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Alternative formulations of the twistor double copy

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Erick Chacón, Silvia Nagy and Chris D. White

Centre for Theoretical Physics, Department of Physics and Astronomy,
Queen Mary University of London,
Mile End Road, London E1 4NS, U.K.

E-mail: e.c.chaconramirez@qmul.ac.uk, s.nagy@qmul.ac.uk, christopher.white@qmul.ac.uk

Abstract: The classical double copy relating exact solutions of biadjoint scalar, gauge and gravity theories continues to receive widespread attention. Recently, a derivation of the exact classical double copy was presented, using ideas from twistor theory, in which spacetime fields are mapped to Cech cohomology classes in twistor space. A puzzle remains, however, in how to interpret the twistor double copy, in that it relies on somehow picking special representatives of each cohomology class. In this paper, we provide two alternative formulations of the twistor double copy using the more widely-used language of Dolbeault cohomology. The first amounts to a rewriting of the Cech approach, whereas the second uses known techniques for discussing spacetime fields in Euclidean signature. The latter approach indeed allows us to identify special cohomology representatives, suggesting that further application of twistor methods in exploring the remit of the double copy may be fruitful.

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1 Introduction

The study of (quantum) field theories in recent years has been characterised by a relentless search for common underlying structures. An example of this endeavour is the *double copy*, a set of ideas for relating various quantities in a number of different theories. Inspired by previous work in string theory [1], the double copy was first formulated for scattering amplitudes in gauge and gravity theories [2, 3], both with and without supersymmetry. It was subsequently extended to exact classical solutions in ref. [4], which focused on the special family of *Kerr-Schild* solutions in gravity. Follow-up work (see e.g. refs. [5–14]) has attempted to see whether this family of solutions can be extended, and the development of different techniques is also useful in this regard. Reference [15] (see also refs. [16–18]) presented an alternative exact classical double copy, that uses the spinorial formalism of General Relativity and related field theories, and which is known as the *Weyl double copy*. This is complementary to the Kerr-Schild approach of ref. [4], agreeing where they overlap. Alternative formalisms offer complementary insights [19–30], and it is also known how to double-copy classical solutions order-by-order in the coupling constants of given physical theories, at the price of giving up exactness (see e.g. refs. [31–39]). This may offer new calculational tools for astrophysical observables, including those related to gravitational waves. However, it is also important to probe the origins of the double copy, given that a fully nonperturbative understanding of its scope and applicability is still missing.[1]

Recently, refs. [41, 42] provided a derivation of the Weyl double copy using twistor theory [43–46] (see e.g. refs. [47–51] for pedagogical reviews of this subject, and refs. [20, 52] for related work on twistor approaches to the double copy). Basic ideas from the latter include that points in our own spacetime are mapped non-locally to geometric objects in an abstract twistor space $\mathbb{T}$, and vice versa. Furthermore, physical fields in spacetime map to cohomological data in twistor space. In more pedestrian terms, one may write solutions of the field equation for massless free spacetime fields as a certain contour integral in twistor space known as the *Penrose transform*. The integrand contains a holomorphic function of twistor variables, which is defined up to contributions that vanish upon performing the integral. The freedom to redefine twistor functions in this manner is expressed by saying that they are cohomology classes (elements of a cohomology group), and in the traditional twistor theory approach pioneered by refs. [43–45], these are *sheaf cohomology groups*, which can be suitably approximated by *Čech cohomology groups*.

The Čech approach was used by refs. [41, 42] to derive the Weyl double copy, which led to an interesting puzzle. The spacetime relationship embodied by the Weyl double copy turns into a simple product of functions in the integrand of the Penrose transform in twistor space. As remarked above, however, these are not actually functions, but representatives of cohomology classes, which are meant to be subjectable to the above-mentioned redefinitions. Any non-linear relationship is incompatible with first performing such redefinitions, and thus it seems that the twistor approach demands certain "special" representatives of each cohomology class be chosen, with no useful guidance of how to make such a choice. All that is needed to derive the Weyl double copy in position space is simply to find suitable representatives in twistor space that do the right job. But it would be nice to know if the double copy can be given a more genuinely twistorial interpretation, by fixing a procedure for choosing appropriate representatives.

Another potential issue with refs. [41, 42] is that the Čech approach is not so widely used in contemporary works on twistor approaches to field theory. Instead, it is more common to use the language of differential forms, where the ambiguities inherent in the Penrose transform can be characterised by *Dolbeault cohomology* [53, 54]. That this is equivalent to the Čech approach follows from known isomorphisms between Čech and Dolbeault cohomology groups. Thus, if the double copy has a genuinely twistorial expression, then it must be possible to describe it using the Dolbeault language. Preliminary and very useful comments in this regard were made in ref. [55], which presented a classical double copy defined at asymptotic infinity in spacetime, and showed that it could be used to constrain Dolbeault representatives in the twistor formalism (see ref. [56] for earlier related work). Our aim in this paper is to explore the relationship between the Dolbeault and Čech approaches in more detail, and also to go beyond the purely radiative spacetimes considered in ref. [55]. We will present two different incarnations of the Dolbeault double copy. The first is ultimately a rewriting of the Čech approach, using a known approach for turning

[1]For recent proofs of the double copy in a perturbative field theory context, see refs. [26, 27, 40].

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representatives of Čech cohomology groups into Dolbeault representatives. We will see that a product structure in twistor space indeed emerges in the Dolbeault framework, which is ultimately not surprising given that this is essentially inherited from the Čech double copy. Furthermore, this first technique for constructing a Dolbeault double copy will suffer from the same inherent ambiguities as the Čech approach, namely that it is not clear what the recipe is for picking out a special representative of each cohomology class. Motivated by this puzzle, we will then present a second Dolbeault double copy, which uses known techniques for writing Dolbeault representatives associated with spacetime fields in Euclidean signature. We will argue that the spacetime double copy is again associated with a certain product of functions in twistor space. In this case, however, special representatives of each cohomology class are indeed picked out: they are the *harmonic* representatives, which are uniquely defined for each spacetime field. We hope that our results provide further motivation for the use of twistor methods in understanding the classical double copy. They may also prove useful in relating the classical double copy with the original BCJ double copy for scattering amplitudes, given that twistor methods have appeared naturally in the study of latter (see e.g. refs. [57–61]).

The structure of our paper is as follows. In section 2, we review the twistor double copy of refs. [41, 42], using the Čech formalism, and also relevant aspects of differential forms and Dolbeault cohomology needed for what follows. In section 3, we provide a first example of the twistor double copy in the Dolbeault language, and demonstrate its close relation to the Čech approach. In section 4, we provide a second incarnation, and argue that it allows us to identify special representatives of each cohomology class. We discuss the implications of our results in section 5.

2 Review of necessary concepts

In this section, we will review those details of twistor theory that are needed for what follows, including relevant aspects of Čech and Dolbeault cohomology. We will also describe the twistor double copy of refs. [41, 42], which was formulated in the Čech language. All of these ideas rely on the spinorial formalism of field theory, in which any spacetime tensor field[2] can be converted to a multi-index spinor upon contracting with *Infeld-van-der-Waerden* symbols $\{\sigma^a_{AA'}\}$, defined in a suitable basis[3] e.g.

$$V_{AA'} = V_a \sigma^a_{AA'}. \tag{2.1}$$

Spinor indices A and A' run from 0 to 1, and can be raised and lowered with the 2-dimensional Levi-Civita symbols ϵ^{AB}, $\epsilon^{A'B'}$ etc. The spinorial formalism makes many nice properties of field theory manifest. In particular, any multi-index spinor can be decomposed into sums of fully symmetric spinors multiplied by Levi-Civita symbols. For massless free fields in spacetime, one may write separate spinors $\phi_{AB\ldots C}$ and $\bar{\phi}_{A'B'\ldots C'}$ for the anti-self-dual and self-dual parts of the field respectively. These obey the general *massless free field*

[2]Throughout, we will use lower-case Latin letters for spacetime indices, upper-case Latin letters for spinor indices, and Greek letters for the twistor indices to be defined in what follows.

[3]A common choice results in the identity matrix for $\sigma^0_{AA'}$, and the Pauli matrices for $\sigma^i_{AA'}$.

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equation

$$\nabla^{AA'}\bar{\phi}_{A'\ldots C'} = 0, \quad \nabla^{AA'}\phi_{AB\ldots C} = 0, \tag{2.2}$$

where $\nabla^{AA'}$ is the spinorial translation of the covariant derivative, and there are $2n$ indices for a spin-n field. The cases $n = 0$, 1 and 2 correspond to solutions of a scalar theory, gauge theory and gravity, respectively. Then ref. [15] showed that, for vacuum solutions of Petrov type D, the corresponding spinors were related by the *Weyl double copy* formula

$$\phi_{A'B'C'D'}(x) = \frac{\phi^{(1)}_{(A'B'}(x)\phi^{(2)}_{C'D')}(x)}{\phi(x)}. \tag{2.3}$$

Here $\phi^{(1,2)}_{A'B'}$ are two potentially different electromagnetic spinors, and the brackets denote symmetrisation over indices. Follow-up work — including the use of the twistor methods to be outlined below — has established the validity of eq. (2.3) for other Petrov types [18, 41, 42], albeit at linearised level only in some cases: it is only for types D and N that Minkowski-space solutions of eq. (2.2) correspond to exact solutions of the field equations.

