Published for SISSA by Springer

Received: *December 6, 2021*
Accepted: *March 16, 2022*
Published: *March 30, 2022*

Cosmologies, singularities and quantum extremal surfaces

Kaberi Goswami, K. Narayan and Hitesh K. Saini

Chennai Mathematical Institute,
SIPCOT IT Park, Siruseri 603103, India

E-mail: goswamikaberi500@gmail.com, narayan@cmi.ac.in, hkuniphyeng@gmail.com

Abstract: Following [1], we study quantum extremal surfaces in various families of cosmologies with Big-Crunch singularities, by extremizing the generalized entropy in 2-dimensional backgrounds which can be thought of as arising from dimensional reduction. Focussing first on the isotropic AdS Kasner case, introducing a spatial regulator enables relating the locations in time of the quantum extremal surface and the observer. This shows that the quantum extremal surface lags behind the observer location. A potential island-like region, upon analysing more closely near the island boundary, turns out to be inconsistent. Similar results arise for other holographic cosmologies. We then study certain families of null Kasner singularities where we find that the quantum extremal surface can reach the near singularity region although the on-shell generalized entropy is generically singular. We also study other cosmologies including de Sitter (Poincare slicing) and FRW cosmologies under certain conditions.

Keywords: 2D Gravity, AdS-CFT Correspondence

ArXiv ePrint: 2111.14906

https://doi.org/10.1007/JHEP03(2022)201

JHEP03(2022)201

Contents

1 Introduction

In [1], we studied aspects of entanglement and quantum extremal surfaces (QES) in various families of holographic spacetimes exhibiting cosmological singularities. This is inspired by the exciting discoveries made recently on the black hole information paradox [2–6], unravelled via the study of entanglement, quantum extremal surfaces and islands: by now there is a large body of literature on various aspects of these issues, reviewed in e.g. [7–10]. Quantum extremal surfaces are extrema of the generalized entropy [11, 12] obtained by incorporating the bulk entanglement entropy of matter alongwith the classical area of the entangling RT/HRT surface [13–16]. These lead to various new insights on black holes. Explicit calculation is possible in effective 2-dimensional models where the bulk entanglement entropy can be studied through 2-dim CFT techniques.

It is interesting to ask if quantum extremal surfaces might be used to probe cosmological, Big-Crunch or -Bang, singularities. While the vicinity of the singularity is expected to be rife with severe stringy/quantum gravity effects, one might hope to gain some insight into how these extremal surfaces probe such singularities. Some interesting recent work on QES and cosmologies appears in [17, 18] and also e.g. [19–30].

The investigations in [1] pertained to various Big-Crunch singularities, in particular the isotropic AdS Kasner spacetime. These spacetimes have no horizons and no significant entropy, so they are somewhat unlike black hole horizons. Further, we are considering closed universes with no entanglement with "elsewhere" (e.g. other universes). Part of the goal here is to gain some understanding of how quantum extremal surfaces probe such spacetime singularities in closed universes with no horizons and no entanglement with regions external to these universes. The time-dependence implies that the classical extremal RT/HRT surface dips into the bulk radial and as well as time directions. Explicitly analysing the extremization equations in the semiclassical region far from the singularity can be carried out in detail: we find the surface bends in the direction away from the singularity. In the 2-dim cosmologies [31] obtained by dimensional reduction of these and other singularities, quantum extremal surfaces can be studied by extremizing the generalized entropy, with the bulk matter taken to be in the ground state (which is reasonable in the semiclassical region far from the singularity). The resulting extremization shows the quantum extremal surfaces to always be driven to the semiclassical region far from the singularity. In section 2, we review the analysis in [1]. The 2-dim dilaton gravity theories in these cases are somewhat more complicated than Jackiw-Teitelboim gravity and are not "near JT" in essential ways. The cosmological solutions here are sourced by an extra scalar which descends from the scalar in the higher dimensional theory. These theories capture a subset of the observables of the higher dimensional theory and so are best regarded as models of "effective holography" [32], UV-incomplete in totality but adequate for capturing various aspects including entanglement. Since the quantum extremal surfaces are driven to the semiclassical region far from the singularity, the approximation of using the 2-dimensional theory is consistent and the other higher dimensional modes do not make any significant contribution.

In this paper, we continue our investigations there and develop them further: wherever possible we look for quantum extremal surfaces spacelike-separated from the observer location. We first do a careful study of QES focussing on AdS Kasner singularities (section 3), by introducing a spatial regulator. This enables relating the locations in time of the observer on the holographic boundary and the QES with the bulk matter central charge and the regulator. In the semiclassical region, this shows that the quantum extremal surface lags behind the observer location (in the direction away from the singularity). A potential island-like region, upon analysing in detail near the island boundary, turns out to be inconsistent. We then extend this to more general singularities admitting a holographic interpretation, which exhibit similar behaviour. In section 4, we study certain families of null Kasner Big-Crunch singularities: these exhibit a certain "holomorphy" due to special properties of null backgrounds. Further they are also distinct in the behaviour of the QES, which now can reach the singularity (although the generalized entropy continues to be singular). We then discuss aspects of 2-dimensional effective theories involving dimensional reduction of other cosmologies in section 5, including de Sitter space (Poincare slicing) and FRW cosmologies under certain conditions. Section 6 contains some conclusions. Some details appear in two appendices.

2 Review: Big-Crunches & quantum extremal surfaces

There is a long history of studying cosmological singularities in string theory and holography: see [31] for a partial list of references in this regard, and e.g. [43, 44] for reviews of cosmological singularities in string theory. In [1], various families of cosmological spacetimes

JHEP03(2022)201

with spacelike Big-Crunch singularities were considered: the higher dimensional space and its reduction ansatz are of the form

$$ds_D^2 = g_{\mu\nu}^{(2)} dx^\mu dx^\nu + \phi^{\frac{2}{d_i}} d\sigma_{d_i}^2 ; \qquad g_{\mu\nu} = \phi^{\frac{d_i-1}{d_i}} g_{\mu\nu}^{(2)} , \qquad D = d_i + 2 . \tag{2.1}$$

d_i is the dimension of the transverse space. The Weyl transformation from $g_{\mu\nu}^{(2)}$ to the 2-dim metric $g_{\mu\nu}$ ensures that the dilaton kinetic energy vanishes and the action is

$$S = \frac{1}{16\pi G_2} \int d^2x \sqrt{-g} \left(\phi \mathcal{R} - U(\phi, \Psi) - \frac{1}{2}\phi(\partial\Psi)^2 \right) . \tag{2.2}$$

The dilaton potential $U(\phi, \Psi)$ potentially couples the dilaton ϕ to Ψ. Certain aspects of generic dilaton gravity theories of this kind (and these 2-dim cosmological backgrounds), dimensional reduction and holography were discussed in [32] (see also [33]): these theories are more complicated than JT gravity and are not "near JT". they capture a subset of the observables of the higher dimensional theory and so are best regarded as UV-incomplete models of "effective holography". There is nontrivial dynamics in the theory (2.2) driven by the extra scalar Ψ which descends from the scalar in the higher dimensional theory. In particular there are nontrivial cosmological singularity solutions here, which were analysed in [31]. See appendix A for some details. The power-law scaling ansatze for the 2-dim fields and the corresponding higher dimensional spacetimes are

$$\phi = t^k r^m, \quad e^f = t^a r^b, \quad e^\Psi = t^\alpha r^\beta \quad \to \quad ds_D^2 = \frac{e^f}{\phi^{(d_i-1)/d_i}} (-dt^2 + dr^2) + \phi^{2/d_i} dx_i^2 . \tag{2.3}$$

The universality (A.3) implies that $k = 1$. Note that $r = 0$ is the asymptotic (holographic) boundary. The equations of motion (A.2) then lead to algebraic relations between the various exponents above, which can then be solved for, leading to nontrivial families of cosmological solutions [31]. A prototypical example is AdS Kasner and its reduction to 2-dimensions,

$$\begin{aligned}
&U = 2\Lambda\phi^{1/d_i} , \qquad \Lambda = -\frac{1}{2} d_i(d_i+1) , \qquad p = \frac{1}{d_i} , \qquad \alpha = \sqrt{\frac{2(d_i-1)}{d_i}} ,\\
&ds^2 = \frac{R^2}{r^2}(-dt^2 + dr^2) + \frac{t^{2p} R^2}{r^2} dx_i^2 , \qquad e^\Psi = t^\alpha , \qquad d_i p^2 = 1 - \frac{1}{2}\alpha^2 ,\\
&\to \quad \phi = \frac{t\, R^{d_i}}{r^{d_i}} , \qquad ds^2 = \frac{t^{(d_i-1)/d_i} R^{d_i+1}}{r^{d_i+1}} (-dt^2 + dr^2) , \qquad e^\Psi = t^{\sqrt{2(d_i-1)/d_i}} .
\end{aligned} \tag{2.4}$$

R is the AdS length scale. We are suppressing an implicit Kasner scale t_K: e.g. $t^{2p} \to (t/t_K)^{2p}$. We will reinstate this as required.

The higher dimensional spacetimes and their dual field theories were in fact studied long back in [34–37] as certain kinds of time-dependent deformations of AdS/CFT with the hope of gaining insights via gauge/gravity duality into cosmological (Big-Bang or -Crunch) singularities: some aspects of these were reviewed in [31]. See also [39–42] for further investigations on some of these. While the bulk spacetime develops a cosmological Big-Crunch (or -Bang) singularity and breaks down, the holographic dual field theory (in

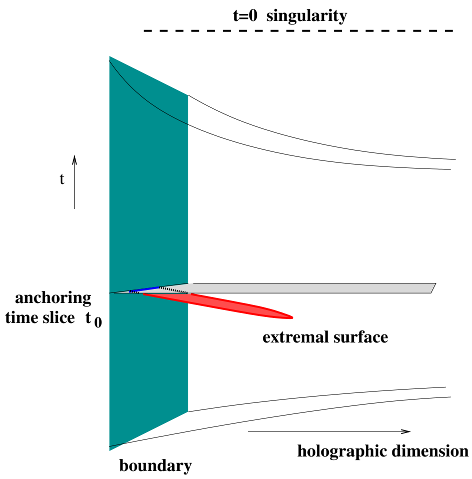

Figure 1. Cartoon of extremal surfaces in AdS Kasner spacetime, anchored on a boundary time slice t_0 (extended as the grey horizontal plane in the bulk). The extremal surface (red) bends away from the singularity at $t = 0$ (dotted line), i.e. $t_* > t_0$, with (t_*, r_*) the turning point.

the AdS_5 case), living on a space that itself crunches, is subject to a severe time-dependent gauge coupling $g_{\rm YM}^2 = e^{\Psi}$ and may be hoped to provide insight into the dual dynamics. In this case the scalar Ψ controls the gauge/string coupling. Generically it was found by analysing at weak coupling that the gauge theory response also ends up appearing singular [37]: however null singularities appear better-behaved admitting weakly coupled CFT descriptions in certain variables [35] (a string worldsheet analysis in related null Kasner singularities appears in [38]). There is a large family of such backgrounds exhibiting cosmological singularities found long back: in these the deformations of the metric and string dilaton Ψ are constrained, suggesting that the dual CFT state is likely nontrivial, with nontrivial non-generic initial conditions required to create Big-Crunch singularities which are perhaps qualitatively different from black holes (note that generic severe time-dependent deformations on the vacuum state are expected to thermalize on long timescales, dual to black hole formation in the bulk). Some of these backgrounds have the technical feature of spatial isotropy which allows studying these backgrounds from a possibly simpler perspective, by carrying out a dimensional reduction on the spatial directions with the ansatz (2.1). This enables the 2-dim dilaton gravity perspective (2.2) formulated in [31], and also helps uncover new cosmologies of the form (2.3) including ones with nonrelativistic (hyperscaling violating Lifshitz) asymptotics, reviewed briefly in appendix A.

Extremal surfaces can be studied as codim-2 surface probes of these cosmological spacetimes. This is reliable if the surface is anchored on a boundary subregion in the semiclassical region far from the singularity where stringy or quantum gravity effects are not large. This can be analysed in great detail as in [1]. The time-dependence of the cosmology implies that the RT/HRT surface dips into the time direction also, besides the radial (holographic) direction. The resulting picture (focussing on strip-shaped subregions consistent with the symmetries here) is as in figure 1. The surface is parametrized as $(t(r), x(r))$ stretching in all x_i directions except $x \in \{x_i\}$ which represents the width (size) direction of the strip with $\Delta x = l$. The anchoring time slice is $t(0) = t_0$: some details appear in appendix A. The extremization of the time function $t(r)$ is more complicated due to the time-dependence and gives a second order nonlinear differential equation for $t(r)$. In the semiclassical region, we expect the time-dependence to be mild so that $\frac{dt}{dr} \equiv t' \ll 1$. This leads to a slightly simpler, but still nonlinear, equation, which however can be shown to admit power-series solutions,

$$t(r) = t_0 + \sum_n c_n r^n \,, \quad c_n \sim \frac{1}{t_0^{\#}} \quad \Rightarrow \quad t_* > t_0 \,, \tag{2.5}$$

which can then be shown to satisfy $t_* \equiv t(r_*) > t_0$, in other words, the surface bends in the direction away from the singularity. This is straightforward to see (although involved) in the regime of small subregion width (where $A = \frac{t_*}{r_*^2} \gtrsim \frac{1}{t_0^2}$). The analysis is a little more delicate in the IR limit where the subregion becomes the whole space (and $A \to 0$): here we find

$$r_* \to \infty \,, \quad t_0 \to \infty \,, \quad \frac{t_0}{r_*} \lesssim 1; \qquad t_* \gtrsim t_0 \,. \tag{2.6}$$

Thus the RT/HRT surface is driven to the region far from the singularity (see also [45] for similar observations in a different context): in the IR limit this effectively means infinitely far from the singularity since $t_* \gtrsim t_0 \to \infty$.

