

Maximal transcendental weight contribution of scattering amplitudes

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ABSTRACT: Feynman integrals in quantum field theory evaluate to special functions and numbers that are usefully described by the notion of transcendental weight. In this paper, we propose a way of projecting a given dimensionally-regularised Feynman integral, for example contributing to a scattering amplitudes, onto its maximal weight part. The method uses insights into the singularity structure of space-time loop integrands, and is complementary to usual generalised unitarity approaches. We describe the method and give a proof-of-principle application to the two-loop scattering amplitudes $gg \rightarrow H$ in the heavy top-quark mass limit, which involves both planar and non-planar Feynman integrals. We also comment on further possible applications and discuss subtleties related to evanescent integrand terms.

KEYWORDS: Duality in Gauge Field Theories, Scattering Amplitudes

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1 Introduction

Observables computed in quantum field theory often depend on interesting transcendental numbers and special functions that arise from Feynman integrals. Understanding the structure of the perturbative expansion, and finding efficient ways of computing it are important challenges.

In practice, the integrated answers can be organised according to the complexity of special functions and numbers appearing in them. For example, in anomalous dimensions one encounters, among others, (multiple) zeta values $\zeta_n = \sum_{k \geq 1} 1/k^n$. This concept also extends naturally to functions written in terms of iterated integrals, see e.g. [1]. At a given loop order, there is a heuristic bound for the maximal value of n , also referred to as transcendental weight, or ‘transcendentality’ [2], namely $2L$ for L -loop integrals in four dimensions (cf. [3] and references therein). A remarkable conjecture states that for certain quantities, the maximal weight terms agree between the maximally supersymmetric Yang-Mills theory ($\mathcal{N} = 4$ sYM) and quantum chromodynamics (QCD) [2]. Maximal weight refers to the terms with $n = 2L$, where L is the loop order. This observation was instrumental in obtaining three-loop anomalous dimensions of twist-two anomalous dimensions in $\mathcal{N} = 4$ sYM [2] from the QCD results. More recently, it was used to predict the planar four-loop value of the planar cusp anomalous dimension in $\mathcal{N} = 4$ sYM [4].

Unfortunately, the one-to-one correspondence between maximal weight terms in $\mathcal{N} = 4$ sYM and QCD is restricted to certain observables only. In the original case, it can be

argued that it comes about thanks to special properties of the DGLAP evolution equation that governs twist-two anomalous dimensions [5], namely that the gluonic contributions give the relevant contributions in this case. For more general observables, this does not seem to be the case, and, therefore, the maximal weight terms can differ between $\mathcal{N} = 4$ sYM and QCD. Nevertheless, one can ask: is there a good way of obtaining the maximal weight contributions of an observable?

Insight into this question comes from studies in $\mathcal{N} = 4$ sYM. It was noticed, heuristically, that observables computed in this theory are expressed in terms of maximal weight functions (see e.g. the review [6] and references therein). It turns out that this property can be anticipated by looking at the types of Feynman integrals that appear in that theory [7, 8]. What was noticed is that integrals in this theory seem to have the special property that their integrands can be written as a ‘ $d \log$ ’ form. In particular, this means that they do not have any double pole on any (generalised unitarity) cuts. This observation was crucial, among other things, to bootstrap integrands of the theory, in combination with generalised unitarity or related techniques. In some cases, the $d \log$ property can be proven via loop level recursion relations [9].

Although initially observed in $\mathcal{N} = 4$ sYM, the idea of $d \log$ integrands and their connection to maximal weight functions turned out to be dramatically useful in the computation of generic Feynman integrals. In the state-of-the-art approach, one uses integration-by-parts (IBP) relations [10–12] to write scattering amplitudes in terms of a basis of Feynman integrals, called master integrals (MIs), where a major problem and bottleneck is in their computation. MIs are known to satisfy (a system of) differential equations, but the latter is typically complicated, and depends on the set of kinematic invariants and on the dimensional regulator ϵ , where e.g. $D = 4 - 2\epsilon$. Moreover, the form of the differential equations depends on the choice of integral basis. A key observation is that if a basis of maximal (and uniform) weight integrals is chosen, then the differential equations take a simple, canonical form [13]. In that form, the differential equations can be solved easily in a series expansion in the dimensional regulator ϵ . These ideas have led to numerous new calculations relevant for phenomenology.

In this paper, we build on the insights of loop integrands. We wish to understand directly at integrand level what terms lead to the ‘maximal weight’ part of the answer, avoiding the (often computationally heavy) IBP machinery. To do this, we propose a way of systematically extracting the ‘maximal weight’ contributions of Feynman integrands, without the need to evaluate the integrals.

The procedure we propose is the following. For a given scattering process, one first considers the denominator structure of the Feynman integrals. Firstly, for each denominator structure, one determines the most general set of $d \log$ integrands that can be written down. This is an algorithmic procedure, in principle. Secondly, one imagines a decomposition of the original integrand in terms of the $d \log$ basis just constructed, plus other terms. The latter have, by construction, at least a double pole, and are therefore expected to produce less than maximal weight terms only. The third step then consists in projecting the amplitude onto the $d \log$ basis only. We show how the coefficients of these integrals are obtained via a residue calculation (making use of algebraic geometry techniques). In this way, we define a maximal weight projection of the original integrand. The method we propose bears

several connections and was in part inspired by unitarity-based methods [14–29], and can be considered as a complementary tool.

In what situations do we expect the procedure we propose to apply? Let us discuss two possible caveats. The first one is related to the class of functions that can appear in Feynman integrals, and the second one has to do with dimensional regularisation [30, 31]. Let us discuss these two issues in turn.

Firstly, we expect the method to work as described in cases where the integrals evaluate to multiple poly-logarithms, for which the concept of transcendental weight is well understood (at least, to the satisfaction of physicists). For integrals evaluating to more general transcendental functions, such as elliptic poly-logarithms and beyond, modifications are likely to be needed, which however go beyond the scope of this work.

Secondly, in practice it is often desirable to compute Feynman integrals in dimensional regularisation, with e.g. $D = 4 - 2\epsilon$. Of course, it is much easier to perform the integrand analysis in integer dimensions, and often this is in fact sufficient (see e.g. a discussion in [32]). The distinction between integer and non-integer dimension can become relevant in cases that involve many independent momenta, because then certain Gram determinants vanish in e.g. $D = 4$ dimensions, but not in $D = 4 - 2\epsilon$. See e.g. [33], where a more refined $d \log$ analysis is performed, using Baikov representations [34–38]. For the purposes of the present paper, we wish to restrict ourselves to a four-dimensional integrand analysis. This means that we tacitly assume that maximal weight terms can be extracted from the knowledge of naive four-dimensional integrands. Since this may not be the case in all situations, we therefore study this important point in this paper. We show that if one is interested in knowing one-loop amplitudes up-to and including the finite part only, then considering naive four-dimensional integrands is sufficient. On the other hand, the same is not true for evanescent terms in the dimensional regulator — those terms may contribute maximal weight terms at five points, for example. The latter affect the analysis at higher loops, so that one can hope to extract the maximal weight terms there only when considering appropriately renormalised observables. We discuss explicit examples of these features for two-loop five-point scattering amplitudes.

In order to validate our method, we illustrate it for a two-loop scattering process in QCD, namely $H \rightarrow gg$ amplitudes at the two-loop order. In order to do so, we first generate all Feynman diagrams, which include non-planar cases. We then construct an explicit basis of $d \log$ integrands for all relevant Feynman diagram topologies. With this basis at our disposal, we project the general integrands onto our $d \log$ basis. This new expression is expected to give the maximal weight part of the QCD process. We test this hypothesis by explicitly comparing the original and the projected expression after an IBP reduction. We also plug in the explicit values of MIs to illustrate the effect of the projection at the level of the ϵ -expansion.

The outline of this paper is as follows. Section 2 contains a review of $d \log$ representations, and discusses the representations that will be needed for the purpose of this paper. In section 3, we describe in more detail the idea of the maximal weight projection, before applying it in section 4 to $H \rightarrow gg$ amplitudes at the two-loop order. Then, in section 5, we discuss in which cases subtleties with evanescent terms can be expected, and discuss ideas on avoiding them. Finally, we give our conclusion and outlook in section 6.

2 $d \log$ basis for $H \rightarrow gg$ integrands at two loops

In this section, we briefly review how to find Feynman integrands that admit a $d \log$ representation in four dimensions. Many examples of such Feynman integrals can be found in the literature, especially at one loop, see e.g. [39]. This provides very useful information for constructing the higher-loop $d \log$ representations that we will need for the $H \rightarrow gg$ that we wish to study. The reason is that in many cases, one can obtain a higher-loop representation using a loop-by-loop approach. The representations obtained in this way have the additional advantage that they tend to involve few terms, and often can be written in terms of single terms only.

In order to consistently apply the maximal weight projection method, it is important that we classify *all* $d \log$ integrands for a given Feynman integral topology, as otherwise we might miss relevant contributions. In order to address this issue, we can profit from an algorithmic method for identifying $d \log$ integrands [40, 41]. Ref. [41] also provides a public implementation, the MATHEMATICA package DLOG. This algorithm takes as input a given Feynman diagram topology (i.e., the propagator structure) and considers, as starting point, an ansatz for the most general numerator that can lead to $d \log$ integrands. The ansatz is finite because the numerator is restricted to obey certain power counting rules [41]. The algorithm then employs a convenient parametrisation of the four-dimensional loop momenta, similar to what we will use later in this section. The resulting integrand is a rational function in $4L$ integration variables, and depends on the external kinematics, and on the parameters of the ansatz. The algorithm then analyses, iteratively, residues that can be taken in each integration variable, until one arrives at the leading singularities, i.e. the maximal residues of the integrand. Whenever a double or higher pole is encountered, a condition is placed on the ansatz parameter to nullify the expression. In this way, a complete list of $d \log$ integrands for a given integral topology is obtained.

