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Five-brane current algebras in type II string theories

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ABSTRACT: We study the current algebras of the NS5-branes, the Kaluza-Klein (KK) five-branes and the exotic 5_2^2 -branes in type IIA/IIB superstring theories. Their worldvolume theories are governed by the six-dimensional $\mathcal{N} = (2, 0)$ tensor and the $\mathcal{N} = (1, 1)$ vector multiplets. We show that the current algebras are determined through the S- and T-dualities. The algebras of the $\mathcal{N} = (2, 0)$ theories are characterized by the Dirac bracket caused by the self-dual gauge field in the five-brane worldvolumes, while those of the $\mathcal{N} = (1, 1)$ theories are given by the Poisson bracket. By the use of these algebras, we examine extended spaces in terms of tensor coordinates which are the representation of ten-dimensional supersymmetry. We also examine the transition rules of the currents in the type IIA/IIB supersymmetry algebras in ten dimensions. Based on the algebras, we write down the section conditions in the extended spaces and gauge transformations of the supergravity fields.

KEYWORDS: D-branes, p-branes, String Duality

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1 Introduction

Dualities are key ingredients to understand string theory and M-theory. In particular, T-duality is the most fundamental duality in string theory. This is a distinct feature of the string as an extended object and it never shows up from a point particle picture. Indeed, T-duality is interpreted as the canonical transformation in the worldsheet theory of the fundamental string [1]. There are several formulations of string theory with manifest T-duality. The doubled space was introduced in [2] and double field theory (DFT) [3–7], which realizes manifest T-duality with the help of the doubled space, has been intensively studied recently. The doubled space is characterized by the spacetime (the Kaluza-Klein) coordinate x^μ together with the T-dualized winding coordinate $\tilde{x}_\mu$ [8, 9]. This is generalized to the U-duality manifest formulation of supergravity known as exceptional field theory (EFT) based on the exceptional or extended geometries [10–14]. They are extensions of the generalized geometry [15, 16] that possesses the manifest T-duality.

In addition to the fundamental strings, there are a vast number of extended objects in string theory. For example, the D-branes are the most famous extended objects in string theory. One of the authors studied the current algebras of Dp-branes in type II string theories [17]. The analysis is generalized to the extended objects in M-theory in

eleven dimensions. The current algebras of the M2-brane [18] and the M5-brane [19] are studied in detail. A remarkable fact about these works is that the U-duality groups such as $SL(5)$ in seven dimensions and $SO(5, 5)$ in six dimensions are explained by the canonical symmetries in these branes. Even more, with the Courant brackets on the basis introduced in these algebras, one finds the structure of the generalized geometries associated with the $U(T)$ -dualities. This fact indicates that the current algebras of branes dictate the duality covariant gauge symmetries in the generalized (doubled or exceptional) geometries.

One of the remaining brane family is the NS5-brane and the Kaluza-Klein (KK) 5-brane. The former is the magnetic dual of the fundamental string while the latter is obtained by the T-duality transformation of the NS5-branes along the transverse direction to the brane worldvolume. They are also indispensable elements in string theory. Exploiting this procedure further, one finds another kind of branes. One can continue to perform further T-duality transformations on the KK5-brane. The resulting object is a five-brane of codimension two whose geometry is characterized by the $O(2, 2)$ monodromy [20]. It is known as the exotic 5_2^2 -brane or the Q-brane [21]. Their local geometries are patched together not by diffeomorphism or the gauge transformation, but the $O(2, 2)$ T-duality transformation. Therefore the background space of the 5_2^2 -brane loses the conventional meaning of geometries. This kind of background is known as the T-fold or globally non-geometric space [22]. The existence of the exotic branes are necessary in string theory. Indeed, they appear as the higher dimensional origins of the exotic states in the U-duality multiplets in lower dimensions [23, 24].

The purpose of this paper is to write down the current algebras of the branes in the remaining cornerstones of string theory. They include the NS5-branes, the KK5-branes and the exotic 5_2^2 -branes in type II string theories. They are related by the several duality transformations and dimensional reductions of the algebras in the M5-brane and Dp-branes. We will explicitly perform these manipulations and complete the collections of current algebras for brane theories. It is worthwhile to note that the effective worldvolume theories of several exotic branes have been constructed in this way [25–27].

The organization of this paper is as follows. In the next section, we introduce the current algebra of the M5-brane in eleven dimensions. We perform the direct dimensional reduction of the worldvolume theory of the M5-brane and write down the current algebra of the type IIA NS5-brane. We then perform the T-duality transformations of the IIA NS5-brane along the transverse directions and discuss the algebras of the KK5- and the exotic 5_2^2 -branes. Since the T-duality transformation requires an isometry of geometries, one of the scalar fields in the brane worldvolume theories loses its meaning as the geometric fluctuation mode along x^μ . This mode is supplemented, in the T-duality transformation, by another scalar field associated with the fluctuation along the dual winding coordinates $\tilde{x}_\mu$. We will utilize this fact to write down the algebras of the KK5- and the exotic 5_2^2 -branes. These worldvolume theories are governed by the six-dimensional $\mathcal{N} = (2, 0)$ tensor multiplet. In Section 3, we study the current algebras of five-branes whose effective theories are given by the $\mathcal{N} = (1, 1)$ vector multiplet. They stem from that of the D5-brane in type IIB theory. In Section 4, by using the current algebras, we study the supersymmetry algebras in ten dimensions. We determine the central charges in type IIA/IIB supergravities. Based on

the spatial diffeomorphism constraints, we also write down the section conditions in the extended spaces. Section 5 is devoted to the conclusion and discussions.

2 Current algebras in $\mathcal{N} = (2, 0)$ theories

In this section, we derive the current algebras of the five-branes whose worldvolume effective theories are governed by the six-dimensional $\mathcal{N} = (2, 0)$ tensor multiplet whose bosonic components consist of the five scalar fields and the self-dual 2-form gauge field. These include the type IIA NS5-brane, the IIB KK5-brane and the IIA 5_2^2 -brane in ten dimensions.

We first introduce the current algebra of the M5-brane and then perform the direct dimensional reduction to obtain the algebra of the type IIA NS5-brane. The worldvolume effective theory of the M5-brane is governed by the six-dimensional $\mathcal{N} = (2, 0)$ tensor multiplet. The bosonic sector of the M5-brane is given by the Pasti-Sorokin-Tonin (PST) action [28, 29]. The action consists of the Dirac-Born-Infeld (DBI), the self-dual field and the Wess-Zumino (WZ) parts:

$$S = T \int d^6\sigma (\mathcal{L}_{\text{DBI}} + \mathcal{L}_{\text{SD}} + \mathcal{L}_{\text{WZ}}), \quad (2.1)$$

where T is the tension of the M5-brane and each part of the Lagrangian is defined by

$$\begin{aligned} \mathcal{L}_{\text{DBI}} &= -\sqrt{-h_{\hat{\mathcal{F}}}}, \quad h_{\hat{\mathcal{F}}} = \det(h_{\hat{i}\hat{j}} + \tilde{F}_{\hat{i}\hat{j}}), \quad h_{\hat{i}\hat{j}} = \partial_{\hat{i}}x^m \partial_{\hat{j}}x^n \hat{G}_{mn}, \quad \tilde{\mathcal{F}}_{\hat{i}\hat{j}} = h_{\hat{i}\hat{k}} h_{\hat{j}\hat{k}'} \tilde{\mathcal{F}}^{\hat{k}\hat{k}'}, \\ \mathcal{L}_{\text{SD}} &= \frac{\sqrt{-h}}{4} \tilde{\mathcal{F}}^{\hat{i}\hat{j}} \mathcal{F}_{\hat{i}\hat{j}\hat{k}} n^{\hat{k}}, \quad h = \det h_{\hat{i}\hat{j}}, \quad \hat{i} = 0, 1, 2, \dots, 5, \\ \mathcal{L}_{\text{WZ}} &= \epsilon^{\hat{i}_1 \dots \hat{i}_6} \left(\frac{1}{6!} \hat{C}_{\hat{i}_1 \dots \hat{i}_6}^{[6]} + \frac{1}{2 \cdot 3!^2} \mathcal{F}_{\hat{i}_1 \hat{i}_2 \hat{i}_3} \hat{C}_{\hat{i}_4 \hat{i}_5 \hat{i}_6}^{[3]} \right), \\ F_{\hat{i}\hat{j}\hat{k}} &= \partial_{\hat{i}} A_{\hat{j}\hat{k}} + \partial_{\hat{j}} A_{\hat{k}\hat{i}} + \partial_{\hat{k}} A_{\hat{i}\hat{j}}, \quad \mathcal{F}_{\hat{i}\hat{j}\hat{k}} = F_{\hat{i}\hat{j}\hat{k}} - \partial_{\hat{i}} x^m \partial_{\hat{j}} x^n \partial_{\hat{k}} x^p \hat{C}_{mnp}^{[3]}. \end{aligned} \quad (2.2)$$

Here $\hat{G}_{mn}$, $\hat{C}^{[3]}$, $\hat{C}^{[6]}$ are the spacetime metric, the 3-form and its magnetic dual in eleven-dimensional supergravity. x^m ($m = 0, 1, \dots, 10$) are the spacetime coordinates represented by scalar fields in the six-dimensional worldvolume while $\sigma^{\hat{i}}$ ($\hat{i} = 0, 1, \dots, 5$) is the worldvolume coordinate. $F^{[3]} = dA^{[2]}$ is the field strength of the self-dual 2-form $A^{[2]}$ in six dimensions. The spacetime fields on the worldvolume are evaluated as the pullback like $P[\hat{C}^{[3]}]_{\hat{i}\hat{j}\hat{k}} = \hat{C}_{\hat{i}\hat{j}\hat{k}}^{[3]} = \partial_{\hat{i}} x^m \partial_{\hat{j}} x^n \partial_{\hat{k}} x^p \hat{C}_{mnp}^{[3]}$. We have also defined the following quantities:

$$\tilde{\mathcal{F}}^{\hat{i}\hat{j}} = \frac{1}{3! \sqrt{-h}} \epsilon^{\hat{i}\hat{j}\hat{k}\hat{k}_1\hat{k}_2\hat{k}_3} n_{\hat{k}} \mathcal{F}_{\hat{k}_1\hat{k}_2\hat{k}_3}, \quad n_{\hat{k}} = \frac{\partial_{\hat{k}} a}{\sqrt{-h^{\hat{i}\hat{j}} \partial_{\hat{i}} a \partial_{\hat{j}} a}}. \quad (2.3)$$

Here a is the auxiliary field. The worldvolume indices $\hat{i}, \hat{j}, \dots$ are contracted by the induced metric $h_{\hat{i}\hat{j}} = P[\hat{G}]_{\hat{i}\hat{j}} = \partial_{\hat{i}} x^m \partial_{\hat{j}} x^n \hat{G}_{mn}$ and its inverse $h^{\hat{i}\hat{j}}$.

Based on the PST action, the current algebra of the M5-brane was derived [19]. The result is

$$\begin{aligned}
 \{Z_M(\sigma), Z_N(\sigma')\}_D &= i\rho_{MN}^i \partial_i \delta(\sigma - \sigma'), \\
 \rho_{MN}^i &= \begin{bmatrix} 0 & 2E^{ji} \partial_j x^{[n_1} \delta_m^{n_2]} r_m^{i:n_1 \dots n_5} \\ 2E^{ji} \partial_j x^{[m_1} \delta_n^{m_2]} r_\ell^{i:m_1 m_2 n_1 n_2 \ell} & 0 \\ r_n^{i:m_1 \dots m_5} & 0 & 0 \end{bmatrix}, \\
 Z_M &= \begin{bmatrix} Z_m \\ Z^{[2]m_1 m_2} \\ Z^{[5]m_1 \dots m_5} \end{bmatrix} = \begin{bmatrix} p_m \\ 2E^{i_1 i_2} \partial_{i_1} x^{m_1} \partial_{i_2} x^{m_2} \\ T \epsilon^{i_1 \dots i_5} \partial_{i_1} x^{m_1} \partial_{i_2} x^{m_2} \partial_{i_3} x^{m_3} \partial_{i_4} x^{m_4} \partial_{i_5} x^{m_5} \end{bmatrix}, \\
 E^{ij} &= \frac{1}{4} \epsilon^{ij i_1 i_2 i_3} \partial_{i_1} A_{i_2 i_3}, r_m^{i:n_1 \dots n_5} = \frac{1}{4!} \epsilon^{i i_1 i_2 i_3 i_4} \partial_{i_1} x^{[n_1} \dots \partial_{i_4} x^{n_4} \delta_m^{n_5]}, \quad (2.4)
 \end{aligned}$$

where $M, N, \dots = (m, m_1 m_2, m_1 \dots m_5)$ are the indices in the extended space, $\{\cdot, \cdot\}_D$ is the Dirac bracket, $p_m, E^{ij} = \frac{1}{4} \epsilon^{ij i_1 i_2 i_3} \partial_{i_1} A_{i_2 i_3}$ are the canonical momenta conjugate to the worldvolume fields x^m and A_{ij} , respectively. The worldvolume delta function is abbreviated as $\partial_i \delta(\sigma - \sigma') = \frac{\partial}{\partial \sigma^i} \delta^5(\sigma - \sigma')$. We have defined the spatial indices $i, j, \dots = 1, 2, \dots, 5$ in the worldvolume and $\epsilon^{0 i_1 i_2 \dots i_5} = \epsilon^{i_1 i_2 \dots i_5}$. The explicit form of the momentum p_m is given by

$$p_m = \tilde{p}_m - \hat{G}_{mn} t^n + \hat{C}_m^{[6]} - \frac{\epsilon^{i_1 \dots i_5}}{4!} \left(2F_{i_1 i_2 i_3} - \hat{C}_{i_1 i_2 i_3}^{[3]} \right) \partial_{i_4} x^{m_1} \partial_{i_5} x^{m_2} \hat{C}_{m_1 m_2 m}^{[3]}, \quad (2.5)$$

where we have defined

$$\begin{aligned}
 \tilde{p}_m &= \frac{\partial \mathcal{L}_{\text{DBI}}}{\partial (\partial_0 x^m)}, \\
 t^n &= \partial_{i_5} x^n \frac{\epsilon^{i_1 \dots i_5}}{8} \tilde{\mathcal{F}}_{i_1 i_2} \tilde{\mathcal{F}}_{i_3 i_4} = \partial_i x^n \frac{1}{2} h^{ij} \mathcal{F}_{ji i_2} \tilde{E}^{i_1 i_2}, \\
 \tilde{E}^{ij} &= \frac{2}{4!} \epsilon^{ij i_1 i_2 i_3} \mathcal{F}_{i_1 i_2 i_3}, \hat{C}_{i_1 i_2 i_3}^{[3]} = \partial_{i_1} x^{n_1} \partial_{i_2} x^{n_2} \partial_{i_3} x^{n_3} \hat{C}_{n_1 n_2 n_3}^{[3]}, \\
 \hat{C}_n^{[6]} &= \frac{1}{5!} \epsilon^{i_1 \dots i_5} \partial_{i_1} x^{n_1} \partial_{i_2} x^{n_2} \partial_{i_3} x^{n_3} \partial_{i_4} x^{n_4} \partial_{i_5} x^{n_5} \hat{C}_{n n_1 n_2 n_3 n_4 n_5}^{[6]}. \quad (2.6)
 \end{aligned}$$

The momentum from the DBI part is explicitly given by

$$\begin{aligned}
 \tilde{p}_m &= -\frac{1}{2\sqrt{-h_{\tilde{\mathcal{F}}}}} \left[h'_m - \epsilon^{i_1 \dots i_5} \left(\frac{1}{6} \partial_{i_1} x^{m_1} \partial_{i_2} x^{m_2} \partial_{i_3} x^{m_3} \tilde{\mathcal{F}}_{i_4 i_5} \hat{G}_{m_1 n_1} \hat{G}_{m_2 n_2} \hat{G}_{m_3 n_3} \hat{G}_{m n_4} \mathcal{T}_1^{n_1 n_2 n_3 n_4} \right. \right. \\
 &\quad \left. \left. + \frac{1}{8} \partial_{i_1} x^{m_1} \tilde{\mathcal{F}}_{i_2 i_3} \tilde{\mathcal{F}}_{i_4 i_5} \hat{G}_{m_1 n_1} \hat{G}_{m n_2} \mathcal{T}_2^{n_1 n_2} \right) \right. \\
 &\quad \left. + \epsilon^{\hat{i}_1 \dots \hat{i}_6} \left(\frac{1}{4!} \partial_{\hat{i}_1} x^{m_1} \dots \partial_{\hat{i}_4} x^{m_4} \tilde{\mathcal{F}}'_{\hat{m} \hat{i}_5 \hat{i}_6} \hat{G}_{m_1 n_1} \dots \hat{G}_{m_4 n_4} \mathcal{T}_1^{n_1 n_2 n_3 n_4} \right. \right. \\
 &\quad \left. \left. + \frac{1}{8} \partial_{\hat{i}_1} x^{m_1} \partial_{\hat{i}_2} x^{m_2} \tilde{\mathcal{F}}'_{\hat{m} \hat{i}_3 \hat{i}_4} \tilde{\mathcal{F}}_{\hat{i}_5 \hat{i}_6} \hat{G}_{m_1 n_1} \hat{G}_{m_2 n_2} \mathcal{T}_2^{n_1 n_2} + \frac{1}{4!} \frac{1}{4^2} \tilde{\mathcal{F}}'_{\hat{m} \hat{i}_1 \hat{i}_2} \tilde{\mathcal{F}}_{\hat{i}_3 \hat{i}_4} \tilde{\mathcal{F}}_{\hat{i}_5 \hat{i}_6} \mathcal{T}_3 \right) \right]. \quad (2.7)
 \end{aligned}$$

