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Fractional black p -branes on orbifold $\mathbb{C}^n/\mathbb{Z}_n$

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ABSTRACT: The recent discovery of an explicit solution of a black hole on the resolved orbifold $\mathbb{C}^n/\mathbb{Z}_n$ makes it possible to investigate the existence of p -branes on the orbifold. In particular, it is possible with reasonable precision to verify the prediction that an M2-brane on $\mathbb{C}^4/\mathbb{Z}_4$ in eleven dimensions and a D3-brane on $\mathbb{C}^3/\mathbb{Z}_3$ in ten dimensions have a family of black p -branes on the orbifold $\mathbb{C}^n/\mathbb{Z}_n$. These solutions are extremal and have regular horizons $S^{2n-1}/\mathbb{Z}_n$ without any naked singularity, with near horizon geometries $\text{AdS}_{p+2} \times S^{2n-1}/\mathbb{Z}_n$.

KEYWORDS: p -branes, Black Holes

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1 Introduction

In recent developments involving string theory and supergravities, p -branes carrying higher-dimensional charges have played crucial roles to understand their non-perturbative dynamics [1–8]. The terminology “ p -brane” generally should be used to indicate a classical solution which is extended in p directions in the context of a gravity theory. p -brane solutions have p spacelike translational Killing vectors and carry the charge of an antisymmetric tensor field. Since they have the mass saturating a lower bound given by the charge, they are called extremal, which are also called as Bogomol’nyi-Prasad-Sommerfield (BPS) states [9–11]. Four-dimensional charged black holes are prototypes of p -branes for the case of $p = 0$. The p -branes are also called solitons in the sense that they are classical solutions. In many instances it is important to know about black hole solutions of such objects. For example, a class of extended black hole solutions can be found simply by taking the product of $(D - d)$ -dimensional flat space and the d -dimensional Schwarzschild solution. Since these spacetimes are Ricci flat, we will solve the field equations of low-energy effective theory of string theory. They give a $p (= D - d)$ brane surrounded by an event horizon, which is a black p -brane. These solutions are described by a physical parameter which can be interpreted as the tension, i.e., the mass per unit p -volumes of the p -branes.

The p -branes can be put in more general manifolds. In particular, they gain a lot of interesting properties when these objects are located at an orbifold singularity [12, 13], such as a D3-brane on $\mathbb{C}^3/\mathbb{Z}_3$ in type-II string theory [14–16] and an M2-brane on $\mathbb{C}^4/\mathbb{Z}_4$ in M-theory explicitly constructed in refs. [17, 18]. One of characteristic features is that at the orbifold singularities, the mass and charge of the p -branes become fractional, and thus they cannot move from the orbifold singularity freely due to the Dirac’s quantization condition. Such fractional charges of solitons stucked at orbifold singularities are a common feature among Yang-Mills instantons [19, 20], vortices [21], and D-branes [12, 13, 22] on orbifold singularities. Another interesting property is that orbifold singularities are resolved in the D-brane world volume theory [12, 13]. Thus, one can expect that the spacetime itself

becomes regular without any naked singularity once p -branes are placed at the orbifold singularities. This is important when one finds hints that the gravity theory giving a p -brane on the orbifold is actually an already known and well-defined supergravity, thus giving a handle on the strong coupling dynamics of string theory.

An example of a theory for which a cosmological model such as a brane world scenario [23, 24] was found owing to the properties of the orbifold geometry is the strongly coupled $E_8 \times E_8$ heterotic string theory in ten dimensions. This is equivalent to the eleven-dimensional limit of M-theory compactified on an $S^1/\mathbb{Z}_2$ orbifold. A set of E_8 gauge fields is located at each ten-dimensional orbifold fixed plane. The concrete solution in five-dimensional theory was later constructed by an orbifold compactification [25, 26]. The idea that our universe may be a 3-brane in a higher dimensional spacetime gives also a supersymmetric realization of the brane-world in type IIB supergravity [27, 28].