In this way, the algorithm [41] informs us of the total number of $d \log$ integrals that we need to consider, and it also suggests a basis for them. In what follows, we also need to know the explicit $d \log$ representation. This feature has not been implemented in the DLOG procedure for performance reasons, but could be incorporated easily. However, a typical $d \log$ representation would involve many terms, even for simple integrands, whereas we know that often, one can find a one-term $d \log$ representation, which is clearly advantageous. For this reason, we prefer, for the present purpose, to construct more compact $d \log$ forms by the loop-by-loop approach alluded to earlier.

Thus, we begin this section with the study of one-loop $d \log$ integrands that we will need later in our two-loop analysis. It turns out that the classes of integrands we need contain either Feynman or eikonal propagators.

2.1 One-loop $d \log$ integrands

Let us illustrate the procedure for constructing $d \log$ representations for one-loop Feynman integrands. We do so for convenience of readers, and in order to keep this paper self-contained. Of course, similar discussions can be found in the literature, see e.g. [39, 40].

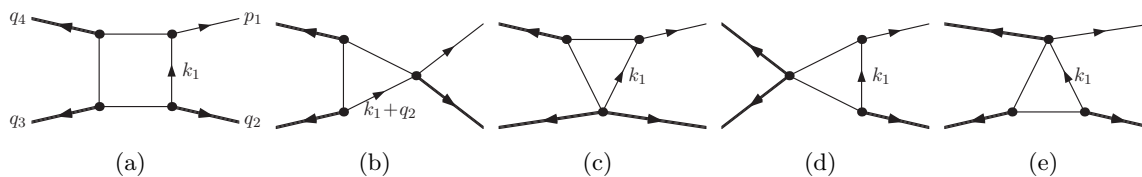


Figure 1. $d \log$ basis for integrand family (2.3).

As was mentioned earlier, we take the loop momentum to be four-dimensional for this analysis. It is useful to choose a convenient parametrisation in a four-dimensional basis $\mathcal{E} = \{e_1, e_2, e_3, e_4\}$, such that we can write

$$k_i^\mu = \alpha_{i,1} e_1^\mu + \alpha_{i,2} e_2^\mu + \alpha_{i,3} e_3^\mu + \alpha_{i,4} e_4^\mu. \quad (2.1)$$

If there are at least two independent massless momenta, say p_1 and p_2 . In this case, a particularly convenient choice of the basis vectors is as follows,

$$e_1^\mu = p_1^\mu, \quad e_2^\mu = p_2^\mu, \quad e_3^\mu = \langle 1 | \gamma^\mu | 2 \rangle / 2, \quad e_4^\mu = \langle 2 | \gamma^\mu | 1 \rangle / 2. \quad (2.2)$$

Here, angle and square brackets refer to the spinor-helicity formalism [42].

Let us apply this method to the four-point integrals shown in figure 1(a). It stands for an integrand the ones constructed only from Feynman propagators and consider the family of integrands generated from the following set of propagators,

$$\mathcal{I} = \frac{d^4 k_1 \mathcal{N}}{(k_1 - p_1)^2 k_1^2 (k_1 + q_2)^2 (k_1 + q_2 + q_3)^2}, \quad (2.3)$$

where $p_i^2 = 0$ and q_i can be considered as massless ($q_i = p_i$) or massive ($q_i^2 \neq 0$), depending on the case we wish to consider, with the momentum conservation $p_1 + q_2 + q_3 + q_4 = 0$. Therefore, the kinematic invariants can be cast as, $s = (p_1 + q_2)^2$ and $t = (q_2 + q_3)^2$.

$d \log$ forms for one-loop integrands with massless propagators. Given the integrand (2.3), and the parametrisation (2.1), we would now like to generate a $d \log$ basis for the family of integrands (2.3). To this end, we rely on the MATHEMATICA package DLOG, where, after using its built-in routines, we find a set of five one-loop $d \log$ integrands. This set of $d \log$ integrands, depicted in figure 1, consists of one scalar box, and four scalar triangles.

Having found the $d \log$ basis, let us now illustrate how one can write down an explicit $d \log$ representation. For example, consider the triangle integral depicted in figure 1(d), where for simplicity we take $q_2 = p_2$ to be massless,

$$\mathcal{I} = \frac{d^4 k_1}{(k_1 - p_1)^2 k_1^2 (k_1 + p_2)^2}. \quad (2.4)$$

A short calculation shows that, upon plugging in the parametrisations (2.1), (2.2), and taking the Jacobian factor into account, one obtains,

$$\mathcal{I} = \frac{d\alpha_1 d\alpha_2 d\alpha_3 d\alpha_4}{s((\alpha_1 - 1)\alpha_2 - \alpha_3\alpha_4)(\alpha_1\alpha_2 - \alpha_3\alpha_4)(\alpha_1(\alpha_2 + 1) - \alpha_3\alpha_4)}, \quad (2.5)$$

where we have dropped an irrelevant (because kinematic-independent) overall factor. Notice that, to simplify the notation in this section, we define the loop components of the loop momentum parametrisation (2.1) as $\alpha_{1,i} = \alpha_i$.

The integrand (2.4) contains three denominator factors, and it appears natural to change variables $\alpha_{1,2,3} \rightarrow \tau_{1,2,3}$ according to

$$\begin{aligned}\tau_1 &= ((\alpha_1 - 1)\alpha_2 - \alpha_3\alpha_4) s, \\ \tau_2 &= (\alpha_1\alpha_2 - \alpha_3\alpha_4) s, \\ \tau_3 &= (\alpha_1(\alpha_2 + 1) - \alpha_3\alpha_4) s.\end{aligned}\tag{2.6}$$

Remarkably, upon doing so, one finds the following simple expression,

$$\mathcal{I} = \frac{1}{s} d \log \tau_1 d \log \tau_2 d \log \tau_3 d \log \alpha_4.\tag{2.7}$$

At this stage we emphasise that it is remarkable that eq. (2.7) consists of a single term only. We will in fact find similar forms for the other integrals considered in this paper as well. To appreciate that this is by no means guaranteed, let us write an equivalent expression in the α_i variables,

$$\begin{aligned}\omega^{1m\text{-tri}}(k_1; p_1, p_2) &= -d \log(\alpha_4) d \log(\alpha_2) d \log(\alpha_3) d \log(\alpha_1\alpha_2 - \alpha_3\alpha_4) \\ &\quad - d \log(\alpha_4) d \log(\alpha_2) d \log(\alpha_2^2 + \alpha_2 + \alpha_3\alpha_4) d \log[\alpha_1(\alpha_2 + 1) - \alpha_3\alpha_4] \\ &\quad + d \log(\alpha_4) d \log(\alpha_2) d \log(\alpha_2^2 + \alpha_2 + \alpha_3\alpha_4) d \log(\alpha_1\alpha_2 - \alpha_2 - \alpha_3\alpha_4) \\ &\quad + d \log(\alpha_4) d \log(\alpha_2) d \log(\alpha_3) d \log[\alpha_1(\alpha_2 + 1) - \alpha_3\alpha_4].\end{aligned}\tag{2.8}$$

Having made this comment, let us return to the simpler representation of eq. (2.7). Taking into account the explicit definitions for τ_i and α_4 in terms of denominators and scalar products, we note that this integrand can be re-expressed as follows,

$$\mathcal{I} = \frac{1}{s} d \log(k_1 - p_1)^2 d \log k_1^2 d \log(k_1 + p_2)^2 d \log(2 k_1 \cdot e_3).\tag{2.9}$$

In this form it is clear that the overall factor $1/s$ corresponds to the four-dimensional leading singularity of the scalar triangle \mathcal{I} . Hence, accounting for the latter, we can define the $d \log$ integrand,

$$\mathcal{I} \left(\text{triangle diagram} \right) \equiv \omega^{1m\text{-tri}}(k_1; p_1, p_2),\tag{2.10}$$

with,

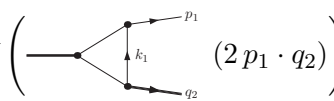
$$\omega^{1m\text{-tri}}(k_1; p_1, p_2) = d \log(k_1 - p_1)^2 d \log k_1^2 d \log(k_1 + p_2)^2 d \log(2 k_1 \cdot e_3).\tag{2.11}$$

In eq. (2.10), the l.h.s., $\mathcal{I}(\dots)$ stands for the integrand of the loop topology, whose Feynman propagators can be read off from the diagram. Thus, we have found a $d \log$ form $\omega^{1m\text{-tri}}(k_1; p_1, p_2)$ for the scalar “one-mass triangle” integral. In the pictorial representation


of integrands, here and in the following, external thin and thick lines represent, respectively, massless (on-shell) and massive (or off-shell) momenta.

As illustrated for the scalar triangle in eq. (2.4), finding an explicit $d \log$ representation for a given integrand family, say (2.3), is a procedure at integrand level. Relations that arise as consequence of dimensional regularisation at integral level are not taken into account at this stage. For instance, in the case where all external momenta are massless, we have five $d \log$ basis integrands, but upon using IBP relations, only three of them are independent. In fact, it is important for the consistency of our method that a complete integrand basis is constructed.

