Heterotic complex structure moduli stabilization for elliptically fibered Calabi-Yau manifolds

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ABSTRACT: Holomorphicity of vector bundles can stabilize complex structure moduli of a Calabi-Yau threefold in $N = 1$ supersymmetric heterotic compactifications. In principle, the Atiyah class determines the stabilized moduli. In this paper, we study how this mechanism works in the context of elliptically fibered Calabi-Yau manifolds where the complex structure moduli space contains two kinds of moduli, those from the base and those from the fibration. Defining the bundle with spectral data, we find three types of situations when bundles' holomorphicity depends on algebraic cycles exist only for special loci in the complex structure moduli, which allows us to stabilize both of these two moduli. We present concrete examples for each type and develop practical tools to analyze the stabilized moduli. Finally, by checking the holomorphicity of the four-flux and/or local Higgs bundle data in F-theory, we briefly study the dual complex structure moduli stabilization scenarios.

Contents

1 Introduction

Compactification of $E_8 \times E_8$ heterotic string [1, 2] on a Calabi-Yau threefold with a vector bundle is a promising approach to derive the Standard Model of particle physics in the four-dimensional spacetime [3–11]. To preserve supersymmetry, the bundle should be holomorphic, slope zero, and poly-stable. Fluctuations of the complex structure and Kähler form of the base manifold and fluctuations of the bundle appear as moduli particles in the effective theory. Holomorphicity of the bundle has been used to stabilize the complex structure moduli in these models [12–16]. The idea is that only some of the deformations of the complex structure are consistent with the supersymmetry. The others will force the bundle to become non-holomorphic. This will generate non-trivial F-terms for such deformations in the effective theory, and the complex structure moduli corresponding to them are stabilized. As shown in [14], this mechanism, together with a similar one due to bundle poly-stability [17–21] is an effective way to stabilize geometric moduli in heterotic Calabi-Yau compactification.

Deformations of the complex structure-preserving bundle holomorphy are described by the Atiyah class [22]. The computation of it is, in general, challenging. However, for

holomorphic bundles in which the necessary ingredients in the construction have a complex structure dependence, these deformations can be calculated efficiently by a "bottom-up" approach. For example, extension bundles [5] are defined by specifying an element in the Ext group. As shown in [12, 13], the Ext group jumps in dimension for a special choice of the complex structure. This signals that some of the complex structure moduli are stabilized locally over the particular locus in moduli space. Instead of the Atiyah class, the stabilized moduli can be identified quickly by a computation of "jumping" Ext. As a result, the "bottom-up" approaches are equivalent to the calculation of the Atiyah class and are efficient tools for the complex structure moduli stabilization.

In this work, we will focus on Calabi-Yau threefolds with an elliptic fibration[1] [23–30], and study how its complex structure moduli can be stabilized by bundle holomorphy. Bundles on an elliptically fibered Calabi-Yau threefold can be constructed by spectral data [31–34], which can be mapped to the extension or monad bundles by a Fourier-Mukai transformation. The reason to work with the spectral cover bundle is that poly-stability, a difficult property to check for other constructions, is straightforward to study in terms of spectral data, at least as long as the spectral cover is smooth. Besides, since the spectral data specify the Heterotic models, the Heterotic model's moduli stabilization process can be translated to an associated process in the dual F-theory model. Hence the relation with moduli stabilization scenarios in F-theory/type II models will be manifest.

Complex structure moduli of an elliptically fibered Calabi-Yau threefold X can be either the complex structure moduli of the base B or the complex structure moduli of the fibration F. As we will see later, both of them can be stabilized using holomorphic bundles. Specifically, we find three types of approaches to do that but, from the perspective of spectral cover bundles, they all correspond to emergent algebraic cycles in X. In more technical terms, we look for the Noether-Lefschetz loci in the complex structure moduli and specify the spectral data so that the associated vector bundle stays holomorphic only on these loci. For type I case, we consider a $SU(2)$ extension bundle with Ext group jumping on a special locus in complex structure moduli space of B. One can observe that variations of the complex structure away from the special locus in moduli space will make the algebraic curve disappear. The spectral cover is not algebraic anymore. Hence the corresponding vector bundle becomes non-holomorphic, and the associated complex structure moduli of B are stabilized. We will get some insight into how the stabilized moduli can be identified by studying the spectral data directly.

In the type II approach, we find that the Picard group of B jumps when the complex structure of the base takes some "special" form. Suppose the holomorphic bundle is defined with the new algebraic cycles in the "jumping" Picard group. Then, variations of the complex structure that does not preserve this "special" form of complex structure will make the bundle non-holomorphic, and the corresponding moduli are stabilized. Besides the base's complex structure moduli, holomorphic bundles can also stabilize the moduli of the fibration. The type III approach is similar to the type II one, in which taking a

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[1]The Calabi-Yau manifolds considered here are all in the Weierstrass form. This paper's ideas will be valid for any elliptic fibration, as long as there is at least one holomorphic section. However, when all of the sections are rational, one will need more careful analysis [33, 34].

"special" complex structure gives new algebraic cycles in the base. The difference is that this time, the algebraic curves appear in the total space of the fibration. Suppose the spectral data are defined in terms of these emergent algebraic cycles. Then, variations of the complex structure away from the "special" form will make algebraic cycles non-algebraic, the bundle becomes non-holomorphic, and the associated moduli are stabilized. Notice that the last two approaches do not require the bundle to be constructed as an extension and are applicable to more general bundles. We give explicit examples for all three approaches. Like "jumping" Ext, we develop practical tools to identify variations of the complex structure that does not preserve the "special" form used above and thus obtain the stabilized moduli in type II and type III approaches.[2]

The image of these moduli stabilization scenarios is also interesting. In the type I example, the flux derived from the spectral line bundle is responsible for the moduli stabilization. Finally, the four-flux in type II example is the horizontal algebraic four-flux constructed in [35]. A recent work on complex structure moduli stabilization also used the idea of algebraic cycles is discussed in [36].

This paper is organized as follows: in section 2, we review how a vector bundle's holomorphicity can be used for complex structure moduli stabilization in Calabi-Yau compactification. Specially we review the Atiyah class and illustrate it with a SU(2) extension bundle. In section 3, we introduce three approaches to stabilize the complex structure moduli of an elliptically fibered Calabi-Yau threefold with holomorphic bundles in spectral cover construction. In section 4, we discuss how these approaches in heterotic compactification manifest themselves in F-theory. In section 5, we conclude our work. Some of the technical discussions are presented in the appendices.

2 The Atiyah class

Let X be a Calabi-Yau threefold with a vector bundle V. To preserve $N = 1$ supersymmetry, the bundle should have a connection satisfying the following (zero slope) Hermitian Yang-Mills equations,

$$F_{ab} = F_{\bar{a}\bar{b}} = 0, \ g^{a\bar{b}}F_{a\bar{b}} = 0, \tag{2.1}$$

where F is the gauge field strength associated with that connection and a and $\bar{b}$ are holomorphic and anti-holomorphic indices on the Calabi-Yau manifold. The first equation in (2.1) implies that the bundle is holomorphic [37]. It is the property that will be used for moduli stabilization. The second equation in (2.1) means that the bundle is poly-stable [38, 39]. We will always keep V poly-stable when the complex structure of X is varied. As shown in [12, 13], one can do that as long as the bundle is holomorphic.

We start with a supersymmetric vacuum where the initial Kähler, complex structure, and bundle moduli are chosen to obey the Hermitian Yang-Mills equation in (2.1). Notice that the initial objects will be denoted with a superscript "(0)" and their deformation

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[2]It is interesting to see how many of the moduli fields can be stabilized in the phenomenologically interesting models using the methods presented in this paper. However, we only explain the general ideas here and leave these questions for a future project.

will be denoted with a δ in front. From the deformation theory of compact complex manifolds [22, 40–43], the space of simultaneous holomorphic deformations of X and V is measured by $H^1(\mathcal{Q})$ with Q satisfying the Atiyah sequence

$$0 \to V \otimes V^\vee \to \mathcal{Q} \xrightarrow{\pi} TX \to 0 \ . \tag{2.2}$$

The extension class is

$$\alpha = [F^{(0)1,1}] \in H^1(V \otimes V^\vee \otimes TX^\vee) \tag{2.3}$$

the $(1,1)$ component of the initial field strength $F^{(0)}$ of the background, referred to as the "Atiyah class" [22]. From the long exact sequence of (2.2), we have

$$0 \to H^1(V \otimes V^\vee) \to H^1(\mathcal{Q}) \xrightarrow{d\pi} H^1(TX) \xrightarrow{\alpha} H^2(V \otimes V^\vee) \to \dots \ . \tag{2.4}$$

The simultaneous holomorphic deformations space $H^1(\mathcal{Q})$ is given by

$$H^1(\mathcal{Q}) = H^1(V \otimes V^\vee) \oplus \mathrm{Ker}(\alpha) \ . \tag{2.5}$$

Notice that we have used the fact that $H^0(TX) = 0$ for the stable bundle TX in (2.4).

As we can see in (2.5), all bundle moduli in $H^1(V \otimes V^\vee)$ belong to $H^1(\mathcal{Q})$ while for complex structure moduli, only those that are in the kernel of the Atiyah class are elements of $H^1(\mathcal{Q})$ and should be regarded as the "true" moduli in the effective theory. The other ones have non-zero images in $H^2(V \otimes V^\vee)$, which lead to non-vanishing field strengths $F^{0,2}$ in the background and make V non-holomorphic. Thus, variations of the complex structure along those directions are stabilized by non-trivial F-terms in superpotential. The number of stabilized moduli is counted by the dimension of $\mathrm{Im}(\alpha)$. From the sequence in (2.4), it is bounded by $h^2(V \otimes V^\vee)$, so there are at most $h^2(V \otimes V^\vee)$ complex structure moduli that can be stabilized by the holomorphic bundle.

The above mechanism can be made more precise if we write the elements of $H^1(TX)$ explicitly as $\nu = \delta\mathfrak{z}^I v_{I\bar{a}}^c$ where v_I are tangent bundle valued harmonic one-forms and $\delta\mathfrak{z}^I$ are variations of the complex structure moduli $\mathfrak{z}^I$. By the sequence (2.4), elements in the kernel of Atiyah class should satisfy,

$$\delta\mathfrak{z}^I v_{I[\bar{a}}^c F^{(0)}_{|c|\bar{b}]} = D^{(0)}_{[\bar{a}} \Lambda_{\bar{b}]} \tag{2.6}$$

where Λ is a bundle-valued one-form and $D^{(0)}$ is the covariant derivative with respect to the initial connection $A^{(0)}$. Notice that the right hand side of the equation above is an exact 2-form in $H^2(X, V \otimes V^\vee)$. In fact, the bundle-valued one-form Λ can be understood as deformation of the connection $\Lambda_{\bar{b}} = \delta A_{\bar{b}}$. Thus, for deformations of complex structure $\delta\mathfrak{z}^I$ that are in $\mathrm{Ker}(\alpha)$, there exist a deformation of the connection δA such that the equation (2.6) is satisfied and V can remain holomorphic. However, other deformations will make V non-holomorphic, and the associated moduli are stabilized.