Another notable example with similar behaviour is [46], where the Hartman-Maldacena surfaces exhibit a limiting surface in the black hole interior.

Quantum extremal surfaces. Now we briefly review the discussion of quantum extremal surfaces in [1]. Quantum extremal surfaces are extrema of the generalized entropy $S_{\text{gen}} = S_{\text{cl}} + S_{\text{bulk}}$, the leading classical term being the area of the extremal surface while the second term is the entropy of the bulk matter in the region enclosed by the extremal surface and the boundary. In 2-dim theories, the bulk entropy can be calculated by using 2-dim CFT techniques. For instance, if the bulk matter is approximated by a CFT in a curved space and is taken to be in the ground state, then the bulk entropy can be obtained by a generalization of the Calabrese-Cardy replica formulation [47, 48] for a single spacetime interval Δ^2, giving (see appendix B)

$$S_{\text{gen}} = \frac{\phi}{4G} + \frac{c}{12} \log\left(\Delta^2 e^f|_{(t,r)} e^f|_{(t_0,r_0)}\right); \qquad 1 \ll c \ll \frac{1}{G} \,. \tag{2.7}$$

The last condition arises from requiring that the bulk matter entropy is non-negligible but not so large as to destabilize the leading classical area contribution. The 2-dim space is of the form $ds^2 = e^f \eta_{\mu\nu} dx^\mu dx^\nu$ and the Weyl factors above arise from the conformal transformation of the twist operator 2-point function in the replica formulation, the twist operators located at the endpoints of the interval in question (between the boundary and the extremal surface).

As a simple time-independent example consider the 2-dim dilaton-gravity background obtained from the dimensional reduction (2.1) of AdS_{d_i+2} with metric in the Poincare slicing $ds^2_{\text{AdS}_{d_i+2}} = \frac{R^2}{r^2}(-dt^2 + dr^2) + \frac{R^2}{r^2} dx_i^2$ (R is the AdS scale). Some aspects of such generic

2-dim dilaton gravity theories have been discussed in [32]. This 2-dim background, the corresponding generalized entropy (2.7) and its extremization give

$$\phi = \frac{R^{d_i}}{r^{d_i}}\,, \qquad\qquad ds^2 = \frac{R^{d_i+1}}{r^{d_i+1}}(-dt^2+dr^2)\,, \tag{2.8}$$
$$S_{\rm gen} = \frac{\phi_r}{4G}\frac{R^{d_i}}{r^{d_i}} + \frac{c}{12}\log\left(\frac{r^2/\epsilon_{\rm UV}^2}{(r/R)^{d_i+1}}\right) \quad\Rightarrow\quad \partial_r S_{\rm gen} = -\frac{d_i\phi_r R^{d_i}}{4G\, r^{d_i+1}} - \frac{c}{6}\left(\frac{d_i-1}{2}\right)\frac{1}{r} = 0\,.$$

(We have written $S_{\rm bulk}$ using the rules of boundary CFT since the effective space is the half-line with one end of the interval at the boundary $r=0$: see appendix B. A useful resource for QES calculations in time-independent cases is [49].) We see that both terms are of the same sign since $c>0$ and $d_i>1$. Thus the solution is $r\equiv r_*\to\infty$ for the location of the QES: this leads to the entire Poincare wedge which is the expected answer (also in the higher dimensional point AdS_D when the subsystem becomes the whole space). Thus in this case, there are no islands, i.e. regions disconnected from the boundary defined e.g. by a finite location of the quantum extremal surface.[1]

One way to understand this is in terms of the violation of the Bekenstein bound, as discussed in [18]: if the classical dilatonic term is overpowered by the subleading bulk entropy contribution, we may expect islands. To see this, note that (2.8) can be recast as

$$S_{\rm gen} = \frac{\phi}{4G} + \frac{c}{12}\,\frac{d_i-1}{d_i}\,\log\phi\,, \tag{2.9}$$

with a relative plus sign in the two contributions, retaining only terms relevant for extremization. As long as ϕ is not too small, the bulk entropy term scaling as $\log\phi$ is subdominant to the classical area term scaling as ϕ. If we entangle the bulk matter with "elsewhere" then $S_{\rm bulk}$ could increase possibly leading to nontrivial competition with the classical area term and thereby islands, as is the case in [50] and in various cases in [18].

Now we will study quantum extremal surfaces in the 2-dim cosmological backgrounds reviewed earlier. We focus first on the 2-dim cosmology obtained by reduction of the AdS_D Kasner spacetime (2.4), restricting attention to the observer at the holographic boundary at $r=0$. We carry out the extremization in the reliable semiclassical region far from the singularity at $t=0$: the observer is at $(t_0,0)$. Assuming for simplicity that the QES lies on the same time slice as the observer i.e. $t=t_0$, equivalently that the QES is maximally spacelike separated from the observer, it turns out that (2.7) can be recast as

$$t=t_0:\quad S_{\rm gen} = \frac{\phi}{4G}+\frac{c}{6}\frac{d_i-1}{d_i}\log\phi\,, \qquad\qquad \phi=\frac{t}{r^{d_i}}\,,$$
$$\partial_r S_{\rm gen}\sim -\left(\frac{\phi_r}{4G}\frac{d_i\,t}{r^{d_i+1}}+\frac{c}{12}\frac{d_i-1}{r}\right)=0,\quad \partial_t S_{\rm gen}\sim\frac{\phi_r}{4G}\frac{1}{r^{d_i}}+\frac{c}{12}\frac{d_i-1}{d_i\,t}=0\,. \tag{2.10}$$

[1] In e.g. [50], a flat non-gravitating (bath) region was appended beyond the boundary $r=0$ of an AdS_2 region, giving the generalized entropy $S_{\rm gen}\sim\frac{\phi_r}{4G}\frac{1}{r}+\frac{c}{6}\log((r+r')^2\,\frac{1}{r})$. The interval in question has endpoints $r\in AdS_2$ and r' in the flat space region beyond the boundary: the warp factor at the r' end does not contribute since it is trivial in that flat region. Both $r,r'>0$ in this parametrization: the space is not a half-line now. Setting $r'\sim 0$ for simplicity and extremizing gives $-\frac{\phi_r}{4G}\frac{1}{r^2}+\frac{c}{6}\frac{1}{r}=0$: the competition between the two terms leads to a finite value $r_*\sim\frac{\phi_r}{Gc}$ for the QES location, i.e. an island.

Since $c > 0$ and $d_i > 1$, both contributions in both derivative expressions appear with the same sign. Note also that in this entire discussion, we are on one side (the past) of the singularity at $t = 0$, so the range of the time variable is $t \equiv |t| \geq 0$. Then it is clear that the only QES solution (t_*, r_*) to extremization is [1]

$$t \sim t_0 \,, \qquad r \equiv r_* \to \infty \,, \qquad t \equiv t_* \to \infty \,; \qquad t_* \lesssim r_* \,, \tag{2.11}$$

i.e. the quantum extremal surface is driven to the semiclassical region, infinitely far from the singularity at $t = 0$. A more general analysis vindicates this. Further, in this semiclassical region the dilaton is not too small so there are no islands here since the Bekenstein bound is not violated, as in (2.9).

3 AdS Kasner, quantum extremal surfaces, regulated

In what follows we will study various 2-dim backgrounds given by the dilaton ϕ and the 2-dim metric e^f and analyse quantum extremal surfaces obtained from the extremization of the generalized entropy (2.7): in general these are of the form

$$\begin{aligned} &S_{\rm gen} = \frac{\phi}{4G} + \frac{c}{12}\log\left(\Delta^2\, e^f|_{(t,r)}\right) , &&\Delta^2 = r^2 - (t-t_0)^2 \,, \\ &\frac{\partial_r\phi}{4G} + \frac{c}{6}\frac{r}{\Delta^2} + \frac{c}{12}\partial_r f = 0 \,, &&\frac{\partial_t\phi}{4G} - \frac{c}{6}\frac{t-t_0}{\Delta^2} + \frac{c}{12}\partial_t f = 0 \,, \end{aligned} \tag{3.1}$$

where we have retained only terms relevant for extremization. These are all spaces with a holographic boundary so we are using the corresponding expression for the generalized entropy in appendix B.

We would like to understand the dependence of the quantum extremal surface (t_*, r_*) on the observer location $(t_0, r_0) \equiv (t_0, 0)$: we will focus on the observer at the holographic boundary $r = 0$. Here we study the AdS Kasner case: we will put back the AdS scale R and the Kasner scale t_K in (2.4) so the lengthscales are manifest. Then the dilaton and 2-dim metric become

$$\phi = \frac{t/t_K}{(r/R)^{d_i}} \,, \qquad ds^2 = \frac{(t/t_K)^{(d_i-1)/d_i}}{(r/R)^{d_i+1}}(-dt^2 + dr^2) \,. \tag{3.2}$$

Towards understanding quantum extremal surfaces, let us study (3.1) with the scales put in explicitly as in (3.2). If we assume $t = t_0$, we obtain (2.10), (2.11), which are structurally similar to the AdS case (2.8). We will instead attempt solving for t as a function of t_0. Then the extremization equations are (introducing ϕ_r as bookkeeping for now)

$$\frac{c}{6}\frac{r}{\Delta^2} = \frac{\phi_r}{4G}\frac{d_i\, t/t_K}{r^{d_i+1}/R^{d_i}} + \frac{c}{12}\frac{d_i+1}{r} \,, \qquad \frac{c}{6}\frac{t-t_0}{\Delta^2} = \frac{\phi_r}{4G}\frac{1/t_K}{r^{d_i}/R^{d_i}} + \frac{c}{12}\frac{d_i-1}{d_i\, t} \,. \tag{3.3}$$

Note that each term now has dimensions of inverse length manifestly. In the parametrization of these cosmologies (2.4), the singularity is at $t = 0$: regarding this as a Big-Crunch, we take the time coordinate t to represent $|t|$ so that $t > 0$ in our entire discussion.

JHEP03(2022)201

We require that the QES is spacelike-separated from the observer, consistent with the interpretation of these extremal surfaces as holographic entanglement. This implies

$$\Delta^2 > 0 \quad \Rightarrow \quad t_* > t_0 \,, \qquad\qquad [\Delta^2 = r^2 - (\Delta t)^2] \tag{3.4}$$

from the t-equation in (3.3). This means that the QES always lags behind the observer, in the direction away from the singularity ($t = 0$).

Let us now look in more detail at QES solutions near the semiclassical solution (2.11), where $\Delta t \sim 0$ and $r, t \to \infty$. Let us first rewrite the r-extremization equation in (3.3) as

$$\frac{3\phi_r}{Gc}\frac{d_i t/t_K}{r^{d_i+1}/R^{d_i}} + \left(\frac{d_i+1}{r} - \frac{2r}{\Delta^2}\right) = \frac{3\phi_r}{Gc}\frac{d_i t/t_K}{r^{d_i+1}/R^{d_i}} + \frac{d_i+1}{r}\left(\frac{\frac{d_i-1}{d_i+1}r^2 - (\Delta t)^2}{r^2-(\Delta t)^2}\right) = 0 \tag{3.5}$$

As long as Δt is small, i.e. $\Delta^2 \sim r^2$, the second term is positive: thus both terms are positive, the only solution to this being $r \equiv r_* \to \infty$. This is very similar to the time-independent AdS case in (2.8), giving the entire Poincare wedge as the entanglement wedge: there are no islands.

Analysing the t-extremization equation is rendered tricky with $r_* \to \infty$ strictly. Towards obtaining insight into the t_0 dependence of t_*, let us regulate as $r_* = R_c \sim \infty$ with some large but finite spatial cutoff R_c that represents the boundary of the entanglement wedge. Then the t-equation in (3.3) becomes

$$\frac{\Delta t}{R_c^2 - (\Delta t)^2} = \frac{1}{2K_c} + \frac{d_i - 1}{2d_i\, t}\,, \qquad\qquad \frac{1}{K_c} = \frac{3\phi_r}{Gc}\frac{1/t_K}{R_c^{d_i}/R^{d_i}}\,. \tag{3.6}$$

This expression is manifestly satisfied semiclassically as in (2.11). Taking these regulated equations as containing finite terms we can solve for t_*: with $\Delta t \ll R_c$, we obtain the approximate regulated expression

$$\frac{\Delta t}{R_c^2} \sim \frac{1}{2K_c} + \frac{d_i - 1}{2d_i\, t_0}\,, \qquad\qquad \Delta t = t_* - t_0\,, \tag{3.7}$$

where we have approximated $\Delta^2 \sim R_c^2$ and set $t \sim t_0$ in the last expression (with t_0 large, as in (2.11)) (there is some similarity with the semiclassical expansion (2.5)). We see that the QES (3.7) lags behind the observer, in the direction away from the singularity. We now see that as t_0 decreases, Δt increases, i.e. the lag of the QES is increasing: see the top part of figure 2 for a heuristic depiction (the lag is exaggerated!).

