

Holographic striped superconductor

Yi Ling^{a,b,c} and Meng-He Wu^{a,b}

^a*Institute of High Energy Physics, Chinese Academy of Sciences,
Beijing 100049, China*

^b*School of Physics, University of Chinese Academy of Sciences,
Beijing 100049, China*

^c*Shanghai Key Laboratory of High Temperature Superconductors,
Shanghai 200444, China*

E-mail: lingyi@ihep.ac.cn, mhwu@ihep.ac.cn

ABSTRACT: We construct a holographic model describing the striped superconductor (SSC), which is characterized by the presence of pair density waves (PDW). We explicitly demonstrate that the SSC phase is implemented as the intertwined phase of charge density waves (CDW) order and uniform superconducting (SC) order. The interplay of PDW order, CDW order as well as the uniform SC order in SSC phase is studied. It is found that the PDW order is prominent when both CDW order and uniform SC order are balanced. The critical temperature of CDW becomes higher in the presence of the uniform SC order, but its charge density amplitude is suppressed. On the other hand, the SC order is not sensitive to the presence of CDW order. We also demonstrate that among all the possible solutions, the black hole in SSC phase has the lowest free energy and thus is thermodynamically favored.

KEYWORDS: AdS-CFT Correspondence, Black Holes, Holography and condensed matter physics (AdS/CMT)

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1 Introduction

The striped superconductor (SSC) is a special kind of high- T_c superconductors characterized by the presence of pair density waves (PDW), which is described by a spatially modulated order parameter [1–5]. The SSC phase plays a key role in understanding the mechanism of high-temperature superconductivity because in many materials SSC appears as the intertwined phase of charge density wave (CDW) phase and the uniform superconducting (SC) phase. The former is due to the spontaneous breaking of the translational symmetry, while the latter is due to the spontaneous breaking of U(1) gauge symmetry. The interplay between CDW phase and SC phase has been studied in high- T_c superconductors for decades [6–13]. They exhibit very peculiar and complicated relations which may be both cooperative and competitive. However, due to the strongly coupled nature of the system, the theoretical mechanism leading to these relations maintains mysterious, which prevents us from understanding the abundant phase structure of high- T_c superconductors. Now, more and more experimental evidences for PDW have been accumulated such as in cuprate $Bi_2Sr_2CaCu_2O_{8+x}$, where the PDW order exhibits the same period as CDW [14]. Therefore, investigating the formation of SSC and its features from a theoretical point of view will improve our understanding on the relationship between CDW phase and SC phase.

The holographic duality, also known as AdS/CFT correspondence, has been applied to analyze the fundamental problems in strongly coupled system in condensed matter physics [15–17]. The key point is that a quantum operator involving dynamics with strong interactions can be holographically dual to a field in one-dimension higher spacetime, whose

dynamics is well described by the classical theory of gravity. In particular, recent progress on AdS/CMT has provided us robust foundation to investigate the relationship between SC phase and CDW phase. On one hand, various holographic models have been built for SC since the seminal work in [18–20], where the occurrence of superconducting phase transition is signaled by the formation of the scalar hair in the bulk. On the other hand, the holographic description of CDW has also been established by spontaneously breaking the translational invariance [21–24].

In the early stage of AdS/CMT duality, some references were motivated to construct holographic models for SSC [25, 26]. However, in these models the spatially modulated phase was sourced by a chemical potential, which means the translational invariance was broken explicitly instead of spontaneously. Moreover, the full backreaction to the background was not taken into account. Later, some efforts have been made to implement the phase diagram with SC and CDW where the translational symmetry is spontaneously broken [27, 28], and then attempted to construct a holographic model for PDW [29, 30]. Perhaps it should be stressed that in these papers the holographic superconductor is achieved by means of Stückelberg mechanism, rather than the standard U(1) symmetry breaking in which the scalar field is the modular of the complex field and thus must be positive definite. Furthermore, the translational symmetry breaking is characterized by the same order parameter of superconductivity such that these two different symmetries must be broken simultaneously [29, 30]. In another word, the CDW phase and PDW always coexist and can not be separated such that SSC as the intertwined phase of CDW and SC is not transparently demonstrated.

In this paper, we intend to construct a holographic model for SSC that is achieved as the intertwined phase of CDW phase and SC phase. In contrast to the previous holographic work in literature, we insist that the CDW phase is implemented by breaking the translational invariance spontaneously and the SC phase is implemented by the standard U(1) symmetry breaking, rather than Stückelberg mechanism. More importantly, we will introduce different order parameters for above symmetry breaking such that the CDW and SC phase can exist individually. Moreover, the interplay between the CDW phase and SC phase can be manifestly demonstrated such that the SSC as the intertwined phase of these two phases is manifestly observed in the phase diagram. In another word, the PDW order is induced due to the coexistence of CDW and SC. Therefore, the SSC is characterized by the coexistence of three orders, namely the CDW, the SC and the PDW order, which has been experimentally observed in ref. [13]. We investigate the interplay of these three orders in SSC phase and explore how the features of PDW could reflect the relationship between the CDW and the SC.

Nevertheless, the main motivation of the current paper comes from our recent work in ref. [31], in which we have successfully constructed a holographic model demonstrating that the superconductivity can be induced by CDW solely. To achieve this, one needs to separate the CDW from free charges such that the electric chemical potential is set to zero. In this paper we intend to turn on the electric chemical potential and investigate the relationship between CDW and SC which is induced by the normal free charges. Therefore, in this paper we introduce a doping parameter x , which is defined as the ratio of two chemical

potentials associated with two gauge fields A and B respectively. In current paper we will obtain a different kind of CDW in contrast to the CDW constructed in ref. [31], and demonstrate that the CDW exhibits a quite different relationship with SC. More explicitly, we find that in the presence of free charges, only the even orders of the Fourier modes exist in the expansion of the charge density, while in ref. [31] only the odd orders exist. Based on the results obtained in this paper and those in ref. [31], we intend to conjecture that the complicated relationship between CDW and SC observed in experiments would be the combinational effects of these two different kinds of CDWs.

The paper is organized as follows. In section 2, we present the holographic setup for the striped superconductor based on doubly charged black holes. In section 3, we investigate the instability of the system under the perturbations with spatially modulated modes, and obtain the phase diagram for CDW with the critical temperature as the function of the doping parameter x . In section 4, we investigate the superconducting condensate in the absence of the translational symmetry breaking. Then we focus on the SSC phase in section 5, which results from the coexistence of CDW phase and SC phase. The full background with charge density waves and superconductivity will be numerically obtained and the relations among these three orders will be analyzed. We present our conclusions and discussions in the last section.

2 The holographic setup

We consider a holographic model in four dimensional spacetime, in which gravity is coupled to a dilaton field, two U(1) gauge fields and a complex scalar field. The action is given by,

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left[R - \frac{1}{2} (\nabla\Phi)^2 - V(\Phi) - \frac{1}{4} Z_A(\Phi) F^2 - \frac{1}{4} G^2 - |(\nabla - ieB)\Psi|^2 - m_v^2 \Psi\Psi^* \right], \tag{2.1}$$

where $F = dA$, $G = dB$, $Z_A(\Phi) = 1 - \frac{\beta}{2} L^2 \Phi^2$, $V(\Phi) = -\frac{1}{L^2} + \frac{1}{2} m_s^2 \Phi^2$. The β -term is introduced to induce the instability of the homogeneous background such that the translational symmetry can be spontaneously broken. The dilaton field Φ is real and its leading order near the boundary will be treated as the order parameter of translational symmetry breaking. We also introduce two U(1) gauge fields A and B such that the black hole could be doubly charged, but we will treat B as the electromagnetic field and study the transport properties of its dual field. Moreover, to break U(1) gauge symmetry spontaneously we introduce the complex scalar field Ψ as the order parameter of condensation. We redefine the complex scalar field Ψ as $\eta e^{i\theta}$, where $\eta > 0$ and θ will be set $\theta = 0$ as a gauge fixing. The equations of motion are given by,