In conclusion, with Atiyah class defined in (2.3), we can determine which deformations of the complex structure in $H^1(TX)$ can keep V holomorphic and which are not. The number of these independent deformations are given by the dimension of $\mathrm{Ker}(\alpha)$ and $\mathrm{Im}(\alpha)$ respectively. In general, they can be computed directly from the long exact sequence

in (2.4) known as the "top-down" approach. However, for some holomorphic bundles defined properly only with some special complex structure, there is a quick way to determine the $\mathrm{Ker}(\alpha)$ and $\mathrm{Im}(\alpha)$. This "bottom-up" approach is equivalent[3] to the direct computation of $H^1(\mathcal{Q})$ but is much easier computationally [13]. We will illustrate this method with a simple extension bundle in the following subsection.

2.1 A rank two extension bundle

We assume that the Calabi-Yau threefold X is defined as a hypersurface in a product of projective spaces $\mathcal{A} = \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_m}$. We will further ask that it is "favorable," i.e. the Picard group of X is spanned by the restriction to X of ambient divisors, D_i, associated with the hyperplane class in $\mathbb{P}^{n_i}$. The defining polynomial of X is given by $p_0 \in H^0(\mathcal{A}, N)$. Here N is the normal bundle of the hypersurface. Notice that the defining polynomial p_0 is a redundant description of the complex structure. The independent ones can be obtained from the computation of $H^1(X, TX)$. We will discuss this in section 3.

Consider a rank two bundle V defined as an extension of a line bundle L and its dual

$$0 \to L \to V \to L^\vee \to 0 \tag{2.7}$$

with the extension class $\phi \in Ext^1(L^\vee, L) = H^1(L^2)$. If $\phi \neq 0$, the extension is non-trivial and the bundle defined is a $SU(2)$ indecomposable bundle, otherwise, the extension is trivial, and the bundle splits into $V = L \oplus L^\vee$. However, the non-trivial extension does not always exist on X because the Ext group $H^1(L^2)$ that the extension class takes value in depends on the choice of complex structure. We will be interested in a situation where $H^1(L^2)$ is non-trivial only for some special complex structure of X and $H^1(L^2) = 0$ otherwise. Thus, the indecomposable $SU(2)$ bundle is well-defined on a special locus in the moduli space, and it splits when the complex structure is deformed away from this locus. This setting will play a key role in the moduli stabilization discussed later.

Let's assume that the initial complex structure is chosen such that the $H^1(L^2)$ jumps in dimension and the bundle defined in (2.7) is an indecomposable $SU(2)$ bundle. At the same time, one needs to choose the initial Kähler form ω of X so that V is poly-stable. Although poly-stability is in general difficult to check, for this simple extension bundle, it can be determined relatively easier[4] by the following criterion [19]. V is poly-stable if the Kähler form ω satisfies $\mu(L) < 0$ where $\mu(L)$ is the slope of L defined by

$$\mu(L) = \frac{1}{rank(V)} \int_X c_1(L) \wedge \omega \wedge \omega. \tag{2.8}$$

With this condition, we will fix the initial Kähler moduli of X so that V is poly-stable in this example.

Now, let's consider the complex structure deformations with respect to this background. From the equation (2.3), deformations that are consistent with the holomorphic

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[3]For the rank two vector bundles constructed as an extension of a line bundle and its dual, it is proved in [13] that the Atiyah class computation is rigorously equivalent to an analysis of "Jumping" of Ext group. This is an example of the "bottom-up" approach.

[4]It have been shown in [19], it is a sufficient condition for V being poly-stable.

bundle is measured by the kernel of the Atiyah class $\alpha \in H^1(V \otimes V^\vee \otimes TX^\vee)$. For this example, it can be shown [19] that α is an element of $H^1(X, L^2 \otimes TX^\vee)$, a subset of $H^1(V \otimes V^\vee \otimes TX^\vee)$. Thus, the part of the sequence in (2.4) containing α becomes

$$\alpha : H^1(X, TX) \to H^2(X, L^2). \tag{2.9}$$

By Bott-Borel-Weil Theorem [37], one can express the source $H^1(X, TX)$ and target $H^2(X, L^2)$ in terms of homogeneous polynomials. The kernel of the map α can be found by a direct computation of these polynomials. However, the non-trivial extension does not always exist on X because the Ext group $H^1(L^2)$ that the extension class takes value in depends on the choice of the complex structure.

The idea of the "bottom-up" approach [19] is that instead of the direct calculation of $\mathrm{Ker}(\alpha)$ outlined above, the same information can be obtained by studying the "jumping" of Ext group $H^1(X, L^2)$. Specifically, recall that the indecomposable bundle V considered here is defined on the special locus in the moduli space where Ext group $H^1(X, L^2)$ jumps. Consider the deformation of complex structure $p_0 \to p_0 + \delta p$. If δP is within the special locus of moduli space, the V is still indecomposable and holomorphic. However, if δP is generic, the Ext group becomes $H^1(X, L^2) = 0$. It seems that in this case, the only choice for extension class is $\phi = 0$ and the bundle becomes a direct sum $L \oplus L^\vee$. In fact, the extension class $\phi \in H^1(X, L^2)$, does not change when we vary the complex structure moduli. Instead, ϕ is no longer a closed $(0,1)$-form with respect to the new complex structure. It means that V after generic deformations δp is not holomorphic anymore. In this example, we find that deformations of the complex structure that make the Ext group "jumping" are the same deformations that keep V holomorphic. In contrast, deformations of the complex structure that leads to the trivial Ext group correspond to ones that make V non-holomorphic. Thus, we see clearly that the calculation of "jumping" Ext is equivalent to the Atiyah computation.

Next, let's briefly describe how to identify the special locus where the Ext group $H^1(X, L^2)$ jumps in the moduli space. By the Koszul sequence, [37], the line bundle L^2 on X can be expressed in terms of bundles on the ambient space $\mathcal{A}$. For the manifold considered here, it is

$$0 \to N^\vee \otimes L_\mathcal{A}^2 \xrightarrow{p_0} L_\mathcal{A}^2 \to L^2 \to 0, \tag{2.10}$$

where $N^\vee$ is the dual of the normal bundle of X. The cohomology $H^1(X, L^2)$ can be computed via the associated long exact sequence

$$0 \to H^1(X, L^2) \to H^2(\mathcal{A}, N^\vee \otimes L_\mathcal{A}^2) \xrightarrow{p_0} H^2(\mathcal{A}, L_\mathcal{A}^2) \to H^2(X, L^2) \to 0. \tag{2.11}$$

Here p_0 is the defining polynomial of X. Thus, we have $H^1(X, L^2) = \mathrm{Ker}(p_0)$ and $H^1(X, L^{\vee 2}) = \mathrm{Coker}(p_0)$. For generic defining polynomials, both $\mathrm{Ker}(p_0)$ and $\mathrm{Coker}(p_0)$ are trivial. While for some special choice of p_0, they can "jump" together so that the index $\mathrm{Ind}(L^2) = -h^1(X, L^2) + h^2(X, L^2)$ is preserved [13]. Note that, for the variation of the complex structure $p \to p_0 + \delta p$, the number of independent variations that make $h^1(X, L^2)$ jumps is the same as the dimension of $\mathrm{Ker}(\alpha)$ computed from the equation (2.9) while the other variations of complex structure with $h^1(X, L^2) = 0$ corresponds to $\mathrm{Im}(\alpha)$ in the Atiyah computation.

This moduli stabilization mechanism can also be understood in the low-energy effective theory. Let $\mathfrak{z}_0$ be the initial complex structure moduli chosen within the special locus of moduli space where Ext group of the bundle defined in (2.7) jumps. The non-trivial extension defines a SU(2) bundle. The low-energy gauge group is E_7. However, if the extension class is trivial, V splits and the low-energy gauge group is enhanced by an anomalous U(1) factor to $E_7 \times$ U(1) [17–19]. The bundle moduli (and matter fields) are charged under the enhanced U(1) symmetry denoted by

$$C_+^i \in H^1(L^2), \qquad C_-^j \in H^1(L^{\vee 2}), \tag{2.12}$$

with the subscript $\pm$ indicating the U(1) charge. We will take the extension class $\langle C_+^i \rangle \neq 0$ not far from the zero in $H^1(X, L^2)$ and keep $\langle C_-^j \rangle = 0$ in the following.

The nontrivial F-term comes from the Gukov-Vafa-Witten superpotential [48, 49]. For bundle considered in (2.7), one can show that superpotential from the contribution of bundle moduli is [13]

$$W = \lambda_{ij}(\mathfrak{z}_0) C_+^i C_-^j + \dots \tag{2.13}$$

where the coefficients λ_{ij} depends on the complex structure moduli $\mathfrak{z}_0^I \in H^1(X, TX)$ explicitly. Note that for the generic complex structure, the holomorphic function $\lambda_{ij}(\mathfrak{z}_0) \neq 0$. As a result, the fields C_+^i and C_-^j are massive. However, on the special locus in complex structure moduli space, $\lambda_{ij}(\mathfrak{z}_0) = 0$, which indicates that C_+^i and C_-^j are massless. The non-trivial VEVs of them define the non-Abelian SU(2) bundle [19]. This corresponds to the "jumping" of the Ext group described above.

The F-terms of the bundle moduli follows directly from the superpotential above. In particular, the F-terms of C_+^i vanish while the other one becomes

$$F_{C_-} = \frac{\partial W}{\partial C_-^j} = \lambda_{ij}(\mathfrak{z}_0) \langle C_+^i \rangle . \tag{2.14}$$

As discussed above, since the initial complex structure is chosen in the special locus of the moduli space, hence, $\lambda_{ij}(\mathfrak{z}_0) = 0$. Consider fluctuations of $\mathfrak{z}_0^I \to \mathfrak{z}_0^I + \delta\mathfrak{z}$. If these fluctuations are within the sub-locus where $H^1(X, L^2) \neq 0$ (denoted by $\mathfrak{z}_\parallel^a$), the new complex structure is still in the sub-locus and hence $\lambda_{ij}(\mathfrak{z}_0) = 0$ and F_{C_-} vanishes. However, for the fluctuations away from this sub-locus (denoted by $\mathfrak{z}_\perp^a$), the new complex structure is a generic element in the moduli space and $\lambda_{ij}(\mathfrak{z}_0) \neq 0$. Up to the quadratic order, we have the non-zero F-term of F_{C_-} and the potential is given by

$$V = |\frac{\partial \lambda_{ij}(\mathfrak{z}_0)}{\partial \mathfrak{z}_\perp^I} \langle C_+^i \rangle|^2 |\delta\mathfrak{z}_\perp^I|^2 + \dots . \tag{2.15}$$

This gives mass to the moduli $\mathfrak{z}_\perp^a$ and these degrees of freedom are stabilized. The number of these moduli is the same as the Im(α) computed in the Atiyah sequence.

In summary, we review the complex structure moduli stabilization in Calabi-Yau compactification due to the presence of holomorphic bundles. The "true" moduli are those that keep the bundle holomorphic, while the other moduli are stabilized. With a simple rank two extension bundle (2.7), we demonstrate that Ker(α) can be calculated by an analysis of

"jumping" Ext group, which is equivalent to the direct computation from the sequence (2.4) and much easier computationally. Next, we will focus on the moduli stabilization on an elliptically fibered threefold and study how this mechanism works in that context.