The on-shell generalized entropy (3.1) in the semiclassical regime where $\Delta^2 \sim R_c^2$ becomes

$$S_{\rm gen}^{\rm o.s.} \sim \frac{\phi_r}{4G}\frac{t_*/t_K}{(R_c/R)^{d_i}} + \frac{c}{12}\log\left(\frac{R_c^2}{\epsilon_{\rm UV}^2}\frac{(t_*/t_K)^{(d_i-1)/d_i}}{(R_c/R)^{d_i+1}}\right)\,, \tag{3.8}$$

with t_* in (3.7). Since $t_* \gtrsim t_0$ and R_c is large, $S_{\rm gen}^{\rm o.s.}$ is not dramatically different structurally from the AdS value (2.8), without the t_*/t_K factors. In more detail, we see that the on-shell AdS expression (2.8) with $r_* = R_c$ and $\phi|_{r_*} = \phi_*$ becomes $S^{\rm o.s.} = \frac{\phi_*}{4G} + \frac{c}{12}\log\left(\frac{R^2}{\epsilon_{\rm UV}^2}\left(\frac{\phi_*}{\phi_r}\right)^{(d_i-1)/d_i}\right)$ so the log vanishes when its argument becomes $O(1)$, i.e. when ϕ_* is sufficiently small. At

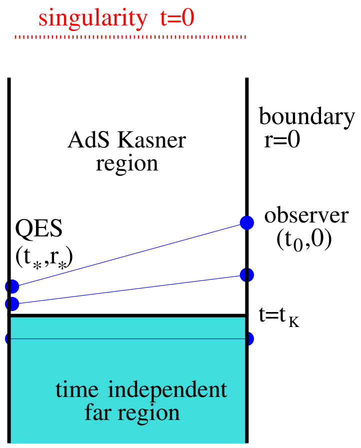

Figure 2. Cartoon of the 2-dim AdS Kasner geometry (singularity at $t = 0$), the holographic boundary at $r = 0$ and the QES at (t_*, r_*), with a time-independent AdS space appended for $t > t_K$. The boundary observer $(t_0, 0)$ moves in time from the time-independent region to the AdS Kasner region. The QES lags behind in time, i.e. $t_* > t_0$, when t_0 is in the Kasner region.

this point, $S^{\text{o.s.}} \sim \frac{\phi_*}{4G} \sim 0$, in accord with the physical expectation that the AdS ground state has zero entropy. In this sense the spatial regulator R_c has physical meaning as the effective physical boundary of the entanglement wedge, where ϕ_* becomes small enough to be comparable with $(\frac{\epsilon_{\text{UV}}}{R})^\#$. Note that we can recast $S^{\text{o.s.}}$ as (2.9) exactly setting $\frac{1}{\phi_r^{(d_i-1)/d_i}} \frac{R^2}{\epsilon_{\text{UV}}^2} \sim 1$ thus fixing ϕ_r, which can possibly be regarded as renormalizing $\frac{\phi_r}{G} \equiv \frac{1}{G_r}$ (and rendering S_{gen} finite). The above expression (3.8) is similar when the t_*/t_K factors are $O(1)$ so the above arguments apply, and the overall entropy is not appreciable.

As a further check, note that this QES solution vindicates the maximin property.[2]

Naively it appears that $\frac{\Delta t}{R_c^2} \sim \frac{1}{t_0}$ shows a growth as t_0 decreases. Rewriting (3.6) and solving as a quadratic, taking $\Delta t > 0$, gives

$$\frac{\Delta t}{R_c} = \frac{1}{R_c} \left(\sqrt{\frac{1}{(\frac{1}{K_c} + \frac{d_i-1}{d_i\, t})^2} + R_c^2} - \frac{1}{\frac{1}{K_c} + \frac{d_i-1}{d_i\, t}} \right) , \tag{3.9}$$

showing a slow growth in Δt as t decreases, for fixed regulator R_c. Extrapolating and setting $t_0 = 0$ shows that $t = 0$ is not a solution (this can also be seen in (3.3)).

Our analysis is best regarded as valid in the semiclassical regime, far from the singularity, approximating bulk matter to be in the ground state. However perhaps the qualitative feature of the quantum extremal surfaces and the associated entanglement wedge excluding the near singularity region (depicted schematically in the top, AdS Kasner, part of figure 2) will remain as a reliable result even with better near singularity bulk entropy models.

[2] In the semiclassical regime, the second derivatives $\partial_t^2 S_{\text{gen}}|_* \sim -\frac{c}{12} \frac{d_i-1}{d_i\, t_*^2} - \frac{c}{6} \frac{1}{\Delta_*^2} - \frac{c}{3} \frac{(t_*-t_0)^2}{\Delta_*^4} < 0$ and $\partial_r^2 S_{\text{gen}}|_* \sim \frac{\phi_r}{4G} \frac{d_i\,(d_i+1)\, t_*\, R_i^d}{t_k\, R_c^{d_i+2}} + \frac{c}{12} \frac{d_i+1}{R_c^2} + \frac{c}{6} \frac{1}{\Delta_*^2} \left(1 - \frac{2R_c^2}{\Delta_*^2}\right) > 0$ confirm time-maximization and spatial minimization, with the regulator R_c finite.

We note that the $S_{\rm gen}^{\rm o.s.}$ (3.8) decreases with time evolution towards the singularity (this is reminiscent of e.g. [51, 52] revealing low complexity in such singularities). Recasting this semiclassical value in the form (2.9), we note that as long as ϕ is not too small, the bulk entropy term is subleading to the area term. Thus the Bekenstein bound is not violated and there are no spatially disconnected islands of the kind noted in black holes. In some qualitative sense, it is tempting to regard the excluded near singularity region as a timelike separated island-like region: it would be interesting to understand this better.

3.1 Searching for islands

Looking now at (3.5), we see that for

$$\frac{d_i-1}{d_i+1}r^2 < (\Delta t)^2 < r^2\,, \tag{3.10}$$

a spacelike-separated island appears to emerge. Unlike the semiclassical region with $\Delta t \ll r$ (where both terms are the same sign), the numerator in the term in brackets in (3.5) now changes sign indicating a large but finite $r \sim (\frac{\phi_r}{Gc})^{\#}$ solution leading to a disconnected region: there is some structural similarity to the discussion in [50] (see Footnote 1). Towards exploring this in detail, first, note that the ∂_r-equation in (3.3) can be rewritten as

$$\Delta^2 = \frac{2r^2}{d_i+1}\frac{1}{1+\frac{d_i}{d_i+1}\frac{t}{K}}\,, \qquad \frac{1}{K} = \frac{3\phi_r}{Gc}\frac{1/t_K}{r^{d_i}/R^{d_i}}\,, \tag{3.11}$$

so

$$\Delta^2 = r^2 - (\Delta t)^2 \quad \Rightarrow \quad \frac{\Delta t}{r} = \sqrt{\frac{\frac{d_i-1}{d_i+1} + \frac{d_i}{d_i+1}\frac{t}{K}}{1+\frac{d_i}{d_i+1}\frac{t}{K}}}\,. \tag{3.12}$$

The potential island arises at large finite r in (3.5) when

$$(\Delta t)^2 \gtrsim \frac{d_i-1}{d_i+1}r^2 \tag{3.13}$$

so that Δt is not small but in fact scales as r which is large. Expanding (3.12) in the vicinity of (3.13) gives

$$\frac{\Delta t}{r^2} \sim \sqrt{\frac{d_i-1}{d_i+1}}\frac{1}{r}\left(1+\frac{d_i}{d_i^2-1}\frac{t}{K}+\ldots\right). \tag{3.14}$$

Now the ∂_t-equation (3.3) as an exact quadratic can be solved to obtain (choosing $\Delta t > 0$)

$$\frac{\Delta t}{r} = \sqrt{\frac{\frac{d_i^2}{(d_i-1)^2}\frac{t^2}{r^2}}{(\frac{d_i}{d_i-1}\frac{t}{K}+1)^2}+1} - \frac{\frac{d_i}{d_i-1}\frac{t}{r}}{\frac{d_i}{d_i-1}\frac{t}{K}+1}\,, \tag{3.15}$$

with K defined in (3.11). For a nontrivial island-like solution, this expression for $\frac{\Delta t}{r}$ must match that in (3.12) in the vicinity of the island boundary (3.13). With $\frac{t}{K} \sim \epsilon$ being small, we expand and obtain at leading order

$$\sqrt{1+x^2} - x = \sqrt{\frac{d_i-1}{d_i+1}} \quad \xrightarrow{\text{solving}} \quad x \equiv \frac{d_i}{d_i-1}\frac{t}{r} = \frac{1}{\sqrt{d_i^2-1}} \tag{3.16}$$

This gives

$$\Delta t \gtrsim \sqrt{\frac{d_i - 1}{d_i + 1}}\, r \sim d_i\, t\,. \tag{3.17}$$

The last condition $\Delta t \gtrsim d_i\, t$ is clearly impossible with $\Delta t = t - t_0$ for any $d_i > 1$.

In addition, using the leading term matching condition (3.17) and expanding (3.15) about the potential island boundary (3.13) shows that the first subleading term in $\frac{t}{K}$ is $\frac{1}{d_i}\sqrt{\frac{d_i+1}{d_i-1}}\frac{t}{K}$ which does not match the first subleading term in (3.14).

We have been looking for an island-like solution in the vicinity of the potential island boundary (3.13) emerging continuously from the semiclassical region where $r_* \to \infty$ as discussed after (3.5). So we require a simultaneous solution to the extremization equations (3.3) recast as (3.12) and (3.15), just inside the island region. Then at the very least the leading and first subleading terms in the expansions of (3.12) and (3.15) near (3.13) must agree, which is not the case. Thus this potential island solution is inconsistent.

One could ask if there are nontrivial islands further away, towards the singularity (although they may not be physically reliable). In this regard, we can write $\Delta t = t - t_0$ and expand out the r- and t-extremization equations (3.5), (3.6) : this leads to two cubic equations in t. However, taking them as simultaneously true (and e.g. eliminating the t^3 term), it appears that there are no consistent finite r, t solutions to these, i.e. no islands.

3.2 Appending a time-independent far region

Let us now consider appending the AdS Kasner space with a time-independent AdS region far from the singularity, joined at the Kasner scale $t = t_K$. See figure 2. So we have AdS Kasner for $t < t_K$ and the time-independent AdS space for $t > t_K$, i.e.

$$\begin{aligned} \phi &= \frac{t/t_K}{(r/R)^{d_i}}\,, \qquad ds^2 = \frac{(t/t_K)^{(d_i-1)/d_i}}{(r/R)^{d_i+1}}(-dt^2 + dr^2) \qquad [t < t_K]\,, \\ \phi &= \frac{1}{(r/R)^{d_i}}\,, \qquad ds^2 = \frac{1}{(r/R)^{d_i+1}}(-dt^2 + dr^2) \qquad [t > t_K]\,. \end{aligned} \tag{3.18}$$

The spaces are joined continuously at $t = t_K$ but the joining is not smooth. Now the extremization equations must be analysed separately as the observer at t_0 moves through each region. The generalized entropy and its extremization (3.1) applied to the background profiles (3.18) in both regions give

$$\begin{aligned} t_0 > t_K : \quad & \frac{c}{6}\frac{r}{\Delta^2} = \frac{\phi_r}{4G}\frac{d_i}{r^{d_i+1}/R^{d_i}} + \frac{c}{12}\frac{d_i+1}{r}\,, \quad \frac{c}{6}\frac{t-t_0}{\Delta^2} = 0\,; \\ t_0 < t_K : \quad & \frac{c}{6}\frac{r}{\Delta^2} = \frac{\phi_r}{4G}\frac{d_i\, t/t_K}{r^{d_i+1}/R^{d_i}} + \frac{c}{12}\frac{d_i+1}{r}\,, \quad \frac{c}{6}\frac{t-t_0}{\Delta^2} = \frac{\phi_r}{4G}\frac{1/t_K}{r^{d_i}/R^{d_i}} + \frac{c}{12}\frac{d_i-1}{d_i\, t}\,. \end{aligned} \tag{3.19}$$

In the time-independent region $t > t_K$ we see it is physically reasonable to set $t_* = t_0$, i.e. the QES lies on the same time slice as the observer. This follows from time-translation invariance in that region at least for $t_0 \gg t_K$ (far from the junction at t_K). Since the joining slice t_K is in the semiclassical region far from the singularity, it is adequate to use (3.7) with the regulator to study the time evolution of the QES in the Kasner region. The lagging (or repulsive) feature of the QES thus begins once the observer transits into the Kasner region (the sharp joining at t_k implies that the lag does not evolve smoothly).

To see this in more detail, consider the time $t_0 = t_K - \delta t_0$ when the observer is just entering the Kasner region: then we expect that the quantum extremal surface is just a little away from the observer time slice t_0. To quantify this, let us compare δt_* in (3.7) with δt_0 (and K_c defined in (3.6)): we have

$$\delta t_0 = t_K - t_0 > 0\,; \qquad \frac{\delta t_*}{R_c^2} = \frac{t_* - t_0}{R_c^2} \sim \frac{1}{2K_c} + \frac{d_i - 1}{2d_i\, t_K}\left(1 + \frac{\delta t_0}{t_K}\right), \tag{3.20}$$

so that for small δt_0 i.e. $t_0 \sim t_K$, the quantum extremal surface ends up being pushed to the time-independent region ($t_* > t_K$). Of course as the observer moves in time further, the QES enters the Kasner region as well. To see this further, let us compare the QES location with the Kasner scale: with $t_0 \lesssim t_K$, we have

$$t_* \lesssim t_K \quad \Rightarrow \quad \frac{t_K - t_0}{R_c^2} \gtrsim \frac{1}{2K_c} + \frac{d_i - 1}{2d_i\, t_0} \tag{3.21}$$

In other words, the quantum extremal surface is within the Kasner region if the observer is sufficiently further within. The cross-over of the QES to the Kasner region occurs when $t_* \sim t_K$, i.e. when the above inequality is saturated (giving $\frac{t_0 - t_K}{R_c^2} \sim -\frac{1}{2K_c} - \frac{d_i-1}{2d_i t_K}$).