Here each quantity is found to be

$$\begin{aligned}
 h'_m &= -\frac{2}{5!} \epsilon^{\hat{i}_1 \dots \hat{i}_5} \partial_{\hat{i}_1} x^{m_1} \dots \partial_{\hat{i}_5} x^{m_5} \hat{G}_{m_1 n_1} \dots \hat{G}_{m_5 n_5} \hat{G}_{mn_6} (\epsilon^{\hat{j}_1 \dots \hat{j}_6} \partial_{\hat{i}_1} x^{n_1} \dots \partial_{\hat{i}_6} x^{n_6}), \\
 \tilde{\mathcal{F}}'_{m\hat{i}\hat{j}} &= (\delta_{0\hat{i}} \partial_{\hat{k}} x^p + \delta_{0\hat{k}} \partial_{\hat{i}} x^p) \hat{G}_{mp} h_{\hat{j}\hat{k}'} \tilde{\mathcal{F}}^{\hat{k}\hat{k}'} + h_{\hat{i}\hat{k}} \left(\delta_{0\hat{j}} \partial_{\hat{k}'} x^p + \delta_{0\hat{k}'} \partial_{\hat{j}} x^p \right) \hat{G}_{mp} \tilde{\mathcal{F}}^{\hat{k}\hat{k}'} \\
 &\quad + h_{\hat{i}\hat{k}} h_{\hat{j}\hat{k}'} \left[\frac{1}{3!} \frac{1}{2} (-h)^{-3/2} h'_m \epsilon^{\hat{k}\hat{k}' \hat{\ell}\hat{\ell}_1 \hat{\ell}_2 \hat{\ell}_3} n_{\hat{\ell}} \mathcal{F}_{\hat{\ell}_1 \hat{\ell}_2 \hat{\ell}_3} \right. \\
 &\quad - \frac{1}{3! \sqrt{-h}} \epsilon^{\hat{k}\hat{k}' \hat{\ell}\hat{\ell}_1 \hat{\ell}_2 \hat{\ell}_3} (-h^{\hat{i}\hat{j}} \partial_{\hat{i}} a \partial_{\hat{j}} a)^{-3/2} \partial_{\hat{\ell}} a \partial_{\hat{k}_1} a \partial_{\hat{k}_2} a h^{\hat{k}_1 0} h^{\hat{k}_2 \hat{k}_3} \partial_{\hat{k}_3} x^n \hat{G}_{mn} \mathcal{F}_{\hat{\ell}_1 \hat{\ell}_2 \hat{\ell}_3} \\
 &\quad \left. + \frac{1}{2\sqrt{-h}} \epsilon^{kk' \ell_1 \ell_2 \ell_3} n_{\ell_1} \partial_{\ell_2} x^p \partial_{\ell_3} x^q \hat{C}_{mpq}^{[3]} \right], \\
 \mathcal{T}_1^{m_1 \dots m_4} &= \epsilon^{\hat{i}_1 \dots \hat{i}_6} \partial_{\hat{i}_1} x^{m_1} \partial_{\hat{i}_2} x^{m_2} \partial_{\hat{i}_3} x^{m_3} \partial_{\hat{i}_4} x^{m_4} \tilde{\mathcal{F}}_{\hat{i}_5 \hat{i}_6}, \\
 \mathcal{T}_2^{m_1 m_2} &= \epsilon^{\hat{i}_1 \dots \hat{i}_6} \partial_{\hat{i}_1} x^{m_1} \partial_{\hat{i}_2} x^{m_2} \tilde{\mathcal{F}}_{\hat{i}_3 \hat{i}_4} \tilde{\mathcal{F}}_{\hat{i}_5 \hat{i}_6}, \quad \mathcal{T}_3 = \epsilon^{\hat{i}_1 \dots \hat{i}_6} \tilde{\mathcal{F}}_{\hat{i}_1 \hat{i}_2} \tilde{\mathcal{F}}_{\hat{i}_3 \hat{i}_4} \tilde{\mathcal{F}}_{\hat{i}_5 \hat{i}_6}. \tag{2.8}
 \end{aligned}$$

We note that all the background fields $\hat{G}_{mn}, \hat{C}^{[3]}, \hat{C}^{[6]}$ in eleven-dimensional supergravity are included in $\tilde{p}_m$ in (2.5).

For later convenience, we decompose the relation (2.4) as

$$\begin{aligned}
 \{Z_m(\sigma), Z_n(\sigma')\}_D &= 0, \\
 \{Z_m(\sigma), Z^{[2]n_1 n_2}(\sigma')\}_D &= 2i E^{ji} \partial_j x^{[n_1} \delta_m^{n_2]} \partial_i \delta(\sigma - \sigma'), \\
 \{Z_m(\sigma), Z^{[5]n_1 \dots n_5}(\sigma')\}_D &= i r_m^i : n_1 \dots n_5 \partial_i \delta(\sigma - \sigma'), \\
 \{Z^{[2]m_1 m_2}(\sigma), Z^{[2]n_1 n_2}(\sigma')\}_D &= i r_\ell^i : m_1 m_2 n_1 n_2 \partial_i \delta(\sigma - \sigma'), \\
 \{Z^{[2]m_1 m_2}(\sigma), Z^{[5]n_1 \dots n_5}(\sigma')\}_D &= 0, \\
 \{Z^{[5]m_1 \dots m_5}(\sigma), Z^{[5]n_1 \dots n_5}(\sigma')\}_D &= 0. \tag{2.9}
 \end{aligned}$$

We stress that the algebra (2.9) is characterized by the Dirac bracket. This stems from the fact that the worldvolume theory of the M5-brane is governed by the $\mathcal{N} = (2, 0)$ tensor multiplet. In which, the self-duality of the 2-form gauge field gives the second class constraint [19].

2.1 IIA NS5-brane

We now perform the direct dimensional reduction of the PST action to ten dimensions and obtain the effective action of the type IIA NS5-brane. We define $x^{10} = Y$ and perform the dimensional reduction in this direction. The KK ansatz for the eleven-dimensional metric is

$$\hat{G}_{mn} = \begin{bmatrix} e^{-\frac{2}{3}\phi} (G_{\mu\nu} + e^{2\phi} C_\mu^{[1]} C_\nu^{[1]}) & e^{\frac{4}{3}\phi} C_\mu^{[1]} \\ e^{\frac{4}{3}\phi} C_\nu^{[1]} & e^{\frac{4}{3}\phi} \end{bmatrix}. \tag{2.10}$$

Here $G_{\mu\nu}$, ϕ , $C_\mu^{[1]}$ are the spacetime metric, the dilaton and the RR 1-form in ten-dimensional type IIA supergravity. The spacetime indices $\mu, \nu, \dots$ run from 0 to 9.

The potentials in eleven-dimensional supergravity is decomposed as

$$\begin{aligned}\hat{C}^{[3]} &= C^{[3]} - B \wedge dY, \\ \hat{C}^{[6]} &= B^{[6]} + C^{[5]} \wedge dY + \frac{1}{2}C^{[5]} \wedge C^{[1]} + \frac{1}{2}C^{[3]} \wedge B \wedge dY,\end{aligned}\quad (2.11)$$

where B and $B^{[6]}$ are the NSNS B -field and its magnetic dual while $C^{[5]}, C^{[3]}$ are the RR 5- and 3-form potentials in ten dimensions. Then the PST action of the M5-brane reduces to that of the NS5-brane in type IIA string theory [30]:

$$\begin{aligned}S &= -T_{\text{NS5}} \int d^6\sigma e^{-2\phi} \sqrt{-\det(P[G]_{\hat{i}\hat{j}} + \lambda^2 e^{2\phi} F_{\hat{i}} F_{\hat{j}})} \sqrt{\det\left(\delta_{\hat{i}}^{\hat{j}} + \frac{\lambda e^{\phi} (P[G]_{\hat{i}\hat{k}} + \lambda^2 e^{2\phi} F_{\hat{i}} F_{\hat{k}})}{\mathcal{N} \sqrt{\det(\delta_{\hat{l}}^{\hat{m}} + \lambda^2 e^{2\phi} F_{\hat{l}} F_{\hat{m}})}} H^{*\hat{j}\hat{k}}\right)} \\ &\quad - \frac{\lambda^2}{4} T_{\text{NS5}} \int d^6\sigma \sqrt{-G} \frac{1}{\mathcal{N}^2} H^{*\hat{i}\hat{j}} H_{\hat{i}\hat{j}\hat{k}} \left(P[G]^{\hat{k}\hat{l}} - \frac{e^{2\phi} \lambda^2 F^{\hat{k}} F^{\hat{l}}}{1 + \lambda^2 e^{2\phi} F^2} \right) \frac{\partial_{\hat{l}} a}{\sqrt{(\partial a)^2}} \\ &\quad + \mu_5 \int_{M_6} \left(P[B^{[6]}] + \lambda P[C^{[5]} \wedge dY] + \frac{1}{2} P[C^{[5]} \wedge C^{[1]}] + \frac{\lambda}{2} P[C^{[3]} \wedge B \wedge dY] \right. \\ &\quad \left. - \frac{\lambda}{2} F^{[3]} \wedge P[C^{[3]}] + \frac{\lambda}{2} F^{[3]} \wedge P[B \wedge dY] \right).\end{aligned}\quad (2.12)$$

where T_{NS5} and μ_5 is the tension and the NSNS charges of the NS5-brane, P denotes the pullback from the ten-dimensional spacetime to the six-dimensional worldvolume and $\lambda = 2\pi\alpha'$ is the string slope parameter. We have also defined the following quantities.

$$\begin{aligned}F_{\hat{i}} &= \partial_{\hat{i}} Y + \lambda^{-1} P[C^{[1]}]_{\hat{i}}, \quad \mathcal{N} = \left[1 - \frac{\lambda^2 e^{2\phi} (F \partial a)^2}{(\partial a)^2 (1 + \lambda^2 e^{2\phi} F^2)} \right]^{1/2}, \quad G = \det P[G], \\ (\partial a)^2 &= P[G]^{\hat{i}\hat{j}} \partial_{\hat{i}} a \partial_{\hat{j}} a, \quad H_{\hat{i}\hat{j}\hat{k}} = F_{\hat{i}\hat{j}\hat{k}}^{[3]} - \lambda^{-1} P[C^{[3]}]_{\hat{i}\hat{j}\hat{k}} - (P[B] \wedge dY)_{\hat{i}\hat{j}\hat{k}}, \\ H^{*\hat{i}\hat{j}\hat{k}} &= \frac{1}{3! \sqrt{-G}} \epsilon^{\hat{i}\hat{j}\hat{k}\hat{m}\hat{n}\hat{p}} H_{\hat{m}\hat{n}\hat{p}}, \quad H^{*\hat{i}\hat{j}} = \frac{1}{\sqrt{(\partial a)^2}} H^{*\hat{i}\hat{j}\hat{k}} \partial_{\hat{k}} a.\end{aligned}\quad (2.13)$$

The action (2.12) enables one to write down the current algebra. This is indeed obtained by the direct dimensional reduction of (2.9). The Z_M in the M5-brane algebra is decomposed as

$$Z_m \begin{cases} Z_{\mu} \\ Z_Y = Z_{\text{RR}}^{[0]} \end{cases}, \quad Z^{[2]m_1 m_2} \begin{cases} Z^{[2]\mu\nu} = Z_{\text{RR}}^{[2]\mu\nu} \\ Z^{[2]\mu Y} = Z_{\text{NS}}^{[1]\mu} \end{cases}, \quad Z^{[5]m_1 \dots m_5} \begin{cases} Z^{[5]\mu_1 \dots \mu_5} \\ Z^{[5]\mu_1 \dots \mu_4 Y} = Z_{\text{RR}}^{[4]\mu_1 \dots \mu_4} \end{cases}$$

Here, NS and RR denote that Z belongs to the NS- and the RR-sector with each other. The rank five anti-symmetric tensors are linear combinations of the self-dual and anti self-dual tensors, where the self-dual and anti self-dual tensors are $Z^{[5\pm]\mu_1 \dots \mu_5} = (Z^{[5]\mu_1 \dots \mu_5} \pm \frac{1}{5!} \epsilon^{\mu_1 \dots \mu_{10}} Z^{[5]\mu_6 \dots \mu_{10}})/2$. The KK5-brane and the NS5-brane currents are $Z_{\text{KK}}^{[5(+)]} = (Z^{[5+]} + Z^{[5-]})/2$ and $Z_{\text{NS}}^{[5(-)]} = (Z^{[5+]} - Z^{[5-]})/2$. Then we find that the non-zero components of the type IIA NS5-brane algebra are

$$\left\{ Z_{\mu}(\sigma), Z_{\text{RR}}^{[2]\nu\sigma}(\sigma') \right\}_D = 2i E^{ji} \partial_j x^{[\nu} \delta_{\mu}^{\sigma]} \partial_i \delta(\sigma - \sigma'),$$

$$\begin{aligned}
 \{Z_\mu(\sigma), Z_{\text{NS}}^{[1]\nu}(\sigma')\}_D &= -2iE^{ji}\partial_j Y \delta_\mu^\nu \partial_i \delta(\sigma - \sigma'), \\
 \{Z_{\text{RR}}^{[0]}(\sigma), Z_{\text{NS}}^{[1]\nu}(\sigma')\}_D &= 2iE^{ji}\partial_j x^\nu \partial_i \delta(\sigma - \sigma'), \\
 \{Z_\mu(\sigma), Z_{\text{RR}}^{[5]\nu_1 \dots \nu_5}(\sigma')\}_D &= i\left(\frac{1}{4!}\epsilon^{ii_1 \dots i_4} \partial_{i_1} x^{[\nu_1} \dots \partial_{i_4} x^{\nu_4} \delta_\mu^{\nu_5]}\right) \partial_i \delta(\sigma - \sigma'), \\
 \{Z_\mu(\sigma), Z_{\text{RR}}^{[4]\nu_1 \dots \nu_4}(\sigma')\}_D &= i\left(\frac{1}{3!}\epsilon^{ii_1 \dots i_4} \partial_{i_1} Y \partial_{i_2} x^{[\nu_1} \partial_{i_3} x^{\nu_2} \partial_{i_4} x^{\nu_3} \delta_\mu^{\nu_4]}\right) \partial_i \delta(\sigma - \sigma'), \\
 \{Z_{\text{RR}}^{[0]}(\sigma), Z_{\text{RR}}^{[4]\nu_1 \dots \nu_4}(\sigma')\}_D &= i\left(\frac{1}{4!}\epsilon^{ii_1 \dots i_4} \partial_{i_1} x^{[\nu_1} \dots \partial_{i_4} x^{\nu_4]}\right) \partial_i \delta(\sigma - \sigma'), \\
 \{Z_{\text{RR}}^{[2]\mu_1 \mu_2}(\sigma), Z_{\text{RR}}^{[2]\nu_1 \nu_2}(\sigma')\}_D &= i\frac{7}{4!}\epsilon^{ii_1 \dots i_4} \partial_{i_1} x^{[\mu_1} \partial_{i_2} x^{\mu_2} \partial_{i_3} x^{\nu_1} \partial_{i_4} x^{\nu_2]} \partial_i \delta(\sigma - \sigma'), \\
 \{Z_{\text{NS}}^{[1]\mu_1}(\sigma), Z_{\text{RR}}^{[2]\nu_1 \nu_2}(\sigma')\}_D &= -i\frac{7}{6}\epsilon^{ii_1 \dots i_4} \partial_{i_1} Y \partial_{i_2} x^{[\mu_1} \partial_{i_3} x^{\nu_1} \partial_{i_4} x^{\nu_2]} \partial_i \delta(\sigma - \sigma'). \quad (2.14)
 \end{aligned}$$

Here each component is given by

$$\begin{bmatrix} Z_\mu \\ Z_{\text{RR}}^{[0]} \\ Z_{\text{RR}}^{[2]\mu_1 \mu_2} \\ Z_{\text{NS}}^{[1]\mu} \\ Z_{\text{RR}}^{[5]\mu_1 \dots \mu_5} \\ Z_{\text{RR}}^{[4]\mu_1 \dots \mu_4} \end{bmatrix} = \begin{bmatrix} p_\mu \\ p_Y \\ 2E^{i_1 i_2} \partial_{i_1} x^{\mu_1} \partial_{i_2} x^{\mu_2} \\ 2E^{i_1 i_2} \partial_{i_1} x^{\mu_1} \partial_{i_2} Y \\ \epsilon^{i_1 \dots i_5} \partial_{i_1} x^{\mu_1} \partial_{i_2} x^{\mu_2} \partial_{i_3} x^{\mu_3} \partial_{i_4} x^{\mu_4} \partial_{i_5} x^{\mu_5} \\ \epsilon^{i_1 \dots i_5} \partial_{i_1} x^{\mu_1} \partial_{i_2} x^{\mu_2} \partial_{i_3} x^{\mu_3} \partial_{i_4} x^{\mu_4} \partial_{i_5} Y \end{bmatrix}. \quad (2.15)$$

The explicit form of the momenta p_μ , p_Y , E^{ij} are given by (2.4) but written by the ten-dimensional quantities (2.10) and (2.11). We note that the algebra (2.15) is again characterized by the Dirac bracket. This is obvious since the worldvolume supermultiplet, the $\mathcal{N} = (2, 0)$ tensor multiplet, is inherited from the M5-brane. The first and the second class constraints coming from the self-dual property of the 2-form $A^{[2]}$ are taken over to the type IIA NS5-brane.