$$\begin{aligned} R_{\mu\nu} - T_{\mu\nu}^\Phi - T_{\mu\nu}^A - T_{\mu\nu}^B - T_{\mu\nu}^\eta &= 0, \\ \nabla^2\Phi - \frac{1}{4} Z'_A F^2 - V' &= 0, \\ \nabla^2\eta - m_v^2\eta - (eB)^2\eta &= 0, \\ \nabla_\mu(Z_A F^{\mu\nu}) &= 0, \\ \nabla_\mu G^{\mu\nu} - 2e^2\eta^2 B^\nu &= 0, \end{aligned} \tag{2.2}$$

where

$$\begin{aligned}
 T_{\mu\nu}^{\Phi} &= \frac{1}{2}\nabla_{\mu}\Phi\nabla_{\nu}\Phi + \frac{1}{2}Vg_{\mu\nu}, \\
 T_{\mu\nu}^A &= \frac{Z_A}{2}\left(F_{\mu\rho}F_{\nu}^{\rho} - \frac{1}{4}g_{\mu\nu}F^2\right), \\
 T_{\mu\nu}^B &= \frac{1}{2}\left(G_{\mu\rho}G_{\nu}^{\rho} - \frac{1}{4}g_{\mu\nu}G^2\right), \\
 T_{\mu\nu}^{\eta} &= \nabla_{\mu}\eta\nabla_{\nu}\eta + e^2\eta^2B_{\mu}B_{\nu} + \frac{1}{2}m_v^2\eta^2g_{\mu\nu},
 \end{aligned}
 \tag{2.3}$$

and the prime ' denotes the derivative with respect to the dilaton field Φ . We consider the formation of CDW and superconductivity over a doubly charged AdS-RN black hole. The ansatz for the background can be written as:

$$\begin{aligned}
 ds^2 &= \frac{1}{z^2}\left[-(1-z)p(z)Q_{tt}dt^2 + \frac{Q_{zz}dz^2}{(1-z)p(z)} + Q_{xx}\left(dx + z^2Q_{xz}dz\right)^2 + Q_{yy}dy^2\right], \\
 A_t &= \mu_1(1-z)a, \quad B_t = \mu_2(1-z)b, \quad \Phi = z\phi, \quad \eta = z\zeta,
 \end{aligned}
 \tag{2.4}$$

where $p(z) = 4\left(1+z+z^2 - \frac{(\mu_1^2+\mu_2^2)z^3}{16}\right)$, μ_1 and μ_2 are the chemical potential of gauge field A and B , respectively. $Q_{tt}, Q_{zz}, Q_{xx}, Q_{yy}, Q_{xz}, \psi, a, b, \phi, \zeta$ are functions of x and z . Obviously, if we set $Q_{tt} = Q_{zz} = Q_{xx} = Q_{yy} = a = b = 1$ and $Q_{xz} = \phi = \zeta = 0$, the background is a doubly charged version of AdS-RN black hole. Throughout this paper we shall set the AdS radius $l^2 = 6L^2 = 1/4$, the masses of the dilaton and the condensation $m_s^2 = m_v^2 = -2/l^2 = -8$, the coupling constant $\beta = -129$. In addition, we require that $\frac{l^2}{2\kappa^2} \gg 1$ such that the large N limit of the dual field theory is guaranteed. Moreover, we will take μ_1 as the unit and define $x = \mu_2/\mu_1$ as the doping parameter, as proposed in ref. [27]. Obviously, with a larger doping parameter, the system contains more effective carriers and its transport properties are expected to vary correspondingly. Finally, the Hawking temperature of the black hole is simply given by $T/\mu_1 = (48 - \mu_1^2 - \mu_2^2)/(16\pi\mu_1)$ and this expression is also applicable for striped black holes due to the Einstein-DeTurck method [32–37].

We remark that the holographic model considered in this paper is different from what has previously been studied in refs. [24, 31], where a coupling term between gauge field A and B , namely $\gamma\Phi FG$, is introduced into the action such that the dome of the unstable region could shift to a position with non-zero k_c . This coupling term is harmless there since the gauge field B is turned off, such that the order parameter of translational symmetry, namely the dilaton field Φ , always has a zero solution before the translational symmetry breaking. However, in current paper we intend to introduce free charges with finite density associated with the gauge field B , thus B is turned on to form a doubly charged AdS-RN black hole as the background. At this situation, one needs to turn off the coupling term, otherwise it would contribute a non-trivial term to the equation of dilaton field Φ in eq. (2.2), such that the order parameter would not have a zero solution even prior to the symmetry breaking. Therefore, in current paper we drop off this coupling term and demonstrate that it will lead to interesting phenomenon for the interplay between CDW and SC, which is dramatically different from those observed in [24, 31].

Near the AdS boundary $z \rightarrow 0$, we obtain the following asymptotic form of the fields,

$$\begin{aligned}
 Q_{tt} &= 1 + q_{tt}(x)z^3 + o(z^4), & Q_{zz} &= 1 + o(z^4), \\
 Q_{xx} &= 1 + q_{xx}(x)z^3 + o(z^4), & Q_{yy} &= 1 + q_{yy}(x)z^3 + o(z^4), \\
 Q_{xz}(z) &= o(z^2), & A_t &= \mu_1 - \rho_A(x)z + o(z^2), \\
 B_t &= \mu_2 - \rho_B(x)z + o(z^2),
 \end{aligned}
 \tag{2.5}$$

where $\rho_B(x)$ is the charge density we are interested in, and the form of Fourier expansion is given by

$$\rho_B(x) = \rho_B^{(0)} + \rho_B^{(1)} \cos(kx) + \rho_B^{(2)} \cos(2kx) + \dots .
 \tag{2.6}$$

In section 3, 4 and 5, we will firstly obtain the phase diagram for the system by perturbative analysis and then solve all the coupled equations with full backreaction numerically. Before this we argue that over a doubly charged black hole background, both translational symmetry and U(1) gauge symmetry could be spontaneously broken when the temperature drops down, giving rise to the CDW phase and SC phase, respectively. We will explicitly show that, which one breaks prior to the other depends on the doping parameter x .

3 The charge density wave phase ($\Phi \neq 0, \eta = 0$)

In this section we consider the formation of CDW in the absence of superconductivity, which can be done by turning off the condensate field, namely setting $\eta = 0$ always. The holographic CDW is formed by spontaneously breaking the translational symmetry. Such instability is caused by the violation of the BF bound of AdS_2 , which is the near horizon geometry in the extremal limit. Below the critical temperature T_c , the AdS-RN black hole becomes unstable and the spatially modulated modes may appear to form a striped black hole. Specifically, we turn on the following linear perturbation to examine the instability of the black hole,¹

$$\delta\Phi = \delta\phi(z) \cos(k_c x).
 \tag{3.1}$$

By substituting (3.1) into the equations of motion (2.2), we obtain the linearized equation for $\delta\Phi$. Moreover, we demand the regular boundary condition at the horizon $z = 1$, while the asymptotic expansion of $\delta\phi(z)$ near the boundary is

$$\delta\phi \approx \delta\phi_1 z + \delta\phi_2 z^2 + \dots ,
 \tag{3.2}$$

where $\delta\phi_1$ is treated as the source and $\delta\phi_2$ as the expectation value in the dual theory. Since one expects the translational invariance is spontaneously breaking, we set $\delta\phi_1 = 0$. In figure 1, we plot the critical temperature as the function of the wave number for $x = 0, 1, 2, 3$. Below the curves the AdS-RN black hole becomes unstable and the spatially modulated modes will appear with the allowed wave number as illustrated in figure 1.

Without loss of generality, we take the wave number $k_c/\mu_1 = 0.34$ throughout the paper. For a given k_c/μ_1 and doping parameter x , one can drop down the temperature of

¹Unlike the case in ref. [31], here one need not turn on the perturbation of gauge field B since at linear level they are decoupled to each other.