One of the most interesting issues of this kind, and the main focus in this paper concern p -branes with vanishing scalar fields in D dimensions. The eleven-dimensional supergravity theory is for instance believed to have a gravity and four-form field strength for the bosonic sector if the higher-order terms are negligible. One can characterize this by saying that it has a charged black hole (M-brane) without any naked singularity in a neighborhood and everywhere outside of the event horizon [29]. The physical mass and charge for the black hole of these M-branes has been worked out in detail [8, 30]. The mass formula is determined by the Arnowitt-Deser-Misner (ADM) formalism [30–32] and so it will be enough in this paper to find a p -brane solution.

What makes these questions accessible for study is that an explicit and amazingly simple description of a black hole on the resolved orbifold $\mathbb{C}^n/\mathbb{Z}_n$ has recently been given in the Einstein-Maxwell system [33] as a generalization of $D = 5$ black hole on the Eguchi-Hanson space [34, 35] (see refs. [36–38] for another higher dimensional generalization). The basic two-form field strength has a 1-form gauge potential since it was seen in ref. [33] to carry a point charge, but on backgrounds does not give any naked singularity. Underlying geometry of the resolved orbifold $\mathbb{C}^n/\mathbb{Z}_n$ is to formulate this manifold as a complex line bundle over $\mathbb{CP}^{n-1}$ admitting a Ricci-flat Kähler metric [39–41].

In this paper, we construct black p -brane solutions on the resolved orbifold $\mathbb{C}^n/\mathbb{Z}_n$ in a $D = 2n + p + 1$ dimensional $(p + 1)$ -form Einstein system. The mass and charge are proportional to each other and the solution is extremal or BPS. The near horizon geometry is $\text{AdS}_{p+2} \times S^{2n-1}/\mathbb{Z}_n$ space. The case of $p = 0$, $D = 5$ ($n = 2$) reduces to a black hole on the Eguchi-Hanson space [34], while the one of $p = 0$ and $D = 2n + 1$ is for a black hole on the orbifold $\mathbb{C}^n/\mathbb{Z}_n$ [33]. As previously known p -brane solutions, our solutions reduce in the case of $p = 3$ and $D = 10$ to a D3-brane on $\mathbb{C}^3/\mathbb{Z}_3$ in type-II string theory with a near horizon geometry $\text{AdS}_5 \times S^5/\mathbb{Z}_3$ [14–16], and in the case of $p = 2$ and $D = 11$ to an M2-brane on $\mathbb{C}^4/\mathbb{Z}_4$ in M-theory with a near horizon geometry $\text{AdS}_4 \times S^7/\mathbb{Z}_4$ [17, 18]. However, our solutions with general p and D do not rely on any string theory origin, and thus have a potential application to more general cases such as brane-world scenarios.

This paper is organized as follows. In section 2, we show that simple field equations exist as an almost immediate consequence of the ansatz for fields we will impose. To see the p -brane on the orbifold $\mathbb{C}^n/\mathbb{Z}_n$ with $n \geq 1$ requires a more elaborate construction

to which we then turn in section 3. In the process, $(p+2)$ -dimensional AdS spacetime AdS_{p+2} makes an expected appearance near horizon limit of the D -dimensional p -brane background. A reasonably compelling argument for the existence of these vacua will be presented. We then go on in section 4 to summarize our results and comments about some of the other p -branes.

2 Classical p -brane solutions

We now go on to the formulation of finding p -brane solutions. In this section, we will present a general action in D dimensions. Here, we consider a theory including gravity, scalar field, and antisymmetric tensor fields of arbitrary rank $(p+1)$ with its field strength of rank $(p+2)$. Then, the most general action is the following [8, 42]

$$S = \frac{1}{2\kappa^2} \int d^D x \sqrt{-g} \left[R - \frac{1}{2 \cdot (p+2)!} F_{(p+2)}^2 \right], \quad (2.1)$$

where κ^2 states again the D -dimensional gravitational constant. The reduction to the cases expressed in eq. (2.1) is straightforward. If we consider $D = 11$ supergravity, we only have a 4-form field strength.