2.1 Twistor space and the incidence relation

Twistor theory provides an alternative viewpoint on solutions of eq. (2.2). We start by defining twistor space $\mathbb{T}$ as the set of solutions of the *twistor equation*

$$\nabla^{(A}_{A'}\Omega^{B)} = 0, \tag{2.4}$$

where Ω^B is a spinor field. Until further notice, we will work in *complexified Minkowski space* $\mathbb{M}_C$, which can be thought of as $\mathbb{C}^4$ equipped with the metric $\eta_{ab} = \mathrm{diag}(1,-1,-1,-1)$, such that the line element in (complex) Cartesian coordinates takes the form

$$ds^2 = \eta_{ab}dx^a dx^b = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2, \quad x^i \in \mathbb{C}. \tag{2.5}$$

Given a vector $x^a = (x^0, x^1, x^2, x^3)$, its spinorial representation following from eq. (2.1) is

$$x^{AA'} = \frac{1}{\sqrt{2}}\begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}, \tag{2.6}$$

where we have defined the Infeld-van-der-Waerden symbols as in refs. [47, 48]. We may then write the general solution to the twistor equation of eq. (2.4) as

$$\Omega^A = \omega^A - ix^{AA'}\pi_{A'}, \tag{2.7}$$

where ω^A, $\pi_{A'}$ are constant (in spacetime) spinors, that we may combine to make a four-component *twistor*

$$Z^\alpha = (\omega^A, \pi_{A'}). \tag{2.8}$$

The "location" of a twistor in spacetime is defined to be such that the field Ω^A in eq. (2.7) vanishes, which sets up a non-local map between spacetime and twistor space known as the *incidence relation*

$$\omega^A = ix^{AA'}\pi_{A'}. \tag{2.9}$$

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Given the invariance of this relation under rescalings $Z^\alpha \to \lambda Z^\alpha$, $\lambda \in \mathbb{C} \setminus \{0\}$, twistors satisfying the incidence relation correspond to points in *projective twistor space* $\mathbb{PT}$. Points in spacetime correspond to complex lines (Riemann spheres) in projective twistor space. For a point x in complexified Minkowski space, we denote the corresponding Riemann sphere by $X \cong \mathbb{CP}^1$. As the components of $\pi_{A'}$ vary for a given $x^{AA'}$, they trace out all points on the Riemann sphere X, so that a given point on X is completely specified by a given $\pi_{A'}$. To specify all points on X, we must cover it with at least two coordinate patches, which we will label by U_0 and U_1 in what follows.

2.2 The Penrose transform and Čech cohomology

A key result of twistor theory is the *Penrose transform*, that relates massless free fields obeying eq. (2.2) to cohomological data in twistor space. More precisely, the original formulation of the Penrose transform [44] expresses spacetime fields as contour integrals in projective twistor space:

$$\bar{\phi}_{A'B'\ldots C'}(x) = \frac{1}{2\pi i} \oint_\Gamma \langle \pi d\pi \rangle \pi_{A'} \pi_{B'} \ldots \pi_{C'} \check{f}(Z)|_X, \tag{2.10}$$

where the measure contains the inner product

$$\langle \pi d\pi \rangle = \pi^{A'} d\pi_{A'} = \epsilon^{A'B'} \pi_{B'} d\pi_{A'}, \tag{2.11}$$

and the notation $|_X$ denotes restriction to the twistor line $X \simeq \mathbb{CP}^1$ associated with x by the incidence relation (2.9). Furthermore, the contour Γ is defined on X, such that it separates any poles of $\check{f}(Z^\alpha)$. All of the functions considered in this paper will have at most two poles, and we may parametrise the sphere such that they are contained in two regions N and S around the north and south poles, as shown in figure 1. Let us then define the coordinate patches

$$U_0 = X \setminus N, \quad U_1 = X \setminus S. \tag{2.12}$$

That is, U_0 (U_1) consists of the sphere X excluding the region N (S), and thus contains the pole P_0 (P_1). By construction, the twistor function $\check{f}(Z^\alpha)$ is holomorphic on the intersection $U_0 \cap U_1$. Also, for eq. (2.10) to make sense as an integral in *projective* twistor space, the integrand plus measure must be invariant under rescalings of Z^α. Thus, $\check{f}(Z^\alpha)$ must be homogeneous of degree $-2n-2$. However, one is clearly free to redefine the function $\check{f}(Z^\alpha)$ up to contributions that vanish upon performing the contour integral. That is, if h_0 (h_1) is a holomorphic function on U_0 (U_1), then the Penrose transform (2.10) is invariant under

$$\check{f} \to \check{f} + h_0 - h_1. \tag{2.13}$$

In simple terms, this corresponds to adding additional functions with poles on only one side of the contour Γ, such that one may always choose to close the contour in a region with no poles. The freedom of eq. (2.13) means that it is not correct to regard $\check{f}(Z^\alpha)$ as a function, but as a representative of a *Čech cohomology class*. We refer the reader to refs. [48, 49] for excellent pedagogical reviews of Čech cohomology in the present context, with a brief summary as follows. Given an open cover $\{U_i\}$ of some space X, one may consider a

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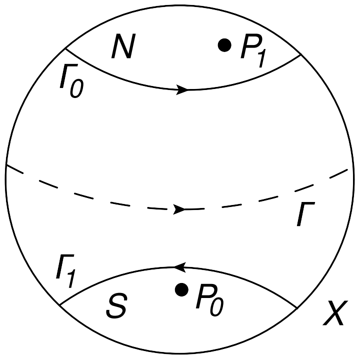

Figure 1. The Riemann sphere X in twistor space corresponding to a spacetime point x. We consider a twistor function $\check{f}(Z^\alpha)$ with poles P_i contained in the regions N and S around the north and south poles. The contour Γ separates these two poles.

p-cochain $f_{i_0 i_1 \ldots i_p}$, consisting of (for our purposes) a function living on the intersection $U_{i_0} \cap U_{i_1} \ldots \cap U_{i_p}$, where an ordering of the intersection of the sets is implied, such that $f_{i_0 i_1 \ldots i_p}$ is defined to be antisymmetric in all indices. Note that f_i (a 0-cochain) is simply a function defined in the single patch U_i. We may further define the *coboundary operator* δ_p, that acts on the set of p-cochains to make $(p+1)$-cochains:

$$\delta_p(\{f_{i_0 \ldots i_p}\}) = \{(p+1)\rho_{[i_{p+1}} f_{i_0 \ldots i_p]}\}, \tag{2.14}$$

where square brackets denote antisymmetrisation over indices, and ρ_i denotes the restriction of a quantity to the patch U_i. The $(p+1)$-cochains generated in this manner are referred to as *coboundaries*. Furthermore, cochains satisfying $\delta_p f_{i_0 \ldots i_p} = 0$ are called *cocycles*, and we can reinterpret the transformation of eq. (2.13) in this language. First, note that all the quantities that appear are holomorphic functions on the intersection $U_0 \cap U_1$. Restricting to this intersection, we may write the first term on the right-hand side, with Čech indices made explicit, as $\check{f}_{01}$. Given that there are no triple intersections for our cover, one automatically has $\delta_p \check{f}_{01} = 0$, so that $\check{f}_{01}$ is in fact a cocycle.[4] The function h_0 (defined on the intersection) stems from a function that is holomorphic throughout the whole of U_0, such that it has the form

$$h_0 = \rho_1 H_0, \tag{2.15}$$

where H_0 is a 0-cochain on U_0. Similar reasoning applies to h_1, such that we may rewrite eq. (2.13) with formal Čech indices made explicit:

$$\check{f}_{01} \to \check{f}_{01} + \rho_1 H_0 - \rho_0 H_1. \tag{2.16}$$

Comparison of the latter two terms with eq. (2.14) shows that this transformation consists of modifying the 1-cocycle $\check{f}_{01}$ by a coboundary.

[4] For more general covers, consistency of closed contour integrals on triple intersections implies that the cocycle condition $\delta_p f_{ij} = 0$ indeed holds for all such quantitites appearing in eq. (2.10). See e.g. ref. [48] for a discussion of this point.

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Cocycles and coboundaries both form groups, denoted by $Z^p(\{U_i\}, \mathcal{S})$ and $B^p(\{U_i\}, \mathcal{S})$, where $\mathcal{S}$ denotes the type of function.[5] We will be concerned with holomorphic functions of homogeneity $(-2n-2)$ for a spin-n spacetime field, which we denote by $\mathcal{S} = \mathcal{O}(-2n-2)$. Then one may define the $p^{\rm th}$ *Čech cohomology group*

$$\check{H}^p(\{U_i\}, \mathcal{S}) = \frac{Z^p(\{U_i\}, \mathcal{S})}{B^p(\{U_i\}, \mathcal{S})}. \tag{2.17}$$

Elements of this group are *Čech cohomology classes*, consisting of cocycles that are equivalent up to addition of coboundaries. Equation (2.16) tells us that the quantity appearing in the Penrose transform of eq. (2.10) is indeed a representative of a Čech cohomology class, and thus an element of the group[6] $\check{H}^1(\mathbb{PT}, \mathcal{O}(-2n-2))$.