3 Complex structure moduli stabilization mechanisms

An elliptically fibered Calabi-Yau threefold X consists of a complex two-surface B and an analytic map

$$\pi : X \to B \tag{3.1}$$

with the property that for a generic point $b \in B$, the fiber $E_b = \pi^{-1}(b)$ is an elliptic curve. In addition, we will require that there exist a global section $\sigma : B \to X$ that assigns to every point $b \in B$ the zero element $\sigma(b) = p \in E_b$ discussed below. Let K_B be the canonical bundle of B. The Weierstrass form describes the elliptic fibration as

$$y^2 - x^3 - fxz^4 - gz^6 = 0, \tag{3.2}$$

where f and g are sections of K_B^{-4} and K_B^{-6}, and (x, y, z) are global sections of $\mathcal{O}_X(2\sigma) \otimes K_B^{-2}$, $\mathcal{O}_X(3\sigma) \otimes K_B^{-3}$, $\mathcal{O}_X(\sigma)$ respectively.

Consider a SU(n) vector bundle V on X. The Chern character of V can be written as the following general form,

$$ch(V) = n + \sigma\pi^*\eta + \omega[f] + \frac{1}{2}c_3(V), \tag{3.3}$$

where n is the rank of V, σ is the section of the fibration, $\eta \in H^2(B, \mathbb{Z})$ is a divisor in the base, ω in an element in $H^{2,2}(B, \mathbb{Z})$, and $[f]$ is the fiber class of the elliptic fibration. The bundle V can be constructed in terms of a spectral cover $\mathcal{S}_n$ and a spectral line bundle $\mathcal{N}$ [31, 32]. The spectral cover is a n sheeted cover of B given by a finite morphism $\pi_S : \mathcal{S}_n \to B$. In affine coordinates, where $z = 1$, we can write it as

$$\mathcal{S}_n = a_0 + a_2 x + a_3 y + a_4 x^2 + a_5 x^2 y + \cdots + a_n x^{n/2} = 0 \tag{3.4}$$

Note that the last term is $a_n x^{(n-3)/2} y$ when n is odd. $\mathcal{N}$ is a line bundle over $\mathcal{S}_n$ (or a rank one coherent sheaf generally). The set of data $(\mathcal{S}, \mathcal{N})$ is called spectral data. It is shown [31, 32] that there is a one-to-one correspondence between holomorphic vector bundles and the spectral data,

$$V \leftrightarrow (\mathcal{S}_n, \mathcal{N}). \tag{3.5}$$

For an elliptically fibered Calabi-Yau threefold, the complex structure moduli space contains the moduli from the base B and the moduli from the fibration. In fact, one can show that the moduli space $H^1(B, TB)$ is a subset of $H^1(X, TX)$. The detail of the derivation can be found in appendix A. The moduli space $H^1(B, TB)$ can be computed from the Euler sequence and adjunction formula. Suppose B is defined as a hypersurface of an ambient space A, where A can be the direct product of projective spaces $\mathbb{P}^{m_1} \times \ldots \times \mathbb{P}^{m_n}$

or in general toric varieties. The defining polynomial of B is $p_0 \in H^0(A, N)$ for the ample line bundle N. Then the tangent bundle is described by

$$0 \to O_B^{\oplus n} \overset{l_1}{\to} \bigoplus_{i=1,\ldots,n} O_B(D_i)^{\oplus(n_i+1)} \to TA|_B \to 0 \tag{3.6}$$

$$0 \to TB \to TA|_B \overset{l_2}{\to} N \to 0$$

where D_i are the restriction of the hyperplane divisors of each projective factor of the ambient space A, and the polynomial maps l_1, l_2 satisfy $l_2 \circ l_1 = p_0$. From the long exact sequence of cohomology groups, we have

$$0 \to H^0(B, TB) \to H^0(B, TA) \overset{l_2}{\to} H^0(B, N) \to H^1(B, TB) \to H^1(B, TA) \overset{l_2}{\to} H^1(B, N) \to \ldots$$

In general, it contains the polynomial deformations and the non-polynomial ones. We will only focus on the polynomial part $H^1_{\text{poly}}(B, TB) \subset H^1(B, TB)$ which is given by

$$H^1_{\text{poly}}(B, TB) = \frac{H^0(B, N)}{\text{Im}(l_2)} \ . \tag{3.7}$$

The fibration F of the Calabi-Yau threefold is taken to be in the Weierstrass form. The complex structure moduli space of F can be determined following a similar calculation.

We wish to use the spectral data to find new examples of the complex structure moduli stabilization. There are three types of examples that we are going to consider. In the first type, some complex structure deformations are obstructed simply because the spectral cover cannot be defined globally everywhere in moduli space. In the second type, however, the Picard group of the base space jumps over some loci in the complex structure moduli, and if the spectral data depends explicitly on these new divisors, the deformations normal to this loci is obstructed. In the third type, the Picard group of the Calabi-Yau threefold jumps. More clearly, some specific algebraic curves inside the ambient four-fold will lie inside the Calabi-Yau threefold. For some specific bundles, these algebraic curves lie inside the spectral cover.[5]

3.1 Type I: "jumping" extension group

Consider the rank two extension bundle V defined in (2.7). As discussed in section 2, on special loci in complex structure moduli space, cohomology $H^2(X, V \otimes V^\vee)$ jumps in dimension and the rank of Atiyah class α in (2.4) jumps as well. Hence, complex structure deformations perpendicular to this special loci will make V non-holomorphic, and the associated moduli are stabilized [14]. In this subsection, we consider a similar set-up over an elliptically fibered Calabi-Yau threefold $\pi : X \to B$, and by analyzing such bundles using spectral cover/Fourier-Mukai construction, one gets a better insight about how exactly such bundles become non-holomorphic.

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[5]Notation: from now on, we will denote the divisors in the base without writing the pullback π^* on the front, e.g., we will write η instead of $\pi^*\eta$. We also denote the spectral cover simply by $[\mathcal{S}]$ or $\mathcal{S}$ without emphasizing its degree.

As in [33] we take $\mathcal{L} = \mathcal{O}_X(-\sigma + D_b)$. The extension group of V is given by

$$Ext^1_X(\mathcal{L}^\vee, \mathcal{L}) = H^1(X, \mathcal{O}_X(-2\sigma + 2D_b)) \tag{3.8}$$
$$\simeq H^0(B, \mathcal{O}_B(2D_b + K_B)) \oplus H^0(B, \mathcal{O}_B(2D_b - K_b)),$$

where the last isomorphism is derived by the Leray spectral sequence relative to the morphism π. It is also not hard to show [14] that the (holomorphic) deformations of V i.e., $H^1(X, V^\vee \otimes V) \simeq H^2(X, V^\vee \otimes V)$ depend on the extension group above.

On the other hand, over the elliptically fibered Calabi-Yau X the vector bundle V can be parameterized by the spectral data $(\mathcal{S}, \mathcal{N})$ [31, 32], where $\mathcal{S}$ is a double cover of the base B and $\mathcal{N}$ is a rank one coherent sheaf over $\mathcal{S}$. When $\mathcal{S}$ is smooth, $\mathcal{N}$ is simply a line bundle. The divisor class of $\mathcal{S}$ is determined by the topology of V,

$$[\mathcal{S}] = 2\sigma + \eta, \tag{3.9}$$
$$\eta = 2D_b + c_1(B). \tag{3.10}$$

Also the general form of the algebraic equation of $\mathcal{S}$ (when X is given by the Weierstrass fibration) is given by,

$$\mathcal{S} = a_2 Z^2 + a_0 X, \tag{3.11}$$
$$a_0 \in H^0(B, \mathcal{O}_B(c_1(B) + 2D_b)), \quad a_2 \in H^0(B, \mathcal{O}_B(-c_1(B) + 2D_b)). \tag{3.12}$$

By computing the Fourier-Mukai transformation of V, it is shown [33] that the polynomials a_2 and a_0 can be identified with the elements of the extension group (3.8). Therefore in the spectral cover language, the loci in the complex structure moduli where the group $H^2(X, V^\vee \otimes V)$ jumps correspond to the loci where it is possible to add new terms in the polynomials a_2 and a_0. Once one adds such terms in the spectral cover, the complex structure moduli are stabilized. The complex structure deformations that turn off such terms make the spectral cover non-algebraic, hence the associated vector bundle becomes non-holomorphic.

Example. To be more concrete, consider a Weierstrass elliptically fibered Calabi-Yau threefold X over the following surface,

$$B = \begin{bmatrix} \mathbb{P}^1_x & 2 \\ \mathbb{P}^2_y & 2 \end{bmatrix}, \quad c_1(B) = H_2. \tag{3.13}$$

Let us choose $D_b = -H_1 + 2H_2$, where H_1 and H_2 are hyperplanes in $\mathbb{P}^1_x$ and $\mathbb{P}^2_y$ respectively. Therefore

$$a_0 \in H^0(B, \mathcal{O}_B(-2, 5)), \tag{3.14}$$
$$a_2 \in H^0(B, \mathcal{O}_B(-2, 3)). \tag{3.15}$$

Generic defining equation P for B can be written as

$$P = x_1^2 P_1(y) + x_2^2 P_2(y) + x_1 x_2 P_3(y), \tag{3.16}$$

where P_1, P_2, P_3 are degree two homogeneous polynomials in y. However, line bundles' cohomology over B can jump over codimension one loci in complex structure moduli of the base B. In particular the cohomology of $\mathcal{O}_B(-2, 3)$ and $\mathcal{O}_B(-2, 5)$ can jump as,[6]

$$h^0(B, O(-2, 3)) = \begin{cases} 0, & P \quad \text{generic} \\ 3, & P = x_1^2 P_1(y) + x_2^2 P_2(y) \end{cases}, \tag{3.17}$$

$$h^0(B, O(-2, 5)) = \begin{cases} 9, & P \quad \text{generic} \\ 13, & P = x_1^2 P_1(y) + x_2^2 P_2(y) \end{cases}. $$

The details of the computations are in the appendix C. This means the spectral cover, in general, can be written as,

$$\mathcal{S} = \begin{cases} a_0 Z^2, & P \quad \text{generic} \\ a_0 Z^2 + a_2 X, & P = x_1^2 P_1(y) + x_2^2 P_2(y), \end{cases} \tag{3.18}$$

where in the second case a_0 is a rational polynomial [50] which is holomorphic over the hypersurface $P = 0$,

$$a_2 = \frac{P_1(y)}{x_2^2} y_1, \frac{P_1(y)}{x_2^2} y_2, \frac{P_1(y)}{x_2^2} y_3. \tag{3.19}$$

Similarly, the extra terms for the special defining equation $P = 0$ is generally of form $\frac{P_1(y)}{x_2^2} f_3(y)$. Where f_3 is degree three polynomial in y_1, y_2, y_3, but we don't need the detailed form of it. Therefore in the complex structure locus where $P = x_1^2 P_1(y) + x_2^2 P_2(y)$ the extension group (3.8) jumps, and it is possible turn on these new terms. This corresponds to add the new terms in a_0 and a_2 in the spectral cover. We consider the new terms in a_0 and a_2 in turn.

First consider the a_2 term. When we turn on terms (3.19), the spectral cover is smooth. As it is shown in [31–33] generally the spectral line bundle $\mathcal{N}$ is

$$\mathcal{N} = \mathcal{O}_S(\sigma + D_b). \tag{3.20}$$

Therefore in general $\mathcal{N}$ depends on the divisor $\sigma|_S$. The image of this curve on B is the curve $(a_2 = 0)$. When one turns on the term $x_1 x_2 P_3(y)$ in (3.16), this curve "disappears". In other words, away form the locus (in the complex structure moduli space) where $P_3(y)$ in (3.16) is nonzero the divisor $-2H_1 + 3H_2$ is not effective anymore. This means after this complex structure deformation the curve $(a_2 = 0)$ mixes with $(2, 0)$ or $(0, 2)$ cycles. Therefore the corresponding curve $\sigma|_S$ on the spectral cover is not algebraic anymore and mixes with $(2, 0)$ or $(0, 2)$ cycles. Hence the corresponding vector bundle becomes non-holomorphic.