2.2 IIB KK5-brane

We next determine the current algebra of the type IIB Kaluza-Klein (KK) 5-brane. This is obtained by the T-duality transformation along a transverse direction to the type IIA NS5-brane. The T-duality transformation of backgrounds is given by the famous Buscher rule [1, 2, 31, 32]. For example, the T-duality transformation of the spacetime metric and the NSNS B -field along the x^9 direction is given by

$$\begin{aligned}
 G'_{\mu\nu} &= G_{\mu\nu} - \frac{G_{9\mu}G_{9\nu} - B_{9\mu}B_{9\nu}}{g_{99}}, & G'_{9\mu} &= \frac{B_{9\mu}}{G_{99}}, & G'_{99} &= \frac{1}{G_{99}}, \\
 B'_{\mu\nu} &= B_{\mu\nu} - \frac{G_{9,\mu}B_{9\nu} - g_{9\nu}B_{9\mu}}{g_{99}}, & B'_{9\mu} &= \frac{G_{9\mu}}{g_{99}},
 \end{aligned} \quad (2.16)$$

where $\mu, \nu \neq 9$. The analogous transformations in the RR sector are also available [33]. The geometry in the transverse directions to the KK5-brane is given by the Taub-NUT space. This is obtained by applying the Buscher rule (2.16) to the NS5-brane background. The worldvolume effective action of the type IIB KK5-brane is obtained by applying the Buscher rule to the background of the type IIA NS5-brane effective action and exchanging

the isometry fluctuation x^9 to the one in the dual direction $\tilde{x}^9$. The latter is nothing but the Lagrange multiplier in deriving the rule (2.16) in the worldsheet theory of the fundamental strings [1, 2, 31, 32]. For example, this is implemented in worldvolume theories as the transformation of the pullbacks of the backgrounds:

$$P[G]_{\hat{i}\hat{j}} = G_{\mu\nu}\partial_{\hat{i}}x^\mu\partial_{\hat{j}}x^\nu \longrightarrow \sum_{\mu,\nu\neq 9} \left[G_{\mu\nu} - \frac{G_{9\mu}G_{9\nu} - B_{9\mu}B_{9\nu}}{G_{99}} \right] \partial_{\hat{i}}x^\mu\partial_{\hat{j}}x^\nu + \sum_{\mu\neq 9} \frac{B_{9\mu}}{G_{99}} \left(\partial_{\hat{i}}\tilde{x}^9\partial_{\hat{j}}x^\mu + \partial_{\hat{i}}x^\mu\partial_{\hat{j}}\tilde{x}^9 \right) + \frac{1}{G_{99}}\partial_{\hat{i}}\tilde{x}^9\partial_{\hat{j}}\tilde{x}^9. \quad (2.17)$$

Indeed, the worldvolume effective actions of the KK5-branes have been obtained with this replacement [26, 34]. It is obvious that the worldvolume effective theory of the IIB KK5-brane is again governed by the $\mathcal{N} = (2, 0)$ tensor multiplet. Correspondingly, the current algebra of the type IIB KK5-brane is again given by the Dirac bracket. This is written down by applying the T-duality transformation to that of the IIA NS5-brane. Now it is straightforward to write down the current algebra of the IIA KK5-brane. We assume that the x^9 direction is a transverse direction to the NS5-brane and perform the T-duality transformation along this direction in the NS5-brane algebra. Furthermore, the type IIA currents Z_M is decomposed and translated into the type IIB currents. This is given as follows. The left hand sides are the IIA RR currents and the right hand sides are the IIB RR currents:

$$Z_{\text{RR}}^{[0]} = Z_{\text{RR}}^{[1]9}, \quad Z_{\text{RR}}^{[2]} \begin{cases} Z_{\text{RR}}^{[2]\mu 9} = Z_{\text{RR}}^{[1]\mu} \\ Z_{\text{RR}}^{[2]\mu\nu} = Z_{\text{RR}}^{[3]\mu\nu 9} \end{cases}, \quad Z_{\text{RR}}^{[4]} \begin{cases} Z_{\text{RR}}^{[4]\mu_1\mu_2\mu_3 9} = Z_{\text{RR}}^{[3]\mu_1\mu_2\mu_3} \\ Z_{\text{RR}}^{[4]\mu_1\cdots\mu_4} = Z_{\text{RR}}^{[5+] \mu_1\cdots\mu_4 9} \end{cases}$$

where $\mu, \nu, \dots \neq 9$. As a result, the non-zero components of the IIB KK5-brane algebra is

$$\begin{aligned} \{Z_\mu(\sigma), Z_{\text{RR}}^{[3]\nu\sigma 9}(\sigma')\}_D &= 2iE^{ji}\partial_j x^{[\nu}\delta_\mu^{\sigma]}\partial_i\delta(\sigma - \sigma'), \\ \{Z_\mu(\sigma), Z_{\text{RR}}^{[1]\nu}(\sigma')\}_D &= -2iE^{ji}\partial_j\tilde{x}^9\delta_\mu^\nu\partial_i\delta(\sigma - \sigma'), \\ \{Z_9(\sigma), Z_{\text{RR}}^{[1]\nu}(\sigma')\}_D &= \{Z_{\text{RR}}^{[1]9}(\sigma), Z_{\text{NS}}^{[1]\nu}(\sigma')\}_D = 2iE^{ji}\partial_j x^\nu\partial_i\delta(\sigma - \sigma'), \\ \{Z_9(\sigma), Z_{\text{NS}}^{[1]9}(\sigma')\}_D &= -2iE^{ji}\partial_j Y\partial_i\delta(\sigma - \sigma'), \\ \{Z_\mu(\sigma), Z_{\text{NS}}^{[1]\nu}(\sigma')\}_D &= -2iE^{ji}\partial_j Y\delta_\mu^\nu\partial_i\delta(\sigma - \sigma'), \\ \{Z_{\text{RR}}^{[1]9}(\sigma), Z_{\text{NS}}^{[1]9}(\sigma')\}_D &= 2iE^{ji}\partial_j\tilde{x}^9\partial_i\delta(\sigma - \sigma'), \\ \{Z_\mu(\sigma), Z^{[5]\nu_1\cdots\nu_5}(\sigma')\}_D &= i\frac{1}{4!}\epsilon^{ii_1\cdots i_4}\partial_{i_1}x^{[\nu_1}\cdots\partial_{i_4}x^{\nu_4}\delta_\mu^{\nu_5]}\partial_i\delta(\sigma - \sigma'), \\ \{Z_\mu(\sigma), Z^{[5]\nu_1\cdots\nu_4 9}(\sigma')\}_D &= i\frac{1}{3!}\epsilon^{ii_1\cdots i_4}\partial_{i_1}\tilde{x}^9\partial_{i_2}x^{[\nu_1}\partial_{i_3}x^{\nu_2}\partial_{i_4}x^{\nu_3}\delta_\mu^{\nu_4]}\partial_i\delta(\sigma - \sigma'), \\ \{Z_9(\sigma), Z^{[5]\nu_1\cdots\nu_4 9}(\sigma')\}_D &= \{Z_{\text{RR}}^{[1]9}(\sigma), Z_{\text{RR}}^{[5+] \nu_1\cdots\nu_4 9}(\sigma')\}_D \\ &= i\frac{1}{4!}\epsilon^{ii_1\cdots i_4}\partial_{i_1}x^{[\nu_1}\cdots\partial_{i_4}x^{\nu_4]}\partial_i\delta(\sigma - \sigma'), \\ \{Z_\mu(\sigma), Z_{\text{RR}}^{[5+] \nu_1\cdots\nu_4 9}(\sigma')\}_D &= i\frac{1}{3!}\epsilon^{ii_1\cdots i_4}\partial_{i_1}Y\partial_{i_2}x^{[\nu_1}\partial_{i_3}x^{\nu_2}\partial_{i_4}x^{\nu_3}\delta_\mu^{\nu_4]}\partial_i\delta(\sigma - \sigma'), \end{aligned}$$

$$\begin{aligned}
 \left\{ Z_9(\sigma), Z_{\text{RR}}^{[3]\nu_1\nu_2\nu_3}(\sigma') \right\}_D &= \mathfrak{i} \frac{1}{3!} \epsilon^{ii_1 \dots i_4} \partial_{i_1} Y \partial_{i_2} x^{[\nu_1} \partial_{i_3} x^{\nu_2} \partial_{i_4} x^{\nu_3]} \partial_i \delta(\sigma - \sigma'), \\
 \left\{ Z_\mu(\sigma), Z_{\text{RR}}^{[3]\nu_1\nu_2\nu_3}(\sigma') \right\}_D &= -\mathfrak{i} \frac{1}{2!} \epsilon^{ii_1 \dots i_4} \partial_{i_1} Y \partial_{i_2} \tilde{x}^9 \partial_{i_3} x^{[\nu_1} \partial_{i_4} x^{\nu_2} \delta_\mu^{\nu_3]} \partial_i \delta(\sigma - \sigma'), \\
 \left\{ Z_{\text{RR}}^{[1]9}(\sigma), Z_{\text{RR}}^{[3]\nu_1\nu_2\nu_3}(\sigma') \right\}_D &= \mathfrak{i} \frac{1}{3!} \epsilon^{ii_1 \dots i_4} \partial_{i_1} x^{[\nu_1} \partial_{i_2} x^{\nu_2} \partial_{i_3} x^{\nu_3]} \partial_{i_4} \tilde{x}^9 \partial_i \delta(\sigma - \sigma'), \\
 \left\{ Z_{\text{RR}}^{[3]\mu_1\mu_2 9}(\sigma), Z_{\text{RR}}^{[3]\nu_1\nu_2 9}(\sigma') \right\}_D &= \mathfrak{i} \frac{7}{4!} \epsilon^{ii_1 \dots i_4} \partial_{i_1} x^{[\mu_1} \partial_{i_2} x^{\mu_2} \partial_{i_3} x^{\nu_1} \partial_{i_4} x^{\nu_2]} \partial_i \delta(\sigma - \sigma'), \\
 \left\{ Z_{\text{RR}}^{[1]\mu_1}(\sigma), Z_{\text{RR}}^{[3]\nu_1\nu_2 9}(\sigma') \right\}_D &= -\mathfrak{i} \frac{7}{6} \epsilon^{ii_1 \dots i_4} \partial_{i_1} \tilde{x}^9 \partial_{i_2} x^{[\mu_1} \partial_{i_3} x^{\nu_1} \partial_{i_4} x^{\nu_2]} \partial_i \delta(\sigma - \sigma'), \\
 \left\{ Z_{\text{NS}}^{[1]\mu_1}(\sigma), Z_{\text{RR}}^{[3]\nu_1\nu_2 9}(\sigma') \right\}_D &= -\mathfrak{i} \frac{7}{6} \epsilon^{ii_1 \dots i_4} \partial_{i_1} Y \partial_{i_2} x^{[\mu_1} \partial_{i_3} x^{\nu_1} \partial_{i_4} x^{\nu_2]} \partial_i \delta(\sigma - \sigma'), \\
 \left\{ Z_{\text{NS}}^{[1]\mu_1}(\sigma), Z_{\text{RR}}^{[1]\nu_1}(\sigma') \right\}_D &= -\mathfrak{i} \frac{7}{2} \epsilon^{ii_1 \dots i_4} \partial_{i_1} Y \partial_{i_2} \tilde{x}^9 \partial_{i_3} x^{[\mu_1} \partial_{i_4} x^{\nu_1]} \partial_i \delta(\sigma - \sigma'), \\
 \left\{ Z_{\text{NS}}^{[1]9}(\sigma), Z_{\text{RR}}^{[3]\nu_1\nu_2 9}(\sigma') \right\}_D &= -\mathfrak{i} \frac{7}{2} \epsilon^{ii_1 \dots i_4} \partial_{i_1} Y \partial_{i_2} \tilde{x}^9 \partial_{i_3} x^{[\nu_1} \partial_{i_4} x^{\nu_2]} \partial_i \delta(\sigma - \sigma'). \quad (2.18)
 \end{aligned}$$

Here $\mu, \nu, \dots \neq 9$ and we have shown only the non-zero contributions. Each component is given by (2.15) but all the background fields in the momenta $p_\mu (\mu \neq 9), p_9, p_Y$ are replaced by the T-dualized ones. We note that the KK5-brane possesses a particular transverse direction that corresponds to the isometry in the Taub-NUT space. Reflecting this fact, we have a distinguished scalar field $\tilde{x}^9$ which contrasts the algebra (2.18) with (2.14).

2.3 IIA 5_2^2 -brane

We apply the second T-duality transformation in the type IIB KK5-brane along another transverse, say, the x^8 -direction. The resulting object is known as the 5_2^2 -brane or the exotic Q-brane or the T-fold. It is an object of the codimension two. Although it loses the role of the consistent global solution as a stand alone object, it is nevertheless allowed as a local solution in supergravity [20]. The background fields of the 5_2^2 -brane have a peculiar structure, namely, they are not single valued functions of spacetime but are patched together by the non-trivial $O(2, 2)$ monodromy. In this sense, the background space of the 5_2^2 -brane ceases to be a conventional geometry and the brane is called the (globally) non-geometric object.

The worldvolume effective action of the IIB 5_2^2 -brane is obtained by the same procedure discussed in the previous subsection [26]. The effective theory is again characterized by the six-dimensional $\mathcal{N} = (2, 0)$ tensor multiplet where the scalar field x^8 in the isometry direction is replaced by its dual $\tilde{x}^8$. We find that the non-zero components of the current algebra in type IIA 5_2^2 -brane are

$$\begin{aligned}
 \left\{ Z_\mu(\sigma), Z_{\text{RR}}^{[2]\nu\sigma}(\sigma') \right\}_D &= 2\mathfrak{i} E^{ji} \partial_j x^{[\nu} \delta_\mu^{\sigma]} \partial_i \delta(\sigma - \sigma'), \\
 \left\{ Z_\mu(\sigma), Z_{\text{RR}}^{[2]\nu 8}(\sigma') \right\}_D &= -2\mathfrak{i} E^{ji} \partial_j \tilde{x}^8 \delta_\mu^\nu \partial_i \delta(\sigma - \sigma'), \\
 \left\{ Z_\mu(\sigma), Z_{\text{RR}}^{[2]\nu 9}(\sigma') \right\}_D &= -2\mathfrak{i} E^{ji} \partial_j \tilde{x}^9 \delta_\mu^\nu \partial_i \delta(\sigma - \sigma'), \\
 \left\{ Z_9(\sigma), Z_{\text{RR}}^{[2]\nu 9}(\sigma') \right\}_D &= \left\{ Z_8(\sigma), Z_{\text{RR}}^{[2]\nu 8}(\sigma') \right\}_D = \left\{ Z_{\text{RR}}^{[0]}(\sigma), Z_{\text{NS}}^{[1]\nu}(\sigma') \right\}_D \\
 &= 2\mathfrak{i} E^{ji} \partial_j x^\nu \partial_i \delta(\sigma - \sigma'),
 \end{aligned}$$