The model (3.18) is just meant as a simple toy model for gaining some insight into the evolution of the quantum extremal surface as the observer transits from the time-independent far region into the time-dependent AdS Kasner region towards the singularity. The existence of the time-independent far region suggests that one can prepare the initial state as the ground state via a Euclidean continuation. Putting this on firmer footing is however more tricky. There is a discontinuity at the $t = t_K$ slice perhaps reflecting the fact that the Kasner time-dependence does not switch off at t_K: this might imply additional concerns in smooth time evolution into the Kasner region (without any external energy-momentum inflow). More detailed analysis of this requires detailed understanding of the junction conditions for joining up the spacetimes at t_K. Perhaps rather than a sharp time slice at t_K, it would be more physical to find a thickened spacetime region interpolating smoothly between the time-independent far region and the Kasner region: then the QES lag is likely to evolve smoothly. We will leave these questions for the future.

3.3 More general 2-dim cosmologies, QES, regulated

In the previous subsections, we studied AdS-Kasner cosmologies and their 2-dim reflections obtained by dimensional reduction (2.4), and quantum extremal surfaces. Now we will extend this to more general 2-dim cosmologies (2.3). We have the 2-dim dilaton and metric fields of the form

$$\phi = t r^m\,, \quad e^f = t^a r^b\,, \qquad a > 0, \quad m < 0, \quad b < 0\,. \tag{3.22}$$

Note that we have taken the time exponent of the dilaton in accord with the universality (A.3) of the near singularity region found in [31]. We take $a > 0$ to simulate a Big-Crunch

JHEP03(2022)201

singularity at $t = 0$. Further we assume $m, b < 0$ in accord with the intuition that the dilaton and the 2-dim metric grow towards the holographic boundary at $r = 0$.

The generalized entropy (3.1) and its extremization with r, t, give

$$\frac{c}{6}\frac{r}{\Delta^2} = \frac{\phi_r}{4G}\frac{|m|t}{r^{|m|+1}} + \frac{c}{12}\frac{|b|}{r}\,, \qquad \frac{c}{6}\frac{t-t_0}{\Delta^2} = \frac{\phi_r}{4G}\frac{1}{r^{|m|}} + \frac{c}{12}\frac{a}{t}\,, \tag{3.23}$$

analogous to (3.3), except that we have suppressed length scales analogous to R, t_K here. Firstly, requiring the spacelike condition $\Delta^2 > 0$ implies $t_* > t_0$, analogous to (3.4): this means the QES lags behind the observer, in the direction away from the singularity at $t = 0$.

As noted already in [1], it is clear that the QES solution to these extremization equations is again of the form (2.11), i.e. $r_* \to \infty$, $t_* \sim t_0 \to \infty$ with $t_* \lesssim r_*$. In the vicinity of the semiclassical region, analogous to the AdS Kasner case (3.5) we can recast the r-equation as

$$\frac{3\phi_r}{Gc}\frac{|m|t}{r^{|m|+1}} + \left(\frac{|b|}{r} - \frac{2r}{\Delta^2}\right) = \frac{3\phi_r}{Gc}\frac{|m|t}{r^{|m|+1}} + \frac{|b|}{r}\left(\frac{\frac{|b|-2}{|b|}r^2 - (\Delta t)^2}{r^2 - (\Delta t)^2}\right) = 0\,. \tag{3.24}$$

As in that case, with Δt small, i.e. $\Delta^2 \sim r^2$, both terms are positive and the only solution to this is $r_* \to \infty$, giving the entire Poincare wedge as the entanglement wedge: there are no islands. Now, the t-equation becomes

$$\frac{\Delta t}{R_c^2 - (\Delta t)^2} = \frac{3\phi_r}{2Gc}\frac{1/t_K}{R_c^{d_i}/R^{d_i}} + \frac{d_i - 1}{2d_i\, t}\,, \tag{3.25}$$

analogous to (3.6). As before, we are regulating the QES solution as $r_* = R_c \sim \infty$ with some large but finite spatial cutoff R_c representing the boundary of the entanglement wedge. Taking these regulated equations as containing finite terms we can solve for t_*, obtaining an approximate regulated expression analogous to (3.7) after setting $\Delta^2 \sim R_c^2$ and $t \sim t_0$. The resulting semiclassical picture is similar to the discussion in the AdS Kasner case, with the QES lag increasing as t_0 decreases.

Now let us look for island-like solutions in these more general holographic cosmologies, analogous to section 3.1. The corresponding island boundary here, analogous to (3.13), is

$$(\Delta t)^2 \gtrsim \frac{|b|-2}{|b|}r^2\,. \tag{3.26}$$

Analogous to (3.12) and (3.15) in the AdS Kasner case, we obtain, respectively,

$$\Delta^2 = r^2 - (\Delta t)^2 \quad \Rightarrow \quad \frac{\Delta t}{r} = \sqrt{\frac{\frac{|b|-2}{|b|} + \frac{|m|}{|b|}\frac{t}{K}}{1 + \frac{|m|}{|b|}\frac{t}{K}}}\,, \qquad \frac{1}{K} = \frac{3\phi_r}{Gc}\frac{1}{r^{|m|}}\,, \tag{3.27}$$

rearranging (3.24), and

$$\frac{\Delta t}{r} = \sqrt{\frac{\frac{1}{a^2}\frac{t^2}{r^2}}{(\frac{1}{a}\frac{t}{K}+1)^2} + 1} - \frac{\frac{1}{a}\frac{t}{r}}{\frac{1}{a}\frac{t}{K}+1}\,, \tag{3.28}$$

from the ∂_t-equation in (3.23) regarded as a quadratic, choosing $\Delta t > 0$.

For a nontrivial island-like solution emerging in the vicinity of (3.26), these two expressions for $\frac{\Delta t}{r}$ must match: expanding, the leading order terms give

$$x \equiv \frac{1}{a}\frac{t}{r} : \quad \sqrt{1+x^2} - x = \sqrt{\frac{|b|-2}{|b|}} \quad \xrightarrow{\text{solving}} \quad \frac{t}{r} = \frac{a}{\sqrt{|b|(|b|-2)}}\,, \tag{3.29}$$

while matching the first subleading terms requires

$$\frac{1}{a\sqrt{|b|(|b|-2)}}\left(1 - \frac{1}{\sqrt{|b|(|b|-2)+1}}\right)\frac{t}{K} = \frac{|m|/|b|}{\sqrt{|b|(|b|-2)}}\,\frac{t}{K} \tag{3.30}$$

i.e.

$$\frac{a|m|}{|b|} = 1 - \frac{1}{\sqrt{|b|(|b|-2)+1}} \tag{3.31}$$

For the AdS Kasner values $a = \frac{d_i-1}{d_i}\,,\ m = -d_i\,,\ b = -(d_i+1)$, these agree with the conditions obtained in section 3.1, which were not consistent as we saw. The condition (3.29) gives $\Delta t = \frac{|b|-2}{a}t$: this is impossible in the AdS Kasner case (3.17) as we saw. For the hyperscaling violating cosmologies (A.5), this condition can again be shown to be impossible to satisfy (a takes its maximum value for $\gamma = 0$). The hyperscaling violating Lifshitz cosmologies in [31] require $a = |b| - 2\,,\ m = -1$ (reviewed very briefly after (A.5)). This gives $\Delta t = \frac{|b|-2}{a}t = t$, which is satisfied for $t_0 = 0$, but this is the location of the singularity which is unreliable (the condition (3.31) becomes $\frac{2}{b} = \frac{1}{\sqrt{|b|(|b|-2)+1}}$ giving $b = -2, a = 0$). Thus overall, these more general holographic cosmologies appear qualitatively similar to the AdS Kasner case.

The conditions (3.22) on the exponents are motivated by the more general investigations on 2-dimensional cosmologies in [31]. These investigations employ fairly general and minimal assumptions on the effective action governing such cosmological spacetimes: the resulting space of cosmologies is quite rich, including ones with nonrelativistic (e.g. hyperscaling violating Lifshitz) asymptotics and boundary conditions, and they all satisfy the conditions (3.22). However it would be interesting to explore the space of such cosmologies, possibly enlarging them (including those that do not admit reduction to 2-dimensions), towards understanding the behaviour of quantum extremal surfaces with regard to the Big-Crunch (-Bang) singularities they may exhibit.

4 Null cosmologies and quantum extremal surfaces

We consider cosmological spacetimes with null time-dependence in this section: there are parallels with the discussions in [34, 35, 38], as well as e.g. [53–57]. If we further require that the higher dimensional spacetime admits dimensional reduction (2.1) to 2-dimensions, this reduces to a restricted family of 2-dimensional backgrounds of the form

$$ds^2 = -dx^+dx^-\,, \qquad \phi = \phi(x^+)\,, \qquad \Psi = \Psi(x^+)\,, \qquad x^{\pm} = t \pm r\,. \tag{4.1}$$

The 2-dim metric can always be coordinate-transformed to be flat if we only have x^+-dependence in ϕ, e^f in the reduction ansatz (2.1), leading to the above. The upstairs

JHEP03(2022)201

spacetime (2.3) then is

$$ds^2 = -\phi^{-(d_i-1)/d_i} dx^+ dx^- + \phi^{2/d_i} dy_i^2 \,, \qquad x_i = \{r, y_i\} \,. \tag{4.2}$$

This comprises various higher dimensional backgrounds with null singularities e.g.

$$ds^2 = (x^+)^a(-dx^+dx^-) + (x^+)^b dy_i^2 \tag{4.3}$$

which however are somewhat special, given the restriction to the 2-dimensional reduction ansatz (2.1): thus it also does not include the null holographic AdS cosmologies in [34, 35, 55, 56] which are of the form $ds^2 = \frac{R^2}{r^2}[e^{f(x^+)}(-dx^+dx^- + dx_i^2) + dr^2]$. There are qualitative parallels however. The exponents a, b in (4.3) are related by the Einstein equations. These are a bit similar to the null Kasner backgrounds considered in [38], except that the 2-dim restriction implies that $e^f \equiv (x^+)^a$ can be absorbed by redefining the null time variable $x^+ \to X^+ = \int e^f dx^+$. In writing the 2-dim backgrounds (4.1) we have effectively redefined the lightcone variables $x^\pm$ in this manner. These backgrounds are likely supersymmetric.

Now the equations of motion (A.1) simplify tremendously since there is only null-time dependence in the background ansatze (4.1): for instance all nontrivial contractions of the form $g^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi \sim g^{+-}\partial_+\Psi\partial_-\Psi$ vanish since there is no x^--dependence. We also have $\mathcal{R} = 0$ since the 2-dim space is flat. Thus the equations of motion give

$$\begin{aligned} &(++): \quad -\partial_+^2\phi - \frac{\phi}{2}(\partial_+\Psi)^2 = 0\,; \\ &(\phi): \quad \frac{\partial U}{\partial\phi} = \mathcal{R} - \frac{1}{2}(\partial\Psi)^2 = 0\,; \qquad (\Psi): \quad \frac{\partial U}{\partial\Psi} = \partial_\mu(\phi g^{\mu\nu}\partial_\nu\Psi) = 0\,. \end{aligned} \tag{4.4}$$

These imply that the dilaton potential is trivial and give a single nontrivial condition from the $(++)$ equation relating ϕ, Ψ. We want to consider a Big-Crunch singularity arising at $x^+ = 0$ as a future null singularity, so we take $x^+ < 0$ in our entire discussion below. Then

$$\begin{aligned} &\phi = (-x^+)^k \,, \qquad \Psi = \Psi(x^+) \qquad\quad \Rightarrow \qquad (\partial_+\Psi)^2 = -2\frac{\partial_+^2\phi}{\phi} = -\frac{2k(k-1)}{(x^+)^2}\,, \\ \Rightarrow \quad &0 < k \le 1\,, \qquad \phi = (-x^+)^k\,, \qquad e^\Psi = (-x^+)^{\pm\sqrt{2k(1-k)}}\,. \end{aligned} \tag{4.5}$$

While $k > 0$ gives vanishing dilaton as $x^+ \to 0$, the exponent of e^Ψ could have either sign. The single ϕ, Ψ-relation allows extrapolating ϕ, Ψ above to asymptotically constant functions i.e. flat space. This 2-dim background implies the upstairs background (4.2) with ϕ as above: this is of the form (4.3) with $a = -\frac{k(d_i-1)}{d_i}$ and $b = \frac{2k}{d_i}$. These have $R^i{}_{+i+} = \frac{k(1-k)}{d_i\,(x^+)^2}$ so tidal forces diverge (all curvature invariants vanish due to the null nature of the backgrounds). To see this in more detail, consider a null geodesic congruence propagating along x^+ with cross-section along some y^i-direction: the geodesic equation then gives

$$\frac{dx^+}{d\lambda^2} + \Gamma^+_{++}\left(\frac{dx^+}{d\lambda}\right)^2 = 0 \quad \to \quad \lambda = \frac{(x^+)^{a+1}}{a+1}\,, \tag{4.6}$$

where $\Gamma^+_{++} = \frac{a}{x^+}$ is the only nonvanishing Γ^+_{ij} component. Solving this leads to the affine parameter above and the tangent vector becomes $\xi = \partial_\lambda = (\frac{dx^+}{d\lambda})\partial_+$ so $\xi^+ = (x^+)^a$.

The relative acceleration of neighbouring geodesics then is $a^M = R^M{}_{CDB}\xi^C\xi^D n^B$ with $n = n^B\partial_B$ the unit normalized cross-sectional separation vector. Then it can be seen that $a^i = R^i{}_{+i+}(\xi^+)^2 n^i$ so $|a^i|^2$ diverges for all $0 < k < 1$ leading to diverging tidal forces, somewhat similar to the corresponding discussion in [38]. For $k = 1$ the spacetimes (4.2) have all Riemann components vanishing: these can be recast as $ds^2 = -dX^+dx^- + (X^+)^2 dy_i^2$ which can be shown to be flat space in null Milne coordinates (redefining $Y_i = X^+y_i$, $y^- = x^- + y_i^2 X^+$).

