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Exact β -function of Yang-Mills theory in 2+1 dimensions

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ABSTRACT: To set the stage, I discuss the β -function of the massless $O(N)$ model in three dimensions, which can be calculated exactly in the large N limit. Then, I consider $SU(N)$ Yang-Mills theory in 2+1 space-time dimensions. Relating the β -function to the expectation value of the action in lattice gauge theory, and the latter to the trace of the energy-momentum tensor, I show that $\frac{d \ln g^2 / \mu}{d \ln \mu} = -1$ for all g and all N in one particular renormalization scheme. As a consequence, I find that the Yang-Mills β -function in three dimensions must have the same sign for all finite and positive bare coupling parameters in any renormalization scheme, and all non-trivial infrared fixed points are unreachable in practice.

KEYWORDS: Lattice field theory simulation, Non-perturbative renormalization

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1 Introduction

The running of the coupling in Yang-Mills theory in $3+1$ dimensions is one of the cornerstones of modern physics. It has spawned a whole discipline of renormalization group frameworks that have led to important discoveries in a variety of physical systems, and is central to lattice gauge theory simulations [1, 2].

In standard textbooks on quantum field theory (QFT) such as refs. [3, 4], the non-trivial running of the coupling in $3+1$ dimensions is related to the presence of logarithmic divergences. It is these logarithmic divergences that cannot be avoided in any regularization scheme, and which require the dependence of coupling parameters on energy scale. The presence of logarithmic divergences also implies that the “bare” or “fundamental” coupling parameters introduced in the Lagrangian of the theory get replaced (renormalized) by effective coupling parameters. Since only the effective (renormalized) coupling enters in observables, the bare coupling is typically not considered to be an interesting quantity by itself. Similar considerations apply to quantum field theories in $1+1$ dimensions, where many features similar to $3+1$ dimensions arise.

However, the situation is considerably different in $2+1$ dimensions, which is the subject of the present work. In $2+1$ dimensional quantum field theories, logarithmic divergences familiar from $1+1$ and $3+1$ dimensions are often absent. In these cases, renormalization of the bare coupling parameter is not required, and hence the coupling does not depend on the momentum scale in an “anomalous” fashion (the coupling only runs trivially with its bare mass dimension). Also, it implies that for these cases the bare coupling parameter enters directly into observables, and can thus be regarded as “fundamental”. In effect, quantum field theories in $2+1$ dimensions are more similar to quantum mechanics ($0+1$ dimensional quantum field theory) than to quantum field theories in $1+1$ and $3+1$ dimensions.

The above introduction, which is perfectly standard whenever perturbative quantum field theory is considered, is sometimes met with fierce objection by researchers who are adhering to a renormalization group (RG) point of view. Specifically, in the RG literature, one encounters the statement that the running of the coupling is unrelated to the presence or absence of divergences. This statement can be reconciled with the above QFT view

by noting that the running of the coupling is renormalization scheme-dependent, and that by changing the scheme it is possible (but not necessary) to have a non-trivial running coupling even if logarithmic divergences are absent. (Of course, physical observables will be independent on the choice of scheme; therefore, whether or not the coupling runs non-trivially is only a matter of interpretation.) I will make the resolution of the apparent tension between the perturbative QFT and RG view explicit when considering the critical $O(N)$ model in section 2 below.

This note is meant to provide explicit and exact calculations for quantum field theories in 2+1 dimensions as a means to demonstrating that what I said in the introduction makes quantitative sense. Specifically I am considering the scalar $O(N)$ model with quartic interaction in the large N limit and Yang-Mills theory. (QED with many flavors of electrons, which I am not considering here, would also have the same properties). I stress that none of the results discussed here make use of perturbation theory or strong coupling expansions.

Throughout the calculations, I am assuming that the total volume of the systems I am considering is large, but finite. This implies that there is an infrared cut-off that can be arbitrarily small,¹ but non-zero.

