RECEIVED: February 5, 2020

REVISED: February 14, 2020

ACCEPTED: February 17, 2020

PUBLISHED: March 24, 2020

Charged dilaton black hole with multiple Liouville potentials and gauge fields

Yen-Kheng Lim

*Department of Mathematics, Xiamen University Malaysia,
Jalan Sunsuria, Sepang, Malaysia*

E-mail: yenkheng.lim@gmail.com

ABSTRACT: A solution to an Einstein-Maxwell-dilaton-type theory with M Liouville potentials and N gauge fields is presented, where M and N are arbitrary integers. This exact solution interpolates between the Lifshitz black hole and the topological dilaton black hole. The thermodynamic behaviour of the solution is found to be similar to that of the Lifshitz black hole, where a phase transition may occur for sufficiently small charge in the canonical ensemble, or sufficiently small potential in the grand canonical ensemble.

KEYWORDS: Classical Theories of Gravity, AdS-CFT Correspondence, Black Holes

ARXIV EPRINT: [1904.02297](https://arxiv.org/abs/1904.02297)

Contents

1	Introduction	1
2	Derivation of the solution	4
3	Physical and geometrical properties	14
4	Thermodynamics	19
4.1	The thermodynamic variables	19
4.2	Canonical ensemble	20
4.3	Grand canonical ensemble	23
4.4	Electrical stability	24
5	Conclusion	26

1 Introduction

In the various applications of General Relativity, spacetimes which are non-asymptotically flat have gained increasing interest. Perhaps the most notable is the asymptotically Anti-de Sitter spacetimes which play an important role in the gauge/gravity correspondence, string theory, and quantum gravity. Extensions of these and other related ideas have subsequently led to spacetimes of other asymptotics.

For instance, attention has recently been turned to spacetimes which may serve as a gravitational dual to non-relativistic field theories [1, 2]. For a more detailed review, see, e.g., [3–5]. Here we shall briefly recall the essential ideas that are most relevant to this paper. Such non-relativistic field theories in d -dimensions require an anisotropic scaling between space and time in the form

$$t \rightarrow \lambda^z t, \quad \vec{x} \rightarrow \lambda \vec{x}, \quad (1.1)$$

where λ is a real constant, t denotes the time coordinate and $\vec{x}$ denotes the $(d-1)$ -dimensional spatial coordinates. The real constant z is referred to as the *Lifshitz exponent*.

The scaling requirement leads the authors of [2, 6] to consider gravity duals taking the form of the *Lifshitz spacetime*. Concretely, a particular form of the Lifshitz spacetime in $D = (d+1)$ -dimensions is given by

$$ds^2 = -\frac{r^{2z}}{L^2} dt^2 + \frac{L^2}{r^2} dr^2 + r^2 d\vec{x}_{(d-1)}^2, \quad (1.2)$$

where L is a constant curvature scale. Clearly this spacetime is invariant under the scaling

$$t \rightarrow \lambda^z t, \quad r \rightarrow \frac{r}{\lambda}, \quad \vec{x} \rightarrow \lambda \vec{x}, \quad (1.3)$$

which satisfies eq. (1.1), along with an additional coordinate r representing the one extra dimension of the holographic dual. Furthermore, duals to systems at finite temperature requires the presence of a black hole. Hence on the gravity side, a suitable holographic dual would be the *Lifshitz black hole*, which are black holes in spacetimes which are asymptotically (1.2).

On the gravity side, work has been done to find an appropriate action which supports the Lifshitz spacetime as a solution to its equations of motion. Such a solution require the presence of various matter fields. The approach adopted by [2, 7–9] is to consider gravity a coupled to one-form and two-form gauge fields. Construction of the solutions using Einstein-Proca theory was achieved in [10, 11]. The Lifshitz black hole can also be constructed in Lovelock gravity [12] and higher-curvature gravity [13, 14].

Of relevance to this paper is the thread where an Einstein-Maxwell-dilaton (EMd)-type action is used. An EMd-type action with multiple U(1) fields with a single dilaton was used by Tarrío and Vandoren to construct a charged Lifshitz black hole with a spherical horizon [15]. From their solution, it was found that distinct U(1) fields are required to support various structures required by the spacetime geometry. Subsequently, Alishahiha, Colgáin, and Yavartanoo shows that the same action, but with just two U(1) fields, supports a hyperscaling-violating planar Lifshitz black hole solution [16]. A hyperscaling-violating solution can also be constructed by generalising the action by further including multiple dilaton fields [17].

To elaborate on this idea, we take a look at the action used by Tarrío and Vandoren [15], for which the bulk term in $D = (d + 1)$ -dimensions is

$$I = \frac{1}{16\pi G} \int d^D x \sqrt{-g} \left(R - 2\Lambda - \frac{1}{2} (\nabla\psi)^2 - \frac{1}{4} \sum_{i=1}^N e^{-2a_i\psi} F_i^2 \right), \quad (1.4)$$

where G is the D -dimensional gravitational constant and Λ is the cosmological constant. The gauge fields F_i represent a distinct U(1) field for each i , each of them coupled to the same dilaton field ψ through their respective coupling parameter a_i . It was known since [3] that at least one U(1) field (say, F_1) is necessary to support the Lifshitz spacetime (1.2) as a solution, even in the absence of the black hole. More specifically, if F_1 vanishes, eq. (1.2) will reduce to the Anti-de Sitter spacetime with $z = 1$. If one wishes to include a black hole with a spherical horizon, a second U(1) field F_2 is needed. Finally, a third gauge field F_3 is used to charge up the black hole itself. Subsequent inclusions of $F_4, F_5, \dots, F_N$ will contribute to different U(1) charges on the black hole. The value of z fixes F_1 , while the positive curvature of the black hole's spherical horizon fixes F_2 . Hence, out of N distinct gauge fields, only $N - 2$ of them ($F_3, F_4, \dots, F_N$) are freely parameterised (within constraints) charges carried by the black hole itself. The holographic and thermodynamic consequences of these solutions were studied in [18, 19], in addition to the original analysis performed by [15].

Solutions for EMd-type theories have also been of interest in the context of string theory and supergravity, especially those involving dimensional compactification [20, 21].

These studies involve bulk actions of the form

$$I = \frac{1}{16\pi G} \int d^D x \sqrt{-g} \left(R - \frac{1}{2} (\nabla\psi)^2 - \frac{1}{4} e^{-2a\psi} F^2 - 2\Lambda_1 e^{2b_1\psi} - 2\Lambda_2 e^{2b_2\psi} \right). \quad (1.5)$$

In other words, this action contains a single U(1) gauge field F , and the scalar dilaton ψ has a Liouville-type potential consisting of exponential terms $\mathcal{V}(\psi) = 2\Lambda_1 e^{2b_1\psi} + 2\Lambda_2 e^{2b_2\psi}$, where Λ_1 , Λ_2 , b_1 and b_2 are constants. Black hole solutions under this action have been studied, for instance, in refs. [22–26], along with multi-black holes in a cosmological background [27], as well as magnetised universes with and without a black hole [28, 29].

For these solutions, a similar intuition can be carried over from the Lifshitz black hole with multiple gauge fields. Though in this case the situation is more straightforward; the value of Λ_1 determines whether the horizon has positive, negative, or zero curvature. Here, we take the horizon to be the surface which separates two regions of a spacetime, in which time-like and null curves are unable to escape from one region into the other. (See, e.g., [25, 30–32].) Therefore, the value of Λ_1 determines the curvature of this surface. Finally, F contributes to the charge of the black hole. Further properties and thermodynamics of these solutions were studied by [25, 26, 29]. More general features of spacetimes under this potential were considered in refs. [33–35].

In light of these intuitions, it may be possible to construct spacetimes of various desired properties by considering an action with the appropriate combinations of gauge fields and Liouville potentials. For instance, as we have briefly alluded to above, the authors of [16] considered an action with two U(1) gauge fields and a single Liouville potential to construct planar Lifshitz black holes with hyperscaling-violation. In that context, the two U(1) fields are used to charge the black hole and support the Lifshitz asymptotics, respectively. The Liouville potential serves to produce a hyperscaling-violating factor in the metric.

In the present paper, we wish to study the possible solution of an action that contains the ingredients of both (1.4) and (1.5), so that the Lifshitz and dilaton black hole solutions are contained within a single framework. Specifically, the action is an EMd-type action with M Liouville potentials and N gauge fields, for any number M and N . The first hints of this possibility may come from noticing that the Lifshitz black hole and dilaton black hole solution have certain features in common with each other. For example, that the dilaton ψ in both cases is proportional to $\ln r$, and diverges at the boundary. In the context of Lifshitz holography, this corresponds to a running scalar coupling. On a similar note, the gauge fields which support the Lifshitz geometry (A_1 and A_N in our solution below) also diverges at the boundary. These suggest that the Lifshitz symmetry (1.1) is broken in the dual field theory [4]. Nevertheless, the dual theory may possibly obey some generalised scaling behaviour.

The solution presented in this paper is spacetime which contains the Lifshitz and dilaton black holes, with a parameter ν that interpolates between the two. The horizon of the black hole may have planar, spherical, or hyperbolic topology, depending on the relative values of the gauge fields and Liouville potentials. Particularly, in the case of the planar horizon, the black hole asymptotically approaches (1.2). Another consequence of

the solution is that the Lifshitz black hole metric of [36] can be reproduced by replacing one U(1) gauge field by a Liouville potential.

This paper is organised as follows. In section 2, the equations of motion are presented and the exact solution is derived. Some physical and geometrical properties of the solution will be studied in section 3, followed by an elementary thermodynamic analysis in section 4. The paper concludes in section 5. We will be using units where the speed of light, Planck's constant, and Boltzmann's constant equals unity, $c = \hbar = k = 1$. We shall also take the Lorentzian signature for the spacetime metric to be $(-, +, \dots, +)$.

2 Derivation of the solution

Our goal is to consider an action for which a solution that extremises this action is a spacetime which reduces to the Lifshitz black hole as well as the dilaton black hole. The former solution is supported by a EMD-type action with N gauge fields and no dilaton potential [36], as sketched in eq. (1.4), whereas the dilaton black holes of [24] are supported by an EMD-type action with a single gauge field with two dilaton potentials for ψ , sketched in eq. (1.5).

Hence, in combining the features of eqs. (1.4) and (1.5), we consider an EMD-type theory consisting of N different U(1) gauge fields with a dilaton potential $\mathcal{V}(\psi)$,

$$I = \frac{1}{16\pi G} \int_{\mathcal{M}} d^D x \sqrt{-g} \left(R - \frac{1}{2} (\nabla \psi)^2 - \frac{1}{4} \sum_{i=1}^N e^{-2a_i \psi} F_i^2 - \mathcal{V}(\psi) \right) + \frac{1}{8\pi G} \int_{\partial \mathcal{M}} d^{D-1} x \sqrt{-\gamma} K. \quad (2.1)$$

where F_i are the N two-form fields which arise from the exterior derivative of their respective one-form potentials, $F_i = dA_i$. We have also denoted $F_i^2 = (F_i)_{\mu\nu} (F_i)^{\mu\nu}$. Each of these fields are coupled to the scalar field ψ via their respective coupling parameters a_i .¹ As in eq. (1.4) and (1.5), G denotes the D -dimensional gravitational constant.

The second term in the action is the Gibbons-Hawking-York boundary term [37, 38]. This term serves to fix the metric on the boundary and to ensure that the variation of the action vanishes on-shell. A time-like boundary $\partial \mathcal{M}$ is chosen so that a Hamilton-Jacobi-type analysis can be applied to the action. Concretely, let the metric on the boundary be $\gamma_{\mu\nu}$. The boundary stress tensor is obtained from the variation of action with respect to $\gamma_{\mu\nu}$. According to the analysis by Brown and York [39], the time-like components of this boundary stress tensor is interpreted as the quasi-local energy contained within the slice of $\partial \mathcal{M}$ that is orthogonal to the time-like direction. For a concrete example, consider the Schwarzschild solution in its usual coordinates (t, r, θ, ϕ) . For a spherical volume $r < r_b$ for some chosen r_b , the quasilocal energy at some fixed time is the energy contained within that spherical region.