Now we analyze quantum extremal surfaces. These cosmologies have no holographic boundary: introducing a bookkeeping ϕ_r, the generalized entropy (appendix B) is

$$S_{\text{gen}} = \frac{\phi_r}{4G}(-x^+)^k + \frac{c}{6}\log(-\Delta x^+\Delta x^-)\,, \tag{4.7}$$

where $\Delta x^\pm = x^\pm - x_0^\pm$ characterizes the spacetime interval between the observer O and the QES (see figure 3). Strictly speaking, there is a null Kasner scale t_N here appearing as $\phi = (\frac{-x^+}{t_N})^k$ so ϕ is dimensionless: however since the 2-dim metric is flat in these variables, t_N can be absorbed into the definition of ϕ_r above: so we will suppress this (unlike the spacelike cases in section 3 earlier). The extremization with respect to x^- and x^+ gives

$$\partial_- S_{\text{gen}} = \frac{c}{6}\frac{-\Delta x^+}{-\Delta x^+\Delta x^-} = 0\,, \qquad \partial_+ S_{\text{gen}} = -\frac{\phi_r}{4G}\frac{k}{(-x^+)^{1-k}} + \frac{c}{6}\frac{\partial_+\Delta^2}{\Delta^2} = 0\,. \tag{4.8}$$

With $0 < k < 1$, the classical extremization ($c = 0$) gives $x^+ \to \infty$: in full, we have

$$\Delta^2 = -\Delta x^+\Delta x^- > 0\,, \quad \Delta x^- = x^- - x_0^- \to -\infty\,, \quad -\frac{1}{(-x^+)^{1-k}} + \frac{2Gc}{3\phi_r k}\frac{1}{x^+ - x_0^+} = 0\,, \tag{4.9}$$

so

$$\Delta x^+ > 0, \quad x_*^+ = x_0^+ + \frac{2Gc}{3k\phi_r}(-x_*^+)^{1-k} > x_0^+\,; \quad \Delta x^- < 0, \quad x_*^- \to X_c^- \sim -\infty\,. \tag{4.10}$$

This is best visualized as in figure 3: we describe this further below. From (2.7), we have $Gc \ll 1$ so that $x^+ \sim x_0^+$ upto small corrections (with $k \neq 0$). Thus employing perturbation theory in Gc, we obtain

$$x_*^+ \sim x_0^+ + \frac{2Gc}{3k\phi_r}(-x_0^+)^{1-k}\,, \tag{4.11}$$

i.e. the QES is almost on the same null-time (x^+) slice as the observer, but just a little *towards* the null singularity (using absolute values gives $|x^+| - |x_0^+| \sim -\frac{2Gc}{3k\phi_r}|x_0^+|^{1-k}$). The location of the QES as being towards the singularity rather than away as in the spacelike cases may look surprising at first sight. However from figure 3, drawing constant x^+ and x^- slices, it is clear that the location of the QES with $\Delta x^+ > 0$ and $\Delta x^- \to -\infty$ is geometrically reasonable and expected if the QES and the observer are to be spacelike separated ($\Delta x^+ < 0$ gives timelike separation between the QES and the observer). In terms of the (t, r)-coordinates (4.1), figure 3 can be taken to depict the region with $x^+ = t + r < 0$ in the (t, r)-plane, with the singularity locus being $t + r = 0$ and the timelike observer worldline having some fixed r_0 with $t_0 < 0$. The description in figure 3 continues to hold

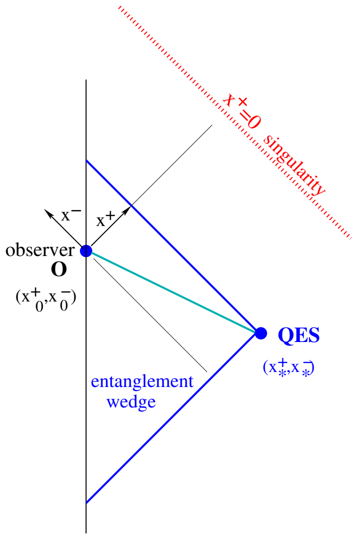

Figure 3. Cartoon of the 2-dim geometry with the null singularity at $x^+ = 0$, the worldline (x_0^+, x_0^-) of a timelike observer (vertical trajectory, representing for simplicity a fixed spatial location), and the quantum extremal surface at (x_*^+, x_*^-). As can be seen, the QES is spacelike separated from the observer ($\Delta^2 > 0$) if $\Delta x^+ > 0$ and $\Delta x^- \sim -\infty$, and lies towards the singularity in terms of x^+-slices. The entanglement wedge defined by the QES is shown as the blue wedge.

as long as the observer remains timelike: it also holds if the observer is moving along a null trajectory along x^+ with fixed x^-. As a further check, we see that this extremization exhibits time-maximization with null time (x^+): we have, using (4.10),

$$\partial_+^2 S_{\text{gen}} = -k(1-k)\frac{\phi_r}{4G}(-x^+)^{k-2} - \frac{c}{6}\frac{1}{(x^+ - x_0^+)^2} \quad \to \quad \partial_+^2 S_{\text{gen}}|_* < 0 \,. \tag{4.12}$$

Note however that $\partial_-^2 S_{\text{gen}} = -\frac{c}{6}\frac{1}{(\Delta x^-)^2} \to 0^-$ from (4.8). This should not be surprising: the 2-dim space here is flat and the absence of the bulk gravitational field makes it quite different from AdS-like spaces (e.g. an expression like $S \sim \log r$ gives $\partial_r^2 S \sim -\frac{1}{r^2} \to 0^-$).

As examples of (4.7), we see that for a nearly smooth space e.g. with $k = \epsilon \ll 1$, (4.10) gives $x_*^+ \sim (1 - \frac{2Gc}{3\epsilon\phi_r})x_0^+$. The case $k = \frac{2}{3}$ gives the cubic

$$x_*^+ = -t^3 \;: \qquad t^3 + \frac{Gc}{\phi_r}t - |x_0^+| = 0\,, \tag{4.13}$$

which can be shown to have one real root which satisfies $\Delta x^+ > 0$ and agrees with (4.11) in perturbation theory in Gc. For generic k values, recasting using $x^+ = -y^{\frac{1}{1-k}}$, it can be seen numerically that there is one real root satisfying $\Delta x^+ > 0$. Along these lines, for values such as $k = \frac{1}{2}$ we choose the positive root of the resulting quadratic in continuity with neighbouring k values, which then again gives $\Delta x^+ > 0$.

Note that these null cosmological singularities are somewhat different from the spacelike ones: for instance the extremization (4.10) shows that the singularity locus $x^+ = 0$ is in fact an allowed QES solution when $x_0^+ = 0$. The behaviour near $x^+ = 0$ can be seen explicitly in examples including (4.13), e.g. numerically. Thus these null singularities appear to not be excluded from the entanglement wedge of the observer. However the on-shell generalized entropy (4.7) continues to be singular generically in the vicinity of the singularity: (4.10) gives

$$S_{\rm gen}^{\rm o.s.} = \frac{\phi_r}{4G}(-x_*^+)^k + \frac{c}{6}\log\left(\frac{2Gc}{3k\phi_r}\,\frac{(-x_*^+)^{1-k}\,|X_c^-|}{\epsilon_{\rm UV}^2}\right). \tag{4.14}$$

Thus although formal extrapolation to the singularity appears possible, the above implies that the QES (4.10) is only reliable in the semiclassical regime with large x_*^+ and $Gc \ll 1$ (where the Bekenstein bound does not appear violated). Also since $S_{\rm gen}^{\rm o.s.}$ appears singular, further subleading contributions beyond the bulk entropy term presumably also must be considered. It was observed in [38] that strings become highly excited in the vicinity of a null Kasner Big-Crunch singularity (see also [53, 57]). It is likely that this will be true for (4.3) as well. In this regard, note that the backgrounds (4.1) necessarily require the extra scalar e^Ψ to be nontrivial: interpreting this as the string coupling $g_s = e^\Psi$ and choosing the negative sign exponent for e^Ψ in (4.5) suggests large string interactions in the vicinity of the singularity $x^+ = 0$. It is conceivable however that in some appropriate double-scaling limit $x_*^+ \to 0$, $X_c^- \to -\infty$, with $\frac{2Gc}{3k\phi_r}\frac{(-x_*^+)^{1-k}|X_c^-|}{\epsilon_{\rm UV}}$ held fixed, the generalized entropy can be rendered nonsingular. It would be nice to explore this more carefully, perhaps dovetailing with the positive sign exponent for e^Ψ in (4.5) and suppressed string interactions.

It is interesting to note that there is an entire function-worth of nontrivial null backgrounds in (4.1), as (4.4) shows. This is a special feature of 2-dim spacetimes that have a "holomorphic" structure, as is the case here with solely x^+-dependence: for instance the backgrounds (4.3) can be recast by redefining the null-time variable to give (4.1), so that the 2-dim metric is flat in these $x^\pm$-coordinates.[3] Spacelike cosmological singularities generically do not exhibit any such "holomorphy" and cannot generically be recast in flat coordinates and the metric factor e^f lingers. This holomorphy shows up in the extremization equations (4.8), (4.10), where the $x^\pm$ sectors decouple (in contrast with e.g. (3.3) in the AdS Kasner case, and more generally (3.1)). In fact, considering generic 2-dim backgrounds (4.1), extremizing the generalized entropy gives

$$\partial_+ S_{\rm gen} = 0 \quad\to\quad \partial_+\phi + \frac{2Gc}{3\phi_r}\frac{1}{x^+ - x_0^+} = 0 \quad\to\quad x^+ - x_0^+ = -\frac{2Gc}{3\phi_r}\frac{1}{\partial_+\phi}, \tag{4.15}$$

again exhibiting this holomorphicity. From the logic in figure 3 with $\Delta^2 > 0$ and $\Delta x^+ > 0$, $\Delta x^- \to -\infty$, this implies that the quantum extremal surface must lie in the direction of decreasing dilaton, i.e. $\partial_+\phi < 0$. This is consistent with our earlier discussion since the dilaton Crunches towards decreasing x^+.

[3] Instead of these "flat" variables, had we taken the background to be

$$e^f = (X^+)^\alpha\,, \quad \phi = (X^+)^K\,, \quad\to\quad (\partial_+\Psi)^2 = \frac{2K(\alpha - K + 1)}{(X^+)^2} \quad\to\quad 0 < K \le \alpha + 1\,.$$

In other words, the exponent k earlier is related as $k = \frac{K}{\alpha+1}$. Now the generalized entropy contains the metric factor $e^{f/2}|_*$, thus appearing singular.

5 Other cosmologies and QES

In this section, we will study other cosmological backgrounds, in particular de Sitter space in the Poincare slicing and FRW universes under certain conditions. One might take these to have natural asymptotics at future or past timelike infinity, and are thus quite different from the previous discussions on AdS-like or null cosmologies where the asymptotics are at spatial or null infinity. As we will see (and has been noted previously), the extremal surface structure is rather different in these cases below: in some sense we are simply extending our previous investigations in some formal way to the cosmologies below, with the hope that better understanding will emerge over time.

5.1 de Sitter, Poincare

de Sitter space dS_{d_i+1} in the Poincare slicing and its 2-dim reduction are

$$ds^2 = \frac{R^2}{\tau^2}(-d\tau^2+dx^2+dy_i^2) \quad \rightarrow \quad \phi = \frac{R^{d_i}}{(-\tau)^{d_i}}\,, \quad ds^2 = \frac{R^{d_i+1}}{(-\tau)^{d_i+1}}(-d\tau^2+dx^2)\,. \tag{5.1}$$

We are parametrizing the upper Poincare patch with the future boundary I^+ at $\tau = 0$ and the past horizon at $\tau \rightarrow -\infty$, and $-\infty < \tau < 0$ generically so the minus signs are explicitly retained. As τ increases to the future, the dilaton grows. There is a singularity at $\tau \rightarrow -\infty$ in the effective 2-dim space: the space is conformally dS_2 (there are some parallels with the discussions of AdS_D reductions in [32]).

In this inflationary patch, we take the observer to be in the ground state, so the bulk entropy is given by the ground state expression. Then the generalized entropy for a bulk observer on a static worldline at say (x_0, τ_0) is (see appendix B)

$$S_{\text{gen}} = \frac{\phi_r}{4G}\frac{R^{d_i}}{(-\tau)^{d_i}} + \frac{c}{6}\log\left(\Delta^2\,\frac{R^{(d_i+1)/2}}{(-\tau)^{(d_i+1)/2}}\right)\,, \tag{5.2}$$

retaining only terms relevant for extremization. Then extremization gives

$$\frac{c}{3}\frac{\Delta x}{\Delta^2} = 0\,, \qquad -\left(-\frac{d_i\phi_r}{4G}\frac{R^{d_i}}{(-\tau)^{d_i+1}} - \frac{c}{12}\frac{d_i+1}{(-\tau)}\right) - \frac{c}{3}\frac{\tau-\tau_0}{\Delta^2} = 0\,. \tag{5.3}$$

One solution to this is

$$\Delta x = 0\,, \quad \Delta^2 = -(\tau-\tau_0)^2\,; \qquad \frac{d_i\phi_r}{4G}\frac{R^{d_i}}{(-\tau)^{d_i+1}} + \frac{c}{12}\frac{d_i+1}{(-\tau)} + \frac{c}{3}\frac{1}{\tau-\tau_0} = 0\,. \tag{5.4}$$

For a late time observer with $\tau_0 \sim 0$, we have

$$\Delta x = 0\,, \quad \frac{d_i\phi_r}{4G}\frac{R^{d_i}}{(-\tau)^{d_i+1}} + \frac{c}{12}\frac{3-d_i}{\tau} = 0 \quad \rightarrow \quad \tau_* = -R\left(\frac{d_i}{3-d_i}\,\frac{3\phi_r}{Gc}\right)^{1/d_i} \tag{5.5}$$

Note that we are looking for a solution with $\tau < 0$ as per our parametrization: so for $d_i \geq 3$ (i.e. dS_5 and higher) the only real QES solution is $\tau \rightarrow -\infty$. For $d_i = 1$ this matches the result in [17] ((5.2) matches eq. 6.7 there).