2 Warm-up: critical $O(N)$ model in $D=3, 4$ dimensions

As a warm-up, let me consider the $O(N)$ model with quartic interaction in D dimensions. Finite temperature T is introduced by considering imaginary time x_0 that is compactified on a circle of radius T^{-1} . The Euclidean action then reads

$$S_E = \int d^D x \left[\frac{1}{2} (\partial_\mu \vec{\phi}) \cdot (\partial_\mu \vec{\phi}) + \frac{m_0^2}{2} \vec{\phi}^2 + \frac{\lambda_0}{N} (\vec{\phi}^2)^2 \right], \quad (2.1)$$

where the coupling λ_0 is dimensionless in $D = 4$, but has mass dimension one in $D = 3$. The partition function is given by the path integral $Z = \int \mathcal{D}\phi e^{-S_E}$ which can be conveniently re-written as $(\vec{\phi}^2)^2 = \sigma^2$ using the auxiliary fields σ, ζ :

$$1 = \int \mathcal{D}\sigma \delta(\sigma - \vec{\phi}^2) = \int \mathcal{D}\sigma \mathcal{D}\zeta e^{i \int \zeta (\sigma - \vec{\phi}^2)}.$$

σ may be integrated out exactly, leaving only ζ . In the large N limit, only the zero mode $\bar{\zeta}$ contributes to the partition function. As a consequence, the dependence of the action on the scalar fields $\vec{\phi}$ is quadratic, and the path integral over $\vec{\phi}$ may be performed as

$$\int \mathcal{D}\phi e^{-\int d^D x \left[-\frac{1}{2} \vec{\phi} \partial_\mu \partial_\mu \vec{\phi} + i \bar{\zeta} \vec{\phi}^2 + \frac{m_0^2}{2} \vec{\phi}^2 \right]} = e^{-\frac{NV}{T} \frac{T}{2} \sum_n \int_{\mathbf{k}} \ln(\omega_n^2 + \mathbf{k}^2 + m_0^2 + 2i\bar{\zeta})}, \quad (2.2)$$

where $\omega_n = 2\pi nT$, $n \in \mathbb{Z}$ are the Matsubara frequencies and V is the $D - 1$ dimensional “volume” of the space-time.

¹In the case of the $O(N)$ model considered below, this means that I consider first the limit $N \rightarrow \infty$ and then send the infrared cut-off to zero, cf. the discussion in ref. [5].

Integration over the remaining zero mode $\bar{\zeta}$ and these may be evaluated exactly from the saddle point of the action. Writing $2i\bar{\zeta} = z^*$, the saddle point is given by

$$\frac{z^*}{\lambda_0} = 4T \sum_n \int \frac{d^{D-1}\mathbf{k}}{(2\pi)^{D-1}} \frac{1}{\omega_n^2 + \mathbf{k}^2 + m_0^2 + z^*} = 2 \int \frac{d^{D-1}\mathbf{k}}{(2\pi)^{D-1}} \frac{1 + 2n_B(\sqrt{\mathbf{k}^2 + m_0^2 + z^*})}{\sqrt{\mathbf{k}^2 + z^* + m_0^2}}, \quad (2.3)$$

where $n_B(x) = \frac{1}{e^{x/T} - 1}$ is the thermal distribution for bosons.

Let me first discuss what happens in dimensional regularization, which is the de-facto standard regularization scheme for analytic calculations. In this case, accessing the massless (critical) theory is easy and simply amounts to setting $m_0 = 0$. In dimensional regularization, one finds

$$\int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{1}{\sqrt{k^2 + z^*}} = \frac{1}{(4\pi)^{\frac{D-1}{2}}} \frac{\Gamma(\frac{2-D}{2})}{\Gamma(\frac{1}{2})} (z^*)^{\frac{D-2}{2}} = \begin{cases} -\frac{z^*}{8\pi^2\epsilon} + \mathcal{O}(1), & D = 4 - 2\epsilon, \\ -\frac{\sqrt{z^*}}{2\pi} + \mathcal{O}(\epsilon), & D = 3 - 2\epsilon, \end{cases} \quad (2.4)$$

which is a well-known result in large N field theory [6].