¹For comparison with Tarrío and Vandoren's action [15], we have $\lambda_i = -2a_i$ where λ_i is the coupling parameter used in their paper.

We shall consider potentials $\mathcal{V}(\psi)$ where it is the sum of M exponential terms,

$$\mathcal{V}(\psi) = \sum_{j=1}^M 2\Lambda_j e^{2b_j\psi}, \quad (2.2)$$

where b_j and Λ_j are constants. Therefore, the case of $N > 0$ and $M = 0$ corresponds to eq. (1.4) under which the Lifshitz black hole was obtained [15], where at least $N = 3$ is needed to have a charged Lifshitz black hole with a spherical horizon.² The case $N = 1$ and $M = 2$ corresponds to eq. (1.5) where the dilaton black hole was found [24]. Therefore, we should consider at least $M = N = 3$ to be the simplest non-trivial solution to go beyond the cases already contained in [15] and [24].

Extremising the action gives the Einstein-Maxwell-dilaton equations

$$R_{\mu\nu} = \frac{1}{2}\nabla_\mu\psi\nabla_\nu\psi + \frac{1}{D-2}\mathcal{V}g_{\mu\nu} + \sum_{i=1}^N \left[\frac{1}{2}e^{-2a_i\psi}(F_i)_{\mu\lambda}(F_i)_{\nu}{}^\lambda - \frac{1}{4(D-2)}e^{-2a_i\psi}F_i^2 g_{\mu\nu} \right], \quad (2.3)$$

$$\nabla_\mu \left(e^{-2a_i\psi} F_i^{\mu\nu} \right) = 0, \quad (2.4)$$

$$\nabla^2\psi = 2\frac{d\mathcal{V}}{d\psi} - \frac{1}{2}\sum_{i=1}^N a_i e^{-2a_i\psi} F_i^2. \quad (2.5)$$

As briefly alluded to in the Introduction, the general form of the Lifshitz and dilaton black hole share many similar features. In particular, their metrics in both cases take the form

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2\tilde{\gamma}_{ab}dy^a dy^b, \quad (2.6)$$

where $f(r)$ and $h(r)$ are functions depending only on r . The metric $\tilde{\gamma}_{ab}dy^a dy^b$ is a $(D-2)$ -dimensional space of constant unit curvature $k = 0, \pm 1$. Our present goal can then be focused into seeking a solution which also takes the form (2.6) and find an appropriate $f(r)$ and $h(r)$ that extremises the action (2.1). We shall also take the ansatz where the gauge potentials only have ‘electric’ radial components,

$$A_i = \chi_i(r) dt, \quad (2.7)$$

where $\chi_i(r)$ are scalar functions that depend only on r . Similarly we shall also take the ansatz where $\psi = \psi(r)$ is a scalar function that depends only on r .

In the following paragraphs we shall show the details of how the solution is derived. The reader uninterested in such details may skip ahead to the final solution shown in eqs. (2.35), (2.37), and (2.36).

²Note the $N = 2$ case in section 2.1 of [15] gives a black hole with a planar horizon.

Substitution of the ansatz into eq. (2.3) gives

$$-\frac{1}{2\sqrt{fh}} \left(\frac{f'}{\sqrt{fh}} \right)' - \frac{(D-2)f'}{2rfh} = \sum_{j=1}^M \frac{2\Lambda_j e^{2b_j\psi}}{D-2} - \sum_{i=1}^N \frac{D-3}{2(D-2)} \frac{1}{fh} e^{-2a_i\psi} \chi_i'^2, \quad (2.8)$$

$$-\frac{1}{2\sqrt{fh}} \left(\frac{f'}{\sqrt{fh}} \right)' + \frac{(D-2)h'}{2rh^2} = \sum_{j=1}^M \frac{2\Lambda_j e^{2b_j\psi}}{D-2} - \sum_{i=1}^N \frac{D-3}{2(D-2)} \frac{1}{fh} e^{-2a_i\psi} \chi_i'^2 + \frac{1}{2h} \psi'^2, \quad (2.9)$$

$$\frac{D-3}{r^2} \left(k - \frac{1}{h} \right) + \frac{1}{2rh} \left(\frac{h'}{h} - \frac{f'}{f} \right) = \sum_{j=1}^M \frac{2\Lambda_j e^{2b_j\psi}}{D-2} + \sum_{i=1}^N \frac{1}{2(D-2)} \frac{1}{fh} e^{-2a_i\psi} \chi_i'^2, \quad (2.10)$$

where primes denote derivatives with respect to r . Furthermore, putting the ansatz into the Maxwell equation (2.4) and dilaton equation (2.5) give

$$\left(\frac{r^{D-2}}{\sqrt{fh}} e^{-2a_i\psi} \chi_i' \right)' = 0, \quad i = 1, \dots, N, \quad (2.11)$$

$$\frac{1}{\sqrt{fhr}^{D-2}} \left(\sqrt{\frac{f}{h}} r^{D-2} \psi' \right)' = \sum_{j=1}^M 4b_j \Lambda_j e^{2b_j\psi} + \sum_{i=1}^N \frac{a_i}{fh} e^{-2a_i\psi} \chi_i'^2. \quad (2.12)$$

Taking the difference between (2.8) and (2.9) leads to

$$\frac{D-2}{r} \left(\frac{h'}{h} + \frac{f'}{f} \right) = \psi'^2. \quad (2.13)$$

To make progress towards obtaining a solution, we shall follow [16, 24, 25, 40] and consider a specific ansatz for h as

$$h = \frac{r^{2n}}{f}, \quad (2.14)$$

for some constant n . While this does not rule out the possibility that there exist solutions for other forms of h , the ansatz taken here will nevertheless lead to a solution for f that interpolates between the Lifshitz and dilaton black hole cases. With this ansatz for (2.14), eq. (2.13) becomes $\psi'^2 = 2(D-2)n/r^2$ and can be solved to give³

$$\psi = -\delta \ln r + \psi_0, \quad \delta = \sqrt{2(D-2)n}, \quad (2.15)$$

where ψ_0 is an integration constant. From the above expression for δ , it is clear that we require $n \geq 0$. It turns out that in the context of the hyperscaling-violating solution in [16], this condition coincides with satisfying the Null Energy Condition, and also for the

³Following [15, 25, 40], we shall choose the negative sign upon taking the square root. As we will see below, solving the equations of motion for this choice of sign will lead to a positive sign for the coupling parameter a_i when the solution is reduced to that of [40] and [15]. Furthermore this will be consistent if the terms $e^{-2a_i\psi} F_i^2$ of the action is a result of a dimensional compactification [20, 21, 23], in which case a_i is a positive number that depends on the number of reduced dimensions.

charged Lifshitz black hole of [15]. As we will show below, this continues to hold true for our present solution. It is worth noting that besides the context of dilaton and Lifshitz black hole solutions, scalar fields of the form (2.15) are solutions to asymptotically-AdS spacetimes in various models [41–43].

The Maxwell equations for each U(1) gauge field (2.11) can then be immediately integrated once to give

$$\chi'_i = \lambda_i e^{a_i \psi_0} r^{-(D-2-n+2a_i \delta)}, \quad (2.16)$$

where each λ_i are the integration constants parametrising the strength of their respective U(1) charges.

Substituting eqs. (2.14), (2.15), and (2.16) into eqs. (2.8), (2.10), and (2.12) leads to

$$-\frac{1}{2r^n} \left(\frac{f'}{r^n} \right)' - \frac{(D-2)f'}{2r^{2n+1}} = \frac{2}{D-2} \sum_{j=1}^M V_j r^{-2b_j \delta} - \frac{D-3}{2(D-2)} \sum_{i=1}^N \lambda_i^2 r^{-2a_i \delta - 2(D-2)}, \quad (2.17a)$$

$$(D-3)kr^{-2} - (D-3-n)r^{-2-2n}f - r^{-1-2n}f' = \frac{2}{D-2} \sum_{j=1}^M V_j r^{-2b_j \delta} + \frac{1}{2(D-2)} \sum_{i=1}^N \lambda_i^2 r^{-2a_i \delta - 2(D-2)}, \quad (2.17b)$$

$$-\delta r^{-(D-2+n)} (r^{D-3-n}f)' = 4 \sum_{j=1}^M b_j V_j r^{-2b_j \delta} + \sum_{i=1}^N a_i \lambda_i^2 r^{-2a_i \delta - 2(D-2)}, \quad (2.17c)$$

where we have denoted

$$V_j = \Lambda_j e^{2b_j \psi_0}. \quad (2.18)$$

What remains is to determine f . We shall motivate our ansatz by noting that, for example, the Reissner-Nordström-AdS solution has the form $f = k - \frac{\mu}{r^{D-3}} + \frac{q^2}{r^{2(D-2)}} + \frac{r^2}{\ell^2}$, particularly that the term $\frac{q^2}{r^{2(D-2)}}$ is associated with the U(1) gauge field and the term $\frac{r^2}{\ell^2}$ is associated with the cosmological constant. Either of these terms vanish if their respective gauge field or cosmological constant is switched off. A negative cosmological constant allows for $k = 0$ or $k = -1$. A similar situation occurs for the charged Lifshitz black hole with N U(1) gauge fields [15], where their function f has a separate term for each of the N gauge fields. One of the gauge fields is required in order to have a spherical black hole. In the dilaton black holes of [24, 26], there is a separate term for each Liouville potential. Even for hyperscaling-violating solutions such as the one in [16], one U(1) gauge field is to charge the black hole and a second U(1) field helps to support the background geometry.

We then carry these considerations into our present context, which has N gauge fields and M Liouville potentials. To this end, first we rewrite the sums appearing in the right-

hand sides of eq. (2.17) explicitly as three parts, for instance,

$$\sum_{i=1}^N \lambda_i^2 r^{-2a_i\delta-2(D-2)} = \lambda_N^2 r^{-2a_N\delta-2(D-2)} + \sum_{l=1}^p \lambda_l^2 r^{-2a_l\delta-2(D-2)} + \sum_{i=p+1}^{N-1} \lambda_i^2 r^{-2a_i\delta-2(D-2)}, \quad (2.19a)$$

$$\sum_{j=1}^M \frac{2V_j}{D-2} r^{-2b_j\delta} = \frac{2V_M}{D-2} r^{-2b_M\delta} + \sum_{l=1}^p \frac{2V_l}{D-2} r^{-2b_l\delta} + \sum_{j=p+1}^{M-1} \frac{2V_j}{D-2} r^{-2b_j\delta}. \quad (2.19b)$$

Next, we take our ansatz for f to have $M + N - p$ terms,

$$f = Br^\gamma - \mu r^{c_0} + \sum_{l=1}^p \frac{r^{\theta_l}}{L_l^2} + \sum_{i=p+1}^{N-1} q_i^2 r^{\omega_i} + \sum_{j=p+1}^{M-1} \frac{r^{\sigma_j}}{\ell_j^2}, \quad (2.20)$$

where the constant coefficients $\frac{1}{L_l^2}$, q_i^2 , $\frac{1}{\ell_j^2}$, B , and constant exponents c_0 , θ_l , ω_i , σ_j , and γ are to be determined. The reason why we consider $M + N - p$ terms is to include the possibility of overlap; namely that p of the terms receive simultaneous contributions from a gauge field and Liouville potential, for some p . We will also let the term Br^γ be sourced by the N -th gauge field and M -th Liouville potential. The term $-\mu r^{c_0}$ is a Schwarzschild-like term and is not sourced by any gauge field or Liouville potential.