In this paper, we focus on a particular case of the action (2.1), where there is a single $(p+2)$ -form field strength. In this setup, we can impose several symmetries in the background, which will allow us to constrain the fields in such a way that it will be possible to obtain a solution to the field equations derived from the action. The solutions found in this procedure will then give useful methods when we construct more general solutions in issues with less symmetries. It is also useful to consider the configurations with several branes like the multiply charged solutions. The matter fields presented in this paper is well-known as giving the single brane solutions [1, 5, 7, 8].

Now we can write the equations of motion and Bianchi identities for this simpler case, containing only gravity, a $(p+2)$ -form field strength and the scalar field. The equations of motion from eq. (2.1) can be expressed as

$$R_{MN} = \frac{1}{2 \cdot (p+2)!} \left[(p+2) F_{MA \dots B} F_N^{A \dots B} - \frac{p+1}{D-2} g_{MN} F_{(p+2)}^2 \right], \quad (2.2a)$$

$$d \left[*F_{(p+2)} \right] = 0, \quad (2.2b)$$

where $*$ denotes the Hodge dual operator in the D -dimensional spacetime. The $(p+2)$ -form field strength $F_{(p+2)}$ coming from $(p+1)$ -form gauge potential $A_{(p+1)}$ has to obey the Bianchi identities

$$dF_{(p+2)} = 0. \quad (2.3)$$

The three equations (2.2), (2.3), are the ones which we will use in this section to find the p -brane solutions.

We now impose the symmetries of the background in order to simplify the field equations by restricting us to particular field configurations. A p -brane solution living in a D -dimensional spacetime is in general specified by the fact that p spacelike directions can

be described longitudinal to the p -brane while the remaining $D - p - 1$ spacelike direction are considered transverse to the brane. When we consider single p -brane solutions carrying a single charge, it is natural to assume that p longitudinal directions are all equivalent. Although the timelike direction can be also longitudinal to the world-volume of the p -brane, it will not be considered as equivalent to the other longitudinal directions in the general case.

Since the p -brane is a uniform object, it should not single out particular points or regions of its volume. Then, it has the p spacelike longitudinal directions which is defined as many translational invariant directions of the solution. For a static p -brane, as we will discuss in this paper, there is another translationally invariant direction which is the timelike one. Since the p -brane is localized at a point in the transverse space, invariance under translations is thus broken [8]. If we assume that the p -brane is static with vanishing angular momentum, we can postulate spherical symmetry in the transverse space.

In this paper, the further restrictions that we will implement on the fields will specify the solution to be in a class, which is extremal p -brane because field equations are easily solved, and also these extremal p -branes turn out to be physically interesting, being identified in some cases to fundamental objects in the string theory as well as the general relativity.

Extremal p -brane roughly corresponds to the description that the mass of the p -brane is equal or proportional to its charge. This states that the p -brane is fully characterized by only the p -brane charge in the absence of angular momenta. From the viewpoint of the worldvolume of the p -brane, it is seen as a configuration carrying no energy at all [43]. It seems thus to describe this configuration as the flat space. If there is any mass excitation, or any departure from extremality in D -dimensional theory, the Lorentz invariance is broken in $(p + 1)$ -dimensional worldvolume spacetime [8].

Now we set a metric of extremal p -brane in D dimensions

$$ds^2 = h^a(y)q_{\mu\nu}(x)dx^\mu dx^\nu + h^b(y)u_{ij}(y)dy^i dy^j, \quad (2.4)$$

where we split the coordinates accordingly in two sets, x^μ and y^i with $\mu = 0, \dots, p$ and $i = 1, \dots, D - p - 1$, and $q_{\mu\nu}(x)$ and $u_{ij}(y)$ are metrics of $(p + 1)$ -dimensional spacetime, $(D - p - 1)$ -dimensional space depending only on x^μ and y^i , respectively. We assume that the constants a and b are given by

$$a = -\frac{2}{p+1}, \quad b = \frac{2}{D-p-3}. \quad (2.5)$$

The coordinates x^μ span the directions longitudinal to the brane, and the y^i states the coordinates of the transverse space. We have chosen the timelike direction $x^0 = t$. The function h depends only on the coordinates of the transverse space to the p -brane.