2.3 The Penrose transform and Dolbeault cohomology

An alternative formulation for the Penrose transform exists, which uses the language of differential forms [53] (see refs. [46, 50] for pedagogical reviews). In general on a complex manifold $\mathcal{M}$ with complex coordinates z^i, one may decompose differential forms into (anti-)holomorphic parts, such that $\Omega^{p,q}(\mathcal{M})$ denotes the space of so-called (p, q) forms

$$\omega = \omega_{a_1 \ldots a_p \bar{a}_1 \ldots \bar{a}_q} dz^{a_1} \wedge \ldots \wedge dz^{a_p} \wedge d\bar{z}^{\bar{a}_1} \wedge \ldots \wedge d\bar{z}^{\bar{a}_q}, \tag{2.18}$$

where the bar on coordinates denotes complex conjugation. The exterior derivative operator d can then be split as follows:

$$\mathrm{d} = \partial + \bar{\partial}, \tag{2.19}$$

where the *Dolbeault operators* ∂ and $\bar{\partial}$ act on a (p, q) form to give a $(p+1, q)$ and $(p, q+1)$ form respectively, and are individually nilpotent ($\partial^2 = \bar{\partial}^2 = 0$). Projective twistor space is a complex manifold, where the precise definition of complex conjugation depends on the signature of the spacetime we are working in. That is, different real slices of complexified Minkowski spacetime lead to different types of conjugation in twistor space. However, once a given choice has been made, we may introduce the Dolbeault operator

$$\bar{\partial} = d\bar{Z}^\alpha \frac{\partial}{\partial \bar{Z}^\alpha}, \tag{2.20}$$

and use it to define holomorphic quantities h by $\bar{\partial} h = 0$. Then the Penrose transform may be written as [53]

$$\phi_{A'B'\ldots C'}(x) = \frac{1}{2\pi i} \int_X \langle \pi d\pi \rangle \wedge \pi_{A'} \pi_{B'} \ldots \pi_{C'} f(Z)|_X, \tag{2.21}$$

which has a number of differences in comparison with eq. (2.10). The quantity $d\pi_{A'}$ appearing in the measure is now to be regarded as a (1,0) form, and the integration is over the

[5]More precisely, $\mathcal{S}$ denotes the so-called sheaf to which the functions belong. The Čech cohomology described here is an approximation to sheaf cohomology, that is sufficient for our purposes.

[6]Our notation here reminds us that we are considering cohomology groups defined on projective twistor space, but suggests that they will be independent of the particular cover $\{U_i\}$ used. That this is indeed the case follows from the fact that the cover $\{U_i\}$ used throughout the paper is a so-called *Leray cover*, such that Čech cohomology groups are isomorphic to the relevant sheaf cohomology groups.

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whole Riemann sphere X, rather than over a contour. For this integral to make sense, the holomorphic quantity $f(Z)$ must be a (0,1) form which, given we are in projective twistor space as before, must again have homogeneity $-2n-2$ for a spin-n field:

$$f \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O}(-2n-2)), \qquad f(\lambda Z) = \lambda^{-2n-2} f(Z), \qquad \bar{\partial} f = 0. \tag{2.22}$$

Similarly to eq. (2.10), there is a redundancy in how one chooses $f(Z)$: if we redefine it according to

$$f \to f + \bar{\partial} g, \tag{2.23}$$

for some $g \in \Omega^0(\mathbb{PT}, \mathcal{O}(-2s-2))$, and where $\bar{\partial} g$ is globally defined on X, the second term will vanish as a total derivative on the Riemann sphere when inserted in eq. (2.21). Furthermore, the additional term preserves the holomorphic property $\bar{\partial} f(Z) = 0$, by nilpotency of $\bar{\partial}$. In general, the set of (p,q) forms on a manifold $\mathcal{M}$ satisfying $\bar{\partial} h = 0$ are called $\bar{\partial}$-closed, and form a group under addition denoted by $Z^{p,q}_{\bar{\partial}}(\mathcal{M})$. Forms of the form $h = \bar{\partial} g$ are called $\bar{\partial}$-exact, and form the group $B^{p,q}_{\bar{\partial}}(\mathcal{M})$. One may then define the $(p,q)^{\text{th}}$ *Dolbeault cohomology group*

$$H^{p,q}_{\bar{\partial}}(\mathcal{M}) = \frac{Z^{p,q}_{\bar{\partial}}(\mathcal{M})}{B^{p,q}_{\bar{\partial}}(\mathcal{M})}. \tag{2.24}$$

Elements of this group are *Dolbeault cohomology classes*, namely $\bar{\partial}$-closed (p,q) forms that are defined only up to arbitrary additions of $\bar{\partial}$-exact forms. It follows from these definitions that the twistor (0,1)-form $f(Z)$ appearing in eq. (2.21) is a representative element of the Dolbeault cohomology group

$$H^{0,1}_{\bar{\partial}}(\mathbb{PT}, \mathcal{O}(-2n-2)), \tag{2.25}$$

where our enhanced notation relative to eq. (2.24) makes clear that we are considering holomorphic $(0,1)$ forms of a certain homogeneity only.

2.4 Connection between Dolbeault and Čech descriptions

The previous sections provide two different descriptions of the cohomological identification of spacetime fields implied by the Penrose transform. Let us now examine the relationship between them, where we will follow the arguments presented in e.g. refs. [62, 63]. We will consider explicitly the situation of figure 1 in the Dolbeault approach, so that $f(Z)|_X$ is a (0,1) form that is holomorphic everywhere apart from singularities at P_0 and P_1. Then the Dolbeault representative $f(Z)$ associated with a given Čech representative $\check{f}(Z)$ may be defined as follows. First, given our cover (U_0, U_1) of X, we may choose a partition of unity $\{\eta_i\}$, where each η_i is defined in U_i, subject to

$$\sum_i \eta_i = 1. \tag{2.26}$$

We may thus write

$$\eta_0 = \eta, \quad \eta_1 = 1 - \eta, \tag{2.27}$$

and also define

$$f_i = \sum_j \check{f}_{ij} \eta_j \tag{2.28}$$

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in U_i, so that we have explicitly

$$f_0 = (1-\eta)\check{f}_{01}, \quad f_1 = \eta \check{f}_{10} = -\eta \check{f}_{01}. \tag{2.29}$$

Then the desired Dolbeault representatives are given by

$$f(Z) = \{\bar{\partial} f_i\}. \tag{2.30}$$

This satisfies $\bar{\partial} f(Z) = 0$ by construction. Furthermore, on the intersection $U_0 \cap U_1$, one has (via eq. (2.29))

$$\bar{\partial} f_0 - \bar{\partial} f_1 = \bar{\partial} \check{f}_{01} = 0, \tag{2.31}$$

so that $f(Z)$ is indeed uniquely defined globally on X. To check these identifications, it is instructive to see how the Penrose transform of eq. (2.21) reduces to that of eq. (2.10). First, note that one may write the integral over the Riemann sphere $X = U_0 \cup U_1$ in eq. (2.21) as

$$\left[\int_{U_0} + \int_{U_1} - \int_{U_0 \cap U_1}\right] \langle \pi d\pi \rangle \wedge \pi_{A'}\pi_{B'} \ldots \pi_{C'} f(Z)|_X.$$

In the third term, the integrand will contain $\bar{\partial}\check{f}_{01}$ evaluated on the intersection (i.e. away from the poles at P_0 and P_1), which is zero. Thus, we need only consider the first two terms. Substituting the results of eq. (2.29), we may rewrite them using Stokes' theorem to give

$$\begin{aligned}&\left((1-\eta)\oint_{\Gamma_0} \langle \pi d\pi \rangle \pi_{A'}\pi_{B'} \ldots \pi_{C'} \check{f}_{01}(Z)\right)\\ &- \left(\eta \oint_{\Gamma_1} \langle \pi d\pi \rangle \pi_{A'}\pi_{B'} \ldots \pi_{C'} \check{f}_{01}(Z)\right),\end{aligned} \tag{2.32}$$

where Γ_i is the oriented boundary of U_i. These boundaries are depicted in figure 1, and the absence of poles between each Γ_i and Γ imply that one may write

$$\oint_{\Gamma_0} \equiv \oint_{\Gamma}, \quad \oint_{\Gamma_1} \equiv -\oint_{\Gamma},$$

where one must take the opposite orientation of Γ_1 on the sphere into account. The remaining integral over $d\pi$ can be interpreted as a conventional contour integral, such that eq. (2.32) reproduces the Penrose transform of eq. (2.10), as required. We have here shown how to go from the Čech representative of a twistor function to the corresponding Dolbeault representative. For a discussion of how to go the other way, we refer the reader to e.g. refs. [62, 63].