In the following, we shall fix the indices l , i , and j to run according to the range implied in eq. (2.19) and (2.20). Substitution of eq. (2.20) into (2.17a), along with splitting the sums according to (2.19) gives

$$\begin{aligned} & -\frac{1}{2}(D-3-n+\gamma)\gamma Br^{\gamma-2-2n} - \frac{1}{2} \sum_{l=1}^p (D-3-n+\theta_l) \frac{\theta_l}{L_l^2} r^{\theta_l-2-2n} \\ & - \frac{1}{2} \sum_{i=p+1}^{N-1} (D-3-n+\omega_i) \omega_i q_i^2 r^{\omega_i-2-2n} - \frac{1}{2} \sum_{j=p+1}^{M-1} (D-3-n+\sigma_j) \frac{\sigma_j}{\ell_j^2} r^{\sigma_j-2-2n} \\ & - \frac{1}{2}(D-3-n+c_0)c_0\mu r^{c_0} \\ & = \frac{2V_M}{D-2} r^{-2b_M\delta} - \frac{(D-3)\lambda_M^2}{2(D-2)} r^{-a_M\delta-2(D-2)} \\ & + \sum_{l=1}^p \left[\frac{2V_l}{D-2} r^{-2b_l\delta} - \frac{(D-3)\lambda_l^2}{2(D-2)} r^{-2a_l\delta-2(D-2)} \right] \\ & - \sum_{i=p+1}^{N-1} \frac{(D-3)\lambda_i^2}{2(D-2)} r^{-2a_i\delta-2(D-2)} + \sum_{j=p+1}^{M-1} \frac{2V_j}{D-2} r^{-2b_j\delta}. \end{aligned} \quad (2.21)$$

Substitution of eq. (2.20) into (2.17b) and splitting the sums gives

$$\begin{aligned} & (D-3)kr^{-2} - (D-3-n+\gamma)Br^{\gamma-2-2n} \\ & - \sum_{l=1}^p (D-3-n+\theta_l)L_l^{-2} r^{\theta_l-2-2n} - \sum_{i=p+1}^{N-1} (D-3-n+\omega_i)q_i^2 r^{\omega_i-2-2n} \end{aligned}$$

$$\begin{aligned}
 & - \sum_{j=p+1}^{M-1} (D-3-n+\sigma_j) \ell_j^{-2} r^{\sigma_j-2-2n} \\
 & - (D-3-n-c_0) \mu r^{c_0-2-2n} \\
 & = \frac{2V_M}{D-2} r^{-2b_M\delta} + \frac{\lambda_N^2}{2(D-2)} r^{-2a_N-2(D-2)} \\
 & + \sum_{l=1}^p \left[\frac{2V_l}{D-2} r^{-2b_l\delta} + \frac{\lambda_l^2}{2(D-2)} r^{-2a_l\delta-2(D-2)} \right] \\
 & + \sum_{i=p+1}^{N-1} \frac{\lambda_i^2}{2(D-2)} r^{-2a_i\delta-2(D-2)} + \sum_{j=p+1}^{M-1} \frac{2V_j}{D-2} r^{-2b_j\delta}. \quad (2.22)
 \end{aligned}$$

Substitution of eq. (2.20) into (2.17c) and splitting the sums gives

$$\begin{aligned}
 & - \delta(D-3-n+\gamma) B r^{\gamma-2n-2} \\
 & - \sum_{l=1}^p \delta(D-3-n+\theta_l) L_l^{-2} r^{\theta_l-2n-2} - \sum_{i=p+1}^{N-1} \delta(D-3-n+\omega_i) q_i^2 r^{\omega_i-2n-2} \\
 & - \sum_{j=p+1}^{M-1} \delta(D-3-n+\sigma_j) \ell_j^{-2} r^{\sigma_j-2n-2} - \delta(D-3-n+c_0) \mu r^{c_0-2n-2} \\
 & = 4b_M V_M r^{-2b_M\delta} + a_N \lambda_N^2 r^{-2a_N\delta-2(D-2)} \\
 & + \sum_{l=1}^p \left[4b_l V_l r^{-2b_l\delta} + a_l \lambda_l^2 r^{-2a_l\delta-2(D-2)} \right] \\
 & + \sum_{i=p+1}^{N-1} a_i \lambda_i^2 r^{-2a_i\delta-2(D-2)} + \sum_{j=p+1}^{M-1} 4b_j V_j r^{-2b_j\delta}. \quad (2.23)
 \end{aligned}$$

Firstly, if $c_0 = -(D-3-n)$, then the terms involving μ vanishes from the equations and. In f , it appears as

$$-\mu r^{-(D-3-n)}, \quad (2.24)$$

which is the Schwarzschild-like term and is associated with the presence of the black hole.

The rest of the equations are solved via term-by-term comparisons. The remaining terms in the left-hand sides of these equations involve the coefficients k , B , $\frac{1}{L_l^2}$, q_i^2 , and $\frac{1}{\ell_j^2}$, while the right-hand sides of the equations involve $\lambda_1, \dots, \lambda_p, \lambda_{p+1}, \dots, \lambda_{N-1}, \lambda_N$, and $V_1, \dots, V_p, V_{p+1}, \dots, V_{M-1}, V_M$. We match the terms on both sides of each equations by the following scheme:

$$\begin{aligned}
 & (\text{terms involving } L_l^{-2}) \leftrightarrow (\text{terms involving } \lambda_l, V_l), \quad l = 1, \dots, p, \\
 & (\text{terms involving } q_i) \leftrightarrow (\text{terms involving } \lambda_i), \quad i = p+1, \dots, N-1 \\
 & (\text{terms involving } \ell_j^{-2}) \leftrightarrow (\text{terms involving } V_j), \quad j = p+1, \dots, M-1, \\
 & (\text{terms involving } k, B) \leftrightarrow (\text{terms involving } \lambda_N, V_M). \quad (2.25)
 \end{aligned}$$

For example, matching terms involving q_i with terms containing λ_i in eq. (2.21), (2.22), and (2.23) respectively gives

$$-\frac{1}{2}(D-3-n+\omega_i)\omega_i q_i^2 r^{\omega_i-2-2n} = -\frac{(D-3)\lambda_i^2}{2(D-2)} r^{-a_i\delta-2(D-2)}, \quad (2.26)$$

$$-(D-3-n+\omega_i)q_i^2 r^{\omega_i-2-2n} = \frac{\lambda_i^2}{2(D-2)} r^{-2a_i\delta-2(D-2)}, \quad (2.27)$$

$$-\delta(D-3-n+\omega_i)q_i^2 r^{\omega_i-2-2n} = a_i\lambda_i^2 r^{-2a_i\delta-2(D-2)}. \quad (2.28)$$

Demanding that the coefficients and exponents of r on both sides of each equations to be equal gives

$$\begin{aligned} \omega_i &= -2(D-3), \quad a_i\delta = n, \\ \lambda_i &= \sqrt{2(D-2)(D-3+n)}q_i. \end{aligned} \quad (2.29)$$

Similarly, matching the terms involving $\frac{1}{L_l^2}$ with the terms containing λ_l and V_l gives three algebraic equations. Matching the coefficients and exponents of r , we find

$$\begin{aligned} \theta_l &= 2+2n-2\nu_l, \quad a_l\delta = \nu_l - (D-2), \\ \lambda_l^2 &= 2(D-1+n-2\nu_l)(n-\nu_l)\frac{1}{L_l^2}, \\ 2V_l &= 2\Lambda_l e^{2b_l\psi_0} = -(D-1+n-2\nu_l)(D-2+n-\nu_l)\frac{1}{L_l^2}, \end{aligned} \quad (2.30)$$

where we have introduced the parametrisation $b_l\delta = \nu_l$. Next, matching the terms containing $\frac{1}{\ell_j^2}$ with terms containing V_j gives

$$\begin{aligned} \sigma_j &= 2, \quad b_j\delta = n, \\ 2V_j &= 2\Lambda_j e^{2b_j\psi_0} = -(D-2)(D-1-n)\frac{1}{\ell_j^2}, \end{aligned} \quad (2.31)$$

whereas matching the terms involving B with terms containing λ_N and V_M results in equations which are solved by

$$\begin{aligned} \gamma &= 2n, \quad b_M\delta = 1, \quad a_N\delta = -(D-3), \\ 2V_M &= 2\Lambda_M e^{2b_M\psi_0}, \quad B = \frac{(D-3)^2 k - 2\Lambda_M e^{2b_M\psi_0}}{(D-3+n)^2} = \frac{(D-3)k - (D-3+n)\rho^2}{(1-n)(D-3+n)}, \end{aligned} \quad (2.32)$$

where $\lambda_N = \sqrt{2(D-3+n)}\rho$.

The gauge potentials are obtained upon integrating the Maxwell equations again, which gives

$$\chi_l = \Phi_l + e^{\alpha_l \psi_0} \sqrt{\frac{2(n - \nu_l)}{D - 1 + n - 2\nu_l}} \frac{r^{D-1+n-2\nu_l}}{L_l}, \quad (2.33a)$$

$$\chi_i = \Phi_i - e^{\alpha_i \psi_0} \sqrt{\frac{2(D-2)}{D-3+n}} q_i r^{-(D-3+n)}, \quad (2.33b)$$

$$\chi_N = \Phi_N + e^{\alpha_N \psi_0} \sqrt{\frac{2}{D-3+n}} \rho r^{D-3+n}. \quad (2.33c)$$

where, Φ_l , Φ_i , and Φ_N above are the integration constants.

We collect and summarise our results: reiterating for the benefit of the reader who has skipped the details of the derivation, the indices l , i , and j , appearing in the equations below are taken to run as follows

$$l \in \{1, \dots, p\}, \quad i \in \{p+1, \dots, N-1\}, \quad j \in \{p+1, \dots, M-1\}. \quad (2.34)$$

The metric, gauge potentials and dilaton field which solves the equations of motion (2.3), (2.4), and (2.5) are

$$ds^2 = -f dt^2 + h dr^2 + r^2 \tilde{\gamma}_{ab} dy^a dy^b, \quad (2.35a)$$

$$f = Br^{2n} - \frac{\mu}{r^{D-3-n}} + \sum_{l=1}^p \frac{r^{2+2n-2\nu_l}}{L_l^2} + \sum_{i=p+1}^{N-1} \frac{q_i^2}{r^{2(D-3)}} + \sum_{j=p+1}^{M-1} \frac{r^2}{\ell_j^2}, \quad (2.35b)$$

$$h = \frac{r^{2n}}{f}, \quad (2.35c)$$

$$A_l = \left(\Phi_l + e^{a_l \psi_0} \sqrt{\frac{2(n - \nu_l)}{D - 1 + n - 2\nu_l}} \frac{r^{D-1+n-2\nu_l}}{L_l} \right) dt, \quad (2.35d)$$

$$A_i = \left(\Phi_i - e^{-a_i \psi_0} \sqrt{\frac{2(D-2)}{D-3+n}} q_i r^{-(D-3+n)} \right) dt, \quad (2.35e)$$

$$A_N = \left(\Phi_N + e^{a_N \psi_0} \sqrt{\frac{2}{D-3+n}} \rho r^{D-3+n} \right) dt, \quad (2.35f)$$

$$\psi = -\sqrt{2(D-2)n} \ln r + \psi_0, \quad (2.35g)$$

provided that the Liouville potential strengths are related to B , $\frac{1}{L_l^2}$, $\frac{1}{\ell_j^2}$ by

$$2\Lambda_l e^{2b_l \psi_0} = -\frac{(D-1+n-2\nu_l)(D-2+n-\nu_l)}{L_l^2}, \quad (2.36a)$$

$$2\Lambda_j e^{2b_j \psi_0} = -\frac{(D-2)(D-1-n)}{\ell_j^2}, \quad (2.36b)$$

$$B = \frac{(D-3)^2 k}{(D-3+n)^2} - \frac{2\Lambda_M e^{2b_M \psi_0}}{(D-3+n)^2} = \frac{(D-3)k - (D-3+n)\rho^2}{(1-n)(D-3+n)}, \quad (2.36c)$$

and that the coupling constants are related to the parameters ν_l and n by

$$\begin{aligned} a_l &= \frac{\nu_l - (D-2)}{\sqrt{2(D-2)n}}, & a_i &= \sqrt{\frac{n}{2(D-2)}}, & a_N &= -\sqrt{\frac{D-3}{2(D-2)n}}, \\ b_l &= \frac{\nu_l}{\sqrt{2(D-2)n}}, & b_j &= \sqrt{\frac{n}{2(D-2)}}, & b_M &= \frac{1}{\sqrt{2(D-2)n}}. \end{aligned} \quad (2.37)$$