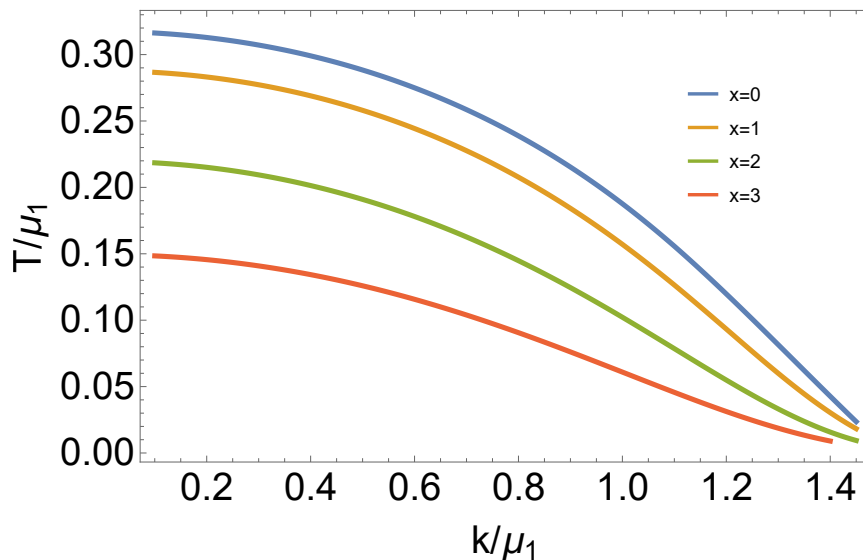


Figure 1. The critical temperature as the function of wave number for $x = 0, 1, 2, 3$. The region below each curve becomes unstable and CDW can be generated.

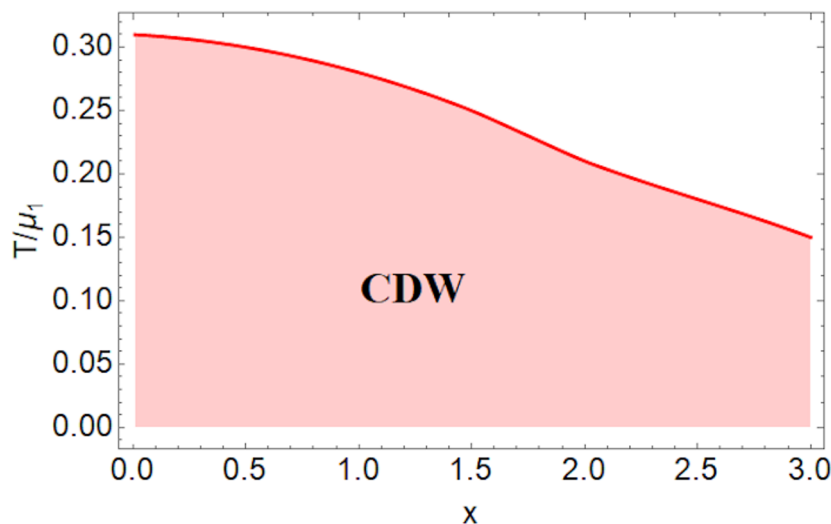


Figure 2. The CDW phase diagram in (x, T) plane.

the black hole, then the instability of the background is signaled by the presence of non-trivial solution for $\delta\Phi$ at some temperature, which is estimated as the critical temperature T_c . As illustrated in figure 2, the curve in red depicts the critical temperature T_c of CDW phase as the function of x . In general, we find the critical temperature goes down with the increase of the doping parameter x . This is not surprising since the CDW is an insulating phase, with the increase of the carriers, the formation of CDW phase becomes harder.

Next, we explicitly construct the background with CDW by numerically solving all the equations of motion in the absence of condensation field. In particular we obtain the numerical results for the charge density. It is found that only *even* orders of the Fourier

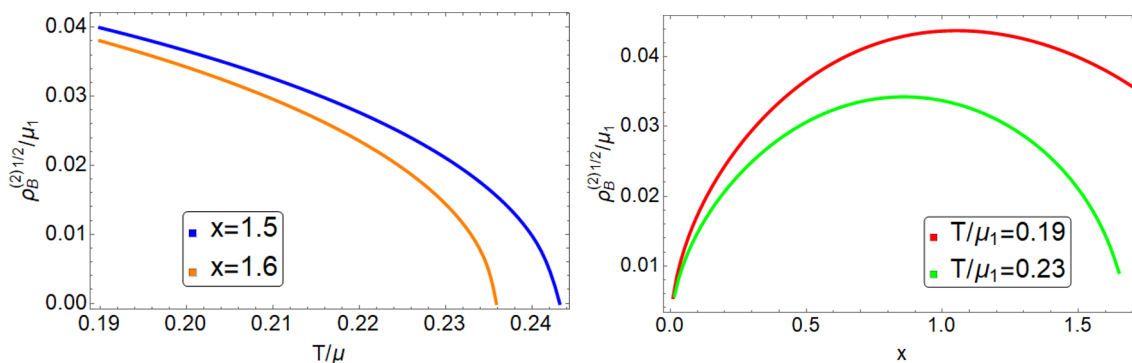


Figure 3. The CDW amplitude $\rho_B^{(2)}$ as the functions of temperature T (left) and doping parameter x (right).

series of $\rho_B(x)$ appear, namely

$$\rho_B(x) = \rho_B^{(0)} + \rho_B^{(2)} \cos(2k_c x) + \dots \quad (3.3)$$

It is quite interesting to compare the above result with that obtained in [31], where only *odd* orders of the Fourier series exist, namely

$$\rho_B(x) = \rho_B^{(1)} \cos(k_c x) + \rho_B^{(3)} \cos(3k_c x) + \dots \quad (3.4)$$

This key difference results from the following two facts. First, in [31] the chemical potential is always set to zero, namely $x = 0$, then the leading term of charge density has to be zero, namely $\rho_B^{(0)} = 0$. Second, the coupling term ΦFG is absent in current paper, thus even in the limit $x \rightarrow 0$, the striped black hole background will not go back to the solutions in [31], instead the charge density maintains the form as in eq. (3.3) and goes to zero in the limit $x \rightarrow 0$.

To justify this we may plot the constant term of the charge density $\rho_B^{(0)}$ as well as the CDW amplitude $\rho_B^{(2)}$ as the function of temperature T and doping parameter x , as illustrated in figure 4 and figure 3. From the left of figure 4, it is obvious to see that $\rho_B^{(0)}$ has the same tendency with the temperature even in the presence of CDW. In another word, the presence of CDW does not change the value of $\rho_B^{(0)}$ at a given temperature. Moreover, with the increase of doping parameter, $\rho_B^{(0)}$ increases as well, as illustrated in the right plot of figure 4. In the left plot of figure 3, we find the amplitude of CDW $\rho_B^{(2)}$ increases and then intends to be saturated when dropping down the temperature. In the right plot of figure 3, $\rho_B^{(2)}$ exhibits interesting behavior with the doping parameter x . In the limit $x \rightarrow 0$, both $\rho_B^{(0)}$ and $\rho_B^{(2)}$ are vanishing such that the gauge field B has the zero solution only, while for large x we find $\rho_B^{(2)}$ becomes smaller again because the phase transition is suppressed by the increase of carriers, as illustrated in the phase diagram figure 2.