As well as showing how Dolbeault representatives may be defined from their Čech counterparts, we may also reinterpret the cohomological freedom. It follows from eq. (2.28) that redefining a Čech representative by

$$\check{f}_{ij} \to \check{f}_{ij} + h_i - h_j$$

amounts to redefining the Dolbeault representative according to

$$f(Z) \to f(Z) - \bar{\partial}\left(\sum_i h_i \eta_i\right). \tag{2.33}$$

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Above, we have used an arbitrary partition of unity on our cover (U_0, U_1). We can simplify things, however, by choosing a trivial partition in which $\eta = 0$. Then the Dolbeault Penrose transform can be carried out by integrating solely over the patch U_0, even though this does not cover the entire sphere.

2.5 The twistor double copy

Having reviewed various aspects of the Penrose transform, let us now turn our attention to the Weyl double copy of eq. (2.3), connecting scalar, gauge and gravity fields in spacetime. As was recently presented in refs. [41, 42], it is possible to derive this relationship from the Penrose transform of eq. (2.10) (i.e. in the Čech language). The procedure involves choosing holomorphic twistor quantities $\check{f}(Z)$, $\check{f}^{(i)}_{\rm EM}(Z)$ of homogeneity -2 and -4 respectively, such that they correspond to scalar and EM fields in spacetime respectively. One may then form the product

$$\check{f}_{\rm grav.}(Z) = \frac{\check{f}^{(1)}_{\rm EM}(Z)\check{f}^{(2)}_{\rm EM}(Z)}{\check{f}(Z)}, \tag{2.34}$$

which has homogeneity -6 by construction. This corresponds to a gravity field in spacetime, and refs. [41, 42] presented choices for the various functions appearing on the right-hand side of eq. (2.34) such that the spacetime fields obtained from eq. (2.10) obey the Weyl double copy of eq. (2.3). For the original type D Weyl double copy of ref. [15], it is sufficient to choose functions of the form

$$\check{f}_m(Z) = \left[Q_{\alpha\beta} Z^\alpha Z^\beta\right]^{-m}, \tag{2.35}$$

for some constant dual twistor $Q_{\alpha\beta}$, and where $m = 1$ and 2 for the scalar and EM cases respectively. This is a quadratic form in twistor space, and implies the presence of two poles on the Riemann sphere X corresponding to a given spacetime point x. The scalar, gauge and gravity fields linked by eq. (2.34) then share the same poles. These poles give rise to the principal spinors of their respective spacetime fields, so that one obtains a geometric interpretation of how kinematic information is inherited between different theories in spacetime. Furthermore, ref. [42] provided examples of non-type D solutions (albeit at linearised level due to the limitations of the Penrose transform), showing that at the very least the twistor double copy provides a highly convenient book-keeping device for constructing spacetime examples of the Weyl double copy.

However, there is an obvious deficiency of eq. (2.34), discussed in detail in ref. [42]. As reviewed above, the quantities appearing in eq. (2.34) are not in fact functions, but representatives of cohomology classes, which may in principle be subjected to the equivalence of transformations of eq. (2.13). The product of eq. (2.34), in being a non-linear relationship, clearly violates this invariance. Upon redefining the scalar and gauge theory quantities $\check{f}(Z)$ and $\check{f}^{(i)}_{\rm EM}$ *before* forming the product, one would obtain a different gravity solution in general. This does not matter from the point of view of deriving the Weyl double copy: all that is required is that we find suitable quantitites in twistor space that correspond to the desired spacetime relationship. However, if the classical double copy is to be given a genuinely twistorial interpretation, we need a prescription for picking a

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"special" representative for each cohomology class, that eliminates any ambiguity in the double copy procedure. This has been discussed recently in ref. [55], which focused on purely radiative spacetimes, namely those that can be completely determined by data at future null infinity. It is known that this characteristic data can be used, in either gauge theory or gravity, to uniquely fix a Dolbeault representative in the Penrose transform [56]. Thus, for such spacetimes a natural mechanism arises for fixing the ambiguities inherent in eq. (2.34). It is not immediately clear, however, how to generalise this argument to non-radiative spacetimes, and thus we will present alternative arguments in what follows.

3 The twistor double copy in the Dolbeault approach

In the previous section, we reviewed the twistor double copy of eq. (2.34), based on the Penrose transform of eq. (2.10), in which all twistor functions are to be interpreted as representatives of Čech cohomology classes. Let us now see how one can instead formulate the same idea within the framework of Dolbeault cohomology. We will begin by studying a particularly simple example of solutions of eq. (2.2).

3.1 Momentum eigenstates

Momentum eigenstates in spacetime are characterised by a given null momentum with spinorial translation $p_a \to \tilde{p}_A p_{A'}$, and are a special case of plane waves. The solution of eq. (2.2) corresponding to such a wave can then be written as

$$\bar{\phi}_{A'B'\ldots C'}(x) = p_{A'}p_{B'}\ldots p_{C'}e^{ip\cdot x}, \tag{3.1}$$

where the basic phase factor $e^{ip\cdot x} = e^{i\tilde{p}_A p_{A'} x^{AA'}}$ is dressed by an appropriate number of spinors $p_{A'}$, according to the spin of the relevant field. We must then be able to find a (0,1)-form $f(Z)$ in projective twistor space that, when restricted to the Riemann sphere X and substituted into eq. (2.21), yields the spacetime field of eq. (3.1) for a given spin. This (0,1) form will be defined only up to the addition of an arbitrary $\bar{\partial}$-closed (0,1) form, and a suitable Dolbeault representative for a plane wave of helicity h can be written as (see e.g. ref. [50])

$$f^{[h]} = \left(\frac{\langle ap\rangle}{\langle a\pi\rangle}\right)^{-2h+1} \bar{\delta}(\langle \pi p\rangle)\exp\left[\frac{\langle ap\rangle}{\langle a\pi\rangle}[\omega\tilde{p}]\right], \tag{3.2}$$

where $a_{A'}$ is an arbitrary constant Weyl spinor, and we have introduced a holomorphic delta function to be inserted into our Penrose transform integral, which may be further decomposed using the useful identity[7]

$$\bar{\delta}(u) = \bar{\partial}\left(\frac{1}{u}\right). \tag{3.3}$$

We have also introduced the inner product

$$[\omega\tilde{p}] = \omega^A \tilde{p}_A, \tag{3.4}$$

[7]Our convention for the holomorphic delta function differs from that of ref. [50] in that we have not included a factor of $(2\pi i)^{-1}$. The reason is that this has already been included in our Penrose transform definition of eq. (2.21).

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where ω^A is the Weyl spinor appearing in Z^α according to eq. (2.8). To see that eq. (3.2) indeed reproduces eq. (3.1), regardless of the choice of $a_{A'}$, we may parametrise X in eq. (2.21) by choosing

$$\pi_{A'} = b_{A'} + z a_{A'} \quad \Rightarrow \quad \langle \pi d\pi \rangle = -\langle ab \rangle dz, \quad \pi_{A'} = \frac{\langle ab \rangle}{\langle ap \rangle} p_{A'}, \tag{3.5}$$

where $b_{A'}$ is another constant spinor such that $\langle ab \rangle \neq 0$, and we have used the delta function condition in the third equation. Without loss of generality, let us choose this parametrisation to correspond to the patch U_0 discussed in the previous section, such that this contains the support of the holomorphic delta function. Equation (3.2) then becomes, after restriction to the Riemann sphere X,

$$f^{[h]}\Big|_X = \frac{\langle ap \rangle^{-1}}{2\pi i} \left(\frac{\langle ap \rangle}{\langle ab \rangle} \right)^{-2h+1} \bar{\partial} \left(\frac{1}{z + \frac{\langle bp \rangle}{\langle ap \rangle}} \right) \exp \left[i x^{AA'} p_{A'} \tilde{p}_A \right], \tag{3.6}$$

where the incidence relation of eq. (2.9) has been used. Substituting this into eq. (2.21), one finds

$$p_{A'} \ldots p_{C'} \exp \left[i x^{AA'} p_{A'} \tilde{p}_A \right] \frac{1}{2\pi i} \int_{U_0} dz \wedge \bar{\partial} \left(\frac{1}{z + \frac{\langle bp \rangle}{\langle ap \rangle}} \right),$$

so that carrying out the integral using Stoke's theorem yields eq. (3.1) as required. Note that we integrated only over U_0 above, rather than the complete Riemann sphere X. In order to complete the latter, as per the discussion in section 2.4, one should also integrate over a second patch U_1. However, by construction this has been taken so as not to contain the support of the holomorphic delta function, and thus the further integration will not affect the above result, as expected given that we have already recovered the plane wave spacetime field of eq. (3.1).