$$\begin{aligned}
 \left\{ Z_9(\sigma), Z_{\text{RR}}^{[2]89}(\sigma') \right\}_D &= \left\{ Z_{\text{RR}}^{[0]}(\sigma), Z_{\text{NS}}^{[1]8}(\sigma') \right\}_D = 2i E^{ji} \partial_j \tilde{x}^8 \partial_i \delta(\sigma - \sigma'), \\
 \left\{ Z_8(\sigma), Z_{\text{RR}}^{[2]98}(\sigma') \right\}_D &= \left\{ Z_{\text{RR}}^{[0]}(\sigma), Z_{\text{NS}}^{[1]9}(\sigma') \right\}_D = 2i E^{ji} \partial_j \tilde{x}^9 \partial_i \delta(\sigma - \sigma'), \\
 \left\{ Z_9(\sigma), Z_{\text{NS}}^{[1]9}(\sigma') \right\}_D &= \left\{ Z_8(\sigma), Z_{\text{NS}}^{[1]8}(\sigma') \right\}_D = -2i E^{ji} \partial_j Y \partial_i \delta(\sigma - \sigma'), \\
 \left\{ Z_\mu(\sigma), Z_{\text{NS}}^{[1]\nu}(\sigma') \right\}_D &= -2i E^{ji} \partial_j Y \delta_\mu^\nu \partial_i \delta(\sigma - \sigma'), \\
 \left\{ Z_\mu(\sigma), Z^{[5]\nu_1 \dots \nu_5}(\sigma') \right\}_D &= i \frac{1}{4!} \epsilon^{ii_1 \dots i_4} \partial_{i_1} x^{[\nu_1} \dots \partial_{i_4} x^{\nu_4} \delta_\mu^{\nu_5]} \partial_i \delta(\sigma - \sigma'), \\
 \left\{ Z_\mu(\sigma), Z^{[5]\nu_1 \dots \nu_4 8}(\sigma') \right\}_D &= i \frac{1}{3!} \epsilon^{ii_1 \dots i_4} \partial_{i_1} \tilde{x}^8 \partial_{i_2} x^{[\nu_1} \partial_{i_3} x^{\nu_2} \partial_{i_4} x^{\nu_3} \delta_\mu^{\nu_4]} \partial_i \delta(\sigma - \sigma'), \\
 \left\{ Z_\mu(\sigma), Z^{[5]\nu_1 \dots \nu_4 9}(\sigma') \right\}_D &= i \frac{1}{3!} \epsilon^{ii_1 \dots i_4} \partial_{i_1} \tilde{x}^9 \partial_{i_2} x^{[\nu_1} \partial_{i_3} x^{\nu_2} \partial_{i_4} x^{\nu_3} \delta_\mu^{\nu_4]} \partial_i \delta(\sigma - \sigma'), \\
 \left\{ Z_\mu(\sigma), Z^{[5]\nu_1 \nu_2 \nu_3 98}(\sigma') \right\}_D &= -i \frac{1}{2!} \epsilon^{ii_1 \dots i_4} \partial_{i_1} \tilde{x}^8 \partial_{i_2} \tilde{x}^9 \partial_{i_3} x^{[\nu_1} \partial_{i_4} x^{\nu_2} \delta_\mu^{\nu_3]} \partial_i \delta(\sigma - \sigma'), \\
 \left\{ Z_9(\sigma), Z^{[5]\nu_1 \dots \nu_4 9}(\sigma') \right\}_D &= \left\{ Z_8(\sigma), Z^{[5]\nu_1 \dots \nu_4 8}(\sigma') \right\}_D = \left\{ Z_{\text{RR}}^{[0]}(\sigma), Z_{\text{RR}}^{[4]\nu_1 \dots \nu_4}(\sigma') \right\}_D \\
 &= i \frac{1}{4!} \epsilon^{ii_1 \dots i_4} \partial_{i_1} x^{[\nu_1} \dots \partial_{i_4} x^{\nu_4]} \partial_i \delta(\sigma - \sigma'), \\
 \left\{ Z_9(\sigma), Z^{[5]\nu_1 \nu_2 \nu_3 89}(\sigma') \right\}_D &= \left\{ Z_{\text{RR}}^{[0]}(\sigma), Z_{\text{RR}}^{[4]\nu_1 \nu_2 \nu_3 8}(\sigma') \right\}_D \\
 &= i \frac{1}{3!} \epsilon^{ii_1 \dots i_4} \partial_{i_1} x^{[\nu_1} \partial_{i_2} x^{\nu_2} \partial_{i_3} x^{\nu_3]} \partial_{i_4} \tilde{x}^8 \partial_i \delta(\sigma - \sigma'), \\
 \left\{ Z_8(\sigma), Z^{[5]\nu_1 \nu_2 \nu_3 98}(\sigma') \right\}_D &= \left\{ Z_{\text{RR}}^{[0]}(\sigma), Z_{\text{RR}}^{[4]\nu_1 \nu_2 \nu_3 9}(\sigma') \right\}_D \\
 &= i \frac{1}{3!} \epsilon^{ii_1 \dots i_4} \partial_{i_1} x^{[\nu_1} \partial_{i_2} x^{\nu_2} \partial_{i_3} x^{\nu_3]} \partial_{i_4} \tilde{x}^9 \partial_i \delta(\sigma - \sigma'), \\
 \left\{ Z_\mu(\sigma), Z_{\text{RR}}^{[4]\nu_1 \dots \nu_4}(\sigma') \right\}_D &= i \frac{1}{3!} \epsilon^{ii_1 \dots i_4} \partial_{i_1} Y \partial_{i_2} x^{[\nu_1} \partial_{i_3} x^{\nu_2} \partial_{i_4} x^{\nu_3} \delta_\mu^{\nu_4]} \partial_i \delta(\sigma - \sigma'), \\
 \left\{ Z_\mu(\sigma), Z_{\text{RR}}^{[4]\nu_1 \nu_2 \nu_3 8}(\sigma') \right\}_D &= -i \frac{1}{2!} \epsilon^{ii_1 \dots i_4} \partial_{i_1} Y \partial_{i_2} \tilde{x}^8 \partial_{i_3} x^{[\nu_1} \partial_{i_4} x^{\nu_2} \delta_\mu^{\nu_3]} \partial_i \delta(\sigma - \sigma'), \\
 \left\{ Z_\mu(\sigma), Z_{\text{RR}}^{[4]\nu_1 \nu_2 \nu_3 9}(\sigma') \right\}_D &= -i \frac{1}{2!} \epsilon^{ii_1 \dots i_4} \partial_{i_1} Y \partial_{i_2} \tilde{x}^9 \partial_{i_3} x^{[\nu_1} \partial_{i_4} x^{\nu_2} \delta_\mu^{\nu_3]} \partial_i \delta(\sigma - \sigma'), \\
 \left\{ Z_\mu(\sigma), Z_{\text{RR}}^{[4]\nu_1 \nu_2 89}(\sigma') \right\}_D &= -i \epsilon^{ii_1 \dots i_4} \partial_{i_1} Y \partial_{i_2} \tilde{x}^9 \partial_{i_3} \tilde{x}^8 \partial_{i_4} x^{[\nu_1} \delta_\mu^{\nu_2]} \partial_i \delta(\sigma - \sigma'), \\
 \left\{ Z_8(\sigma), Z_{\text{RR}}^{[4]\nu_1 \nu_2 \nu_3 8}(\sigma') \right\}_D &= \left\{ Z_9(\sigma), Z_{\text{RR}}^{[4]\nu_1 \nu_2 \nu_3 9}(\sigma') \right\}_D \\
 &= i \frac{1}{3!} \epsilon^{ii_1 \dots i_4} \partial_{i_1} Y \partial_{i_2} x^{[\nu_1} \partial_{i_3} x^{\nu_2} \partial_{i_4} x^{\nu_3]} \partial_i \delta(\sigma - \sigma'), \\
 \left\{ Z_8(\sigma), Z_{\text{RR}}^{[4]\nu_1 \nu_2 98}(\sigma') \right\}_D &= i \frac{1}{2!} \epsilon^{ii_1 \dots i_4} \partial_{i_1} Y \partial_{i_2} \tilde{x}^9 \partial_{i_3} x^{[\nu_1} \partial_{i_4} x^{\nu_2]} \partial_i \delta(\sigma - \sigma'), \\
 \left\{ Z_9(\sigma), Z_{\text{RR}}^{[4]\nu_1 \nu_2 89}(\sigma') \right\}_D &= i \frac{1}{2!} \epsilon^{ii_1 \dots i_4} \partial_{i_1} Y \partial_{i_2} \tilde{x}^8 \partial_{i_3} x^{[\nu_1} \partial_{i_4} x^{\nu_2]} \partial_i \delta(\sigma - \sigma'), \\
 \left\{ Z_{\text{RR}}^{[0]}(\sigma), Z_{\text{RR}}^{[4]\nu_1 \nu_2 89}(\sigma') \right\}_D &= i \frac{1}{2!} \epsilon^{ii_1 \dots i_4} \partial_{i_1} x^{[\nu_1} \partial_{i_2} x^{\nu_2]} \partial_{i_3} \tilde{x}^8 \partial_{i_4} \tilde{x}^9 \partial_i \delta(\sigma - \sigma'), \\
 \left\{ Z_{\text{RR}}^{[2]\mu_1 \mu_2}(\sigma), Z_{\text{RR}}^{[2]\nu_1 \nu_2}(\sigma') \right\}_D &= i \frac{7}{4!} \epsilon^{ii_1 \dots i_4} \partial_{i_1} x^{[\mu_1} \partial_{i_2} x^{\mu_2} \partial_{i_3} x^{\nu_1} \partial_{i_4} x^{\nu_2]} \partial_i \delta(\sigma - \sigma'), \\
 \left\{ Z_{\text{RR}}^{[2]\mu_1 8}(\sigma), Z_{\text{RR}}^{[2]\nu_1 \nu_2}(\sigma') \right\}_D &= -i \frac{7}{6} \epsilon^{ii_1 \dots i_4} \partial_{i_1} \tilde{x}^8 \partial_{i_2} x^{[\mu_1} \partial_{i_3} x^{\nu_1} \partial_{i_4} x^{\nu_2]} \partial_i \delta(\sigma - \sigma'), \\
 \left\{ Z_{\text{RR}}^{[2]\mu_1 9}(\sigma), Z_{\text{RR}}^{[2]\nu_1 \nu_2}(\sigma') \right\}_D &= -i \frac{7}{6} \epsilon^{ii_1 \dots i_4} \partial_{i_1} \tilde{x}^9 \partial_{i_2} x^{[\mu_1} \partial_{i_3} x^{\nu_1} \partial_{i_4} x^{\nu_2]} \partial_i \delta(\sigma - \sigma'),
 \end{aligned}$$

$$\begin{aligned}
 \left\{ Z_{\text{RR}}^{[2]89}(\sigma), Z_{\text{RR}}^{[2]\nu_1\nu_2}(\sigma') \right\}_D &= -\frac{7}{2}\epsilon^{ii_1\cdots i_4}\partial_{i_1}\tilde{x}^9\partial_{i_2}\tilde{x}^8\partial_{i_3}x^{[\nu_1}\partial_{i_4}x^{\nu_2]}\partial_i\delta(\sigma-\sigma'), \\
 \left\{ Z_{\text{RR}}^{[2]\mu_18}(\sigma), Z_{\text{RR}}^{[2]\nu_19}(\sigma') \right\}_D &= -\frac{7}{2}\epsilon^{ii_1\cdots i_4}\partial_{i_1}\tilde{x}^8\partial_{i_2}\tilde{x}^9\partial_{i_3}x^{[\mu_1}\partial_{i_4}x^{\nu_1]}\partial_i\delta(\sigma-\sigma'), \\
 \left\{ Z_{\text{NS}}^{[1]\mu_1}(\sigma), Z_{\text{RR}}^{[2]\nu_1\nu_2}(\sigma') \right\}_D &= -\frac{7}{6}\epsilon^{ii_1\cdots i_4}\partial_{i_1}Y\partial_{i_2}x^{[\mu_1}\partial_{i_3}x^{\nu_1}\partial_{i_4}x^{\nu_2]}\partial_i\delta(\sigma-\sigma'), \\
 \left\{ Z_{\text{NS}}^{[1]\mu}(\sigma), Z_{\text{RR}}^{[2]\nu8}(\sigma') \right\}_D &= \left\{ Z_{\text{NS}}^{[1]8}(\sigma), Z_{\text{RR}}^{[2]\mu\nu}(\sigma') \right\}_D \\
 &= -\frac{7}{2}\epsilon^{ii_1\cdots i_4}\partial_{i_1}Y\partial_{i_2}\tilde{x}^8\partial_{i_3}x^{[\mu}\partial_{i_4}x^{\nu]}\partial_i\delta(\sigma-\sigma'), \\
 \left\{ Z_{\text{NS}}^{[1]\mu}(\sigma), Z_{\text{RR}}^{[2]\nu9}(\sigma') \right\}_D &= \left\{ Z_{\text{NS}}^{[1]9}(\sigma), Z_{\text{RR}}^{[2]\mu\nu}(\sigma') \right\}_D \\
 &= -\frac{7}{2}\epsilon^{ii_1\cdots i_4}\partial_{i_1}Y\partial_{i_2}\tilde{x}^9\partial_{i_3}x^{[\mu}\partial_{i_4}x^{\nu]}\partial_i\delta(\sigma-\sigma'), \\
 \left\{ Z_{\text{NS}}^{[1]\mu}(\sigma), Z_{\text{RR}}^{[2]89}(\sigma') \right\}_D &= 7i\epsilon^{ii_1\cdots i_4}\partial_{i_1}Y\partial_{i_2}\tilde{x}^9\partial_{i_3}\tilde{x}^8\partial_{i_4}x^\mu\partial_i\delta(\sigma-\sigma'), \\
 \left\{ Z_{\text{NS}}^{[1]8}(\sigma), Z_{\text{RR}}^{[2]\nu9}(\sigma') \right\}_D &= 7i\epsilon^{ii_1\cdots i_4}\partial_{i_1}Y\partial_{i_2}\tilde{x}^8\partial_{i_3}\tilde{x}^9\partial_{i_4}x^\nu\partial_i\delta(\sigma-\sigma'), \\
 \left\{ Z_{\text{NS}}^{[1]9}(\sigma), Z_{\text{RR}}^{[2]\nu8}(\sigma') \right\}_D &= 7i\epsilon^{ii_1\cdots i_4}\partial_{i_1}Y\partial_{i_2}\tilde{x}^9\partial_{i_3}\tilde{x}^8\partial_{i_4}x^\nu\partial_i\delta(\sigma-\sigma'). \tag{2.19}
 \end{aligned}$$

Each component is given by (2.15), also where the indices $\mu, \nu \cdots \neq 8, 9$. All the background fields in the momenta p_μ ($\mu \neq 8, 9$), p_8, p_9, p_Y are replaced by the T-dualized ones.

Again, due to the self-duality constraints for $A^{[2]}$ in the $\mathcal{N} = (2, 0)$ tensor multiplet, the algebra is governed by the Dirac bracket. We have two distinguished directions $\tilde{x}^8$ and $\tilde{x}^9$ in the 5_2^2 -brane. We stress that the complicated structure of the worldvolume theory is substantially encoded into the form of the momenta p_μ, p_8, p_9, p_Y .

3 Current algebras in $\mathcal{N} = (1, 1)$ theories

In this section, we derive the current algebras of type IIB NS5-brane, the IIA KK5-brane and the IIB 5_2^2 -brane. The current algebras of these branes are characterized by the $\mathcal{N} = (1, 1)$ vector multiplet whose bosonic components are four scalar fields and a vector field. In order to write down these algebras, we first introduce the algebra of the D5-brane in type IIB theory [35, 36]. We then perform the S-duality transformation to obtain the current of the type IIB NS5-brane.

The action for a Dp-brane is given by the sum of the DBI action and the WZ term as

$$\begin{aligned}
 S &= S_{\text{DBI}} + S_{\text{WZ}}, & S_{\text{DBI}} &= T_{Dp} \int_M d^p\sigma \mathcal{L}_{\text{DBI}}, \\
 \mathcal{L}_{\text{DBI}} &= -e^{-\phi} \sqrt{-h_F}, & h_F &= \det h_{F\hat{i}\hat{j}}, \\
 S_{\text{WZ}} &= T_{Dp} \int_M e^{\mathcal{F}} P[C^{\text{RR}}], \\
 h_{F\hat{i}\hat{j}} &= \partial_{\hat{i}}x^\mu \partial_{\hat{j}}x^\nu G_{\mu\nu} + \mathcal{F}_{\hat{i}\hat{j}}, \\
 F_{\hat{i}\hat{j}} &= \partial_{\hat{i}}A_{\hat{j}} - \partial_{\hat{j}}A_{\hat{i}}, & \mathcal{F}_{\hat{i}\hat{j}} &= F_{\hat{i}\hat{j}} + \partial_{\hat{i}}x^\mu \partial_{\hat{j}}x^\nu B_{\mu\nu}. \tag{3.1}
 \end{aligned}$$

Here T_{Dp} is the tension of the Dp-brane, C^{RR} is the RR polyform potential, $F = dA$ is the field strength of the worldvolume vector $A = A_i d\sigma^i$. The effective theory is governed by the six-dimensional $\mathcal{N} = (1, 1)$ vector multiplet.