Most notably, the above QES solution is timelike-separated from the observer: so $\Delta^2 < 0$ (unlike e.g. (3.4), (4.9)) and the generalized entropy (5.2) has an imaginary part from $\log(-1)$. In the spirit of our earlier discussions, it is interesting to look for quantum extremal surfaces that are spacelike-separated from the observer: towards this, note that there is a distinct family of solutions which by construction are spacelike-separated, along the lines of the discussions in the cosmologies earlier with a regulator. Then (5.3) gives

$$\Delta^2 \sim R_c^2\,, \qquad \frac{d_i\phi_r}{4G}\,\frac{R^{d_i}}{(-\tau)^{d_i+1}} + \frac{c}{12}\,\frac{d_i+1}{(-\tau)} \sim \frac{c}{3}\,\frac{\tau-\tau_0}{R_c^2}\,, \tag{5.6}$$

regulating the QES as before with a spatial cutoff R_c. First, note that if we remove the regulator so $R_c \to \infty$, then we obtain (with $t \equiv -\tau > 0$)

$$\frac{d_i\phi_r}{4G}\,\frac{R^{d_i}}{t^{d_i+1}} + \frac{c}{12}\,\frac{d_i+1}{t} = 0\,. \tag{5.7}$$

Both terms have the same sign so the only real QES is at $\tau \to -\infty$. This is spacelike separated only if the observer is also localized at sufficiently early times.

With a finite spatial regulator R_c, we see that in general $\tau > \tau_0$, i.e. the QES lies on time slices later than the observer. As a first approximation, note that in the classical limit $c \to 0$, the solution is $\tau \to -\infty$: this is the location where the dilaton is minimized. For early times also, the solution is similar, i.e.

$$|\tau_0| \gg R: \qquad \tau \to -\infty\,, \quad \tau \sim \tau_0\,, \tag{5.8}$$

i.e. the QES is in the far past when the observer is also in the far past. This can be seen to exhibit time-maximization. Let us analyze (5.6) for 4-dim de Sitter upstairs ($d_i = 2$): then

$$\Delta^2 \sim R_c^2\,, \qquad \frac{\phi_r}{2G}\,\frac{R^2}{(-\tau)^3} + \frac{c}{12}\,\frac{3}{(-\tau)} \sim \frac{c}{6}\,\frac{\tau-\tau_0}{R_c^2}\,. \tag{5.9}$$

With $t \equiv -\tau$, we can rewrite this as

$$\Delta^2 \sim R_c^2\,, \qquad \frac{6\phi_r}{Gc}\,R^2R_c^2 + 3R_c^2\,t^2 \sim 2t^3(t_0-t) \quad \rightarrow \quad t^4 - t_0t^3 + \frac{3R_c^2}{2}\,t^2 + \frac{3\phi_r}{Gc}\,R^2R_c^2 \sim 0\,. \tag{5.10}$$

Clearly as $t_0 \to 0$, there is no real QES solution since all terms are positive (there appears to be a critical t_0 where the real QES (5.8) stops existing). dS_{d_i+1} can be seen to exhibit similar behaviour.

Overall, in some essential sense, the physical interpretation of the generalized entropy in these cases is not transparent, for instance as holographic entanglement in the dual boundary theory, along the lines of the AdS cases (even from a bulk point of view alone, the timelike separation is unconventional compared with the usual formulations of entanglement on a spatial slice). However our discussion appears to corroborate previous work on classical de Sitter extremal surfaces. Taking the future boundary as a natural anchor in dS, there are either complex extremal surfaces [58–60] or future-past (timelike) extremal surfaces [61, 62]. The latter future-past surfaces perhaps suggest some new "temporal entanglement" between

$I^{\pm}$: taking the area of such surfaces to be real is effectively removing an overall i-factor which would arise from rotating a spatial extremal surface to a timelike one (this is also vindicated by the complex generalized entropy (5.2), (5.4), generalizing [17] for dS_2). Overall perhaps this suggests new interpretations towards entanglement in de Sitter space based on the future boundary and dS/CFT [63–65]. The dS/CFT dictionary $\Psi_{dS} = Z_{CFT}$ suggests that boundary entanglement is not bulk entanglement (quite unlike $Z_{bulk} = Z_{CFT}$ in AdS). Bulk observables require $|\Psi_{dS}|^2$ suggesting two copies of the dual CFT: this is reflected in the future-past extremal surfaces [61, 62] alluded to above. Other recent perspectives on extremal surfaces anchored on the de Sitter horizon include e.g. [66].

5.2 FRW cosmologies, 2-dim gravity and QES

Consider FRW cosmologies with flat spatial sections sourced by a scalar field Ψ (general reviews include e.g. [67, 68][4]): we choose one of the spatial directions to be noncompact and perform dimensional reduction on the others to obtain a 2-dim background

$$ds^2 = -dt^2 + a(t)^2 dx_i^2 \quad \rightarrow \quad \phi = a^{d_i}\,, \quad ds^2 = a^{d_i+1}(-d\tau^2 + dx^2)\,, \tag{5.11}$$

as a solution to (2.2), (A.1), (A.2). The energy-momentum conservation equation gives $dE + pdV = d(\rho\, a^{d_i+1}) + pd(a^{d_i+1}) = 0$, i.e. $\dot{\rho} + (d_i+1)H(\rho+p) = 0$. This along with the Friedmann equation and the equation of state $p = w\rho$ gives FRW cosmologies with

$$p = w\rho\,, \quad a \sim t^k\,, \quad k = \frac{2}{(1+d_i)(1+w)} \qquad \left[\rho = \frac{1}{2}\dot{\Psi}^2 - V\,, \quad p = \frac{1}{2}\dot{\Psi}^2 + V\right] \tag{5.12}$$

Now using conformal time τ gives

$$d\tau = \frac{dt}{a(t)} \quad \rightarrow \quad \tau \sim t^{1-k} \quad \rightarrow \quad a(\tau) \sim \left(\frac{\tau}{\tau_F}\right)^{\frac{k}{1-k}} \equiv \left(\frac{\tau}{\tau_F}\right)^{\nu}, \tag{5.13}$$

introducing the FRW scale τ_F so the scale factor becomes dimensionless: τ_F controls the strength of time-dependence in these backgrounds, analogous to t_K in (3.2). Note that the above FRW description is slightly different from focussing on the vicinity of the singularity as in [31]: taking dominant time derivatives implies $\dot{\Psi}^2 \gg V$ so $p \sim \rho$, i.e. $w \sim 1$, giving $\nu = \frac{1}{d_i}$ so $\phi = a^{d_i} \sim \tau$ in agreement with the universality (A.3). More generally the physical bounds on the equation of state parameter w translate to corresponding regimes for ν:

$$\nu = \frac{2}{(1+d_i)(1+w)-2}\,; \qquad -1 \le w \le 1 \quad \Rightarrow \quad \nu > \frac{1}{d_i} \quad \text{or} \quad \nu \le -1\,. \tag{5.14}$$

Now we analyse quantum extremal surfaces here. In general the bulk matter entropy corresponds to some excited state, such as the thermal state. A variety of such studies for FRW cosmologies including entanglement with auxiliary universes appears in [18], revealing islands in various cases. Our discussion here will be limited to simply extending the earlier de Sitter QES solutions to certain FRW cases, which correspond to matter in the ground

[4]See also lectures by D. Baumann at http://cosmology.amsterdam/.

JHEP03(2022)201

state (as may arise for pressureless matter with $w = 0$). The generalized entropy for an observer in such a background is (see appendix B)

$$S_{\rm gen} = \frac{a^{d_i}}{4G} + \frac{c}{6}\log\left(\Delta^2\, a^{(d_i+1)/2}|_{(\tau,r)}\right)\,, \qquad \Delta^2 = (\Delta x)^2 - (\tau-\tau_0)^2\,, \tag{5.15}$$

where we are using conformal time τ in the 2-dim theory. Now extremization gives

$$\begin{aligned} \partial_x S_{\rm gen} = \frac{c}{6}\frac{\partial_x\Delta^2}{\Delta^2} = 0\,, \qquad & \partial_\tau S_{\rm gen} = \frac{d_i a^{d_i-1}\,\partial_\tau a}{4G} + \frac{c}{12}\frac{(d_i+1)\partial_\tau a}{a} + \frac{c}{6}\frac{\partial_\tau\Delta^2}{\Delta^2} = 0\,, \\ \longrightarrow \qquad \frac{c}{3}\frac{\Delta x}{\Delta^2} = 0\,, \qquad & \frac{d_i\nu}{4G}\frac{\tau^{\nu d_i-1}}{\tau_F^{\nu d_i}} + \frac{c}{12}\frac{(d_i+1)\nu}{\tau} - \frac{c}{3}\frac{\tau-\tau_0}{\Delta^2} = 0\,, \end{aligned} \tag{5.16}$$

using (5.13), (5.14). First considering timelike separated QES, we have

$$\Delta x = 0\,, \quad \Delta^2 = -(\tau-\tau_0)^2\,; \qquad \frac{d_i\nu}{4G}\frac{\tau^{\nu d_i-1}}{\tau_F^{\nu d_i}} + \frac{c}{12}\frac{(d_i+1)\nu}{\tau} + \frac{c}{3}\frac{1}{\tau-\tau_0} = 0\,, \tag{5.17}$$

analogous to (5.4) in the de Sitter case $\nu = -1$: for $\nu < -1$ the nature of these timelike QES is similar. For $\nu d_i > 1$, taking the first term to be dominant over the second gives

$$\nu > \frac{1}{d_i}\ : \qquad \frac{d_i\nu}{4G}\frac{\tau^{\nu d_i-1}}{\tau_F^{\nu d_i}} + \frac{c}{3}\frac{1}{\tau-\tau_0} \sim 0 \qquad [\tau \gtrsim \tau_F]\,, \tag{5.18}$$

which is structurally similar to (4.9), with corresponding QES solutions (with $\tau_* - \tau_0 < 0$), valid for τ large compared to τ_F. Since these are timelike-separated, the on-shell generalized entropy acquires an imaginary part from $\log(-1)$ in $\Delta^2 < 0$, similar to (5.4).

Alternatively, looking for spacelike separated QES along the lines of (5.6) gives

$$\Delta^2 \sim R_c^2\,, \qquad \frac{d_i\nu}{4G}\frac{\tau^{\nu d_i-1}}{\tau_F^{\nu d_i}} + \frac{c}{12}\frac{(d_i+1)\nu}{\tau} \sim \frac{c}{3}\frac{\tau-\tau_0}{R_c^2}\,. \tag{5.19}$$

We are looking in the region of slow time evolution i.e. large $\tau \gtrsim \tau_F$ (far from the singularity at $\tau = 0$), towards understanding the evolution of the QES with the observer time τ_0. Then for any $\nu > 0$, we have $\tau^{\nu-1} > \tau^{-1}$ so we can approximate the time extremization equation as

$$\frac{d_i\nu}{4G}\frac{\tau^{\nu d_i-1}}{\tau_F^{\nu d_i}} \sim \frac{c}{3}\frac{\tau-\tau_0}{R_c^2} \quad\rightarrow\quad \tau^{\nu d_i-1} \sim \frac{4Gc}{3d_i\nu}\frac{\tau_F^{\nu d_i}}{R_c^2}\,(\tau-\tau_0). \tag{5.20}$$

This equation while tricky in general does have solutions at least for specific families of ν. For instance pressureless dust has $w = 0$ so using (5.14) we have

$$\begin{aligned} w = 0 \ \text{ i.e. } \ \nu = \frac{2}{d_i-1} \ &\xrightarrow{d_i=2}\ \tau^3 \sim \frac{Gc\,\tau_F^4}{3R_c^2}\,(\tau-\tau_0)\,, \\ &\xrightarrow{d_i=3}\ \tau^2 \sim \frac{4Gc\,\tau_F^3}{9R_c^2}\,(\tau-\tau_0)\,, \end{aligned} \tag{5.21}$$

both of which admit real solutions as long as $\frac{Gc\tau_F^{\nu d_i}}{R_c^2}$ lies in appropriate regimes with regard to τ_0. For instance the $d_i = 3$ case requires $\frac{4Gc\,\tau_F^3}{9R_c^2} > 4\tau_0$ for reality. Since the spatial regulator $R_c \to \infty$ strictly speaking, it is clear that these solutions only make sense in a limit where we take c small and R_c large holding the above condition fixed: so the existence of these spacelike-separated QES solutions is not generic.

For generic scalar configurations, it is more appropriate to consider bulk entropy contributions that are not those pertaining to the ground state: then $S_{\rm gen} = \frac{a^{d_i}}{4G} + S_b$ gives $\frac{d_i\,\partial_\tau a}{4G} + \partial_\tau S_b = 0$. Discussions of this sort have previously appeared in e.g. [18]. When S_b overpowers the classical area term, the Bekenstein bound is violated and islands can arise if further conditions hold. For S_b representing bulk matter in some mixed state, one might imagine some auxiliary purifying universe "elsewhere" which could then lead to islands. We will not discuss this further here.