The near boundary expansion of the CDW order parameter is $\phi = \phi_2(x)z^2$, and we numerically find ϕ_2 behaves as

$$\phi_2(x) = \phi_2^{(1)} \cos(k_c x) + \phi_2^{(3)} \cos(3k_c x) \dots \quad (3.5)$$

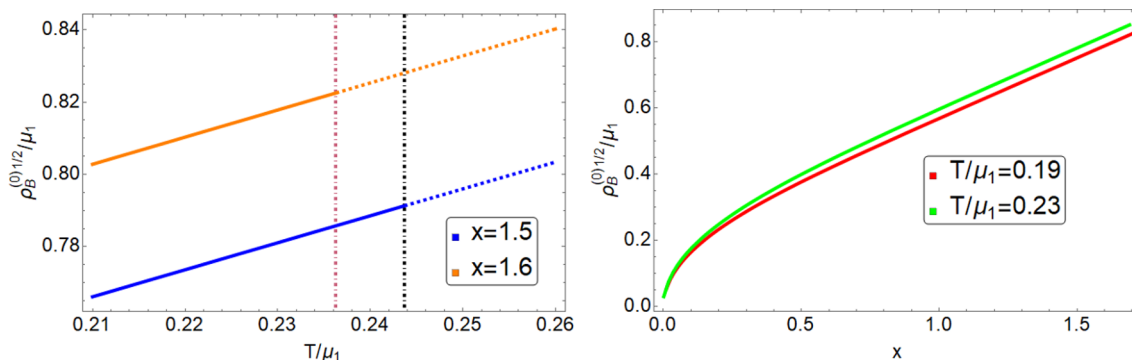


Figure 4. The charge density $\rho_B^{(0)}$ as the function of temperature T (left) and doping parameter x (right). The solid lines in color stand for $\rho_B^{(0)}$ in black hole with CDW, while the dashed lines in color stand for $\rho_B^{(0)}$ in RN black hole. The vertical lines denote the location of CDW phase transition.

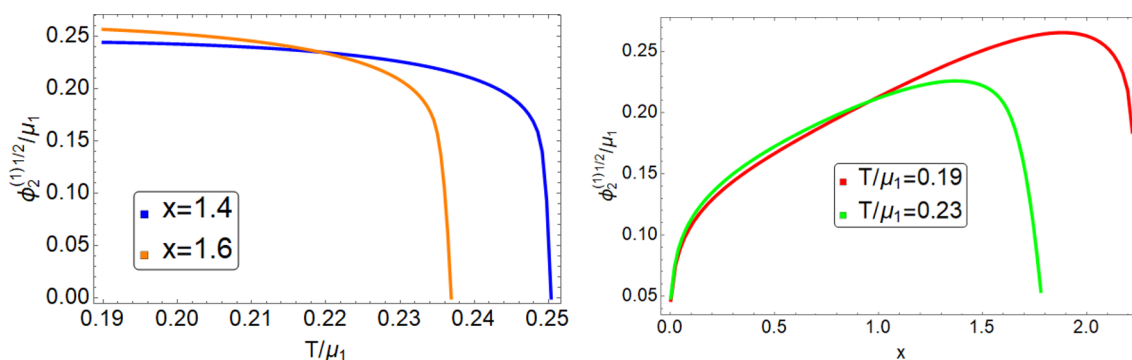


Figure 5. The order parameter $\phi_2^{(1)}$ as the function of temperature T (left) and doping parameter x (right).

We plot the magnitude of the order parameter $\phi_2^{(1)}$ in figure 5. It is also noticed that $\phi_2^{(1)}$ goes to zero in the limit $x \rightarrow 0$, while for large x , $\phi_2^{(1)}$ drops down again which behaves similarly as $\rho_B^{(2)}$.

As a summary of this section, we find that the critical temperature T_c of CDW decreases with the increase of the doping parameter x , which means achieving such an insulating phase becomes harder with the doping. However, the Fourier series of the charge density $\rho_B(x)$ grow up with x , indicating that the CDW phase benefits from the increase of carriers.

4 The uniform superconducting phase ($\eta \neq 0, \Phi = 0$)

In this section we consider the condensate of superconductivity in the absence of CDW, which can be done by setting $\Phi = 0$ always. The SC phase is characterized by the condensation of the complex scalar fields when U(1) symmetry is spontaneously broken. In order to evaluate the critical temperature for the condensation, we may consider the perturbation of the scalar field η over a fixed background. The perturbative equation of motion is given by:

$$-\nabla^2\eta + m_v^2\eta = -e^2B^2\eta. \tag{4.1}$$

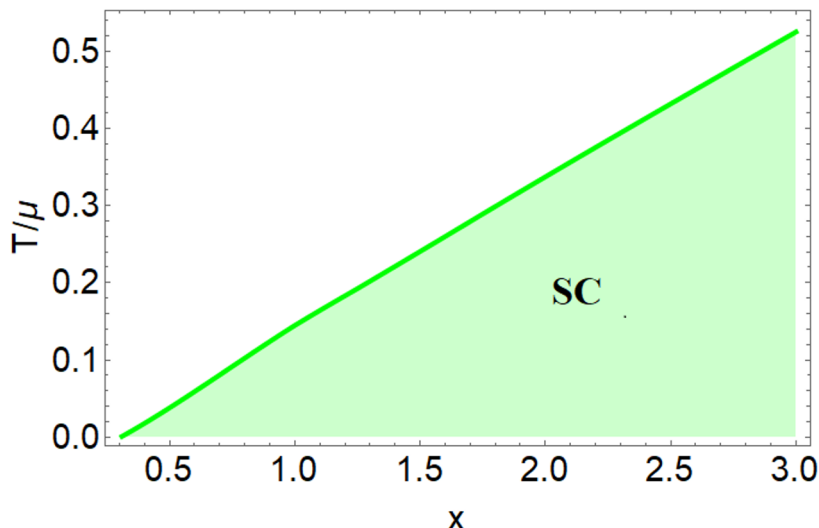


Figure 6. The SC phase diagram in (x, T) plane.

As investigated in [35, 38], this is a positive self-adjoint eigenvalue problem for e^2 . We demand regularity on the horizon $z = 1$. And the asymptotical behavior near $z = 0$ is given by

$$\eta \approx \eta_1 z + \eta_2 z^2 + \dots \quad (4.2)$$

Since U(1) symmetry should be broken spontaneously, we set the source term $\eta_1 = 0$. The condensation of the scalar field will be signaled by the appearance of non-vanishing solution for η_2 . For a fixed e^2 , the corresponding critical temperature T_c for SC depends on the doping parameter x . As illustrated in figure 6, the curve in green depicts the critical temperature T_c of SC phase as the function of x for $e = 4$. In contrast to CDW phase, we find that with the increase of the doping parameter x , the critical temperature T_c for SC phase goes up, indicating that the condensation of Cooper pairs becomes easier with the increase of carriers.

Next we construct the background with condensation by solving all the equations of motions. Note that, as compared to section 3, the order parameter $\eta_2(x)$ of SC and the charge density $\rho_B(x)$ are now uniform in x -direction, namely $\eta_2(x) = \eta_2^{(0)}$ and $\rho_B(x) = \rho_B^{(0)}$. We plot $\rho_B^{(0)}$ and $\eta_2^{(0)}$ as the function of T and x in figure 8 and figure 7. Obviously, one finds that below the critical temperature, the charge density $\rho_B^{(0)}$ in SC phase exhibits an opposite behavior with the temperature and is much larger than that in normal phase. In addition, we find the charge density as well as the critical temperature of SC increases with the doping parameter x . In the plot of figure 7, we find the saturated value of the condensation increases as well with the doping parameter.

As a summary of this section we conclude that the condensation of uniform SC benefits from the doping mechanism. The critical temperature T_c increases with the doping parameter x . The charge density $\rho_B^{(0)}$ and the saturated value $\eta_2^{(0)}$ of the condensation grow up with x as well.

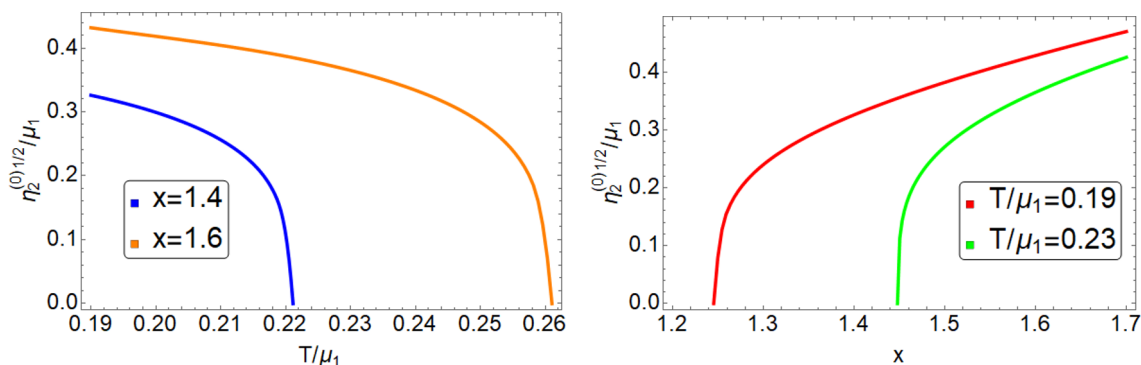


Figure 7. The order parameter $\eta_2^{(0)}$ of uniform SC as the function of temperature T (left) and doping parameter x (right).