The canonical momenta conjugate to x^μ and A_i are defined as

$$\begin{aligned} p_\mu &= -\sqrt{-h_F} \left(\frac{1}{2} h_F^{(\hat{i}0)} G_{\mu\nu} + \frac{1}{2} h_F^{[\hat{i}0]} B_{\mu\nu} \right) \partial_i x^\nu + \frac{\partial \mathcal{L}_{\text{WZ}}}{\partial(\partial_0 x^\mu)}, \\ E^i &= -\sqrt{-h_F} \frac{1}{2} h_F^{[i0]} + \frac{\partial \mathcal{L}_{\text{WZ}}}{\partial F_{0i}}, \quad i = 1, \dots, p. \end{aligned} \quad (3.2)$$

The Hamiltonian is given by [35, 36]

$$\begin{aligned} \mathcal{H} &= p_\mu \partial_0 x^\mu + E^i \partial_0 A_i - \mathcal{L} \\ &= -\frac{1}{\sqrt{-h} h^{00}} \mathcal{H}_\perp - \frac{h^{0i}}{h^{00}} \mathcal{H}_i - A_0 \Phi, \end{aligned} \quad (3.3)$$

where we have defined

$$\begin{aligned} \mathcal{H}_\perp &= \frac{1}{2} e^\phi \left(\tilde{p}_\mu G^{\mu\nu} \tilde{p}_\nu + \tilde{E}^i h_{ij} \tilde{E}^j + e^{-2\phi} \det h_{Fij} \right) = 0, \\ \mathcal{H}_i &= \tilde{p}_\mu \partial_i x^\mu + \mathcal{F}_{ij} \tilde{E}^j = p_\mu \partial_i x^\mu + F_{ij} E^j = 0, \\ \Phi &= \partial_i E^i = 0, \end{aligned} \quad (3.4)$$

with the following quantities

$$\begin{aligned} \tilde{p}_\mu &= p_\mu - B_{\mu\nu} E^i \partial_i x^\nu - \frac{\partial \mathcal{L}_{\text{WZ}}}{\partial(\partial_0 x^\mu)}, \\ &= -e^{-\phi} \sqrt{-h_F} \frac{1}{2} h_F^{(\hat{i}0)} G_{\mu\nu} \partial_i x^\nu, \\ \tilde{E}^i &= E^i - \frac{\partial \mathcal{L}_{\text{WZ}}}{\partial F_{0i}} = -e^{-\phi} \sqrt{-h_F} \frac{1}{2} h_F^{[i0]}. \end{aligned} \quad (3.5)$$

For the D5-brane in type IIB theory, $\mathcal{H}_\perp$ is written by the sum of bilinears [35, 36],

$$\mathcal{H}_\perp = \frac{1}{2} Z_M \mathcal{M}^{MN} Z_N, \quad (3.6)$$

where

$$Z_M = \begin{pmatrix} p_\mu \\ Z_{\text{NS}}^{[1]\mu} \\ Z_{\text{RR}}^{[1]\mu} \\ Z_{\text{RR}}^{[3]\mu_1\mu_2\mu_3} \\ Z_{\text{RR}}^{[5]\mu_1\mu_2\mu_3\mu_4\mu_5} \end{pmatrix}. \quad (3.7)$$

Each component is explicitly given by

$$\begin{aligned} Z_{\text{NS}}^{[1]\mu} &= E^i \partial_i x^\mu, \\ Z_{\text{RR}}^{[1]\mu} &= \epsilon^{i_1 i_2 i_3 i_4 i_5} F_{i_1 i_2} F_{i_3 i_4} \partial_{i_5} x^\mu, \\ Z_{\text{RR}}^{[3]\mu_1\mu_2\mu_3} &= \epsilon^{i_1 i_2 i_3 i_4 i_5} F_{i_1 i_2} \partial_{i_3} x^{\mu_1} \partial_{i_4} x^{\mu_2} \partial_{i_5} x^{\mu_3}, \\ Z_{\text{RR}}^{[5]\mu_1\mu_2\mu_3\mu_4\mu_5} &= \epsilon^{i_1 i_2 i_3 i_4 i_5} \partial_{i_1} x^{\mu_1} \partial_{i_2} x^{\mu_2} \partial_{i_3} x^{\mu_3} \partial_{i_4} x^{\mu_4} \partial_{i_5} x^{\mu_5}. \end{aligned} \quad (3.8)$$

The current algebra of the D5-brane is therefore given by

$$\{Z_M(\sigma), Z_N(\sigma')\} = \mathfrak{i} \rho_{MN}^i \partial_i \delta(\sigma - \sigma')$$

$$\rho_{MN}^i = \left(\begin{array}{cc|ccc} 0 & E^i \delta_\mu^\nu & \rho_{13} & \rho_{14} & \rho_{15} \\ E^i \delta_\nu^\mu & 0 & \rho_{23} & \rho_{24} & 0 \\ \hline \rho_{13} & \rho_{23} & & & \\ \rho_{14} & \rho_{24} & & & \\ \rho_{15} & 0 & & & \end{array} \right) \quad (3.9)$$

where each non-zero component is given by

$$\begin{aligned} \rho_{13} &= \epsilon^{ii_1 \dots i_4} F_{i_1 i_2} F_{i_3 i_4} \delta_\mu^\nu \\ \rho_{14} &= 3 \epsilon^{ii_1 \dots i_4} F_{i_1 i_2} \partial_{i_3} x^{[\nu_1} \partial_{i_4} x^{\nu_2} \delta_\mu^{\nu_3]} / 3! \\ \rho_{15} &= 5 \epsilon^{ii_1 \dots i_4} \partial_{i_1} x^{[\nu_1} \partial_{i_2} x^{\nu_2} \partial_{i_3} x^{\nu_3} \partial_{i_4} x^{\nu_4} \delta_\mu^{\nu_5]} / 5! \\ \rho_{23} &= 4 \epsilon^{ii_1 \dots i_4} F_{i_1 i_2} \partial_{i_3} x^\mu \partial_{i_4} x^\nu \\ \rho_{24} &= 2 \epsilon^{ii_1 \dots i_4} \partial_{i_1} x^\mu \partial_{i_2} x^{\nu_1} \partial_{i_3} x^{\nu_2} \partial_{i_4} x^{\nu_3}. \end{aligned} \quad (3.10)$$

3.1 IIB NS5-brane

The effective action of the type IIB NS5-brane is obtained by the S-duality transformation of the D5-brane action [34] and it is governed by the $\mathcal{N} = (1, 1)$ vector multiplet. The S-duality transformation rules of the background fields are

$$\begin{aligned} \tau &\xrightarrow{S} -\frac{1}{\tau}, & C^{[2]} &\xrightarrow{S} B, & B &\xrightarrow{S} -C^{[2]}, & g_{\mu\nu} &\xrightarrow{S} |\tau| g_{\mu\nu}, \\ C^{[4]} &\xrightarrow{S} C^{[4]} + C^{[2]} \wedge B, & C^{[6]} &\xrightarrow{S} -B^{[6]} + \frac{1}{2} B \wedge C^{[2]} \wedge C^{[2]}, \end{aligned} \quad (3.11)$$

where $\tau = C^{[0]} + \mathfrak{i} e^{-\phi}$ is the complex axion-dilaton field and $B^{[6]}$ is the magnetic dual of B . The algebra of the type IIB NS5-brane is obtained by performing the S-duality transformation of that of the D5-brane (3.10). The non-zero components of the current algebra in the type IIB NS5-brane are therefore

$$\begin{aligned} \{Z_\mu(\sigma), Z_{\text{NS}}^{[1]\nu}(\sigma')\} &= \mathfrak{i} E^i \delta_\mu^\nu \partial_i \delta(\sigma - \sigma'), \\ \{Z_\mu(\sigma), Z_{\text{RR}}^{[1]\nu}(\sigma')\} &= \mathfrak{i} \epsilon^{ii_1 \dots i_4} F_{i_1 i_2} F_{i_3 i_4} \delta_\mu^\nu \partial_i \delta(\sigma - \sigma'), \\ \{Z_\mu(\sigma), Z_{\text{RR}}^{[3]\nu_1 \nu_2 \nu_3}(\sigma')\} &= \frac{\mathfrak{i}}{2!} \epsilon^{ii_1 \dots i_4} F_{i_1 i_2} \partial_{i_3} x^{[\nu_1} \partial_{i_4} x^{\nu_2} \delta_\mu^{\nu_3]} \partial_i \delta(\sigma - \sigma'), \\ \{Z_\mu(\sigma), Z_{\text{RR}}^{[5]\nu_1 \dots \nu_5}(\sigma')\} &= \frac{\mathfrak{i}}{4!} \epsilon^{ii_1 \dots i_4} \partial_{i_1} x^{[\nu_1} \dots \partial_{i_4} x^{\nu_4} \delta_\mu^{\nu_5]} \partial_i \delta(\sigma - \sigma'), \\ \{Z_{\text{NS}}^{[1]\mu}(\sigma), Z_{\text{RR}}^{[1]\nu}(\sigma')\} &= 4 \mathfrak{i} \epsilon^{ii_1 \dots i_4} F_{i_1 i_2} \partial_{i_3} x^\mu \partial_{i_4} x^\nu \partial_i \delta(\sigma - \sigma'), \\ \{Z_{\text{NS}}^{[1]\mu}(\sigma), Z_{\text{RR}}^{[3]\nu_1 \nu_2 \nu_3}(\sigma')\} &= 2 \mathfrak{i} \epsilon^{ii_1 \dots i_4} \partial_{i_1} x^\mu \partial_{i_2} x^{\nu_1} \partial_{i_3} x^{\nu_2} \partial_{i_4} x^{\nu_3} \partial_i \delta(\sigma - \sigma'). \end{aligned} \quad (3.12)$$

Here each component is given by (3.10) but all the background fields in the momentum p_μ are replaced by the S-dualized ones.

Compared with the type IIA NS5-brane, the algebra (3.12) is characterized by the Poisson bracket $\{\cdot, \cdot\}$. This apparently comes from the fact that the theory is governed by the $\mathcal{N} = (1, 1)$ vector multiplet. There is no second class constraint in the effective theory and we do not care about the restricted phase space.

3.2 IIA KK5-brane

The type IIA KK5-brane is obtained by the T-duality transformation along a transverse direction to the IIB NS5-brane. The transformations of the background fields are given by the Buscher rule (2.16). Given the worldvolume action of the type IIB NS5-brane, we obtain the worldvolume effective action of the type IIA KK5-brane [34]. We also recombine the type IIB currents Z_M into those in the type IIA. The left hand sides are the IIB RR currents and the right hand sides are the IIA RR currents:

$$Z_{\text{RR}}^{[1]} \begin{cases} Z_{\text{RR}}^{[1]\mu} = Z_{\text{RR}}^{[2]\mu 9} \\ Z_{\text{RR}}^{[1]9} = Z_{\text{RR}}^{[0]} \end{cases}, \quad Z_{\text{RR}}^{[3]} \begin{cases} Z_{\text{RR}}^{[3]\mu\nu\rho} = Z_{\text{RR}}^{[4]\mu\nu\rho 9} \\ Z_{\text{RR}}^{[3]\mu\nu 9} = Z_{\text{RR}}^{[2]\mu\nu} \end{cases}, \quad Z_{\text{RR}}^{[5+]\mu_1 \dots \mu_4 9} = Z_{\text{RR}}^{[4]\mu_1 \dots \mu_4}.$$

We find that the non-zero components of the IIA KK5-brane algebra are given by

$$\begin{aligned} \{Z_\mu(\sigma), Z_{\text{NS}}^{[1]\nu}(\sigma')\} &= iE^i \delta_\mu^\nu \partial_i \delta(\sigma - \sigma'), \\ \{Z_9(\sigma), Z_{\text{NS}}^{[1]9}(\sigma')\} &= iE^i \partial_i \delta(\sigma - \sigma'), \\ \{Z_\mu(\sigma), Z_{\text{RR}}^{[2]\nu 9}(\sigma')\} &= i\epsilon^{ii_1 \dots i_4} F_{i_1 i_2} F_{i_3 i_4} \delta_\mu^\nu \partial_i \delta(\sigma - \sigma'), \\ \{Z_9(\sigma), Z_{\text{RR}}^{[0]}(\sigma')\} &= i\epsilon^{ii_1 \dots i_4} F_{i_1 i_2} F_{i_3 i_4} \partial_i \delta(\sigma - \sigma'), \\ \{Z_\mu(\sigma), Z_{\text{RR}}^{[4]\nu_1 \nu_2 \nu_3 9}(\sigma')\} &= \frac{i}{2!} \epsilon^{ii_1 \dots i_4} F_{i_1 i_2} \partial_{i_3} x^{[\nu_1} \partial_{i_4} x^{\nu_2} \delta_\mu^{\nu_3]} \partial_i \delta(\sigma - \sigma'), \\ \{Z_\mu(\sigma), Z_{\text{RR}}^{[2]\nu_1 \nu_2}(\sigma')\} &= i\epsilon^{ii_1 \dots i_4} F_{i_1 i_2} \partial_{i_3} \tilde{x}^9 \partial_{i_4} x^{[\nu_1} \delta_\mu^{\nu_2]} \partial_i \delta(\sigma - \sigma'), \\ \{Z_9(\sigma), Z_{\text{RR}}^{[2]\nu_1 \nu_2}(\sigma')\} &= \frac{i}{2!} \epsilon^{ii_1 \dots i_4} F_{i_1 i_2} \partial_{i_3} x^{[\nu_1} \partial_{i_4} x^{\nu_2]} \partial_i \delta(\sigma - \sigma'), \\ \{Z_\mu(\sigma), Z_{\text{RR}}^{[4]\nu_1 \dots \nu_4}(\sigma')\} &= \frac{i}{3!} \epsilon^{ii_1 \dots i_4} \partial_{i_1} \tilde{x}^9 \partial_{i_2} x^{[\nu_1} \partial_{i_3} x^{\nu_2} \partial_{i_4} x^{\nu_3} \delta_\mu^{\nu_4]} \partial_i \delta(\sigma - \sigma'), \\ \{Z_9(\sigma), Z_{\text{RR}}^{[4]\nu_1 \dots \nu_4}(\sigma')\} &= \frac{i}{4!} \epsilon^{ii_1 \dots i_4} \partial_{i_1} x^{[\nu_1} \dots \partial_{i_4} x^{\nu_4]} \partial_i \delta(\sigma - \sigma'), \\ \{Z_{\text{NS}}^{[1]\mu}(\sigma), Z_{\text{RR}}^{[2]\nu 9}(\sigma')\} &= 4i\epsilon^{ii_1 \dots i_4} F_{i_1 i_2} \partial_{i_3} x^\mu \partial_{i_4} x^\nu \partial_i \delta(\sigma - \sigma'), \\ \{Z_{\text{NS}}^{[1]\mu}(\sigma), Z_{\text{RR}}^{[0]}(\sigma')\} &= 4i\epsilon^{ii_1 \dots i_4} F_{i_1 i_2} \partial_{i_3} x^\mu \partial_{i_4} \tilde{x}^9 \partial_i \delta(\sigma - \sigma'), \\ \{Z_{\text{NS}}^{[1]9}(\sigma), Z_{\text{RR}}^{[2]\nu 9}(\sigma')\} &= 4i\epsilon^{ii_1 \dots i_4} F_{i_1 i_2} \partial_{i_3} \tilde{x}^9 \partial_{i_4} x^\nu \partial_i \delta(\sigma - \sigma'), \\ \{Z_{\text{NS}}^{[1]\mu}(\sigma), Z_{\text{RR}}^{[4]\nu_1 \nu_2 \nu_3 9}(\sigma')\} &= 2i\epsilon^{ii_1 \dots i_4} \partial_{i_1} x^\mu \partial_{i_2} x^{\nu_1} \partial_{i_3} x^{\nu_2} \partial_{i_4} x^{\nu_3} \partial_i \delta(\sigma - \sigma'), \\ \{Z_{\text{NS}}^{[1]9}(\sigma), Z_{\text{RR}}^{[4]\nu_1 \nu_2 \nu_3 9}(\sigma')\} &= 2i\epsilon^{ii_1 \dots i_4} \partial_{i_1} \tilde{x}^9 \partial_{i_2} x^{\nu_1} \partial_{i_3} x^{\nu_2} \partial_{i_4} x^{\nu_3} \partial_i \delta(\sigma - \sigma'), \\ \{Z_{\text{NS}}^{[1]\mu}(\sigma), Z_{\text{RR}}^{[2]\nu_1 \nu_2}(\sigma')\} &= 2i\epsilon^{ii_1 \dots i_4} \partial_{i_1} x^\mu \partial_{i_2} x^{\nu_1} \partial_{i_3} x^{\nu_2} \partial_{i_4} \tilde{x}^9 \partial_i \delta(\sigma - \sigma'), \end{aligned} \quad (3.13)$$

where each component is defined by (3.10) but all the background fields in the momenta p_μ, p_9 ($\mu \neq 9$) are replaced by the T-dualized ones.