D=4 in $\overline{\text{MS}}$: the Wilson-Fisher fixed point. Therefore, in $D = 4 - 2\epsilon$, the integral (2.4) has a pole for $\epsilon \rightarrow 0$, and the saddle point condition reads (see e.g. ref. [7] for details)

$$0 = z^* \left(\frac{1}{\lambda_0} + \frac{1}{4\pi^2\epsilon} \right) + \text{finite \& T-dependent}, \quad D = 4 - 2\epsilon. \quad (2.5)$$

In order for the action (2.1) to be dimensionless in $D = 4 - 2\epsilon$, λ_0 must have mass dimension 2ϵ . Thus writing

$$\frac{1}{\lambda_0} = \frac{Z\mu^{D-4}}{\lambda_R(\mu)}, \quad (2.6)$$

where μ is the renormalization point in the $\overline{\text{MS}}$ scheme and λ_R is the renormalized coupling, the Z -factor in $\overline{\text{MS}}$ needs to be chosen as $Z = 1 - \frac{\lambda_R}{4\pi^2\epsilon}$ in order for the saddled point condition (2.5) to be finite for all temperatures. Since λ_0 does not depend on the scale μ , it follows that

$$\mu \frac{\partial \lambda_R^{-1}}{\partial \mu} = \frac{2\epsilon}{\lambda_R} - \frac{1}{2\pi^2}, \quad \beta_{\overline{\text{MS}}}(\lambda_R) \equiv \frac{\partial \lambda_R}{\partial \ln \mu} = \lambda_R \left(\frac{\lambda_R}{2\pi^2} - 2\epsilon \right), \quad D = 4 - 2\epsilon \quad \text{in } \overline{\text{MS}}. \quad (2.7)$$

As a consequence, $\beta_{\overline{\text{MS}}}$ has two zeros (fixed points), namely the free theory point located at $\lambda_R = 0$ and the Wilson-Fisher fixed point located at $\lambda_R = 4\pi^2\epsilon$ [8]. This result is exact in the large N limit. Note that at the Wilson-Fisher fixed point $Z = 0$ such that the bare coupling λ_0 is singular. This implies that for finite and positive-definite bare coupling $\lambda_0 < \infty$, the Wilson-Fisher fixed point can never be reached. In some sense, the Wilson-Fisher fixed point is more of an endpoint, since for $\lambda_0 \in [0, \infty)$ the renormalized coupling only takes on values $\lambda_R \in [0, 4\pi^2\epsilon)$. In the limit $\epsilon \rightarrow 0$, this shows that the theory is trivial (but the theory is perfectly well defined as a cut-off theory when keeping ϵ finite).

D=3 in $\overline{\text{MS}}$: no apparent Wilson-Fisher fixed point. However, the situation encountered for $D = 3 - 2\epsilon$ dimensions is very different. In this case, the integral (2.4) does not contain a logarithmic divergence, so the saddle-point condition is finite without renormalization [9]. Hence $Z = 1$ in $\overline{\text{MS}}$ and one finds

$$\mu \frac{\partial \lambda_R^{-1}}{\partial \mu} = \frac{1}{\lambda_R}, \quad \beta_{\overline{\text{MS}}}(\lambda_R) = -\lambda_R, \quad D = 3 - 2\epsilon \quad \text{in } \overline{\text{MS}}. \quad (2.8)$$