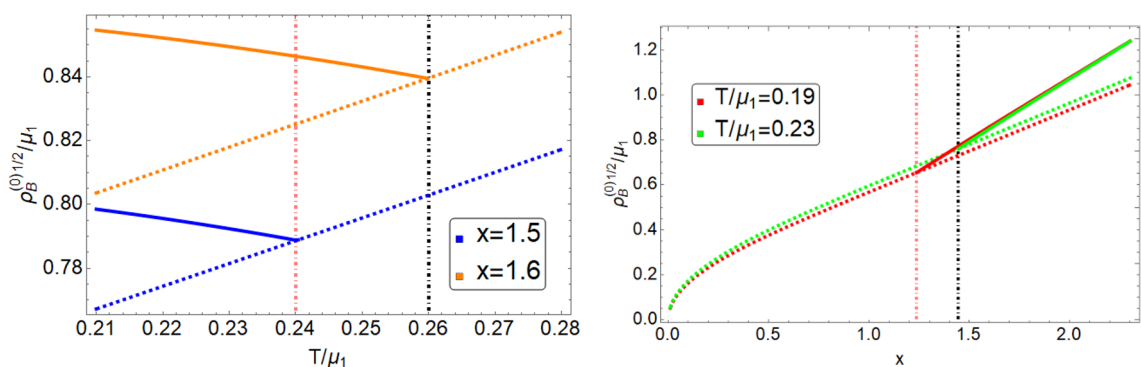


Figure 8. The charge density $\rho_B^{(0)}$ as the function of temperature T (left) and doping parameter x (right). The solid lines in color stand for $\rho_B^{(0)}$ after condensation, while the dashed lines in color stand for $\rho_B^{(0)}$ in RN black hole. The vertical lines denote the location of SC phase transition.

5 The striped SC phase ($\Phi \neq 0, \eta \neq 0$)

5.1 The phase diagram

In this section we focus on the striped superconductivity due to the coexistence of CDW and uniform SC, namely $\eta \neq 0$ and $\Phi \neq 0$. When both the translational symmetry and U(1) symmetry are spontaneously broken, a new phase is formed which is called the striped superconducting phase. Now the asymptotic behavior of η_2 becomes x -dependent and can be expanded as

$$\eta_2(x) = \eta_2^{(0)} + \eta_2^{(1)} \cos(k_c x) + \eta_2^{(2)} \cos(2k_c x) + \dots, \quad (5.1)$$

where among coefficients $\eta_2^{(i)}$ ($i = 1, 2, \dots$) of the modulated modes, the leading orders, for instance $\eta_2^{(1)}$ or $\eta_2^{(2)}$, could be treated as the order parameter of the striped SC phase. Obviously, the presence of the striped phase results from the coexistence of the CDW phase and the SC phase. Usually this novel phase is also called pair density waves (PDW) phase implying the periodic distribution of Cooper pairs in spatial directions. In a parallel way, we may plot the phase diagram by perturbative analysis, as illustrated in figure 9. The brown region stands for SSC phase, and there are three orders in this region, namely

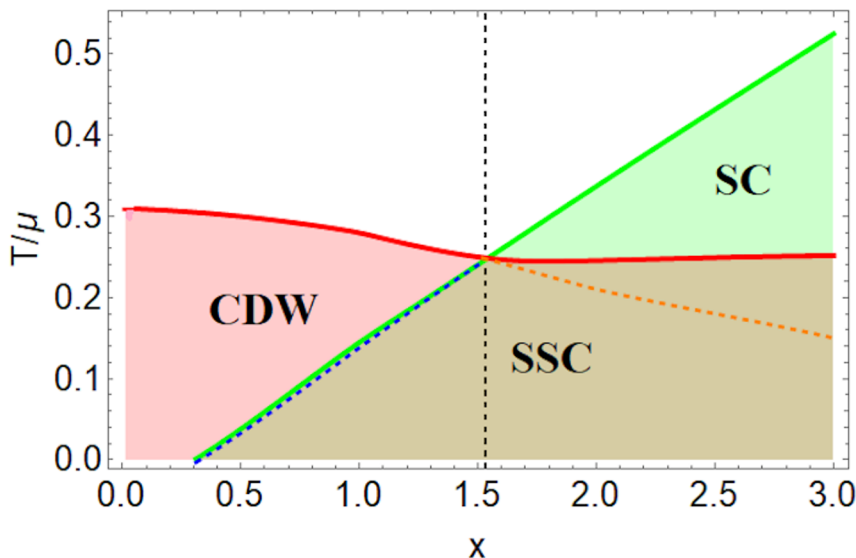


Figure 9. The phase diagram in (x, T) plane. The dashed vertical line corresponds to $x_c \approx 1.53$.

CDW order, uniform SC order and PDW order, where $x_c \approx 1.53$ is the critical doping parameter. The dashed curve in orange depicts the critical temperature T_c of CDW phase in the absence of SC phase, while the curve in red depicts T_c of CDW in the presence of SC phase. On the left-hand side of x_c , the CDW phase is formed prior to the SC phase, thus these two curves are overlapped, while on the right-hand side of x_c , the SC phase is formed prior to the CDW phase, thus we find these two curves are different. In particular, we find the critical temperature of CDW in the presence of SC is always higher than that in the absence of superconductivity. This is because the constant term of the charge density becomes larger in the presence of SC such that the formation of CDW becomes easier. Furthermore, the curve in green depicts the critical temperature T_c of SC phase in the absence of CDW phase, while the dashed curve in blue depicts T_c of SC in the presence of CDW phase. It is quite interesting to notice that on the left-hand side of x_c , the green curve is almost overlapped with the dashed blue curve, indicating that the presence of CDW does not change the critical temperature of SC. This result will further be justified by our analysis on the background with full backreactions, as presented below.

5.2 The PDW order in SSC phase

Now we take the backreactions into account and numerically solve all the equations for the background with CDW and SC, focusing on the temperature behavior of the charge density wave $\rho_B(x)$, the scalar condensate $\eta_2(x)$ and the CDW order parameter $\phi_2(x)$.

In figure 10, we perform the contour plot for the charge density wave $\rho_B(x)$ as the function of x and temperature T for $x = 1.5$ and $x = 1.6$, respectively. In parallel, we perform the contour plot for the condensate $\eta_2(x)$ as the function of x and temperature T for $x = 1.5$ and $x = 1.6$ in figure 11, respectively. One finds that $\eta_2(x)$ becomes spatially modulated when the CDW is involved, indicating that the striped SC is formed.

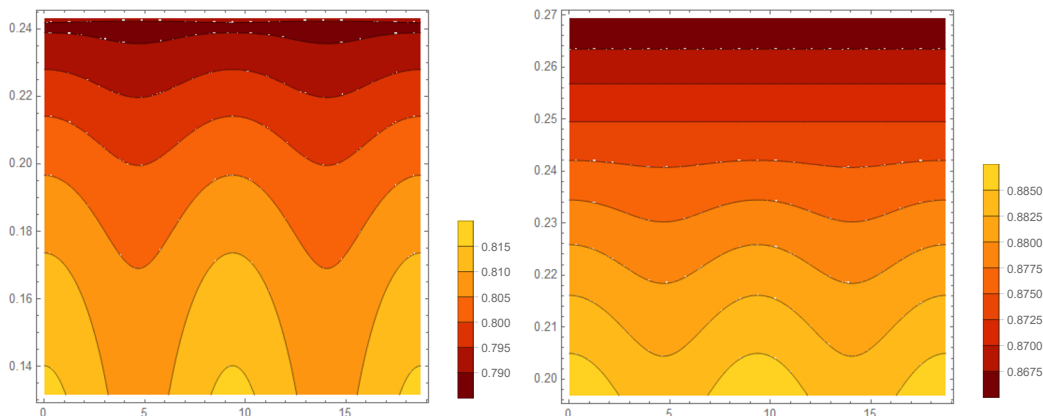


Figure 10. The contour plot for the charge density wave $\rho_B(x)$ as a function of x and temperature T for $x = 1.5$ and $x = 1.6$, respectively.