Now let us examine plane waves of different helicity, and note that we may choose to rewrite eq. (3.6) (before restriction to X) as

$$f^{[h]} = \bar{\partial} F^{[h]}, \tag{3.7}$$

where

$$F^{[h]} = \langle ap \rangle^{-1} \left(\frac{\langle ap \rangle}{\langle ab \rangle} \right)^{-2h+1} \frac{1}{z + \frac{\langle bp \rangle}{\langle ap \rangle}} \exp \left[i \frac{\langle ap \rangle}{\langle ab \rangle} [\omega \tilde{p}] \right]. \tag{3.8}$$

It is then straightforward to verify the relationship

$$F^{[h+h']} = \frac{F^{[h]} F^{[h']}}{F^{[0]}}, \tag{3.9}$$

which can be interpreted as follows. Choosing $h = h' = 1$, one finds that eq. (3.7) applied to $F^{[0]}$ and $F^{[1]}$ yields Dolbeault representatives associated with scalar and gauge theory respectively, such that the Penrose transform of eq. (2.21) gives spacetime scalar and photon plane waves. Equation (3.9), after substitution into eq. (3.7), yields a Dolbeault representative for a gravity wave. From eq. (3.1), the resulting spacetime fields are then precisely

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related by the Weyl double copy of eq. (2.3). Thus, eq. (3.9) is a twistor-space expression of the Weyl double copy, that can be used to generate the Dolbeault representative for a gravity solution, from similar representatives in scalar and gauge theory. Of course, plane waves are very special solutions, and it is perhaps not clear that the procedure of eq. (3.9) generalises to a wider class of solutions. That it indeed generalises in fact follows from the ideas reviewed in section 2.4, as we now discuss.

3.2 The twistor double copy from Dolbeault representatives

The previous section suggests the following general prescription. Consider Dolbeault representatives

$$f(Z) = \bar{\partial} F(z), \quad f^{(l)}_{\rm EM}(Z) = \bar{\partial} F^{(l)}_{\rm EM} \tag{3.10}$$

defined locally on some patch U_i, corresponding to scalar and electromagnetic fields respectively. Then one can form a gravitational Dolbeault representative on U_i by

$$f_{\rm grav.} = \bar{\partial}\left[\frac{F^{(1)}_{\rm EM}(Z) F^{(2)}_{\rm EM}(Z)}{F(Z)}\right]. \tag{3.11}$$

Our claim is then that suitable representatives may be chosen so that the corresponding spacetime fields obtained from eq. (2.21) are related by the Weyl double copy of eq. (2.3). To see why, note that one may choose Čech representatives in eq. (2.34) so as to obtain a gravitational Čech representative, where the corresponding fields obey the Weyl double copy. We may then convert each Čech representative to a Dolbeault representative using eqs. (2.28), (2.30). To simplify this procedure, we may choose a trivial partition of unity, such that $\eta = 0$. We may then carry out the Penrose transform of eq. (2.21) by integrating only over the patch U_0, and such that the functions appearing in eq. (3.10) are simply given by

$$F(z) = \check{f}(Z), \quad F^{(l)}_{\rm EM} = \check{f}^{(l)}_{\rm EM}. \tag{3.12}$$

Thus, the Dolbeault double copy formula of eq. (3.11) is ultimately a rewrite of the Čech formula, where the latter is converted to a (0,1) form by the action of the Dolbeault operator $\bar{\partial}$. Upon integrating the Penrose transform over U_0, a non-zero result survives provided the quantity in the square brackets in eq. (3.11) has a pole in the patch U_0. To clarify our rather abstract discussion, we now present some illustrative examples.

3.2.1 Elementary states

In the Čech language, *elementary states* are holomorphic twistor functions consisting of simple ratios of factors such as $(A_\alpha Z^\alpha)$, where A_α is a constant dual twistor. They were originally studied as potential twistor-space wavefunctions for scattering particles (see e.g. refs. [44, 48]), and have since been reinterpreted as giving rise to novel knotted solutions of gauge and gravity theory [16, 64–68]. In refs. [41, 42], elementary states were used to construct examples of Weyl double copies where the gravity solution had arbitrary Petrov type, albeit at linearised level in certain cases. Given such an elementary state, we may

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form a suitable Dolbeault representative as in eqs. (3.10), (3.12), and thus consider the following family of (0,1) forms:

$$f^{(a,b)}(Z)=\bar{\partial}\left(\frac{1}{(A_\alpha Z^\alpha)^{a+1}(B_\beta Z^\beta)^{b+1}}\right), \tag{3.13}$$

where $2n=a+b$ for a spin-n field, and the dual twistors $A_\beta=(A_A,A^{A'}),B_\alpha=(B_B,B^{B'})$. Upon restricting to the Riemann sphere X of a given spacetime point x, let us choose a cover (U_0,U_1) such that the pole in $A\cdot Z$ $(B\cdot Z)$ lies in the patch U_0 but not U_1 (U_1 but not U_0). In U_0, eq. (3.13) may then be written as

$$f^{(a,b)}(Z)\Big|_{U_0}=\frac{1}{(B_\beta Z^\beta)^{b+1}}\bar{\partial}\left(\frac{1}{(A_\alpha Z^\alpha)^{a+1}}\right). \tag{3.14}$$

The product between the twistor Z^β and the dual twistor A_β is given by

$$A_\beta Z^\beta|_X=ix^{AA'}A_A\pi_{A'}+A^{A'}\pi_{A'}=(ix^{AA'}A_A+A^{A'})\pi_{A'}\equiv\langle\mathcal{A}\pi\rangle, \tag{3.15}$$

where we have introduced the *Robinson field*

$$\mathcal{A}^{A'}=ix^{AA'}A_A+A^{A'}, \tag{3.16}$$

and used eq. (2.9). Similarly, one may write

$$B_\alpha Z^\alpha|_X\equiv\langle\mathcal{B}\pi\rangle,\quad \mathcal{B}^{A'}=ix^{AA'}B_A+B^{A'}, \tag{3.17}$$

such that eq. (3.14) becomes

$$f(Z)\Big|_{U_0}=\frac{1}{\langle\mathcal{B}\pi\rangle^{b+1}}\bar{\partial}\left(\frac{1}{\langle\mathcal{A}\pi\rangle^{a+1}}\right). \tag{3.18}$$

Under the Penrose transform, and using our trivial partition of unity, we have

$$\phi_{A'_1\dots A'_{2n}}(x)=\frac{1}{2\pi i}\int_{U_0}\langle\pi d\pi\rangle\wedge\pi_{A'_1}\dots\pi_{A'_{2n}}\frac{1}{\langle\mathcal{B}\pi\rangle^{b+1}}\bar{\partial}\left(\frac{1}{\langle\mathcal{A}\pi\rangle^{a+1}}\right), \tag{3.19}$$

which may be explicitly evaluated by making the following parametrisation for U_0:

$$\pi_{A'}(z)=\mathcal{A}_{A'}+z\mathcal{B}_{A'}. \tag{3.20}$$

Note that our requirement that U_0 does not contain the pole at $\langle\mathcal{B}\pi\rangle=0$ implies $\langle\mathcal{B}\mathcal{A}\rangle\neq 0$. Substituting eq. (3.20) into eq. (3.19) then yields

$$\begin{aligned}\phi_{A'_1\dots A'_{2n}}(x)&=\frac{1}{2\pi i}\frac{(-1)^a}{\langle\mathcal{B}\mathcal{A}\rangle^{a+b+1}}\int_{U_0}dz\,\pi_{A'_1}(z)\dots\pi_{A'_{2n}}(z)\bar{\partial}\left(\frac{1}{z^{a+1}}\right)\\&=\frac{1}{2\pi i}\frac{(-1)^a}{\langle\mathcal{B}\mathcal{A}\rangle^{a+b+1}}\oint_{\partial U_0}\frac{dz}{z^{a+1}}\,\pi_{A'_1}(z)\dots\pi_{A'_{2n}}(z),\end{aligned} \tag{3.21}$$

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where Stokes' theorem has been used in the second line. Taking the residue of the pole at $z = 0$, one finds

$$\begin{aligned}\phi_{A'_1\ldots A'_{2n}}(x) &= \frac{(-1)^a}{\langle\mathcal{B}\mathcal{A}\rangle^{a+b+1}}\frac{1}{a!}\lim_{z\to 0}\frac{d^a}{dz^a}\left[\pi_{A'_1}(z)\ldots\pi_{A'_{2n}}(z)\right]\\ &= \frac{(-1)^a}{\langle\mathcal{B}\mathcal{A}\rangle^{a+b+1}}\binom{a+b}{a}\mathcal{A}_{(A'_1}\ldots\mathcal{A}_{A'_b}\mathcal{B}_{A'_{b+1}}\ldots\mathcal{B}_{A'_{2n})}.\end{aligned} \tag{3.22}$$

Special cases of this family of spacetime fields indeed obey the Weyl double copy of eq. (2.3), as already discussed in refs. [41, 42]. Indeed, this will be the case whenever scalar, EM and gravity fields are chosen such that eqs. (3.10), (3.11) are obeyed for their Dolbeault representatives in twistor space, as may be easily verified.