We go on characterizing which components of the $(p+2)$ -form field strength are relevant to the p -brane solutions. In this paper, we try to formulate the following ansatz for a $(p+2)$ -form field strength. A $(p + 2)$ -form satisfies an electric ansatz if it is of the form

$$F_{(p+2)} = \sqrt{\frac{2(D-2)}{(p+1)(D-p-3)}} d(h^{-1}) \wedge \Omega(X), \quad (2.6)$$

where “electric” means the ansatz (2.6) solves trivially its Bianchi identities (2.3), and we have defined

$$\Omega(X) = \sqrt{-q} dx^0 \wedge dx^1 \wedge \cdots \wedge dx^p. \quad (2.7)$$

Here, q states the determinant of the $(p+1)$ -dimensional metric and X denotes the $(p+1)$ -dimensional spacetime which is longitudinal to the world-volume of the brane. It is straightforward to find that with such an ansatz the Bianchi identities (2.3) are trivially satisfied. Taking into account the metric (2.4), the field equation (2.2b) for the antisymmetric tensor becomes

$$\Delta_Y h = 0, \quad (2.8)$$

where Δ_Y states the Laplace operator constructed from the metric $u_{ij}(y)$, and Y is the $(D-p-1)$ -dimensional space transverse to the p -brane.

We are now ready to collect all the above results and to express the Einstein equations for the metric (2.4) and the $(p+2)$ -form field strength (2.6). The Einstein equations are totally diagonal form in these coordinates, and we write here the equations in the following:

$$R_{\mu\nu}(X) + \frac{1}{p+1} q_{\mu\nu}(x) h^{-\frac{2(D-2)}{(p+1)(D-p-3)}-1} \Delta_Y h = 0, \quad (2.9a)$$

$$R_{ij}(Y) - \frac{1}{D-p-3} u_{ij}(y) h^{-1} \Delta_Y h = 0, \quad (2.9b)$$

where $R_{\mu\nu}(X)$ and $R_{ij}(Y)$ denote the Ricci tensor on X spacetime and Y space with respect to the metrics $q_{\mu\nu}(x)$ and $u_{ij}(y)$, respectively. If we require that the Ricci tensor $R_{\mu\nu}(X)$ depends only on the coordinates x^μ , these equations can be reduced to

$$R_{\mu\nu}(X) = 0, \quad R_{ij}(Y) = 0, \quad \Delta_Y h = 0. \quad (2.10)$$

To summarize, for any h of the form $h = h(y)$, and $(p+2)$ -form $F_{(p+2)}$ on Y space satisfying $\Delta_Y h = 0$, the metric (2.4) with $R_{\mu\nu}(X) = 0$, $R_{ij}(Y) = 0$, and the field strength given by (2.6) yield a solution to the D -dimensional theory [6–8, 29, 42].

3 Black p -branes on the orbifold $\mathbb{C}^n/\mathbb{Z}_n$

The solution of field equation is thus fully characterized by the equation $\Delta_Y h = 0$. Now we show that the a solution to eq. (2.10) has a complex line bundle over the complex projective space $\mathbb{CP}^{n-1}$. This is a generalization of the solution discussed in refs. [33, 34].