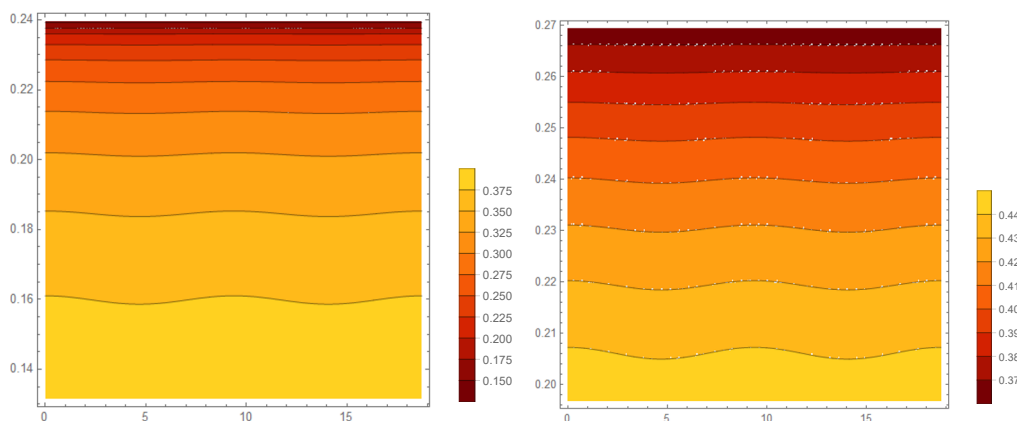


Figure 11. The contour plot for the condensate $\eta_2(x)$ as a function of x and temperature T for $x = 1.5$ and $x = 1.6$, respectively.

Numerically, in terms of Fourier series, we find that $\rho_B(x)$, $\eta_2(x)$, and $\phi_2(x)$ have the following form

$$\begin{aligned}
 \rho_B(x) &= \rho_B^{(0)} + \rho_B^{(2)} \cos(2k_c x) + \dots, \\
 \eta_2(x) &= \eta_2^{(0)} + \eta_2^{(2)} \cos(2k_c x) + \dots, \\
 \phi_2(x) &= \phi_2^{(1)} \cos(k_c x) + \phi_2^{(3)} \cos(3k_c x) \dots.
 \end{aligned}
 \tag{5.2}$$

The formation of SSC is signaled by the appearance of non-zero $\eta_2^{(2)}$, that we treat as the order parameter of PDW. Obviously, the formation of PDW results from the coexistence of CDW $\phi_2^{(1)}$ and the uniform SC $\eta_2^{(0)}$, thus the SSC is implemented as the intertwined phase of CDW and the uniform SC phase and is characterized by the coexistence of three orders, namely CDW order, uniform SC order and PDW order. Moreover, it is interesting to notice that the condensate $\eta_2(x)$ shares the same period with the charge density wave $\rho_B(x)$.

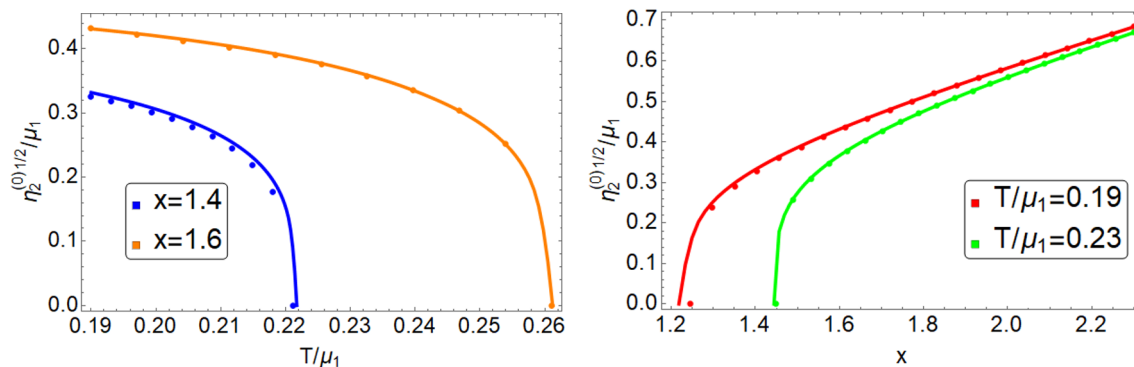


Figure 12. The uniform SC order parameter $\eta_2^{(0)}$ as the function of temperature T (left) and doping parameter x (right). The solid curves in color stand for $\eta_2^{(0)}$ in SSC phase, while the dotted curves in color stand for $\eta_2^{(0)}$ in the absence of CDW.

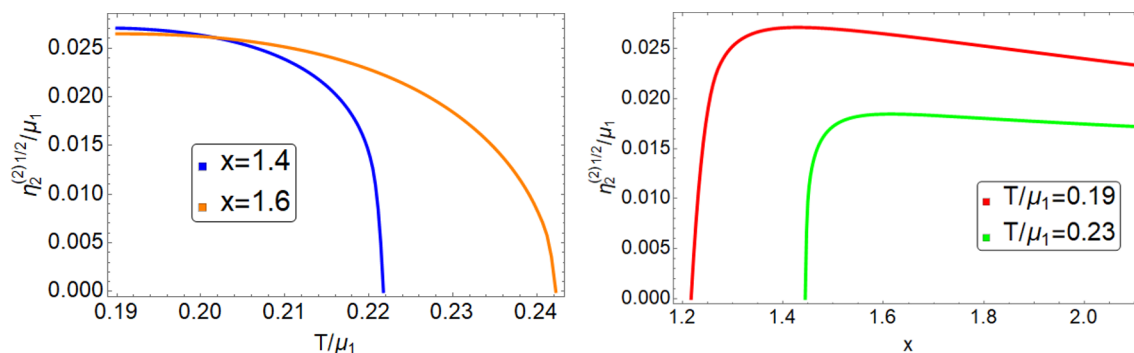


Figure 13. The PDW order parameter $\eta_2^{(2)}$ as the function of temperature T (left) and doping parameter x (right).

5.3 The interplay of three orders

Next, we demonstrate the temperature behavior of the order parameters and charge density in details, which is helpful for us to understand the structure of the phase diagram obtained in figure 9. First of all, we are concerned with the order parameter of PDW, namely $\eta_2^{(2)}$. The non-zero value of $\eta_2^{(2)}$ implies that the Cooper pairs develop a periodic structure with spatial oscillation due to the presence of the CDW. In figure 13, we plot $\eta_2^{(2)}$ as the function of T and x , respectively. The left plot shows that below the critical temperature the PDW grows rapidly as T drops down. The right plot shows that the PDW becomes prominent with the increase of the doping parameter x at first, and then gradually decays with the doping parameter x .

