Yang-Yang functions, monodromy and knot polynomials

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ABSTRACT: We derive a structure of $\mathbb{Z}[t, t^{-1}]$ -module bundle from a family of Yang-Yang functions. For the fundamental representation of the complex simple Lie algebra of classical type, we give explicit wall-crossing formula and prove that the monodromy representation of the $\mathbb{Z}[t, t^{-1}]$ -module bundle is equivalent to the braid group representation induced by the universal R-matrices of $U_h(g)$. We show that two transformations induced on the fiber by the symmetry breaking deformation and respectively the rotation of two complex parameters commute with each other.

KEYWORDS: Chern-Simons Theories, Quantum Groups, Spontaneous Symmetry Breaking, Wilson, 't Hooft and Polyakov loops

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Contents

1	Introduction	1
2	Yang-Yang function, monodromy of its associated bundle and main theorems	2
2.1	Yang-Yang function and its critical point	2
2.2	$V_\lambda \otimes V_\lambda$ realized as the fiber of $\mathbb{Z}[t, t^{-1}]$ -module bundle	6
2.2.1	$V_\lambda \otimes V_\lambda$ and the thimble space of Yang-Yang functions	6
2.2.2	The thimble space as a fiber bundle of $\mathbb{Z}[t, t^{-1}]$ -module	7
2.3	Monodromy representation of the $\mathbb{Z}[t, t^{-1}]$ -module fiber bundle	8
3	Wall-crossing formula and monodromy representation	9
3.1	A simple example of wall-crossing phenomena	9
3.2	Variations of critical points, wall-crossing formula and monodromy representation	11
3.2.1	A_n	13
3.2.2	B_n	15
3.2.3	C_n	24
3.2.4	D_n	29
3.3	Similarity transformation Q	35
4	Commutation of two parameter deformations	36

1 Introduction

Yang-Yang function was named by N. Nekrasov, A. Rosly and S. Shatashvili in [1]. Originated from C.N. Yang and C.P. Yang's paper [2, 3], it was used for the analysis of the non-linear Schrödinger model. Behind this function hides a quantum integrable system [1, 4], thus it aroused much interest. Yang-Yang function can also be realized as the power in the correlation function of the free field realization of Virasoro vertex operators [5]. D. Gaiotto and E. Witten used this realization to derive Jones polynomial for knots from four dimensional Chern-Simons gauge theory. Our interest is to figure out the structure underlying the derivation of knot invariants from the Yang-Yang function. Similar to the work of V.G. Drinfeld and T. Kohno [6–8], where the monodromy of Knizhnik-Zamolodchikov system was proved to be equivalent to the braid group representation, induced by the universal R-matrices of the quantum enveloping algebra $U_h(g)$ of semisimple Lie algebra g , we prove that the monodromy representation of the $\mathbb{Z}[t, t^{-1}]$ -module bundle constructed from a family of Yang-Yang functions associated with the fundamental representation of the classical complex simple Lie algebra is equivalent to the braid group representation,

induced by the universal R-matrices of $U_h(g)$. Furthermore, transformations induced on the fiber by two parameter deformations, the symmetry breaking parameter c from $c = 0$ to $c \rightarrow \infty$ and respectively the rotation of two singular complex parameters z_1 and z_2 , commute with each other. By studying the monodromy, creation and annihilation matrices from the parameter deformations, one can derive the HOMFLY-PT polynomial and Kauffman polynomial for knots. We expect the existence of new knot invariants, different from HOMFLY-PT and Kauffman polynomials, from general representations of Lie algebras. It will be investigated elsewhere.

This paper is organized as follows. Firstly we give the definition of Yang-Yang function in general case and derive from it the $\mathbb{Z}[t, t^{-1}]$ -module bundle structure, then state two main theorems in section 2. In section 3, the parameter rotation for Yang-Yang function is considered. Four types of variations of critical point of Yang-Yang function are studied in detail. For the fundamental representation of classical complex simple Lie algebra, general wall-crossing formula and monodromy representation are derived. The first main theorem follows. In section 4, we discuss the relation between two monodromy representations of $c = 0$ and $c \rightarrow +\infty$, which leads to a proof of the second main theorem.

2 Yang-Yang function, monodromy of its associated bundle and main theorems

2.1 Yang-Yang function and its critical point

Let g be a finite dimensional complex simple Lie algebra with Chevalley generators $\{E_i, F_i, H_i\}, i = 1, \dots, n$ and Cartan matrix $C_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$, where α_i primary roots of g and $(,)$ the inner product induced from Killing form. Let P^+ be the positive Weyl chamber in the weight space of g . Fundamental weights ω_i are a set of bases of weight space satisfying $\frac{2(\omega_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}$. Weyl vector is defined as $\rho = \sum_{i=1}^n \omega_i$, thus $\frac{2(\rho, \alpha_i)}{(\alpha_i, \alpha_i)} = 1$ for any i ($1 \leq i \leq n$). Let $\lambda = (\lambda_1, \dots, \lambda_m)$ be a sequence of weights $\lambda_a \in P^+$. Denote by V_λ the representation $V_{\lambda_1} \otimes \dots \otimes V_{\lambda_m}$ and Ω_λ the set of weights in V_λ . $l = (l_1, \dots, l_n)$ a sequence of nonnegative integers is just a n partition of $l = \sum_{i=1}^n l_i$ and $\mathbf{0} = (0, \dots, 0)$.

Yang-Yang function is defined by

$$\begin{aligned} W(\mathbf{w}, \mathbf{z}, \lambda, l) = & \sum_{j=1}^l \sum_{a=1}^m (\alpha_{i_j}, \lambda_a) \log(w_j - z_a) - \sum_{1 \leq j < k \leq l} (\alpha_{i_j}, \alpha_{i_k}) \log(w_j - w_k) \\ & - \sum_{1 \leq a < b \leq m} (\lambda_a, \lambda_b) \log(z_a - z_b). \end{aligned} \quad (2.1)$$

It is a function of complex variables $\mathbf{w} = (w_1, \dots, w_l)$, distinct complex parameters $\mathbf{z} = (z_1, \dots, z_m)$, weights λ and n partition l of l . To each z_a and w_j , a dominant integral weight $\lambda_a \in P^+$ and a primary root α_{i_j} are associated respectively, where $i_j \in \{1, \dots, n\}$. Let $l_k = \#\{j \mid 1 \leq j \leq l, i_j = k\}$ be the number of α_k in $\{\alpha_{i_j}\}_{j=1, \dots, l}$. Denote by $\alpha(l)$ the sum of all these primary roots $\alpha(l) = l_1 \alpha_1 + \dots + l_n \alpha_n$. Then $W(\mathbf{w}, \mathbf{z}, \lambda, \mathbf{0})$ is a constant function of $\mathbf{w}$.

The critical points of $\mathbf{W}(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, l)$ satisfy

$$\frac{\partial \mathbf{W}(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, l)}{\partial w_j} = 0, j = 1, \dots, l \quad (2.2)$$

equivalently,

$$\sum_a \frac{(\alpha_{i_j}, \lambda_a)}{w_j - z_a} = \sum_{s \neq j} \frac{(\alpha_{i_j}, \alpha_{i_s})}{w_j - w_s}, j = 1, \dots, l. \quad (2.3)$$

By definition, if $\mathbf{w}$ is a solution of $\frac{\partial \mathbf{W}}{\partial w_j} = 0$ and $(\alpha_{i_j}, \alpha_{i_s}) \neq 0$, then $w_j \neq w_s$. Also if $(\alpha_{i_j}, \lambda_a) \neq 0$, then $w_j \neq z_a$. Under the permutation of all the coordinates w_j associated with the same primary root, the critical point equation above is invariant, thus we do not distinguish these critical points.

The critical points of Yang-Yang function have a close relation with the singular vectors of the tensor product space $V_{\boldsymbol{\lambda}} = V_{\lambda_1} \otimes V_{\lambda_2} \otimes \dots \otimes V_{\lambda_m}$. Let $SingV = \{v \in V \mid E_i v = 0, \forall i\}$ be the subspace of singular vectors in V . The following fact is known.

Theorem 1. *If $\lambda_1 + \dots + \lambda_m - \alpha(l) \in P^+$, then for all nondegenerate critical points of Yang-Yang function $\mathbf{W}(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, l)$, there exists a singular vector in $SingV_{\boldsymbol{\lambda}}$.*

If $\lambda_1 + \dots + \lambda_m - \alpha(l) \in P^+$, for each critical point of Yang-Yang function, Bethe vector can be constructed [9]. Shapovarov form of Bethe vector is proved to be equal to Hessian of Yang-Yang function at the critical point [4]. Thus nondegeneration of critical point guarantees Bethe vector is nonzero. By theorem 11.1 in [10], each nonzero Bethe vector belongs to $SingV_{\boldsymbol{\lambda}}$. For details of proof, we refer to [4, 9, 10]. The master function in the reference is just the exponential of Yang-Yang function $\mathbf{W}(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, l)$.

Formally denote by w_0 the critical point of constant function $\mathbf{W}(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, 0)$.

Theorem 2. *Let g be a classical complex simple Lie algebra. If $\mathbf{z} = (z_1, z_2)$, $\boldsymbol{\lambda} = (\omega_1, \omega_1)$, the nondegenerate critical points of the family of Yang-Yang functions*

$$\{\mathbf{W}(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, l) \mid V_{2\omega_1 - \alpha(l)} \subset V_{\boldsymbol{\lambda}}\}$$

together with the formal degenerate critical point w_0 of $\mathbf{W}(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, 0)$ one to one correspond to the weight vectors in $SingV_{\boldsymbol{\lambda}}$.

Proof. By Littelmann-Littlewood-Richardson rule [11], the following direct sum decompositions can be derived for each case of classical Lie algebras.

$$\begin{aligned} A_n : \quad & V_{\omega_1} \otimes V_{\omega_1} = V_{2\omega_1} \oplus V_{2\omega_1 - \alpha_1} \\ B_n : \quad & V_{\omega_1} \otimes V_{\omega_1} = V_{2\omega_1} \oplus V_{2\omega_1 - \alpha_1} \oplus V_{2\omega_1 - 2\alpha_1 - \dots - 2\alpha_n} \\ C_n : \quad & V_{\omega_1} \otimes V_{\omega_1} = V_{2\omega_1} \oplus V_{2\omega_1 - \alpha_1} \oplus V_{2\omega_1 - 2\alpha_1 - \dots - 2\alpha_{n-1} - \alpha_n} \\ D_n : \quad & V_{\omega_1} \otimes V_{\omega_1} = V_{2\omega_1} \oplus V_{2\omega_1 - \alpha_1} \oplus V_{2\omega_1 - 2\alpha_1 - \dots - 2\alpha_{n-2} - \alpha_{n-1} - \alpha_n} \end{aligned} \quad (2.4)$$

For B_n Lie algebra, the singular vectors of $SingV_{\omega_1} \otimes V_{\omega_1}$ are

$$v_{2\omega_1}, v_{2\omega_1 - \alpha_1}, v_{2\omega_1 - 2\alpha_1 - \dots - 2\alpha_n}.$$

By theorem 1, there is an injective map between the set of the nondegenerate critical points and $SingV_{\omega_1} \otimes V_{\omega_1}$. Corresponding to each singular vector, the explicit solution of the critical point equation can be derived by lemma in section 3.2. They are nondegenerate except when $l = 0$. Thus the map is bijective on $SingV_{\omega_1} \otimes V_{\omega_1} \setminus v_{2\omega_1}$. When $l = 0$, the formal degenerate critical point w_0 corresponds to the singular vector $v_{2\omega_1}$ in $SingV_{\omega_1} \otimes V_{\omega_1}$. Therefore, one to one correspondence is proved. For A_n, C_n, D_n , the proofs are similar. $\square$

With an additional deformation parameter $c \in \mathbb{R}$, symmetry breaking Yang-Yang function is defined by

$$\begin{aligned} \mathbf{W}_c(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, \mathbf{l}) = & \sum_{j,a} (\alpha_{i_j}, \lambda_a) \log(w_j - z_a) - \sum_{j < k} (\alpha_{i_j}, \alpha_{i_k}) \log(w_j - w_k) \\ & - \sum_{a < b} (\lambda_a, \lambda_b) \log(z_a - z_b) - c \sum_j (\rho, \alpha_{i_j}) w_j + c \sum_a (\rho, \lambda_a) z_a. \end{aligned} \quad (2.5)$$

It is clear that $\mathbf{W}_0(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, \mathbf{l}) = \mathbf{W}(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, \mathbf{l})$. In fact, when $c \in \mathbb{Z}_{\geq 0}$, $\mathbf{W}_c(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, \mathbf{l})$ is just a limitation of the parameter deformation of η with an additional parameter $z_\infty = 1$ and the dominant integral weight $\lambda_\infty = c\rho \in P^+$ associated with it.

$$\begin{aligned} \mathbf{W}_c(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, \mathbf{l}) = & \lim_{\eta \rightarrow 0} \left[\sum_{j,a} (\alpha_{i_j}, \lambda_a) \log(w_j - z_a) + \sum_j \left(\frac{c\rho}{\eta}, \alpha_{i_j} \right) \log(1 - \eta w_j) \right. \\ & - \sum_{j < k} (\alpha_{i_j}, \alpha_{i_k}) \log(w_j - w_k) - \sum_{0 < a < b} (\lambda_a, \lambda_b) \log(z_a - z_b) \\ & \left. - \sum_a \left(\frac{c\rho}{\eta}, \lambda_a \right) \log(1 - \eta z_a) \right]. \end{aligned} \quad (2.6)$$

The critical point equation of $\mathbf{W}_c(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, \mathbf{l})$ is

$$\sum_{a=1}^m \frac{(\alpha_{i_j}, \lambda_a)}{w_j - z_a} = \sum_{s \neq j} \frac{(\alpha_{i_j}, \alpha_{i_s})}{w_j - w_s} + c(\rho, \alpha_{i_j}), j = 1, \dots, l. \quad (2.7)$$

Lemma 1. *If w_j is the coordinate of the critical point of $\mathbf{W}_c(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, \mathbf{l})$, then*

$$\lim_{c \rightarrow +\infty} w_j \in \{z_a\}_{a=1, \dots, m}.$$

Proof. Because $(\rho, \alpha_{i_j}) > 0$, it is clear for any j ,

$$\lim_{c \rightarrow +\infty} w_j \in \{z_a\}_{a=1, \dots, m} \cup \{w_k\}_{k \neq j}.$$

Divide the set $\{w_1, \dots, w_l\}$ into a disjoint union of $Z_1, \dots, Z_m$ and M , where $Z_a, a = 1, \dots, m$ contains the coordinates w_j which tend to z_a and M the rest of them. Assume $M = \{w_{m_1}, \dots, w_{m_p}\} \neq \emptyset$, then $p > 1$. Sum the equations of $j \in M$

$$\sum_{j=m_1}^{m_p} \sum_a \frac{(\alpha_{i_j}, \lambda_a)}{w_j - z_a} = \sum_{j=m_1}^{m_p} \sum_{s \neq j} \frac{(\alpha_{i_j}, \alpha_{i_s})}{w_j - w_s} + c \sum_{j=m_1}^{m_p} (\rho, \alpha_{i_j}). \quad (2.8)$$

When $c \rightarrow +\infty$, the left hand side of (2.8) is bounded.

$$\sum_{j=m_1}^{m_p} \sum_{s \neq j} \frac{(\alpha_{i_j}, \alpha_{i_s})}{w_j - w_s} = \sum_{w_j \in M} \sum_{w_s \in Z_1 \cup \dots \cup Z_m} \frac{(\alpha_{i_j}, \alpha_{i_s})}{w_j - w_s},$$

where the summation in M is canceled, thus also bounded. Eq. (2.8) leads to contradiction. Therefore, $M = \emptyset$. The lemma follows. $\square$

Denote also by w_0 the critical point of the constant function $\mathbf{W}_c(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, \mathbf{0})$. For $\mathbf{z} = z_1$ and $\boldsymbol{\lambda} = \omega_1$, we have the following theorem

Theorem 3. *Let g be a classical complex simple Lie algebra. If $c \in \mathbb{Z}_{\geq 0}$ and $c \geq 2$, the nondegenerate critical points of the family of Yang-Yang functions*

$$\{\mathbf{W}_c(\mathbf{w}, z_1, \omega_1, \mathbf{l})\}_{\omega_1 - \alpha(\mathbf{l}) \in \Omega_{\omega_1}}$$

together with the formal degenerate critical point w_0 one to one correspond to the weight vectors in V_{ω_1} .