This result is exact in the large N limit. The running of the coupling is trivially given by the mass dimension of λ .^{2,3} The β function has a free theory fixed point located at $\lambda_R = 0$ in D=3 in the $\overline{\text{MS}}$ renormalization scheme. For finite values of λ_R (corresponding to finite values of the bare coupling $\lambda_0 \in [0, \infty)$), there is no apparent Wilson-Fisher fixed point or any other non-trivial infrared fixed point to leading order in large N. However, in the limit where $\lambda_R \rightarrow \infty$, the derivative $\frac{\partial \lambda_R^{-1}}{\partial \ln \mu} \rightarrow 0$. This “fixed point” at $\lambda_R \rightarrow \infty = \lambda^*$ can be used to calculate a critical exponent

$$\omega = \left. \frac{\partial}{\partial \lambda_R^{-1}} \frac{\partial \lambda_R^{-1}}{\partial \ln \mu} \right|_{\lambda^*} = 1. \quad (2.9)$$

(Note that since $\beta_{\overline{\text{MS}}}$ does not possess a fixed-point for finite coupling, it can not be used to define ω in the $\overline{\text{MS}}$ scheme).

It has been brought to my attention that some readers may not trust results derived using dimensional regularization. So let me investigate what happens to (2.3) when using cut-off regularization instead of dimensional regularization for $D = 3$. For the case of cut-off regularization, I will *not* set the bare mass parameter m_0 to zero, for reasons that will become clear in a moment. The saddle point condition (2.3) in cut-off regularization becomes

$$\frac{z^*}{\lambda_0} = 2 \int_0^\Lambda \frac{dk k}{2\pi} \frac{1}{\sqrt{k^2 + z^* + m_0^2}} = \frac{\Lambda}{\pi} - \frac{\sqrt{z^* + m_0^2}}{\pi} + \mathcal{O}(\Lambda^{-1}) + \text{finite}, \quad (2.10)$$

where Λ is the momentum cut-off. Obviously, there is a linear divergence in cut-off regularization. However, the divergence cannot be absorbed into a renormalized coupling, because the only dependence on the coupling constant is in the term $\frac{z^*}{\lambda_0}$. Instead, note that the relevant combination appearing in the propagator is the combination $z^* + m_0^2$, which reads

$$\frac{z^* + m_0^2}{\lambda_0} = \frac{m_R^2}{\lambda_0} - \frac{\sqrt{z^* + m_0^2}}{\pi} + \text{finite}, \quad (2.11)$$

where I have introduced the renormalized mass $m_R^2(\mu)$ as

$$m_R^2(\mu) = m_0^2 + \frac{\Lambda \lambda_0}{\pi}. \quad (2.12)$$

²As a consequence, the theory is invariant under the combined scale transformations $x^\mu \rightarrow x^\mu/\alpha$, $\lambda_0 \rightarrow \alpha \times \lambda_0$ with α the scale parameter. Note that this does *not* imply scale invariance of the theory, because the bare coupling needs to be rescaled in addition to the space-time coordinates, cf. refs. [10, 11].

³The trivial running is unrelated to the saddle point technique used here. For instance, it has been shown in ref. [12] that the perturbative series for the O(N) model in three dimensions is Borel-summable, recovering the trivial running to all orders in perturbation theory.

One may tune m_0^2 to the critical theory defined by $m_R^2 = 0$, such that the cut-off regularized saddle point condition becomes identical to the one in dimensional regularization. Therefore, one has again $Z = 1$ for the coupling, but a non-trivial running for the renormalized mass. As in dimensional regularization, there are no non-trivial infrared fixed points in the β -function in $\overline{\text{MS}}$ for $\lambda_0 \in [0, \infty)$ at large N .

D=3 in other renormalization schemes: recovering the fixed point? There is a school of thought that would have the large N Wilson-Fisher fixed point persists to three dimension by simply using the result (2.7) derived for $D = 4 - 2\epsilon$ and sending $\epsilon \rightarrow \frac{1}{2}$ at the end. I find this view somewhat confusing, even though it is to some extent only a semantic difference. Putting $2\epsilon = 1$ in a calculation where terms of order $\mathcal{O}(\epsilon)$ have been neglected is at best an approximation, but not an exact result. This approximation may be useful in many contexts where no other non-perturbative solution exists, but for theories that are exactly solvable in $D=3$, the exact result (2.4) does not possess a logarithmic divergence; therefore, using $2\epsilon = 1$ in (2.7) no longer corresponds to the $\overline{\text{MS}}$ renormalization scheme, because non-divergent terms are subtracted.