3.2.2 General type D vacuum solutions

To go further than the previous section, we may consider the Dolbeault representatives

$$f_m(Z) = \bar{\partial}F_m(Z), \quad F_m(Z) = \check{f}_m(Z), \tag{3.23}$$

where $\check{f}_m(Z)$ consists of an inverse power of a quadratic form, shown explicitly in eq. (2.35). On the Riemann sphere X, let us parametrise U_0 by

$$\pi_{A'} = (1,\xi) \tag{3.24}$$

i.e. such that $\pi_{0'} \neq 0$. The quadratic form will assume the general form

$$Q_{\alpha\beta}Z^\alpha Z^\beta = N^{-1}(x)(\xi-\xi_0)(\xi-\xi_1) \tag{3.25}$$

where the normalisation factor $N^{-1}(x)$ inherits its spacetime dependence from the incidence relation, and the pole ξ_i is taken to lie exclusively in U_i, as shown in figure 1. The Penrose transform of eq. (2.21) for each m then evaluates to

$$\begin{aligned}\phi_{A'\ldots D'}(x) &= -\frac{N^m(x)}{2\pi i}\int_{U_0} d\xi\wedge\frac{(1,\xi)_{A'}\ldots(1,\xi)_{D'}}{(\xi-\xi_1)^m}\bar{\partial}\left(\frac{1}{(\xi-\xi_0)^m}\right)\\ &= -\frac{N(x)}{2\pi i}\oint_{\partial U_0} d\xi\frac{(1,\xi)_{A'}\ldots(1,\xi)_{D'}}{(\xi-\xi_0)^m(\xi-\xi_1)^m}.\end{aligned} \tag{3.26}$$

Taking the residue of the pole at $\xi = \xi_0$, one finds spacetime fields (for m=1, 2 and 3 respectively)

$$\phi(x) = -\frac{N(x)}{\xi_0-\xi_1}, \quad \phi_{A'B'}(x) = \frac{N^2(x)}{(\xi_0-\xi_1)^3}\alpha_{(A'}\beta_{B')}, \quad \phi_{A'B'C'D'} = -\frac{N^3(x)}{(\xi_0-\xi_1)^5}\alpha_{(A'}\alpha_{B'}\beta_{C'}\beta_{D')}, \tag{3.27}$$

where we have introduced the spinors

$$\alpha = (1,\xi_0), \quad \beta = (1,\xi_1). \tag{3.28}$$

The fields of eq. (3.27) clearly obey the Weyl double copy of eq. (2.3). Furthermore, as has been pointed out in ref. [69], use of a general dual twistor $Q_{\alpha\beta}$ allows one to span the

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complete space of vacuum type D solutions. We thus recover the results of refs. [41, 42] in the Čech approach, but this was in any case guaranteed by our general argument. The examples of alternative Petrov types presented in refs. [41, 42] will also generalise to the Dolbeault approach.

To summarise, in this section we have introduced a general procedure for obtaining twistor double copies in the Dolbeault formalism, which is essentially a rewriting of the twistor double copy in the Čech approach to twistor theory. Alas, this means that the Dolbeault approach also suffers from the same apparent ambiguities as the Čech double copy, which we discuss in the following section.

3.3 Cohomology and the Dolbeault double copy

As we reviewed in section 2, the twistor quantities appearing in the Penrose transforms of eq. (2.10) and (2.21) are representatives of cohomology classes. In eq. (2.10), these are Čech cohomology classes, and we saw that the product of eq. (2.34) is incompatible in general with the ability to redefine each representative according to the equivalence transformations of eq. (2.13). In the Dolbeault language, this freedom translates to the ability to add a $\bar{\partial}$-exact form to each representative, as expressed in eq. (2.23). One may then investigate whether the prescription of eq. (3.11) respects the ability to redefine each Dolbeault representative according to eq. (2.23), and it is straightforward to see that it does not.

To show this, recall that the functions appearing in eq. (3.11) are straightforwardly related to their corresponding Čech representatives by eq. (3.12). Redefining the latter according to eq. (2.13) amounts, from eq. (2.33) and our partition of unity with $\eta_0 = 0$, to adding $\bar{\partial} h_0(Z)$ to the corresponding Dolbeault representative. Here $h_0(Z)$ has poles only in U_1. In eq. (3.11), the conversion of the square brackets to a (0,1) form happens *after* the double copy product has already taken place. For our purposes, it is sufficient to consider equivalence transformations of the functions appearing in the numerator of eq. (3.11), such that one replaces eq. (3.11) with

$$\begin{aligned} f_{\text{grav.}}(Z) &\to \bar{\partial}\left[\frac{(F^{(1)}_{\text{EM}}(Z)+h^{(1)}_0(Z))(F^{(2)}_{\text{EM}}(Z)+h^{(2)}_0(Z))}{F(Z)}\right] \\ &= f_{\text{grav.}}(Z)+\bar{\partial}\left[\frac{h^{(1)}_0(Z)F^{(2)}_{\text{EM}}(Z)+h^{(2)}_0(Z)F^{(1)}_{\text{EM}}(Z)+h^{(1)}_0(Z)h^{(2)}_0(Z)}{F(Z)}\right]. \end{aligned} \tag{3.29}$$

We stress that, despite appearances, this replacement does not have the same form as the equivalence transformation of eq. (2.23): in the latter, the second term is defined over the whole Riemann sphere, and thus vanishes as a total derivative when integrated. By contrast, in eq. (3.29) the second term is defined only locally within the patch U_0, and thus gives a potentially non-zero result after integration. Indeed, Stoke's theorem implies that the contribution of the second term in eq. (3.29) to the Penrose transform integral is

$$\frac{1}{2\pi i}\oint_{\partial U_0}\langle\pi d\pi\rangle\pi_{A'}\pi_{B'}\pi_{C'}\pi_{D'}\left[\frac{h^{(1)}_0(Z)F^{(2)}_{\text{EM}}(Z)+h^{(2)}_0(Z)F^{(1)}_{\text{EM}}(Z)+h^{(1)}_0(Z)h^{(2)}_0(Z)}{F(Z)}\right].$$

The terms in the square brackets have poles in U_0 in general, and thus this integral will be non-zero. Thus, as in the Čech double copy of refs. [41, 42], redefining the scalar and EM fields *before* forming the twistor space product results in a different spacetime field in

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general. As in that case, this is not a problem when it comes to deriving the form and scope of the Weyl double copy, where one must simply find suitable representatives for each field in twistor space so as to reproduce the desired spacetime relationship. However, it would be nice if there were a systematic way to decide which representative should be chosen. We give one such method in the following section.

4 Dolbeault representatives in Euclidean signature

As we have seen, both the Čech and Dolbeault double copies involve forming apparently ambiguous products of twistor functions, where the non-linear nature of this relationship is at odds with the fact that these functions are actually representatives of cohomology classes. It is then natural to ponder whether there are any natural ways to choose "special" representatives of each class, so that the procedure can be made unambiguous. One such procedure has been presented recently in ref. [55], which focused on radiative spacetimes. Here we give a different procedure that works for all examples considered in this paper, provided one uses Euclidean signature in spacetime. This allows the use of known methods from complex analysis that can indeed pick out special representatives of Dolbeault cohomology classes. For reviews of twistor theory in Euclidean signature, see refs. [46, 50], the latter of which inspires our review of relevant material below.

The spinorial translation of a spacetime point x^a has been given in eq. (2.6). One may impose Euclidean signature by defining the hat-operation

$$\hat{x}^{AA'} = \frac{1}{\sqrt{2}} \begin{pmatrix} \bar{x}^0 - \bar{x}^3 & -\bar{x}^1 + i\bar{x}^2 \\ -\bar{x}^1 - i\bar{x}^2 & \bar{x}^0 + \bar{x}^3 \end{pmatrix}, \tag{4.1}$$

where the bar denotes complex conjugation. Demanding that $x^{AA'} = \hat{x}^{AA'}$ yields the constraints $x^0 \in \mathbb{R}$ and

$$x^l = iy^l, \quad y_l \in \mathbb{R}, \quad l \in \{1,2,3\},$$

such that

$$x_a x^a = (x^0)^2 + (y^1)^2 + (y^2)^2 + (y^3)^2 \tag{4.2}$$

as required. The hat operation in turn induces the following conjugation on 2-spinors:

$$\omega^A = (a,b) \to \hat{\omega}^A = (-\bar{b}, \bar{a}), \qquad \pi_{A'} = (c,d) \to \hat{\pi}_{A'} = (-\bar{d}, \bar{c}), \tag{4.3}$$

such that this operation acts on twistors as follows:

$$Z^\alpha = (\omega^A, \pi_{A'}) \to \hat{Z}^\alpha = (\hat{\omega}^A, \hat{\pi}_{A'}). \tag{4.4}$$

Using this notation, the Dolbeault operator discussed in section 2.3 takes the form

$$\bar{\partial} = d\hat{Z}^\alpha \frac{\partial}{\partial \hat{Z}^\alpha} = d\hat{\omega}^A \frac{\partial}{\partial \hat{\omega}^A} + d\hat{\pi}_{A'} \frac{\partial}{\partial \hat{\pi}_{A'}}. \tag{4.5}$$

As explained in e.g. ref. [50], it is convenient to rewrite this by introducing a particular basis for anti-holomorphic vectors and (0,1) forms on the appropriate projective twistor space (which we denote by $\mathbb{PT}(\mathbb{R}^4)$). That is, we may write the tangent space of anti-holomorphic fields as

$$T^{0,1}_{\mathbb{PT}(\mathbb{R}^4)} = \text{span}\left\{ \bar{\partial}_2 = \langle \pi\hat{\pi}\rangle \pi^{A'} \frac{\partial}{\partial \hat{\pi}^{A'}}, \quad \bar{\partial}_A = \pi^{A'} \frac{\partial}{\partial x^{AA'}} \right\}, \tag{4.6}$$