Next, suppose $a_2 = 0$. Instead, we turn on the extra terms in a_0. In this case the spectral cover[7] $S = a_0 Z^2$. Similar to the previous case if a_0 depends on the terms $\frac{P_1(y)}{x_2^2} f_3(y)$, the corresponding curve a_0 becomes non-algebraic after these complex structure deformations. Therefore the corresponding vector bundle V becomes non-holomorphic.

---

[6]The surface B is still smooth over this special complex structure locus.

[7]In this case the spectral cover is a union of two copies of the zero section $Z = 0$ (which are "infinitesimally close" to each other) and the vertical surface $a_2 = 0$.

The usual stabilization scenario of complex structure moduli by rank two vector bundles [14, 16] translates to whether the coefficients of the spectral cover equation exists globally. Therefore one may generalize this type of stabilization to vector bundles of arbitrary ranks simply by asking if a holomorphic vector bundle with the Chern character $ch(V) = n + \sigma\eta + \omega[f]$ is given, then does the divisor $[S] = n\sigma + \eta$ have global section everywhere in the complex structure moduli?

Now, we will see how many complex structure moduli are stabilized in this example. If the initial defining polynomial is fixed by equation (C.7), the complex structure moduli space $H^1(B, TB)$ can be computed by the equation (3.6) and (3.7). It follows that the space of polynomial deformations $H^1_{\text{poly}}(B, TB)$ has dimension 6 with basis $\{d_i\}, i = 1, 2, \ldots, 6$. As discussed above, if any monomials in a polynomial deformation of P contains $x_0 x_1$ term and spectral cover depends on the extra terms in (3.17), then $\mathcal{S}$ will be non-algebraic after the deformation, and V is not holomorphic any more. It turns out that there are 4 such deformations in $\{d_1, d_2, \ldots, d_6\}$.[8] Therefore, we find that 4 out of 6 complex structure moduli of B are stabilized.

3.2 Type II: "jumping" Picard group

For an elliptically fibered Calabi-Yau threefold, the Picard group of the base B can jump in dimension, which can be used for moduli stabilization. The idea is that look for the loci in the complex structure moduli that the Picard group jumps (Noether-Lefschetz loci) and choose the spectral data to depend on the "new" elements of the Picard group. Therefore deforming the complex structure back is impossible because the corresponding vector bundle becomes non-holomorphic. We will explain this approach using an example.

Example. Consider the following surface

$$
B^0 = \begin{array}{cccccc|c}
x_1 & x_2 & z_1 & z_2 & u_1 & u_2 & P \\
\hline
1 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 3 \\
0 & 1 & 0 & 1 & 1 & 1 & 3
\end{array}
\tag{3.21}
$$

where P is generically given by

$$
x_1(z_1^3 f_3(u) + z_1^2 z_2 f_2(u) + z_1 z_2^2 f_1 + z_2^3 f_0) + x_2(z_1^2 g_2(u) + z_1 z_2 g_1(u) + z_2^2 g_0) = 0. \tag{3.22}
$$

Here, $f_d(u)$ and $g_d(u)$ represent generic degree d homogeneous polynomials in $\{u_1, u_2\}$. This is actually the blow-up of $\mathbb{F}_1$ at the following six points

$$
z_1^3 f_3(u) + z_1^2 z_2 f_2(u) + z_1 z_2^2 f_1 + z_2^3 f_0 = z_1^2 g_2(u) + z_1 z_2 g_1(u) + z_2^2 g_0 = 0. \tag{3.23}
$$

However the Picard number of B is three ($\rho = 3$), but $h^{1,1} = 8$. In other words, this surface is not favorable. This is because the six points are blown up "at the same time".

---

[8]Note that by an argument from linear algebra, this result does not change by choice of basis in $H^1_{\text{poly}}(B, TB)$.

Note that the exceptional divisor can be identified by $-J_1 + 2J_2 + 2J_3$ is effective and has self-intersection (-6). The global section of the corresponding line bundle is given as

$$e = \frac{z_1^2 g_2(u) + z_1 z_2 g_1(u) + z_2^2 g_0}{x_1}. \tag{3.24}$$

The zero locus $e = 0$ corresponds to a union of six $\mathbb{P}^1$'s each one is a (-1) curve.

Let us tune the complex structure of the base P to[9]

$$x_1(z_1^3 f_3(u) + z_1^2 z_2 f_2(u) + z_1 z_2^2 f_1 + z_2^3 f_0) + x_2(z_1 h_1(u) + z_2 h_0)(z_1 l_1(u) + z_2 l_0) = 0. \tag{3.25}$$

Now there are two independent sets of three curves over

$$z_1^3 f_3(u) + z_1^2 z_2 f_2(u) + z_1 z_2^2 f_1 + z_2^3 f_0 = z_1 h_1(u) + z_2 h_0 = 0,$$

and over

$$z_1^3 f_3(u) + z_1^2 z_2 f_2(u) + z_1 z_2^2 f_1 + z_2^3 f_0 = z_1 l_1(u) + z_2 l_0 = 0. \tag{3.26}$$

So the Picard number jumps to 4. To make this more clear, one can rewrite the same geometry in the following way

$$B^1 = \begin{array}{cccccccc|cc} y_1 & y_2 & x_1 & x_2 & z_1 & z_2 & u_1 & u_2 & P_1 & P_2 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 1 & 1 & 0 & 0 & 2 & 3 \\ 0 & 2 & 0 & 1 & 0 & 1 & 1 & 1 & 2 & 3 \end{array}. \tag{3.27}$$

$$P_1 = y_1 x_2(z_1 h_1(u) + z_2 h_0) + y_2 x_1 = 0,$$
$$P_2 = -y_1(z_1^3 f_3(u) + z_1^2 z_2 f_2(u) + z_1 z_2^2 f_1 + z_2^3 f_0) + y_2(z_1 l_1(u) + z_2 l_0) = 0. \tag{3.28}$$

The geometry above is equivalent to (3.25) by eliminating y_1 and y_2. The two (-1)-curves and their global sections can be identified as

$$e_1 = -J_1 + J_3 + J_4, \qquad\qquad a_1 = \frac{z_1 l_1(u) + z_2 l_0}{y_1}, \tag{3.29}$$

$$e_2 = J_1 - J_2 + J_3 + J_4, \qquad\qquad a_2 = \frac{y_1(z_1 h_1(u) + z_2 h_0)}{x_1}. \tag{3.30}$$

Now, we want to use this jump in $Pic(B)$ to stabilize the complex structure deformation that makes the divisor $e_1 + e_2$ irreducible. To do this, consider, for example, a holomorphic vector bundle with Chern character,

$$ch(V) = n - (\sigma(\eta_0 + e_1) + \omega[f]) + \dots, \tag{3.31}$$

where η_0 depends on $e_1 + e_2$. Due to the special form of $\eta = \eta_0 + e_1$, the spectral cover of V is reducible. Therefore the spectral sheaf $\mathcal{N}$ is therefore defined as an extension,

$$0 \to \mathcal{L}_1 \to \mathcal{N} \to \mathcal{L}_2 \to 0, \tag{3.32}$$

---

[9]The hypersurface is still smooth after this tuning.

where $\mathcal{L}_1$ is a line bundle supported on $\pi^* e_1$, and $\mathcal{L}_2$ is a line bundle supported over smooth surface in the divisor class $n\sigma + \eta_0$. After a calculation similar to [33] one can show[10]

$$c_1(\mathcal{L}_2) = \frac{1}{2}(c_1(B) - [\mathcal{S}_0]) + \lambda \left(n\sigma - \eta_0 + nc_1(B)\right) + \lambda_1 e_1. \tag{3.33}$$

The details on how to derive it is left in appendix B. What is important here is the dependence of the spectral sheaf on e_1. This means that deforming the complex structure back to the point where $e_1 + e_2$ is irreducible is impossible for two reasons. First the spectral cover $[\mathcal{S}] = [\mathcal{S}_0] + e_1$ becomes non-effective, i.e. one cannot define the spectral cover globally. Second, if $\lambda_1 \neq 0$ the line bundle $\mathcal{L}_2$ over $\mathcal{S}_0$ becomes non-holomorphic as we deform the complex structure.

We will see how the complex structure moduli of B^0 are stabilized in this example. Set the initial defining polynomial $P^{(0)}$ to be the special polynomial defined in (3.25). To simply the notation, let's write it as $P^{(0)} = x_1 F_1^{(0)} + x_2 F_2^{(0)}$ where $F_1^{(0)} = z_1^3 f_3(u) + z_1^2 z_2 f_2(u) + z_1 z_2^2 f_1 + z_2^3 f_0$ and $F_2^{(0)}$ is a reducible polynomial given by

$$F_2^{(0)} = (A_1 u_1 z_1 + A_2 u_2 z_1 + A_3 z_2)(B_1 u_1 z_1 + B_2 u_2 z_1 + B_3 z_2), \tag{3.34}$$

with A_i's and B_i's the coefficients. As discussed above, the base B^0 admits two algebraic divisors e_1 and e_2 given by equation (3.29). With them, one can define a holomorphic bundle V.

Now, consider the polynomial deformations of $P^{(0)}$ given by $\delta P = x_1 \delta F_1 + x_2 \delta F_2$. At first, we keep our discussion in general and allow δP to be any elements in $H^0(B^0, N)$. Eventually, we will restrict them to be the deformations in $H^1_{\text{poly}}(B^0, TB^0)$. To keep V holomorphic, the term $F_2^{(0)} + \delta F_2$ in deformed defining polynomial $P^{(0)} + \delta P$ should remain reducible. Deformations satisfying this condition are called *reducible deformations.* Before studying complex structure moduli stabilization, one needs to find all the reducible deformations.

Obviously, δP with $\delta F_2 = 0$ are reducible deformations because they have no effect to $F_2^{(0)}$. Let $M^1_{\ r}$ be the space of all such deformations. For deformations with $\delta F_2 \neq 0$, it is not clear if $F_2^{(0)} + \delta F_2$ remain reducible. However, we know that the deformed F_2 term, if reducible, should be able to be written in the form of (3.34). The first-order deformation δF_2 just shifts the initial coefficients in (3.34) to a set of new coefficients $A_i + \delta A_i$ and $B_i + \delta B_i$. We will see this in details in appendix D. Notice that if δF_2 is a deformations that keeps of $F_2^{(0)} + \delta F_2$ reducible, then $\delta P = x_2 \delta F_2$ is a reducible deformation of $P^{(0)}$.

To obtain the reducible deformations of $F_2^{(0)}$ it is sufficient to calculate the δF_2 obtained by a shift of each coefficient. For example, the reducible deformation corresponding to a shift of the first coefficient $A_1 + \delta A_1$ is obtained by

$$\begin{aligned} F_2 &= ((A_1 + \delta A_1)u_1 z_1 + A_2 u_2 z_1 + A_3 z_2)(B_1 u_1 z_1 + B_2 u_2 z_1 + B_3 z_2) \\ &= F_2^{(0)} + \delta A_1 u_1 z_1 (B_1 u_1 z_1 + B_2 u_2 z_1 + B_3 z_2) \\ &= F_2^{(0)} + \delta A_1 e_1^A. \end{aligned}$$

---

[10]The term $\lambda_1 e_1$ in $c_1(\mathcal{L}_2)$ have to be compensated with a term $-\lambda_1 \mathcal{S}_0$ in $c_1(\mathcal{L}_1)$. They contribute to $c_3(V)$ and ω, but since these are not important for the main purpose of this paper, we ignore further details.

