta(2)\right)\right).
\end{aligned} \tag{B.5}$$

To lowest order, the mass of the gravitating quadrupole, as computed within the same approach as the other solutions in this work, is

$$M = \frac{64L\epsilon^2}{5}. \tag{B.6}$$

References

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