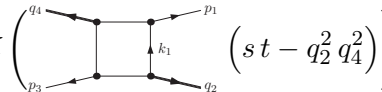
Let us return to the construction of one-loop $d \log$ representations. Following the same strategy adopted for the one-loop scalar triangle, we find algebraic expressions for the following $d \log$ forms (which correspond to different kinematic configurations of integrals figure 1). For the two-mass triangle, we have

$$\begin{aligned} \omega^{2m\text{-tri}}(k_1; p_1, q_2) &\equiv \mathcal{I} \left(\text{Diagram} \right) & (2.12) \\ &= d \log (k_1 - p_1)^2 d \log k_1^2 d \log (k_1 + q_2)^2 d \log (2k_1 \cdot e_3), \end{aligned}$$


For the “two-mass-hard” and “two-mass-easy” boxes, we have [39],

$$\begin{aligned} \omega^{2mh\text{-box}}(k_1; p_1, p_2, q_3) & \\ &\equiv \mathcal{I} \left(\text{Diagram} \right) & (2.13) \\ &= \mp d \log \frac{(k_1 - p_1)^2}{(k_1 - k_1^\pm)^2} d \log \frac{k_1^2}{(k_1 - k_1^\pm)^2} d \log \frac{(k_1 + p_2)^2}{(k_1 - k_1^\pm)^2} d \log \frac{(k_1 + p_2 + q_3)^2}{(k_1 - k_1^\pm)^2}, \end{aligned}$$


and

$$\begin{aligned} \omega^{2me\text{-box}}(k_1; p_1, q_2, p_3) & \\ &\equiv \mathcal{I} \left(\text{Diagram} \right) & (2.14) \\ &= \mp d \log \frac{(k_1 - p_1)^2}{(k_1 - k_1^\pm)^2} d \log \frac{k_1^2}{(k_1 - k_1^\pm)^2} d \log \frac{(k_1 + q_2)^2}{(k_1 - k_1^\pm)^2} d \log \frac{(k_1 + q_2 + p_3)^2}{(k_1 - k_1^\pm)^2}, \end{aligned}$$


respectively. Here k^\pm are the two solutions of the maximal cut conditions,

$$\left(k_1^\pm - p_1\right)^2 = \left(k_1^\pm\right)^2 = \left(k_1^\pm + q_2\right)^2 = \left(k_1^\pm + q_2 + p_3\right)^2 = 0. \quad (2.15)$$

Let us emphasise that the additional prefactors that appears in the definition of integrands correspond to four-dimensional leading singularities of the scalar Feynman integrand. Also, let us note that although the $d \log$ form of the four-point Feynman integrands involves the auxiliary factor $(k_1 - k_1^\pm)^2$, the integrand does not have a residue at $k_1 = k_1^\pm$. We refer interested readers to [39] for more details.

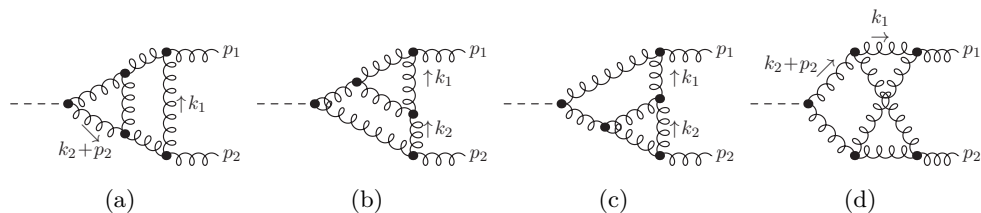


Figure 2. Representative two-loop (parent) topologies with seven propagators present in the scattering amplitude $H \rightarrow gg$.

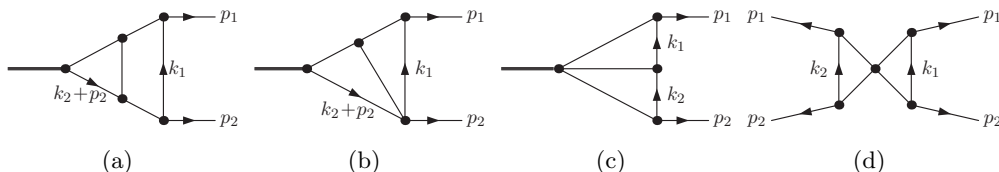


Figure 3. Representative two-loop planar triangles that admit a $d \log$ representation.

derivation of planar and non-planar $d \log$ three-point Feynman integrands at two loops and also discuss representative four-point $d \log$ forms.

We are interested in finding a $d \log$ basis for the process $H \rightarrow gg$ at two loops. This involves the generation of planar and non-planar Feynman diagrams, as shall be illustrated in detail in section 4. The complete set of Feynman diagrams contributing to this scattering amplitude can be grouped into four ‘parent’ topologies with six propagators, as depicted in figure 2. Let us discuss them in turn.

In order to study all possible planar three-point Feynman integrands at two loops, which correspond to figures 2(a), 2(b), 2(c), it is convenient to use the following notation,

$$\mathcal{I} = \frac{d^4 k_1 d^4 k_2 \mathcal{N}}{k_1^2 k_2^2 (k_1 - k_2)^2 (k_1 - p_1)^2 (k_2 - p_1)^2 (k_1 + p_2)^2 (k_2 + p_2)^2}, \quad (2.21)$$

to account for all possible propagators (see direction flow of the loop momenta in figure 2).

Different choices of \mathcal{N} correspond to the integral families mentioned previously. The advantage of the formulation in terms of eq. (2.21) is that they can all be described simultaneously.

In order to determine all numerator factors \mathcal{N} that lead to $d \log$ forms, we employ the DLOG program of [41]. This readily gives us the following list of nine $d \log$ forms:

- two versions of figure 3(a), obtained by flipping $k_1 \leftrightarrow k_2$,
- four versions of figure 3(b), obtained by flipping $k_1 \leftrightarrow k_2$, and by flipping the graph along the vertical axis,
- two versions of figure 3(c), obtained by flipping $k_1 \leftrightarrow k_2$,
- the squared triangle integrand of figure 3(d).

Recall that we operate at integrand level, and hence, relations at integral level, e.g. sector symmetries and IBP identities, are not considered. Taking the latter into account would lead to four independent Feynman master integrals, which correspond to the ones depicted in figure 3, but we work with the basis of nine $d \log$ integrands listed above.

Let us now turn to the computation of algebraic expressions for the $d \log$ integrands of figure 3. Thus, we begin the discussion with the ladder three-point scalar integrand depicted in figure 3(a). We notice that, due to the way how this integrand is constructed, one can decompose it in terms of two one-loop integrands. The first one corresponds to a scalar box with two off-shell external momenta, whereas the second one, that is generated from the former, can be understood as a one-mass scalar triangle,

$$\begin{aligned}
 \omega_{p,(a)}^{(2)} &= \mathcal{I} \left(\text{Diagram 1} \quad s^2 \right) \\
 &= \mathcal{I} \left(\text{Diagram 2} \quad k_2^2 s \right) \times \mathcal{I} \left(\text{Diagram 3} \quad s \right). \tag{2.22}
 \end{aligned}$$

Notice that in l.h.s. of eq. (2.22) we consider the integrand of figure 3(a) with its appropriate four-dimensional leading singularity, which was provided by DLOG and is needed to have a complete $d \log$ form for this integrand. Then, to follow a loop-by-loop approach, we start by considering the subset of Feynman propagators, in the two-loop integrand, that depends on k_1 . These propagators generate a one-loop scalar box with two off-shell, $-k_2 - p_2$ and $k_2 - p_1$, and two on-shell, p_1 and p_2 , momenta, whose $d \log$ form, after taking into account the prefactor that comes from the leading singularity of the integrand, was considered in eq. (2.13). Notably, this prefactor, $s k_2^2$, allows to analyse the part of integrand that only contains k_2 but not k_1 , in such a way that a one-mass scalar triangle, together with its $d \log$ representation, is straightforwardly generated.

Hence, the two-loop $d \log$ integrand $\omega_{p,(a)}^{(2)}$ of eq. (2.22) is expressed as a product of two one-loop $d \log$ integrands,

$$\omega_{p,(a)}^{(2)} = \omega^{2\text{mh-box}}(k_1; p_2, p_1, k_2 - p_1) \omega^{1\text{m-tri}}(k_2; p_1, p_2), \tag{2.23}$$

whose explicit algebraic expressions were given in eqs. (2.11) and (2.13).

A similar analysis and decomposition of the remaining two-loop $d \log$ integrands of figure 3 can be carried out with the help of the building blocks discussed in the previous section. We find,

$$\begin{aligned}
 \omega_{p,(b)}^{(2)} &= \mathcal{I} \left(\text{Diagram 4} \quad s \right) \\
 &= \mathcal{I} \left(\text{Diagram 5} \quad (-2p_1 \cdot k_2) \right) \times \mathcal{I} \left(\text{Diagram 6} \quad s \right), \\
 &= \omega^{2\text{m-tri}}(k_1; p_1, -k_2) \omega_{\text{Eikonal } 1}^{1\text{m-tri}}(k_2; p_1, p_2), \tag{2.24}
 \end{aligned}$$

$$\begin{aligned}
 \omega_{p,(c)}^{(2)} &= \mathcal{I} \left(\text{Diagram 1} \right) \\
 &= \mathcal{I} \left(\text{Diagram 2} \right) \times \mathcal{I} \left(\text{Diagram 3} \right), \\
 &= \omega^{2m\text{-tri}}(k_1; p_1, -k_2) \omega_{\text{Eikonal } 2}^{1m\text{-tri}}(k_2; p_1, p_2), \tag{2.25}
 \end{aligned}$$

$$\begin{aligned}
 \omega_{p,(d)}^{(2)} &= \mathcal{I} \left(\text{Diagram 4} \right) \\
 &= \mathcal{I} \left(\text{Diagram 5} \right) \times \mathcal{I} \left(\text{Diagram 6} \right), \\
 &= \omega^{1m\text{-tri}}(k_1; p_1, p_2) \omega^{1m\text{-tri}}(k_2; p_1, p_2). \tag{2.26}
 \end{aligned}$$

The remaining d log forms for the full set of nine integrands are simply given by relabelling of loop momenta, as discussed above. Hence, for the purpose of illustrating the discussion in section 4, we express $\omega'_{p,(i)}$ as reflection of $\omega_{p,(i)}$, namely, $\omega'_{p,(i)} = \omega_{p,(i)}|_{k_1 \leftrightarrow k_2}$.