We recover known solutions by the following choices of the parameters:

- **Lifshitz spherical/planar black hole:** the Lifshitz black hole is recovered by setting $\nu_l = 0$ and $\Lambda_j = 0$. The latter condition is tantamount to having $\ell_j^2 \rightarrow \infty$ and all the terms proportional to r^2 in f will vanish. Then, with $\nu_l = 0$, the p gauge fields are redundant since they all give rise to the same term proportional to r^{2+2n} , so we might well consider $p = 1$. If we further let $z = n + 1$, the function f can be rewritten as

$$f = r^{2z} \left[\frac{(D-3)^2 k}{(D-3+n)^2 r^2} + \frac{1}{L^2} - \frac{\mu}{r^{D-2-z}} + \sum_{i=2}^{N-1} q_i^2 r^{-2(D-3+z)} \right]. \quad (2.38)$$

We see that the term in the square brackets is precisely the function denoted as b_k in [15] and that z is the familiar Lifshitz exponent. Along with rest of the solution, we have recovered the Lifshitz spherical/planar black hole derived by [15].

- **Charged black brane with hyperscaling-violating factor:** if, instead we set $\rho = k = 0$, and $\ell_j^2 \rightarrow 0$, we effectively turn off $M - p$ scalar potentials and one U(1) field. The choice $k = 0$ then corresponds to a planar horizon. If we further have $p = 1$ and $N = 2$, we are left with two U(1) fields and one scalar potential. This solution corresponds to the charged planar black hole with hyperscaling violation found in [16]. To see this explicitly, we reparametrise n and ν_1 by

$$n = \frac{(z-1)(D-2) - \theta}{D-2-\theta}, \quad \nu_1 = -\frac{\theta}{D-2-\theta}, \quad (2.39)$$

and our function f is written as

$$f = r^{\frac{2z(D-2)-2\theta}{D-2-\theta}} \left(-\mu r^{-\frac{(D-2)(D-2+z-\theta)}{D-2-\theta}} + \frac{1}{L_1^2} + q_2^2 r^{-\frac{2(D-2)(D-3+z-\theta)}{D-2+z-\theta}} \right). \quad (2.40)$$

Further introducing a coordinate transformation

$$r = \varrho^{\frac{D-2-\theta}{D-2}}, \quad (2.41)$$

the solution becomes

$$ds^2 = \varrho^{-\frac{2\theta}{D-2}} \left(-\varrho^{2z} U dt^2 + \frac{d\varrho^2}{\varrho^2 U} + \varrho^2 d\vec{x}_{(D-2)}^2 \right), \quad (2.42)$$

$$U = -\frac{\mu}{\varrho^{D-2+z-\theta}} + \frac{1}{L_1^2} + \frac{q_2^2}{\varrho^{D-3+z-\theta}}, \quad (2.43)$$

where we recover the form of the solution given in [16], up to a rescaling of parameters and coordinates.

- **The dilaton black hole:** if we set $\nu_l = n$, then the gauge fields χ_l where $l = 1, \dots, p$ are switched off. Then, eq. (2.36a) becomes the same as eq. (2.36b), $b_l = b_j = \sqrt{\frac{n}{2(D-2)}}$. Then $M - 1$ terms in the dilaton potential become identical and hence redundant, because

$$\begin{aligned} \mathcal{V} &= \sum_{l=1}^p 2\Lambda_l e^{2b_l \psi} + \sum_{i=p+1}^{M-1} 2\Lambda_i e^{2b_i \psi} + 2\Lambda_M e^{2b_M \psi} \\ &= 2 \left(\sum_{j=1}^p \Lambda_j + \sum_{i=p+1}^{M-1} \Lambda_i \right) e^{\sqrt{\frac{2n}{D-2}} \psi} + 2\Lambda_M e^{2b_M \psi}. \end{aligned} \quad (2.44)$$

We can then rename $\sum_{j=1}^p \Lambda_j + \sum_{i=p+1}^{M-1} \Lambda_i = \Lambda_1$, and $M = 2$ suffices. We then have

$$2\Lambda_1 e^{2b_1 \psi_0} = -(D-2)(D-1-n)\ell^{-2}, \quad b_1 = \sqrt{\frac{n}{2(D-2)}}. \quad (2.45)$$

Further introducing the transformation

$$r = \varrho^{\frac{1}{1+n}}, \quad t = (1+n)\hat{t}, \quad (2.46)$$

the solution becomes

$$ds^2 = -U d\hat{t}^2 + U^{-1} d\varrho^2 + \varrho^{\frac{1}{1+n}} d\Omega_{k, (D-2)}^2, \quad (2.47)$$

$$U = (1+n)^2 \left(B \varrho^{\frac{2n}{1+n}} - \mu \varrho^{\frac{n-(D-3)}{1+n}} + \sum_{i=1}^{N-1} q_i^2 \varrho^{-\frac{2(D-3)}{1+n}} + \frac{\varrho^{\frac{2}{1+n}}}{\ell^2} \right). \quad (2.48)$$

At the moment, the solution still contains $N - 1$ non-zero $U(1)$ charges. If we set all but one of them to zero, we recover the charged dilaton black hole with two exponential potentials originally obtained in ref. [24].

The intuition we have learned in solving the equations of motion with this ansatz is that all but two terms of f in (2.35b) require a source which is either a Liouville potential, gauge field, or both. Explicitly, the terms $q_i^2 / r^{2(D-3)}$ are supported by $U(1)$ gauge fields, r^2 / ℓ_j^2 are supported by Liouville potentials, and $r^{2+2n-2\nu_l} / L_l^2$ are simultaneously supported by gauge fields and Liouville potentials. Turning off any of the fields or potentials appropriately will set its corresponding terms to zero.

The first two terms of f are $B r^{2n} - \mu / r^{D-3-n}$, clearly this reduces to the usual Schwarzschild-Tangherlini form $1 - \mu / r^{D-3}$ when all the matter fields are turned off. The second term here is related to the black hole mass and is not affected by the presence of the matter fields, though the presence of the M -th potential and N -th gauge field modifies the particular value of B in accordance to eq. (2.36c).

Before closing this section, let us briefly check whether the solution satisfies the Null Energy Condition. For an arbitrary null vector k^μ , the Null Energy Condition is satisfied

if $R_{\mu\nu}k^\mu k^\nu \geq 0$. To check this, we require the explicit Ricci tensor components of the solution, which are

$$R_{tt} = -f\mathcal{E}_1, \quad R_{rr} = \frac{r^{2n}}{f}\mathcal{E}_2, \quad R_{ij} = r^2\tilde{\gamma}_{ij}\mathcal{E}_3, \quad (2.49)$$

where

$$\mathcal{E}_1 = -\frac{1}{2r^n} \left(\frac{f'}{r^n} \right)' - \frac{D-2}{2} \frac{1}{r^{2n+1}} f', \quad (2.50a)$$

$$\mathcal{E}_2 = -\frac{1}{2r^n} \left(\frac{f'}{r^n} \right)' - \frac{D-2}{2} \frac{1}{r^{2n+1}} f' + \frac{(D-2)n}{r^{2n+2}} f, \quad (2.50b)$$

$$\mathcal{E}_3 = \frac{D-3}{r^2} \left(k - \frac{1}{r^{2n}} f \right) + \frac{n}{r^{2n+2}} f - \frac{1}{r^{2n+1}} f'. \quad (2.50c)$$

In the present coordinates, an arbitrary null vector can be written in the form $k^\mu = (k^t, k^r, k^i)$, for $i = 1, \dots, D-2$. Therefore, we have

$$R_{\mu\nu}k^\mu k^\nu = \frac{r^{2n}}{f} (\mathcal{E}_2 - \mathcal{E}_1) (k^r)^2 + (\mathcal{E}_3 - \mathcal{E}_1) r^2 \tilde{\gamma}_{ij} k^i k^j, \quad (2.51)$$

where we have used the null property $k_\mu k^\mu = 0$ to eliminate k^t . The Null Energy Condition is satisfied for any arbitrary null vector if $\mathcal{E}_2 - \mathcal{E}_1 \geq 0$ and $\mathcal{E}_3 - \mathcal{E}_1 \geq 0$ are satisfied separately. Firstly, we see that

$$\mathcal{E}_2 - \mathcal{E}_1 = \frac{(D-2)n}{r^{2n+2}} f. \quad (2.52)$$

This will be positive outside the horizon for $n \geq 0$. For the second condition, we find by direct computation

$$\mathcal{E}_3 - \mathcal{E}_1 = \frac{(D-3+n)\rho^2}{r^2} + \sum_{i=p+1}^{N-1} \frac{(D-2)(D-3+n)q_i^2}{r^{2(D-2-n)}} + \sum_{l=1}^p \frac{r^{2+2n-2\nu_l}}{L_l^2}. \quad (2.53)$$

Clearly, every term in the expression is positive, thus $\mathcal{E}_3 - \mathcal{E}_1 \geq 0$. Therefore, provided that $n \geq 0$, the Null Energy Condition is satisfied.

3 Physical and geometrical properties

For concreteness in the rest of the paper, we shall henceforth consider the simplest charged non-trivial case $M = N = 3$. Loosely speaking, this is the least number of fields which has a charged solution with a spherical horizon and non-flat/non-AdS asymptotics which contain the afore-mentioned Lifshitz black hole and dilaton black hole.