3.1 Black p -brane solution

Let us consider the metric $u_{ij}(y)$ on the $(2n)$ -dimensional space Y in eq. (2.4)

$$u_{ij}(y) dy^i dy^j = dr^2 + r^2 \left[\left\{ d\rho + \sin^2 \xi_{n-1} \left(d\psi_{n-1} + \frac{1}{2(n-1)} \omega_n \right) \right\}^2 + ds_{\mathbb{CP}^{n-1}}^2 \right]. \quad (3.1)$$

Here r is a radial coordinate, ρ is a coordinate of S^1 , ξ_{n-1} and ψ_{n-1} are coordinates of the $\mathbb{CP}^{n-1}$ space with the ranges $0 \leq \xi_{n-1} \leq \pi/2$, $0 \leq \psi_{n-1} \leq 2\pi$. ω_{n-1} and $ds_{\mathbb{CP}^{n-1}}^2$ state a

one-form and a metric on the $\mathbb{CP}^{n-1}$ space, respectively, recursively defined as [36–38]

$$ds_{\mathbb{CP}^{n-1}}^2 = 2n \left[d\xi_{n-1}^2 + \sin^2 \xi_{n-1} \cos^2 \xi_{n-1} \left\{ d\psi_{n-1} + \frac{1}{2(n-1)} \omega_{n-2} \right\}^2 + \frac{1}{2(n-1)} \sin^2 \xi_n ds_{\mathbb{CP}^{n-2}}^2 \right], \quad (3.2)$$

and

$$\omega_{n-2} = 2(n-1) \sin^2 \xi_{n-2} \left[d\psi_{n-2} + \frac{1}{2(n-2)} \omega_{n-3} \right], \quad (3.3a)$$

$$ds_{\mathbb{CP}^1}^2 = 4 \left(d\xi_1^2 + \sin^2 \xi_1 \cos^2 \xi_1 d\psi_1^2 \right), \quad (3.3b)$$

$$\omega_1 = 4 \sin^2 \xi_1 d\psi_1, \quad (3.3c)$$

where (r, ρ) describes a complex line, and ρ together with $\mathbb{CP}^n$ denote a $(2n-1)$ -dimensional sphere $S^{2n-1}/\mathbb{Z}_n = S^{D-p-2}/\mathbb{Z}_n$, which is actually an event horizon.

If we set $h = h(r)$, the equation $\Delta_Y h = 0$ gives the solution

$$h(r) = c_0 + \frac{c_1}{r^{D-p-3}}, \quad (3.4)$$

where c_0 and c_1 denote constants. The D -dimensional metric then writes

$$ds^2 = \left(c_0 + \frac{c_1}{r^{D-p-3}} \right)^{-\frac{2}{p+1}} q_{\mu\nu}(x) dx^\mu dx^\nu + \left(c_0 + \frac{c_1}{r^{D-p-3}} \right)^{\frac{2}{D-p-3}} u_{ij}(y) dy^i dy^j. \quad (3.5)$$

The solution (3.5) has the same form as a standard non-dilatonic p -brane solution with Ricci flat transverse space [6, 8, 44] due to the structure of the field equations (2.10).

As a particular example, we consider the case $q_{\mu\nu} = \eta_{\mu\nu}$ where $\eta_{\mu\nu}$ is the $(p+1)$ -dimensional Minkowski metric. Now we further define a new coordinate $\bar{r}$ by $\bar{r} = r^\alpha$, where α is expressed as

$$\alpha = \frac{D-p-3}{p+1}. \quad (3.6)$$

Keeping the values of these coordinates finite, the metric in the limit $\bar{r} \rightarrow 0$ then becomes

$$ds^2 = c_1^{-\frac{2}{p+1}} \bar{r}^2 \eta_{\mu\nu} dx^\mu dx^\nu + \left(\frac{c_1^{\frac{1}{D-p-3}}}{\alpha} \right)^2 \frac{d\bar{r}^2}{\bar{r}^2} + c_1^{\frac{2}{D-p-3}} \left[\left\{ d\rho + \sin^2 \xi_{n-1} \left(d\psi_{n-1} + \frac{1}{2(n-1)} \omega_n \right) \right\}^2 + ds_{\mathbb{CP}^{n-1}}^2 \right]. \quad (3.7)$$

Hence, D -dimensional metric becomes an $\text{AdS}_{p+2} \times S^{D-p-2}/\mathbb{Z}_n = \text{AdS}_{p+2} \times S^{2n-1}/\mathbb{Z}_n$ space.