Let us continue our analysis with d log integrands that arise in non-planar sector of the three-point two-loop Feynman integrands. This Feynman diagram has six propagators (see figure 3(d)), and therefore we look for d log integrands of the form,

$$\mathcal{I} = \frac{d^4 k_1 d^4 k_2 \mathcal{N}}{k_1^2 (k_1 - p_1)^2 (k_2 - p_1)^2 (k_1 - k_2)^2 (k_1 - k_2 - p_2)^2 (k_2 + p_2)^2}, \tag{2.27}$$

for some (in general, loop-momentum-dependent) numerator \mathcal{N} . We introduce the quantity k_2^2 , in addition to the denominator factors in eq. (2.27), to express this numerator in terms of irreducible scalar products.

Thus, by making use of the built-in routines of DLOG, we find eight d log integrands:

- two versions of figure 3(b), obtained by shrinking either the first or second propagator of (2.27),
- two versions of the reflection figure 3(b), obtained by shrinking either the fourth or fifth propagator of (2.27),
- two versions of figure 3(c), obtained by shrinking either the third or sixth propagator of (2.27),
- one version of the non-planar triangle of figure 4(a),
- one version of figure 4(b), which corresponds to the non-planar triangle with the irreducible scalar product k_2^2 .

Notice that in this d log basis, obtained from the non-planar sector through the integrand family (2.27), six of these d log integrands were already considered in the planar sector. This pattern was expected due to the isomorphisms between sub-topologies coming from planar and non-planar sectors, and leads to identities after integration. However, for our

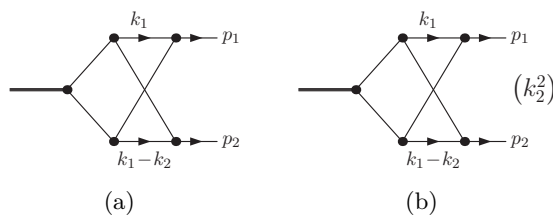


Figure 4. Two-loop non-planar triangles that admit a $d\log$ form. The factor (k_2^2) in the $d\log$ integrand (b) corresponds to an irreducible scalar product in the numerator.

integrand analysis of this non-planar class of diagrams, it is important that we write down the planar integrals from the sub-sectors in the notation of eq. (2.27) (for more details see section 3).

In the following, to simplify the notation for non-planar $d\log$ integrands by keeping in mind the isomorphism between different sub-topologies, we introduce the shorthand notation $\omega_{\text{np},(i);j}$ in the $d\log$ forms to express that the loop topology of this integrand is isomorphic to the graph class (i) in the planar sector (see figure 3) when the j th propagator in the integrand family (2.27) is removed.

Since the $d\log$ expressions for the planar topologies can be obtained from the formulas given above by a simple change of variables, let us focus on the genuine non-planar integrals of figure 4.

We have the loop-by-loop decomposition,

$$\begin{aligned}
 \omega_{\text{np},(e)}^{(2)} &= \mathcal{I} \left(\text{Diagram (a)} \quad s^2 \right) \\
 &= \mathcal{I} \left(\text{Diagram (b)} \quad (2e_3 \cdot k_2)(2e_4 \cdot k_2) \right) \times \mathcal{I} \left(\text{Diagram (c)} \quad s \right) \\
 &= \omega^{2\text{me-box}}(k; p_1, -k_2, p_2) \omega_{\text{Eikonal } 1}^{\text{Box}}(k_2; p_1, p_2) , \tag{2.28}
 \end{aligned}$$

and

$$\begin{aligned}
 \omega_{\text{np},(f)}^{(2)} &= \mathcal{I} \left(\text{Diagram (a)} \quad s k_2^2 \right) \\
 &= \mathcal{I} \left(\text{Diagram (b)} \quad (2e_3 \cdot k_2)(2e_4 \cdot k_2) \right) \times \mathcal{I} \left(\text{Diagram (c)} \quad s k_2^2 \right) \\
 &= \omega^{2\text{me-box}}(k; p_1, -k_2, p_2) \omega_{\text{Eikonal } 2}^{\text{Box}}(k_2; p_1, p_2) . \tag{2.29}
 \end{aligned}$$

To summarise, in this section, we studied the algebraic structure of $d\log$ integrands at one and two loops, focusing on integrals needed for the $gg \rightarrow H$ process, which includes planar and non-planar Feynman diagrams. We found a complete list of $d\log$ integrands with the help of the algorithm DLOG, and then provided, for each of the basis elements,

expressions in a $d \log$ form. For studies at integrand level, a loop-by-loop approach turned out to be sufficient, and very useful. This approach allowed us to easily find $d \log$ forms with a single term only.

The main results of this section are the two-loop $d \log$ representations given in eqs. (2.23), (2.24), (2.25), (2.26), (2.28), (2.29). The complete list of expressions for the $d \log$ planar and non-planar integrand basis (9 and 8 integrands, respectively) is obtained from these by symmetry/reparametrisation, as explained above. These expressions are the main input for the maximal weight projection method that is discussed in section 3.

3 Maximal weight projection

3.1 Description of the method

As discussed in the introduction, we are interested in the following general question: can we extract the maximal transcendental weight piece of a quantity in perturbative quantum field theory, such as scattering amplitudes or correlation functions? Our starting point is an expression of the quantity of interest in terms of Feynman diagrams. The idea is to analyse the expression for the Feynman integrand, i.e. prior to an (often computationally-heavy) analysis of relations between integrated functions.

Recall that we expect the maximal weight of an L -loop integral in four dimensions to be $2L$ [3]. Moreover, the concept of weight can be applied to the Laurent series of integrals in dimensional regularisation, with $D = 4 - 2\epsilon$. In this context, one assigns weight -1 to ϵ [13]. In this way, a uniform and maximal weight function at L loops is one where each term in the ϵ expansion has weight $2L$.

Based on experience from $\mathcal{N} = 4$ sYM and related insights into differential equations that Feynman integrals satisfy, we expect that one can predict which terms in the integrand give maximal weight contributions, and which do not. In the case of Feynman integrals evaluating to multiple poly-logarithms, which is our focus in this paper, the idea is that Feynman integrals whose integrands can be written in $d \log$ form contribute to the maximal weight piece. Therefore, the first step in our method is to obtain a basis of such $d \log$ integrands for the quantity of interest. Let us denote the set of m $d \log$ integrands at L by

$$d \log \text{ integrand basis : } \{ \mathcal{I}_i^{(L)} \}, \quad i \in \{1, \dots, m\}. \quad (3.1)$$

In addition to knowing what the basis elements are, we also need to know an explicit expression of their integrand in $d \log$ form, i.e. an expression of the form

$$\mathcal{I}_i^{(L)} = \sum_j b_{ij} \prod_{k=1}^{4L} d \log \alpha_{ijk}. \quad (3.2)$$

Here, we have assumed that we consider integrals in four dimensions, but one could also consider other integer dimensions.

To give an explicit example of (3.2), consider the one-loop triangle integral from eq. (2.11). In this case, $L = 1$, and the sum over j in eq. (3.2) involves a single term only. This fact is a special property that can sometimes be achieved [39], and is the case for all

integrals considered in this paper. We find it likely that this property is closely related to novel geometric interpretations of scattering amplitudes and Feynman integrals, see e.g. [43–45]. Having single-term $d\log$ representations will be useful in the following, but is not essential to our method.

One procedure for obtaining a basis (3.1), and the explicit representations (3.2) was discussed in section 2, and a basis for planar and non-planar $d\log$ integrands relevant for the process $gg \rightarrow H$ at two loops was given. Remarkably, in the representations we found, the sum over j in eq. (3.2) has a single term only.

Let us assume that a basis and explicit expressions of $d\log$ integrands have been found. Further, let $\mathcal{A}^{(L)}$ be the quantity we are interested in, for which we have a representation in terms of a loop integrand $\omega^{(L)}$,

$$\mathcal{A}^{(L)} = \int \omega^{(L)}. \tag{3.3}$$

As mentioned in the introduction, we will consider ω in four dimensions, even if \mathcal{A} is computed in $D = 4 - 2\epsilon$ dimensions. Section 5 will address in more detail questions related to evanescent terms.¹

The next step is then to imagine a decomposition of $\omega^{(L)}$ in terms of our $d\log$ basis, plus other terms,

$$\omega^{(L)} = \sum_{i=1}^m c_i \mathcal{I}_i^{(L)} + \dots \tag{3.4}$$

The ellipses correspond to terms with at least a double pole. These are the terms that we wish to drop when the projection onto the $d\log$ basis is performed. The reason is that we expect such terms to contribute less-than-maximal weight terms only. In other words, given eq. (3.4) we define the projection operator \mathcal{P} at integrand level as follows,

$$\mathcal{P}(\omega^{(L)}) = \sum_{i=1}^m c_i \mathcal{I}_i^{(L)}. \tag{3.5}$$

Let us remark that in this decomposition no relations at integral level are needed. This is essential for the analysis we carry out throughout this paper. We stress that the expectation that the projection (3.5) captures all maximal weight terms of the original expression (up-to the caveats mentioned earlier) is a conjecture.