Explicitly, the solution for $M = N = 3$ is given by

$$ds^2 = -f dt^2 + r^{2n} f^{-1} dr^2 + r^2 \tilde{\gamma}_{ab} dy^a dy^b, \quad (3.1a)$$

$$f = Br^{2n} - \frac{\mu}{r^{D-3-n}} + \frac{r^{2+2n-2\nu}}{L^2} + \frac{q^2}{r^{2(D-3)}} + \frac{r^2}{\ell^2}, \quad (3.1b)$$

$$\psi = -\sqrt{2(D-2)n} \ln r + \psi_0, \quad (3.1c)$$

$$A_1 = \left(\Phi_1 + e^{a_1 \psi_0} \sqrt{\frac{2(n-\nu)}{D-1+n-2\nu}} \frac{r^{D-1+n-2\nu}}{L} \right) dt, \quad (3.1d)$$

$$A_2 = \left(\Phi_2 - e^{a_2 \psi_0} \sqrt{\frac{2(D-2)}{D-3+n}} q r^{-(D-3+n)} \right) dt, \quad (3.1e)$$

$$A_3 = \left(\Phi_3 + e^{a_3 \psi_0} \sqrt{\frac{2}{D-3+n}} \rho r^{D-3+n} \right) dt, \quad (3.1f)$$

$$2\Lambda_1 e^{2b_1 \psi_0} = -(D-1+n-2\nu)(D-2+n-\nu)L^{-2}, \quad (3.1g)$$

$$2\Lambda_2 e^{2b_2 \psi_0} = -(D-2)(D-1-n)\ell^{-2}, \quad (3.1h)$$

$$B = \frac{(D-3)^2 k - 2\Lambda_3 e^{2b_3 \psi_0}}{(D-3+n)^2} = \frac{(D-3)k - (D-3+n)\rho^2}{(1-n)(D-3+n)}, \quad (3.1i)$$

with

$$\begin{aligned} a_1 &= \frac{\nu - (D-2)}{\sqrt{2(D-2)n}}, & a_2 &= \sqrt{\frac{n}{2(D-2)}}, & a_3 &= -\frac{D-3}{\sqrt{2(D-2)n}}, \\ b_1 &= \frac{\nu}{\sqrt{2(D-2)n}}, & b_2 &= \sqrt{\frac{n}{2(D-2)}}, & b_3 &= \frac{1}{\sqrt{2(D-2)n}}. \end{aligned} \quad (3.2)$$

First, let us account for the independent parameters of this solution. The first two of these are straightforwardly L^2 and ℓ^2 , which fixes Λ_1 and Λ_2 via eq. (3.1g) and (3.1h). If we wish to consider the cases where Λ_1 and/or Λ_2 are positive, we replace $L^2 \rightarrow -L^2$ and/or $\ell^2 \rightarrow -\ell^2$. Next, we have n which controls the strength of ψ . The parameter ν determines the strength of the gauge potential A_1 together with n . The parameter q determines the strength of A_2 , and is this particular U(1) field that behaves like a Reissner-Nordström charge in terms of the spacetime and its thermodynamics.

The quantities B , Λ_3 , and ρ are intertwined in the two equalities in (3.1i). Hence there should be one free parameter among them. (Since n has already been accounted for.) Solving for Λ_3 , or ρ , we find

$$2\Lambda_3 e^{2b_3 \psi_0} = \frac{(D-3+n)^2 \rho^2 - n(D-2)(D-3)k}{1-n}, \quad (3.3)$$

$$\rho^2 = \frac{(1-n)2\Lambda_3 e^{2b_3 \psi_0} + n(D-2)(D-3)k}{(D-3+n)^2}. \quad (3.4)$$

Without loss of generality, let us choose Λ_3 as the independent parameter. Subsequently ρ will be determined from eq. (3.4). With these two quantities will then fix B via eq. (3.1i).

Finally, μ is not related to any of the matter fields and is obviously the Schwarzschild-like term responsible for the mass of the black hole, as we will see in further detail below.

To identify the event horizon, we consider $r = \text{constant}$ hypersurfaces. The vector normal to the hypersurface is $\partial_\mu r$, and its norm is $\partial_\mu r \partial^\mu r = g^{rr} = \frac{1}{r^{2n}} f$. We see that $r = r_+$, where $f(r_+) = 0$ is a null hypersurface where time-like and null curves in $r < r_+$ cannot escape to $r > r_+$. This can be seen by transforming to Eddington-Finkelstein-like coordinates and observing that light cones at $r < r_+$ are tilted inwards, indicating that future-directed null geodesics goes in the direction of decreasing r . A calculation of the curvature invariants (see eqs. (3.23) and (3.24)) show that the spacetime is regular at the horizon. For the purposes of analysis, it will be convenient to express μ in terms of r_+ via

$$\mu = \frac{r_+^{D-1+n-2\nu}}{L^2} + \frac{r_+^{D-1-n}}{\ell^2} + Br_+^{D-3+n} + q^2 r_+^{-(D-3+n)}, \quad (3.5)$$

and we shall take r_+ as one of the parameters of the solution, where μ can be determined from r_+ using eq. (3.5).

Having the parameters accounted for, we shall regard our solution (3.1) as being parametrised by the following seven quantities:

$$(r_+, q, L^2, \ell^2, \Lambda_3, n, \nu), \quad (3.6)$$

where eq. (3.1d) indicates that ν must take the range $\nu \leq n$. Also, eq. (3.3) forbids $n = 1$, unless $\Lambda_3 = 0$. In the ranges where all the parameters are well-defined, the function f typically has three roots, one of which is our parameter r_+ as intended. The other two roots are denoted r_- and r_c , where

$$r_- \leq r_+ < r_c, \quad (3.7)$$

where r_- is non-zero if q is non-zero and r_c is finite if Λ_1 and/or Λ_2 are positive. The spacetime will be static with Lorentzian signature $(-, +, \dots, +)$ in the range $r_+ < r < r_c$ and $r < r_-$. As we will show below, the latter region contains a curvature singularity at $r = 0$, and hence we shall mainly be interested in physical quantities measurable by observers at $r_+ < r < r_c$, where the curvature singularity is hidden by the horizon r_+ . This range is understood to include the case $r_c = \infty$ for negative Λ_1 and Λ_2 .

For the various physical and thermodynamic quantities below, we first evaluate quantities in the spacetime region

$$r_+ < r < r_b. \quad (3.8)$$

The boundary $r = r_b$ shall be denoted as $\partial\mathcal{M}$ with unit normal

$$n^\mu = r_b^{-n} \sqrt{f(r_b)} \delta_r^\mu. \quad (3.9)$$

The spacetime carries a time-like Killing vector $\xi^\mu \propto \delta_t^\mu$. It can be explicitly checked that ξ^μ solves the Killing equation if the proportionality factor is a constant. We shall choose the proportionality constant such that $\xi_\mu \xi^\mu = -1$ at the boundary. Therefore,

$$\xi^\mu = \frac{1}{\sqrt{f(r_b)}} \delta_t^\mu. \quad (3.10)$$

To calculate the mass, we adopt the boundary stress tensor procedure of [44] (see also [39]), where we consider the $r = r_b$ unit normal given by eq. (3.9). The strategy is as follows: the procedure is to calculate the boundary stress tensor which is associated with the quasi-local energy contained in the volume inside the boundary where r_b at this stage is large but finite. Next one has to perform an appropriate subtraction to remove the contribution of the ‘background’ spacetime. (Here, we shall take the ‘background’ as the spacetime corresponding to $\mu = 0$ in the metric.) The resulting stress tensor can then be regarded as coming purely from the black hole’s contribution. Furthermore, upon subtraction there will no longer be terms that grow with r_b . The limit $r_b \rightarrow \infty$ can then be taken safely. Note that the scalar invariant $\xi_\mu \xi^\mu = -1$ remains finite at any r_b , by construction.

The induced metric at the boundary is $\gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$. The boundary stress tensor is the variation of the action with respect to $\gamma_{\mu\nu}$ [39]:

$$T_{\mu\nu} = \frac{2}{\sqrt{|\gamma|}} \frac{\delta I}{\delta \gamma_{\mu\nu}} = -\frac{1}{8\pi G} (K_{\mu\nu} - K\gamma_{\mu\nu}), \quad (3.11)$$

evaluated at some $r = r_b$. In general, the energy calculated from this stress tensor will grow with r_b , as it accounts for the energy of the entire spacetime. Hence we shall subtract the contribution from an appropriately chosen background and the remaining quantity is independent of r_b and is to be regarded as the contribution from the black hole alone.

To this end, we shall take our background $g_{\mu\nu}^0$ as the $\mu = 0$ case of the metric. The time coordinate of the background is scaled appropriately so that $g_{\tau\tau}^0 = g_{tt}$ at the boundary [44]. We then repeat the procedure of the above paragraph for this spacetime to obtain its corresponding stress tensor ${}^0T_{\mu\nu}$. Indeed, the choice of background is not unique and is made to agree when the parameters coincide with cases known in earlier literature, especially refs. [15] and [26]. Our background-subtracted stress tensor is obtained by

$$\hat{T}_{\mu\nu} = T_{\mu\nu} - {}^0T_{\mu\nu}. \quad (3.12)$$

As explained in [39], if the Killing vector is normalised according to (3.10), the conserved charge associated with the time-like Killing vector ξ^μ agrees with the quasi-local energy of the spacetime. Upon taking $r_b \rightarrow \infty$, the resulting constant quantity is interpreted as the ‘mass’ of the black hole and is given explicitly by

$$M = \oint_{\partial\mathcal{M}} d^{D-2}x \sqrt{\gamma} \xi^\mu \xi^\nu \hat{T}_{\mu\nu} = \frac{\Omega}{16\pi G} (D-2)\mu, \quad (3.13)$$

where $\Omega = \int d^{D-2}\sigma \sqrt{\gamma}$ is the volume of the part of the spacetime described by $\tilde{\gamma}_{ab}$. It is worth emphasising that all intermediate calculations are performed at finite r_b , and the final expression for the scalar M given in (3.13) is ultimately independent of r_b and therefore the limit $r_b \rightarrow \infty$ can be taken safely. In the topological dilaton black hole case with $\nu = n$, this agrees with the mass obtained by the counter-term method by Cai and Ohta [26], and also agrees in the Lifshitz case ($\nu = 0$) with the calculation by Tarrio and Vandoren [15] wherein the Komar integral method was used.

In the canonical ensemble which we will consider in the thermodynamic analysis, the charge of the spacetime is fixed. In that case a more appropriate background would be the extremal spacetime $g_{\mu\nu}^e$, which is the case where r_- coincides with r_+ . Let μ_e be the mass parameter such that $r_- = r_+$. Then the mass of the black hole measured against this extremal background is then

$$\Delta M = \frac{\Omega}{16\pi G}(D-2)(\mu - \mu_e). \quad (3.14)$$

The respective U(1) charges are calculated by

$$Q_i = \frac{1}{16\pi G} \oint d^{D-2}y \sqrt{\tilde{\gamma}} e^{-2a_i\psi} (F_i)_{\mu\nu} n^\mu \xi^\nu, \quad (3.15)$$

for $i = 1, 2$, and 3 . Calculating the Maxwell tensors from the potentials given in eqs. (3.1d), (3.1e) and (3.1f), the charges are explicitly

$$Q_1 = \frac{\Omega}{16\pi G} e^{-a_1\psi_0} \sqrt{2(n-\nu)(D-1+n-2\nu)} \frac{1}{L}, \quad (3.16)$$

$$Q_2 = \frac{\Omega}{16\pi G} e^{-a_2\psi_0} \sqrt{2(D-2)(D-3+n)} q, \quad (3.17)$$

$$Q_3 = \frac{\Omega}{16\pi G} e^{-a_3\psi_0} \sqrt{2(D-3+n)} \rho. \quad (3.18)$$

We shall also fix the respective gauge potentials so that each A_i are zero at the horizon. This gives

$$\Phi_1 = -e^{\alpha_1\psi_0} \sqrt{\frac{2(n-\nu)}{D-1+n-2\nu}} \frac{r_+^{D-1+n-2\nu}}{L}, \quad (3.19)$$

$$\Phi_2 = e^{\alpha_2\psi_0} \sqrt{\frac{2(D-2)}{D-3+n}} \frac{q}{r_+^{D-3+n}}, \quad (3.20)$$

$$\Phi_3 = -e^{\alpha_3\psi_0} \sqrt{\frac{2}{D-3+n}} \rho r_+^{D-3+n}. \quad (3.21)$$

The horizon area, which will be crucial in the thermodynamic analysis in the next section, is

$$\mathcal{A} = r_+^{D-2} \Omega. \quad (3.22)$$

Briefly looking at some curvature invariants, the Ricci and Kretschmann scalars are

$$R = -\frac{1}{r^n} \left(\frac{f'}{r^n} \right)' + \frac{(D-2)}{r^{1+n}} \left(\frac{2n}{r} f - 2f' \right) + \frac{(D-3)(D-2)}{r} \left(k - \frac{f}{r^n} \right), \quad (3.23)$$

$$\begin{aligned} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} &= \frac{1}{r^{2n}} \left[f''^2 - \frac{2n}{r} f'' f' + \frac{n^2}{r^2} f'^2 \right] + \frac{D-2}{r^{2n+2}} f^2 \left(\frac{2f'^2}{f^2} + \frac{4n^2}{r^2} - \frac{4n}{r} \frac{f'}{f} \right) \\ &\quad + \frac{2(D-2)(D-3)}{r^4} \left(k - \frac{1}{r^{2n}} f \right)^2, \end{aligned} \quad (3.24)$$

we find that a curvature singularity occurs for $r = 0$, and it persists in the ‘no black hole’ case $\mu = 0$, which is similar to the pure Lifshitz spacetime, as well as the zero mass limit of the dilaton black hole with Liouville potentials. Furthermore, we see that the curvature invariants are regular at the horizon $r = r_+$.