However, the idea of modifying the renormalization scheme is certainly a valid one: after all, the $\overline{\text{MS}}$ -scheme is just a convention, and one may just as well define a “renormalization” scheme by absorbing finite terms in the definition of the coupling. For instance, putting $Z = 1 - \lambda/c$ in eq. (2.8) where c is a finite constant leads to

$$\mu \frac{\partial \lambda^{-1}}{\partial \mu} = \frac{1}{\lambda} - \frac{1}{c}, \quad \beta = \lambda \left(\frac{\lambda}{c} - 1 \right), \quad D = 3 - 2\epsilon \quad \text{NOT in } \overline{\text{MS}}. \quad (2.13)$$

Now there is an apparent non-trivial fixed point in the β function located at $\lambda = c$. The critical exponent $\omega = \frac{\partial}{\partial \lambda^{-1}} \frac{\partial \lambda^{-1}}{\partial \ln \mu} = 1$ matches the result (2.9). However, the scheme is no longer $\overline{\text{MS}}$, so one could object that the fixed point thus found is not associated with a physical property of the system. For this reason, let me discuss a “physical” definition of the coupling instead of (2.13).

“Physical coupling definition” in $D=3$ and “unreachable” fixed points. A popular scheme to define the running coupling of the theory is via a physical process such as two-by-two scattering. In this case, one has to calculate the four-point function of the theory, which can be done exactly in the large N limit. Specifically, one finds for the connected amputated four-point function $\Gamma = \langle \vec{\phi}^2(P) \vec{\phi}^2(-P) \rangle$ [13]

$$\Gamma = \left[A(P) - \frac{2\lambda_0}{N} A^2(P) + \left(\frac{2\lambda_0}{N} A^2(P) \right)^2 + \dots \right] = \frac{A(P)}{1 + \frac{2\lambda_0}{N} A(P)},$$

$$A(P) = 2NT \sum_n \int \frac{d^2 \mathbf{k}}{(2\pi)^2} G(K) G(P - K), \quad (2.14)$$

where $G(K) = \frac{1}{\omega_n^2 + \mathbf{k}^2 + z^*}$ and z^* is the solution to the gap equation (2.3) for $D = 3$. In the zero temperature limit,

$$\lim_{T \rightarrow 0} A(P) = 2N \int \frac{d^3 K}{(2\pi)^3} G(K) G(P - K) = \frac{N}{4\sqrt{P^2}}. \quad (2.15)$$

Therefore, it is possible to define an effective coupling $\lambda_{\text{eff}}(P)$ as

$$\frac{\lambda_{\text{eff}}(P)}{2} = 1 - \frac{\Gamma}{A(P)} = \frac{\lambda_0}{2\sqrt{P^2 + \lambda_0}} = \frac{\lambda_0}{2\sqrt{P^2}} - \left(\frac{\lambda_0}{2\sqrt{P^2}}\right)^2 + \dots \quad (2.16)$$

Putting $\mu = \sqrt{P^2}$, this effective coupling has an effective β -function, which can be calculated as

$$\mu \frac{\partial \lambda_{\text{eff}}(\mu)}{\partial \mu} = -\lambda_{\text{eff}} \left(1 - \frac{\lambda_{\text{eff}}}{2}\right). \quad (2.17)$$

In the large N limit, this result is exact. Taken at face value, the result implies the existence of a non-trivial fixed point located at $\lambda_{\text{eff}} = 2$. As in the example of eq. (2.13), the renormalization scheme used to define λ_{eff} is not $\overline{\text{MS}}$. It seems that depending on the choice of renormalization scheme, there is or isn't a fixed point in the β -function. What is going on?