Taking into account eq. (3.2), the integrand of the amplitude $\omega^{(L)}$ must have the form

$$\mathcal{P}(\omega^{(L)}) = \sum_{i=1}^n \tilde{c}_i \prod_{j=1}^{4L} d\log(\tau_{i,j}), \tag{3.6}$$

with $\tau_{i,j}$ being some (often rational) functions depending on the loop components α of parametrisation (2.1).

¹A related comment is that we do not expect the regularisation scheme to play a role when extracting the maximal transcendental weight of a given scattering amplitude.

Let us note that in r.h.s. of eq. (3.6) in general, $n \geq m$. For instance, if we take ω to be the integrand of the one-mass triangle, then we have $n = 1$, but according to eq. (2.8) we may have $m = 4$. This is related to the fact that $d \log$ representations are by no means unique.

Our goal is to determine the m coefficients c_i . As mentioned above, this is a problem that is very close to what is usually done when using generalised unitarity methods, see e.g. [46, 47]. In fact, the set of n coefficients \tilde{c}_i contains the necessary information we are looking for. In practice, if we can compute a subset of \tilde{c}_i whose map to the c_i has rank m , then we can invert the system.

In order to extract the coefficients \tilde{c}_i in eq. (3.6), we make use of the Cauchy residue theorem, similar to the way how it is carried out in generalised unitarity. Thus, the locus where residues are extracted is determined once and for all by the set of $d \log$ forms present in the decomposition of scattering amplitude. Then, we can naively extract \tilde{c}_i as,

$$\tilde{c}_i = \oint_{\tau_{i,1}=0} \dots \oint_{\tau_{i,4L}=0} \omega^{(L)}. \tag{3.7}$$

Let us explain what we mean by ‘naively’. The way residues are extracted in uni-variate rational functions is unambiguous. However, for multi-variate rational functions, the extraction of residues in a variable-by-variable approach, is more subtle. In particular, residues in $\tau_{i,j} = 0$ do not commute.

This subtlety can, in principle, be overcome by employing the Grothendieck residue [48]. However, for the calculations carried out in this paper, we make use of methods based on Gröbner basis and polynomial division.² Indeed, the computation of multi-variate residues [48, 50], has been applied to the calculation of multi-loop scattering amplitudes in the context of generalised unitarity [22, 51–55]. Let us also comment that the study of multi-variate polynomial division has been employed to perform reductions of multi-loop scattering amplitudes at integrand level [17–24], inspired by the well-known Ossola-Papadopoulos-Pittau (OPP) decomposition [16]. In the present work, we rely on the partial-fraction decomposition introduced by Leinartas [56, 57], which allows for an unambiguous extraction of certain coefficients of poles. Improved versions of the Leinartas’ decomposition have been recently provided in refs. [58–62], where an extensive use of multivariate polynomial division, or, said differently, reduction of polynomials, through Gröbner basis is carried out. In this paper, we use of the MATHEMATICA package MULTIVARIATEAPART provided together with ref. [62].

Before applying the method to a quantum field theory calculation, as an invitation let us first provide a toy example.

3.2 Beta function toy example: an invitation

In order to illustrate the decomposition of a given integrand in terms of $d \log$ forms as well as the extraction of the maximal weight contribution of a given integrand, let us start with the one-dimensional example of the beta function. In view of its compact structure, one

²For a pedagogical and detailed discussion of techniques to perform multi-variate polynomial division, we refer the reader to ref. [49].

can easily cast interesting properties and the main objective of this manuscript that in the following sections is extended to scattering amplitudes.

Let us define the one-form

$$\omega_{a,b} = dz z^{-1-b+\epsilon} (1-z)^{-1-a+\epsilon}, \tag{3.8}$$

where ϵ is the analog of the dimensional regularisation parameter. With this, we can define,

$$I_{a,b} = \int_0^1 \omega_{a,b} = \frac{\Gamma(-a+\epsilon)\Gamma(-b+\epsilon)}{\Gamma(-a-b+2\epsilon)}. \tag{3.9}$$

Thanks to the presence of ϵ , the integral is well-defined for integer values of a, b .

In analogy with the dimensional regularisation case, let us now study d log forms for $\epsilon = 0$. We find that there are two forms, namely,

$$\{d \log z, -d \log(1-z)\}. \tag{3.10}$$

These $d \log$ forms correspond to the integrands $\omega_{0,-1}$ and $\omega_{-1,0}$.

Due to a $z \leftrightarrow 1-z$ symmetry, we have that, after integration, $I_{a,b} = I_{b,a}$, so that in particular $I_{-1,0} = I_{0,-1}$. Inspecting the result (3.9), and expanding in ϵ , we find,

$$I_{0,-1} = I_{-1,0} = \frac{1}{\epsilon} - \frac{\pi^2}{6}\epsilon + 2\zeta_3\epsilon^2 + \mathcal{O}(\epsilon^3). \tag{3.11}$$

We see that all terms in the expansion in ϵ of $I_{-1,0}$ have uniform transcendental weight 1, if one assigns weight -1 to ϵ [13].

Let us now consider a generic integrand that can be expressed in terms of $\omega_{a,b}$. We would like to use our projection method to extract the leading weight contribution of the integral. To do this, we apply decomposition (3.6) in terms of the $d \log$ integrands (3.10),

$$\omega_{a,b} = c_0 [d \log z] + c_1 [-d \log(1-z)] + \dots, \tag{3.12}$$

where once again ellipsis corresponds to terms with at least a double pole, which are assumed to be irrelevant for the $d \log$ decomposition, and c_i are unknown coefficients that we need to determine.

It is clear from eq. (3.12) that c_0 is calculated by considering a contour integral around $z = 0$. Applying this to the definition of $\omega_{a,b}$ in eq. (3.8), we have,

$$c_0 = \oint_{z=0} \omega_{a,b}|_{\epsilon=0} = \oint_{z=0} \frac{dz}{z^{a+1}(1-z)^{b+1}} = \binom{a+b}{a}. \tag{3.13}$$

Likewise, c_1 is determined from a residue at $z = 1$. In this particular example, it turns to be $c_1 = c_0$. Taking into account the relation $I_{-1,0} = I_{0,-1}$, we find the maximal weight projection

$$\begin{aligned} I_{a,b} &= 2 \binom{a+b}{a} I_{-1,0} + \text{weight drop terms} \\ &= \binom{a+b}{a} \left(\frac{2}{\epsilon} - \frac{\pi^2}{3}\epsilon + 4\zeta_3\epsilon^2 \right) + \mathcal{O}(\epsilon^3) + \text{weight drop terms}. \end{aligned} \tag{3.14}$$

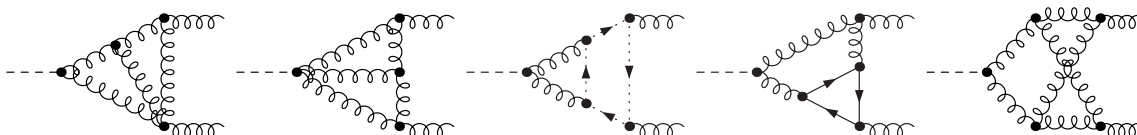


Figure 5. Representative two-loop Feynman diagrams for the scattering amplitude $H \rightarrow gg$. Curly, dashed, and straight lines represent, respectively, gluon, ghost, and massless fermions particles.

In this specific case the leading weight piece, $I_{-1,0}$ has weight 1, see eq. (3.11). This means that “weight drop terms” in this specific case are terms with transcendental weight 0 or less.

In order to have a very explicit example, let us consider the values $a = 5$ and $b = 3$, whose integrated expression, after expanding up-to second order in ϵ , becomes,

$$\begin{aligned}
 I_{5,3} = & \frac{112}{\epsilon} - \frac{56}{3} \pi^2 \epsilon + 224 \zeta_3 \epsilon^2 \\
 & - \frac{2216}{15} - \frac{19468}{225} \epsilon + \epsilon^2 \left(-\frac{234554}{3375} + \frac{1108}{45} \pi^2 \right) + \mathcal{O}(\epsilon^3) .
 \end{aligned} \tag{3.15}$$

Here, the first line of eq. (3.15) corresponds exactly to the maximal weight contribution, as predicted by our decomposition (i.e., after setting $a = 5$ and $b = 3$ in eq. (3.14)), while the terms in the second line all have less transcendental weight compared to the first line.

4 Application: $H \rightarrow gg$ at two loops

In the previous section, we provided a procedure to extract maximal leading transcendental weight terms of Feynman integrals. This is done by applying a unique partial fractioning of the Feynman integrand via Leinartas’ decomposition method. In this way, given the knowledge of $d \log$ forms for a given multi-loop scattering amplitude, an algebraic decomposition, as the one displayed in eq. (3.6), can straightforwardly be achieved.

In this section we, wish to apply this new method to the two-loop scattering amplitude $H \rightarrow gg$ in the large top quark mass limit. This tests the method for the multi-variate decomposition in a non-trivial setting that involves both planar and non-planar Feynman diagrams. We have already provided a basis of $d \log$ forms for this process in section 3. We now work out the $d \log$ decomposition of planar and non-planar integrands.