4 Thermodynamics

4.1 The thermodynamic variables

In this section, we shall consider the thermodynamic behaviour of the black hole in the case $M = N = 3$. We observe that, similar to the situation in ref. [15], the potentials A_1 and A_3 diverge at the boundary. Though the former is required for a spherical horizon while the latter supports the asymptotic structure of the spacetime. These two fields, along with their associated conserved charges Q_1 and Q_3 , do not affect the thermodynamics of the black hole. Therefore, strictly speaking, we are considering the fixed charge ensemble if Q_1 and Q_3 are regarded as thermodynamic variables. To keep these charges fixed, we should to the action (2.1) the boundary terms [45]

$$I_{Q_1} + I_{Q_3} = \frac{1}{16\pi G} \int d^{D-1}x \sqrt{-\gamma} n_\nu \left[(A_1)_\mu e^{-2a_1\psi} (F_1)^{\mu\nu} + (A_2)_\mu e^{-2a_2\psi} (F_2)^{\mu\nu} \right], \quad (4.1)$$

where n_μ is unit normal at $\partial\mathcal{M}$ as defined in eq. (3.9) and $\gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$, as in the previous section. This ensures we have a action with a well-defined variational principle. Furthermore, the condition for a spherical horizon $k = 1$ is obtained by adjusting two parameters, namely ρ and Λ_3 in eq. (3.1i). Therefore it is possible to set $\rho = 0$ by choosing an appropriate Λ_3 and A_3 vanishes altogether. Henceforth, A_1 , A_3 , Q_1 , and Q_3 do not play any role as thermodynamic variables.

On the other hand, A_2 and its associated conserved charge Q_2 play the role of the Reissner-Nordström-type potentials and charges, and do contribute to the thermodynamic behaviour of the solution. As they are the only U(1) gauge fields relevant to the thermodynamic analysis, in this section we shall drop the subscripts and simply refer to them as A and Q . In the thermodynamic analysis, the constant ψ_0 only modifies the values of Q and A by an overall factor. Hence may set $\psi_0 = 0$ without loss of generality. Therefore, the charge Q shall be explicitly given as

$$Q = \frac{\Omega}{8\pi G} \sqrt{\frac{1}{2}(D-2)(D-3+n)}q, \quad (4.2)$$

The temperature of the black hole can be obtained by ensuring the Euclidean time $\tau = it$ has the appropriate periodicity so that the Euclideanised metric is regular at the horizon [46]. The temperature T is then obtained as the inverse of this periodicity, where the resulting expression is $T = \frac{1}{4\pi} r_+^{-n} f'(r_+)$. In terms of the spacetime parameters, the temperature is explicitly

$$T = \frac{1}{4\pi} \left[(D-1+n-2\nu)L^2 r_+^{1+n-2\nu} + (D-1-n)\ell^{-2} r_+^{1-n} + (D-3+n)B r_+^{-1+n} - (D-3+n)q^2 r_+^{-(2D-5+n)} \right]. \quad (4.3)$$

The case of extremal black holes is where the temperature is zero. The value of horizon radius which gives $T = 0$ is $r_+ = r_e$, where r_e satisfies the equation

$$q^2 r_e^{-(D-3+n)} = \frac{D-1+n-2\nu}{D-3+n} L^{-2} r_e^{D-1+n-2\nu} + \frac{D-1-n}{D-3+n} \ell^{-2} r_e^{D-1-n} + B r_e^{D-3+n}. \quad (4.4)$$

The corresponding mass parameter in the extremal case is

$$\mu_e = \frac{2(D-2+n-\nu)}{D-3+n} L^{-2} r_e^{D-1+n-2\nu} + \frac{2(D-2)}{D-3+n} \ell^{-2} r_e^{D-1-n} + 2B r_e^{D-3+n}. \quad (4.5)$$

The entropy of the black hole is, in the units of the present paper, is

$$S = \frac{\mathcal{A}}{4G}, \quad (4.6)$$

where $\mathcal{A}$ is the horizon area given by eq. (3.22).

In the grand canonical ensemble, the gauge potential is held fixed at the boundary at value Φ , and serves as the variable conjugate to the charge Q ,

$$\Phi = \sqrt{\frac{2(D-2)}{D-3+n}} q r_+^{-(D-3+n)}. \quad (4.7)$$

The energy E in this ensemble is taken to be the black hole mass M as given by (3.13). With these quantities, it can be checked that the first law of thermodynamics hold:

$$dE = TdS + \Phi dQ. \quad (4.8)$$

For the canonical ensemble, the charge Q is held fixed. Therefore $dQ = 0$ and we take its conjugate variable to be the difference of the potential with the extremal case, $\Phi - \Phi_e$, where $\Phi_e = \sqrt{\frac{D-2}{2(D-3+n)}} q r_e^{-(D-3+n)}$. Also, the energy should appropriately be measured against the background of the extremal spacetime, where its horizon has zero temperature. Hence we shall take the energy of the canonical ensemble to be $\Delta E = \Delta M$, where ΔM is given in eq. (3.14). In this case, the first law can be checked to read

$$d(\Delta E) = TdS. \quad (4.9)$$

4.2 Canonical ensemble

The relevant thermodynamic potential in the canonical ensemble is the Helmholtz free energy

$$\begin{aligned} \mathcal{F} &= \Delta E - TS \\ &= \frac{\Omega}{16\pi G} \left[- (1+n-2\nu) \ell^{-2} r_+^{D-1+n-2\nu} - (1-n) \ell^{-2} r_+^{D-1-n} \right. \\ &\quad \left. + (1-n) B r_+^{D-3+n} + (2D-5+n) q^2 r_+^{-(D-3+n)} \right. \\ &\quad \left. - (D-2) \left(\frac{D-1+n-2\nu}{D-3+n} L^{-2} r_e^{D-1+n-2\nu} + \frac{D-1-n}{D-3+n} \ell^{-2} r_e^{D-1-n} + B r_e^{D-3+n} \right) \right]. \end{aligned} \quad (4.10)$$

Let us also write the temperature here explicitly for convenient reference:

$$T = \frac{1}{4\pi} \left[(D-1+n-2\nu)L^{-2}r_+^{1+n-2\nu} + (D-1-n)\ell^{-2}r_+^{1-n} + (D-3+n)Br_+^{-1+n} - (D-3+n)q^2r_+^{-(2D-5+n)} \right]. \quad (4.11)$$

The heat capacity at constant charge is

$$C_Q = T \left(\frac{\partial S}{\partial T} \right)_Q = \frac{\partial E / \partial r_+}{\partial T / \partial r_+}, \quad (4.12)$$

where

$$\begin{aligned} \frac{\partial E}{\partial r_+} &= \frac{\Omega(D-2)}{4G} r_+^{D-3} T, \\ \frac{\partial T}{\partial r_+} &= \frac{1}{4\pi} r_+^{-1} \left[(1+n-2\nu)(D-1+n-2\nu)L^{-2}r_+^{1+n-2\nu} + (1-n)(D-1-n)\ell^{-2}r_+^{1-n} - (1-n)(D-3+n)Br_+^{-(1-n)} + (2D-5+n)(D-3+n)q^2r_+^{-(2D-5+n)} \right]. \end{aligned} \quad (4.13)$$

$$(4.14)$$

Recall that for fixed n , the parameter ν ranges from $\nu = 0$ to $\nu = n$. The former case being the Lifshitz black hole and the latter case corresponds to a charged dilaton black hole with two exponential potentials. We find that both cases share similar thermodynamic behaviour connected by the continuous parameter ν .

In particular, for sufficiently small charge, there exist a possible range where three branches of solutions with different r_+ share the same temperature. This is reflected in eq. (4.11) where the equation $T_0 = T(r_+)$ has possibly multiple roots for some T_0 . As q is increased until q_{crit} , the roots will coalesce to a single point, after which for $q > q_{\text{crit}}$ there will be a unique r_+ for every T . Given some spacetime parameters (n, ν, L^2, ℓ^2, B) at some dimensionality D , the value of q_{crit} is determined by the condition

$$\frac{\partial T}{\partial r_+} = \frac{\partial^2 T}{\partial r_+^2} = 0. \quad (4.15)$$

Figure 1 shows a representative example for the case $n = 0.6$, $\nu = 0.3$, $L^2 = 2$, $\ell^2 = 2$, and $D = 4$. In this case, solving eq. (4.15) gives $q_{\text{crit}} = 0.0588$. For a charge less than q_{crit} , there are three co-existing branches of black holes with temperature $0.3931 \leq T \leq 0.4508$, which can be seen in the solid curve in the left-hand plot in figure 1. The plot of $\mathcal{F}$ vs. T for this charge is shown in the solid curve on the right-hand plot of figure 1, where we can see its characteristic swallowtail structure. Recall that the thermodynamically favoured phase is the one with lower Helmholtz free energy. Hence for a given T with multiple values of $\mathcal{F}$, the lower branch is preferred. At $q = q_{\text{crit}}$, the three branches collapse to a point and the swallowtail in the $\mathcal{F}$ - T curve shrinks to a single kink. Further increasing beyond $q > q_{\text{crit}}$, the kink disappears and $\mathcal{F}$ becomes a smooth function of T .

Figure 1. Plots of T vs. r_+ (left) and $\mathcal{F}$ vs. T (right) for the case $n = 0.6$, $\nu = 0.3$, $L^2 = \ell^2 = 2$, and $D = 4$. In this case, the critical charge has the value $q_{\text{crit}} = 0.0588$.

Figure 2. Plots of T vs. r_+ (left) and $\mathcal{F}$ vs. T (right) for $q = \sqrt{0.3}q_{\text{crit}}$ in the case $n = 0.4$, $\nu = 0.3$, $L^2 = \ell^2 = 2$, and $D = 4$. In this case, the critical charge has the value $q_{\text{crit}} = 0.0868$.

Let us explore another case of $q < q_{\text{crit}}$ in further detail, this time with $n = 0.4$, $\nu = 0.3$, $L^2 = \ell^2 = 2$ and $D = 4$. The three branches are labelled explicitly in figure 2. Branch 1 is a low-temperature solution which starting from the extremal case $r_+ = r_e$ up to $r_+ = 0.2209$. whereas Branch 2 is an unstable one at $0.2209 < r_+ < 0.5920$, and Branch 3 is for $r_+ > 0.5920$. From figure 2, we see that the three branches can co-exist in the temperature range $0.3290 \leq T \leq 0.3606$, and that a first-order phase transition may occur in this range [15, 47].

This behaviour is further corroborated by the heat capacity. The left-hand plot of figure 3 shows the heat capacity C_Q vs. r_+ . The heat capacities of Branches 1 and 3 are positive while it is negative for Branch 2, and they are separated by discontinuities. The plot of C_Q vs. T is shown in the right-hand plot of figure 3, where we observe the co-existing branches in the aforementioned temperature range.

The thermodynamics typically show similar qualitative behaviour for $0 \leq \nu \leq n$, where ν determines the specific value of q_{crit} . In figure 4, some values of q_{crit} as a function of ν are

Figure 3. Plots of C_Q vs. r_+ (left) and C_Q vs. T (right). The parameter choices are the same as for figure 2.