A p -brane solution without any scalar field near the horizon describes AdS space is a consequence of the fact that if the $r \rightarrow 0$ limit is taken (also called the near horizon limit), the contribution of field strength becomes strong and the spacetime is close to the Freund-Rubin type of compactification [45] with a $(p+2)$ -form gauge field strength while it will vanish in the limit $r \rightarrow \infty$. Then, it is possible then to write a theory of these “low-energy” modes, and the $(p+2)$ -dimensional Minkowski spacetime is recovered in the asymptotic limit.

The case of $p = 0$, $D = 5$ reduces to a black hole on the Eguchi-Hanson space [34], generalized to $p = 0$ and arbitrary $D = 2n + 1$ for a black hole on the orbifold $\mathbb{C}^n/\mathbb{Z}_n$ [33]. On the other hand, the case of $p = 3$ and $D = 10$ reduces to a D3-brane on $\mathbb{C}^3/\mathbb{Z}_3$ in type-II string theory having near horizon geometry $\text{AdS}_5 \times S^5/\mathbb{Z}_3$ [14–16], and the case of $p = 2$ and $D = 11$ reduces to an M2-brane on $\mathbb{C}^4/\mathbb{Z}_4$ in M-theory having near horizon geometry $\text{AdS}_4 \times S^7/\mathbb{Z}_4$ [17, 18].

3.2 The mass and charge of the black p -brane

We discuss the mass and charge of the p -brane giving rise to a metric like (2.4). The mass of the p -brane can be simply calculated using the ADM formalism [31, 32]. This has already been constructed for p -branes in [8, 30]. Here we use the ADM formula in our case, and derive the expression for the ADM mass

$$M = \frac{V_p V_{S^{2n-1}/\mathbb{Z}_n}}{\kappa^2} \frac{c_1}{c_0} (D - p - 3) \left(\frac{p}{p+1} c_0^{-\frac{1}{p+1}} + \frac{D-p-2}{D-p-3} c_0^{\frac{1}{D-p-3}} \right), \quad (3.8)$$

where V_p is the volume of the “longitudinal” space spanned by the world volume of the p -brane and $V_{S^{2n-1}/\mathbb{Z}_n}$ is the volume of $S^{2n-1}/\mathbb{Z}_n$.

The charge density per a unit p -volume is defined by

$$q = \frac{1}{2\kappa^2} \int_{S^{2n-1}/\mathbb{Z}_n} *F_{(p+2)}. \quad (3.9)$$

Then, the total amount of the charge carried by the solution becomes

$$\begin{aligned} |Q| &= V_p |q| = V_p \sqrt{\frac{2(D-2)(D-p-3)}{p+1}} \frac{c_1 V_{S^{2n-1}/\mathbb{Z}_n}}{2\kappa^2} \\ &= c_0 \sqrt{\frac{D-2}{2(p+1)(D-p-3)}} \left(\frac{p}{p+1} c_0^{-\frac{1}{p+1}} + \frac{D-p-2}{D-p-3} c_0^{\frac{1}{D-p-3}} \right)^{-1} M. \end{aligned} \quad (3.10)$$

Since the mass M and the charge Q are proportional to each other, this is extremal or BPS.

A black p -brane in flat space $\mathbb{C}^n$ has a ADM mass and charge density, given by

$$M_1 = \frac{V_p V_{S^{2n-1}}}{\kappa^2} \frac{c_1}{c_0} (D - p - 3) \left(\frac{p}{p+1} c_0^{-\frac{1}{p+1}} + \frac{D-p-2}{D-p-3} c_0^{\frac{1}{D-p-3}} \right), \quad (3.11)$$

and

$$\begin{aligned} |Q_1| &= V_p |q_1| = V_p \sqrt{\frac{2(D-2)(D-p-3)}{p+1}} \frac{c_1 V_{S^{2n-1}}}{2\kappa^2} \\ &= c_0 \sqrt{\frac{D-2}{2(p+1)(D-p-3)}} \left(\frac{p}{p+1} c_0^{-\frac{1}{p+1}} + \frac{D-p-2}{D-p-3} c_0^{\frac{1}{D-p-3}} \right)^{-1} M_1, \end{aligned} \quad (3.12)$$

respectively. Here we define the charge density per a unit p -volume:

$$q_1 = \frac{1}{2\kappa^2} \int_{S^{2n-1}} *F_{(p+2)}. \quad (3.13)$$

This black p -brane can live also on the orbifold as well. The geometry is not significantly modified if it is far away from the orbifold singularity where the geometry is asymptotically flat.