This scattering amplitude, because of Lorentz and gauge invariance, admits the decomposition,

$$\mathcal{A}_{Hgg}^{(2)} = g_S^4 g_{\text{EFT}} \left(g^{\mu_1 \mu_2} - \frac{2p_1^{\mu_2} p_2^{\mu_1}}{s_{12}} \right) \delta^{a_1 a_2} \epsilon_{1,a_1}^{\mu_1} \epsilon_{2,a_2}^{\mu_2} A_1^{(2)} , \tag{4.1}$$

with g_S and g_{EFT} stand for strong and Hgg -effective coupling constants, respectively, and a_i correspond to colour indices.

In order to carry out the calculation of $A_1^{(2)}$, we consider as internal particles: gluons, ghosts and massless fermions, as depicted for representative diagrams in figure 5. The generation of Feynman diagrams and construction of their integrands were carried out with aid of the MATHEMATICA packages FEYNARTS [63], FEYNCALC [64, 65], and FEYNRULES [66].

Since our approach strongly relies on algebraic manipulations at integrand level, we perform a grouping of diagrams according to the propagators present in the integrand families (2.21) and (2.27), for planar and non-planar sectors, respectively. Within those, four parent topologies, or Feynman integrands with six propagators, are found and displayed in figure 3.

Thus, with the integrands obtained in the common set of propagators, we can project the latter onto the $d \log$ basis studied in section 2.2, by following the procedure described in section 3. In effect, for this form factor, we expect to have the decomposition,

$$\mathcal{P}\left(A_1^{(2)}\right) = \sum_{i=1}^9 c_{p,i} \omega_{p,i}^{(2)} + \sum_{i=1}^8 c_{np,i} \omega_{np,i}^{(2)}. \tag{4.2}$$

Here $\mathcal{P}(\dots)$ corresponds to the projection of a given integrand onto the two-loop $d \log$ basis, and the labels ‘p’ and ‘np’ stand for planar and non-planar, respectively. See section 2.2 for the definition of the nine planar and eight non-planar forms $\omega_{p,i}$ and $\omega_{np,i}$, respectively.

To proceed with the extraction of coefficients for each $d \log$ integrand, we plug the parametrisation of loop momenta (2.1) and expand out the various four-dimensional scalar products between internal and external momenta that appear in the form factor at integrand level. In other words, according to the notation of eq. (3.6), the integrand for this scattering amplitude becomes,

$$\omega^{(2)} = \mathcal{I}(D_s, s; \{\alpha\}). \tag{4.3}$$

Here, $D_s = (g_{[D_s]})^\mu{}_\mu$ corresponds to the dimension where internal particles, e.g. gluons, live in, and is related with the dimension in the momentum integration as follows,

$$D_s = D + n_\epsilon = 4 - 2\epsilon + n_\epsilon. \tag{4.4}$$

In the four-dimensional helicity scheme, $D_s = 4$ and $n_\epsilon = 2\epsilon$, while in conventional dimensional regularisation, $D_s = D$ and $n_\epsilon = 0$ [67].

In eq. (4.3), $\{\alpha\} = \{\alpha_{1,1}, \dots, \alpha_{1,4}, \alpha_{2,1}, \dots, \alpha_{2,4}\}$ accounts for the set of loop momentum components, according to parametrisation (2.1). These algebraic manipulations are performed with the aid of the MATHEMATICA packages S@M [68] and T@M [69] that explicitly work out the various spinor products in terms of a reduced set of variables [70–72]. For this particular example, we only need to deal with the single kinematic invariant $s = (p_1 + p_2)^2$ and we set the dimension where internal particles live to four (cf. the discussion in section 5).

We extract the coefficient of each $d \log$ integrand by appropriately taking the residue, making use of the package MULTIVARIATEAPART. In more detail, say we would like to identify \tilde{c}_i in eq. (3.6). We choose variables $\tau_{i,j}$ with $j = 1 \dots 4L$. Then, we apply MULTIVARIATEAPART, after which the integrand has the form $\tilde{c}_i / (\prod_{j=1}^{4L} \tau_{i,j}) + R$, where R is the remainder of the polynomial division. In this form, one can read off the coefficient \tilde{c}_i .

Let us remark that in the case studied here, the leading singularities \tilde{c}_i are rational functions in the kinematic variables. This is not the case in general. These \tilde{c}_i can involve square roots that may arise from the change of variables from the loop momenta to the

$\tau_{i,j}$. For instance, this occurs in the case of the two-dimensional massive bubble integral. However, the \tilde{c}_i are constant from the point of view of the partial fractioning (since the latter only concerns the $\tau_{i,j}$, but not the external kinematics). Therefore, the appearance of square roots in them does not represent a conceptual problem for our procedure.

For the planar integrands, we find,

$$\begin{aligned}
 \mathcal{P} \left(\text{Diagram 1} \right) &= \left[-2\omega_{p,(a)} + \frac{3}{2} \left(\omega_{p,(b)} + \omega'_{p,(b)} \right) \right] s, \\
 \mathcal{P} \left(\text{Diagram 2} \right) &= \left[-\frac{3}{4}\omega_{p,(b)} - \frac{1}{2}\omega_{p,(c)} \right] s, \\
 \mathcal{P} \left(\text{Diagram 3} \right) &= \left[-\frac{3}{4}\omega'_{p,(b)} - \frac{1}{2}\omega_{p,(c)} \right] s,
 \end{aligned} \tag{4.5}$$

whose algebraic expressions are reported in section 2.

The projections (4.5) allow us to extract the maximal weight contributions in the planar sector,

$$\mathcal{P} \left(A_1^{(2)} \right) \Big|_{\text{planar}} = \left[-2\omega_{p,(a)} + \frac{3}{4} \left(\omega_{p,(b)} + \omega'_{p,(b)} \right) - \omega_{p,(c)} \right] s. \tag{4.6}$$

Let us emphasize once again that in this decomposition no relations at integral level were involved and that is exactly the reason for the presence of the two $d \log$ forms $\omega_{p,(b)}$ and $\omega'_{p,(b)}$, whose individual structure at integrand level is different, but after integration they turn out to be the same,

$$\int_{\ell_1, \ell_2} \omega'_{p,(b)} = \int_{\ell_1, \ell_2} \omega_{p,(b)}, \tag{4.7}$$

where, here and in the following, we use the shorthand notation,

$$\int_{\ell_s} \bullet \equiv e^{(4-D)\gamma_E/2} \int \frac{d^D \ell_s}{i \pi^{D/2}} \bullet, \tag{4.8}$$

with $D = 4 - 2\epsilon$.

Hence, with the additional relation (4.7), we can directly compare our result, obtained purely in four dimensions, against the one that involves relations at integral level. To do so, we generate IBP identities through the publicly available software REDUZE [73], where, in the integrand constructed for the form factor, we substitute the integral relations generated by the latter and choose the master integrals that appear in figure 3. Thus, setting $D_s = 4$ in the reduced integrand, we recover the very same result of eq. (4.6), after taking into account relation (4.7).

that leads the form factor to be expressed as,

$$\int_{\ell_1, \ell_2} \mathcal{P}(A_1^{(2)}) = \int_{\ell_1, \ell_2} \left[-2\omega_{p,(a)} - \frac{1}{2}\omega_{np,(e)} \right] s, \quad (4.13)$$

finding full agreement with approaches that rely on relations at integral level.

Let us now put all results together. Since the main purpose of our procedure is to extract the maximal transcendental weight piece of a given scattering amplitude, let us work this out explicitly. Using the known expressions for the ϵ -expansion of the master integrals (see e.g. [74, 75]), we find,

$$\int_{\ell_1, \ell_2} \mathcal{P}(A_1^{(2)}) = \left(-\frac{1}{\epsilon^4} + \frac{\pi^2}{12\epsilon^2} + \frac{25\zeta_3}{6\epsilon} + \frac{7\pi^4}{120} \right) s + \mathcal{O}(\epsilon). \quad (4.14)$$

This result is in perfect agreement (up-to an irrelevant normalisation factor) with the maximal leading terms of eq. (17) of ref. [76].

Let us discuss this result. We see that the projection method gives us a preliminary insight of the structure of the scattering amplitude under consideration. In particular, thanks to the decomposition into $d \log$ forms, it is straightforward to extract the maximal transcendental weight contribution of a given scattering amplitude. The procedure proposed in section 3 allows us to unambiguously calculate multi-variate residues, which is essential to extract the coefficients of $d \log$ integrands in the form factor decomposition (4.2). We find this procedure very efficient. In particular, because of the approach we follow is at integrand level, the calculation of IBP identities in four dimensions is replaced by a multi-variate polynomial division modulo Gröbner basis. Remarkably, contrary to the traditional generation of the former, there is no obstacle when increasing the rank in the numerator for a given integrand, as depicted in eq. (4.10) for a monomial with degree 20 since the polynomial division is straightforwardly performed.

5 Analysis of evanescent numerator terms

In the previous sections, we discussed an algebraic procedure to extract maximal transcendental weight of multi-loop scattering amplitude. In the present paper, we implemented this approach in a purely four-dimensional formulation, as this has many practical advantages. A natural and important question is to what extent subtleties can arise due to this approximation. In this section, we address this in more detail.