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and the space of (0,1) forms as

$$\Omega^{0,1}(\mathbb{PT}(\mathbb{R}^4)) = \text{span}\left\{ \bar{e}^2 = \frac{\langle\hat{\pi} d\hat{\pi}\rangle}{\langle\pi\hat{\pi}\rangle^2}, \quad \bar{e}^A = \frac{\hat{\pi}_{A'} dx^{AA'}}{\langle\pi\hat{\pi}\rangle} \right\}, \tag{4.7}$$

such that the Dolbeault operator of eq. (4.5) is recast as

$$\bar{\partial} = \bar{e}^2\bar{\partial}_2 + \bar{e}^A\bar{\partial}_A. \tag{4.8}$$

We may write the Penrose transform following eq. (2.21), where the twistor function that appears must be an element of the Dolbeault cohomology group $H^{0,1}_{\bar{\partial}}(\mathbb{PT}(\mathbb{R}^4), \mathcal{O}(-2n-2))$, and may be expanded in the above basis as

$$f = f_2\bar{e}^2 + f_A\bar{e}^A. \tag{4.9}$$

The convenience of this basis then becomes apparent. Upon restriction to the Riemann sphere X corresponding to fixed $x^{AA'}$, only the term involving $\bar{e}^2$ survives, and one thus finds

$$\phi_{A'B'\ldots C'}(x) = \frac{1}{2\pi i}\int_X \langle\pi d\pi\rangle \wedge \pi_{A'}\pi_{B'}\ldots\pi_{C'} f_2|_X \bar{e}^2. \tag{4.10}$$

Let us now return to the problem of how to pick special representatives of Dolbeault cohomology classes for given spacetime fields, such that the twistor double copy prescription of eq. (3.11) becomes more meaningful. First, let us recall that on a complex manifold M, one may define a positive definite inner product between two (p,q) forms α, $\beta \in \Omega^{p,q}(M)$ according to

$$(\alpha, \beta) \equiv \int_M \alpha \wedge *\bar{\beta}. \tag{4.11}$$

We may then define the *adjoint Dolbeault operators* $\partial^\dagger$, $\bar{\partial}^\dagger$ via

$$(\alpha, \partial\beta) = (\partial^\dagger\alpha, \beta), \quad (\alpha, \bar{\partial}\beta) = (\bar{\partial}^\dagger\alpha, \beta). \tag{4.12}$$

Then, the *Hodge decomposition theorem* says that, if M is compact, one may write an arbitrary (p,q) form $\omega \in \Omega^{p,q}(M)$ as

$$\omega = \bar{\partial}\alpha + \bar{\partial}^\dagger\beta + \gamma, \tag{4.13}$$

where $\alpha \in \Omega^{p,q-1}(M)$, $\beta \in \Omega^{p,q+1}(M)$ and $\gamma \in \Omega^{p,q}(M)$. The form γ is called the *harmonic part* of ω and satisfies $\bar{\partial}\omega = \bar{\partial}^\dagger\omega = 0$. We denote by $\text{Harm}^{p,q}_{\bar{\partial}}(M)$ the set of all such harmonic forms, and there is a known isomorphism between the set $\text{Harm}^{p,q}_{\bar{\partial}}(M)$ and the Dolbeault cohomology group $H^{0,1}_{\bar{\partial}}(M)$, which is straightforward to understand: elements of the latter are cohomology classes; each cohomology class has a unique harmonic representative, namely an element of the former. For a $\bar{\partial}$-closed form ($\bar{\partial}\omega = 0$), one finds $\bar{\partial}\bar{\partial}^\dagger\beta = 0$. Consideration of

$$\langle\beta, \bar{\partial}\bar{\partial}^\dagger\beta\rangle = \langle\bar{\partial}^\dagger\beta, \bar{\partial}^\dagger\beta\rangle \geq 0$$

then reveals $\bar{\partial}^\dagger\beta = 0$.

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In our present context, we are concerned with (0,1) forms on the Riemann sphere X corresponding to a given spacetime point x. As mentioned above, after restriction to X one has $\bar{\partial}|_X \equiv \bar{e}^2\bar{\partial}_2$. The above comments imply that a $\bar{\partial}$-closed (0,1) form can be written as

$$f|_X = \bar{e}^2\bar{\partial}_2 g + f_\Delta, \tag{4.14}$$

where g is a function and

$$f_\Delta \in \mathrm{Harm}^{0,1}_{\bar{\partial}}(\mathbb{CP}^1, \mathcal{O}(-2n-2))$$

is a $\bar{\partial}$-harmonic (0,1)-form on $X \simeq \mathbb{CP}^1$, where we have also labelled the homogeneity required for a spin-n field. From eq. (4.14), picking a Dolbeault representative for a given field to correspond to the purely harmonic part amounts to imposing the requirement[8]

$$\bar{\partial}_2^\dagger f|_X = 0, \tag{4.15}$$

and the Penrose transform then assumes the form

$$\phi_{A'B'\ldots C'}(x) = \frac{1}{2\pi i}\int_X \langle\pi d\pi\rangle \wedge \pi_{A'}\pi_{B'}\ldots\pi_{C'} f_\Delta(Z)|_X. \tag{4.16}$$

To address the relationship with the Weyl double copy, it is worthwhile pointing out that there is an explicit mechanism to generate harmonic Dolbeault representatives in twistor space [46]. Given a spacetime spinor field $\phi_{A'B'\ldots C'}(x)$, one may construct a twistor function on X as follows:

$$\Phi_\phi = \frac{1}{\langle\pi\hat{\pi}\rangle^{2n+1}}\phi_{A'B'\ldots C'}(x)\hat{\pi}^{A'}\hat{\pi}^{B'}\ldots\hat{\pi}^{C'}. \tag{4.17}$$

One may then construct the (0,1) form

$$f_\phi = \hat{\partial}\Phi_\phi = \frac{2n+1}{\langle\pi\hat{\pi}\rangle^{2n}}\phi_{A'B'\ldots C'}\hat{\pi}^{A'}\pi^{B'}\ldots\pi^{C'}\bar{e}^2, \tag{4.18}$$

where we have introduced the operator

$$\hat{\partial} \equiv d\hat{\pi}^{A'}\frac{\partial}{\partial\pi'_A}, \tag{4.19}$$

and used the basis of eq. (4.7). Equation (4.18) indeed turns out to be harmonic. Conversely, using f_ϕ in the Penrose transform of eq. (4.16) reveals $\phi_{A'B'\ldots C'}$ to be the spacetime field associated with the twistor one-form f_ϕ. To see this, one may write the Penrose transform out in full as

$$\begin{aligned}\phi_{A'\ldots C'}(x) &= \frac{1}{2\pi i}\int_X \frac{\langle\pi d\pi\rangle\wedge\langle\hat{\pi}d\hat{\pi}\rangle}{\langle\pi\hat{\pi}\rangle^2}\pi_{A'}\ldots\pi_{C'}\frac{2n+1}{\langle\pi\hat{\pi}\rangle^{2n}}\phi_{D'\ldots E'}(x)\hat{\pi}^{D'}\ldots\hat{\pi}^{E'}\\ &= \frac{2n+1}{2\pi i}\phi_{D'\ldots E'}(x)\int_X \omega\frac{\pi_{A'}\ldots\pi_{C'}\hat{\pi}^{D'}\ldots\hat{\pi}^{E'}}{\langle\pi\hat{\pi}\rangle^{2n}}\\ &= \phi_{D'\ldots E'}(x)\delta^{D'}_{(A'}\ldots\delta^{E'}_{C')},\end{aligned} \tag{4.20}$$

[8]Equation (4.15) also occurs when describing Yang-Mills gauge fields in twistor space (see e.g. refs. [46, 50, 63, 70–72] for this and related work), when it is referred to as the *harmonic gauge* condition. Our context here is more general, given that the twistor function being referred to may describe a spacetime field in either scalar, gauge or gravity theory.

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where

$$\omega = \frac{\langle \pi d\pi \rangle \wedge \langle \hat{\pi} d\hat{\pi} \rangle}{\langle \pi \hat{\pi} \rangle^2} \tag{4.21}$$

is the volume form on $\mathbb{CP}^1$ and we have used the identity [63]

$$\frac{1}{2\pi i} \int_X \omega \frac{\pi_{A'} \dots \pi_{C'} \hat{\pi}^{D'} \dots \hat{\pi}^{E'}}{\langle \pi \hat{\pi} \rangle^{2n}} = \frac{1}{2n+1} \delta^{D'}_{(A'} \dots \delta^{E'}_{C')}. \tag{4.22}$$

Thus, f_ϕ is the harmonic Dolbeault representative for the field $\phi_{A'B'\dots C'}$.