3.3 IIB 5₂²-brane

A further T-duality transformation to another transverse isometric direction to the type IIB KK5-brane results in the exotic 5₂²-brane in type IIB theory. The worldvolume effective action of the type IIB 5₂²-brane has been obtained in [25, 26]. The non-zero components of the type IIB 5₂²-brane algebra are

$$\begin{aligned}
 \{Z_\mu(\sigma), Z_{\text{NS}}^{[1]\nu}(\sigma')\} &= iE^i \delta_\mu^\nu \partial_i \delta(\sigma - \sigma'), \\
 \{Z_9(\sigma), Z_{\text{NS}}^{[1]9}(\sigma')\} &= \{Z_8(\sigma), Z_{\text{NS}}^{[1]8}(\sigma')\} = iE^i \partial_i \delta(\sigma - \sigma'), \\
 \{Z_\mu(\sigma), Z_{\text{RR}}^{[1]\nu}(\sigma')\} &= i\epsilon^{i_1 \dots i_4} F_{i_1 i_2} F_{i_3 i_4} \delta_\mu^\nu \partial_i \delta(\sigma - \sigma'), \\
 \{Z_9(\sigma), Z_{\text{RR}}^{[1]9}(\sigma')\} &= \{Z_8(\sigma), Z_{\text{RR}}^{[1]8}(\sigma')\} = i\epsilon^{i_1 \dots i_4} F_{i_1 i_2} F_{i_3 i_4} \partial_i \delta(\sigma - \sigma'), \\
 \{Z_\mu(\sigma), Z_{\text{RR}}^{[3]\nu_1 \nu_2 \nu_3}(\sigma')\} &= \frac{i}{2!} \epsilon^{i_1 \dots i_4} F_{i_1 i_2} \partial_{i_3} x^{[\nu_1} \partial_{i_4} x^{\nu_2} \delta_\mu^{\nu_3]} \partial_i \delta(\sigma - \sigma'), \\
 \{Z_\mu(\sigma), Z_{\text{RR}}^{[3]\nu_1 \nu_2 9}(\sigma')\} &= i\epsilon^{i_1 \dots i_4} F_{i_1 i_2} \partial_{i_3} \tilde{x}^9 \partial_{i_4} x^{[\nu_1} \delta_\mu^{\nu_2]} \partial_i \delta(\sigma - \sigma'), \\
 \{Z_\mu(\sigma), Z_{\text{RR}}^{[3]\nu_1 \nu_2 9}(\sigma')\} &= i\epsilon^{i_1 \dots i_4} F_{i_1 i_2} \partial_{i_3} \tilde{x}^8 \partial_{i_4} x^{[\nu_1} \delta_\mu^{\nu_2]} \partial_i \delta(\sigma - \sigma'), \\
 \{Z_9(\sigma), Z_{\text{RR}}^{[3]\nu_1 \nu_2 9}(\sigma')\} &= \{Z_8(\sigma), Z_{\text{RR}}^{[3]\nu_1 \nu_2 8}(\sigma')\} \\
 &= \frac{i}{2!} \epsilon^{i_1 \dots i_4} F_{i_1 i_2} \partial_{i_3} x^{[\nu_1} \partial_{i_4} x^{\nu_2]} \partial_i \delta(\sigma - \sigma'), \\
 \{Z_\mu(\sigma), Z_{\text{RR}}^{[3]\nu_1 8 9}(\sigma')\} &= i\epsilon^{i_1 \dots i_4} F_{i_1 i_2} \partial_{i_3} \tilde{x}^8 \partial_{i_4} \tilde{x}^9 \delta_\mu^{\nu_1} \partial_i \delta(\sigma - \sigma'), \\
 \{Z_9(\sigma), Z_{\text{RR}}^{[3]\nu_1 8 9}(\sigma')\} &= i\epsilon^{i_1 \dots i_4} F_{i_1 i_2} \partial_{i_3} x^{\nu_1} \partial_{i_4} \tilde{x}^8 \partial_i \delta(\sigma - \sigma'), \\
 \{Z_8(\sigma), Z_{\text{RR}}^{[3]\nu_1 9 8}(\sigma')\} &= i\epsilon^{i_1 \dots i_4} F_{i_1 i_2} \partial_{i_3} x^{\nu_1} \partial_{i_4} \tilde{x}^9 \partial_i \delta(\sigma - \sigma'), \\
 \{Z_\mu(\sigma), Z_{\text{RR}}^{[5]\nu_1 \dots \nu_5}(\sigma')\} &= \frac{i}{4!} \epsilon^{i_1 \dots i_4} \partial_{i_1} x^{[\nu_1} \dots \partial_{i_4} x^{\nu_4} \delta_\mu^{\nu_5]} \partial_i \delta(\sigma - \sigma'), \\
 \{Z_\mu(\sigma), Z_{\text{RR}}^{[5]\nu_1 \dots \nu_4 9}(\sigma')\} &= \frac{i}{3!} \epsilon^{i_1 \dots i_4} \partial_{i_1} \tilde{x}^9 \partial_{i_2} x^{[\nu_1} \partial_{i_3} x^{\nu_2} \partial_{i_4} x^{\nu_3} \delta_\mu^{\nu_4]} \partial_i \delta(\sigma - \sigma'), \\
 \{Z_\mu(\sigma), Z_{\text{RR}}^{[5]\nu_1 \dots \nu_4 8}(\sigma')\} &= \frac{i}{3!} \epsilon^{i_1 \dots i_4} \partial_{i_1} \tilde{x}^8 \partial_{i_2} x^{[\nu_1} \partial_{i_3} x^{\nu_2} \partial_{i_4} x^{\nu_3} \delta_\mu^{\nu_4]} \partial_i \delta(\sigma - \sigma'), \\
 \{Z_9(\sigma), Z_{\text{RR}}^{[5]\nu_1 \dots \nu_4 9}(\sigma')\} &= \{Z_8(\sigma), Z_{\text{RR}}^{[5]\nu_1 \dots \nu_4 8}(\sigma')\} = \frac{i}{4!} \epsilon^{i_1 \dots i_4} \partial_{i_1} x^{[\nu_1} \dots \partial_{i_4} x^{\nu_4]} \partial_i \delta(\sigma - \sigma'), \\
 \{Z_\mu(\sigma), Z_{\text{RR}}^{[5]\nu_1 \nu_2 \nu_3 8 9}(\sigma')\} &= \frac{i}{2!} \epsilon^{i_1 \dots i_4} \partial_{i_1} \tilde{x}^8 \partial_{i_2} \tilde{x}^9 \partial_{i_3} x^{[\nu_1} \partial_{i_4} x^{\nu_2} \delta_\mu^{\nu_3]} \partial_i \delta(\sigma - \sigma'), \\
 \{Z_9(\sigma), Z_{\text{RR}}^{[5]\nu_1 \nu_2 \nu_3 8 9}(\sigma')\} &= \frac{i}{3!} \epsilon^{i_1 \dots i_4} \partial_{i_1} x^{[\nu_1} \partial_{i_2} x^{\nu_2} \partial_{i_3} x^{\nu_3]} \partial_{i_4} \tilde{x}^8 \partial_i \delta(\sigma - \sigma'), \\
 \{Z_8(\sigma), Z_{\text{RR}}^{[5]\nu_1 \nu_2 \nu_3 9 8}(\sigma')\} &= \frac{i}{3!} \epsilon^{i_1 \dots i_4} \partial_{i_1} x^{[\nu_1} \partial_{i_2} x^{\nu_2} \partial_{i_3} x^{\nu_3]} \partial_{i_4} \tilde{x}^9 \partial_i \delta(\sigma - \sigma'), \\
 \{Z_{\text{NS}}^{[1]\mu}(\sigma), Z_{\text{RR}}^{[1]\nu}(\sigma')\} &= 4i\epsilon^{i_1 \dots i_4} F_{i_1 i_2} \partial_{i_3} x^\mu \partial_{i_4} x^\nu \partial_i \delta(\sigma - \sigma'), \\
 \{Z_{\text{NS}}^{[1]\mu}(\sigma), Z_{\text{RR}}^{[1]9}(\sigma')\} &= 4i\epsilon^{i_1 \dots i_4} F_{i_1 i_2} \partial_{i_3} x^\mu \partial_{i_4} \tilde{x}^9 \partial_i \delta(\sigma - \sigma'), \\
 \{Z_{\text{NS}}^{[1]9}(\sigma), Z_{\text{RR}}^{[1]\nu}(\sigma')\} &= 4i\epsilon^{i_1 \dots i_4} F_{i_1 i_2} \partial_{i_3} \tilde{x}^9 \partial_{i_4} x^\nu \partial_i \delta(\sigma - \sigma'), \\
 \{Z_{\text{NS}}^{[1]\mu}(\sigma), Z_{\text{RR}}^{[1]8}(\sigma')\} &= 4i\epsilon^{i_1 \dots i_4} F_{i_1 i_2} \partial_{i_3} x^\mu \partial_{i_4} \tilde{x}^8 \partial_i \delta(\sigma - \sigma'),
 \end{aligned}$$

$$\begin{aligned}
 \{Z_{\text{NS}}^{[1]8}(\sigma), Z_{\text{RR}}^{[1]\nu}(\sigma')\} &= 4i\epsilon^{i_1 \dots i_4} F_{i_1 i_2} \partial_{i_3} \tilde{x}^8 \partial_{i_4} x^\nu \partial_i \delta(\sigma - \sigma'), \\
 \{Z_{\text{NS}}^{[1]9}(\sigma), Z_{\text{RR}}^{[1]8}(\sigma')\} &= 4i\epsilon^{i_1 \dots i_4} F_{i_1 i_2} \partial_{i_3} \tilde{x}^9 \partial_{i_4} \tilde{x}^8 \partial_i \delta(\sigma - \sigma'), \\
 \{Z_{\text{NS}}^{[1]8}(\sigma), Z_{\text{RR}}^{[1]9}(\sigma')\} &= 4i\epsilon^{i_1 \dots i_4} F_{i_1 i_2} \partial_{i_3} \tilde{x}^8 \partial_{i_4} \tilde{x}^9 \partial_i \delta(\sigma - \sigma'), \\
 \{Z_{\text{NS}}^{[1]\mu}(\sigma), Z_{\text{RR}}^{[3]\nu_1 \nu_2 \nu_3}(\sigma')\} &= 2i\epsilon^{i_1 \dots i_4} \partial_{i_1} x^\mu \partial_{i_2} x^{\nu_1} \partial_{i_3} x^{\nu_2} \partial_{i_4} x^{\nu_3} \partial_i \delta(\sigma - \sigma'), \\
 \{Z_{\text{NS}}^{[1]9}(\sigma), Z_{\text{RR}}^{[3]\nu_1 \nu_2 \nu_3}(\sigma')\} &= 2i\epsilon^{i_1 \dots i_4} \partial_{i_1} \tilde{x}^9 \partial_{i_2} x^{\nu_1} \partial_{i_3} x^{\nu_2} \partial_{i_4} x^{\nu_3} \partial_i \delta(\sigma - \sigma'), \\
 \{Z_{\text{NS}}^{[1]\mu}(\sigma), Z_{\text{RR}}^{[3]\nu_1 \nu_2 9}(\sigma')\} &= 2i\epsilon^{i_1 \dots i_4} \partial_{i_1} x^\mu \partial_{i_2} x^{\nu_1} \partial_{i_3} x^{\nu_2} \partial_{i_4} \tilde{x}^9 \partial_i \delta(\sigma - \sigma'), \\
 \{Z_{\text{NS}}^{[1]8}(\sigma), Z_{\text{RR}}^{[3]\nu_1 \nu_2 \nu_3}(\sigma')\} &= 2i\epsilon^{i_1 \dots i_4} \partial_{i_1} \tilde{x}^8 \partial_{i_2} x^{\nu_1} \partial_{i_3} x^{\nu_2} \partial_{i_4} x^{\nu_3} \partial_i \delta(\sigma - \sigma'), \\
 \{Z_{\text{NS}}^{[1]\mu}(\sigma), Z_{\text{RR}}^{[3]\nu_1 \nu_2 8}(\sigma')\} &= 2i\epsilon^{i_1 \dots i_4} \partial_{i_1} x^\mu \partial_{i_2} x^{\nu_1} \partial_{i_3} x^{\nu_2} \partial_{i_4} \tilde{x}^8 \partial_i \delta(\sigma - \sigma'), \\
 \{Z_{\text{NS}}^{[1]\mu}(\sigma), Z_{\text{RR}}^{[3]\nu_1 8 9}(\sigma')\} &= 2i\epsilon^{i_1 \dots i_4} \partial_{i_1} x^\mu \partial_{i_2} x^{\nu_1} \partial_{i_3} \tilde{x}^8 \partial_{i_4} \tilde{x}^9 \partial_i \delta(\sigma - \sigma'), \\
 \{Z_{\text{NS}}^{[1]9}(\sigma), Z_{\text{RR}}^{[3]\nu_1 \nu_2 8}(\sigma')\} &= 2i\epsilon^{i_1 \dots i_4} \partial_{i_1} \tilde{x}^9 \partial_{i_2} x^{\nu_1} \partial_{i_3} x^{\nu_2} \partial_{i_4} \tilde{x}^8 \partial_i \delta(\sigma - \sigma'), \\
 \{Z_{\text{NS}}^{[1]8}(\sigma), Z_{\text{RR}}^{[3]\nu_1 \nu_2 9}(\sigma')\} &= 2i\epsilon^{i_1 \dots i_4} \partial_{i_1} \tilde{x}^8 \partial_{i_2} x^{\nu_1} \partial_{i_3} x^{\nu_2} \partial_{i_4} \tilde{x}^9 \partial_i \delta(\sigma - \sigma'), \tag{3.14}
 \end{aligned}$$

where each component is given by (3.10) but all the background fields in the momenta p_μ, p_8, p_9 ($\mu \neq 8, 9$) are replaced by the T-dualized ones.

We have now written down the current algebras of the five-branes in type II string theories. A comment on the further T-duality transformations is in order. Since there are two yet untouched transverse directions to the five-branes, there remain two further T-dualized five-branes. They are denoted as 5_2^3 - and 5_2^4 -branes and have distinct nature compared with the 5_2^b -branes ($b = 0, 1, 2$) discussed above. We will address these issues in section 5.

In the following, we show that the worldvolume currents of five-branes give rise to the notion of the extended (doubled or exceptional) spaces in DFT and EFT and they are directly tied with the supersymmetry algebras.

4 Extended spaces from current algebras

In this section, after the review of the current algebra approach [3–5, 37] to the double field theory, we extend this approach to the exceptional field theory by using brane current algebras. We focus on brane currents which correspond to brane charges in superalgebras, since a superalgebra includes not only perturbative states but also non-perturbative states representing S-duality. Supersymmetric branes in ten-dimensional $\mathcal{N} = 2$ theories are classified in [38, 39] by the superalgebras of 32 supercharges Q_α

$$\{Q_\alpha, Q_\beta\} = Z_{\alpha\beta} = Z_M \Gamma^M_{\alpha\beta}. \tag{4.1}$$

The index M is decomposed by the form and $(\Gamma^M)_{\alpha\beta}$ is anti-symmetric gamma matrices depending on theories. The bosonic subalgebra Z_M includes not only the momentum and the winding mode but also brane charges, such as KK5-brane, NS5-brane, D5-brane. Exotic branes are variations of them obtained by T-dualities. We consider extended spaces

spanned by Z_M . Coordinates are X^M corresponding to $Z_M = \frac{1}{i} \frac{\partial}{\partial X^M}$. We derive section conditions and generalized Courant brackets from brane current algebras where the Courant bracket from the M5-brane in the five-dimensional torus space is the one for $E_{5(5)}$ [19]. The bracket gives generalized coordinate transformations in the extended space. An analysis on the conserved current in the doubled formalism is found in [40].

4.1 Doubled space from the string current algebra

The string current algebra generated by the current $Z_M(\sigma) = (p_\mu, \partial_\sigma x^\mu)$ is given by

$$\{Z_M(\sigma), Z_N(\sigma')\} = i\eta_{MN}\partial_\sigma\delta(\sigma - \sigma'). \quad (4.2)$$

Virasoro constraints for a string in curved backgrounds are written in bilinear of bosonic currents Z_M and the vielbein $E_A{}^M(X^N)$ as

$$\begin{cases} \mathcal{H}_\perp = \frac{1}{2}Z_M\mathcal{M}^{MN}Z_N = \frac{1}{2}Z_ME_A{}^M\hat{\eta}^{AB}E_B{}^NZ_N \\ \mathcal{H}_\sigma = \frac{1}{2}Z_M\eta^{MN}Z_N \end{cases} \quad (4.3)$$

where $\hat{\eta}^{AB}$ is the doubled D-dimensional Minkowski metric and η^{MN} is the $O(D,D)$ invariant metric. The vielbein satisfies the orthogonal condition $E_A{}^ME_B{}^N\eta_{MN} = \eta_{AB}$. The index M is raised and lowered by η_{MN} and η^{MN} . The vielbein is a coset element of $O(D,D)/SO(D-1,1)^2$ which is parametrized as

$$E_A{}^M = \begin{pmatrix} e_a{}^\nu & 0 \\ 0 & e_\nu{}^a \end{pmatrix} \begin{pmatrix} \delta_\nu{}^\mu & B_{\nu\mu} \\ 0 & \delta_\mu{}^\nu \end{pmatrix}. \quad (4.4)$$

T-duality is $O(D,D)$ rotation which acts linearly in Z_M basis.