In a similar manner, one can obtain the reducible deformations from shifting the other coefficients

$$e_1^A = u_1 z_1 \left(B_1 u_1 z_1 + B_2 u_2 z_1 + B_3 z_2 \right), \quad e_1^B = u_1 z_1 \left(A_1 u_1 z_1 + A_2 u_2 z_1 + A_3 z_2 \right) \quad (3.35)$$
$$e_2^A = u_2 z_1 \left(B_1 u_1 z_1 + B_2 u_2 z_1 + B_3 z_2 \right), \quad e_2^B = u_2 z_1 \left(A_1 u_1 z_1 + A_2 u_2 z_1 + A_3 z_2 \right)$$
$$e_3^A = z_2 \left(B_1 u_1 z_1 + B_2 u_2 z_1 + B_3 z_2 \right), \qquad e_3^B = z_2 \left(A_1 u_1 z_1 + A_2 u_2 z_1 + A_3 z_2 \right).$$

The linear combinations of $\{e_i^A\}$ and $\{e_j^B\}$ give reducible deformations of δF_2. However, there is a relation among them

$$F_2^{(0)} + A_1 e_1^A + A_2 e_2^A + A_3 e_3^A = F_2^{(0)} + B_1 e_1^B + B_2 e_2^B + B_3 e_3^B \quad (3.36)$$

as one can easily check with the equation (3.34) and (3.35). Thus, only 5 of $\{e_i^A\}$ and $\{e_j^B\}$ are independent and, when multiplying them by x_2, they span a vector space denoted by M_r^2 containing all the reducible deformations with $\delta F_2 \neq 0$. Therefore, the space of all reducible deformations for $P^{(0)}$ is $M_r = M_r^1 \cup M_r^2$.

The complex structure moduli space $H^1(B^0, TB^0)$ is determined by the equation (3.6) and (3.7). It follows that the space of polynomial deformations $H^1_{\text{poly}}(B^0, TB^0)$ has dimension 5,[11] spanned by homogeneous polynomials $d_i, i = 1, 2, \ldots, 5$. The intersection $H^1_{\text{poly}}(B^0, TB^0) \cap M_r$ contains all the polynomial deformations from B^0 that keep V holomorphic and contribute to the nontrivial kernel of Atiyah class in (2.4). The dimension of this intersection determines the number of the "true" moduli in the effective theory while the moduli in the complement of the intersection are stabilized. In our example, $M_r \cap H^1_{\text{poly}}(B^0, TB^0)$ is a 4 dimensional space. Thus, 1 complex structure moduli is stabilized.[12]

3.3 Type III: emergent algebraic cycle

In the final example, we find that there exist algebraic curves in an elliptically fibered Calabi-Yau threefold for special complex structure of the fibration. As in the previous example, defining a vector bundle with these curves stabilizes the complex structure moduli in the fibration.

Consider an elliptic fibered Calabi-Yau threefold $\pi : X \to B$ with Weierstrass fibration given by

$$F_W^{(0)} = y^2 - x^3 - f x z^4 - g z^6 = 0, \quad g = \alpha g', \quad (3.37)$$

where $f \in H^0(B, K_B^{-4})$ and $g \in H^0(B, K_B^{-6})$ chosen to be a product of two polynomials α and g' with correct degree. This reducible polynomial g allow us to define an algebraic 2-cycle on X

$$[C] = \{x = y = \alpha = 0\}. \quad (3.38)$$

---

[11] There are also non-polynomials deformations, but we will not consider them here.

[12] One could guess this result by comparing the complex structure of B^0 and B^1. Since the complex structure of B^1 is "frozen" in the sense that it is impossible to "glue" the divisors e_1 and e_2 into a single divisor e, but any other complex structure deformation of B^0 is in a one-one correspondence with the complex structure deformations of B^1. So the space $H^1(TB^1)$ corresponds to the flat (unstabilized) directions. Finally one can check that $h^1_{poly}(B^1, TB^1) - h^1_{poly}(B^0, TB^0) = -1$ as expected.

Note that this algebraic cycle is independent of the intersection of divisors $\sigma \cdot \pi^* D_b$ and the fiber class $[f]$. Now consider the spectral cover of a general rank n vector bundle V_n

$$\mathcal{S}(V_n) = a_0 z^n + a_2 z^{n-2} x + \cdots + a_n x^{n/2}. \tag{3.39}$$

The 2-cycle $[C]$ intersect with spectral cover at a finite number of points. However, after choosing a_0 to be reducible

$$a_0 = \alpha a_0, \tag{3.40}$$

the curve $[C]$ can lie inside the spectral cover. So if the spectral sheaf $\mathcal{N}$ depends on this new divisor inside $\mathcal{S}(V_n)$ we can stabilize the complex structure moduli, simply because any deformation that removes $[C]$ from $\mathcal{S}(V_n)$ makes $\mathcal{N}$ and hence the bundle V_n non-holomorphic. However, a Fourier-Mukai analysis as in [33] shows that if the Chern classes of the vector bundle do not depend on the 2-cycle $[C]$, then the spectral line bundle also does not depend on the new divisor $[C]$ in $\mathcal{S}(V_n)$. Therefore the complex structure moduli cannot be stabilized. So suppose the Chern character of V_n is of the form (See appendix B for details)

$$Ch(V_n) = n - (\sigma \cdot \eta + \omega[f] + [C]) + \frac{1}{2}c_3(V_n). \tag{3.41}$$

By doing similar calculations as in [33] we can derive the topological constraints on the divisor class of $S(V_n)$, Chern classes of V_n and $\mathcal{L}$. We will not repeat them here. However, we just mention the part of the results that we will need,[13]

$$[\mathcal{S}(V_n)] = n\sigma + \eta + \pi_*[C], \tag{3.42}$$

$$c_1(\mathcal{N}) = \frac{1}{2}\left([S(V_n)] - c_1(B)\right) + \lambda\left(n\sigma - \eta + nc_1(B)\right) + \left(\frac{1}{n}\pi_*[C] - [C]\right). \tag{3.43}$$

Therefore $\mathcal{N}$ explicitly depends on the divisor $[C]$ in the spectral cover. This dependence (partially) stabilize both complex structure and vector bundle moduli.

Consider a deformation of Weierstrass fibration $F_W^{(0)} \to F_W^{(0)} + \delta g z^6$, which induces a deformation to function g by

$$\alpha g' \longrightarrow g. \tag{3.44}$$

If the deformed g can be expressed as a product of two polynomials (having the same degree with α and g'), then the algebraic curve $[C]$ defined in (3.38) still exist, and the bundle V remain holomorphic. However, if the function g after deformation is irreducible, the curve $[C]$ is not algebraic anymore, and the vector bundle defined by the spectral line bundle (3.43) which explicitly depends on $[C]$ becomes non-holomorphic. Therefore the associated complex structure moduli of F_W can be stabilized. As we will explain in section 4, these polynomial deformations map to complex structure moduli in the F-theory dual picture, and a four-flux stabilizes them.

---

[13]Note that we should choose α such that $\frac{1}{n}\pi_*[C]$ be integral.

Example. Let $\pi : X \to \mathbb{P}^2$ be an elliptically fibered Calabi-Yau threefold with base $\mathbb{P}^2$. The fibration is Weierstrass introduced in equation (3.2) with $f \in H^0(\mathbb{P}^2, O(12))$ and $g \in H^0(\mathbb{P}^2, O(18))$. Let $\{u_1, u_2, u_3\}$ be the homogeneous coordinates of $\mathbb{P}^2$. Choose the initial fibration to be

$$f^{(0)} = F_{12}(u), \quad g^{(0)} = \alpha F_{16}(u) \tag{3.45}$$

where F_{12} and F_{16} are generic polynomials with degrees 12 and 16 and α is a degree 2 homogeneous polynomial. As shown in (3.38), the fibration with reducible g admits an algebraic curve $[C]$ in $\mathbb{P}^2$, which will be used to define the topology of vector bundles.

Consider a SU(2) extension bundle V over X. The spectral cover construction of V introduced in equation (3.11) is $S = a_2 z^2 + a_0 x$ where x, z are coordinates of the fibration. If $D_b = O(2)$, then a_2 and a_0 should be global sections of $O_B(7)$ and $O_B(1)$. Let's take them to be

$$a_0 = F_1(u_1, u_2, u_3), \quad a_2 = \alpha F_5(u_1, u_2, u_3) \tag{3.46}$$

where F_d is a generic homogeneous polynomial with degree d. As discussed in (3.41) and (3.43), if the Chern character of V depends on the algebraic 2-cycle $[C]$, then the Chern class of spectral sheaf depends on $[C]$ explicitly. Thus, the holomorphy of V depends on the existence of the algebraic cycle $[C]$ that will be used to stabilize complex structure moduli in a second.

We will study how many complex structure moduli can be stabilized in this example. The elliptically fibered Calabi-Yau manifold X can be regarded as

$$X = \begin{array}{ccccccc|c} u_1 & u_2 & u_3 & x & y & z & & P \\ \hline 1 & 1 & 1 & 6 & 9 & 0 & & 18 \\ 0 & 0 & 0 & 2 & 3 & 1 & & 6 \end{array} \tag{3.47}$$

With it, one can compute the complex structure moduli space $H^1(X, TX)$ from the equation (3.6) and (3.7). Again, we will only focus on the polynomial deformations $H^1_{\text{poly}}(X, TX)$ which has dimension 272. Since the base $\mathbb{P}^2$ is rigid, all the complex structure moduli are coming from the fibration, i.e. deformations of f and g.[14]

As discussed above, deformations that keep V holomorphic are those preserve the reducibility of $g = \alpha F_{16}(u)$. Again, we will call them *reducible deformations*. Obviously, any polynomial without z^6 terms is reducible deformation. While, the reducible deformations with z^6 terms can be calculated from the method in appendix D. We find that these deformations can be generated from 159 polynomial deformations $e_i^A, i = 1, 2, \ldots, 6$ and $e_j^B, i = 1, 2, \ldots, 153$, which are obtained by deforming the coefficients in α and $F_{16}(u)$ respectively. In fact, the space of these deformations denoted by M^2_r has dimension 158 because there is a nontrivial relation in (D.8) between $\{e_i^A\}$'s and $\{e_j^A\}$'s.

Finally, we will see how many polynomial deformations $H^1_{\text{poly}}(X, TX)$ are stabilized. One needs to split 272 deformations found earlier into two groups. One group is proportional to z^6, which have no effect on g and should be considered as the moduli of the effective

---

[14]Indeed, the degree of freedom of f and g is $h^0(\mathbb{P}^2, O(12)) + h^0(\mathbb{P}^2, O(18)) = 91 + 190 = 281$. Considering the coordinates redefinition of $\mathbb{P}^2$, one find the independent degree of freedom of f and g is 272, which matches the number of complex structure moduli counted in $h^1(X, TX) = 272$.

theory. There are another 82 of them. These ones contain terms proportional to z^6 and could have an effect on the reducibility of g. It turns out that there are 190 such polynomials. Their coefficients of z^6 is denoted by $d_i, i = 1, 2, \ldots, 190$. Then, one needs to study how many of them, or their linear combinations can give rise to reducibility preserving deformations in M^2_r. Direct computations shows that all 158 polynomials in M^2_r can be expressed in the complex structure moduli $\{d_i\}$. Thus, there are 241 polynomial deformations that preserves the reducible of g while there are 32 complex structure moduli are stabilized.