Proof. By Littelmann-Littlewood-Richardson rule [11], if $c \in \mathbb{Z}_{\geq 0}$ and $c \geq 2$, the following direct sum decompositions can be derived.

$$\begin{aligned} A_n : V_{\omega_1} \otimes V_{c\rho} &= V_{\omega_1+c\rho} \oplus V_{\omega_1+c\rho-\alpha_1} \oplus \dots \oplus V_{\omega_1+c\rho-\alpha_1-\dots-\alpha_n} \\ B_n : V_{\omega_1} \otimes V_{c\rho} &= V_{\omega_1+c\rho} \oplus V_{\omega_1+c\rho-\alpha_1} \oplus \dots \oplus V_{\omega_1+c\rho-\alpha_1-\dots-\alpha_n} \oplus \\ &\quad V_{\omega_1+c\rho-\alpha_1-\dots-2\alpha_n} \oplus V_{\omega_1+c\rho-\alpha_1-\dots-2\alpha_{n-1}-2\alpha_n} \oplus \\ &\quad \dots \oplus V_{\omega_1+c\rho-2\alpha_1-\dots-2\alpha_{n-1}-2\alpha_n} \\ C_n : V_{\omega_1} \otimes V_{c\rho} &= V_{\omega_1+c\rho} \oplus V_{\omega_1+c\rho-\alpha_1} \oplus \dots \oplus V_{\omega_1+c\rho-\alpha_1-\dots-\alpha_n} \oplus \\ &\quad V_{\omega_1+c\rho-\alpha_1-\dots-2\alpha_{n-1}-\alpha_n} \oplus \dots \oplus V_{\omega_1+c\rho-2\alpha_1-\dots-2\alpha_{n-1}-\alpha_n} \\ D_n : V_{\omega_1} \otimes V_{c\rho} &= V_{\omega_1+c\rho} \oplus V_{\omega_1+c\rho-\alpha_1} \oplus V_{\omega_1+c\rho-\alpha_1-\alpha_2} \oplus \dots \oplus \\ &\quad V_{\omega_1+c\rho-\alpha_1-\dots-\alpha_n} \oplus V_{\omega_1+c\rho-\alpha_1-\dots-2\alpha_{n-2}-\alpha_{n-1}-\alpha_n} \oplus \\ &\quad \dots \oplus V_{\omega_1+c\rho-2\alpha_1-\dots-2\alpha_{n-2}-\alpha_{n-1}-\alpha_n} \end{aligned} \quad (2.9)$$

For A_n , it is clear that the singular vectors of $\text{Sing}V_{\omega_1} \otimes V_{c\rho}$

$$v_{\omega_1+c\rho}, v_{\omega_1+c\rho-\alpha_1}, \dots, v_{\omega_1+c\rho-\alpha_1-\dots-\alpha_n},$$

one to one correspond to the weight vectors

$$v_{\omega_1}, v_{\omega_1-\alpha_1}, \dots, v_{\omega_1-\alpha_1-\dots-\alpha_n},$$

in V_{ω_1} . By theorem 1, there is an injective map between the set of the nondegenerate critical points and $\text{Sing}V_{\omega_1} \otimes V_{c\rho}$. Corresponding to each singular vectors, $\mathbf{l}$ satisfies the admissible condition and the explicit solutions of critical points with two parameters $z_1 = 0, z_2 = 1$ are given in [12]. They are nondegenerate except when $\mathbf{l} = 0$. Thus the map is bijective on $\text{Sing}V_{\omega_1} \otimes V_{c\rho} \setminus v_{\omega_1+c\rho}$. When $\mathbf{l} = 0$, the formal degenerate critical point w_0 corresponds to

the singular vector $v_{\omega_1+c\rho}$ in $SingV_{\omega_1} \otimes V_{c\rho}$. By (2.6), the critical point of $\mathbf{W}_c(\mathbf{w}, z_1, \omega_1, \mathbf{l})$ are the limitation of the parameter deformation of those solutions. We give the explicit expressions of them in the lemma 4 in section 3.2.2. Therefore, one to one correspondence is proved. For B_n, C_n, D_n , the proofs are similar. $\square$

Remark 1. *Except for B_n , the direct sum decompositions above are also correct if $c = 1$.*

2.2 $V_\lambda \otimes V_\lambda$ realized as the fiber of $\mathbb{Z}[t, t^{-1}]$ -module bundle

For any dominant integral weight $\lambda \in P^+$, there exists a linear automorphism of $V_\lambda \otimes V_\lambda$ called R-matrix:

$$R : V_\lambda \otimes V_\lambda \rightarrow V_\lambda \otimes V_\lambda$$

satisfying Yang-Baxter equation [13]

$$(R \otimes id_{V_\lambda})(id_{V_\lambda} \otimes R)(R \otimes id_{V_\lambda}) = (id_{V_\lambda} \otimes R)(R \otimes id_{V_\lambda})(id_{V_\lambda} \otimes R).$$

To study R-matrix, $m = 2$ is sufficient. In the following sections, we consider the fundamental representation $\lambda = \omega_1$ of complex simple Lie algebra of classical types. Assume that z_1 and z_2 have the same real part and $Im z_1 > Im z_2$.

2.2.1 $V_\lambda \otimes V_\lambda$ and the thimble space of Yang-Yang functions

By lemma 1, when $c \rightarrow +\infty$, equations of (2.7) splits into two separated sets involving only z_1 or z_2 and its nondegenerate solutions space is just the tensor product of two nondegenerate solutions spaces with only one parameter z_1 or z_2 . Therefore, by theorem 3, when $c \rightarrow +\infty$, the set of nondegenerate critical points of (2.7) together with formal critical point w_0 one to one correspond to the weight vectors in $V_{\omega_1} \otimes V_{\omega_1}$.

The thimble of $\mathbf{W}_c(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, \mathbf{l})$, defined [14] as the cycle formulated by the gradient flow of the real part of Yang-Yang function starting from the critical point is a l dimensional manifold. Therefore, for all vectors $v_{\omega_1-\alpha(\mathbf{k})} \otimes v_{\omega_1-\alpha(\mathbf{l})} \in V_{\omega_1} \otimes V_{\omega_1}$, there exists a unique thimble of $\mathbf{W}_c(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, \mathbf{k} + \mathbf{l})$ denoted by $J_{\mathbf{k}, \mathbf{l}}$ and $J_{\mathbf{k}, \mathbf{l}} = J_{\mathbf{k}} \times J_{\mathbf{l}}$, where $J_{\mathbf{k}}$ and $J_{\mathbf{l}}$ are thimbles of $\mathbf{W}_c(\mathbf{w}, z_1, \omega_1, \mathbf{k})$ and $\mathbf{W}_c(\mathbf{w}, z_2, \omega_1, \mathbf{l})$ respectively. Therefore, we have

Theorem 4. *When $c \rightarrow +\infty$, the basis of $V_{\omega_1} \otimes V_{\omega_1}$ one to one correspond to thimbles generated from a family of Yang-Yang functions*

$$\{\mathbf{W}_c(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, \mathbf{k} + \mathbf{l})\}_{\omega_1-\alpha(\mathbf{k}), \omega_1-\alpha(\mathbf{l}) \in \Omega_{\omega_1}}.$$

Note that the solution of critical point equation (2.7) is not unique. It means that there exist different thimbles corresponding to different critical solutions of the same Yang-Yang function. We will see it in the example of section 3.1.

2.2.2 The thimble space as a fiber bundle of $\mathbb{Z}[t, t^{-1}]$ -module

As in [15, 16] and [17], Yang-Yang function appears naturally as an exponent in the correlation function of Wakimoto realization of Kac-Moody algebra at arbitrary level κ :

$$\begin{aligned} & \int_{\Gamma} \prod_{j,a} (w_j - z_a)^{-\frac{(\alpha_{i_j}, \lambda_a)}{\kappa+h^\vee}} \prod_{j < s} (w_j - w_s)^{\frac{(\alpha_{i_j}, \alpha_{i_s})}{\kappa+h^\vee}} (z_1 - z_2)^{\frac{(\lambda, \lambda)}{\kappa+h^\vee}} \prod_j dw_j \\ &= \int_{\Gamma} e^{-\frac{\mathbf{W}(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, \mathbf{l})}{\kappa+h^\vee}} \prod_j dw_j, \end{aligned} \quad (2.10)$$

where $h^\vee$ is the dual Coxeter number. The crucial problem is to figure out the transformation induced on the thimble space by parameter deformation of $e^{-\frac{\mathbf{W}_c(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, \mathbf{l})}{\kappa+h^\vee}}$, which leads us to study the extra structure of the thimble space.

Thimbles of the real part of Yang-Yang function $\mathbf{W}_c(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, \mathbf{l})$ are the same as thimbles of $\left| e^{-\frac{\mathbf{W}_c(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, \mathbf{l})}{\kappa+h^\vee}} \right|$, except that there are infinite number of pre-images with difference $2\pi i$ in their imaginary parts. Along the thimble of its real part, conservation law [14] of the imaginary part of Yang-Yang function implies global invariance of the phase factor $e^{-\frac{\text{Im} \mathbf{W}_c(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, \mathbf{l})}{\kappa+h^\vee} i}$. By theorem 4, when $c \rightarrow +\infty$, the basis of $V_{\omega_1} \otimes V_{\omega_1}$ one to one correspond to the family of thimbles $\{J_{\mathbf{k}, \mathbf{l}}\}_{\omega_1-\alpha(\mathbf{k}), \omega_1-\alpha(\mathbf{l}) \in \Omega_{\omega_1}}$ of $\left\{ \left| e^{-\frac{\mathbf{W}_c(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, \mathbf{k}+\mathbf{l})}{\kappa+h^\vee}} \right| \right\}_{\omega_1-\alpha(\mathbf{k}), \omega_1-\alpha(\mathbf{l}) \in \Omega_{\omega_1}}$. Each of them has infinite pre-images with different global phases. Choose a branch from infinite pre-images for each thimble. Define $q = e^{\frac{2\pi i}{\kappa+h^\vee}}$. Continuous deformation of the parameters in $e^{-\frac{\mathbf{W}_c(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, \mathbf{l})}{\kappa+h^\vee}}$ will induce a global phase factor variation along the thimble of $\left| e^{-\frac{\mathbf{W}_c(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, \mathbf{l})}{\kappa+h^\vee}} \right|$. $\{J_{\mathbf{k}, \mathbf{l}}\}_{\omega_1-\alpha(\mathbf{k}), \omega_1-\alpha(\mathbf{l}) \in \Omega_{\omega_1}}$ are elements of a certain relative homology group and have a natural integral structure [5]. Let $\mathbb{Z}[t, t^{-1}]$ be the ring of Laurent polynomials of t over $\mathbb{Z}$, then these variations naturally make thimble space a module over ring $\mathbb{Z}[t, t^{-1}]$, where t is some minimal power of q during the variation. From now on, we consider $V_{\omega_1} \otimes V_{\omega_1}$ as a $\mathbb{Z}[t, t^{-1}]$ -module generated by the basis $\{v_{\omega_1-\alpha(\mathbf{k})} \otimes v_{\omega_1-\alpha(\mathbf{l})}\}_{\omega_1-\alpha(\mathbf{k}), \omega_1-\alpha(\mathbf{l}) \in \Omega_{\omega_1}}$. Let $\mathfrak{J} = \mathbb{Z}[t, t^{-1}]\{J_{\mathbf{k}, \mathbf{l}}\}_{\omega_1-\alpha(\mathbf{k}), \omega_1-\alpha(\mathbf{l}) \in \Omega_{\omega_1}}$ be the module over $\mathbb{Z}[t, t^{-1}]$ generated by the corresponding thimbles, then we have a natural decomposition of $V_{\omega_1} \otimes V_{\omega_1}$ as a direct sum of $\mathbb{Z}[t, t^{-1}]$ -submodules.

Lemma 2.

$$V_{\omega_1} \otimes V_{\omega_1} \cong \mathfrak{J} = \bigoplus_{\mathfrak{q}} E^{\mathfrak{q}}, \quad (2.11)$$

where $E^{\mathfrak{q}} = \mathbb{Z}[t, t^{-1}]\{J_{\mathbf{k}, \mathbf{l}}\}_{k+l=\mathfrak{q}}$ is a $\mathbb{Z}[t, t^{-1}]$ -submodule generated by all the $\mathfrak{q}$ dimensional thimbles of $\mathfrak{J}$.

Because $\mathfrak{J}$ depends on the parameters (z_1, z_2) , there exists a bundle $\mathfrak{J} \xrightarrow{\pi} X_2$, where

$$X_2 = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 \neq z_2\} / (z_1, z_2) \sim (z_2, z_1) \quad (2.12)$$

is the configuration space of the two complex parameters z_1 and z_2 . The fiber $\pi^{-1}(z_1, z_2) = \mathfrak{J}(\mathbf{z})$ is defined to be a $\mathbb{Z}[t, t^{-1}]$ -module generated by all the thimbles of

$\left\{ \left| e^{-\frac{W_c(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, \mathbf{k} + \mathbf{l})}{\kappa + h^\vee}} \right| \right\}_{\omega_1 - \alpha(\mathbf{k}), \omega_1 - \alpha(\mathbf{l}) \in \Omega_{\omega_1}}$. Let $E^{\mathfrak{q}} \xrightarrow{\pi^{\mathfrak{q}}} X_2$ be the $\mathbb{Z}[t, t^{-1}]$ -module subbundle of $\mathfrak{J} \xrightarrow{\pi} X_2$. Because of (2.11), we have the following decomposition of the $\mathbb{Z}[t, t^{-1}]$ -module bundle.

Lemma 3.

$$\pi = \bigoplus_{\mathfrak{q}} \pi^{\mathfrak{q}}.$$

In sum, the thimble space of $\left\{ e^{-\frac{W_c(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, \mathbf{k} + \mathbf{l})}{\kappa + h^\vee}} \right\}_{\omega_1 - \alpha(\mathbf{k}), \omega_1 - \alpha(\mathbf{l}) \in \Omega_{\omega_1}}$ as the fiber of $\mathbb{Z}[t, t^{-1}]$ -module bundle $\mathfrak{J} \xrightarrow{\pi} X_2$ gives $V_{\omega_1} \otimes V_{\omega_1}$ a geometric realization. We will use it to derive R matrix.

2.3 Monodromy representation of the $\mathbb{Z}[t, t^{-1}]$ -module fiber bundle

Fixing a point $P = (z_1, z_2) \in X_2$, we consider a continuous parameter transformation $T(s) : X_2 \rightarrow X_2$ defined by

$$T(s) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \frac{1+e^{-i\pi s}}{2} & \frac{1-e^{-i\pi s}}{2} \\ \frac{1-e^{-i\pi s}}{2} & \frac{1+e^{-i\pi s}}{2} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, s \in [0, 1]. \quad (2.13)$$

It is a clockwise rotation around the middle point of z_1 and z_2 :

$$T(1)P = T(1) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_2 \\ z_1 \end{pmatrix} = P.$$

Thus, $T : S^1 \rightarrow X_2$ generates a fundamental group $\pi_1(X_2, P)$ of X_2 with base point P and also induces a $\mathbb{Z}[t, t^{-1}]$ -module transformation

$$\sigma \in \text{End}(\mathfrak{J}(P), \mathbb{Z}[t, t^{-1}]) = \text{End}(V_{\omega_1} \otimes V_{\omega_1}, \mathbb{Z}[t, t^{-1}])$$

on the fiber $\mathfrak{J}(P) \cong V_{\omega_1} \otimes V_{\omega_1}$. σ is called monodromy of the bundle and it generates a monodromy group. Its representation on the fiber space is called monodromy representation. Let $\mathbf{B} : V_{\omega_1} \otimes V_{\omega_1} \rightarrow V_{\omega_1} \otimes V_{\omega_1}$ be the monodromy representation of the bundle $\mathfrak{J} \xrightarrow{\pi} X_2$ induced by $T : S^1 \rightarrow X_2$ and $\mathbf{B}_{U_h(g)}$ the braid group representation induced by the universal R-matrices of the quantum enveloping algebra $U_h(g)$. Our first main theorem is as following:

Theorem 5. *For the fundamental representation V_{ω_1} of classical complex simple Lie algebra g , the monodromy representation $\mathbf{B}_{\mathbf{Y}\mathbf{Y}}$ of the $\mathbb{Z}[t, t^{-1}]$ -module fiber bundle $\mathfrak{J} \xrightarrow{\pi} X_2$ generated by the family of functions*

$$\left\{ e^{-\frac{W_c(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, \mathbf{k} + \mathbf{l})}{\kappa + h^\vee}} \right\}_{\omega_1 - \alpha(\mathbf{k}), \omega_1 - \alpha(\mathbf{l}) \in \Omega_{\omega_1}}$$

is equivalent to $\mathbf{B}_{U_h(g)}$ by a diagonal transformation Q , i.e.

$$\mathbf{B}_{\mathbf{Y}\mathbf{Y}} = Q \mathbf{B}_{U_h(g)} Q^{-1}, Q \in \text{End}(V_{\omega_1} \otimes V_{\omega_1}, \mathbb{Z}[t, t^{-1}]).$$

Remark 2. The advantage of considering thimbles as the basis for $\mathbb{Z}[t, t^{-1}]$ -module $V_{\omega_1} \otimes V_{\omega_1}$ is that the imaginary part of holomorphic function is conserved along the thimble defined by the gradient flow of the real part of the function [14]. When the parameter varies, there is a global variation for the thimble, which can be extracted from the critical point associated with it. Therefore, it is convenient to calculate the monodromy of $\mathfrak{J} \xrightarrow{\pi} X_2$ valued in $\mathbb{Z}[t, t^{-1}]$ only by considering the variation of the critical point of Yang-Yang function.