For a doubled space function $\Phi(X)$ the σ component of the Virasoro operator defines the σ derivative as

$$\partial_\sigma\Phi(X^M) = i\left\{\int d\sigma'\mathcal{H}_\sigma(\sigma'), \Phi(X(\sigma))\right\}. \quad (4.5)$$

As is well known, the σ -translational invariance of closed strings results in the level matching condition for physical states. The section condition on the doubled space functions $\Phi(X)$ and $\Psi(X)$ are imposed to guarantee that the physical quantities are inert under the σ -diffeomorphism

$$\partial_M\eta^{MN}\partial_N = 0 \Rightarrow \partial_M\eta^{MN}\partial_N\Phi(X) = \partial_M\Phi(X)\eta^{MN}\partial_N\Psi(X) = 0. \quad (4.6)$$

This is nothing but the weak and the strong constraints in DFT [4–7]. Introducing components of the doubled coordinate $X^M = (x^\mu, \tilde{x}_\mu)$ the section condition is written as

$$\partial_\mu \frac{\partial}{\partial \tilde{x}_\mu} = 0. \quad (4.7)$$

For two vectors in the doubled space $\Lambda_i{}^M$ $i = 1, 2$ the canonical bracket (4.2) between $\Lambda_i{}^MZ_M$ is calculated as follows

$$\begin{aligned} & \{\Lambda_1{}^MZ_M(\sigma), \Lambda_2{}^NZ_N(\sigma')\} \\ &= \frac{1}{i} \left(\Lambda_{[1}{}^N\partial_N\Lambda_2]{}^M - \frac{1-K}{2}\Lambda_1{}^N\partial^M\Lambda_{2;N} + \frac{1+K}{2}(\partial^M\Lambda_1{}^N)\Lambda_{2;N} \right) Z_M\delta(\sigma - \sigma') \\ &+ i \left(\frac{1-K}{2}\Lambda_1{}^N\Lambda_{2;N}(\sigma) + \frac{1+K}{2}\Lambda_1{}^N\Lambda_{2;N}(\sigma') \right) \partial_\sigma\delta(\sigma - \sigma') \end{aligned} \quad (4.8)$$

where we have introduced an arbitrary real number K . It is known that the structures of the Courant and the Dorfman brackets are extracted from the regular part of the current algebra (4.8) [17]. Namely, the Courant bracket of two vectors $\hat{\Lambda}_i = \Lambda_i^M Z_M$ is obtained from the second line of the current algebra (4.8) with $K = 0$ as

$$[\hat{\Lambda}_1, \hat{\Lambda}_2]_C = \frac{1}{i} \hat{\Lambda}_{12}, \quad \Lambda_{12}^M = \Lambda_{[1}^N \partial_N \Lambda_2]^{M} - \frac{1}{2} \Lambda_{[1} \partial^M \Lambda_2]_{;N}. \quad (4.9)$$

The choice $K = 0$ guarantees the anti-symmetric nature of the Courant bracket. Similarly, the Dorfman bracket is given with $K = 1$ as

$$[\hat{\Lambda}_1, \hat{\Lambda}_2]_D = \frac{1}{i} \hat{\Lambda}_{12}, \quad \Lambda_{12}^M = \Lambda_{[1}^N \partial_N \Lambda_2]^{M} + (\partial^M \Lambda_1^N) \Lambda_{2;N}. \quad (4.10)$$

The generalized coordinate transformation is obtained by these brackets with the gauge parameter. Since the D-dimensional vector component of the vielbein field $E_a^M = (e_a^\mu, e_a^\nu B_{\nu\mu})$ is enough to determine the gauge transformation of the vielbein, we calculate the following canonical commutator between the gauge parameter vector $\Lambda^M = (\lambda^\mu, \lambda_\mu)$ and E_a^M as

$$\begin{aligned} \delta_\lambda E_a^M Z_M(\sigma) &= i \left\{ \int d\sigma' \lambda^N Z_N(\sigma'), E_a^M Z_M(\sigma) \right\} \\ \delta_\lambda E_a^M &= \lambda^N \partial_N E_a^M - E_a^N \partial_N \lambda^M + E_a^N \partial_L \lambda^K \eta_{NK} \eta^{ML}. \end{aligned} \quad (4.11)$$

In component we have

$$\begin{cases} \delta_\lambda e_a^\mu = \mathcal{L}_\lambda e_a^\mu + (\lambda_\nu \partial^\nu e_a^\mu + e_a^\nu \partial^\mu \lambda_\nu) + e_a^\rho B_{\rho\nu} \partial^{[\mu} \lambda^{\nu]} \\ \delta_\lambda B_{\rho\nu} = \mathcal{L}_\lambda B_{\rho\nu} - \partial_{[\rho} \lambda_{\nu]} + (\lambda_\nu \partial^\nu B_{\rho\mu} - B_{\rho\nu} \partial^\nu \lambda_\mu - B_{\nu\mu} \partial^\nu \lambda_\rho) + B_{\rho\nu} B_{\sigma\mu} \partial^{[\nu} \lambda^{\sigma]} \end{cases} \quad (4.12)$$

where $\mathcal{L}_\lambda$ is the usual Lie derivative. After solving the section conditions by setting $\frac{\partial}{\partial \tilde{x}_\mu} = 0$, the generalized coordinate transformations (4.12) reduces to the usual gauge transformations.

4.2 Supersymmetry algebras in ten dimensions

We begin by the eleven-dimensional $\mathcal{N} = 1$ superalgebra which is given by

$$\begin{aligned} \{Q_\alpha, Q_\beta\} &= P_m (C \gamma^m)_{\alpha\beta} + \frac{1}{2!} Z^{[2]mn} (C \gamma_{mn})_{\alpha\beta} \\ &\quad + \frac{1}{5!} Z^{[5]m_1 m_2 m_3 m_4 m_5} (C \gamma_{m_1 m_2 m_3 m_4 m_5})_{\alpha\beta}, \end{aligned} \quad (4.13)$$

where C is the charge conjugation matrix. The number of bosonic charges, P_m , $Z^{[2]}$ and $Z^{[5]}$, is $11 + 55 + 462 = 32 \times 33/2 = 528$. $Z^{[2]}$ gives a M2-brane, while $Z^{[5]}$ gives a M5-brane and a 6-brane.

In $D = 10$ the IIA theory has two 10-dimensional Majorana-Weyl supercharges which have the opposite chirality, $Q_{A\alpha}$ with $A = 1, 2$ and $\alpha = 1, \dots, 16$. The IIA superalgebra is

given by

$$\begin{aligned}
 \{Q_{A\alpha}, Q_{B\beta}\} = & P_\mu \delta_{AB} (C\gamma^\mu)_{\alpha\beta} + \frac{1}{5!} Z_{\text{KK}}^{[5(+)]\mu_1\mu_2\mu_3\mu_4\mu_5} \delta_{AB} (C\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5})_{\alpha\beta} \\
 & + Z_{\text{NS}}^{[1]\mu} (\tau_3)_{AB} (C\gamma_\mu)_{\alpha\beta} + \frac{1}{5!} Z_{\text{NS}}^{[5(-)]\mu_1\mu_2\mu_3\mu_4\mu_5} (\tau_3)_{AB} (C\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5})_{\alpha\beta} \\
 & + Z_{\text{RR}}^{[0]} (i\tau_2)_{AB} C_{\alpha\beta} + \frac{1}{2!} Z_{\text{RR}}^{[2]\mu\nu} (\tau_1)_{AB} (C\gamma_{\mu\nu})_{\alpha\beta} \\
 & + \frac{1}{4!} Z_{\text{RR}}^{[4]\mu_1\mu_2\mu_3\mu_4} (i\tau_2)_{AB} (C\gamma_{\mu_1\mu_2\mu_3\mu_4})_{\alpha\beta}.
 \end{aligned} \tag{4.14}$$

Here notation is followed in [41] in which magnetic charges are not included yet. The number of bosonic charges, P_μ , $Z_{\text{KK}}^{[5(+)]}$, $Z_{\text{NS}}^{[1]}$, $Z_{\text{NS}}^{[5(-)]}$, $Z_{\text{RR}}^{[0]}$, $Z_{\text{RR}}^{[2]}$, $Z_{\text{RR}}^{[4]}$ is $(10 + 126) + (10 + 126) + (1 + 45 + 210) = 528$. Branes include a KK monopole with $Z_{\text{KK}}^{[5(+)]}$, a fundamental string with $Z_{\text{NS}}^{[1]}$, a fundamental 5 brane with $Z_{\text{NS}}^{[5(-)]}$, D0 with $Z_{\text{RR}}^{[0]}$, D2 and D8 with $Z_{\text{RR}}^{[2]}$ and D4 and D6 with $Z_{\text{RR}}^{[4]}$.

In $D = 10$ the IIB theory has two 10-dimensional Majorana-Weyl supercharges which have the same chirality, $Q_{A\alpha}$ with $A = 1, 2$ and $\alpha = 1, \dots, 16$. The IIB superalgebra is given by

$$\begin{aligned}
 \{Q_{A\alpha}, Q_{B\beta}\} = & P_\mu \delta_{AB} (C\gamma^\mu)_{\alpha\beta} + \frac{1}{5!} Z_{\text{KK}}^{[5+]\mu_1\mu_2\mu_3\mu_4\mu_5} \delta_{AB} (C\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5})_{\alpha\beta} \\
 & + Z_{\text{NS}}^{[1]\mu} (\tau_3)_{AB} (C\gamma_\mu)_{\alpha\beta} + \frac{1}{5!} Z_{\text{NS}}^{[5+]\mu_1\mu_2\mu_3\mu_4\mu_5} (\tau_3)_{AB} (C\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5})_{\alpha\beta} \\
 & + Z_{\text{RR}}^{[1]\mu} (\tau_1)_{AB} (C\gamma_\mu)_{\alpha\beta} + \frac{1}{3!} Z_{\text{RR}}^{[3]\mu\nu\rho} (i\tau_2)_{AB} (C\gamma_{\mu\nu\rho})_{\alpha\beta} \\
 & + \frac{1}{5!} Z_{\text{RR}}^{[5+]\mu_1\mu_2\mu_3\mu_4\mu_5} (\tau_1)_{AB} (C\gamma_{\mu_1\mu_2\mu_3\mu_4\mu_5})_{\alpha\beta}.
 \end{aligned} \tag{4.15}$$

The number of bosonic charges, P_m , $Z_{\text{KK}}^{[5+]}$, $Z_{\text{NS}}^{[1]}$, $Z_{\text{NS}}^{[5+]}$, $Z_{\text{RR}}^{[1]}$, $Z_{\text{RR}}^{[3]}$, $Z_{\text{RR}}^{[5+]}$ is $(10 + 126) + (10 + 126) + (10 + 120 + 126) = 528$. Branes include a KK monopole with $Z_{\text{KK}}^{[5+]}$, a fundamental string with $Z_{\text{NS}}^{[1]}$, a fundamental 5 brane with $Z_{\text{NS}}^{[5+]}$, D1 and D9 with $Z_{\text{RR}}^{[1]}$, D3 and D7 with $Z_{\text{RR}}^{[3]}$ and D5 with $Z_{\text{RR}}^{[5+]}$.

T-duality interchanges the IIA theory and the IIB theory. Let us consider the T-duality transformation in the x^9 direction which interchanges the momentum P_9 and the winding mode $Z_{\text{NS}}^{[1]9}$. In order to involve spinors the double space coordinates must be representations of the left and the right Lorentz groups, $P_L^\mu = P^\mu + Z_{\text{NS}}^{[1]\mu}$, $P_R^\mu = P^\mu - Z_{\text{NS}}^{[1]\mu}$, $Q_L = Q_1$ and $Q_R = Q_2$. Under the T-duality transformation double space coordinates are transformed as follows

$$\begin{pmatrix} P_L^{\mu'} \\ P_L^9 \\ P_R^{\mu'} \\ P_R^9 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} P_L^{\mu'} \\ P_L^9 \\ P_R^{\mu'} \\ P_R^9 \end{pmatrix}, \quad \mu' \neq 9. \tag{4.16}$$

The D=10, N=2 supercharges are transformed under the T-duality

$$(Q'_L, Q'_R) = (Q_L, Q_R) \begin{pmatrix} 1 & 0 \\ 0 & \gamma^9 \end{pmatrix} \Rightarrow \begin{cases} \{Q_L, Q_L\} = P_L^\mu C \gamma_\mu + \dots \\ \{Q'_R, Q'_R\} = P_R'^\mu C \gamma_\mu + \dots \\ \{Q_L, Q'_R\} = Z_{RR;M} \Gamma^M \gamma^9 \end{cases} \quad (4.17)$$

which changes the forms of the RR charges by γ^9 .

The RR forms are transformed under the T-duality along x^9 -direction summarized as below.

- **From IIA to IIB** In addition to the interchanging the momentum and the winding mode the five forms are interchanged as

$$P_9 \leftrightarrow Z_{NS}^{[1]9}, \quad Z_{KK}^{[5(+)]9\mu'_1\mu'_2\mu'_3\mu'_4} \leftrightarrow Z_{NS}^{[5(-)]9\mu'_1\mu'_2\mu'_3\mu'_4}, \quad \mu' \neq 9. \quad (4.18)$$

The IIB RR charges are read off from superalgebras, and they are written in terms of the IIA RR charges:

$$\begin{array}{ll} \text{IIB} & \text{IIA} \\ Z_{RR}^{[1]} & (Z_{RR}^{[2]}, Z_{RR}^{[0]}) \\ Z_{RR}^{[3]} & (Z_{RR}^{[4]}, Z_{RR}^{[2]}) \\ Z_{RR}^{[5+]} & Z_{RR}^{[4]} \end{array} \quad (4.19)$$

- **From IIB to IIA**

Analogously for the T-duality from the IIB to IIA the five forms are interchanged as

$$P_9 \leftrightarrow Z_{NS}^{[1]9}, \quad Z_{KK}^{[5+]9\mu'_1\mu'_2\mu'_3\mu'_4} \leftrightarrow Z_{NS}^{[5+]9\mu'_1\mu'_2\mu'_3\mu'_4}, \quad \mu' \neq 9. \quad (4.20)$$

The IIA RR charges are written in terms of the IIB RR charges:

$$\begin{array}{ll} \text{IIA} & \text{IIB} \\ Z_{RR}^{[0]} & Z_{RR}^{[1]} \\ Z_{RR}^{[2]} & (Z_{RR}^{[3]}, Z_{RR}^{[1]}) \\ Z_{RR}^{[4]} & (Z_{RR}^{[3]}, Z_{RR}^{[5+]}) \end{array} \quad (4.21)$$

4.3 Section conditions

Now we go back to brane currents $Z_M(\sigma)$ which are functions of worldvolume coordinates. They correspond to the bosonic charges Z_M of the IIA or IIB theories in (4.14) or (4.15). Extended spaces are defined by current algebras generated by currents $Z_M(\sigma)$ as shown before. In order to recover the physical 10-dimensional space we impose section conditions in the extended space.