Figure 4. Plots of q_{crit} against ν for various n . Note that the allowed ranges of ν are $\nu \leq n$, and hence each curve terminates at their respective $\nu = n$.

shown for different n . Furthermore, as ν ranges from $\nu = 0$ to $\nu = n$, these observations connect the thermodynamics of the Lifshitz black hole [15] to that of a charged dilaton black hole with two Liouville potentials [48, 49].

4.3 Grand canonical ensemble

In the grand canonical ensemble, the gauge potential Φ is held fixed instead of Q . In this case, the relevant thermodynamic potential is the Gibbs free energy

$$\begin{aligned}
 W &= E - TS - \Phi Q \\
 &= \frac{\Omega}{16\pi G} \left[- (1 + n - 2\nu) L^{-2} r_+^{D-1+n-2\nu} - (1 - n) \ell^{-2} r_+^{D-1-n} \right. \\
 &\quad \left. + (1 - n) \left(B - \frac{2(D-3+n)}{D-2} \Phi^2 \right) r_+^{D-3+n} \right]. \quad (4.16)
 \end{aligned}$$

Figure 5. Plots of T vs. r_+ (left) and W vs. T for the case $n = 0.6$, $\nu = 0.3$, $L^2 = \ell^2 = 2$, and $D = 4$. For these values, the critical potential is $\Phi_c = 0.9882$.

In this ensemble, the system should be parametrised by Φ instead of q . Therefore the temperature is expressed as

$$T = \frac{1}{4\pi} \left[(D-1+n-2\nu)L^{-2}r_+^{1+n-2\nu} + (D-1-n)\ell^{-2}r_+^{1-n} + (D-3+n) \left(B - \frac{2(D-3+n)}{D-2}\Phi^2 \right) r_+^{-1+n} \right]. \quad (4.17)$$

The above relations imply the existence of a critical potential Φ_{crit} , where

$$\Phi_{\text{crit}} = \sqrt{\frac{(D-2)B}{2(D-3+n)}}. \quad (4.18)$$

For $\Phi < \Phi_{\text{crit}}$, the third term in eq. (4.17) is negative, and this allows the possibility two branches of solution with the same temperature. the free energies of these two branches meet at a discontinuous point of the W – T curve, as shown in figure 5, where the numerical values are $n = 0.6$, $\nu = 0.3$, $L^2 = \ell^2 = 2$, and $D = 4$. Increasing the potential to Φ_{crit} , we see that the low-temperature branch vanishes by collapsing into the point $(r_+, T) = (0, 0)$ and only one smooth branch W – T curve remains. This behaviour persists for various values of ν in $0 \leq \nu \leq n$, and is similar to the thermodynamics observed in the Reissner-Nordström-Anti-de Sitter black hole (RN-AdS) [47] as well as the spherical Lifshitz black hole [15].

4.4 Electrical stability

In addition to the heat capacity and the analytical behaviour of the free energies, the thermodynamic stability of the solution is also characterised by the isothermal charge susceptibility,

$$\chi = \left(\frac{\partial Q}{\partial \Phi} \right)_T. \quad (4.19)$$

If $\chi \geq 0$, the solution will be stable to electrical fluctuations. (See, e.g., [50]).

To see the electrical stability of the solution, we shall consider isothermal curves of the equation of state $T = T(\Phi, Q)$. To obtain these curves, the charge parameter q should be tuned in eq. (4.3) to keep T fixed. In other words, q is given by

$$q = \frac{r_+^{D-3}}{\sqrt{D-3+n}} \times \sqrt{(D-3+n)Br_+^{2n} + (D-1+n-2\nu)L^{-2}r_+^{2+2n-2\nu} + (D-1-n)\ell^{-2}r_+^2 - 4\pi Tr_+^{1+n}}. \quad (4.20)$$

For this expression of q , the charge and potential at the horizon at fixed temperature is

$$Q = \frac{\Omega}{16\pi G} \sqrt{2(D-2)} r_+^{D-3} \times \sqrt{(D-3+n)Br_+^{2n} + (D-1+n-2\nu)L^{-2}r_+^{2+2n-2\nu} + (D-1-n)\ell^{-2}r_+^2 - 4\pi Tr_+^{1+n}}, \quad (4.21a)$$

$$\Phi = \sqrt{2(D-2)} r_+^{-n} \times \sqrt{(D-3+n)Br_+^{2n} + (D-1+n-2\nu)L^{-2}r_+^{2+2n-2\nu} + (D-1-n)\ell^{-2}r_+^2 - 4\pi Tr_+^{1+n}}. \quad (4.21b)$$

The isothermal charge susceptibility is determined by

$$\chi = \left(\frac{\partial Q}{\partial \Phi} \right)_T = \frac{\partial Q / \partial r_+}{\partial \Phi / \partial r_+}, \quad (4.22)$$

where eqs. (4.21a) and (4.21b) are used to compute the derivatives of Q and Φ with respect to r_+ for constant T .

Firstly, we observe a difference in the present case from the RN-AdS case. In the RN-AdS solution, the isotherms terminate at $Q = 0$, $\Phi = \sqrt{2(D-2)}$ [50].⁴ In our case, due to the r_+^{-n} prefactor in eq. (4.21b), our isotherms do not terminate at a Φ , but rather asymptote to $Q = 0$, $\Phi \rightarrow \infty$ as $r_+ \rightarrow 0$ for $n > 0$. The other qualitative features of the curve are mostly similar to the RN-AdS case. In particular, it depends on whether the temperature is larger or smaller than two special values of T ; firstly, there is a critical temperature T_{crit} where

$$\left(\frac{\partial Q}{\partial \Phi} \right)_{T_{\text{crit}}} = \left(\frac{\partial^2 Q}{\partial \Phi^2} \right)_{T_{\text{crit}}} = 0. \quad (4.23)$$

For temperatures $T \leq T_{\text{crit}}$, we will see that the isotherms consists of two branches, one stable and the other unstable. For $T > T_{\text{crit}}$, there will be two points where $\frac{\partial Q}{\partial \Phi} = 0$,⁵ where the curve in between them has positive slope and hence corresponds to the stable branch.

⁴In the normalised units used in [50], this corresponds to $Q = 0$ and $\Phi = 1$.

⁵Note that when Q is the horizontal axis, $\frac{\partial Q}{\partial \Phi} = 0$ corresponds to a horizontal tangent and $\frac{\partial Q}{\partial \Phi} = \infty$ a vertical tangent.

Another special value of temperature is T_* , where at this temperature, the expression under the square root in eqs. (4.21a) and (4.21b) has a single degenerate root for $r_+ > 0$. For temperatures $T > T_*$, the square root never vanishes and the isotherm is a smooth curve. For $T \leq T_*$, there will be distinct roots for positive r_+ . When the root is reached, Q and Φ vanishes. Therefore the isotherms passes through the origin of the Φ - Q plane.

To clarify the above descriptions, let us consider a representative example of $n = 0.6$, $\nu = 0.3$, $\ell^2 = L^2 = 2$, and $D = 4$, shown in figure 6. For these parameters, up to five significant figures, we find $T_{\text{crit}} \simeq 0.3891$ and $T_* \simeq 0.3934$. The four curves in figure 6 have temperatures $T_1 = 0.95T_{\text{crit}}$, $T_2 = T_{\text{crit}}$, $T_3 = 1.01T_{\text{crit}} < T_*$, and $T_4 = 1.05T_{\text{crit}} > T_*$.

The top curve in figure 6 corresponds to T_1 , which is higher than T_{crit} . In this case, $\frac{dQ}{d\Phi} = \infty$ occurs only at one point, which separates a stable branch (on the right) and an unstable branch (on the left). Aside from the different behaviour at $Q \rightarrow 0$, these are analogous to Branch 3 and Branch 1 in ref. [50]. In the second curve from the top is a curve exactly at the critical temperature. The curve just below corresponds to a temperature $T_{\text{crit}} < T_3 < T_*$. It consists of three branches, separated by two points where $\frac{dQ}{d\Phi} = 0$. The branch that lies between these two points has a positive slope and is a stable branch. This is called Branch 2 in [50]. The rightmost branch (which contains $Q \rightarrow \infty$) is Branch 3 and the leftmost branch is Branch 1. The bottom curve in figure 6 corresponds to $T_4 > T_* > T_{\text{crit}}$. For $T_4 = 1.05T_{\text{crit}}$, the expression under the square root in (4.21a) and (4.21b) vanishes for

$$r_+ = 0, \quad r_+ \simeq 0.3421, \quad r_+ \simeq 0.9210. \quad (4.24)$$

For $r_+ \geq 0.9210$, the curve comes in from $Q = \infty$, and terminates at the origin. For $0.3421 < r_+ < 0.9210$, Q and Φ becomes complex and do not have real points represented on the Q - Φ plane. For $0 \leq r_+ \leq 0.3421$, the curve continues from the origin with positive slope, where it reaches a vertical tangent $\frac{dQ}{d\Phi} = 0$, giving a stable branch lies between these two points. After which, we have an unstable branch where its slope is negative.

5 Conclusion

In summary, we have considered an Einstein-Maxwell-dilaton-type theory consisting of M Liouville potentials and N U(1) gauge fields. With an appropriate ansatz, the equations of motion are solved for arbitrary M and N .

The ansatz was chosen based on the assumption that each term in the metric function g_{tt} is supported by a potential and gauge field, allowing for the possibility that a potential and gauge field may support the same term. By switching off the appropriate potentials and gauge fields, we recover the Lifshitz and dilaton black holes. The solution with $M = N = 3$ corresponds to a metric that interpolates between the two.

Some basic thermodynamic quantities have been studied. In the canonical ensemble, solutions with sufficiently small charge may have three co-existing branches with the same temperature, and that a phase transition may occur among them. In the grand canonical ensemble, solutions with sufficiently small potential, there are two branches with the same temperature. This behaviour shares similar features to the Lifshitz black hole as well as the dilaton black hole to which it interpolates, thus bridging the results of [15] for the Lifshitz

Figure 6. Plots of Φ vs. Q for various fixed temperatures, $0.95T_{\text{crit}}$, T_{crit} , $1.01T_{\text{crit}}$, and T_{crit} , for $n = 0.6$, $\nu = 0.3$, $\ell^2 = L^2 = 2$, and $D = 4$. For these parameters, the critical temperature is $T_{\text{crit}} \simeq 0.3891$.

black hole and [48, 49] for the dilaton black hole. It might be interesting to find exact solutions for other forms of the dilaton potential $\mathcal{V}$, as was shown in [51] for asymptotically-AdS solutions, there are certain forms of potential that lead to thermodynamically stable exact solutions.

Our solution inherits similar problematic features of the Lifshitz and dilaton black holes. In particular, the dilaton ψ diverges at the boundary. It was argued in [15] that this issue might be resolved by an appropriate embedding of the model in string theory. Secondly, the unusual asymptotic structure, being non-asymptotically flat and non-(A)dS complicates the issue of calculations at the boundary. For instance, the choice of background required to renormalise the divergence of the Euclidean action is ambiguous. Instead, a background-independent counter-term was introduced in the Lifshitz case [36] and similarly for the dilaton black hole [26]. However, in the dilaton case, the boundary stress tensor remains ill-defined despite having a finite action [26]. Since the solution of the present paper contains the dilaton black hole as a special case, it inherits the similar issue of the dilaton black hole's boundary stress tensor.

Acknowledgments

This research is supported by Xiamen University Malaysia Research Fund (Grant no. XMUMRF/2019-C3/IMAT/0007).