Denote by $\mathfrak{J}_0 \xrightarrow{\mu} X_2$ the thimble space generated by the family

$$\left\{ e^{-\frac{W_0(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, \mathbf{l})}{\kappa + h^\vee}} \mid V_{2\omega_1 - \alpha}(\mathbf{l}) \subset V_{\omega_1} \otimes V_{\omega_1} \right\}.$$

By theorem 2, it is a fiber bundle of $\mathbb{Z}[t, t^{-1}]$ -module with its fiber $\mathfrak{J}_0(P)$ isomorphic to $Sing V_{\omega_1} \otimes V_{\omega_1}$. Denote also by $\mathbf{B}_{\mathbf{Y}\mathbf{Y}}$ the monodromy representation on it induced by clockwise rotation $T(1)$ in equation (2.13). In section 4, we will define transformation $\mathbf{S} : \mathfrak{J}_0(P) \rightarrow \mathfrak{J}(P)$ induced by the parameter deformations $c \rightarrow +\infty$ from 0. Our second main theorem is as following:

Theorem 6. $\mathbf{S}$ commutes with $\mathbf{B}_{\mathbf{Y}\mathbf{Y}}$ i.e. the following diagram commutes.

$$\begin{array}{ccc} \mathfrak{J}_0(P) & \xrightarrow{\mathbf{S}} & \mathfrak{J}(P) \\ \downarrow \mathbf{B}_{\mathbf{Y}\mathbf{Y}} & & \downarrow \mathbf{B}_{\mathbf{Y}\mathbf{Y}} \\ \mathfrak{J}_0(P) & \xrightarrow{\mathbf{S}} & \mathfrak{J}(P) \end{array}$$

3 Wall-crossing formula and monodromy representation

In this section, we study the monodromy representation $\mathbf{B}_{\mathbf{Y}\mathbf{Y}}$ of $\mathfrak{J} \xrightarrow{\pi} X_2$ and prove the first main theorem. To simplify the notation, we use l to label each Yang-Yang function in the following sections. For D_n Lie algebra, there are two Yang-Yang functions of $\mathbf{l} = (1, \dots, 1, 1, 0)$ and $\mathbf{l} = (1, \dots, 1, 0, 1)$ with the same $l = n - 1$. We will label $(1, \dots, 1, 1, 0)$ by $n - 1$ and $(1, \dots, 1, 0, 1)$ by $n - 1'$ and define an order for them.

3.1 A simple example of wall-crossing phenomena

The key for the derivation of the monodromy representation is to figure out the wall-crossing formula. To illustrate it, we start from an example of A_1 Lie algebra $g = sl(2, \mathbb{C})$, $\dim V_{\omega_1} = 2$ and $\Omega_{\omega_1} = \{\omega_1, \omega_1 - \alpha\}$. The inner product on the weight space is $(a, b) = \frac{a \cdot b}{2}$. For $v_1 \otimes v_0, v_0 \otimes v_1 \in V_{\omega_1} \otimes V_{\omega_1}$, two Yang-Yang functions corresponding to them equal

$$\begin{aligned} W_c(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, 1 + 0) &= W_c(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, 0 + 1) \\ &= \sum_a (\alpha, \omega_1) \log(w - z_a) - (\omega_1, \omega_1) \log(z_1 - z_2) - c \left(w - \frac{1}{2}(z_1 + z_2) \right). \end{aligned} \quad (3.1)$$

Its critical point equation is

$$\frac{1}{w - z_1} + \frac{1}{w - z_2} = c, \quad (3.2)$$

which has two solutions $w^1(c)$ and $w^2(c)$ for $c \geq 2$. Assume

$$\lim_{c \rightarrow +\infty} w^1(c) = z_1, \quad \lim_{c \rightarrow +\infty} w^2(c) = z_2.$$

With large c , let $J_{1,0}$ and $J_{0,1}$ be two thimbles associated respectively to $w^1(c)$ and $w^2(c)$. The continuous clockwise transformation $T(1)$ induces a continuous deformation on the thimble $J_{1,0}$:

$$J_{1,0} \rightarrow q^{-\frac{1}{2}(\omega_1 - \alpha, \omega_1)} J_{0,1} = q^{\frac{1}{4}} J_{0,1},$$

where $q^{-\frac{1}{2}(\omega_1 - \alpha, \omega_1)}$ is from the phase factor difference of the critical values and it is equal to the phase factor difference of $(z_1 - z_2)^{\frac{(\omega_1 - \alpha, \omega_1)}{\kappa + h\sqrt{V}}}$ under the $\frac{1}{2}$ clockwise rotation. The transformation of $J_{0,1}$ is more interesting. In the process of clockwise rotation $T(s)$, z_1 will pass through $J_{0,1}$ from the right hand side of z_2 , when $s = \frac{1}{2}$, the imaginary parts of two critical values equals:

$$\text{Im} \mathbf{W}_c \left(w^1, T \left(\frac{1}{2} \right) P, \boldsymbol{\lambda}, 1 \right) = \text{Im} \mathbf{W}_c \left(w^2, T \left(\frac{1}{2} \right) P, \boldsymbol{\lambda}, 1 \right).$$

There is a gradient flow connecting w^2 to w^1 as shown in figure 14 of [5]. The homotopic class of $J_{0,1}$ after deformation is equivalent to a zig-zag $\mathbb{Z}[t, t^{-1}]$ -linear combination of $J_{1,0}$ and $J_{0,1}$

$$\mathbf{B} J_{0,1} = a J_{1,0} + (c - b) J_{0,1}. \quad (3.3)$$

Because in general situation thimble is of high dimension, it is convenient to just draw the variation of its critical point along the homotopic class of the thimble after rotation, rather than to draw the thimble itself. Here, the zig-zag thimble has three parts and the relations of their critical points are shown in figure 1. In the following, whenever we draw the figure of the variation of the critical point, we are showing the relations between the critical points of the different thimbles in the homotopic class of the thimble after the deformation.

The coefficient a in (3.3) is from the phase factor of $e^{-\frac{\mathbf{W}_c(w^2, T(\frac{1}{2})P, \boldsymbol{\lambda}, 1)}{\kappa + h\sqrt{V}}} \sim (z_1 - z_2)^{\frac{(\omega_1 - \alpha, \omega_1)}{\kappa + h\sqrt{V}}}$. As shown in figure 1, b differs from a by an additional movement of the critical point from the right of z_2 to the right of z_1 and the negative sign before b is from the orientation reverse of the thimble. c differs from b by an anti-clockwise rotation around z_1 . Thus, $a = q^{-\frac{1}{2}(\omega_1, \omega_1 - \alpha)}$, $b = q^{\frac{1}{2}(\omega_1, \alpha)} \cdot a$ and $c = q^{-(\omega_1, \alpha)} \cdot b$.

$$\mathbf{B} J_{0,1} = q^{\frac{1}{4}} J_{1,0} + (q^{-\frac{1}{4}} - q^{\frac{3}{4}}) J_{0,1}. \quad (3.4)$$

The minimal phase factor under T is $t = q^{\frac{1}{4}}$ and therefore the monodromy is valued in $\mathbb{C}[q^{\frac{1}{4}}, q^{-\frac{1}{4}}]$. In sum,

$$\mathbf{B} \begin{pmatrix} J_{1,0} \\ J_{0,1} \end{pmatrix} = \begin{pmatrix} 0 & q^{\frac{1}{4}} \\ q^{\frac{1}{4}} & q^{-\frac{1}{4}} - q^{\frac{3}{4}} \end{pmatrix} \begin{pmatrix} J_{1,0} \\ J_{0,1} \end{pmatrix}. \quad (3.5)$$

The phenomena is called wall-crossing phenomena and the formula (3.4) describing it is called wall-crossing formula. Terms with coefficients b and c are wall-crossing terms. The following properties are clear. Firstly, because critical points of the thimbles in the wall-crossing phenomena are different solutions from the same Yang-Yang function, thus

Figure 1. Variation of the critical point on W plane.

Figure 2. Type I.

the types and total number of primary roots will not be created or annihilated in the process of wall-crossing, but only be transferred from one point to another point. We call this property conservation law of wall-crossing. Secondly, clockwise transformation T make a constraint on the direction of the primary roots transfer: primary roots only move in the direction of positive real axis from z_2 to z_1 . Based on the above two properties, E^q is a σ invariant sub-module. The monodromy representation $\mathbf{B}$ is naturally decomposed into a direct sum of the sub-representations on E^q and all matrices of the sub-representation are triangular and valued in $\mathbb{Z}[t, t^{-1}]$.

3.2 Variations of critical points, wall-crossing formula and monodromy representation

To derive wall-crossing formula, it is necessary to analyze the variation of critical points under the transformation T . As in the previous example, we focus on the homotopic class of the thimble after deformation T and see how the primary roots move from z_2 to z_1 . For the fundamental representation V_{ω_1} of $g \in A_n, B_n, C_n, D_n$, we conclude as following four types of variations of critical points during wall-crossing and the details of derivation can be found in each case in section 3.2.1, 3.2.2, 3.2.3 and 3.2.4.

Type I As shown in figure 2, coordinates of the critical point have the same real parts but different imaginary parts. This variation appears in the case of the finite dimensional irreducible representation of A_1 or in the case of B_n Lie algebra.

Type II Coordinates of the critical point near z_2 have the same imaginary parts formulated as a straight line paralleled to the real axis. The variation is just translation

Figure 3. Type II.

Figure 4. Type III.

as shown in figure 3. The homotopic class of the thimble after deformation is equivalent to three parts. There is only one dimension in the sub-thimble reversing its orientation, thus the sign before the coefficient b is always minus in this case.

Type III Coordinates of the critical point near z_2 have the same imaginary parts formulated as a straight line paralleled to the real axis. The variation from z_2 to z_1 as shown in figure 4 has a clockwise self rotation of π , then followed by a translation. In this case, the orientation of each dimension of the sub-thimble connecting z_2 to z_1 is reversed. Thus, the sign before b is $(-1)^j$, where j is the dimension of the sub-thimble or the number of primary roots moving from z_2 to z_1 . Opposite to the sign before b , it is $(-1)^{j+1}$ before c .

Type IV Coordinates of the critical point near z_2 have the same imaginary parts formulated as a straight line paralleled to the real axis. The variation leaves a trace along the homotopic class of the integration cycle like a “snake” keeping the relative order of the coordinates invariant during the moving. The coefficient b is from the variation of primary roots moving from z_2 to z_1 . c_1 , c_2 and c_3 differs from b respectively by 1, $j - 1$ and j primary roots rotating around z_1 and other primary roots near z_1 in a specific manner, as is shown in the pictures of figure 5. Thus, the sign before b , c_1 , c_2 and c_3 are $(-1)^j$, $(-1)^{j+1}$, $(-1)^{2j-1} = -1$ and $(-1)^{2j} = 1$ respectively.

In the following, we derive the monodromy representations for A_n , B_n , C_n and D_n respectively.

Figure 5. The origin of c_1 , c_2 and c_3 of type IV.

3.2.1 A_n

Denote by $\{\lambda^i\}_{i=0,1,\dots,n}$ the weights of V_{ω_1} , where $\lambda^i = \omega_1 - \sum_{j=1}^i \alpha_j$. For each weight vector $v_{\lambda^j} \in V_{\omega_1}$, the corresponding Yang-Yang function $\mathbf{W}_c(\mathbf{w}, z, \omega_1, j)$ with one parameter z is

$$\begin{aligned} \mathbf{W}_c(\mathbf{w}, z, \omega_1, j) = & \sum_{i=1}^j (\alpha_i, \omega_1) \log(w_i - z) - \sum_{1 \leq i < s \leq j} (\alpha_i, \alpha_s) \log(w_i - w_s) \\ & - c \left(\sum_{i=1}^j (\alpha_i, \rho) w_i - (\omega_1, \rho) z \right). \end{aligned} \quad (3.6)$$

Since $(\omega_1, \alpha_i) = \delta_{i,1}$ and

$$(\alpha_i, \alpha_k) = \begin{cases} 2, & i = k; \\ -1, & |i - k| = 1; \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathbf{W}_c(\mathbf{w}, z, \omega_1, j) = \log(w_1 - z) + \sum_{i=1}^{j-1} \log(w_i - w_{i+1}) - c \left(\sum_{i=1}^j w_i - \frac{n}{2} z \right). \quad (3.7)$$

Its critical point equation is as following:

$$\left\{ \begin{aligned} \frac{1}{w_1 - z} &= \frac{-1}{w_1 - w_2} + c \\ 0 &= \frac{-1}{w_2 - w_1} + \frac{-1}{w_2 - w_3} + c \\ &\dots \\ 0 &= \frac{-1}{w_{j-1} - w_{j-2}} + \frac{-1}{w_{j-1} - w_j} + c \\ 0 &= \frac{-1}{w_j - w_{j-1}} + c. \end{aligned} \right. \quad (3.8)$$

It is obvious that z and the solution $\{w_i = z + \sum_{k=0}^{i-1} \frac{1}{(j-k)c}\}_{i=1,\dots,j}$ are on the same horizontal line of $\mathbf{W}$ plane.

Therefore, for $v_{\lambda^i} \otimes v_{\lambda^j} \in V_{\omega_1} \otimes V_{\omega_1}$, the coordinates of the critical point are distributed respectively along two horizontal straight lines started from z_1 and z_2 on the $\mathbf{W}$ plane, as shown in the first picture of figure 6.

Figure 6. Coefficient a from basic clockwise rotation.

When $i \geq j$, there is no wall-crossing under the transformation T .

$$BJ_{i,j} = q^{-\frac{1}{2}(\lambda^i, \lambda^j)} J_{j,i}.$$

When $i < j$, the variation is moving primary roots $\{\alpha_{i+1}, \alpha_{i+2}, \dots, \alpha_j\}$ from z_2 to z_1 . Let $w_k^a, a = 1, 2$ be the coordinates of α_k . The type of variation can be seen from the deformation of parameter $c \rightarrow +\infty$ from $c = 0$. The critical point equation with two singularities w_i^1 and w_i^2 is as following:

$$\left\{ \begin{array}{l} \frac{1}{w_{i+1} - w_i^1} + \frac{1}{w_{i+1} - w_i^2} = \frac{-1}{w_{i+1} - w_{i+2}} \\ \dots \\ 0 = \frac{-1}{w_k - w_{k-1}} + \frac{-1}{w_k - w_{k+1}} \\ \dots \\ 0 = \frac{-1}{w_j - w_{j-1}} + c, \end{array} \right. \quad (3.9)$$

where $i + 2 \leq k \leq j - 1$.

For $c \neq 0$,

$$\begin{aligned} w_{i+1} &= \frac{2 + cw_i^1 + cw_i^2 \pm \sqrt{4 + c^2(w_i^1 - w_i^2)^2}}{2c}, \\ w_k &= w_{i+1} + \frac{k - i - 1}{c}, \quad k \geq i + 2. \\ \lim_{c \rightarrow 0} w_{i+1} &= \frac{w_i^1 + w_i^2}{2} \quad \text{and} \quad \lim_{c \rightarrow 0} w_k = +\infty, \quad i + 2 \leq k \leq j. \\ \lim_{c \rightarrow +\infty} w_{i+1} &= w_i^1 \quad \text{or} \quad w_i^2. \end{aligned}$$

Note that the process above is independent of the position of w_i^1 and w_i^2 . As shown in figure 7, when $c \rightarrow +\infty$, the coordinates of the critical point at $c = 0$ are continuously moving to w_i^1 or w_i^2 within the same horizontal line on $\mathbf{W}$ plane. In the process of continuous transformation $T(s)$, when $s = 1/2$, the imaginary parts of two critical values equals. Thus there will be two thimbles connecting w_i^2 to w_i^1 from above and below z_1 and they are homotopically equivalent to the thimbles at $c = 0$ with $Imw_i^1 < Imw_i^2$ and $Imw_i^1 > Imw_i^2$ respectively. The variation of $\{w_{i+1}, \dots, w_j\}$ is just a translation and thus a type II variation. As shown in figure 8, the homotopic class of $BJ_{i,j}$ is equivalent to

Figure 7. Variation of critical points as $c \rightarrow +\infty$ from 0.