In parallel, we plot the order parameter of CDW $\phi_2^{(1)}$ and that of the uniform SC $\eta_2^{(0)}$ as functions of T and x in figure 14 and figure 12, respectively. First of all, we are very interested in how these three orders interact in SSC phase. In particular, we are very concerned how the CDW order and the uniform SC order would affect the PDW order in SSC phase. Or conversely, from the PDW order, can one read any information about the CDW order and the uniform SC order? As the intertwined order of CDW and uniform

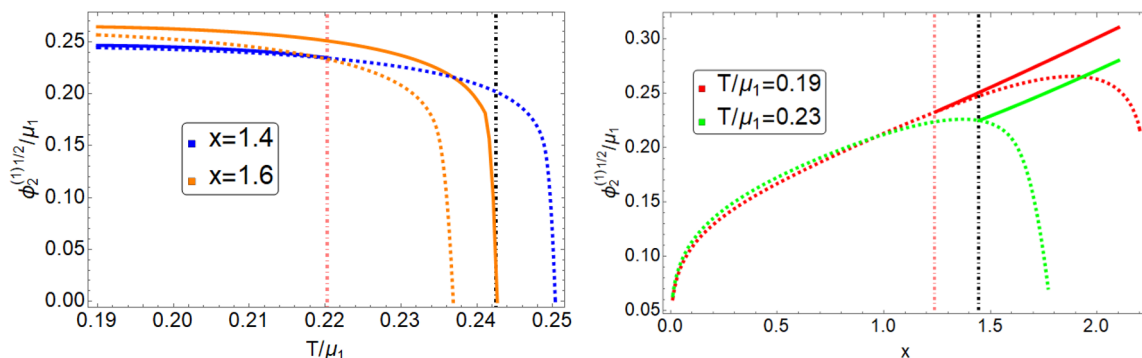


Figure 14. The CDW order parameter $\phi_2^{(1)}$ as the function of temperature T (left) and doping parameter x (right). The solid lines in color stand for $\phi_2^{(1)}$ in SSC phase, while the dashed lines in color stand for $\phi_2^{(1)}$ in the absence of SC. The vertical lines denote the location of SSC phase transition.

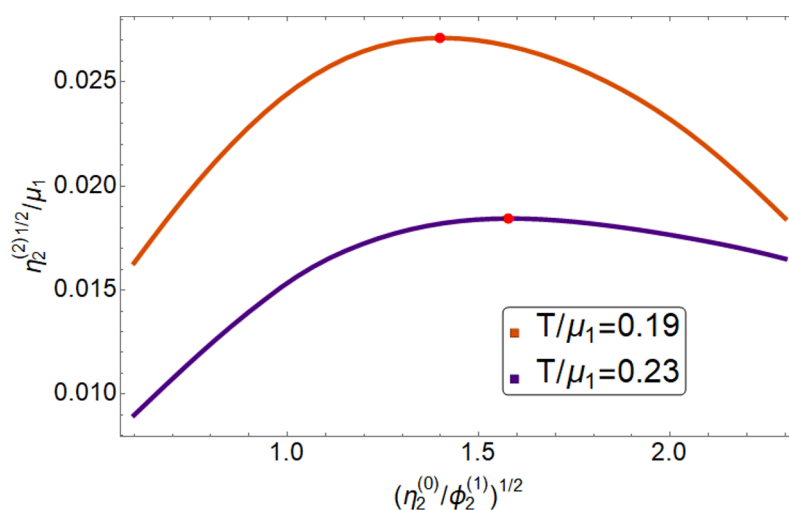


Figure 15. The relation between the PDW order parameter $\eta_2^{(2)}$ and the ratio of uniform SC order parameter to the CDW order parameter $\eta_2^{(0)}/\phi_2^{(1)}$.

SC order, we know that the PDW order $\phi_2^{(1)}$ must be zero when either CDW order $\phi_2^{(1)}$ or uniform SC order $\eta_2^{(0)}$ vanishes. As a result, we intend to plot the relation between $\eta_2^{(2)}$ and $\eta_2^{(0)}/\phi_2^{(1)}$, as illustrated in figure 15. It is interesting to notice that there exists a tip in each curve indeed. Moreover, with the decrease of temperature, we find that $\eta_2^{(0)}/\phi_2^{(1)}$ of the tip approaches to one, which means that the PDW order becomes prominent when the CDW order and uniform SC order have a balanced contribution in the SSC phase.

Next, we turn to investigate how the CDW order and the uniform SC order would affect to each other in SSC phase. When the temperature goes down, there are two possible sequences for the occurrence of phase transition. One is that the CDW order is formed prior to the uniform SC order, while the other is that the uniform SC order is formed prior to the CDW order, which depends on the doping parameter x . In the left plot of figure 14, the blue curve demonstrates that the CDW order is formed prior to the uniform SC order. In comparison with the data in the absence of SC, we find the saturated value of $\phi_2^{(1)}$

grows up a little bit higher, which looks interesting. While the orange curve demonstrates that the formation of uniform SC order is prior to the CDW. We find that the critical temperature of CDW in the presence of SC is higher than that in the absence of SC, which is consistent with the phase diagram in figure 9. Again, its saturated value becomes a little bit higher.

On the other hand, from the left plot of figure 12 we notice that the data of the uniform SC $\eta_2^{(0)}$ in SSC phase are the same as those in the absence of CDW. This means that the CDW order does not affect the formation of uniform SC. The critical temperature of SC is not sensitive to the CDW order either, as illustrated in figure 9. This could be understood as follows, the CDW phase is characterized by $\phi_2^{(1)}$ and $\rho_B^{(2)}$. Numerically such sub-leading orders in Fourier expansion would not change the data of leading orders much. This also interprets why the critical temperature of CDW in SC phase is larger than that in the absence of SC phase in figure 9, since the presence of more carriers would make it easier to distort the crystal lattice leading to the spontaneous breaking of translational symmetry in a doping system.

Next we plot the charge density $\rho_B^{(0)}$ and $\rho_B^{(2)}$ as the function of temperature T and x . In our opinion, in SSC phase these two terms contain the information about both the CDW order and the PDW order. In figure 16 we find $\rho_B^{(0)}$ is sensitive to the SC order only, and independent of the CDW order. However, we insist that if the formation of CDW order is prior to the SC phase, some free charges should be pinned by the lattice effects and exhibit an insulating behavior.² Thus, $\rho_B^{(0)}$ contains two ingredients in SSC phase. One is the average charge density of CDW, and the other is the average charge density of superfluid. In the left plot of figure 17, the blue curve illustrates the case that the CDW order is formed prior to the SC phase. We find $\rho_B^{(2)}$ jumps at the critical temperature of SC, signaling the formation of PDW. While the orange curve illustrates the case that the uniform SC order is formed prior to the CDW phase. One notices that the critical temperature of CDW in the presence of SC is higher than that in the absence of SC, which is consistent with the phase diagram in figure 9. In this case, PDW is formed simultaneously with CDW. Thus from beginning, $\rho_B^{(2)}$ contains the information of these two orders. Obviously, the PDW shares the same period as the CDW.³ Furthermore, it is important to notice that the saturated $\rho_B^{(2)}$ becomes smaller in comparison with that in the absence of SC phase, which implies that the CDW order is suppressed by the presence of SC order. In the right plot of figure 17, we find that at low temperature $\rho_B^{(2)}$ decreases with the presence of the SC order as well.

²Although in current paper we do not introduce the lattice structure for the simplicity.

³This is consistent with phenomenon observed in experiment where the period of PDW is twice of that of CDW, because different order parameters are applied to characterize the superconductivity. Here we take the modular part of the complex scalar field, η_2 , as the order parameter, which is always positive definite (hence $\eta_2^{(0)} > \eta_2^{(2)}$). However, if one takes the real part of the complex scalar field as the order parameter (which could be negative), for instance gauging fixing the imaginary part to zero, then taking the Fourier expansion, one can easily find the period of the modular is just one half of the period of real part of the complex field.