In particular, one can write down evanescent integrand terms, by which we mean integrands that vanish in four dimensions, but are non-zero in $D = 4 - 2\epsilon$ dimensions. For example, given enough momenta in a given scattering process, such terms can be constructed in terms of certain Gram determinants. Closely related to this are ‘ μ ’-terms, which depend on the difference between the D - and four-dimensional part of loop momenta. We wish to study in which situations such terms can impact the maximal weight projection.

We do so by analysing the one-loop case in detail, in subsection 5.1. Based on the conclusions drawn there, we then investigate four- and five-particle processes at two loops in subsection 5.2.

5.1 One-loop analysis

To start this discussion, let us extend the four-dimensional parametrisation (2.1) to D dimensions,

$$k_{i[D]}^\alpha = k_i^\alpha + k_{i[D-4]}^\alpha, \tag{5.1}$$

where we closely follow the notation of ref. [67]. In this decomposition, k_i^α and $k_{i[D-4]}^\alpha$ live in independent sub-spaces, $k_i^\alpha \cdot k_{i[D-4]}^\alpha = 0$, and the extra dimensional product, or evanescent terms, are often defined as, $\mu_{ij} = -k_{i[D-4]} \cdot k_{j[D-4]}$. In effect, the splitting D -dimensional loop momenta (5.1) has been extensively used in the four-dimensional helicity and 't Hooft-Veltman regularisation schemes, in which external momenta are always cast in four dimensions.

Analytic expressions for one-loop helicity amplitudes have been known for a long time. Interestingly, these amplitudes display a simple dependence in terms of the so-called evanescent terms μ_{11} . In effect, any N -point one-loop scattering amplitude in renormalisable theories can be decomposed as follows [77, 78],

$$\begin{aligned} \mathcal{A}_N^{(1)} = & \sum_{i \in \text{pentagons}} e_i I_{5,i}^{(1),D} [1] \\ & + \sum_{i \in \text{boxes}} d_i I_{4,i}^{(1),D} [1] + \sum_{i \in \text{triangles}} c_i I_{3,i}^{(1),D} [1] + \sum_{i \in \text{bubbles}} b_i I_{2,i}^{(1),D} [1] + \sum_{i \in \text{tadpoles}} a_i I_{1,i}^{(1),D} \\ & + \sum_{i \in \text{boxes}} d_{i,1} I_{4,i}^{(1),D} [\mu_{11}] + \sum_{i \in \text{boxes}} d_{i,2} I_{4,i}^{(1),D} [\mu_{11}^2] + \sum_{i \in \text{triangles}} c_{i,1} I_{3,i}^{(1),D} [\mu_{11}] \\ & + \sum_{i \in \text{bubbles}} b_{i,1} I_{2,i}^{(1),D} [\mu_{11}], \end{aligned} \tag{5.2}$$

in which $I_{n,i}^{(1),D}$ corresponds to the D -dimensional i -th n -point Feynman integral whose numerator, at integrand level, is given by the argument of the former. For instance, a three-point topology with an evanescent terms in the numerator is expressed as $I_{3,i}^{(1),D} [\mu_{11}]$. The small letters, $\{a, b, \dots, e\}$, are rational coefficients that only depend on the kinematics of the process under consideration and not on the ϵ parameter.

As a consequence of working in $D = 4 - 2\epsilon$ dimensions, one can relate scalar one-loop pentagon integrands as follows [79–81],

$$\tilde{e}_i I_{5,i}^{(1),D} [1] = I_{5,i}^{(1),D} [\mu_{11}] + \sum_{i \in \text{boxes}} \tilde{d}_i I_{4,i}^{(1),D} [1], \tag{5.3}$$

where, similar to coefficients $\{a, \dots, e\}$ in decomposition (5.2), \tilde{d}_i and \tilde{e}_i are rational functions that only depend on kinematics of the scattering process.

Relation (5.3) is straightforwardly obtained once the loop momentum components α are expressed in terms of the five denominators as well as μ_{11} . The latter is clearly manifest after performing a polynomial division modulo appropriate Gröbner basis. Likewise, by following the latter argument, higher multiplicity scalar integrands, i.e. $N > 5$, can always be reduced to pentagon integrands, since denominators (or inverse of Feynman propagators) are not independent in the former integrand.

Let us now analyse the evanescent terms, i.e. those with μ_{11} numerators. There are questions that we want to answer. First, can they produce leading weight terms, i.e terms of weight two? Second, if so, at what order in ϵ do the leading weight terms appear?

To answer these questions recall that we can trade integrals with an insertion of μ_{11} for higher-dimensional scalar integrals [82]. For one insertion of μ_{11} , one has

$$I_{n,i}^{(1),D}[\mu_{11}] = -\epsilon I_{n,i}^{(1),D+2}[1]. \tag{5.4}$$

At this stage we can use knowledge of the transcendental weight of the six-dimensional integrals appearing on the r.h.s. of eq. (5.4). It turns out that the six-dimensional box integral has maximal weight two, while six-dimensional pentagon and hexagon integrals have maximal weight three. More generally, we expect six-dimensional integrals with $n \geq 5$ to have maximal weight three. Taking into account the overall factor ϵ in eq. (5.4), we conclude that $I_{n,i}^{(1),D}[\mu_{11}]$ has maximal weight one for $n = 4$, and maximal weight two for $n \geq 5$. Moreover, since the six-dimensional integrals are finite, the evanescent terms contribute at $\mathcal{O}(\epsilon)$.

In other words, for $n = 4$ the evanescent terms generated by μ_{11} in the numerator are irrelevant for the maximal weight projection. For $n \geq 5$, this is no longer the case, unless one truncates the ϵ expansion at the finite part.

A similar analysis of transcendental weight drop in one-loop scalar integrals can be carried out by means of integrands constructed from Gram determinants. Let us define

$$G \left(\begin{matrix} k_1, \dots, k_s \\ q_1, \dots, q_s \end{matrix} \right) \equiv \det_{i,j \in s \times s} (2k_i \cdot q_j),$$

$$G(k_1, \dots, k_s) \equiv G \left(\begin{matrix} k_1, \dots, k_s \\ k_1, \dots, k_s \end{matrix} \right). \tag{5.5}$$

Then, a five-point integrand constructed from a Gram determinant, after an analysis at integrand level [80], becomes,

$$I_{5,i}^{(1),D} [G(k_1, p_1, p_2, p_3, p_4)] = -2 G(p_1, p_2, p_3, p_4) I_{5,i}^{(1),D} [\mu_{11}]. \tag{5.6}$$

We emphasise that at integrand level either side of eq. (5.6) vanishes in four dimension but yet gives contributions in D with a weight drop in the transcendental degree, as previously discussed for evanescent terms.

In summary, we find that for one-loop scattering amplitudes with $n \geq 5$, the four-dimensional projection method may miss maximal weight terms starting from the $\mathcal{O}(\epsilon)$ terms. For example, at $n = 5$, the pentagon integral (5.3) with evanescent numerator μ_{11} starts as $\epsilon \times$ (weight three). This means that evanescent terms do not affect the maximal weight piece for n -point one-loop amplitudes up-to and including the finite part. The situation is even better for $n \leq 4$ one-loop amplitudes, where evanescent terms do not lead to maximal weight pieces at all.

5.2 Two-loop analysis at four and five points

In view of the results of the one-loop analysis of the previous section, it is interesting to think about higher loops. In particular, we saw that evanescent terms in one-loop amplitudes can

lead to maximal weight terms starting from $\mathcal{O}(\epsilon)$ for amplitudes with five or more external legs. In two-loop amplitudes, one may have a situation where evanescent terms in one loop are multiplied by divergences originating from another loop (most clearly this happens for products of one-loop amplitudes), and this can lead to maximal weight pieces that are missed, if the evanescent numerators are not kept. It is easy to write down examples of this. Does this mean that the four-dimensional projection method cannot be used starting from two loops and five points?

Of course, if this is the case, one could try to adapt the method by treating the D -dependence of integrands more carefully, as is done for example in Baikov representation [34–38]. However, this is clearly much more work, and it would be desirable to find a positive four-dimensional statement. Hope comes from the fact that in most physical situations, at the end of the day, we wish to compute a (finite) result in four dimensions. Moreover, both ultraviolet and infrared divergences are very well understood, in principle, and satisfy factorisation formulas. Can we profit from these insights?

Let us discuss the divergences in turn. In the case of ultraviolet divergences, we heuristically know that they only involve less-than-maximal weight terms, so we can ignore them for the purpose of this discussion. The situation is different for infrared divergences, as they do involve maximal weight terms. So an infrared double pole, for example, when multiplied to the evanescent pentagon discussed in the last subsection, would produce a divergent maximal weight term at two loops. However, we know from general factorisation properties of infrared divergence that such terms are related to infrared counter-terms, multiplied by lower-loop amplitudes. For this reason, one can hope that such terms cancel when appropriate, infrared renormalised finite terms are considered. Such terms are sometimes called ‘hard functions’ in the QCD literature [33, 83–86].

In this subsection, we therefore discuss in some more detail the issue. We first study the four-point case, where, based on the above discussion, we do not expect any issue due to evanescent terms. Then, we discuss an explicit example of a five-point amplitude in $\mathcal{N} = 4$ super Yang-Mills that illustrates both the issue mentioned above, and its resolution. The aim of this subsection is not to provide a general proof, but rather to collect relevant observations that in our view points towards a more general pattern.