6 Discussion

We have discussed quantum extremal surfaces in various cosmological spacetimes with Big-Crunch singularities, developing further the investigations in [1]. The generalized entropy here is studied in 2-dimensional cosmologies obtainable in part from dimensional reduction of higher dimensional cosmologies: the bulk matter is taken to be in the ground state, which is reasonable in the semiclassical region far from the singularity. First we focussed on the isotropic AdS Kasner spacetime and its reduction to 2-dimensions: the quantum extremal surfaces in [1] were found to be driven to the semiclassical region infinitely far from the Big-Crunch singularities present in these backgrounds (the classical RT/HRT surfaces for finite subregion size bend in the direction away from the singularity, figure 1). Analysing further, the spatial extremization equation (3.5) shows that in the semiclassical region, the QES location leads to the entire Poincare wedge, with no island-like regions. Introducing a spatial regulator in the time extremization equation (3.6) enables understanding the dependence of the QES on the observer's location in time. This shows that the QES lags behind the observer location, in the direction away from the singularity, as in figure 2. The lag can be seen to increase slowly as the observer evolves towards the singularity: extrapolating shows that the singularity $t = 0$ is not a solution to the extremization equations. Thus the entanglement wedge appears to exclude the near singularity region. Removing the regulator recovers the results in [1]. The spatial extremization equation (3.5) shows an island-like region emerging for (3.13). However analysing carefully the extremization equations recast as (3.12), (3.15), in the vicinity of this island boundary reveals that the potential island-like solution is in fact inconsistent. Appending a time-independent far region joined with the AdS Kasner region at the time slice $t = t_K$ as in (3.18) gives further insight on the QES behaviour. This QES analysis in the AdS Kasner case extends to more general singularities admitting a holographic interpretation, with similar QES behaviour (3.24), (3.25), in the semiclassical region, and inconsistencies near a potential island boundary (3.26). These cosmologies include nonrelativistic asymptotics: the assumptions on the exponents (3.22) are fairly general.

In section 4, we studied certain families of null Big-Crunch singularities, which exhibit a certain "holomorphy" due to special properties of null backgrounds. These are distinct in the behaviour of the quantum extremal surface figure 3, which can now reach the

JHEP03(2022)201

singularity: however the on-shell generalized entropy continues to be singular so the vicinity of singularity is not reliable. In all these cases, the QES is manifestly spacelike-separated from the observer (e.g. (3.4), (4.9)), consistent with its interpretation as holographic entanglement. We then discuss aspects of 2-dimensional effective theories involving dimensional reduction of other cosmologies including de Sitter space (Poincare slicing) and FRW cosmologies. In these cases, there are families of QES solutions which are timelike-separated from the observer (5.4), (5.17) (the dS case here is in part a generalization of some results in [17] for the dS_2 case): correspondingly the generalized entropy acquires an imaginary part. We also find real spacelike-separated QES solutions in the presence of finite spatial regulators (5.6). In de Sitter, these real solutions cease to exist for the late-time observer. Overall this perhaps corroborates earlier studies of classical extremal surfaces anchored at the future boundary [58–62]: see the discussion at the end of section 5.1.

Our investigations here have been on using quantum extremal surfaces to gain some insights on cosmological spacetimes containing Big-Crunch singularities: all these admit the form of a 2-dimensional cosmology and thus exclude more general cosmologies that do not admit a reduction to 2-dimensions. Most of our discussions pertain to bulk matter in the ground state, which is reasonable far from the singularities in the cosmologies we have discussed. Overall the cosmologies we have considered are closed universes with no horizons, no appreciable entropy and no additional non-gravitating bath regions: in such cases islands are not generic (there are parallels with some discussions in [69]). This is consistent with previous studies of closed universes with no entanglement with "elsewhere", i.e. regions external to the universes in question which might act as purifiers for mixed states. This is consistent with the Bekenstein bound not being violated, i.e. the bulk entropy does not overpower the classical area in the generalized entropy. Our discussion of de Sitter space pertains only to the Poincare slicing: see e.g. [18, 22] for other discussions of de Sitter. The FRW discussions also must be extended to cases with bulk matter in excited states far from the ground state: in this case islands will appear, corresponding to violations of the Bekenstein bound.

Perhaps the most interesting question pertains to studying more interesting models for bulk matter in the near-singularity spacetime region where the matter might be expected to get highly excited. Presumably incorporating analogs of more "stringy" or quantum entanglement will give more insights into how the near singularity region is accessible via entanglement (with the null singularities perhaps more tractable).

At a more broad brush level, in some essential ways, cosmological singularities in holography are perhaps qualitatively different from black holes. They appear to require nontrivial non-generic initial conditions: generic time-dependent deformations of the CFT vacuum are expected to thermalize on long timescales, leading to black hole formation in the bulk rather than a Big-Crunch. This appears consistent with our finding that e.g. the AdS Kasner and other holographic cosmological singularities are inaccessible via entanglement with conventional ground state bulk matter: perhaps this corroborates the expectation of non-generic holographic dual states (see discussion after (2.4) and also other related studies e.g. [51, 52] of such singularities and complexity). It would be interesting to gain more insights into the role of holographic entanglement, quantum extremal surfaces and islands in cosmology more broadly.

JHEP03(2022)201

Acknowledgments

It is a pleasure to thank Dileep Jatkar and A. Manu for comments on a draft. We also thank Debangshu Mukherjee for some discussions on the FRW cases. This work is partially supported by a grant to CMI from the Infosys Foundation.

A Some details: 2-dim gravity, extremal surfaces

The equations of motion following from the 2-dim effective action (2.2) are

$$g_{\mu\nu}\nabla^2\phi - \nabla_\mu\nabla_\nu\phi + \frac{g_{\mu\nu}}{2}\left(\frac{\phi}{2}(\partial\Psi)^2 + U\right) - \frac{\phi}{2}\partial_\mu\Psi\partial_\nu\Psi = 0\,,$$
$$\mathcal{R} - \frac{\partial U}{\partial\phi} - \frac{1}{2}(\partial\Psi)^2 = 0\,, \qquad \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\,\phi\partial^\mu\Psi) - \frac{\partial U}{\partial\Psi} = 0\,. \tag{A.1}$$

In conformal gauge $g_{\mu\nu} = e^f\eta_{\mu\nu}$ these give

$$\begin{aligned}
(tr) \qquad & \partial_t\partial_r\phi - \frac{1}{2}f'\partial_t\phi - \frac{1}{2}\dot{f}\partial_r\phi + \frac{\phi}{2}\dot{\Psi}\Psi' = 0\,,\\
(rr+tt) \qquad & -\partial_t^2\phi - \partial_r^2\phi + \dot{f}\partial_t\phi + f'\partial_r\phi - \frac{\phi}{2}(\dot{\Psi})^2 - \frac{\phi}{2}(\Psi')^2 = 0,\\
(rr-tt) \qquad & -\partial_t^2\phi + \partial_r^2\phi + e^f U = 0\,,\\
(\phi) \qquad & (\ddot{f} - f'') - \frac{1}{2}(-(\dot{\Psi})^2 + (\Psi')^2) - e^f\frac{\partial U}{\partial\phi} = 0,\\
(\Psi) \qquad & -\partial_t(\phi\partial_t\Psi) + \partial_r(\phi\partial_r\Psi) - e^f\frac{\partial U}{\partial\Psi} = 0\,.
\end{aligned} \tag{A.2}$$

The severe (singular) time-dependence in the vicinity of the singularity implies that time-derivative terms are dominant while other terms, in particular pertaining to the dilaton potential, are irrelevant there: solving these leads to a "universal" subsector

$$\phi \sim t, \qquad e^f \sim t^a, \qquad e^\Psi \sim t^\alpha; \qquad a = \frac{\alpha^2}{2}\,, \tag{A.3}$$

which governs the cosmological singularity. Analysing these equations in more detail can be done using the ansatz (2.3), giving e.g. the AdS Kasner cosmology (2.4) as well as various others, some of which have nonrelativistic (hyperscaling violating Lifshitz) asymptotics. For instance, flat space has $U=0$, giving

$$\phi = t, \quad ds^2 = t^{\alpha^2/2}(-dt^2 + dr^2), \quad e^\Psi = t^\alpha\,. \tag{A.4}$$

With $t = T^{1-p_1}$, these are the reduction of "mostly isotropic" Kasner singularities $ds^2 = -dt^2 + t^{2p_1}dx_1^2 + t^{2p_2}\sum_i dx_i^2$. Hyperscaling violating cosmologies comprise backgrounds (2.3) with exponents and parameters:

$$\begin{aligned}
&U(\phi,\Psi) = 2\Lambda\phi^{\frac{1}{d_i}}e^{\gamma\Psi}\,, \qquad \Lambda = -\frac{1}{2}(d_i+1-\theta)(d_i-\theta), \qquad \gamma = \frac{-2\theta}{\sqrt{2d_i(d_i-\theta)(-\theta)}}\,,\\
&m = -(d_i-\theta)\,, \qquad b = \frac{m(1+d_i)}{d_i}\,, \qquad \beta = -m\gamma\,,\\
&k=1, \quad a = \frac{\alpha^2}{2}\,, \qquad \alpha = -\gamma \pm \sqrt{\gamma^2 + \frac{2(d_i-1)}{d_i}}\,.
\end{aligned} \tag{A.5}$$

JHEP03(2022)201

Here $\theta < 0, \gamma > 0$. The higher dimensional backgrounds here can be obtained as certain kinds of cosmological deformations of reductions of nonconformal branes down to D dimensions.

Still more complicated hyperscaling violating Lifshitz cosmologies (with nontrivial Lifshitz exponents z as well) and their 2-dimensional avatars were also obtained in [31]: these have a more complicated dilaton potential. These are more constrained, requiring the conditions $m = -1,\ a = -b - 2$, as well as further relations between other exponents. A simple example has $\theta = 0,\ z = 2,\ d_i = 2$, and $k = 1,\ m = -1,\ a = \frac{1}{2}\,,\ b = -\frac{5}{2}\,,\ \beta = -\alpha = 1$, and the dilaton potential is $U = \phi^{1/2}(-3 + \frac{1}{\phi^2}e^{-2\Psi})$.

Extremal (RT/HRT) surfaces. The area functional

$$S = \frac{V_{d_i-1}}{4G_{d_i+2}} \int dr\, \phi \sqrt{\frac{e^f}{\phi^{(d_i+1)/d_i}}(1 - (\partial_r t)^2) + (\partial_r x)^2} \tag{A.6}$$

upon extremizing $x(r)$ gives

$$(\partial_r x)^2 = A^2\, \frac{\frac{e^f}{\phi^{(d_i+1)/d_i}}(1 - (\partial_r t)^2)}{\phi^2 - A^2}\,, \qquad S = \frac{V_{d_i-1}}{4G_{d_i+2}} \int dr\, \frac{e^{f/2}\, \phi^{(3-1/d_i)/2}}{\sqrt{\phi^2 - A^2}}\, \sqrt{1 - (\partial_r t)^2}\,. \tag{A.7}$$

In the above expressions, A is the turning point $A = \phi_* = \frac{t_*}{r_*^{d_i}}$ for the AdS Kasner case (2.4). Analysing these extremal surfaces is reliable in the semiclassical region far from the singularity at $t = 0$. In this region, a detailed analysis of the time extremization equation leads to (2.5): the surface lies almost on a constant time slice ($t'' \ll 1$) and can be shown to bend in the direction away from the singularity, as depicted in figure 1.

B Some details on 2d CFT and entanglement entropy

Any 2-dim metric is conformally flat so $ds^2 = e^f \eta_{\mu\nu} dx^\mu dx^\nu$. We can then modify the Calabrese-Cardy result [47, 48], in particular taking the ground state entanglement in flat space and then incorporating the effects of the conformal transformation e^f as in [3]. The twist operator 2-point function scales under a conformal transformation as

$$\langle \sigma(x_1)\, \sigma(x_2) \rangle_{e^f g} = e^{-\Delta_n f/2}|_{x_1}\, e^{-\Delta_n f/2}|_{x_2}\, \langle \sigma(x_1)\, \sigma(x_2) \rangle_g\,, \qquad \Delta_n = \frac{c}{12}\frac{n^2 - 1}{n}\,. \tag{B.1}$$

Since the partition function in the presence of twist operators scales as the twist operator 2-point function, the entanglement entropy becomes

$$S^{12}_{e^f g} = -\lim_{n\to 1} \partial_n \langle \sigma(x_1)\, \sigma(x_2) \rangle_{e^f g} = S^{12}_g + \frac{c}{6} \sum_{\text{endpoints}} \log e^{f/2}\,. \tag{B.2}$$

For a bulk interval, this gives

$$S^{12}_g = \frac{c}{6} \log\left(\frac{\Delta^2}{\epsilon^2_{\text{UV}}}\right) \quad \to \quad S^{12}_{e^f g} = \frac{c}{6} \log\left(\frac{\Delta^2}{\epsilon^2_{\text{UV}}}\, e^{f/2}|_1\, e^{f/2}|_2\right) \tag{B.3}$$

while for a CFT with boundary, we have essentially half the flat space answer (with one end of the interval at the boundary), thus obtaining

$$S_g^{10} = \frac{c}{12}\log\left(\frac{\Delta^2}{\epsilon_{\rm UV}^2}\right) \quad\rightarrow\quad S_{e^f g}^{10} = \frac{c}{12}\log\left(\frac{\Delta^2}{\epsilon_{\rm UV}^2}\, e^f|_1\right) \tag{B.4}$$

We have used the latter in the AdS cases which include the presence of the AdS boundary, while for the bulk cases we use the former expression.

References

[1] A. Manu, K. Narayan and P. Paul, *Cosmological singularities, entanglement and quantum extremal surfaces*, *JHEP* **04** (2021) 200 [arXiv:2012.07351] [INSPIRE].