Consider now a pair of EM fields $\phi^{(1)}_{A'B'}$, $\phi^{(2)}_{A'B'}$, a scalar ϕ and a gravity field $\phi^G_{A'B'C'D'}$ that enter the Weyl double copy of eq. (2.3). From these, we may construct twistor functions $\Phi^{(i)}$, Φ and Φ^G according to eq. (4.17). It is then straightforward to verify that eq. (2.3) implies

$$\Phi^G = \frac{\Phi^{(1)}\,\Phi^{(2)}}{\Phi}. \tag{4.23}$$

For the gravity solution, we thus obtain a harmonic Dolbeault representative

$$f^G = \hat{\partial} \left(\frac{\Phi^{(1)}\,\Phi^{(2)}}{\Phi} \right). \tag{4.24}$$

We therefore see that the spacetime Weyl double copy implies a simple product structure in twistor space. Furthermore, eqs. (4.17), (4.18), (4.23) imply that all Dolbeault representatives occuring in the scalar, gauge and gravity theories are harmonic. This is perhaps the cleanest twistorial incarnation of the double copy that we have yet encountered. Each cohomology class corresponding to a given spacetime field has a unique and minimal representative, namely that (0,1) form which is harmonic. We have seen that it is possible to combine harmonic (0,1) forms in twistor space of homogeneity -2 and -4, in order to obtain a (0,1) form of homogeneity -6 that is also harmonic. This then fixes which gravity solution we are talking about upon performing the double copy.

Note that eq. (4.24) bears a resemblance to eq. (3.11), i.e. to our first Dolbeault double copy obtained as a simple rewriting of the Čech approach. However, there are important differences: the (0,1) form of eq. (3.11) is defined locally, in a single coordinate patch, whereas that of eq. (4.24) is defined globally, as well as involving a different differential operator. We may of course apply eq. (3.11) in Euclidean signature, and it is clear that the Dolbeault representatives defined by eqs. (3.11), (4.24) will not be the same in general. Nevertheless, for gravity fields which obey the Weyl double copy, the two differing representatives correspond to the same spacetime gravity field if the same scalar and gauge fields are chosen. To see this, note that we may Penrose transform each (0,1) form appearing in eqs. (3.10), (3.11) to obtain a spacetime field, which we may then plug into eqs. (4.17), (4.18) to generate harmonic representatives. If the gravity field obtained by Penrose transforming eq. (3.11) obeys the Weyl double copy, we may plug it into eq. (4.17) to give a function satisfying eq. (4.23), as noted above. Then eqs. (3.11) and (4.24) correspond to the same spacetime gravity field, namely to the Weyl double copy of the scalar and gauge fields.

The above discussion implies that there are at least two choices of Dolbeault representatives such that a product structure in twistor space leads to the Weyl double copy in

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position space: those defined separately by eqs. (3.11), (4.24). Unlike the case of harmonic representatives, however, eq. (3.11) does not furnish us with a clear interpretation of which representatives we must choose in order to make the double copy manifest, (i.e. the choice of Čech representatives appears ambiguous). As we have already mentioned above, a third choice has recently appeared in the literature [55], inspired by previous work [56]. The authors considered purely *radiative spacetimes*, namely those that are completely defined by characteristic data at future null infinity $\mathcal{I}^+$. Each point x in Minkowski spacetime is associated with a spherical surface S^2_x, corresponding to where the lightcone of null geodesics at x intersects $\mathcal{I}^+$. Each null geodesic at x, however, corresponds to a point in projective twistor space $\mathbb{PT}$, and the set of all such points forms the Riemann sphere X associated with x. There is then a well-defined map from the sphere S^2_x to X, such that characteristic data on S^2_x can be used to fix a Dolbeault representative on X whose Penrose transform leads to a given radiative spacetime field [56]. As argued in ref. [55], this may be done consistently in scalar, gauge and gravity theories such that a spacetime double copy is obtained. On the face of it, this procedure appears to be different to either of the procedures defined above for choosing Dolbeault representatives in the twistorial double copy, especially given that ref. [55] discussed radiative spacetimes only.

5 Discussion

In this paper, we have considered the classical double copy (specifically the Weyl double copy of ref. [15]), and how one may formulate this in twistor space. This was already considered in refs. [41, 42], which showed that a certain product of twistor functions can be used to derive the Weyl double copy in position space. However, this creates a puzzle, in that one cannot ordinarily multiply twistor functions together in the Penrose transform that turns twistor quantities into spacetime fields. The twistorial quantities associated with any spacetime field can be subjected to equivalence transformations that do not affect the latter, such that they are representatives of cohomology classes. This casts doubt on whether the double copy can be furnished with a genuinely twistorial interpretation, or whether the twistor approach acts merely as a useful book-keeping device, that can be used to efficiently generate instances of the classical double copy. Furthermore, refs. [41, 42] used the language of Čech cohomology groups, and if the twistor picture makes sense then it must also be possible to instead use the more widely used framework of Dolbeault cohomology.

We have herein presented two methods for writing the twistor double copy in the Dolbeault framework. In the first, one may use a well-known procedure for turning Čech representatives into Dolbeault counterparts, in order to recast the Weyl double copy in the Dolbeault language. The product structure that is inherent in the Čech approach then survives in the Dolbeault approach, for obvious reasons. Whilst it is encouraging that this works, it still provides no clue as to how one can somehow pick out special representatives of each cohomology class entering the double copy, so that the procedure becomes unambiguous. To remedy this, we presented a second Dolbeault double copy, which relies on known techniques for treating Euclidean signature spacetime fields [46, 53]. In this approach, the Weyl double copy in position space indeed picks out special cohomology

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class representatives in twistor space, namely those (0,1) forms that are harmonic. This is particularly appealing given that harmonic forms are uniquely defined for each cohomology class, and in some sense minimal. However, it follows from the first approach presented here that choosing harmonic forms is not the only way in which a product in twistor space leads to the same Weyl double copy in position space. Note that is natural to ask what the harmonic condition in the Dolbeault approach translates to in the Čech approach. However, we have been unable to find a simple answer to this question, which perhaps deserves further study. Furthermore, neither of the approaches presented here is obviously equivalent to the arguments of ref. [55], which used characteristic data at future null infinity to fix particular representatives corresponding to radiative spacetimes.

We can perhaps clarify matters by considering the original double copy for scattering amplitudes. In that case, it is only in certain *generalised gauges* (consisting of a choice of gauge and / or field redefinitions) that the double copy — which has a manifest product structure term-by-term in a graphical expansion of the scattering amplitude — is made manifest. It is possible to work in different generalised gauges, but at the expense of losing the simple product form of the double copy [73]. Something like this idea occurs elsewhere in the double copy literature, with a further example being the Kerr-Schild double copy of exact solutions of ref. [4]. In that case, the gravity solution must be in a particular coordinate system in order that its single copy can be taken, which is such that a simple product formula applies between the scalar and gauge fields entering the double copy. Kerr-Schild coordinates are sufficient for this purpose, although there may be other coordinate systems that accomplish this. An approach for copying spacetime fields in arbitrary gauges at linearised level has been developed in the convolutional approach of refs. [21–25, 28, 29], which makes clear the product form of the double copy is not manifest in general. Based on these remarks, we find it highly plausible that the double copy in twistor space can be given a general form, such that the product structure is made manifest only for particular cohomology representatives. That more than one product structure leads to the same position space double copy is not a problem, as there may be more than one choice of representatives that makes the product structure possible. However, it seems unlikely that the product structure would be true in general, given that it is so obviously incompatible with the equivalence transformations that define each cohomology class. This leaves the mystery of how one can choose cohomology representatives *a priori* so that the twistor space product applies. We regard our second Dolbeault double copy as particularly useful, given that it *uniquely* fixes a representative for each of the fields (scalar, gauge and gravity) entering the double copy.

Our above remarks are of course only speculative,[9] and other possibilities remain. For example, one may have different product structures that are possible in twistor space (corresponding to different ways of picking cohomology representatives), but such that these correspond to *different* double copies in position space. In such a case, one could formally define the notion of a (non-unique) *double copy* in twistor space by giving (i) a method for choosing cohomology representatives for scalar, gauge and gravity fields; (ii) a

[9]It should be noted that a direct analogy of having to fix (generalised) gauges for a double copy to be made manifest is curiously unnecessary for the Weyl double copy, which is gauge invariant in spacetime, in the linearised approximation.

product formula (or other map) for combining the chosen representatives. It may then turn out to be the case that only one of these definitions matches the original double copy for amplitudes, but the remaining double copies may nevertheless be useful for something. The relationship between the twistor double copy of refs. [41, 42] and the amplitudes double copy has been very recently addressed in ref. [74], which showed that classical spacetime fields can be obtained as a Penrose transform of scattering amplitudes which have been transformed from momentum to twistor space. The known double copy for amplitudes would then imply a twistor-space double copy, and exactly how this relates to the ideas of this paper would be very interesting to investigate further. Another possibility is that there is no genuine double copy in twistor space at all, and that the results obtained thus far in refs. [41, 42, 55] are coincidental, and do not generalise further. Our present paper gives us hope that this is far too pessimistic a conclusion, but also tells us that further investigation is necessary.

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