Figure 8. In the fundamental representation of A_n Lie algebra, wall-crossing coefficient b is different from a by the translation of $\{\alpha_{i+1}, \alpha_{i+2}, \dots, \alpha_j\}$ from z_2 to z_1 and c different from b by anti-clockwise translation around z_1 and $\{\alpha_1, \alpha_2, \dots, \alpha_i\}$.

$\mathbb{Z}[t, t^{-1}]$ -linear combination of three parts

$$BJ_{i,j} = aJ_{j,i} + (c - b)J_{i,j}$$

and the difference of them coming from the translation of the primary roots $\{\alpha_{i+1}, \alpha_{i+2}, \dots, \alpha_j\}$. In fact, under the basic clockwise rotation as shown in figure 6, a is from the phase factor difference of $e^{-\frac{W_c(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, i+j)}{k+h^v}}$ i.e. that of $(z_1 - z_2)^{(\lambda^i, \lambda^j)}$. Comparing with a , b has an additional phase factor $q^{\frac{1}{2}(\lambda^i, \lambda^i - \lambda^j)}$ of translating $\{\alpha_{i+1}, \alpha_{i+2}, \dots, \alpha_j\}$ from z_2 to z_1 . c differs from b by translation of $\{\alpha_{i+1}, \alpha_{i+2}, \dots, \alpha_j\}$ around z_1 and $\{\alpha_1, \alpha_2, \dots, \alpha_i\}$. Thus

$$c = b \cdot q^{-(\lambda^i, \lambda^i - \lambda^j)}.$$

Therefore, we get the following wall-crossing formula for V_{ω_1} :

$$BJ_{i,j} = \begin{cases} q^{-\frac{1}{2}(\lambda^i, \lambda^j)} J_{j,i}, & i \geq j; \\ q^{-\frac{1}{2}(\lambda^i, \lambda^j)} (J_{j,i} + q^{\frac{1}{2}(\lambda^i, \lambda^i - \lambda^j)} (q^{-(\lambda^i, \lambda^i - \lambda^j)} - 1) J_{i,j}), & j > i. \end{cases}$$

3.2.2 B_n

Denote the weights of the fundamental representation of B_n Lie algebra by

$$\lambda^i = \begin{cases} \omega_1 - \sum_{j=1}^i \alpha_j, & i \leq n; \\ \omega_1 - \sum_{j=1}^n \alpha_j - \sum_{j=2n+1-i}^n \alpha_j, & n < i \leq 2n. \end{cases}$$

Their inner products are:

$$(\lambda^s, \lambda^t) = \begin{cases} 1, & s+t \neq 2n, s=t; \\ 0, & s+t \neq 2n, s \neq t; \\ 0, & s+t = 2n, s=t; \\ -1, & s+t = 2n, s \neq t, \end{cases} \quad (3.10)$$

where $s, t = 0, 1, \dots, 2n$. For any weight vector $v_{\lambda^l} \in V_{\omega_1}$, the corresponding Yang-Yang function is as following:

$$\begin{aligned} \mathbf{W}_c(\mathbf{w}, z, \omega_1, l) = & \sum_{j=1}^l (\alpha_{i_j}, \omega_1) \log(w_j - z) - \sum_{1 \leq j < k \leq l} (\alpha_{i_j}, \alpha_{i_k}) \log(w_j - w_k) \\ & - c \left(\sum_{j=1}^l (\alpha_{i_j}, \rho) w_j - (\omega_1, \rho) z \right), \end{aligned} \quad (3.11)$$

where

$$i_j = \begin{cases} j, & j \leq n; \\ 2n - j + 1, & n < j \leq 2n. \end{cases}$$

By this notation, when $l \geq n + 1$, $\{w_k, w_{2n+1-k}\}_{2n+1-l \leq k \leq n}$ are pairs of symmetric coordinates of α_k in the function. Define $\bar{w}_k = w_k + w_{2n+1-k}$ and $\Delta_k = (w_k - w_{2n+1-k})^2$, $2n + 1 - l \leq k \leq n$.

Lemma 4. *For the fundamental representation V_{ω_1} of B_n Lie algebra, the solutions of the critical point equation (2.7) of the corresponding Yang-Yang functions $\mathbf{W}_c(\mathbf{w}, 0, \omega_1, l)$ are as following:*

When $l < n$,

$$w_j = \sum_{i=1}^j \frac{1}{c(l-i+1)} \quad j = 1, \dots, l. \quad (3.12)$$

When $l = n$,

$$w_j = \sum_{i=1}^j \frac{1}{c(l-i+1/2)} \quad j = 1, \dots, l. \quad (3.13)$$

When $l \geq n + 1$,

$$w_k = \sum_{i=1}^k \frac{1}{c(l-i)} \quad (3.14)$$

for $k = 1, \dots, 2n - l$. For $2n + 1 - l \leq k \leq n$,

$$\begin{aligned} \bar{w}_k &= \frac{1}{c(l-r-1)} + \sum_{j=1}^{2n-l} \frac{2}{c(l-j)} + \sum_{j=2n-l+1}^{k-1} \frac{1}{c(l-j-1)} + \sum_{j=2n-l+1}^{2n-k-1} \frac{1}{c(l-j-1)}, \\ \Delta_k &= \left[\sum_{j=k}^{2n-k-1} \frac{1}{c(l-j-1)} \right]^2 - \frac{1}{c^2(l-n-1)^2}. \end{aligned} \quad (3.15)$$

Figure 9. Coordinates distribution of the B_n critical point on $\mathbf{W}_c$ plane near z .

Proof. The explicit solutions of the critical point equation of Yang-Yang function $\mathbf{W}(\mathbf{w}, z, \boldsymbol{\lambda}, l)$ with $z = (0, 1)$, $\boldsymbol{\lambda} = (\lambda, \omega_1)$ are already known in [12]. If we denote them by $w_1(\lambda), \dots, w_l(\lambda)$, then the solutions $\tilde{w}_1, \dots, \tilde{w}_l$ of the critical point equation (2.7) of Yang-Yang functions $\mathbf{W}_c(\mathbf{w}, 0, \omega_1, l)$ are just the limitation of the critical solutions of $\mathbf{W}(\mathbf{w}, z, \boldsymbol{\lambda}, l)$ with the data $z = (0, \mu)$, $\boldsymbol{\lambda} = (\mu c \rho, \omega_1)$:

$$\tilde{w}_j = \lim_{\mu \rightarrow +\infty} \mu(1 - w_j(\mu c \rho)), \quad j = 1, \dots, l.$$

By the limitation, the explicit solutions of equation (2.7) are straightforward. $\square$

By this lemma, the following property is clear.

Lemma 5. $\Delta_k > 0$ for $k = 2n + 1 - l, \dots, n - 1$ and $\Delta_n < 0$. The coordinates of critical solutions satisfy the following order:

When $l \leq n$,

$$0 < w_1 < w_2 < \dots < w_l; \quad (3.16)$$

When $l \geq n + 1$, assume $w_k < w_{2n+1-k}$ for $k = 2n + 1 - l, \dots, n - 1$, then

$$0 < w_1 < \dots < w_{2n-l} < w_{2n+1-l} < \dots < w_{n-1} < \frac{\bar{w}_n}{2} < w_{n+2} < \dots < w_l. \quad (3.17)$$

If the critical points of $\mathbf{W}_c(\mathbf{w}, 0, \omega_1, l)$ are $\mathbf{w}$, then the critical points of $\mathbf{W}_c(\mathbf{w}, z, \omega_1, l)$ are $z + \mathbf{w}$. By the previous lemma, it is obvious that z and $\{w_k\}_{k \neq n, n+1}$ are on the same horizontal line of $\mathbf{W}_c$ plane except w_n and w_{n+1} vertically symmetrical about the center point $\frac{\bar{w}_n}{2}$. In sum, we draw the distribution of the critical point near z corresponding to $v_{\lambda^i} \in V_{\omega_1}$ in different cases in figure 9.

In the following, we derive monodromy representation for $v_{\lambda^i} \otimes v_{\lambda^j} \in V_{\omega_1} \otimes V_{\omega_1}$. it is convenient to consider the case $i + j \neq 2n$ firstly.

i) $i + j \neq 2n \& i \geq j$

There is no wall-crossing,

$$BJ_{i,j} = q^{-\frac{1}{2}(\lambda^i, \lambda^i)} J_{j,i}.$$

Figure 10. Thimble with $c = 0$ and its critical point w_n^1 and w_n^2 .

ii) $i + j \neq 2n \& i < j \& i \neq n$

By the same method used in the case of A_n , the variation is of type *II*.

$$\begin{aligned} \mathbf{B}J_{i,j} &= q^{-\frac{1}{2}(\lambda^i, \lambda^i)}(J_{j,i} + q^{\frac{1}{2}(\lambda^i, \lambda^i - \lambda^j)}(q^{-(\lambda^i, \lambda^i - \lambda^j)} - 1)J_{i,j}) \\ &= J_{j,i} + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})J_{i,j}. \end{aligned} \quad (3.18)$$

iii) $i + j \neq 2n \& i = n \& j = n + 1$

Lemma 6. Let $\bar{w}_i = w_i^1 + w_i^2$, $i = 1, 2$. Assume that $w_1^1 \neq w_1^2$,

$$\frac{2w_1^1 - \bar{w}_2}{(w_1^1)^2 - \bar{w}_2 w_1^1 + w_2^1 w_2^2} = A, \quad \frac{2w_1^2 - \bar{w}_2}{(w_1^2)^2 - \bar{w}_2 w_1^2 + w_2^1 w_2^2} = -A$$

and

$$A(w_1^1 - w_1^2) \neq 2.$$

Then

$$\bar{w}_2 = \bar{w}_1, \quad w_2^1 w_2^2 = w_1^1 w_1^2 + A^{-1}(w_1^1 - w_1^2).$$

Proof. The equations imply

$$A(w_1^1 - w_1^2)(\bar{w}_1 - \bar{w}_2) = 2(\bar{w}_1 - \bar{w}_2).$$

Then $A(w_1^1 - w_1^2) \neq 2$ implies

$$\bar{w}_2 = \bar{w}_1, \quad w_2^1 w_2^2 = w_1^1 w_1^2 + A^{-1}(w_1^1 - w_1^2).$$

□

By this lemma, the following critical point equation with $c = 0$ can be solved:

$$\begin{cases} \frac{1}{w_n^1 - w_{n-1}^1} + \frac{1}{w_n^1 - w_{n-1}^2} = \frac{1}{w_n^1 - w_n^2} \\ \frac{1}{w_n^2 - w_{n-1}^1} + \frac{1}{w_n^2 - w_{n-1}^2} = \frac{1}{w_n^2 - w_n^1}, \end{cases} \quad (3.19)$$

where w_n^1, w_n^2 are coordinates of α_n and w_{n-1}^1, w_{n-1}^2 coordinates of α_{n-1} . The solution is $\bar{w}_n = \bar{w}_{n-1}$, $w_n^1 w_n^2 = \frac{(\bar{w}_{n-1})^2 - w_{n-1}^1 w_{n-1}^2}{3}$. If $\text{Re} w_{n-1}^1 = \text{Re} w_{n-1}^2$, then $\Delta_n > 0$. The thimble

Figure 11. Homotopic class of the thimble after rotation.

Figure 12. Type *I* wall-crossing of B_n .

and its critical point are shown in figure 10. When $c \rightarrow +\infty$, the condition $i = n \& j = n + 1$ corresponds to the case of w_n^1, w_n^2 tending to w_{n-1}^2 . After rotation of w_{n-1}^1 and w_{n-1}^2 , the homotopic class of the thimble connecting w_{n-1}^2 with ∞ is shown in figure 11. Therefore, as is shown in figure 12, the variation here is of type *I* with only one primary root α_n moving.

$$BJ_{i,j} = aJ_{j,i} + (c_1 + c_2 - b_1 - b_2)J_{i,j}, \quad (3.20)$$

where

$$b_1 + b_2 = a \cdot q^{\frac{1}{2}(\lambda^{n-1}, \alpha_n)} (1 + q^{-\frac{1}{2}(\alpha_n, \alpha_n)})$$

is the sum of phase factors of moving one of two α_n to the right of z_1 and

$$c_1 + c_2 = (b_1 + b_2) \cdot q^{-(\lambda^{n-1}, \alpha_n) + \frac{1}{2}(\alpha_n, \alpha_n)}$$

from anti-clockwise rotation of α_n , 2π around $\{z_1, \alpha_1, \dots, \alpha_{n-1}\}$ and π around α_n . Minus before b_1 and b_2 is from the reverse of the direction of the dimension one sub-thimble connecting z_2 to z_1 . Thus,

$$BJ_{i,j} = J_{j,i} + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})J_{i,j}. \quad (3.21)$$

iv) $i + j \neq 2n \& i = n \& j > n + 1$

The variation is combination of type *I* and *II*. As shown in figure 13, similar to the previous case, there are two possible way of moving α_n from z_2 to z_1 , but α_n is now

Figure 13. Combination of type *I* and *II*.

accompanied by $\{\alpha_{n-1}, \dots, \alpha_{2n+1-j}\}$ horizontally.

$$\begin{aligned}
 a &= q^{-\frac{1}{2}(\lambda^i, \lambda^i)}; \\
 b_1 + b_2 &= a \cdot q^{\frac{1}{2}(\lambda^{n-1}, \alpha_n) + \frac{1}{2}(\lambda^n, \lambda^{n+1} - \lambda^j)} (1 + q^{-\frac{1}{2}(\alpha_n, \alpha_n)}); \\
 c_1 + c_2 &= (b_1 + b_2) \cdot q^{-(\lambda^{n-1}, \alpha_n) + \frac{1}{2}(\alpha_n, \alpha_n) - (\lambda^n, \lambda^{n+1} - \lambda^j)}; \\
 \mathbf{B}J_{i,j} &= aJ_{j,i} + (c_1 + c_2 - b_1 - b_2)J_{i,j} \\
 &= J_{j,i} + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})J_{i,j}.
 \end{aligned} \tag{3.22}$$

Minus is always from reverse of orientation of the dimension one sub-thimble.

In sum, when $i + j \neq 2n$,

$$\mathbf{B}J_{i,j} = \begin{cases} q^{-\frac{1}{2}}J_{j,i}, & i + j \neq 2n \& i = j; \\ J_{j,i}, & i + j \neq 2n \& i > j; \\ J_{j,i} + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})J_{i,j}, & i + j \neq 2n \& i < j. \end{cases} \tag{3.23}$$

When $i + j = 2n$, the situation is more interesting. The critical point equation of two singularities z_1 and z_2 with $c = 0$ is as following:

$$\left\{ \begin{aligned} 0 &= \frac{-1}{w_1^1 - z_1} + \frac{-1}{w_1^1 - z_2} + \frac{2}{w_1^1 - w_1^2} + \frac{-1}{w_1^1 - w_2^1} + \frac{-1}{w_1^1 - w_2^2} \\ 0 &= \frac{-1}{w_1^2 - z_1} + \frac{-1}{w_1^2 - z_2} + \frac{2}{w_1^2 - w_1^1} + \frac{-1}{w_1^2 - w_2^1} + \frac{-1}{w_1^2 - w_2^2} \\ &\dots \\ 0 &= \frac{2}{w_k^1 - w_k^2} + \frac{-1}{w_k^1 - w_{k-1}^1} + \frac{-1}{w_k^1 - w_{k-1}^2} + \frac{-1}{w_k^1 - w_{k+1}^1} + \frac{-1}{w_k^1 - w_{k+1}^2} \\ 0 &= \frac{2}{w_k^2 - w_k^1} + \frac{-1}{w_k^2 - w_{k-1}^1} + \frac{-1}{w_k^2 - w_{k-1}^2} + \frac{-1}{w_k^2 - w_{k+1}^1} + \frac{-1}{w_k^2 - w_{k+1}^2} \\ &\dots \\ 0 &= \frac{1}{w_n^1 - w_n^2} + \frac{-1}{w_n^1 - w_{n-1}^1} + \frac{-1}{w_n^1 - w_{n-1}^2} \\ 0 &= \frac{1}{w_n^2 - w_n^1} + \frac{-1}{w_n^2 - w_{n-1}^1} + \frac{-1}{w_n^2 - w_{n-1}^2}, \end{aligned} \right. \tag{3.24}$$

where $2 \leq k \leq n-1$ and w_i^1, w_i^2 are coordinates of α_i , $1 \leq i \leq n$. Summing up all the equations except the first pair gives

$$\frac{1}{w_1^1 - w_2^1} + \frac{1}{w_1^1 - w_2^2} + \frac{1}{w_1^2 - w_2^1} + \frac{1}{w_1^2 - w_2^2} = 0.$$

Let $A = +\frac{2}{w_1^1 - w_2^1} + \frac{-1}{w_1^1 - w_2^2} + \frac{-1}{w_1^2 - w_2^1}$, then $\frac{2}{w_1^2 - w_2^1} + \frac{-1}{w_1^2 - w_2^2} + \frac{-1}{w_1^1 - w_2^2} = -A$. By using lemma 6 inductively, $\bar{w}_1 = \dots = \bar{w}_n = z_1 + z_2$. Then substituting $\bar{w}_l$ into (3.24), we have the following solution:

$$\bar{w}_1 = \dots = \bar{w}_n = z_1 + z_2, \quad w_l^1 w_l^2 = z_1 z_2 + \frac{(z_1 - z_2)^2 l(2n-l)}{4n^2 - 1}, \quad 1 \leq l \leq n.$$

It is clear that $\Delta_k < 0$, $k = 1, \dots, n-1$ and $\Delta_n > 0$. The distribution of the coordinates of critical point is shown in figure 19. Similar to the case of one singularity in figure 9, the coordinates are symmetric on the same line connecting z_1 and z_2 except w_n^1 and w_n^2 vertically in the middle. Assume $Im w_i^1 > Im w_i^2$, $i = 1, \dots, n-1$. The different imaginary parts of $\{w_i^1, w_i^2\}$ and z_1, z_2 give the coordinates a partial order:

$$z_1 < w_1^1 < \dots < w_{n-1}^1 < w_n^1, w_n^2 < w_{n-1}^2 < \dots < w_1^2 < z_2.$$

Because of the existence of the thimble above, when $c \rightarrow +\infty$, there is a wall-crossing for every $i \neq 0$ during the transformation T and its homotopic class will keep this order.