The renormalization group invariant $\beta \frac{\partial}{\partial \lambda}$ implies that under changes of the definition of the coupling $\lambda \rightarrow \lambda'$, the associated β function should change as

$$\beta(\lambda) = \beta'(\lambda') \frac{d\lambda}{d\lambda'}. \quad (2.18)$$

Fixed points are zeros of the β function, and the above equation seems to imply that the existence of fixed points (but not their location) is independent from the definition of the coupling. This is generally assumed to be true. However, this assumption is violated if the transformation $\lambda \rightarrow \lambda'$ is not invertible, such that the Jacobian $\frac{d\lambda}{d\lambda'}$ becomes singular. The condition for the Jacobian to be regular for all couplings is that the transformation of the coupling is analytic. It is easy to verify for the case at hand that this is *not* the case: taking $\lambda_R = \frac{\lambda_0}{\mu}$ from (2.6) and λ_{eff} from (2.16), one has

$$\lambda_R = \frac{\lambda_{\text{eff}}}{1 - \frac{\lambda_{\text{eff}}}{2}}, \quad (2.19)$$

which is singular at $\lambda_{\text{eff}} = 2$. Note that the location of the singularity in the transformation of the coupling precisely matches the apparent fixed-point found for the effective β -function in (2.17). Because λ_R diverges for $\lambda_{\text{eff}} \rightarrow 2$, the apparent fixed point at $\lambda_{\text{eff}} = 2$ cannot be reached in practice by any positive and finite bare coupling $\lambda_0/\mu \in [0, \infty]$. It is therefore unreachable in practice, and hence the β -function in the “physical coupling definition” has the same sign as $\beta_{\overline{\text{MS}}}$ in (2.8) for all positive and finite bare coupling values.

Lessons from the $O(N)$ model in $D=3$. In all cases discussed above for the massless $O(N)$ model at large N , there is no zero of the β -function for bare couplings $\lambda_0 \in [0, \infty)$, regardless of the renormalization scheme, regularization scheme, or coupling definition. Depending on the scheme, apparent zeros of the β -function may be found, but these fixed points can not be reached with any finite bare coupling parameter. While infrared fixed points in the $O(N)$ model may arise once subleading corrections in $1/N$ are taken into account, the above findings are exact in the large N limit and provide the interpretive stage for the following calculations in Yang-Mills theory.

3 Yang-Mills theory in 2+1 dimensions

The Lagrangian density for Yang-Mills theory in 2+1 dimensional Minkowski space-time in continuum formulation is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^a F^{\mu\nu a}, \quad F_{\mu\nu} = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c, \quad (3.1)$$

where A_μ^a with $a = 1, 2, \dots, N^2 - 1$ is the SU(N) gauge field, f^{abc} are the SU(N) structure constants and g is the (mass dimension one) Yang-Mills coupling. The theory is put at finite temperature using the standard construction in the imaginary time formalism, e.g. by performing the replacement [14]

$$t \rightarrow -ix_0, \quad (3.2)$$

such that

$$iS = i \int dt d^2x \mathcal{L} \rightarrow - \int d^3x \frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} = -S_E, \quad (3.3)$$

where μ, ν are now indices in three dimensional Euclidean space and S_E is the Euclidean action. In the continuum formulation, gauge invariance implies the presence of flat directions when integrating over the gauge fields A^μ , which would render a naive expression for the partition function divergent. A non-perturbative formulation of the partition function is found by replacing the continuum fields $A_\mu(x)$ by link variables $U_\mu(x) = e^{-igaA_\mu^a(x)T^a}$ where T^a are the generators of SU(N) and space-time has been discretized as a cube with lattice spacing a . The discretized Euclidean action then becomes [1]