Evanescent two-loop four-point terms have non-maximal weight. We wish to analyse relevant evanescent terms for planar two-loop four-point amplitudes (see ref. [14] for closely related work). Let us begin by writing down the most general integrand that could appear in a massless Yang-Mills theory. In other words, we write all graphs with cubic vertices, and with at most one loop momentum per vertex. The numerator is then expressed as a linear combination of 1316 monomials $(k_i \cdot q_j)$ with $q_j \in \{k_1, k_2, p_1, \dots, p_3\}$. Then, we project this integrand onto four dimensions through parametrisation (2.1). At this point, all scalar products between external and internal momenta are expressed in terms of loop momentum components, namely, $(k_i \cdot q_i) \rightarrow \{\alpha\}$ with $q_i \in \{k_1, k_2, p_1, p_2, p_3\}$. By asking the latter projection to vanish, we find that our most generic vanishing four-dimensional integrand can be expressed in terms of 31 unknown coefficients. For example, one particular term in our ansatz is just the following Gram determinant, $G(k_1, k_2, p_1, p_2, p_4)$. Let us call the evanescent numerator with the 31 coefficients N_4 .

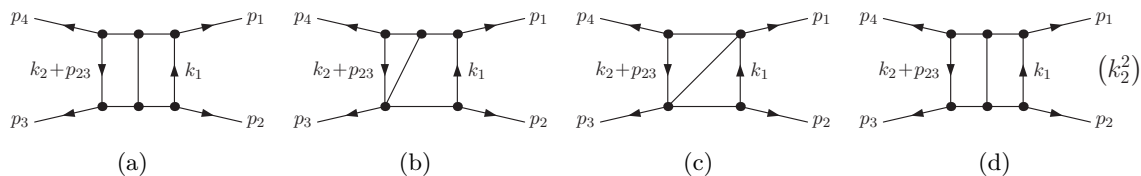


Figure 6. Representative two-loop planar boxes that admit a $d \log$ representation.

Next, we use IBP identities to reduce those integrands to a linear combination of eight master integrals. The latter are chosen such that they admit a $d \log$ representation [40], and correspond to the master integrals displayed in figures 3 and 6. Interestingly, we observe that the generic four-dimensional evanescent satisfies

$$\int_{k_1, k_2} \frac{N_4}{\text{double box propagators}} = \epsilon \sum_{i=1}^8 b_i(\epsilon) \omega_i^{(2)}, \quad (5.7)$$

with b_i being rational functions in ϵ that are regular at $\epsilon = 0$. This equation allows us to conclude that the evanescent two-loop four-point terms lead to a weight drop. In other words, they are irrelevant for the maximal weight projection, at any order in ϵ .

Evidence that infrared-renormalised five-point amplitudes are free of ambiguities. The planar $\mathcal{N} = 4$ super Yang-Mills five-point amplitude [87, 88] provides an instructive case of study of the effect of evanescent terms.

Let us write the amplitude as

$$A_5 = A_{5;\text{tree}} M_5, \quad (5.8)$$

where $A_{5;\text{tree}}$ is the tree-level amplitude. The loop factor,

$$M_5 = 1 + g^2 M_5^{(1)} + g^4 M_5^{(2)} + \dots, \quad (5.9)$$

is a function that depends on kinematic invariants, on the coupling $g^2 = g_{\text{YM}}^2 N_c / (16\pi^2)$, and on the dimensional regulator ϵ .

The one-loop amplitude $M_5^{(1)}$ consists of a sum of five infrared-divergent box integrals, plus exactly the evanescent pentagon integral (5.3) discussed above. Therefore, this constitutes an explicit field-theory example of the issue discussed above. Our projection would give the correct result up-to and including the finite part.

In this particular case, there is additional structure that allows us to trace the effect of the evanescent terms. In fact, the one-loop evanescent terms are parity-odd, while all other one-loop terms are parity even. Interestingly, the duality between scattering amplitudes and Wilson loops states that [89–92]

$$\log M_5 \sim \log W_5 + \mathcal{O}(\epsilon), \quad (5.10)$$

where W_5 is a pentagonal Wilson loop. Here we do not wish to go into the details of this relation, but merely point out that the Wilson loop is parity even. Therefore, if the duality

relation (5.10) holds (and this has been verified), then it implies something about the parity odd terms of the amplitude, and hence about the evanescent terms. Expanding the l.h.s. of eq. (5.10) in the coupling, we learn that apart from $M_5^{(1)}$, the following quantity must be parity even,

$$M_5^{(2)} - \frac{1}{2} \left(M_5^{(1)}\right)^2 + \mathcal{O}(\epsilon). \tag{5.11}$$

However, the product $\left(M_5^{(1)}\right)^2$ clearly involves parity odd terms at $1/\epsilon$, as explained above. Since the whole expression must be parity even, there must be corresponding parity odd terms in $M_5^{(2)}$ to cancel the terms from the product. In other words, certain evanescent terms in $M_5^{(2)}$ cancel the unwanted evanescent terms coming from the product of lower-loop amplitudes, at least to $\mathcal{O}(\epsilon)$.³

How can we interpret this result? The quantity $\log M$, or equivalently, the two-loop expression in eq. (5.11) appears naturally when solving an infrared renormalisation group equation in a conformal field theory (see also refs. [94, 95]). In other words, it takes care of the known partial exponentiation of non-Abelian soft and collinear divergences. In a more general, non-conformal field theory one could write down similar equations, that involve the β function, that take care of infrared sub-divergences. Based on the observations made here, we find it likely that the four-dimensional projection gives the correct maximal weight terms, when applied to properly defined infrared-subtracted ‘hard’ parts of scattering amplitudes. In the present paper, we have shown hints for this statement to be true in the context of the massless planar $\mathcal{N} = 4$ super Yang-Mills two-loop five-point amplitude, but we expect this to be true for general Yang-Mills theories, and also for higher-point amplitudes. It would be very interesting to formally prove such a statement.

6 Conclusion and outlook

In this paper, we proposed a method to extract the maximal weight contribution of scattering amplitudes. The method does not require integration-by-parts identities, but operates directly at the level of the Feynman loop integrand. It draws on insights that Feynman integrals with $d \log$ integrands evaluate to maximal weight functions. For a given scattering process, the idea is to project the Feynman integrand onto a basis of such functions. As such, the method combines well with generalised unitarity approaches.

After working out the method and discussing possible subtleties, we gave a proof-of-principle application, and extracted the maximal weight terms of the two-loop scattering amplitudes $gg \rightarrow H$ (in the heavy top-quark mass limit), which involves both planar and non-planar Feynman diagrams. We found the method to work very efficiently.

For simplicity and convenience, we worked in this paper mostly with four-dimensional integrands, which have obvious advantages. We discussed the scope and limitations of the

³We remark that this is closely related to the observation that the same amplitude can be obtained using a purely four-dimensional construction of loop integrands, see for example refs. [9] and [93]. One possible interpretation of this is that the different integrands (with or without the evanescent terms, in this case) correspond to different scheme choices.

four-dimensional approach. In particular, we showed that certain evanescent terms are irrelevant for one-loop amplitudes up-to and including the finite part. We argued that the same may hold true when considering appropriately defined infrared subtracted finite parts of higher-loop amplitudes, and gave an explicit five-point two-loop example of this.

There are several interesting directions for further research:

1. A natural next step is to apply the method beyond the proof-of-principle cases presented here. Extracting the maximal weight terms of QCD amplitudes would allow to shed a first light on their analytic structure. For example, very recently, four-dimensional $d \log$ basis integrals have been presented for planar two-loop six-particle scattering processes [29], and it would be very interesting (keeping in mind however the discussion regarding evanescent terms) to obtain maximal weight expressions for six-gluon scattering in Yang-Mills theory, for example.
2. In some cases, one may be interested in extracting the maximal weight terms to all orders in the dimensional regulator ϵ . In this case, one does need to keep evanescent terms. In order to do so, one needs to be able to perform the projections discussed in this paper, while keeping some information on the $(D - 4)$ -dimensional components of the loop momentum. A similar issue appears in the construction of uniform weight integrals in the context of the canonical differential equations method, and it was shown how to overcome it in Baikov representation [34–38]. It would be interesting to study our projection method in this setup.
3. The projection method requires the knowledge of a basis of $d \log$ integrands for a given scattering process. There is substantial knowledge and work on solving this problem [33, 58, 96–99], especially since this is also useful input for the canonical differential equations method [13]. It may be interesting to generalise this in at least two directions. Firstly, as was observed in the literature, in many cases one can find $d \log$ representations that consist of a single term only. This is clearly desirable from the point of view of our projection method, but also beyond, see e.g. ref. [100] (direct integration). Can one understand when and how this property can be achieved. Secondly, we know that $d \log$ representations are closely related to geometric interpretations of scattering amplitudes, see e.g. refs. [43, 44]. Is it possible to find a basis of $d \log$ integrands by geometric means?
4. Finally, a further exciting direction is the extension of the projection method beyond leading weight. Amusingly, those “most complicated terms” (following the language of ref. [2]) are the easiest to extract with our method. It should be noted, however, that the terms of lower transcendental weight typically involve more complicated coefficients compared to the maximal weight ones. Therefore, a yet-to-be-devised method for obtaining less-than-maximal weight terms would need to generate those more complicated coefficients. At present, we have a good understanding of which integrands lead to maximal weight functions, and, conversely, which features of integrands lead to a weight drop. But can one quantify the weight drop property —

for example, if the leading weight terms have weight $2L$, is it possible to define a basis of next-to-maximal weight terms that have weight $2L - 1$, and so on? This would provide a very useful new expansion. One hint on how to identify such integrals could come from the canonical differential equations method [13]. In this approach it is especially clear that next-to-maximal weight integrals can be generated by taking a derivative of the leading weight terms (however, these integrals will have non-constant leading singularities).

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