4 Discussion and comments on F-theory dual

In this section, we briefly describe the F-theory translation of the Heterotic examples mentioned before. When the spectral cover is smooth, the duality procedure is well-known [51]. The geometry of the F-theory half-Calabi-Yau fourfold (more precisely dP_8 fibration) is given by (for a SU(n) bundle with $n \leq 5$),

$$Y^2 = X^3 + fXv^4 + gv^6 + uv^{5-n}S_n(X, Y, v), \tag{4.1}$$

where the term in the parenthesis is the spectral cover of a SU(n) vector bundle (with z is replaced by v), and (X, Y, u, v) are the homogeneous coordinates of $\mathbb{WP}^{3211}$. In addition, using the cylinder map, one constructs the four-flux. The details can be found in [51]. It is only sufficient to mention that the part of the $c_1(\mathcal{L})$ that lies in the kernel of $\pi_S : S \to B$, lifts to the four-flux in F-theory language,

$$\left. \begin{array}{c} c_1(\mathcal{L}) = \frac{1}{2}\left(c_1(B) - [S]\right) + \gamma \\ \pi_{S*}\gamma = 0 \end{array} \right\} \Rightarrow \gamma \longrightarrow G_4 \sim P^*_{Cyl}\gamma, \tag{4.2}$$

where P_{Cyl} is the cylinder map which is a $\mathbb{P}^1$ fibration over the spectral cover S_n,

$$P_{Cyl} : \{S_n(X, Y, v) = 0 \subset X_F\} \xrightarrow{\mathbb{P}^1} S_n(x, y, z) \subset X_{Het}. \tag{4.3}$$

In terms of the local 7-brane data, there is a doublet (E, Φ), called Higgs bundles, which consists of a holomorphic bundle E on B (where 7-brane wraps) and a "Higgs" field $\Phi \in H^0(E \otimes E^* \otimes K(\eta))$ which satisfy the Hitchin system equations [52]. One can study the Higgs bundles by spectral data,

$$(E, \Phi) \leftrightarrow (S_H, \mathcal{L}_H), \tag{4.4}$$

where S_H is a finite cover of B defined as a subvariety of $Tot(K)$,

$$S_H = Det\left(\lambda 1 - \Phi\right) = 0, \tag{4.5}$$

where λ is the tautological section of $Tot(K)$, and $\mathcal{L}_H$ is a line bundle over S_H. The Higgs bundle (E, Φ) can be reconstructed by the spectral data as,

$$E = \pi_*\mathcal{L}_H, \quad \Phi = \pi_*\lambda, \tag{4.6}$$

$$\pi_S : S_H \to B. \tag{4.7}$$

The Higgs bundle spectral data is very similar to the Heterotic spectral data derived from the Fourier transform of V. They are indeed related to each other by cylinder construction [53].

Finally, note that what is responsible for the complex structure is the F-term derived from the Gukov-Vafa-Witten like superpotentials, which takes the following forms in Heterotic, F-theory, and 7-brane

$$\text{Heterotic} \quad \leftrightarrow \quad \text{F-theory} \quad \leftrightarrow \quad \text{7-brane}$$

$$W = \int_{X_{Het}} \omega_3 \wedge \Omega_3 \quad \leftrightarrow \quad W = \int_{X_F} G_4 \wedge \Omega_4 \quad \leftrightarrow \quad W = \int_B Tr(F_2 \wedge \Phi)$$

Where Ω corresponds to the complex structure top-form and F_2 is the curvature two-form of E. With this introduction, we move forward to briefly explain the "image" of the moduli stabilization scenarios of the Heterotic string in the dual F-theory. Locally over the 7-brane, the story is similar to the Heterotic version. The deforming away from special loci in complex structure moduli of the base makes the Higgs bundle non-holomorphic. The term $F_2 \wedge \Phi$ in the superpotential makes some components of Φ massive. Generally, since the Higgs field corresponds to the fluctuations of the 7-brane in the normal directions, this means that away from some special loci in the complex structure moduli of B, the 7-brane cannot "wiggle" in specific ways. This can be seen more clearly, by noting that the spectral cover's coefficients can be identified with the Casimir's of the Higgs field.

The above arguments were local, but in the following, we will mainly focus on the effect of these deformations on the G_4 flux. The type III example can only be explained in terms of the global G_4 flux stabilization rather than local two-flux.

Type I. We already know in the extension class of V in (2.7) appeared as the new terms in a_0 and a_2 in (3.8). By the well-known arguments briefly reviewed above, these jumps in a_0 and a_2 correspond to new jumps in Casimir's of Φ, in other words, new zero modes in fluctuations of the 7-brane.

New zero modes in the vector bundle's moduli space in Heterotic string corresponds to new zero modes in deformations of the 7-brane in F-theory.

To construct the G_4 flux, assume the spectral cover is smooth,

$$S = a_2 x + a_0 z^2.$$

Then by the introduction above, G_4 flux is related to the divisor γ in $c_1(\mathcal{N})$. But $\gamma = 0$ for SU(2) bundles.[15] So one may naively expect that the G_4 flux must vanish in this case. But, note that in this case, there are a finite number of "vertical components" (i.e., isolated elliptic curves) inside the spectral cover located above the points $a_0 = a_2 = 0$ in the base B. In the dual F-theory geometry, such vertical components become 4-cycles C_4 by cylinder map, which are (at least topologically) elliptic curves times a $\mathbb{P}^1$.

One can use these cycles to construct G_4 flux,

$$G_4 \longrightarrow C_4 + \text{Corrections}, \tag{4.8}$$

<hr>

[15]γ induces non-zero chirality in the effective theory, but for SU(2) bundles, the chirality must be zero. The vanishing of γ can also be seen from direct spectral cover calculations.

where the corrections are terms we need to add such that

$$G_4 \cdot \sigma \cdot \pi^* D_{\text{base}} = 0. \tag{4.9}$$

Note that this G_4 flux only exists when the terms a_2 and a_0 in the spectral cover are not zero. Therefore, when we move to a generic point in the complex structure moduli where there are no jumps in a_2 and a_0, the G_4 flux constructed above "disappears" i.e., G_4 becomes non-holomorphic, which by the Gukov-Vafa-Witten superpotential makes the corresponding complex structure deformation massive.[16]

Type II. In this case, the spectral cover is smooth, and as explained in subsection 3.2, after the complex structure deformation, the spectral cover is not algebraic anymore. So an argument similar to the one mentioned above is (when $a_2 \neq 0$) also applies in this case. In other words, the dual Higgs bundle spectral data on the F-theory side is also required to be holomorphic, and the F-term equations obstruct these deformations in the 7-brane worldvolume theory.

In addition, γ term in (4.2) can be non-zero in general (for $n > 2$). The general form of γ in [31, 32] is

$$\gamma = \lambda(n\sigma - \eta + nc_1(B))|_S. \tag{4.10}$$

The curves (divisors) $\sigma|_S$ and η_S, and hence γ and G_4 in (4.2), are holomorphic as long as the spectral cover is holomorphic. Therefore, if the spectral cover's coefficients depend on the new terms that appear in the special loci of the complex structure moduli space, then it is impossible to move away from those points since γ. Hence, the corresponding G_4 flux becomes non-holomorphic.

In the example we considered here, γ depends on the new divisor e_1,

$$\gamma = \gamma_0 + \lambda_1 e_1. \tag{4.11}$$

Of course $\pi_{s*}\gamma \neq 0$ now, so one should modify the corresponding four-flux, but the point is that this four-flux is only holomorphic on the jumping locus.

Type III. In this type the γ is modified as[17]

$$\gamma = \lambda\left(n\sigma - \eta + nc_1(B)\right) + \left(\frac{1}{n}\pi_*[C] - [C]\right). \tag{4.12}$$

As mentioned before, on the Heterotic side, the dependence of (3.43) on $[C]$ stabilize both the complex structure moduli and the vector bundle moduli. But in F-theory dual, both (3.37) and (3.40) corresponds to complex structure deformation on the F-theory side.

---

[16]The Calabi-Yau four-fold of F-theory has $(4, 6, 12)$ singularity over the points corresponding to the vertical components of S. Naively one may intend to blow these singularities up, but this is only possible when the Heterotic "bundle" (coherent sheaf) is a direct sum of a vector bundle and small instantons localized on the elliptic curves.

[17]Note that $\pi_{S*}\left(\frac{1}{n}\pi_*[C] - [C]\right) = 0$.

The corresponding (horizontal) four-flux that is responsible for stabilizing the F-theory complex structure is already studied [35]. A quick look at the defining relation for the F-theory geometry (4.1) makes it clear that with this particular constraint over the geometry of the Heterotic Calabi-Yau and spectral cover, there is an algebraic four-cycle,

$$C_4 := \{X = Y = \alpha = 0\} \subset X_F, \qquad (4.13)$$

which can be used to construct an "algebraic four-flux". It is also clear that C_4 is a $\mathbb{P}^1$ fibration (cylinder) over the algebraic curve $[C]$ in the Heterotic geometry. Therefore we can identify C_4 with the $P^*_{Cyl}[C]$ part of the four-flux induced by the spectral cover.

Finally, note that contrary to type I and II, the two-flux localized inside the 7-brane is not responsible for stabilizing the complex structure moduli. This is because the vector bundle in the Heterotic side is identified with the spectral data living in the *compact elliptic fibration* over B. In contrast, the Higgs bundle on the 7-brane is identified with spectral data that live in the *non-compact* fibration $Tot(K_B)$. Therefore in local F-theory/Heterotic duality, only the part of Heterotic spectral data in the patch that contains the zero section is identified with the Higgs bundle spectral data. In other words,

$$(\mathcal{S}_{Het}, \mathcal{N}_{Het})|_{\sigma\text{-patch}} \longleftrightarrow (\mathcal{S}_{\text{Higgs}}, \mathcal{N}_{\text{Higgs}}). \qquad (4.14)$$

However, remember that

$$\sigma \cdot [C] = 0. \qquad (4.15)$$

5 Conclusion

In this paper, we have studied how holomorphic bundles can stabilize complex structure moduli of an elliptically fibered Calabi-Yau threefold in the context of a heterotic compactification. In general, these moduli can come from the complex structure moduli of the base and the complex structure moduli of the fibration. For vector bundles defined in terms of spectral data, we find three approaches that both of these moduli can be stabilized with algebraic cycles existing only on special loci in the moduli space. We develop practical tools to identify the stabilized moduli and present concrete examples for all three approaches. As a result, we show that complex structure moduli of the elliptically fibered Calabi-Yau threefold can be effectively stabilized by the holomorphic bundles.

As mentioned in the introduction, one of the advantages of working with spectral covers is the manifest connection with the F-theory and hence type II models. In other words, the complex structure moduli stabilization scenarios are already well known, at least conceptually, in both type II and Heterotic string, but apparently, they look very different from each other. But both of these theories can be seen as different limits of F-theory. By describing the moduli stabilization in terms of spectral data, we could see the corresponding stabilization scenario in the dual F/type II theory. In particular, we have shown that in the type I/II examples, the dual stabilization is made possible by the flux inside the 7-brane worldvolume. In contrast, in the type III example, the dual F-theory geometry is stabilized by a horizontal algebraic G_4 flux.