To be precise, we redefine the index i and j . Let i be the number of primary roots near z_2 and j be the number of primary roots crossing wall. Assume that

$$B J_{a,b} = \sum_{c,d} B_{a,b}^{c,d} J_{c,d}, \quad (3.25)$$

then from the properties of wall-crossing,

$$B J_{2n-i,i} = \sum_{j=0}^i B_{2n-i,i}^{i-j, 2n-i+j} J_{i-j, 2n-i+j}. \quad (3.26)$$

Phase factor from the rotation gives

$$B_{2n-i,i}^{i, 2n-i} = q^{-\frac{1}{2}(\lambda^i, \lambda^{2n-i})}.$$

The coefficients $B_{2n-i,i}^{i-j, 2n-i+j}$ ($j > 0$) of wall-crossing formula can be computed as in the following cases.

i) $0 < j \leq i \leq n-1$

Keeping the order of the coordinates, moving of $\{\alpha_{i-j+1}, \dots, \alpha_i\}$ from z_2 to z_1 is accompanied by the self clockwise rotation of π , thus the variation is of type III.

$$B J_{2n-i,i} = a J_{i, 2n-i} + ((-1)^j b + (-1)^{j+1} c) J_{i-j, 2n-i+j} + \dots \quad (3.27)$$

$a = q^{-\frac{1}{2}(\lambda^i, \lambda^{2n-i})}$ is from the rotation of z_1 and z_2 .

$$b = a \cdot q^{\frac{1}{4}[(\lambda^{i-j}, \lambda^{i-j} - \lambda^i) + (\lambda^{2n-i}, \lambda^{i-j} - \lambda^i)] + \frac{1}{2}(j-1)}$$

is from moving of $\{\alpha_{i-j+1}, \dots, \alpha_i\}$ from z_2 to z_1 . $c = b \cdot q^{-(\lambda^{2n-i}, \lambda^{i-j} - \lambda^i)}$ is from the anti-clockwise rotation around z_1 and $2n-i$ primary roots near z_1 .

Thus,

$$B_{2n-i,i}^{i-j, 2n-i+j} = (-1)^j (b - c) = (-1)^j q^{\frac{j}{2}} (q^{\frac{1}{2}} - q^{-\frac{1}{2}}). \quad (3.28)$$

Figure 14. $0 < j \leq i = n$.

ii) $0 < j \leq i = n$

$$BJ_{n,n} = aJ_{n,n} + ((-1)^j b + (-1)^{j+1} c)J_{n-j,n+j} + \dots \quad (3.29)$$

$a = q^{-\frac{1}{2}(\lambda^n, \lambda^{2n-n})}$ is from the rotation of z_1 and z_2 .

$$b = a \cdot q^{\frac{1}{4}[(\lambda^{n-j}, \lambda^{n-j} - \lambda^n) + (\lambda^{2n-n}, \lambda^{n-j} - \lambda^n)] + \frac{1}{4}(\alpha_n, \alpha_n) + \frac{1}{2}(j-1)}$$

is from moving of $\{\alpha_{n-j+1}, \dots, \alpha_n\}$ from z_2 to the position indicated by the green dotted circles as shown in figure 14. Additional anti-clockwise rotation to the position indicated by the red dotted circles gives

$$c = b \cdot q^{-(\lambda^{2n-n}, \lambda^{n-j} - \lambda^n) - \frac{1}{2}(\alpha_n, \alpha_n)}.$$

This variation is different from type *III* by the position of two α_n . The variation of α_n is of type *I*. In this sense, we call the variation a combination of type *I* and *III*. Thus,

$$B_{n,n}^{n-j,n+j} = (-1)^j (b - c) = (-1)^j q^{\frac{j}{2}} (1 - q^{-\frac{1}{2}}). \quad (3.30)$$

iii) $i > n \& i - j = n$ As shown in figure 15, because of two α_n , there are two possible cases of moving. They give coefficients b_1 and b_2 .

$$BJ_{2n-i,i} = aJ_{i,2n-i} + ((-1)^j (b_1 + b_2) + (-1)^{j+1} (c_1 + c_2))J_{n,n} + \dots \quad (3.31)$$

$b_1 + b_2 = a \cdot q^{\frac{1}{4}[(\lambda^{i-j}, \lambda^{i-j} - \lambda^i) + (\lambda^{2n-i}, \lambda^{i-j} - \lambda^i)] + \frac{1}{2}(j-1)} (q^{\frac{1}{4}(\alpha_n, \alpha_n)} + q^{-\frac{1}{4}(\alpha_n, \alpha_n)})$. Additional anti-clockwise rotation around z_1 and other $2n - i$ primary roots gives $c_1 + c_2 = (b_1 + b_2) \cdot q^{-(\lambda^{2n-i}, \lambda^{i-j} - \lambda^i)}$. The variation is also a combination of type *I* and *III*. Thus,

$$B_{2n-i,i}^{n,n} = (-1)^j q^{\frac{j}{2} + \frac{1}{4}} (q^{\frac{1}{4}} + q^{-\frac{1}{4}}) (1 - q^{-1}). \quad (3.32)$$

iv) $i > n \& i - j > n$

The variation is of type *III*. $B_{2n-i,i}^{i-j,2n-i+j}$ is exactly the same as (3.28).

$$B_{2n-i,i}^{i-j,2n-i+j} = (-1)^j q^{\frac{j}{2}} (q^{\frac{1}{2}} - q^{-\frac{1}{2}}).$$

Figure 15. $i > n$ & $i - j = n$.

v) $i > n$ & $i - j < n$ & $i - j \neq 2n - i$

The variation is of type *III*, $b = a \cdot q^{\frac{1}{4}[(\lambda^{i-j}, \lambda^{i-j} - \lambda^i) + (\lambda^{2n-i}, \lambda^{i-j} - \lambda^i)] + \frac{1}{2}(j-2)}$ and $c = b \cdot q^{-(\lambda^{2n-i}, \lambda^{i-j} - \lambda^i)}$.

$$B_{2n-i,i}^{i-j, 2n-i+j} = (-1)^j q^{\frac{j}{2}} (1 - q^{-1}). \quad (3.33)$$

vi) $i > n$ & $i - j = 2n - i$ & $i \neq n + 1$ The variation is of type *IV*. $i - j = 2n - i$ means that the moving primary roots are of double multiplicity and symmetrically distributed. Same as type *III*, moving of them from z_2 to z_1 is accompanied by the self clockwise rotation of π . But because of double multiplicity and symmetric distribution, the order of these primary roots will give three additional terms in the wall-crossing formula, as is shown in figure 5.

$$\begin{aligned} BJ_{2n-i,i} &= aJ_{i,2n-i} + ((-1)^j b + (-1)^{j+1} c_1 - c_2 + c_3) J_{2n-i,i} + \dots; \\ b &= a \cdot q^{\frac{1}{4}[(\lambda^{i-j}, \lambda^{i-j} - \lambda^i) + (\lambda^{2n-i}, \lambda^{i-j} - \lambda^i)] + n - i + j - \frac{3}{2}}; \\ c_1 &= b \cdot q^{-(\lambda^{2n-i}, \lambda^{2n-i} - \lambda^{2n-i+1})}; \\ c_2 &= c_3 \cdot q^{-(\lambda^{2n-i}, \lambda^{2n-i} - \lambda^{2n-i+1})}; \\ c_3 &= a \cdot q^{\frac{1}{4}[(\lambda^{i-j}, \lambda^{i-j} - \lambda^i) + (\lambda^{2n-i}, \lambda^{i-j} - \lambda^i)] - (\lambda^{2n-i}, \lambda^{i-j} - \lambda^i)}; \\ B_{2n-i,i}^{2n-i,i} &= ((-1)^j q^{n-i+j-\frac{1}{2}} - 1)(q^{\frac{1}{2}} - q^{-\frac{1}{2}}). \end{aligned} \quad (3.34)$$

vii) $i = n + 1$ & $j = 2$

The variation is of type *I*.

$$\begin{aligned} BJ_{n-1,n+1} &= aJ_{n+1,n-1} + ((-1)^2 b + (-1)^3 (c_1 + c_2) + (-1)^4 c_3) J_{n-1,n+1} + \dots; \\ b &= a \cdot q^{\frac{1}{4}[(\lambda^{i-j}, \lambda^{i-j} - \lambda^i) + (\lambda^{2n-i}, \lambda^{i-j} - \lambda^i)] - \frac{1}{2}}; \\ c_1 + c_2 &= b \cdot q^{-(\lambda^{2n-i}, \lambda^{2n-i} - \lambda^{2n-i+1})} (1 + q^{\frac{1}{2}(\alpha_n, \alpha_n)}); \\ c_3 &= b \cdot q^{-2(\lambda^{2n-i}, \lambda^{2n-i} - \lambda^{2n-i+1}) + \frac{1}{2}(\alpha_n, \alpha_n)}; \\ B_{n-1,n+1}^{n-1,n+1} &= q - 1 - q^{\frac{1}{2}} + q^{-\frac{1}{2}}. \end{aligned} \quad (3.35)$$

In summarization,

$$\begin{aligned}
 & \mathbf{B}_{2n-i,i}^{i-j,2n-i+j} \\
 = & \begin{cases} (-1)^j q^{\frac{j}{2}} (q^{\frac{1}{2}} - q^{-\frac{1}{2}}), & i > n \& i - j > n; \\ & 0 < j \leq i \leq n - 1; \\ (-1)^j q^{\frac{j}{2}} (1 - q^{-\frac{1}{2}}), & 0 < j \leq i = n; \\ (-1)^j q^{\frac{j}{2} + \frac{1}{4}} (q^{\frac{1}{4}} + q^{-\frac{1}{4}}) (1 - q^{-1}), & i > n \& i - j = n; \\ (-1)^j q^{\frac{j-1}{2}} (q^{\frac{1}{2}} - q^{-\frac{1}{2}}), & i > n \& i - j < n \& i - j \neq 2n - i; \\ (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) ((-1)^j q^{n-i+j-\frac{1}{2}} - 1), & i > n \& i - j = 2n - i. \end{cases} \quad (3.36)
 \end{aligned}$$

Monodromy representation is as following:

$$\begin{aligned}
 & \mathbf{B}_{2n-a,a}^{2n-b,b} \\
 = & \begin{cases} (-1)^{a+n} q^{\frac{a-n}{2}} (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) (1 + q^{-\frac{1}{2}}), & a > n \& b = n; \\ (-1)^{a+b} (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) q^{-n + \frac{a+b}{2}}, & a < n \& b > n \text{ or } a > n \& b < n; \\ (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) ((-1)^{a+b} q^{-n + \frac{a+b-1}{2}} - \delta_{a,b}), & a > n \& b > n; \\ (-1)^{n+b} q^{\frac{b-n}{2}} (1 - q^{-\frac{1}{2}}), & a = n \& b > n. \end{cases} \quad (3.37)
 \end{aligned}$$

We have given the description for four types of variations in detail. In the following cases of C_n and D_n , the methods for proofs are similar, so we omit them except for some extraordinary cases.

3.2.3 C_n

Denote the weights of the fundamental representation of C_n Lie algebra by

$$\lambda^i = \begin{cases} \lambda - \sum_{j=1}^i \alpha_j, & i \leq n; \\ \lambda - \sum_{j=1}^n \alpha_j - \sum_{j=2n-i}^n \alpha_j, & i > n. \end{cases}$$

The inner products of them are:

$$(\lambda^s, \lambda^t) = \begin{cases} \frac{1}{2}, & s = t; \\ 0, & s + t \neq 2n - 1, s \neq t; \\ -\frac{1}{2}, & s + t = 2n - 1, \end{cases} \quad (3.38)$$

where $s, t = 0, 1, \dots, 2n - 1$.

For any weight vector $v_{\lambda^l} \in V_{\omega_1}$, Yang-Yang function $\mathbf{W}_c(\mathbf{w}, z, \omega_1, l)$ corresponding to it is as following:

$$\begin{aligned}
 \mathbf{W}_c(\mathbf{w}, z, \omega_1, l) = & \sum_{j=1}^l (\alpha_{i_j}, \omega_1) \log(w_j - z) - \sum_{1 \leq j < k \leq l} (\alpha_{i_j}, \alpha_{i_k}) \log(w_j - w_k) \\
 & - c \left(\sum_{j=1}^l (\alpha_{i_j}, \rho) w_j - (\omega_1, \rho) z \right), \quad (3.39)
 \end{aligned}$$

where

$$i_j = \begin{cases} j, & j \leq n; \\ 2n - j, & n < j \leq 2n - 1. \end{cases}$$

By this notation, when $l \geq n+1$, $\{w_k, w_{2n-k}\}_{2n-l \leq k \leq n-1}$ are pairs of symmetric coordinates of α_k in the function. Define $\bar{w}_k = w_k + w_{2n-k}$ and $\Delta_k = (w_k - w_{2n-k})^2$, $2n-l \leq k \leq n-1$.

Lemma 7. *For the fundamental representation V_{ω_1} of C_n Lie algebra, the solutions of the critical point equation (2.7) of the corresponding Yang-Yang functions $\mathbf{W}_c(\mathbf{w}, 0, \omega_1, l)$ are as following:*

When $l < n$,

$$w_j = \sum_{i=1}^j \frac{1}{c(l-i+1)} \quad j = 1, \dots, l.$$

When $l = n$,

$$w_j = \sum_{i=1}^j \frac{1}{c(n-i+2)} \quad j = 1, \dots, n-1,$$

$$w_n = \frac{1}{c} + \sum_{i=1}^{n-1} \frac{1}{c(n-i+2)}.$$

When $l \geq n+1$,

$$w_k = \sum_{i=1}^j \frac{1}{c(l+2-i)}$$

for $k = 1, \dots, 2n-l-1$,

$$w_n = \sum_{i=1}^{2n-l-1} \frac{1}{c(l+2-i)} + \sum_{i=2n-l}^n \frac{1}{c(l+1-i)}.$$

For $2n-l \leq k \leq n-1$,

$$\begin{aligned} \bar{w}_k = & \sum_{i=1}^{2n-l-1} \frac{2}{c(l-i+2)} + \sum_{i=2n-l}^{k-l} \frac{2}{c(l-i+1)} + \sum_{i=k}^n \frac{1}{c(l-i+1)} \\ & + \sum_{i=k}^n \frac{1}{c(l-2n+i+1)}, \end{aligned} \quad (3.40)$$

$$\begin{aligned} \Delta_k = & \left[\sum_{i=k}^n \frac{1}{c(l-i+1)} + \sum_{i=k}^n \frac{1}{c(l-2n+i+1)} \right] \\ & \times \left[\sum_{i=k}^n \frac{1}{c(l-i+1)} + \sum_{i=k}^n \frac{1}{c(l-2n+i+1)} - \frac{4}{c(2l-2n+1)} \right]. \end{aligned} \quad (3.41)$$

Lemma 8. *For any $2n-l \leq k \leq n-1$, $\Delta_k > 0$. Assume $w_k < w_{2n-k}$ for $k = 2n-l, \dots, n-1$, then for any $l > 0$, the coordinates of critical solutions satisfy the following order:*

$$0 < w_1 < w_2 < \dots < w_l.$$

By this lemma, z and the critical coordinates $\{w_j\}_{j=1, \dots, l}$ of $\mathbf{W}_c(\mathbf{w}, z, \omega_1, l)$ are on the same horizontal line of $\mathbf{W}$ plane as shown in figure 16.