$$S_E = \frac{2N}{g^2 a} \sum_x \sum_\square \left(1 - \frac{1}{N} \text{ReTr} U_\square \right), \quad U_\square = U_i(x) U_j(x+i) U_i^\dagger(x+j) U_j^\dagger(x), \quad (3.4)$$

where the sum is over all sites x of the three-dimensional lattice and over all orientations of plaquettes $U_\square$. The lattice has periodic boundary condition in all directions, and the length of the time-like direction of the lattice is related to the temperature as $T = \frac{1}{N_\tau a}$, where N_τ are the number of lattice sites in the temporal direction. In the lattice formulation, the partition function is given by

$$Z = \int \prod_x dU(x) e^{-S_E}, \quad (3.5)$$

which lends itself to numerical Monte-Carlo simulations via importance sampling.

For the case at hand, it is sufficient to recall that the partition function is related to the pressure p of Yang-Mills theory as

$$p = \frac{T}{V} \ln Z, \quad (3.6)$$

where V is again the “volume” of two-dimensional space. A standard operation in lattice gauge theory thermodynamics is to calculate the derivative of p with respect to the lattice coupling $\beta_L \equiv \frac{2N}{g^2 a}$, e.g.

$$\frac{\partial p/T^3}{\partial \beta_L} = \frac{1}{VT^2} \frac{\int \prod_x dU(x) e^{-S_E} \left(-\frac{\partial S_E}{\partial \beta_L} \right)}{Z} = -\frac{1}{VT^2} \frac{\langle S_E \rangle}{\beta_L}, \quad (3.7)$$

where $\langle S_E \rangle$ is the thermodynamic expectation value of the action, and I have divided the pressure by T^3 to make both sides dimensionless. Using

$$\frac{\partial p/T^3}{\partial \beta_L} = \frac{\partial p/T^3}{\partial T} \left(\frac{\partial \beta_L}{\partial T} \right)^{-1}, \quad (3.8)$$

writing $g^2 a = g^2/(TN_\tau)$, and keeping N_τ fixed (but large) leads to

$$\frac{\partial \beta_L}{\partial T} = -2NN_\tau \left(\frac{g^2}{T} \right)^{-2} \frac{\partial}{\partial T} \left(\frac{g^2}{T} \right), \quad (3.9)$$

such that combining (3.7) and (3.8) gives

$$\frac{\partial p/T^3}{\partial T} = \frac{1}{VT^2} \langle S_E \rangle \frac{\partial \ln(g^2/T)}{\partial T}. \quad (3.10)$$

Basic thermodynamic relations govern the relation between the pressure p , the entropy density s and the energy density ϵ such as

$$\frac{\epsilon + p}{T} = s = \frac{\partial p}{\partial T}. \quad (3.11)$$

Using these one finds

$$\frac{\partial p/T^3}{\partial T} = \frac{\epsilon - 2p}{T^4}. \quad (3.12)$$

However, in thermodynamic equilibrium, the expectation value of the energy-momentum tensor is related to ϵ, p as $\langle T^{\mu\nu} \rangle = \text{diag}(\epsilon, p, p)$ such that $\epsilon - 2p = \langle T_\mu^\mu \rangle$. Therefore, I find the relation

$$\langle T_\mu^\mu \rangle = \frac{T^2}{V} \langle S_E \rangle \frac{\partial \ln(g^2/T)}{\partial T}, \quad (3.13)$$

which is exact and corresponds to the well-known four-dimensional result [1] in $D = 3$. The above relation may be used to calculate the β -function numerically by means of lattice gauge theory simulation, for instance by calculating the expectation value of the action $\langle S_E \rangle$ and similarly calculating the trace of the energy-momentum tensor $\langle T_\mu^\mu \rangle$ e.g. via gradient flow [15]. However, for the case of three dimensions, such calculations may be unnecessary, as I will argue below.