Acknowledgments

A Complex structure moduli space of elliptically fibered Calabi-Yau threefold

In this appendix, we briefly discuss the relation between the complex structure moduli space of an elliptically fibered Calabi-Yau threefold $H^1(X, TX)$ and the complex structure moduli space of its base $H^1(B, TB)$. We assume B is a Fano surface (K_B^{-1} is ample) and $h^1(\mathcal{O}_B) = 0$.

One can use the adjunction sequence for the section, σ, of the fibration to relate the tangent bundle of the base TB with the tangent bundle of the Calabi-Yau manifold X,

$$0 \to TB \to TX|_B \to \mathcal{N}_{B/X} = K_B \to 0. \tag{A.1}$$

From the sequence above, one can see, as long as the base B is simply connected, $H^1(TB) \simeq H^1(B, TX|_B)$. So we should find the relation between $TX|_B$ and TX,

$$0 \to TX \otimes \mathcal{O}_X(-\sigma) \to TX \to TX|_B \to 0. \tag{A.2}$$

The corresponding long exact sequence is,

$$
\begin{aligned}
0 \to H^0(TX \otimes \mathcal{O}_X(-\sigma)) \to H^0(TX) \to H^0(TX|_B) \to \\
\to H^1(TX \otimes \mathcal{O}_X(-\sigma)) \to H^1(TX) \to H^1(TX|_B) \to \\
\to H^2(TX \otimes \mathcal{O}_X(-\sigma)) \to H^2(TX) \to H^2(TX|_B) \to \\
\to H^3(TX \otimes \mathcal{O}_X(-\sigma)) \to H^3(TX) \to 0.
\end{aligned} \tag{A.3}
$$

We will show bellow that as long as K_B^{-1} is ample, the homeomorphism $H^1(X, TX|_B) \simeq H^1(B, TB) \to H^2(TX \otimes \mathcal{O}_X(-\sigma))$ have to be zero. To see this, we will use the following sequences,

$$0 \to TX \otimes \mathcal{O}_X(-\sigma) \to T\mathcal{A}|_X \otimes \mathcal{O}_X(-\sigma) \to \mathcal{N}|_X \otimes \mathcal{O}_X(-\sigma) \to 0,$$
$$0 \to T_{\bar{\pi}}|_X \to T\mathcal{A}|_X \to \pi^* TB \to 0,$$
$$0 \to \mathcal{O}_X \to \mathcal{O}_X(3\sigma - 3K_B) \oplus \mathcal{O}_X(2\sigma - 2K_B) \oplus \mathcal{O}_X(\sigma) \to T_{\bar{\pi}}|_X \to 0,$$

where $\bar{\pi} : \mathcal{A} \to B$ is the ambient space that the Calabi-Yau is embedded inside it.[18] Now, by the last sequence above, one can show

$$\pi_*(T_{\bar{\pi}}|_X \otimes \mathcal{O}_X(-\sigma)) = K_B^{-3} \otimes (K_B^2 \oplus \mathcal{O}_B) \oplus K_B^{-2} \oplus \mathcal{O}_B.$$

On the other hand, one can use the second sequence to show,

$$R\pi_*\left(T\mathcal{A}|_X \otimes \mathcal{O}_X(-\sigma)\right) = \pi_*(T_{\bar{\pi}}|_X \otimes \mathcal{O}_X(-\sigma)) \oplus TB \otimes K_B[-1].$$

---

[18]Note that $\mathcal{A}$ is a $\mathbb{P}^{231}$ fibration.

Finally the first short exact sequence, and the result above leads to,[19]

$$0 \to \pi_*(T_{\bar{\pi}}|_X \otimes \mathcal{O}_X(-\sigma)) \to (\pi_* \mathcal{N}|_X \otimes \mathcal{O}_X(-\sigma)) \to R^1\pi_*(TX \otimes \mathcal{O}_X(-\sigma)) \to TB \otimes K_B \to 0,$$

or, by considering the cokernel of the second map, we can rewrite this sequence as

$$0 \to K_B^{-10} \otimes \mathcal{I}_{24K^2} \to R^1\pi_*(TX \otimes \mathcal{O}_X(-\sigma)) \to TB \otimes K_B \to 0, \qquad (A.4)$$

where, $\mathcal{I}_{24K^2}$ is the ideal sheaf of a finite number of points given by the intersection of divisors $(-6K_B) \cdot (-4K_B)$. Now we can talk about the cohomologies of $TX \otimes \mathcal{O}_X(-\sigma)$ in (A.3). By the Leray spectral sequence,

$$E_2^{p,q} := H^p(B, R^q\pi_*(TX \otimes \mathcal{O}_X(-\sigma))) \Rightarrow H^{p+q}(X, TX \otimes \mathcal{O}_X(-\sigma))).$$

Since $R^0\pi_*(TX \otimes \mathcal{O}_X(-\sigma)) = 0$ then

$$H^p(X, TX \otimes \mathcal{O}_X(-\sigma))) = H^{p-1}(B, R^1\pi_*(TX \otimes \mathcal{O}_X(-\sigma))).$$

The cohomology long exact sequence of the equation (A.4) leads to,

$$
\begin{aligned}
0 \to H^0(K_B^{-10}) &\to H^0(R^1\pi_*(TX \otimes \mathcal{O}_X(-\sigma))) \to H^0(TB \otimes K_B) \to \\
&\to H^1(K_B^{-10}) \to H^1(R^1\pi_*(TX \otimes \mathcal{O}_X(-\sigma))) \to H^1(TB \otimes K_B) \to \\
&\to H^2(K_B^{-10}) \to H^2(R^1\pi_*(TX \otimes \mathcal{O}_X(-\sigma))) \to H^2(TB \otimes K_B) \to 0
\end{aligned}
$$

By the ampleness assumption of K_B^{-1} we expect $H^i(B, K_B^{-10}) = 0$ for $i \neq 0$. So

$$
\begin{aligned}
H^2(X, TX \otimes \mathcal{O}_X(-\sigma))) &= H^1(R^1\pi_*(TX \otimes \mathcal{O}_X(-\sigma))) = H^1(TB \otimes K_B) = H^{1,1}(B), \\
H^3(X, TX \otimes \mathcal{O}_X(-\sigma))) &= H^2(R^1\pi_*(TX \otimes \mathcal{O}_X(-\sigma))) = H^2(TB \otimes K_B) = H^0(B, TB^*).
\end{aligned}
$$

At the end note that in general $h^2(X, TX) = h^{1,1}(X) = 1 + h^{1,1}(B)$, and also from (A.1) we can show $H^2(TX|_B) = H^2(K_B) \oplus H^2(TB) = \mathbb{C} \oplus H^2(TB)$. Then to get the dimension of $H^2(TX)$ in (A.3) correct, all of the elements of $H^2(X, TX \otimes \mathcal{O}_X(-\sigma))) = H^{1,1}(B)$ should inject into $H^2(TX)$. Therefore we get a surjection (at least as long as $H^1(\mathcal{O}_B) = 0$),

$$H^1(TX) \to H^1(TX|_B) \simeq H^1(TB) \to 0.$$

We conclude that $H^1(TB)$ is a subspace of the moduli space $H^1(TX)$ as expected.

---

[19]Note that $\pi_*(TX \otimes \mathcal{O}_X(-\sigma)) = 0$ because TX is a stable bundle, and one can show its restriction on generic fiber of X is given by non-trivial extension of trivial bundles. Therefore $TX \otimes \mathcal{O}_X(-\sigma)$ doesn't have any global section on generic fibers. Naively, we may say $\pi_*(TX \otimes \mathcal{O}_X(-\sigma))$ should be a torsion sheaf on B (probably non-zero only on the discriminant curve of X), but since π is flat and $TX \otimes \mathcal{O}_X(-\sigma)$ locally free, the pushforward $\pi_*(TX \otimes \mathcal{O}_X(-\sigma))$ must be locally free, and hence should vanish everywhere.

B Type II and type III Fourier-Mukai calculations

For completeness, we present some details of the Fourier-Mukai calculations as much as is necessary in this section. Following [33] we can compute the Chern characters of the Fourier-Mukai $\Phi(V)$ of V. In type II example, it is,

$$Ch(\Phi(V)) = -[\mathcal{S}] + \left([\mathcal{S}] \cdot \left(\frac{C_1(B)}{2} \right) + \frac{1}{2} c_3(V)[f] \right), \tag{B.1}$$

where $[\mathcal{S}] = \eta + n\sigma$. In type II example we chose $\eta = \eta_0 + e_1$, and with this choice the algebraic equation of $[\mathcal{S}]$ will be reducible. Therefore the spectral line bundle $\mathcal{N}$ will be given as an extension,

$$0 \longrightarrow i_{e_1 *}\mathcal{L}_1 \longrightarrow \mathcal{N} \longrightarrow i_{S_0 *}\mathcal{L}_2 \longrightarrow 0, \tag{B.2}$$

where i_{e_1} and i_{S_0} are respectively the inclusion morphisms of the surfaces $\pi^* e_1$ and S_0 with divisor class $n\sigma + \eta_0$. Now one can use,

$$\Phi(V) = i_{S*}\mathcal{N}[-1] \implies Ch(\Phi(V)) = -Ch(i_{S*}\mathcal{N}) = -Ch(i_{e_1 *}\mathcal{L}_1) - Ch(i_{S_0 *}\mathcal{L}_2). \tag{B.3}$$

Next we can use the Grothendieck-Riemann-Rock to compute,

$$Ch(i_{e_1 *}\mathcal{L}_1) = [e_1] \cdot \left(\frac{e^{c_1(\mathcal{L}_1)}}{Td(\mathcal{O}(e_1))} \right), \tag{B.4}$$

$$Ch(i_{S_0 *}\mathcal{L}_2) = [S_0] \cdot \left(\frac{s^{c_1(\mathcal{L}_2)}}{Td(\mathcal{O}(S_0))} \right). \tag{B.5}$$

In order by order comparison of the formulas written above leads to the following result,

$$c_1(\mathcal{L}_2) = \frac{1}{2}(c_1(B) - [\mathcal{S}_0]) + \lambda(n\sigma - \eta_0 + nc_1(B)) + \lambda_1 e_1. \tag{B.6}$$

A similar calculation for the type III example would lead to,

$$Ch(\Phi(V)) = -[n\sigma + \eta + \pi^*\pi_*[C]] + [n\sigma + \eta + \pi^*\pi_*[C]] \cdot \left(\frac{c_1(B)}{2} - \frac{1}{2}\pi^*\pi_*[C] \right) + [C] + \dots.$$

Again order by order comparison of this with the Chern character of the spectral sheaf over a smooth surface $i_{S*}\mathcal{N}$ shows that first of all, the divisor class of the spectral cover is

$$[S] = n\sigma + \eta + \pi^*\pi_*[C], \tag{B.7}$$

second Chern class of the line bundle $\mathcal{N}$ must be,

$$c_1(\mathcal{N}) = \frac{1}{2}([S(V_n)] - c_1(B)) + \lambda(n\sigma - \eta + nc_1(B)) + \left(\frac{1}{n}\pi_*[C] - [C] \right). \tag{B.8}$$

C Computation details of "jumping" ext

We show the calculation details about the "jumping" extension group used to stabilize complex structure moduli in subsection 3.1. The base B of the elliptic Calabi-Yau threefold X is defined in (3.13). The defining polynomial p_0 is a degree $(2,2)$ polynomial in $\{x_0, x_1\}$ and $\{y_0, y_1, y_2\}$. The coefficient of p_0 is a redundant description of the complex structure of B. We will extract the independent ones from them later.