To derive monodromy representation, firstly we consider the case $v_{\lambda_i} \otimes v_{\lambda_j} \in V_{\omega_1} \otimes V_{\omega_1}$, $i+j \neq 2n-1$.

Figure 16. Coordinates distribution of the C_n critical point on $\mathbf{W}_c(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, l)$ plane near z .

i) When $i + j \neq 2n - 1$ & $i \geq j$, there is no wall-crossing,

$$\mathbf{B}J_{i,j} = q^{-\frac{1}{2}(\lambda^i, \lambda^j)} J_{j,i}.$$

ii) When $i + j \neq 2n - 1$ & $i < j$, the variation of critical point in wall-crossing is of type II .

$$\begin{aligned} \mathbf{B}J_{i,j} &= q^{-\frac{1}{2}(\lambda^i, \lambda^j)} (J_{j,i} + q^{\frac{1}{2}(\lambda^i, \lambda^i - \lambda^j)} (q^{-(\lambda^i, \lambda^i - \lambda^j)} - 1) J_{i,j}) \\ &= J_{j,i} + (q^{-\frac{1}{2}} - q^{\frac{1}{2}}) J_{i,j}. \end{aligned} \quad (3.42)$$

In sum,

$$\mathbf{B}J_{i,j} = \begin{cases} q^{-\frac{1}{4}} J_{j,i}, & i + j \neq 2n - 1 \text{ \& } i = j; \\ J_{j,i}, & i + j \neq 2n - 1 \text{ \& } i > j; \\ J_{j,i} + (q^{-\frac{1}{4}} - q^{\frac{1}{4}}) J_{i,j}, & i + j \neq 2n - 1 \text{ \& } i < j. \end{cases} \quad (3.43)$$

When $i + j = 2n - 1$, the critical point equation of two singularities z_1 and z_2 with $c = 0$ is as following:

$$\left\{ \begin{aligned} 0 &= \frac{-\frac{1}{2}}{w_1^1 - z_1} + \frac{-\frac{1}{2}}{w_1^1 - z_2} + \frac{1}{w_1^1 - w_1^2} + \frac{-\frac{1}{2}}{w_1^1 - w_2^1} + \frac{-\frac{1}{2}}{w_1^1 - w_2^2} \\ 0 &= \frac{-\frac{1}{2}}{w_1^2 - z_1} + \frac{-\frac{1}{2}}{w_1^2 - z_2} + \frac{1}{w_1^2 - w_1^1} + \frac{-\frac{1}{2}}{w_1^2 - w_2^1} + \frac{-\frac{1}{2}}{w_1^2 - w_2^2} \\ &\dots \\ 0 &= \frac{1}{w_k^1 - w_k^2} + \frac{-\frac{1}{2}}{w_k^1 - w_{k-1}^1} + \frac{-\frac{1}{2}}{w_k^1 - w_{k-1}^2} + \frac{-\frac{1}{2}}{w_k^1 - w_{k+1}^1} + \frac{-\frac{1}{2}}{w_k^1 - w_{k+1}^2} \\ 0 &= \frac{1}{w_k^2 - w_k^1} + \frac{-\frac{1}{2}}{w_k^2 - w_{k-1}^1} + \frac{-\frac{1}{2}}{w_k^2 - w_{k-1}^2} + \frac{-\frac{1}{2}}{w_k^2 - w_{k+1}^1} + \frac{-\frac{1}{2}}{w_k^2 - w_{k+1}^2} \\ &\dots \\ 0 &= \frac{1}{w_{n-1}^1 - w_{n-1}^2} + \frac{-\frac{1}{2}}{w_{n-1}^1 - w_{n-2}^1} + \frac{-\frac{1}{2}}{w_{n-1}^1 - w_{n-2}^2} + \frac{-1}{w_{n-1}^1 - w_n} \\ 0 &= \frac{1}{w_{n-1}^2 - w_{n-1}^1} + \frac{-\frac{1}{2}}{w_{n-1}^2 - w_{n-2}^1} + \frac{-\frac{1}{2}}{w_{n-1}^2 - w_{n-2}^2} + \frac{-1}{w_{n-1}^2 - w_n} \\ 0 &= \frac{-1}{w_n - w_{n-1}^1} + \frac{-1}{w_n - w_{n-1}^2}, \end{aligned} \right. \quad (3.44)$$

where $2 \leq k \leq n - 2$. By lemma 6, its solution is as following:

$$\bar{w}_1 = \bar{w}_k = \bar{w}_{n-1} = z_1 + z_2, \quad \bar{w}_n = \frac{z_1 + z_2}{2}$$

and

$$w_l^1 w_l^2 = z_1 z_2 + \frac{(z_1 - z_2)^2 l(2n + 1 - l)}{4n(n + 1)}, \quad 1 \leq l \leq n - 1.$$

The distribution of the coordinates of critical point is shown in figure 20. Similar to the case of one singularity in figure 16, the coordinates are symmetric on the same line connecting z_1 and z_2 , which gives coordinates of the thimble and singularities z_1 and z_2 an order:

$$z_1 < w_1^1 < \dots < w_{n-1}^1 < w_n < w_{n-1}^2 < \dots < w_1^2 < z_2.$$

Because of the existence of the thimble above, there is a wall-crossing whenever $i \neq 0$ and its homotopic class will keep this order.

For $v_{\lambda^i} \otimes v_{\lambda^j} \in V_{\omega_1} \otimes V_{\omega_1}$, $i + j = 2n - 1$, the coefficients of equation (3.25) are

$$B_{2n-1-i,i}^{i,2n-1-i} = q^{-\frac{1}{2}(\lambda^{2n-1-i}, \lambda^i)} = q^{\frac{1}{4}}$$

and $B_{2n-1-i,i}^{i-j,2n-1-i+j}$ ($j > 0$) can be computed as in the following cases.

i) $0 < j \leq i < n$

The variation is of type III.

$$\begin{aligned} b &= a \cdot q^{\frac{1}{4}[(\lambda^{i-j}, \lambda^{i-j} - \lambda^i) + (\lambda^{2n-1-i}, \lambda^{i-j} - \lambda^i)] + \frac{1}{4}(j-1)}, \\ c &= b \cdot q^{-(\lambda^{2n-1-i}, \lambda^{i-j} - \lambda^i)}, \\ B_{2n-1-i,i}^{i-j,2n-1-i+j} &= (-1)^j q^{\frac{j+1}{4}} (1 - q^{-\frac{1}{2}}). \end{aligned} \quad (3.45)$$

ii) $1 < j \leq i = n$

The variation is of type III.

$$\begin{aligned} b &= a \cdot q^{\frac{1}{4}[(\lambda^{i-j}, \lambda^{i-j} - \lambda^i) + (\lambda^{2n-1-i}, \lambda^{i-j} - \lambda^i)] + \frac{1}{4}j}, \\ c &= b \cdot q^{-(\lambda^{2n-1-i}, \lambda^{i-j} - \lambda^i)}, \\ B_{2n-1-i,i}^{i-j,2n-1-i+j} &= (-1)^j q^{\frac{j+2}{4}} (1 - q^{-\frac{1}{2}}). \end{aligned} \quad (3.46)$$

iii) $i = n \& j = 1$

The variation is of type III.

$$\begin{aligned} b &= a \cdot q^{\frac{1}{4}[(\lambda^{i-j}, \lambda^{i-j} - \lambda^i) + (\lambda^{2n-1-i}, \lambda^{i-j} - \lambda^i)]}, \\ c &= b \cdot q^{-(\lambda^{n-1}, \lambda^{n-1} - \lambda^n)}, \\ B_{n-1,n}^{n-1,n} &= q^{\frac{3}{4}} (-1 + q^{-1}). \end{aligned} \quad (3.47)$$

iv) $n < i \leq 2n - 1 \& i - j \geq n$

Same as (3.45).

v) $n < i \leq 2n - 1 \& i - j = n - 1$

Same as (3.46).

vi) $i > n \& i - j < n - 1 \& i - j \neq 2n - 1 - i$

The variation is of type *III*.

$$\begin{aligned} b &= a \cdot q^{\frac{1}{4}[(\lambda^{i-j}, \lambda^{i-j} - \lambda^i) + (\lambda^{2n-1-i}, \lambda^{i-j} - \lambda^i)] + \frac{1}{4}j}, \\ c &= b \cdot q^{-(\lambda^{2n-1-i}, \lambda^{i-j} - \lambda^i)}, \\ B_{2n-1-i, i}^{i-j, 2n-1-i+j} &= (-1)^j q^{\frac{j+2}{4}} (1 - q^{-\frac{1}{2}}). \end{aligned} \quad (3.48)$$

vii) $i > n \& i - j < n - 1 \& i - j = 2n - 1 - i$

The variation is of type *IV*.

$$\begin{aligned} b &= a \cdot q^{\frac{1}{4}[(\lambda^{i-j}, \lambda^{i-j} - \lambda^i) + (\lambda^{2n-1-i}, \lambda^{i-j} - \lambda^i)] + \frac{1}{2}(n-i+j-1)}, \\ c_1 &= b \cdot q^{-(\lambda^{2n-1-i}, \lambda^{i-j} - \lambda^{i-j+1})}, \\ c_2 &= a \cdot q^{\frac{1}{4}[(\lambda^{i-j}, \lambda^{i-j} - \lambda^i) + (\lambda^{2n-1-i}, \lambda^{i-j} - \lambda^i)]} \\ &\quad \cdot q^{-(\lambda^{2n-1-i}, \lambda^{i-j} - \lambda^i) + (\lambda^{2n-1-i}, \lambda^{i-j} - \lambda^{i-j+1})}, \\ c_3 &= a \cdot q^{\frac{1}{4}[(\lambda^{i-j}, \lambda^{i-j} - \lambda^i) + (\lambda^{2n-1-i}, \lambda^{i-j} - \lambda^i)] - (\lambda^{2n-1-i}, \lambda^{i-j} - \lambda^i)}, \\ B_{2n-1-i, i}^{i-j, 2n-1-i+j} &= (-1)^j q^{\frac{1}{2}(n-i+j) + \frac{1}{4}} (1 - q^{-\frac{1}{2}}) + q^{-\frac{1}{4}} (1 - q^{\frac{1}{2}}). \end{aligned} \quad (3.49)$$

In sum,

$$B_{2n-1-i, i}^{i-j, 2n-1-i+j} = \begin{cases} q^{\frac{j+1}{4}} ((-1)^j + (-1)^{j+1} q^{-\frac{1}{2}}), & 0 < j \leq i < n; \\ q^{\frac{3}{4}} (q^{-1} - 1), & 1 = j \leq i = n; \\ q^{\frac{j+2}{4}} ((-1)^j + (-1)^{j+1} q^{-\frac{1}{2}}), & 1 < j \leq i = n; \\ q^{\frac{j+1}{4}} ((-1)^j + (-1)^{j+1} q^{-\frac{1}{2}}), & n < i \leq 2n-1 \& j > 0 \& i - j \geq n; \\ q^{\frac{j+2}{4}} ((-1)^j + (-1)^{j+1} q^{-\frac{1}{2}}), & n < i \leq 2n-1 \& i - j = n-1; \\ q^{\frac{j+2}{4}} ((-1)^j + (-1)^{j+1} q^{-\frac{1}{2}}), & n < i \leq 2n-1 \\ & \& 2n-1-i < i-j < n-1; \\ (1 - (-1)^j q^{\frac{n-i+j}{2}})(q^{-\frac{1}{4}} - q^{\frac{1}{4}}), & n < i \leq 2n-1 \\ & \& 2n-1-i = i-j < n-1; \\ q^{\frac{j+2}{4}} ((-1)^j + (-1)^{j+1} q^{-\frac{1}{2}}), & n < i \leq 2n-1 \& i - j < 2n-1-i. \end{cases} \quad (3.50)$$

Monodromy representation is as following:

$$B_{2n-1-a, a}^{2n-1-b, b} = \begin{cases} (-1)^{a+b+1} q^{-\frac{2n-a-b-1}{4}} (q^{\frac{1}{4}} - q^{-\frac{1}{4}}), & a \leq n-1 \& b \geq n; \\ & a \geq n \& b \leq n-1; \\ (-1)^{a+b+1} q^{-\frac{2n-a-b-2}{4}} (q^{\frac{1}{4}} - q^{-\frac{1}{4}}) \\ -\delta_{a,b} (q^{\frac{1}{4}} - q^{-\frac{1}{4}}), & a \geq n \& b \geq n. \end{cases} \quad (3.51)$$

3.2.4 D_n

Denote the weights of the fundamental representation of D_n Lie algebra by

$$\lambda^i = \begin{cases} \lambda - \sum_{j=1}^i \alpha_j, & i \leq n; \\ \lambda - \sum_{j=1}^n \alpha_j - \sum_{j=2n-i}^n \alpha_j, & i > n. \end{cases} \quad (3.52)$$

$$\lambda^{n-1'} = \lambda - \sum_{i=1}^{n-2} \alpha_i - \alpha_n. \quad (3.53)$$

For convenience of discussion, we define the order of $i = 0, 1, \dots, n-2, n-1, n-1', n, \dots, 2n-2$ as following

$$o(i) = \begin{cases} i, & i = 0, 1, \dots, n-2, n-1; \\ n, & i = n-1'; \\ i+1, & i = n, n+1, \dots, 2n-2. \end{cases} \quad (3.54)$$

The inner products of them are:

$$(\lambda^s, \lambda^t) = \begin{cases} 1, & o(s) + o(t) \neq 2n-1, s=t; \\ 0, & o(s) + o(t) \neq 2n-1, s \neq t; \\ -1, & o(s) + o(t) = 2n-1, \end{cases} \quad (3.55)$$

where $s, t = 0, 1, \dots, n-2, n-1, n-1', n, n+1, \dots, 2n-2$.

Every weight vector $v_{\lambda^l} \in V_{\lambda}$ has Yang-Yang function $\mathbf{W}_c(\mathbf{w}, z, \omega_1, l)$ corresponding to it. When $l \neq n-1'$,

$$\begin{aligned} \mathbf{W}_c(\mathbf{w}, z, \omega_1, l) = & \sum_{j=1}^l (\alpha_{i_j}, \omega_1) \log(w_j - z) - \sum_{1 \leq j < k \leq l} (\alpha_{i_j}, \alpha_{i_k}) \log(w_j - w_k) \\ & - c \left(\sum_{j=1}^l (\alpha_{i_j}, \rho) w_j - (\omega_1, \rho) z \right), \end{aligned} \quad (3.56)$$

where

$$i_j = \begin{cases} j, & j \leq n; \\ 2n-1-j, & n+1 \leq j \leq 2n-2. \end{cases}$$

When $l = n-1'$,

$$\begin{aligned} \mathbf{W}_c(\mathbf{w}, z, \omega_1, l) = & \sum_{j=1}^{n-1} (\alpha_{i_j}, \omega_1) \log(w_j - z) - \sum_{j < k \leq n-1} (\alpha_{i_j}, \alpha_{i_k}) \log(w_j - w_k) \\ & - c \left(\sum_{j=1}^{n-1} (\alpha_{i_j}, \rho) w_j - (\omega_1, \rho) z \right), \end{aligned} \quad (3.57)$$

where

$$i_j = \begin{cases} j, & j = 1, \dots, n-2; \\ n, & j = n-1. \end{cases}$$

By this notation, when $l \geq n+1$, $\{w_k, w_{2n-1-k}\}_{2n-1-l \leq k \leq n-2}$ are pairs of symmetric coordinates of α_k in the function. Define $\bar{w}_k = w_k + w_{2n-1-k}$ and $\Delta_k = (w_k - w_{2n-1-k})^2$, $2n-1-l \leq k \leq n-2$.