$\mathcal{N} = (2, 0)$ theory. One finds that the worldvolume spatial diffeomorphism constraints are translated into the vanishing condition of the following quantity [17–19]:

$$Z_M \tilde{\rho}^{MN} Z_N = 0, \quad M=(\mu, Y, \mu_1\mu_2, \mu_1Y, \mu_1\cdots\mu_5, \mu_1\cdots\mu_4Y) \quad (4.22)$$

where $\tilde{\rho}^{MN}$ is given by

$$\begin{bmatrix} 0 & 0 & a_{[\nu_1}\delta_{\nu_2]}^\mu & 0 & b_{[\nu_1\cdots\nu_4}\delta_{\nu_5]}^\mu & 0 \\ 0 & 0 & 0 & a_{\nu_1} & 0 & b_{[\nu_1\cdots\nu_4]} \\ a_{[\mu_1}\delta_{\mu_2]}^\nu & 0 & b_{[\mu_1\mu_2\nu_1\nu_2]} & b_{[\mu_1\mu_2\nu_1Y]} & 0 & 0 \\ 0 & a_{\mu_1} & b_{[\mu_1\nu_1\nu_2Y]} & 0 & 0 & 0 \\ b_{[\mu_1\cdots\mu_4}\delta_{\mu_5]}^\nu & 0 & 0 & 0 & 0 & 0 \\ 0 & b_{[\mu_1\cdots\mu_4]} & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (4.23)$$

Here $a_\mu, b_{[\mu_1\mu_2\mu_3\mu_4]}$ are arbitrary constants. The constraint (4.22) is expressed by the derivative representations of the currents. For the decomposition of the currents

$$Z_M = (Z_\mu, Z_{\text{KK}}^{[5(+)]\mu_1\mu_2\mu_3\mu_4\mu_5}, Z_{\text{NS}}^\mu, Z_{\text{NS}}^{[5(-)]\mu_1\mu_2\mu_3\mu_4\mu_5}, Z_{\text{RR}}^{[0]}, Z_{\text{RR}}^{[2]\mu_1\mu_2}, Z_{\text{RR}}^{[4]\mu_1\mu_2\mu_3\mu_4}), \quad (4.24)$$

the coordinates in the extended space are denoted as

$$X^M = (x^\mu, x_{\mu_1\mu_2\mu_3\mu_4\mu_5}, \tilde{x}_\mu, \tilde{x}_{\mu_1\mu_2\mu_3\mu_4\mu_5}, y, y_{\mu_1\mu_2}, y_{\mu_1\mu_2\mu_3\mu_4}). \quad (4.25)$$

Then the Virasoro constraints (4.22) are explicitly written as

$$\begin{cases} Z_\mu Z_{\text{NS}}^{[1]\mu} = 0 \\ Z_\mu Z_{\text{RR}}^{[2]\mu\nu} + Z_{\text{RR}}^{[0]} Z_{\text{NS}}^{[1]\mu} = 0 \\ Z_\mu Z_{\text{RR}}^{[5]\mu\nu_1\nu_2\nu_3\nu_4} + Z_{\text{RR}}^{[0]} Z_{\text{RR}}^{[4]\nu_1\nu_2\nu_3\nu_4} + \frac{1}{8} Z_{\text{RR}}^{[2][\nu_1\nu_2]} Z_{\text{RR}}^{[2]\nu_3\nu_4} = 0 \\ Z_\mu Z_{\text{RR}}^{[4]\nu_1\nu_2\nu_3\nu_4} + \frac{1}{2} Z_{\text{RR}}^{[2][\nu_1\nu_2]} Z_{\text{NS}}^{[1]\nu_3} = 0 \end{cases}. \quad (4.26)$$

The constraint (4.22) is therefore given by the following section conditions in the extended space:

$$\begin{cases} \partial_\mu \frac{\partial}{\partial \tilde{x}_\mu} = 0 \\ \partial_\mu \frac{\partial}{\partial y_{\mu\nu}} - \frac{\partial}{\partial y} \frac{\partial}{\partial \tilde{x}_\nu} = 0 \\ \partial_\mu \frac{\partial}{\partial x_{\mu\nu_1\nu_2\nu_3\nu_4}} + \frac{\partial}{\partial y} \frac{\partial}{\partial y_{\nu_1\nu_2\nu_3\nu_4}} + \frac{1}{8} \frac{\partial}{\partial y_{[\nu_1\nu_2}} \frac{\partial}{\partial y_{\nu_3\nu_4]}} = 0 \\ \partial_\mu \frac{\partial}{\partial y_{\mu\nu_1\nu_2\nu_3\nu_4}} + \frac{1}{2} \frac{\partial}{\partial \tilde{x}_{[\nu_1}} \frac{\partial}{\partial y_{\nu_2\nu_3]}} = 0 \end{cases}. \quad (4.27)$$

We stress that the “section condition” (4.27) is defined in the ten-dimensional (uncompactified) spacetime. Therefore this provides a generalization of the section condition (4.7) in DFT. Indeed, when the target space is restricted to five-dimensions and the index μ runs $1, \dots, 5$, the matrix $\tilde{\rho}^{MN}$ in (4.22) is expanded by $\tilde{\rho}^{MN} = \tilde{\rho}^{\mu MN} A_\mu$ where $A_\mu = (a_\mu, b_{[\mu_1\cdots\mu_4]})$. In this case, one can show that $\tilde{\rho}^{\mu MN}$ are the (5+5)-dimensional gamma matrices and $\tilde{\rho}^{MN}$ becomes the $SO(5, 5)$ covariant tensor [19]. The condition (4.27) then reproduces the section condition in the $SO(5, 5)$ EFT in six dimensions [10–14].

$\mathcal{N} = (1, 1)$ theory. The worldvolume diffeomorphism constraints $\mathcal{H}_i = 0$ is written in a bilinear form by contracting with E^i , $\epsilon^{i_1 \dots i_5} F_{i_1 i_2} F_{i_3 i_4} \partial_{i_5} x^\mu$ and $\epsilon^{i_1 \dots i_5} F_{i_1 i_2} \partial_{i_3} x^{\mu_1} \partial_{i_4} x^{\mu_2} \partial_{i_5} x^{\mu_3}$, $\epsilon^{i_1 \dots i_5} \partial_{i_1} x^{\mu_1} \dots \partial_{i_5} x^{\mu_5}$,

$$Z_M \tilde{\rho}^{MN} Z_N = 0, \quad M=(\mu, \text{NS}\mu, \text{RR}\mu, \mu_1\mu_2\mu_3, \mu_1\dots\mu_5)$$

$$\tilde{\rho}^{MN} = \left(\begin{array}{cc|ccc} 0 & a\delta_\nu^\mu & b\delta_\nu^\mu & \dots & c_{[5]}^\mu \\ a\delta_\mu^\nu & 0 & \dots & c_{[3]\mu} & 0 \\ \hline b\delta_\mu^\nu & \vdots & & & \\ \vdots & c_{[3]\nu} & & & \\ c_{[5]}^\nu & 0 & & & \end{array} \right)$$

$$c_{[5]}^\mu = \gamma_{[\nu_1 \dots \nu_4} \delta_{\nu_5]}^\mu, \quad c_{[3]\mu} = -5C_2 \gamma_{[\nu_1 \dots \nu_3\mu]} \quad (4.28)$$

with arbitrary coefficients a , b , γ . In concretely it is given as

$$\begin{cases} p_\mu Z_{NS}^{[1]\mu} = 0 \\ p_\mu Z_{RR}^{[1]\mu} = 0 \\ p_\mu Z_{RR}^{[3]\mu\mu_1\mu_2} + \frac{1}{2} Z_{NS}^{[1][\mu_1} Z_{RR}^{[1]\mu_2]} = 0 \\ p_\mu Z_{RR}^{[5]\mu\mu_1\mu_2\mu_3\mu_4} + 2 Z_{NS}^{[1][\mu_1} Z_{RR}^{[3]\mu_2\mu_3\mu_4]} = 0 \end{cases} \quad (4.29)$$

The section conditions in the extended space are given by

$$\begin{cases} \partial_\mu \frac{\partial}{\partial \tilde{x}_\mu} = 0 \\ \partial_\mu \frac{\partial}{\partial y_\mu} = 0 \\ \partial_\mu \frac{\partial}{\partial y_{\mu\mu_1\mu_2}} + \frac{1}{2} \frac{\partial}{\partial \tilde{x}_{[\mu_1}} \frac{\partial}{\partial \tilde{x}_{\mu_2]}} = 0 \\ \partial_\mu \frac{\partial}{\partial y_{\mu\mu_1\mu_2\mu_3\mu_4}} + 2 \frac{\partial}{\partial \tilde{x}_{[\mu_1}} \frac{\partial}{\partial y_{\mu_2\mu_3\mu_4]}} = 0 \end{cases} \quad (4.30)$$

Again, this is a generalization of the section condition (4.7) in DFT. As mentioned earlier, the condition (4.30) is defined in the uncompactified, ten-dimensional target space.

4.4 Gauge transformations

Finally, we briefly comment on the gauge transformations of the type II supergravity fields. As we have shown above, the current of the fundamental string generates the gauge transformations of the spacetime vielbein and the B -field. In the following, we derive the gauge transformations of type II supergravity fields by utilizing the current algebras of five-branes discussed so far.

$\mathcal{N} = (2, 0)$ theory. The gauge transformation of gauge fields E_A^M with $E_A^M E_B^N \hat{\eta}^{AB} = \mathcal{M}^{MN}$ and $E_A^M E_B^N \eta^{AB} = \eta^{MN}$ is given as follows. The type IIA gauge fields are represented as a vector in the extended space where the index of a vector V^M runs $M = (\mu, \nu, \dots, \mu_1\mu_2, \mu_1, \dots, \mu_4)$

$$E_a^M = (e_a^\mu, e_a^\nu B_{\nu\mu}, e_a^\nu C_{\nu}^{[1]}, e_a^\nu C_{\nu\mu_1\mu_2}^{[3]}, e_a^\nu C_{\nu\mu_1\mu_2\mu_3\mu_4}^{[5]}). \quad (4.31)$$

The D -dimensional vielbein e_a^μ is invertible. The gauge parameter vector is given as

$$\lambda^M = (\lambda^\mu, \lambda_\mu, \lambda^{[0]}, \lambda_{\mu_1\mu_2}^{[2]}, \lambda_{\mu_1\mu_2\mu_3\mu_4}^{[4]}). \quad (4.32)$$

The gauge transformation rules are given by the commutators of the gauge field vector $E_a^M Z_M$ and the parameter vector $\lambda^M Z_M$ in the extended space as

$$\delta_\lambda E_a^M Z_M(\sigma) = i \int d\sigma' \{ \lambda^N Z_N(\sigma'), E_a^M Z_M(\sigma) \}. \quad (4.33)$$

For this computation the section conditions are used, then the worldvolume coordinate derivatives are calculated by the usual chain rule, $\partial_i \Phi(x^\mu) = \partial_i x^\mu \partial_\mu \Phi$ resulting the closure of the gauge transformation.

The transformation rules are calculated as

$$\left\{ \begin{array}{l} \delta_\lambda e_a^\mu = \mathcal{L}_\lambda e_a^\mu = \lambda^\nu \partial_\nu e_a^\mu - e_a^\nu \partial_\nu \lambda^\mu \\ \delta_\lambda B_{\nu\mu} = \mathcal{L}_\lambda B_{\nu\mu} + \partial_{[\mu} \lambda_{\nu]} \\ \delta_\lambda C_{\nu\mu}^{[1]} = \mathcal{L}_\lambda C_{\nu\mu}^{[1]} + \partial_\mu \lambda^{[0]} \\ \delta_\lambda C_{\mu_1\mu_2\mu_3}^{[3]} = \mathcal{L}_\lambda C_{\mu_1\mu_2\mu_3\mu_4}^{[3]} + \frac{1}{3!} \partial_{[\mu_1} \lambda_{\mu_2\mu_3]}^{[2]} + \frac{1}{3!} B_{[\mu_1\mu_2} \partial_{\mu_3]} \lambda^{[0]} \\ \delta_\lambda C_{\mu_1\mu_2\mu_3\mu_4\mu_5}^{[5]} = \mathcal{L}_\lambda C_{\mu_1\mu_2\mu_3\mu_4\mu_5}^{[5]} + \frac{1}{5!} \partial_{[\mu_1} \lambda_{\mu_2\mu_3\mu_4\mu_5]}^{[4]} \\ \quad + \frac{2}{5!} B_{[\mu_1\mu_2} \partial_{\mu_3} \lambda_{\mu_4\mu_5]}^{[2]} + \frac{2}{5!} C_{[\mu_1\mu_2\mu_3}^{[3]} \partial_{\mu_4} \lambda_{\mu_5]} \end{array} \right. . \quad (4.34)$$

$\mathcal{N} = (1, 1)$ theory The type IIB gauge fields are represented as a vector in the extended space where the index of a vector V^M runs $M = (\mu, \text{NS}\mu, \text{RR}\mu, \mu_1\mu_2\mu_3, \mu_1, \dots, \mu_5)$

$$E_a^M = (e_a^\mu, e_a^\nu B_{\nu\mu}, e_a^\nu C_{\nu\mu}^{[2]}, e_a^\nu C_{\nu\mu_1\mu_2\mu_3}^{[4]}, e_a^\nu C_{\nu\mu_1\mu_2\mu_3\mu_4\mu_5}^{[6]}). \quad (4.35)$$

The gauge parameter vector is given as

$$\lambda^M = (\lambda^\mu, \lambda_\mu, \lambda_\mu^{[1]}, \lambda_{\mu_1\mu_2\mu_3}^{[3]}, \lambda_{\mu_1\mu_2\mu_3\mu_4\mu_5}^{[5]}). \quad (4.36)$$

Figure 1. Type II five-branes related by S- and T-dualities.

The transformation rules are calculated as

$$\left\{ \begin{array}{l} \delta_\lambda e_a^\mu = \mathcal{L}_\lambda e_a^\mu \\ \delta_\lambda B_{\nu\mu} = \mathcal{L}_\lambda B_{\nu\mu} + \partial_{[\mu} \lambda_{\nu]} \\ \delta_\lambda C_{\nu\mu}^{[2]} = \mathcal{L}_\lambda C_{\nu\mu}^{[2]} + \partial_{[\mu} \lambda_{\nu]}^{[1]} \\ \delta_\lambda C_{\mu_1\mu_2\mu_3\mu_4}^{[4]} = \mathcal{L}_\lambda C_{\mu_1\mu_2\mu_3\mu_4}^{[4]} + \frac{1}{3!} \partial_{[\mu_1} \lambda_{\mu_2\mu_3\mu_4]}^{[3]} + \frac{1}{3!} B_{[\mu_1\mu_2} \partial_{\mu_3} \lambda_{\mu_4]} + \frac{1}{3!} B_{[\mu_1\mu_2} \partial_{\mu_3} \lambda_{\mu_4]}^{[1]} \\ \delta_\lambda C_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}^{[6]} = \mathcal{L}_\lambda C_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}^{[6]} + \frac{1}{5!} \partial_{[\mu_1} \lambda_{\mu_2\mu_3\mu_4\mu_5\mu_6]}^{[5]} \\ \quad + \frac{2}{6!} B_{[\mu_1\mu_2} \partial_{\mu_3} \lambda_{\mu_4\mu_5\mu_6]}^{[3]} + \frac{2}{6!} C_{[\mu_1\mu_2\mu_3\mu_4}^{[4]} \partial_{\mu_5} \lambda_{\mu_6]} \end{array} \right. \quad (4.37)$$

5 Conclusion and discussions

In this paper, we studied the current algebras of five-branes in type II string theories. These include the NS5-branes, the KK5-branes and the exotic 5_2^2 -branes in type IIA/IIB string theories. Together with the M2-, M5-branes, fundamental strings and the Dp -branes, they share important roles in string theories. The effective theories of the five-branes are characterized by the six-dimensional $\mathcal{N} = (2, 0)$ tensor and the $\mathcal{N} = (1, 1)$ vector multiplets. The worldvolume effective actions of the $\mathcal{N} = (2, 0)$ tensor multiplet have been derived from the PST action of the M5-brane by the double dimensional reduction and the T-duality transformations. The effective actions of the $\mathcal{N} = (1, 1)$ vector multiplet have been derived from the DBI action of the D5-brane by the S- and T-duality transformations. See Figure 1.

The actions of the KK5-branes and the 5_2^2 -branes contain geometric fluctuation modes x^μ together with the dual scalar modes $\tilde{x}^a$ and the gauge fields in the tensor and the vector multiplets. Indeed, the algebras of the KK5- and the 5_2^2 -branes have specific fluctuation modes $\tilde{x}^a$ ($a = 8, 9$) corresponding to the isometry directions in the transverse space. We explicitly wrote down the current algebras of the five-branes based on these actions. The

existence of these dual scalar modes in the KK5- and the 5_2^2 -branes distinguish their algebras from those of the Dp-branes. Due to the self-duality constraint of the 2-form gauge field, we showed that algebras of the $\mathcal{N} = (2, 0)$ theories are given by the Dirac bracket. On the other hand, the algebras of the $\mathcal{N} = (1, 1)$ theories are given by the Poisson bracket.

The currents Z_M in the algebras are naturally expressed by the differential operators in the extended spaces. The coordinates X^M in the extended spaces are the fundamental basis of the doubled field theory (DFT) and the exceptional field theory (EFT). They include the winding coordinates of various branes. We show that the worldvolume spatial diffeomorphism constraints result in the physical conditions in the extended spaces. The diffeomorphism constraints of five-branes and the corresponding section conditions (4.27), (4.30) are defined in the uncompactified ten-dimensional spacetime. Therefore, in a sense, they can be recognized as a general expressions of the section conditions for EFT in lower-dimensions. We note that the section conditions found in this paper are represented in the 528-dimensional extended space. Whether this extended space fits into the $E_{n(n)}$ languages or not deserves further studies.

Based on the current algebras, we also wrote down the Courant and Dorfman brackets for the gauge parameters in ten-dimensional type II supergravities. We then derive the gauge transformations of the supergravity fields in type IIA and IIB theories. We also discuss the relation between the currents and the central charges in the ten-dimensional supersymmetry algebras.

We have presented the current algebras of the five-branes including the exotic 5_2^2 -branes. However, we note that they are just the entrance to the web of exotic branes. The U-duality multiplets of the 1/2 BPS objects in the T^d compactification have been classified by the duality group $E_{d(d)}$. There are a large number of exotic branes whose tensions are proportional to $g_s^{-\alpha}$ with $\alpha \geq 2$ [42, 50]. Branes whose tensions are proportional to $g_s^{-3}, g_s^{-4}, \dots$ are still mysterious objects. However, this is not the end of the story. There are yet unfamiliar exotic objects in string theory. One can perform further T-duality transformations along the transverse directions to the 5_2^2 -brane worldvolumes. The resulting objects are denote as 5_2^3 - and 5_2^4 -branes and known as the R-branes and space-filling branes [21]. Since their geometries are characterized, not only by the geometrical coordinate x^μ but also the T-dualized one $\tilde{x}_\mu$, they are not solutions to conventional supergravities. This kind of branes are called locally non-geometric objects. They are shown to be solutions to DFT, EFT and deformed supergravities [42, 47–50] and their worldvolume theories are discussed [51].

It is known that the locally non-geometric nature appear in various context of string theory [49, 52–57]. It would be interesting to study the current algebras of these branes. We will come back to these issues in future studies.

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