Since the volumes are $V_{S^{2n-1}/\mathbb{Z}_n} = (1/n)V_{S^{2n-1}}$, our solution has $1/n$ mass and charge densities of the conventional one,

$$M = \frac{1}{n}M_1, \quad |Q| = \frac{1}{n}|Q_1|, \quad (3.14)$$

and thus we call it a *fractional* p -brane. The unit p -brane (M_1, Q_1) , which is supposed to be minimally quantized by the Dirac quantization condition, can exist outside the orbifold singularity of $\mathbb{C}^n/\mathbb{Z}_n$, while fractional one cannot exist and is stacked at the singularity for the consistency with the Dirac quantization condition. Such fractional objects stucked at orbifold singularities are common among Yang-Mills instantons [19, 20], vortices [21], and D-branes [12, 13, 22].

4 Discussions

We conclude with some comments on the properties of the solutions we have found, and potential applications. It is interesting to discuss the supersymmetry of the extremal p -brane solutions. It was pointed out in refs. [2, 3, 6, 8] that the extremal p -brane solutions are supersymmetric. While we have not considered supersymmetry transformation laws, it appears likely that some extremally charged black p -branes on the orbifolds are supersymmetric. This implies the existence of new varieties of solutions for black holes on orbifolds.

We have discussed extended black hole solutions in D dimensions, since this is the interest for supergravity. One can clearly modify the derivation in section 2 to obtain charged black p -branes with a scalar field in any dimension. It is an interesting question whether or not there exist black p -branes in D dimensions if we add non-trivial scalar field. Although there exist spacetimes with this causal structure (taking the product of $(p+2)$ -dimensional spacetime and $(D-p-2)$ -dimensional space in the D -dimensional solution), the associated energy momentum tensor is usually unphysical due to giving rise to naked singularity in the background [8].

There are many open questions regarding the dynamics of black p -branes on the orbifolds. Even more interesting, studying a way in which the classical solutions interact, for example by constructing solutions with intersecting p -branes [46], is relevant for checking the consistency with the interactions of these objects. Another important issue involving intersections of p -branes on the orbifolds is looking for supersymmetric extremal black holes with a non-vanishing horizon area. When one can identify these p -brane configurations in supergravity, mainly systems involving M-brane and D-branes [47], one can obtain a microscopic counting of states definitely giving the semiclassical black hole entropy [48, 49].

In section 2, we have constructed general solutions whose transverse directions are general Ricci-flat manifolds. Thus, replacing the orbifold geometries by more general Ricci-flat manifolds can give further black holes and black p -brane solutions. Recalling a fact that the resolved orbifolds $\mathbb{C}^n/\mathbb{Z}_n$ are complex line bundles over $\mathbb{CP}^{n-1}$, one can

consider Ricci-flat metrics on complex line bundles over other homogeneous Kähler manifolds G/H [39–41]. In fact, D3-branes on six-dimensional resolved and deformed conifolds were constructed in refs. [50, 51]. Black p -branes on higher dimensional deformed conifold [52] should be possible. Also, by replacing $\mathbb{CP}^{n-1}$ in this paper by the quadric surface $G/H = \text{SO}(n)/[\text{SO}(n-2) \times \text{U}(1)]$ [53], one could construct black p -branes on conifolds for which a horizon would be $[\text{SO}(n)/\text{SO}(n-2)]/\mathbb{Z}_{n-2}$. For more general homogeneous Kähler manifolds G/H , one can expect more general geometries with exotic horizons.

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