Lemma 9. For the fundamental representation V_{ω_1} of D_n Lie algebra, the solutions of the critical point equation (2.7) of the corresponding Yang-Yang functions $\mathbf{W}_c(\mathbf{w}, 0, \omega_1, l)$ are as following:

When $l \leq n - 1$,

$$w_j = \sum_{i=1}^j \frac{1}{c(l-i+1)} \quad j = 1, \dots, l;$$

When $l = n - 1'$,

$$w_j = \sum_{i=1}^j \frac{1}{c(n-i)} \quad j = 1, \dots, n-1;$$

When $l = n$,

$$w_j = \sum_{i=1}^j \frac{1}{c(n+1-i)} \quad j = 1, \dots, n-2,$$

and

$$w_{n-1} = w_n = \frac{1}{c} + \sum_{i=1}^{n-2} \frac{1}{c(n+1-i)};$$

When $l \geq n + 1$,

$$w_k = \sum_{i=1}^k \frac{1}{c(l+1-i)}$$

for $k = 1, \dots, 2n - 2 - l$,

$$w_{n-1} = w_n = \sum_{i=1}^{2n-2-l} \frac{1}{c(l+1-i)} + \sum_{i=2n-1-l}^{n-2} \frac{1}{c(l-i)} + \frac{1}{c(l+1-n)}, \quad (3.58)$$

$$\begin{aligned} \bar{w}_k &= \sum_{i=1}^{2n-l-2} \frac{2}{c(l+1-i)} + \sum_{i=2n-1-l}^{k-1} \frac{1}{c(l-i)} + \sum_{i=2n-1-l}^{n-2} \frac{1}{c(l-i)} \\ &+ \frac{2}{c(l-n+1)} + \sum_{i=k}^{n-2} \frac{1}{c(l+i+2-2n)}, \end{aligned} \quad (3.59)$$

and

$$\begin{aligned} \Delta_k &= \left[\sum_{i=k}^{n-2} \frac{1}{c(l-i)} + \frac{2}{c(l-n+1)} + \sum_{i=k}^{n-2} \frac{1}{c(l+i+2-2n)} \right]^2 \\ &+ \sum_{i=k}^{n-2} \frac{1}{c(l-i)} + \frac{2}{c(l-n+1)} + \sum_{i=k}^{n-2} \frac{1}{c(l+i+2-2n)} - \frac{4}{c(2l-2n+1)}, \end{aligned} \quad (3.60)$$

for $2n - 1 - l \leq k \leq n - 2$.

Lemma 10. $\Delta_k > 0$ for $k = 2n - 1 - l, \dots, n - 2$. Assume $w_k < w_{2n-1-k}$, then the coordinates of critical solutions satisfy the following order:

When $l \leq n - 1$,

$$0 < w_1 < w_2 < \dots < w_l;$$

When $l = n - 1'$,

$$0 < w_1 < w_2 < \dots < w_{n-1};$$

When $l \geq n$,

$$0 < w_1 < \dots < w_{n-2} < w_{n-1} = w_n < w_{n+1} < \dots < w_l.$$

By this lemma, z and the critical coordinates $\{w_j\}_{j=1,\dots,l}$ of $\mathbf{W}_c(\mathbf{w}, z, \omega_1, l)$ are on the same horizontal line of $\mathbf{W}$ plane as shown in figure 17.

When $l = 2n - 2$, the critical point equation of two singularities z_1 and z_2 with $c = 0$ is as following:

$$\left\{ \begin{array}{l} 0 = \frac{-1}{w_1^1 - z_1} + \frac{-1}{w_1^1 - z_2} + \frac{2}{w_1^1 - w_1^2} + \frac{-1}{w_1^1 - w_2^1} + \frac{-1}{w_1^1 - w_2^2} \\ 0 = \frac{-1}{w_1^2 - z_1} + \frac{-1}{w_1^2 - z_2} + \frac{2}{w_1^2 - w_1^1} + \frac{-1}{w_1^2 - w_2^1} + \frac{-1}{w_1^2 - w_2^2} \\ \dots \\ 0 = \frac{2}{w_k^1 - w_k^2} + \frac{-1}{w_k^1 - w_{k-1}^1} + \frac{-1}{w_k^1 - w_{k-1}^2} + \frac{-1}{w_k^1 - w_{k+1}^1} + \frac{-1}{w_k^1 - w_{k+1}^2} \\ 0 = \frac{2}{w_k^2 - w_k^1} + \frac{-1}{w_k^2 - w_{k-1}^1} + \frac{-1}{w_k^2 - w_{k-1}^2} + \frac{-1}{w_k^2 - w_{k+1}^1} + \frac{-1}{w_k^2 - w_{k+1}^2} \\ 0 = \frac{2}{w_{n-2}^1 - w_{n-2}^2} + \frac{-1}{w_{n-2}^1 - w_{n-3}^1} + \frac{-1}{w_{n-2}^1 - w_{n-3}^2} \\ \quad + \frac{-1}{w_{n-2}^1 - w_{n-1}^1} + \frac{-1}{w_{n-2}^1 - w_n^1} \\ 0 = \frac{2}{w_{n-2}^2 - w_{n-2}^1} + \frac{-1}{w_{n-2}^2 - w_{n-3}^1} + \frac{-1}{w_{n-2}^2 - w_{n-3}^2} \\ \quad + \frac{-1}{w_{n-2}^2 - w_{n-1}^1} + \frac{-1}{w_{n-2}^2 - w_n^1} \\ 0 = \frac{-1}{w_{n-1} - w_{n-2}^1} + \frac{-1}{w_{n-1} - w_{n-2}^2} \\ 0 = \frac{-1}{w_n - w_{n-2}^1} + \frac{-1}{w_n - w_{n-2}^2}, \end{array} \right. \quad (3.61)$$

where $2 \leq k \leq n - 3$. By lemma 6, the solution is as following:

$$\bar{w}_1 = \bar{w}_k = \bar{w}_{n-2} = z_1 + z_2, w_{n-1} = w_n = \frac{z_1 + z_2}{2}$$

and

$$w_l^1 w_l^2 = z_1 z_2 + \frac{(z_1 - z_2)^2 l(2n - 1 - l)}{4n(n - 1)}, \quad 1 \leq l \leq n - 2.$$

The distribution of the coordinates of critical point is shown in figure 21. Similar to the case of one singularity in figure 17, the coordinates are symmetric on the same line connecting z_1 and z_2 , which gives coordinates of the thimble and singularities z_1 and z_2 a total order:

$$z_1 < w_1^1 < w_k^1 < w_{k+1}^1 < w_n^1 = w_n^2 < w_{k+1}^2 < w_k^2 < w_1^2 < z_2.$$

Figure 17. Coordinates distribution of the D_n critical point on $W_c(w, z, \lambda, l)$ plane near z .

Because of the existence of the thimble above, there is a wall-crossing whenever $o(i) \neq 0$ and its homotopic class will keep this order.

Firstly, we consider the case $v_{\lambda^i} \otimes v_{\lambda^j} \in V_{\omega_1} \otimes V_{\omega_1}$, $o(i) + o(j) \neq 2n - 1$.

- i) $o(i) + o(j) \neq 2n - 1 \& o(i) \geq o(j)$

There is no wall-crossing,

$$BJ_{i,j} = q^{-\frac{1}{2}(\lambda^i, \lambda^i)} J_{j,i}.$$

- ii) $o(i) + o(j) \neq 2n - 1 \& i < j \& i \neq n$

The variation is of type *II*.

$$\begin{aligned}
 BJ_{i,j} &= q^{-\frac{1}{2}(\lambda^i, \lambda^i)} (J_{j,i} + q^{\frac{1}{2}(\lambda^i, \lambda^i - \lambda^j)} (q^{-(\lambda^i, \lambda^i - \lambda^j)} - 1) J_{i,j}) \\
 &= J_{j,i} + (q^{-\frac{1}{2}} - q^{\frac{1}{2}}) J_{i,j}.
 \end{aligned} \tag{3.62}$$

- iii) $o(i) + o(j) \neq 2n - 1 \& i = n \& j = n + 1$

The variation is of type *I*. As is shown in figure 12,

$$b_1 + b_2 = a \cdot q^{\frac{1}{2}(\lambda^{n-1}, \alpha_n)} (1 + q^{-\frac{1}{2}(\alpha_n, \alpha_n)})$$

is the sum of phase factors of moving one of two α_n to the right of z_1 .

$$c_1 + c_2 = (b_1 + b_2) \cdot q^{-(\lambda^{n-1}, \alpha_n) + \frac{1}{2}(\alpha_n, \alpha_n)}$$

is from anti-clockwise rotation of α_n , 2π around $\{z_1, \alpha_1, \dots, \alpha_{n-1}\}$ and π around α_n .

$$\begin{aligned}
 BJ_{i,j} &= aJ_{j,i} + (c_1 + c_2 - b_1 - b_2)J_{i,j} \\
 &= q^{-\frac{1}{2}(\lambda^i, \lambda^i)} (J_{j,i} + q^{\frac{1}{2}(\lambda^{n-1}, \alpha_n)} (1 + q^{-\frac{1}{2}(\alpha_n, \alpha_n)}) (q^{-(\lambda^{n-1}, \alpha_n) + \frac{1}{2}(\alpha_n, \alpha_n)} - 1) J_{i,j}) \\
 &= J_{j,i} + (q^{-\frac{1}{2}} - q^{\frac{1}{2}}) J_{i,j}.
 \end{aligned} \tag{3.63}$$

- iv) $o(i) + o(j) \neq 2n - 1 \& i = n \& j > n + 1$

The variation is a combination of type *I* and *II*. As shown in figure 13, there are two possible ways of moving α_n from z_2 to z_1 . But at this time, α_n is followed by

$$\{\alpha_{n-1}, \dots, \alpha_{2n+1-j}\}.$$

$$\begin{aligned} a &= q^{-\frac{1}{2}(\lambda^i, \lambda^i)}; \\ b_1 + b_2 &= a \cdot q^{\frac{1}{2}(\lambda^{n-1}, \alpha_n) + \frac{1}{2}(\lambda^n, \lambda^{n+1} - \lambda^j)} (1 + q^{-\frac{1}{2}(\alpha_n, \alpha_n)}); \\ c_1 + c_2 &= (b_1 + b_2) \cdot q^{-(\lambda^{n-1}, \alpha_n) + \frac{1}{2}(\alpha_n, \alpha_n) - (\lambda^n, \lambda^{n+1} - \lambda^j)}; \\ \mathbf{B}J_{i,j} &= aJ_{j,i} + (c_1 + c_2 - b_1 - b_2)J_{i,j} \\ &= J_{j,i} + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})J_{i,j}. \end{aligned} \quad (3.64)$$

In sum,

$$\mathbf{B}J_{i,j} = \begin{cases} q^{-\frac{1}{2}}J_{j,i}, & i+j \neq 2n-1 \& i=j; \\ J_{j,i}, & i+j \neq 2n-1 \& i>j; \\ J_{j,i} + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})J_{i,j}, & i+j \neq 2n-1 \& i<j. \end{cases} \quad (3.65)$$

For $v_{\lambda^i} \otimes v_{\lambda^j} \in V_{\omega_1} \otimes V_{\omega_1}$, $o(i) + o(j) = 2n-1$, the coefficients of equation (3.25) are

$$\mathbf{B}_{o^{-1}(2n-1-o(i)),i}^{i,o^{-1}(2n-1-o(i))} = q^{-\frac{1}{2}(\lambda^{o^{-1}(2n-1-o(i))}, \lambda^i)} = q^{\frac{1}{2}}.$$

Let $j > 0$ be the difference of the order, then $\mathbf{B}_{o^{-1}(2n-1-o(i)),i}^{o^{-1}(o(i)-j),o^{-1}(2n-1-o(i)+j)}$ can be computed as in the following cases.

i) $0 < j \leq o(i) < n$ or $o(i) > n \& o(i) - j \geq n$

The variation is of type *III* and j is the number of moving primary roots.

$$\begin{aligned} a &= q^{-\frac{1}{2}(\lambda^{o^{-1}(2n-1-o(i))}, \lambda^i)}; \\ b &= a \cdot q^{\frac{1}{4}[(\lambda^{o^{-1}(o(i)-j)}, \lambda^{o^{-1}(o(i)-j)} - \lambda^i) + (\lambda^{o^{-1}(2n-1-o(i))}, \lambda^{o^{-1}(o(i)-j)} - \lambda^i)] + \frac{j-1}{2}}; \\ c &= b \cdot q^{-(\lambda^{o^{-1}(2n-1-o(i))}, \lambda^{o^{-1}(o(i)-j)} - \lambda^i)}; \\ \mathbf{B}_{o^{-1}(2n-1-o(i)),i}^{o^{-1}(o(i)-j),o^{-1}(2n-1-o(i)+j)} &= (-1)^j q^{\frac{j+1}{2}} (1 - q^{-1}). \end{aligned} \quad (3.66)$$

ii) $n < o(i) \leq 2n-1 \& o(i) - j < n-1 \& 2o(i) - j \neq 2n-1$

The variation is of type *III*. At this time, the number of primary roots moving from z_2 to z_1 is not j , but $j-1$. The sign before b is $(-1)^{j-1}$ and the phase factor from $-\pi$ self rotation of $j-1$ primary roots equals to $q^{\frac{j-2}{2}}$. Thus,

$$\begin{aligned} b &= a \cdot q^{\frac{1}{4}[(\lambda^{o^{-1}(o(i)-j)}, \lambda^{o^{-1}(o(i)-j)} - \lambda^i) + (\lambda^{o^{-1}(2n-1-o(i))}, \lambda^{o^{-1}(o(i)-j)} - \lambda^i)] + \frac{j-2}{2}}; \\ c &= b \cdot q^{-(\lambda^{o^{-1}(2n-1-o(i))}, \lambda^{o^{-1}(o(i)-j)} - \lambda^i)}. \end{aligned} \quad (3.67)$$

$$\mathbf{B}_{o^{-1}(2n-1-o(i)),i}^{o^{-1}(o(i)-j),o^{-1}(2n-1-o(i)+j)} = (-1)^{j-1} q^{\frac{j}{2}} (1 - q^{-1}).$$

iii) $n < o(i) \& o(i) - j < n-1 \& 2o(i) - j = 2n-1$

The variation is of type *IV*. The number of primary roots moving from z_2 to z_1 is $j - 1$. Thus,

$$\begin{aligned}
 b &= a q^{\frac{1}{4}[(\lambda^{o^{-1}(o(i)-j)}, \lambda^{o^{-1}(o(i)-j)} - \lambda^i) + (\lambda^{o^{-1}(2n-1-o(i))}, \lambda^{o^{-1}(o(i)-j)} - \lambda^i)] + n - o(i) + j - 2}, \\
 c_1 &= b \cdot q^{-(\lambda^{o^{-1}(2n-1-o(i))}, \lambda^{o^{-1}(o(i)-j)} - \lambda^{o^{-1}(o(i)-j+1)})}, \\
 c_2 &= a \cdot q^{\frac{1}{4}[(\lambda^{o^{-1}(o(i)-j)}, \lambda^{o^{-1}(o(i)-j)} - \lambda^i) + (\lambda^{o^{-1}(2n-1-o(i))}, \lambda^{o^{-1}(o(i)-j)} - \lambda^i)] \\
 &\quad q^{-(\lambda^{o^{-1}(2n-1-o(i))}, \lambda^{o^{-1}(o(i)-j)} - \lambda^i) + (\lambda^{o^{-1}(2n-1-o(i))}, \lambda^{o^{-1}(o(i)-j)} - \lambda^{o^{-1}(o(i)-j+1)})}, \\
 c_3 &= a \cdot q^{\frac{1}{4}[(\lambda^{o^{-1}(o(i)-j)}, \lambda^{o^{-1}(o(i)-j)} - \lambda^i) + (\lambda^{o^{-1}(2n-1-o(i))}, \lambda^{o^{-1}(o(i)-j)} - \lambda^i)] \\
 &\quad q^{-(\lambda^{o^{-1}(2n-1-o(i))}, \lambda^{o^{-1}(o(i)-j)} - \lambda^i)}.
 \end{aligned} \tag{3.68}$$

$$B_{o^{-1}(2n-1-o(i)), i}^{o^{-1}(o(i)-j), o^{-1}(2n-1-o(i)+j)} = (-1)^{j-1} q^{\frac{j}{2}} (1 - q^{-1}) + q^{-\frac{1}{2}} (1 - q).$$

iv) $n < o(i) \& o(i) - j = n - 1$

The variation is of type *III*. The number of primary roots moving from z_2 to z_1 is $j - 1$. Thus,