It is straightforward to derive the energy-momentum tensor from the Minkowski Lagrangian (3.1), finding [4]

$$T^{\mu\nu} = -F^{\mu\lambda a} F_{\lambda}^{\nu a} + \frac{1}{4} g^{\mu\nu} F_{\rho\lambda}^a F^{\rho\lambda a}. \quad (3.14)$$

The trace of the energy momentum tensor in $D = 3$ dimensions is given by

$$T_\mu^\mu = -\frac{1}{4} F_{\rho\lambda}^a F^{\rho\lambda a} = \mathcal{L}. \quad (3.15)$$

This relation differs from its equivalent in four dimensions. In three dimensions, the trace of the energy-momentum tensor does not vanish. Moreover, rotating into Euclidean time (3.2), the trace of the energy-momentum tensor is related to the Euclidean action as

$$\int d^3x T_\mu^\mu = -S_E = -\int d^3x \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a. \quad (3.16)$$

As a consequence, the expectation value of the action in the exact relation (3.13) may be replaced by the corresponding expectation value of T_μ^μ . While classical, this relation implies that the same operator — discretized on the lattice — controls the action and the trace of the energy momentum tensor. Since the same operator can be used for both, a lattice simulation will give the same expectation value for both quantities, hence the relation (3.16) can be expected to hold outside the classical regime. It is of crucial importance to note that the replacement (3.16) does not work in $D = 4$ because there the classical Lagrangian is not proportional to the trace of the energy-momentum tensor.

Eq. (3.16) does not depend on any details of a lattice discretization. Hence both the continuum and infinite volume limit may be taken, such that $\langle S_E \rangle$ may be evaluated in the continuum theory where

$$\langle S_E \rangle = - \int d^3x \langle T_\mu^\mu \rangle = - \frac{V}{T} \langle T_\mu^\mu \rangle. \quad (3.17)$$

To re-iterate, while quantum effects (such as wave-function renormalization of the fields A_μ) may change each side of this relation individually as to differ from the classical expression, I expect these effects to cancel out in the relation (3.17) because the classical expressions on the left and right are the same, cf. eq. (3.16).

From (3.17), I find

$$\frac{\partial \ln(g^2/T)}{\partial \ln T} = -1, \quad \text{or} \quad \frac{\partial \ln(g^2 a)}{\partial \ln a} = 1. \quad (3.18)$$

This relation is exact for all g and all N , but depends on the choice of scheme.

4 Discussion

Eq. (3.18) implies that the dimensionless coupling g^2/T runs trivially, e.g. there are no non-trivial infrared fixed points for any positive and finite bare lattice coupling β_L in three dimensional Yang-Mills theory. The situation is similar to the $O(N)$ model in $D = 3$ encountered in section 2, but unlike the $O(N)$ model, eq. (3.18) is exact for all N .

It is possible to use a different definition g' for the coupling, e.g. via the scattering amplitude, which will possess a non-trivial fixed point at $g' = g^*$. However, similar to the example of the $O(N)$ model discussed in section 2, in view of (3.18) this fixed point must correspond to a singular Jacobian for the transformation $g \rightarrow g'$. Therefore, it is an unreachable fixed point in the sense that $g' < g^*$ for any positive and finite lattice coupling $\beta_L \in [0, \infty)$.

While employing the scattering amplitude in order to define the lattice spacing a through the lattice coupling β_L is a perfectly well-defined prescription, in three dimensional Yang-Mills theory there is an alternative option: just use the bare coupling. As in the case of a true conformal field theory, one may define a fiducial scale (say $g_0^2 = 1 \text{ GeV}$) after which the physical lattice spacing is given by

$$a = \frac{2N}{\beta_L g_0^2}. \quad (4.1)$$

In light of (3.18) this relation is exact and has been employed in Monte Carlo simulations of lower-dimensional gauge theories motivated by string theory, cf. refs. [16–18]. Using (4.1), scale setting is trivial and does not require the calculation of the string tension or flowed quantities. Of course, both procedures are correct in three dimensional Yang-Mills theory, but only the one from eq. (4.1) is effortless.

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