[2] G. Penington, *Entanglement Wedge Reconstruction and the Information Paradox*, *JHEP* **09** (2020) 002 [arXiv:1905.08255] [INSPIRE].

[3] A. Almheiri, N. Engelhardt, D. Marolf and H. Maxfield, *The entropy of bulk quantum fields and the entanglement wedge of an evaporating black hole*, *JHEP* **12** (2019) 063 [arXiv:1905.08762] [INSPIRE].

[4] A. Almheiri, R. Mahajan, J. Maldacena and Y. Zhao, *The Page curve of Hawking radiation from semiclassical geometry*, *JHEP* **03** (2020) 149 [arXiv:1908.10996] [INSPIRE].

[5] G. Penington, S.H. Shenker, D. Stanford and Z. Yang, *Replica wormholes and the black hole interior*, arXiv:1911.11977 [INSPIRE].

[6] A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian and A. Tajdini, *Replica Wormholes and the Entropy of Hawking Radiation*, *JHEP* **05** (2020) 013 [arXiv:1911.12333] [INSPIRE].

[7] A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian and A. Tajdini, *The entropy of Hawking radiation*, *Rev. Mod. Phys.* **93** (2021) 035002 [arXiv:2006.06872] [INSPIRE].

[8] S. Raju, *Lessons from the information paradox*, *Phys. Rept.* **943** (2022) 2187 [arXiv:2012.05770] [INSPIRE].

[9] B. Chen, B. Czech and Z.-z. Wang, *Quantum information in holographic duality*, *Rept. Prog. Phys.* **85** (2022) 046001 [arXiv:2108.09188] [INSPIRE].

[10] T. Kibe, P. Mandayam and A. Mukhopadhyay, *Holographic spacetime, black holes and quantum error correcting codes: A review*, arXiv:2110.14669 [INSPIRE].

[11] T. Faulkner, A. Lewkowycz and J. Maldacena, *Quantum corrections to holographic entanglement entropy*, *JHEP* **11** (2013) 074 [arXiv:1307.2892] [INSPIRE].

[12] N. Engelhardt and A.C. Wall, *Quantum Extremal Surfaces: Holographic Entanglement Entropy beyond the Classical Regime*, *JHEP* **01** (2015) 073 [arXiv:1408.3203] [INSPIRE].

[13] S. Ryu and T. Takayanagi, *Holographic derivation of entanglement entropy from AdS/CFT*, *Phys. Rev. Lett.* **96** (2006) 181602 [hep-th/0603001] [INSPIRE].

[14] S. Ryu and T. Takayanagi, *Aspects of Holographic Entanglement Entropy*, *JHEP* **08** (2006) 045 [hep-th/0605073] [INSPIRE].

JHEP03(2022)201

[15] V.E. Hubeny, M. Rangamani and T. Takayanagi, *A Covariant holographic entanglement entropy proposal*, *JHEP* **07** (2007) 062 [arXiv:0705.0016] [INSPIRE].

[16] M. Rangamani and T. Takayanagi, *Holographic Entanglement Entropy*, *Lect. Notes Phys.* **931** (2017) 1 [arXiv:1609.01287] [INSPIRE].

[17] Y. Chen, V. Gorbenko and J. Maldacena, *Bra-ket wormholes in gravitationally prepared states*, *JHEP* **02** (2021) 009 [arXiv:2007.16091] [INSPIRE].

[18] T. Hartman, Y. Jiang and E. Shaghoulian, *Islands in cosmology*, *JHEP* **11** (2020) 111 [arXiv:2008.01022] [INSPIRE].

[19] C. Krishnan, *Critical Islands*, *JHEP* **01** (2021) 179 [arXiv:2007.06551] [INSPIRE].

[20] M. Van Raamsdonk, *Comments on wormholes, ensembles, and cosmology*, *JHEP* **12** (2021) 156 [arXiv:2008.02259] [INSPIRE].

[21] V. Balasubramanian, A. Kar and T. Ugajin, *Islands in de Sitter space*, *JHEP* **02** (2021) 072 [arXiv:2008.05275] [INSPIRE].

[22] W. Sybesma, *Pure de Sitter space and the island moving back in time*, *Class. Quant. Grav.* **38** (2021) 145012 [arXiv:2008.07994] [INSPIRE].

[23] S. Choudhury et al., *Circuit Complexity from Cosmological Islands*, *Symmetry* **13** (2021) 1301 [arXiv:2012.10234] [INSPIRE].

[24] R. Bousso and A. Shahbazi-Moghaddam, *Island Finder and Entropy Bound*, *Phys. Rev. D* **103** (2021) 106005 [arXiv:2101.11648] [INSPIRE].

[25] H. Geng, Y. Nomura and H.-Y. Sun, *Information paradox and its resolution in de Sitter holography*, *Phys. Rev. D* **103** (2021) 126004 [arXiv:2103.07477] [INSPIRE].

[26] S. Fallows and S.F. Ross, *Islands and mixed states in closed universes*, *JHEP* **07** (2021) 022 [arXiv:2103.14364] [INSPIRE].

[27] D. Giataganas and N. Tetradis, *Entanglement entropy in FRW backgrounds*, *Phys. Lett. B* **820** (2021) 136493 [arXiv:2105.12614] [INSPIRE].

[28] L. Aalsma, A. Cole, E. Morvan, J.P. van der Schaar and G. Shiu, *Shocks and information exchange in de Sitter space*, *JHEP* **10** (2021) 104 [arXiv:2105.12737] [INSPIRE].

[29] K. Langhoff, C. Murdia and Y. Nomura, *Multiverse in an inverted island*, *Phys. Rev. D* **104** (2021) 086007 [arXiv:2106.05271] [INSPIRE].

[30] S.E. Aguilar-Gutierrez, A. Chatwin-Davies, T. Hertog, N. Pinzani-Fokeeva and B. Robinson, *Islands in multiverse models*, *JHEP* **11** (2021) 212 [arXiv:2108.01278] [INSPIRE].

[31] R. Bhattacharya, K. Narayan and P. Paul, *Cosmological singularities and 2-dimensional dilaton gravity*, *JHEP* **08** (2020) 062 [arXiv:2006.09470] [INSPIRE].

[32] K. Narayan, *Aspects of two-dimensional dilaton gravity, dimensional reduction, and holography*, *Phys. Rev. D* **104** (2021) 026007 [arXiv:2010.12955] [INSPIRE].

[33] D. Grumiller, R. Ruzziconi and C. Zwikel, *Generalized dilaton gravity in 2d*, *SciPost Phys.* **12** (2022) 032 [arXiv:2109.03266] [INSPIRE].

[34] S.R. Das, J. Michelson, K. Narayan and S.P. Trivedi, *Time dependent cosmologies and their duals*, *Phys. Rev. D* **74** (2006) 026002 [hep-th/0602107] [INSPIRE].

[35] S.R. Das, J. Michelson, K. Narayan and S.P. Trivedi, *Cosmologies with Null Singularities and their Gauge Theory Duals*, *Phys. Rev. D* **75** (2007) 026002 [hep-th/0610053] [INSPIRE].

JHEP03(2022)201

[36] A. Awad, S.R. Das, K. Narayan and S.P. Trivedi, *Gauge theory duals of cosmological backgrounds and their energy momentum tensors*, Phys. Rev. D **77** (2008) 046008 [arXiv:0711.2994] [INSPIRE].

[37] A. Awad, S.R. Das, S. Nampuri, K. Narayan and S.P. Trivedi, *Gauge Theories with Time Dependent Couplings and their Cosmological Duals*, Phys. Rev. D **79** (2009) 046004 [arXiv:0807.1517] [INSPIRE].

[38] K. Madhu and K. Narayan, *String spectra near some null cosmological singularities*, Phys. Rev. D **79** (2009) 126009 [arXiv:0904.4532] [INSPIRE].

[39] N. Engelhardt, T. Hertog and G.T. Horowitz, *Holographic Signatures of Cosmological Singularities*, Phys. Rev. Lett. **113** (2014) 121602 [arXiv:1404.2309] [INSPIRE].

[40] N. Engelhardt, T. Hertog and G.T. Horowitz, *Further Holographic Investigations of Big Bang Singularities*, JHEP **07** (2015) 044 [arXiv:1503.08838] [INSPIRE].

[41] N. Engelhardt and G.T. Horowitz, *Holographic Consequences of a No Transmission Principle*, Phys. Rev. D **93** (2016) 026005 [arXiv:1509.07509] [INSPIRE].

[42] N. Engelhardt and G.T. Horowitz, *New Insights into Quantum Gravity from Gauge/gravity Duality*, Int. J. Mod. Phys. D **25** (2016) 1643002 [arXiv:1605.04335] [INSPIRE].

[43] B. Craps, *Big Bang Models in String Theory*, Class. Quant. Grav. **23** (2006) S849 [hep-th/0605199] [INSPIRE].

[44] C.P. Burgess and L. McAllister, *Challenges for String Cosmology*, Class. Quant. Grav. **28** (2011) 204002 [arXiv:1108.2660] [INSPIRE].

[45] N. Engelhardt and G.T. Horowitz, *Entanglement Entropy Near Cosmological Singularities*, JHEP **06** (2013) 041 [arXiv:1303.4442] [INSPIRE].

[46] T. Hartman and J. Maldacena, *Time Evolution of Entanglement Entropy from Black Hole Interiors*, JHEP **05** (2013) 014 [arXiv:1303.1080] [INSPIRE].

[47] P. Calabrese and J.L. Cardy, *Entanglement entropy and quantum field theory*, J. Stat. Mech. **0406** (2004) P06002 [hep-th/0405152] [INSPIRE].

[48] P. Calabrese and J. Cardy, *Entanglement entropy and conformal field theory*, J. Phys. A **42** (2009) 504005 [arXiv:0905.4013] [INSPIRE].

[49] R. Mahajan, *A tutorial on entanglement island computations*, in *ICTS Bangalore String Seminars*, August 26, 2020 [https://youtu.be/7wmpBWGAp44].

[50] A. Almheiri, R. Mahajan and J. Maldacena, *Islands outside the horizon*, arXiv:1910.11077 [INSPIRE].

[51] J.L.F. Barbón and E. Rabinovici, *Holographic complexity and spacetime singularities*, JHEP **01** (2016) 084 [arXiv:1509.09291] [INSPIRE].

[52] P. Caputa, D. Das and S.R. Das, *Path integral complexity and Kasner singularities*, JHEP **01** (2022) 150 [arXiv:2111.04405] [INSPIRE].

[53] G.T. Horowitz and A.R. Steif, *Space-Time Singularities in String Theory*, Phys. Rev. Lett. **64** (1990) 260 [INSPIRE].

[54] B. Craps, S. Sethi and E.P. Verlinde, *A Matrix big bang*, JHEP **10** (2005) 005 [hep-th/0506180] [INSPIRE].

JHEP03(2022)201

[55] C.-S. Chu and P.-M. Ho, *Time-dependent AdS/CFT duality and null singularity*, *JHEP* **04** (2006) 013 [hep-th/0602054] [INSPIRE].

[56] F.-L. Lin and W.-Y. Wen, *Supersymmetric null-like holographic cosmologies*, *JHEP* **05** (2006) 013 [hep-th/0602124] [INSPIRE].

[57] B. Craps, F. De Roo and O. Evnin, *Can free strings propagate across plane wave singularities?*, *JHEP* **03** (2009) 105 [arXiv:0812.2900] [INSPIRE].

[58] K. Narayan, *Extremal surfaces in de Sitter spacetime*, *Phys. Rev. D* **91** (2015) 126011 [arXiv:1501.03019] [INSPIRE].

[59] Y. Sato, *Comments on Entanglement Entropy in the dS/CFT Correspondence*, *Phys. Rev. D* **91** (2015) 086009 [arXiv:1501.04903] [INSPIRE].

[60] M. Miyaji and T. Takayanagi, *Surface/State Correspondence as a Generalized Holography*, *PTEP* **2015** (2015) 073B03 [arXiv:1503.03542] [INSPIRE].

[61] K. Narayan, *On extremal surfaces and de Sitter entropy*, *Phys. Lett. B* **779** (2018) 214 [arXiv:1711.01107] [INSPIRE].

[62] K. Narayan, *de Sitter future-past extremal surfaces and the entanglement wedge*, *Phys. Rev. D* **101** (2020) 086014 [arXiv:2002.11950] [INSPIRE].

[63] A. Strominger, *The dS/CFT correspondence*, *JHEP* **10** (2001) 034 [hep-th/0106113] [INSPIRE].

[64] E. Witten, *Quantum gravity in de Sitter space*, in *Strings 2001: International Conference*, (2001) [hep-th/0106109] [INSPIRE].

[65] J.M. Maldacena, *Non-Gaussian features of primordial fluctuations in single field inflationary models*, *JHEP* **05** (2003) 013 [astro-ph/0210603] [INSPIRE].

[66] E. Shaghoulian, *The central dogma and cosmological horizons*, *JHEP* **01** (2022) 132 [arXiv:2110.13210] [INSPIRE].

[67] M. Trodden and S.M. Carroll, *TASI lectures: Introduction to cosmology*, in *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 2002): Particle Physics and Cosmology: The Quest for Physics Beyond the Standard Model(s)*, pp. 703–793 (2004) [astro-ph/0401547] [INSPIRE].

[68] D. Baumann, *Inflation*, in *Theoretical Advanced Study Institute in Elementary Particle Physics: Physics of the Large and the Small*, pp. 523–686 (2011) [DOI] [arXiv:0907.5424] [INSPIRE].

[69] H. Geng et al., *Inconsistency of islands in theories with long-range gravity*, *JHEP* **01** (2022) 182 [arXiv:2107.03390] [INSPIRE].

JHEP03(2022)201