$$\begin{aligned}
 b &= a \cdot q^{\frac{1}{4}[(\lambda^{o^{-1}(o(i)-j)}, \lambda^{o^{-1}(o(i)-j)} - \lambda^i) + (\lambda^{o^{-1}(2n-1-o(i))}, \lambda^{o^{-1}(o(i)-j)} - \lambda^i)] + \frac{j-2}{2}}, \\
 c &= b \cdot q^{-(\lambda^{o^{-1}(2n-1-o(i))}, \lambda^{o^{-1}(o(i)-j)} - \lambda^i)}.
 \end{aligned} \tag{3.69}$$

$$\begin{aligned}
 B_{o^{-1}(2n-1-o(i)), i}^{o^{-1}(o(i)-j), o^{-1}(2n-1-o(i)+j)} &= B_{o^{-1}(2n-1-o(i)), i}^{n-1, n} \\
 &= (-1)^{j-1} q^{\frac{j}{2}} (1 - q^{-1}).
 \end{aligned} \tag{3.70}$$

v) $o(i) = n \& j \geq 2$

The variation is of type *III*. The number of primary roots moving from z_2 to z_1 is $j - 1$. Thus,

$$\begin{aligned}
 b &= a \cdot q^{\frac{1}{4}[(\lambda^{o^{-1}(o(i)-j)}, \lambda^{o^{-1}(o(i)-j)} - \lambda^i) + (\lambda^{o^{-1}(2n-1-o(i))}, \lambda^{o^{-1}(o(i)-j)} - \lambda^i)] + \frac{j-2}{2}}, \\
 c &= b \cdot q^{-(\lambda^{o^{-1}(2n-1-o(i))}, \lambda^{o^{-1}(o(i)-j)} - \lambda^i)}.
 \end{aligned} \tag{3.71}$$

$$B_{o^{-1}(2n-1-o(i)), i}^{o^{-1}(o(i)-j), o^{-1}(2n-1-o(i)+j)} = (-1)^{j-1} q^{\frac{j}{2}} (1 - q^{-1}).$$

$$\begin{aligned}
 &B_{o^{-1}(2n-1-o(i)), i}^{o^{-1}(o(i)-j), o^{-1}(2n-1-o(i)+j)} \\
 &= \begin{cases} q^{\frac{j+1}{2}} (-1)^j (1 - q^{-1}), & o(i) > n \& o(i) - j \geq n \text{ or } 0 < j \leq o(i) < n; \\ q^{\frac{j}{2}} (-1)^{j-1} (1 - q^{-1}), & o(i) > n \& o(i) - j < n - 1 \& 2o(i) - j \neq 2n - 1; \\ q^{\frac{j}{2}} (-1)^{j-1} (1 - q^{-1}) \\ + q^{-\frac{1}{2}} - q^{\frac{1}{2}}, & o(i) > n \& o(i) - j < n - 1 \& 2o(i) - j = 2n - 1; \\ q^{\frac{j}{2}} (-1)^{j-1} (1 - q^{-1}), & o(i) > n \& o(i) - j = n - 1; \\ q^{\frac{j}{2}} (-1)^{j-1} (1 - q^{-1}), & o(i) = n \& j \geq 2. \end{cases}
 \end{aligned} \tag{3.72}$$

$$\begin{aligned}
 & \mathbf{B}_{o^{-1}(2n-1-o(i)),i}^{o^{-1}(o(i)-j),o^{-1}(2n-1-o(i)+j)} \\
 &= \begin{cases} q^{\frac{j+1}{2}}(-1)^j(1-q^{-1}), & o(i) > n \& o(i) - j \geq n \text{ or } 0 < j \leq o(i) < n; \\ q^{\frac{j}{2}}(-1)^{j-1}(1-q^{-1}) \\ + \delta_{2o(i)-j,2n-1}(q^{-\frac{1}{2}} - q^{\frac{1}{2}}), & o(i) \geq n \& o(i) - j \leq n-1; \end{cases} \quad (3.73)
 \end{aligned}$$

Monodromy representation is as following:

$$\begin{aligned}
 & \mathbf{B}_{o^{-1}(2n-1-o(a)),a}^{o^{-1}(2n-1-o(b)),b} \\
 &= \begin{cases} (-1)^{o(a)+o(b)-1} q^{-n+\frac{o(a)+o(b)+1}{2}} (q^{\frac{1}{2}} - q^{-\frac{1}{2}}), & o(a) \leq n-1 \& o(b) \geq n; \\ & o(a) > n \& o(b) \leq n-1; \\ (-1)^{o(a)+o(b)-2} q^{-n+\frac{o(a)+o(b)}{2}} (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \\ - \delta_{o(a),o(b)} (q^{\frac{1}{2}} - q^{-\frac{1}{2}}), & o(a) \geq n \& o(b) \geq n. \end{cases} \quad (3.74)
 \end{aligned}$$

3.3 Similarity transformation Q

By comparing the monodromy representation with the braid group representation induced by the universal R-matrices (see 7.3B and 7.3C of [6]) of quantum group $U_h(g)$, we have $\mathbf{B}_{\mathbf{Y}\mathbf{Y}} = Q \cdot \mathbf{B}_{U_h(g)} \cdot Q^{-1}$, $Q \in \text{End}(V_{\omega_1} \otimes V_{\omega_1})$.

For A_n , $Q = Id$;

For B_n , $Q = \begin{pmatrix} Id & & \\ & Q_{2n+1} & \\ & & Id \end{pmatrix}$, where

$$Q_{2n+1} = \begin{pmatrix} (-1)^0 & & & & \\ & \cdots & & & \\ & & (-1)^{n-1} & & \\ & & & \frac{(-1)^n}{q^{\frac{1}{4}} + q^{-\frac{1}{4}}} & \\ & & & & (-1)^{n+1} \\ & & & & & \cdots \\ & & & & & & (-1)^{2n} \end{pmatrix}; \quad (3.75)$$

For C_n and D_n , $Q = \begin{pmatrix} Id & & \\ & Q_{2n} & \\ & & Id \end{pmatrix}$, where

$$Q_{2n} = \begin{pmatrix} (-1)^0 & & & & \\ & \cdots & & & \\ & & (-1)^{n-1} & & \\ & & & (-1)^{n-1} & \\ & & & & \cdots \\ & & & & & (-1)^{2n-2} \end{pmatrix}. \quad (3.76)$$

This proves the first main theorem.

Figure 18. Coordinates distribution of the A_n critical point on $\mathbf{W}$ plane at $c = 0$.

Figure 19. Coordinates distribution of the B_n critical point on $\mathbf{W}$ plane at $c = 0$.

4 Commutation of two parameter deformations

In the previous section, we have studied the transformation $T(1)$ of parameters z_1 and z_2 . In this section, we consider the continuous deformation of the real parameter c and its relation with $T(1)$. As shown in theorem 2, for the fundamental representation V_{ω_1} of each classical complex simple Lie algebra, the lowest weight vectors in $SingV_{\omega_1} \otimes V_{\omega_1}$ are

$$\begin{aligned} A_n : & \quad v_{2\omega_1 - \alpha_1} \\ B_n : & \quad v_{2\omega_1 - 2\alpha_1 - \dots - 2\alpha_n} \\ C_n : & \quad v_{2\omega_1 - 2\alpha_1 - \dots - 2\alpha_{n-1} - \alpha_n} \\ D_n : & \quad v_{2\omega_1 - 2\alpha_1 - \dots - 2\alpha_{n-2} - \alpha_{n-1} - \alpha_n} \end{aligned} \tag{4.1}$$

For A_n , the Yang-Yang function with $c = 0$ corresponding to the singular vector $v_{2\omega_1 - \alpha_1}$ is

$$\mathbf{W}(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, 1) = \sum_{a=1}^2 (\alpha_1, \omega_1) \log(w_1 - z_a) - (\omega_1, \omega_1) \log(z_1 - z_2).$$

Its unique critical point is $w_1 = \frac{z_1 + z_2}{2}$. For B_n , C_n and D_n , the corresponding critical point equations (3.24), (3.44), (3.61) are already solved. Thus, corresponding to the lowest weight vector in $SingV_{\omega_1} \otimes V_{\omega_1}$ of each classical Lie algebra, there exists an unique thimble $J \subset \mathfrak{J}_0(P)$ for $\|e^{-\frac{\mathbf{W}}{k+h^v}}\|$ connecting z_1 and z_2 . The distributions of the coordinates of the critical point on $\mathbf{W}$ plane are shown in figures 18, 19, 20 and 21 respectively. As shown in section 2.3, $\mathfrak{J}_0(P)$ is a $\mathbb{Z}[t, t^{-1}]$ -module. Thus, J naturally generates a one dimensional $\mathbb{Z}[t, t^{-1}]$ -submodule of $\mathfrak{J}_0(P)$. We denote also by $\mathbf{B}_{\mathbf{Y}\mathbf{Y}}$ the monodromy representation induced by $T(1)$ on the one dimensional $\mathbb{Z}[t, t^{-1}]$ -submodule generated by J .

Let $\alpha_I = \{\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_m}\}$ be the set of all the primary roots in the lowest weight $2\omega_1 - \alpha_{i_1} - \alpha_{i_2} \dots - \alpha_{i_m}$ and $I = \{i_1, i_2, \dots, i_m\}$ the set of their indexes. Define $SI(\alpha_I) =$

Figure 20. Coordinates distribution of the C_n critical point on $\mathbf{W}$ plane at $c = 0$.

Figure 21. Coordinates distribution of the D_n critical point on $\mathbf{W}$ plane at $c = 0$.

$\sum_{i_j, i_s \in I \& j < s} (\alpha_{i_j}, \alpha_{i_s})$ as the sum of the pairwise inner products of the primary roots in α_I . By a direct calculation for the phase factor difference from the rotation $T(1)$, we have

Lemma 11.

$$\mathbf{B}_{\mathbf{Y}\mathbf{Y}} J = \mathbf{d} J, \quad \mathbf{d} = q^{-\frac{1}{2}[(\omega_1, \omega_1) - 2(\omega_1, \sum_{i_j \in I} \alpha_{i_j}) + SI(\alpha_I)]}. \quad (4.2)$$

For A_n Lie algebra, $m = 1$, $i_1 = 1$,

$$SI(\alpha_I) = 0.$$

$$\text{For } B_n \text{ Lie algebra, } m = 2n, i_j = \begin{cases} j, & j \leq n; \\ 2n + 1 - j, & j > n, \end{cases}$$

$$SI(\alpha_I) = -2(n - 1) + 1 = -2n + 3.$$

$$\text{For } C_n \text{ Lie algebra, } m = 2n - 1, i_j = \begin{cases} j, & j \leq n; \\ 2n - j, & j > n, \end{cases}$$

$$SI(\alpha_I) = -(n - 2) - 1 = -n + 1.$$

$$\text{For } D_n \text{ Lie algebra, } m = 2n - 2, i_j = \begin{cases} j, & j \leq n; \\ 2n - 1 - j, & j > n, \end{cases}$$

$$SI(\alpha_I) = -2(n - 1) = -2n + 2.$$

Although theorem 3 demands that $c \in \mathbb{Z}_{\geq 0}$, the thimble structure can be defined continuously on $c \geq 0$, thus we can consider the continuous deformation of $c \rightarrow +\infty$ from $c = 0$. For any $J \subset \mathfrak{J}_0(P)$, when $c \rightarrow +\infty$, by lemma 1, the coordinates of its critical point

tend either to z_1 or to z_2 , thus it gives several possible thimbles in $J_{a,b} \subset \mathfrak{J}(P)$. Multiply to each thimble $J_{a,b}$ the phase factor generated from the deformation and sum them up. Define the symmetry breaking transformation $\mathbf{S}$ by

$$\begin{aligned} \mathbf{S} : \mathfrak{J}_0(P) &\rightarrow \mathfrak{J}(P) \\ \mathbf{S}J &= \sum_{a,b} e^{a,b} J_{a,b}, \end{aligned} \quad (4.3)$$

where coefficients $e^{a,b}$ are the phase factor difference of $e^{-\frac{\mathbf{W}(\mathbf{w}, \mathbf{z}, \boldsymbol{\lambda}, \mathbf{l})}{\kappa + h^{\vee}}}$ in the process of $c \rightarrow +\infty$ from 0. For $J \in \mathfrak{J}_0(P)$ corresponding to the lowest weight vector in $\text{Sing} V_{\omega_1} \otimes V_{\omega_1}$, by a direct calculation, we have

Lemma 12. *For A_n ,*

$$\mathbf{S}J = q^{-\frac{1}{4}(\omega_1, \alpha_1)} J_{n,0} + q^{\frac{1}{4}(\omega_1, \alpha_1)} J_{0,n} = q^{-\frac{1}{4}} J_{n,0} + q^{\frac{1}{4}} J_{0,n}.$$

For B_n ,

$$\begin{aligned} \mathbf{S}J &= \sum_{i < n} (-1)^i q^{-\frac{n-i}{2} + \frac{1}{4}} J_{2n-i,i} + (-1)^n (q^{-\frac{1}{4} + \frac{1}{4}}) J_{n,n} \\ &\quad + \sum_{i > n} (-1)^i q^{-\frac{n-i}{2} - \frac{1}{4}} J_{2n-i,i}. \end{aligned} \quad (4.4)$$

For C_n ,

$$\mathbf{S}J = \sum_{i < n} (-1)^i q^{-\frac{n-i}{4}} J_{2n-1-i,i} + \sum_{i \geq n} (-1)^i q^{-\frac{n-i-1}{4}} J_{2n-1-i,i}.$$

For D_n ,

$$\begin{aligned} \mathbf{S}J &= \sum_{i < n-1} (-1)^i q^{-\frac{n-i-1}{2}} J_{2n-2-i,i} + \sum_{i \geq n}^{2n-2} (-1)^i q^{-\frac{n-1-i}{2}} J_{2n-2-i,i} \\ &\quad + (-1)^{n-1} (J_{n-1,n-1'} + J_{n-1',n-1}). \end{aligned} \quad (4.5)$$

By the monodromy representation $\mathbf{B}_{a,b}^{c,d}$ in section 3.2 and the lemma above, it is straight forward to prove that $e^{a,b}$ is the eigenvector of the monodromy representation.

Lemma 13.

$$\sum_{a,b} e^{a,b} \mathbf{B}_{a,b}^{c,d} = \mathbf{d} e^{c,d}, \quad (4.6)$$

where $\mathbf{d}$ is defined in (4.2).

By this lemma,

$$\mathbf{B}_{\mathbf{Y}\mathbf{Y}} \mathbf{S}J = \mathbf{B}_{\mathbf{Y}\mathbf{Y}} \sum_{a,b} e^{a,b} J_{a,b} = \mathbf{d} \sum_{a,b,c,d} e^{a,b} \mathbf{B}_{a,b}^{c,d} J_{c,d} = \mathbf{S} \mathbf{B}_{\mathbf{Y}\mathbf{Y}} J.$$

The second main theorem 6 follows.

From the equation (4.3), it is natural to define a $\mathbb{Z}[t, t^{-1}]$ linear operator

$$M : \mathbb{Z}[t, t^{-1}] \rightarrow V_{\omega_1} \otimes V_{\omega_1},$$

satisfying

$$M(1) = \sum_{a,b} e^{a,b} v_{\lambda^a} \otimes v_{\lambda^b}.$$

We can also define a creation matrix associated with M as

$$M^{a,b} = e^{a,b}.$$

Annihilation matrix $M_{a,b}$ is its inverse satisfying

$$\sum_b M_{a,b} M^{b,c} = \delta_a^c.$$

With these two matrices, quantum trace of any (m, m) tensor $T_{i_1 i_2 \dots i_m}^{j_1 j_2 \dots j_m}$ is just as following:

$$Tr_q T = \sum_{i_1, i_2, \dots, i_m, j_1, j_2, \dots, j_m} T_{i_1 i_2 \dots i_m}^{j_1 j_2 \dots j_m} \eta_{j_1}^{i_1} \eta_{j_2}^{i_2} \dots \eta_{j_m}^{i_m},$$

where $\eta_{j_k}^{i_k} = \sum_l M^{i_k, l} M_{l, j_k}$. By Alexander's theorem (p91 I.7 of [18]), the ambient knots invariants defined by contraction of $\mathbf{B}_{a,b}^{c,d}$, $M_{a,b}$ and $M^{a,b}$ in the decomposition of some knot projection diagram coincide with the quantum trace of the (m, m) tensor associated with the braid. In [14] and [19], knots invariants associated with the fundamental representation of A_n Lie algebra and B_n, C_n, D_n Lie algebra are proved to be HOMFLY-PT polynomial and Kauffman polynomial respectively.

Remark 3. *Knots invariant depends on representations of Lie algebra. Different representations may give different knots invariants. Generally, the corresponding knots invariant is not necessary to be HOMFLY-PT polynomial or Kauffman polynomial. We will focus on this point elsewhere.*

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