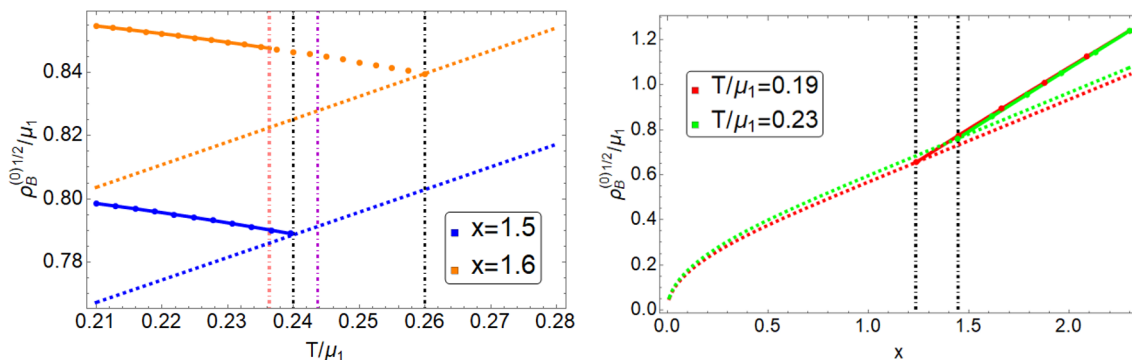


Figure 16. The charge density $\rho_B^{(0)}$ in the presence of CDW as the function of temperature T (left) and doping parameter x (right). The solid curves in color stand for $\rho_B^{(0)}$ in SSC phase, while the dotted curves in color stand for $\rho_B^{(0)}$ in the absence of CDW, and the dashed lines in color stand for $\rho_B^{(0)}$ in RN black hole. The vertical lines in black denote the location of the critical temperature of SC, and the vertical lines in pink or purple denote the location of the critical temperature of CDW.

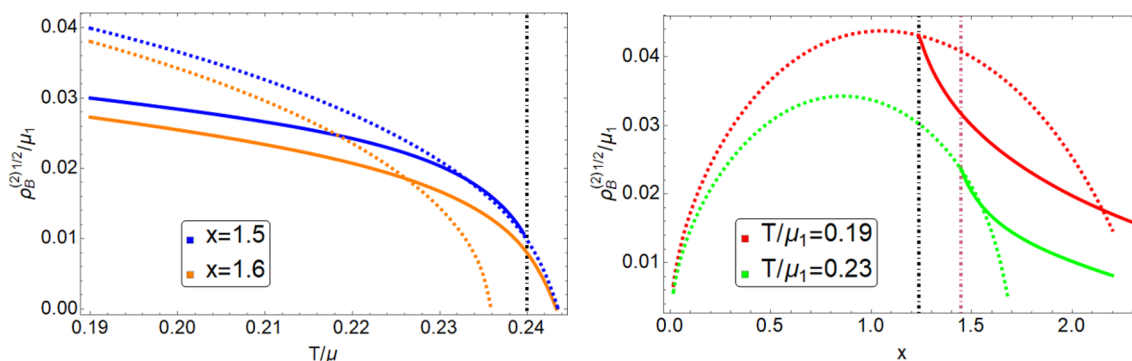


Figure 17. $\rho_B^{(2)}$ as the function of temperature T (left) and doping parameter x (right). The solid curves in color stand for $\rho_B^{(0)}$ in SSC phase, while the dashed lines in color stand for $\rho_B^{(2)}$ in the absence of SC. The vertical lines denote the location of SSC phase transition.

5.4 The free energy of three phases

In the end of this section we intend to verify that among above three phases, namely CDW, SC and SSC phase, the striped superconducting phase is the most stable state, and thus is favored by the system. For this purpose, we compute the averaged free energy of the system. According to refs. [22, 23, 39, 40], the free energy of the dual field on the boundary is identified with the product of the temperature and the on-shell Euclidean bulk action. Specifically, the averaged free energy of the system is given by

$$F = M - \mu_1 Q_A - \mu_2 Q_B - TS, \tag{5.3}$$

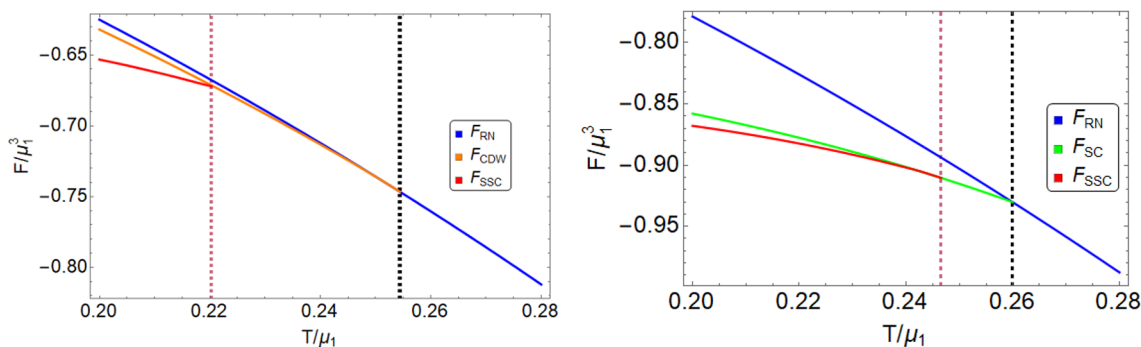


Figure 18. The averaged free energy for RN black hole, CDW/SC black hole and SSC black hole with $x = 1.4$ (left) and $x = 1.6$ (right), respectively.

where

$$\begin{aligned}
 M &= 2 + \frac{\mu_1}{2} + \frac{\mu_2}{2} - \frac{3k_c}{2\pi} \int_0^{2\pi/k_c} q_{tt}(x) dx, \\
 Q_A &= \frac{k_c}{2\pi} \int_0^{2\pi/k_c} \rho_A(x) dx, \\
 Q_B &= \frac{k_c}{2\pi} \int_0^{2\pi/k_c} \rho_B(x) dx, \\
 S &= k \int_0^{2\pi/k_c} \sqrt{Q_{xx}(x, 1) Q_{yy}(x, 1)} dx.
 \end{aligned} \tag{5.4}$$

In figure 18, we plot the averaged free energy for different background solutions with $x = 1.4$ and $x = 1.6$, respectively. The system undergoes the phase transitions from RN black hole to striped black hole with CDW and then to SSC black hole, or from RN black hole to black hole with SC and then to SSC black hole. The plots evidently show that the free energy of black hole with SSC is the lowest one in comparison with other two phases. This indicates that the SSC branch is thermodynamically preferred under the critical temperature, indeed.

6 Discussion

In this paper we have constructed a holographic model for striped superconductor, where both U(1) symmetry and translational symmetry are spontaneously broken. Firstly, we have obtained the phase diagram for the CDW phase and the uniform SC phase separately, and then constructed the phase diagram for the striped superconductor, which contains three orders, namely the CDW order, uniform SC order and PDW order. The PDW order is implemented as the intertwined order of the CDW order and the uniform SC order. It is found that the CDW order is suppressed by the presence of the uniform SC order, while the SC order is not sensitive to the presence of CDW order. Furthermore, PDW shares the same period with CDW. This relation is the same as that observed in experiments [14]. Finally, we have also demonstrated that among all the possible solutions, the black hole in SSC phase has the lowest free energy and thus is favored from the thermodynamical point of view.

It is quite instructive to compare our results obtained in this paper with those in ref. [31]. Two different kinds of CDWs were constructed in these two papers. In current paper CDW contains even orders of the Fourier modes of charge density, while in ref. [31] the CDW contains odd orders only. The different forms of CDW lead to distinct behavior in the interplay of CDW order and SC order. In [31] due to the absence of free charges, the U(1) gauge symmetry is broken by the presence of CDW only, thus the SC order benefits from the existence of CDW. While in current paper the U(1) gauge symmetry is broken by the presence of free charges, and we find the uniform SC order is not sensitive to CDW at all. We conjecture that the CDW observed in practical materials could be the combination of these two kinds of CDW in holographic framework and thus exhibit more abundant interplay behavior with the SC order. Therefore, it is quite desirable to construct a holographic model for CDW which contains both even and odd orders of the charge density.

In current model we have only considered the essential ingredients to generate the striped superconductor as the intertwined phase of CDW order and SC order. It is quite worthwhile to investigate more complicated relations between CDW and SC by introducing various interacting terms among gauge fields and scalar fields, for instance, considering the coupling between the condensation field of SC and the dilaton field of CDW. This may disclose more novel phenomena in high temperature superconductivity and deserve for further investigation.

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