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] D.T. Son, *Toward an AdS/cold atoms correspondence: A Geometric realization of the Schrödinger symmetry*, *Phys. Rev. D* **78** (2008) 046003 [[arXiv:0804.3972](#)] [[INSPIRE](#)].
- [2] K. Balasubramanian and J. McGreevy, *Gravity duals for non-relativistic CFTs*, *Phys. Rev. Lett.* **101** (2008) 061601 [[arXiv:0804.4053](#)] [[INSPIRE](#)].
- [3] M. Taylor, *Non-relativistic holography*, [arXiv:0812.0530](#) [[INSPIRE](#)].
- [4] M. Taylor, *Lifshitz holography*, *Class. Quant. Grav.* **33** (2016) 033001 [[arXiv:1512.03554](#)] [[INSPIRE](#)].
- [5] C. Park, *Review of the holographic Lifshitz theory*, *Int. J. Mod. Phys. A* **29** (2014) 1430049 [[INSPIRE](#)].
- [6] S. Kachru, X. Liu and M. Mulligan, *Gravity duals of Lifshitz-like fixed points*, *Phys. Rev. D* **78** (2008) 106005 [[arXiv:0808.1725](#)] [[INSPIRE](#)].
- [7] K. Balasubramanian and J. McGreevy, *An Analytic Lifshitz black hole*, *Phys. Rev. D* **80** (2009) 104039 [[arXiv:0909.0263](#)] [[INSPIRE](#)].
- [8] U.H. Danielsson and L. Thorlacius, *Black holes in asymptotically Lifshitz spacetime*, *JHEP* **03** (2009) 070 [[arXiv:0812.5088](#)] [[INSPIRE](#)].
- [9] R.B. Mann, *Lifshitz Topological Black Holes*, *JHEP* **06** (2009) 075 [[arXiv:0905.1136](#)] [[INSPIRE](#)].
- [10] G. Bertoldi, B.A. Burrington and A. Peet, *Black Holes in asymptotically Lifshitz spacetimes with arbitrary critical exponent*, *Phys. Rev. D* **80** (2009) 126003 [[arXiv:0905.3183](#)] [[INSPIRE](#)].
- [11] H.-S. Liu and H. Lü, *Thermodynamics of Lifshitz Black Holes*, *JHEP* **12** (2014) 071 [[arXiv:1410.6181](#)] [[INSPIRE](#)].
- [12] M.H. Dehghani and R.B. Mann, *Lovelock-Lifshitz Black Holes*, *JHEP* **07** (2010) 019 [[arXiv:1004.4397](#)] [[INSPIRE](#)].
- [13] E. Ayon-Beato, A. Garbarz, G. Giribet and M. Hassaine, *Lifshitz Black Hole in Three Dimensions*, *Phys. Rev. D* **80** (2009) 104029 [[arXiv:0909.1347](#)] [[INSPIRE](#)].
- [14] Y. Gim, W. Kim and S.-H. Yi, *The first law of thermodynamics in Lifshitz black holes revisited*, *JHEP* **07** (2014) 002 [[arXiv:1403.4704](#)] [[INSPIRE](#)].
- [15] J. Tarrio and S. Vandoren, *Black holes and black branes in Lifshitz spacetimes*, *JHEP* **09** (2011) 017 [[arXiv:1105.6335](#)] [[INSPIRE](#)].
- [16] M. Alishahiha, E. Ó Colgáin and H. Yavartanoo, *Charged Black Branes with Hyperscaling Violating Factor*, *JHEP* **11** (2012) 137 [[arXiv:1209.3946](#)] [[INSPIRE](#)].
- [17] L. Li, *Hyperscaling Violating Solutions in Generalised EMD Theory*, *Phys. Lett. B* **767** (2017) 278 [[arXiv:1608.03247](#)] [[INSPIRE](#)].
- [18] B.W. Langley and P.W. Phillips, *Quantum Critical Diffusion and Thermodynamics in Lifshitz Holography*, [arXiv:1812.08164](#) [[INSPIRE](#)].
- [19] Z. Dayyani and A. Sheykhi, *Critical behavior of Lifshitz dilaton black holes*, *Phys. Rev. D* **98** (2018) 104026 [[arXiv:1805.00368](#)] [[INSPIRE](#)].
- [20] P. Hořava and E. Witten, *Eleven-dimensional supergravity on a manifold with boundary*, *Nucl. Phys. B* **475** (1996) 94 [[hep-th/9603142](#)] [[INSPIRE](#)].

- [21] A. Lukas, B.A. Ovrut, K.S. Stelle and D. Waldram, *The Universe as a domain wall*, *Phys. Rev. D* **59** (1999) 086001 [[hep-th/9803235](#)] [[INSPIRE](#)].
- [22] R. Bousso, *Charged Nariai black holes with a dilaton*, *Phys. Rev. D* **55** (1997) 3614 [[gr-qc/9608053](#)] [[INSPIRE](#)].
- [23] S.J. Poletti and D.L. Wiltshire, *The Global properties of static spherically symmetric charged dilaton space-times with a Liouville potential*, *Phys. Rev. D* **50** (1994) 7260 [Erratum *ibid.* **D 52** (1995) 3753] [[gr-qc/9407021](#)] [[INSPIRE](#)].
- [24] K.C.K. Chan, J.H. Horne and R.B. Mann, *Charged dilaton black holes with unusual asymptotics*, *Nucl. Phys. B* **447** (1995) 441 [[gr-qc/9502042](#)] [[INSPIRE](#)].
- [25] R.-G. Cai, J.-Y. Ji and K.-S. Soh, *Topological dilaton black holes*, *Phys. Rev. D* **57** (1998) 6547 [[gr-qc/9708063](#)] [[INSPIRE](#)].
- [26] R.-G. Cai and N. Ohta, *Surface counterterms and boundary stress energy tensors for asymptotically nonAnti-de Sitter spaces*, *Phys. Rev. D* **62** (2000) 024006 [[hep-th/9912013](#)] [[INSPIRE](#)].
- [27] T. Maki and K. Shiraishi, *Multi-black hole solutions in cosmological Einstein-Maxwell dilaton theory*, *Class. Quant. Grav.* **10** (1993) 2171 [[arXiv:1403.1320](#)] [[INSPIRE](#)].
- [28] E. Radu and R.J. Slagter, *Melvin solution with a dilaton potential*, *Class. Quant. Grav.* **21** (2004) 2379 [[gr-qc/0311075](#)] [[INSPIRE](#)].
- [29] M. Agop, E. Radu and R. Slagter, *On Ernst black holes with a dilaton potential*, *Mod. Phys. Lett. A* **20** (2005) 1077 [[INSPIRE](#)].
- [30] R.-G. Cai and Y.-Z. Zhang, *Black plane solutions in four-dimensional space-times*, *Phys. Rev. D* **54** (1996) 4891 [[gr-qc/9609065](#)] [[INSPIRE](#)].
- [31] L. Vanzo, *Black holes with unusual topology*, *Phys. Rev. D* **56** (1997) 6475 [[gr-qc/9705004](#)] [[INSPIRE](#)].
- [32] D. Birmingham, *Topological black holes in Anti-de Sitter space*, *Class. Quant. Grav.* **16** (1999) 1197 [[hep-th/9808032](#)] [[INSPIRE](#)].
- [33] C. Charmousis, *Dilaton space-times with a Liouville potential*, *Class. Quant. Grav.* **19** (2002) 83 [[hep-th/0107126](#)] [[INSPIRE](#)].
- [34] C. Charmousis, B. Gouteraux and J. Soda, *Einstein-Maxwell-Dilaton theories with a Liouville potential*, *Phys. Rev. D* **80** (2009) 024028 [[arXiv:0905.3337](#)] [[INSPIRE](#)].
- [35] S. Abdolrahimi and A.A. Shoom, *Geometric properties of static Einstein-Maxwell dilaton horizons with a Liouville potential*, *Phys. Rev. D* **83** (2011) 104023 [[arXiv:1103.1171](#)] [[INSPIRE](#)].
- [36] J. Tarrio, *Asymptotically Lifshitz Black Holes in Einstein-Maxwell-Dilaton Theories*, *Fortsch. Phys.* **60** (2012) 1098 [[arXiv:1201.5480](#)] [[INSPIRE](#)].
- [37] J.W. York Jr., *Role of conformal three geometry in the dynamics of gravitation*, *Phys. Rev. Lett.* **28** (1972) 1082 [[INSPIRE](#)].
- [38] G.W. Gibbons and S.W. Hawking, *Action Integrals and Partition Functions in Quantum Gravity*, *Phys. Rev. D* **15** (1977) 2752 [[INSPIRE](#)].
- [39] J.D. Brown and J.W. York Jr., *Quasilocal energy and conserved charges derived from the gravitational action*, *Phys. Rev. D* **47** (1993) 1407 [[gr-qc/9209012](#)] [[INSPIRE](#)].

- [40] K.C.K. Chan and R.B. Mann, *Static charged black holes in $(2+1)$ -dimensional dilaton gravity*, *Phys. Rev. D* **50** (1994) 6385 [Erratum *ibid.* **D 52** (1995) 2600] [[gr-qc/9404040](#)] [[INSPIRE](#)].
- [41] A. Anabalón and D. Astefanesei, *Black holes in ω -deformed gauged $N = 8$ supergravity*, *Phys. Lett. B* **732** (2014) 137 [[arXiv:1311.7459](#)] [[INSPIRE](#)].
- [42] A. Aceña, A. Anabalón, D. Astefanesei and R. Mann, *Hairy planar black holes in higher dimensions*, *JHEP* **01** (2014) 153 [[arXiv:1311.6065](#)] [[INSPIRE](#)].
- [43] A. Anabalón, D. Astefanesei, A. Gallerati and M. Trigiante, *Hairy Black Holes and Duality in an Extended Supergravity Model*, *JHEP* **04** (2018) 058 [[arXiv:1712.06971](#)] [[INSPIRE](#)].
- [44] R.C. Myers, *Stress tensors and Casimir energies in the AdS/CFT correspondence*, *Phys. Rev. D* **60** (1999) 046002 [[hep-th/9903203](#)] [[INSPIRE](#)].
- [45] S.W. Hawking and S.F. Ross, *Duality between electric and magnetic black holes*, *Phys. Rev. D* **52** (1995) 5865 [[hep-th/9504019](#)] [[INSPIRE](#)].
- [46] S.W. Hawking, *The path integral approach to quantum gravity*, in *General Relativity: An Einstein Centenary Survey*, *Advanced Series in Astrophysics and Cosmology*, volume 8, World Scientific (1993), pp. 163–206.
- [47] A. Chamblin, R. Emparan, C.V. Johnson and R.C. Myers, *Charged AdS black holes and catastrophic holography*, *Phys. Rev. D* **60** (1999) 064018 [[hep-th/9902170](#)] [[INSPIRE](#)].
- [48] A. Sheykhi, *Thermodynamics of charged topological dilaton black holes*, *Phys. Rev. D* **76** (2007) 124025 [[arXiv:0709.3619](#)] [[INSPIRE](#)].
- [49] S.H. Hendi, A. Sheykhi, S. Panahiyan and B. Eslam Panah, *Phase transition and thermodynamic geometry of Einstein-Maxwell-dilaton black holes*, *Phys. Rev. D* **92** (2015) 064028 [[arXiv:1509.08593](#)] [[INSPIRE](#)].
- [50] A. Chamblin, R. Emparan, C.V. Johnson and R.C. Myers, *Holography, thermodynamics and fluctuations of charged AdS black holes*, *Phys. Rev. D* **60** (1999) 104026 [[hep-th/9904197](#)] [[INSPIRE](#)].
- [51] D. Astefanesei, D. Choque, F. Gómez and R. Rojas, *Thermodynamically stable asymptotically flat hairy black holes with a dilaton potential*, *JHEP* **03** (2019) 205 [[arXiv:1901.01269](#)] [[INSPIRE](#)].