The rank two extension bundle V is introduced in (2.7) with extension group $Ext(L, L^\vee) = H^1(X, L^2)$. If the threefold X admits an elliptic fibration, this group can be expressed as

$$H^1(X, L^2) = H^0(B, L_1 \oplus L_2), \tag{C.1}$$

where $L_1 = O(D_b - C_1(B))$ and $L_2 = O(D_b + C_1(B))$ are line bundles and D_b is a divisor on the base. Thus, the Ext group of V is a direct sum of $H^0(B, L_1)$ and $H^0(B, L_2)$. We will study the "jumping" of them in the following.

The line bundle cohomology on the base B follows from a similar procedure discussed in subsection 2.1. As before, we can write the L_1 and L_2 in terms of the line bundles of the ambient space by Koszul sequence [37]. The associated long exact sequence is

$$0 \to H^0(B, L) \to H^1(\mathcal{A}, N^\vee \otimes L_\mathcal{A}) \xrightarrow{p_0} H^1(\mathcal{A}, L_\mathcal{A}) \to H^1(B, L) \to 0 . \tag{C.2}$$

Notice that, instead of $H^1(X, L^2)$, we are interested in the $H^0(B, L)$ here. For our example considered in section 3, $L_1 = O(-2, 3)$ and $L_2 = O(-2, 5)$. From the equation (C.2), we find

$$H^0(X, L_1) = \text{Ker}\left[H^1(A, O(-4, 1)) \xrightarrow{p_0} H^1(A, O(-2, 3))\right] \tag{C.3}$$

$$H^0(X, L_2) = \text{Ker}\left[H^1(A, O(-4, 3)) \xrightarrow{p_0} H^1(A, O(-2, 5))\right] \tag{C.4}$$

To determine the kernel, we express the cohomology of the source and the target in terms of their Bott-Borel-Weil polynomial representations. For the first one in equation (C.3), they are

$$H^1(\mathbb{P}^1 \times \mathbb{P}^2, O(-4, 1)) = \left\{\frac{y_0}{x_0^2}, \frac{y_1}{x_0^2}, \dots \frac{y_0}{x_0 x_1}, \dots \frac{y_2}{x_1^2}\right\} \tag{C.5}$$

$$H^1(\mathbb{P}^1 \times \mathbb{P}^2, O(-2, 3)) = \{y_0^3, y_0^2 y_1, \dots y_2^3\} \tag{C.6}$$

and p_0 is the degree $(2,2)$ defining polynomial. Note that the multiplication of $\frac{y_0}{x_0 x_1}$ and $x_0^2 y_0^2$ maps to zero in the target because the resulting monomials have the wrong degree and should not be considered as an appropriate element of the target. The same is for a map like $x_1^2 y_0^2$. Thus, the most general form of the p_0 that have a non-trivial kernel is given by

$$p_0 = x_1^2 P_1(y) + x_2^2 P_2(y), \tag{C.7}$$

where $P_1(y)$ and $P_2(y)$ are arbitrary degree 2 polynomials in $\{y_0, y_1, y_2\}$. There are three independent monomials with the denominator $x_0 x_1$. Therefore, the dimension of the kernel "jumps" to 3 when the defining polynomial has the form in (C.7), and it vanishes when p_0 is a generic polynomial. By the same approach, we also compute the "jumping" of $H^0(X, L_2)$. The results are summarized in equation (3.17) that is used for complex structure moduli stabilization in section 3.

D Practical approach to identify reducible deformations

In subsection 3.2, we find that the space of reducible deformations M^2_r is generated by 5 out of 6 polynomials given in (3.35). There, we use the fact that polynomial deformation δF_2, if reducible, is equivalent to shifts of the coefficients in $F_2^{(0)}$. We will see why it is true and verify that M^2_r contains all the possible reducible deformations. In addition, we will generalize the method introduced in subsection 3.2 to find reducible deformations of other initial polynomials.

Let's expand the initial $F_2^{(0)}$ defined in (3.34) as

$$F_2^{(0)} = (au_1^2 + bu_1u_2 + cu_2^2)z_1^2 + (du_1 + eu_2)z_1z_2 + fz_2^2 . \tag{D.1}$$

where coefficients a, b, c, d, e, f are given by

$$a = A_1B_1 , \qquad b = A_1B_2 + A_2B_1 , \qquad c = A_2B_2 , \tag{D.2}$$
$$d = A_1B_3 + A_3B_1 , \qquad e = A_2B_3 + A_3B_2 , \qquad f = A_3B_3$$

with $A_1, A_2, A_3, B_1, B_2, B_3$ the coefficient of $F_2^{(0)}$ in the factored form. It means that given a polynomial with coefficients a, b, c, d, e, f, if one can find a set of $\{A_i\}$ and $\{B_i\}$ solves the equation (D.2), the polynomial is reducible. This is the reducible condition that will be used later. Obviously, $F_2^{(0)}$ is expressed by the following 6 monomials

$$(u_1^2z_1^2, u_1u_2z_1^2, u_2^2z_1^2, u_1z_1z_2, u_2z_1z_2, z_2^2) . \tag{D.3}$$

Thus, there are at most 6 polynomial deformations for $F_2^{(0)}$. Each of those is given by the deformation of a monomial. We will study which of these monomial deformations or their combinations δF_2 can keep $F_2^{(0)} + \delta F_2$ reducible.

Suppose we deform the initial $F_2^{(0)}$ by the first monomial $\delta au_1^2z_1^2$. The coefficients of the deformed polynomial $F_2^{(0)} + \delta au_1^2z_1^2$ becomes $\{a+\delta a, b, c, d, e, f\}$, which does not satisfy the reducible condition in (D.2). Thus, the deformed polynomial is not reducible. To restore the reducibility, one needs to adjust the coefficients $\{A_i\}$ and $\{B_i\}$ so that they give rise to the new parameters $\{a+\delta a, b, c, d, e, f\}$ via the condition (D.2). Since $a = A_1B_1$, it can be done by simply redefining A_1 to be $A_1' = A_1 + \frac{\delta a}{B_1}$. Now, $a + \delta a = A_1'B_1$. To satisfy the reducible condition (D.2), one needs to replace the A_1 with A_1' in all other equations involving A_1, which are $b = A_1'B_2 + A_2B_1$ and $d = A_1'B_3 + A_3B_1$. The redefinition is equivalent to shift $b \to b + (\frac{B_2}{B_1})\delta a$ and $d \to d + (\frac{B_3}{B_1})\delta a$. That means, to preserve the reducibility, whenever we deform the F by $\delta au_1^2z_1^2$, we have to add the extra terms $(\frac{B_2}{B_1})u_1u_2z_1^2\delta a$ and $(\frac{B_3}{B_1})u_1z_1z_2\delta a$ as well. The reducible deformation after a redefinition of A_1 is

$$e_1 = u_1^2z_1^2 + \left(\frac{B_2}{B_1}\right)u_1u_2z_1^2 + \left(\frac{B_3}{B_1}\right)u_1z_1z_2, \tag{D.4}$$

which, by a factor of B_1, is the same deformations $e_1^A = B_1e_1$ as the one obtained by a shift of A_1 in (3.35).

Besides the redefinition of A_1, to absorb the deformation $\delta au_1^2z_1^2$, one can also consider the redefinition of the B_1 as $B_1' = B_1 + \frac{\delta a}{A_1}$. From the relation (D.2), one has to redefine

the b and d by shifting $b \to b + (\frac{A_2}{A_1})\delta a$ and $d \to d + (\frac{A_3}{A_1})\delta a$, which is given by

$$e_2 = u_1^2 z_1^2 + \left(\frac{A_2}{A_1}\right) u_1 u_2 z_1^2 + \left(\frac{A_3}{A_1}\right) u_1 z_1 z_2. \tag{D.5}$$

Again, up to a factor of A_1, it is the same deformation obtained by a shift of B_1. Following the same analysis, we consider all 6 monomial deformations in (D.3). We find that the shifts of the coefficients can absorb all these deformations $\{A_i\}$ and $\{B_i\}$, which gives the same reducible deformations as ones in $M^2{}_r$.

The reducible deformations considered above are obtained by the redefinition of one coefficients. For example, e_1 and e_2 are generated by a shift of A_1 and B_1. One may suspect that can we get more reducible deformations by redefining A_1 and B_1 at the same time? For example, given the monomial deformation $\delta a u_1^2 z_1^2$, one can absorb it by shifting the A_1 and B_1 at the same time by $\delta A_1 = \frac{\delta a}{2B_1}$ and $\delta B_1 = \frac{\delta a}{2A_1}$. It is easy to check that, to first-order, the reducible condition (D.2) is satisfied. In particular, $a + \delta a = (A_1 + \delta A_1)(B_1 + \delta B_1)$. The corresponding reducible deformations E follow by shifting the A_1 and B_1 in all other equations in (D.2). One can show that $E = \frac{1}{2}e_1 + \frac{1}{2}e_2$, which means that E is included in $M^2{}_r$ and redefining multiple coefficients at the same time does not give new reducible deformations. Therefore, we verify that $M^2{}_r$ contains all the possible reducible deformations, and they are generated by the shifts of each coefficient in (3.34) independently.

We can generalize the above method to study the reducible deformations of a general initial $F_2^{(0)}$, which is a product of a polynomial with m terms and a polynomial with n terms like below

$$F_2^{(0)} = (A_1 m_1^A + A_2 m_2^A + \ldots + A_r m_r^A)(B_1 m_1^B + B_2 m_2^B + \ldots + B_s m_s^B). \tag{D.6}$$

Here $\{m_i^A\}, i = 1, 2, \ldots, r$ and $\{m_j^B\}, j = 1, 2, \ldots, s$ are monomials and $\{A_i\}$ and $\{B_j\}$ are the corresponding coefficients. As before, the reducible deformations can be obtained by shifts of each coefficient in $\{A_i\}$ and $\{B_j\}$,

$$
\begin{aligned}
e_1^A &= (B_1 m_1^B + B_2 m_2^B + \ldots + B_s m_s^B)m_1^A, \quad & e_1^B &= (A_1 m_1^A + A_2 m_2^A + \ldots + A_r m_r^A)m_1^B \quad \text{(D.7)} \\
e_2^A &= (B_1 m_1^B + B_2 m_2^B + \ldots + B_s m_s^B)m_2^A, \quad & e_2^B &= (A_1 m_1^A + A_2 m_2^A + \ldots + A_r m_r^A)m_2^B, \\
&\ \ \vdots & &\ \ \vdots \\
e_r^A &= (B_1 m_1^B + B_2 m_2^B + \ldots + B_s m_s^B)m_r^A, \quad & e_s^B &= (A_1 m_1^A + A_2 m_2^A + \ldots + A_r m_r^A)m_s^B.
\end{aligned}
$$

There are total $(r + s)$ such polynomials. As the equation (3.36), one can verify that these deformations satisfy the relation

$$F_2^{(0)} + \left(A_1 e_1^A + A_2 e_2^A + \ldots + A_r e_r^A\right) = F_2^{(0)} + \left(B_1 e_1^B + B_2 e_2^B + \ldots + B_s e_s^B\right). \tag{D.8}$$

Thus, there are $(r + s - 1)$ independent polynomials, and they form a basis of $M^2{}_r$.

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